Mathematical Knowledge for Teaching Proof in Secondary Mathematics Teacher Preparation

A DISSERTATION

SUBMITTED TO THE FACULTY OF THE

UNIVERSITY OF MINNESOTA

BY

Foster Graif

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

Erin E. Baldinger, Advisor

July 2020

Acknowledgements

I would first like to thank my advisor, Dr. Erin Baldinger. I could not have done this without you. You have a gift for getting the best out of your advisees and that is a constant theme I hear from others who have worked with you. You pushed me every step of the way, provided valuable feedback, asked the right questions, and never let me believe I could not do this.

I would also like to thank the other members of my committee: Dr. Bodong Chen, Dr. Terry Wyberg, and Dr. Jonathan Rogness. You each brought a lens to this work that I greatly appreciate. To all of the other instructors I had at the University of Minnesota, you helped me grow in the areas of education and research in ways that I did not know were possible when I started here. I must also call out Dr. Matthew Campbell of West Virginia University, who played a key role in my development as a researcher and a writer.

To my family, particularly my parents, thank you for the support and encouragement you always offered me. To my friends, specifically those I met at the STEM center, thank you for the memories and lessons along the way. I am grateful for the times that I have been able to visit with Kyle, Nate, and James since their graduations.

Finally, I want to thank Alyssa. I cannot sum up what your love and support means to me. The joy you bring me in life gets me through every day. You encouraged me more than anyone, you never let me off the hook, and you checked my writing each time I asked. More than anything, you provided all of the support I needed at home. Whenever I needed time to work on this, you found a way to make it happen. You are everything. Thank you.

Dedication

To Kennedy and Finley—

I will always associate your coming into this world with this part of my life…

Never give up.

Abstract

Proof is considered foundational for mathematical understanding and has received increased attention in mathematics education over the last two decades. This mixed methods research study explores opportunities to develop mathematical knowledge for teaching proof during secondary mathematics teacher preparation. I used the mathematical knowledge for teaching proof framework (Lesseig, 2011) to develop a survey distributed to secondary mathematics methods instructors. This survey provided data pertaining to each instructor's learning goals around proof and instructional strategies they use to support opportunities to develop their teacher candidates' mathematical knowledge for teaching proof. In addition, interviews were conducted with five participants to provide further details on their survey responses and their instructional strategies. The responses related to learning goals were often focused on providing opportunities to develop common content knowledge for proof. The findings also indicated that factors such as educational level and departmental assignment were not associated with providing opportunities intended to support the development of mathematical knowledge for teaching proof. Instead, a teacher educator's approach towards proof in their methods course(s) is influenced by their view of what counts as proof. This view varied across all participants and is not unlike the variation discovered in previous research. Further research must explore reasonable expectations for what counts as proof at the secondary level and must identify specific strategies for drawing connections between common content knowledge for proof and the work of teaching.

Keywords: proof, proving, mathematical knowledge for teaching, teacher preparation

Table of Contents

List of Tables

List of Figures

List of Abbreviations

Chapter 1: Introduction

Proof has received increased attention in mathematics education (e.g., Ellis, Bieda, & Knuth, 2012; A. J. Stylianides, Bieda, & Morselli, 2016; Stylianides, Stylianides, & Weber, 2017) and is considered the foundation of mathematical understanding amongst researchers and educational leaders (e.g., Ball & Bass, 2003; G. J. Stylianides, 2008; Stylianides, Stylianides, & Weber, 2017). However, despite the attention devoted to proof, there exists a range of views on what counts as proof and what role proof plays in mathematics education (Stylianides, Stylianides, & Weber, 2017). While there also exists a range of views on the various roles that proof may play in the field of mathematics, there is widespread agreement that the *primary* role is to verify mathematical assertions as true (e.g., Alibert, 1988; Hanna, 1983; Kline, 1973; Volmink, 1990; Wilder, 1944). In the field of mathematics education, verification serves as just one potential role and there is much debate over the primary purpose proof serves, whether it be discovery, convincing, explaining, or problem-solving (de Villiers, 1990, 2012; Hanna, 2018; Hersh, 1993; Stylianides, Stylianides, & Weber, 2017). Although disagreements exist related to the exact purpose of proof, it is widely recognized that proof plays an important role in mathematics education (e.g., Ball & Bass, 2003; Hanna, 1990; Hersh, 1993; A. J. Stylianides, Bieda, & Morselli, 2016; Stylianides, Stylianides, & Weber, 2017).

Despite an existing understanding that proof is an important topic in mathematics education, evidence suggests that students and teachers have struggled with this topic (e.g., Healy & Hoyles, 2000; Knuth, 2002a, 2002b; Knuth, Choppin, & Bieda, 2009; Senk, 1989). However, there is also evidence that students are capable of proof if their teachers present proof in ways that are accessible to them (e.g., Reid, 2002; A. J. Stylianides, 2007). Teachers must be equipped with strategies to support students with the topic of proof and knowledge of these strategies lie within Shulman's (1986) notion of *pedagogical content knowledge* (PCK), which is content knowledge specifically related to the work of teaching.

The notion of PCK has been adapted to the field of mathematics teaching in the way of *mathematical knowledge for teaching* (MKT) (Ball, Hill, & Bass, 2005; Hill, Ball, & Schilling, 2008). This work has broken down the specific mathematical knowledge required for teaching from both a PCK standpoint and purely mathematical knowledge, which is subject matter knowledge (SMK). Lesseig (2011) further built on this work to develop a framework around mathematical knowledge for teaching proof (MKT-P). The MKT-P framework serves as a more specific tool to bring out examples of what mathematics educators must be able to know and do to support students towards proficiency in reasoning and proof.

The research around MKT-P is limited (e.g., Lessieg, 2016; Steele & Rogers, 2012; A. J. Stylianides & Ball, 2008) and has not yet focused on the development of MKT-P within teacher preparation. It is likely that pre-service teachers engage in proof quite often while enrolled in their content courses (e.g., Larsen & Zandieh, 2007; Mejia-Ramos, Fuller, Weber, Rhoads, & Samkoff, 2011). However, research suggests (e.g., Graham, Li, & Buck, 2000; Tatto, 2018; Wasserman, 2018) that connections to teaching in these content courses are often implicit, or even nonexistent. To capture the range of knowledge that MKT-P encompasses, methods courses may be the most appropriate setting to consider knowledge that is connected to teaching (Graham, Li, & Buck, 2000).

The current research landscape has not yet fully explored the topic of proof in secondary mathematics methods courses. This study attempts to gather evidence related to opportunities for teacher candidates to develop MKT-P in their teacher preparation programs, specifically in methods courses.

To gather this evidence, I used a survey distributed to secondary mathematics teacher educators. This survey collected personal and professional background information and asked participants about specific topics that may be covered in their methods course(s). Additionally, participants provided specific examples of learning goals and instructional strategies related to proof. Interview participants were selected from the survey respondents if they were identified as providing consistent opportunities to develop MKT-P or if they were identified as having the potential to offer a unique perspective on proof. These interviews provided further details on these opportunities and allowed for the contribution of additional specific examples. This data was used to advance the purpose of this study, which is to:

1. Identify ways in which secondary mathematics methods courses provide opportunities to support the development of MKT-P.
2. Identify instructional techniques that teacher educators use to provide opportunities to support the development of MKT-P.

This dissertation begins with a summary of the relevant literature used to motivate my study, presented in chapter two. Chapter three describes methods used to complete the study, including details related to survey design, data collection, and data analysis. Chapters four and five present the findings used to answer the research questions, split into survey findings and interview findings, respectively. I close this dissertation with chapter six, which discusses the contributions the findings bring to the literature outlined in chapter two and offers implications and directions for further research.

Chapter 2: Literature Review

In this chapter, I outline the relevant literature used to frame this study. I first outline literature related to proof and proving, including summaries around defining proof, the role of proof in mathematics and mathematics education, and current research around teachers' and students' experience with proof. I then summarize the development of the mathematical knowledge for teaching framework and how it was used to develop a framework around mathematical knowledge for teaching proof (Lessieg, 2011). I close with literature related to teacher education, in which there is minimal work done in relation to the development of mathematical knowledge for teaching proof. This gap in the literature serves as the motivation for this study and prompted the development of my research questions, which are presented at the end of this chapter.

Proof and Proving

Proof has received increased attention in mathematics education, both in research and practice (e.g., Ellis, Bieda, & Knuth, 2012; A. J. Stylianides, Bieda, & Morselli, 2016; Stylianides, Stylianides, & Weber, 2017) throughout the last two decades. The National Council of Teachers of Mathematics (NCTM) Principles and Standards (2000) includes *reasoning and proof* as one of its five process standards and the Common Core State Standards for Mathematics (CCSSM) (2010) also advocates for proof to be embedded as a tool for learning at all levels of mathematics. This increased emphasis comes from a belief amongst researchers and educational leaders that proof lies as the foundation of mathematical understanding (e.g., Ball & Bass, 2003; G. J. Stylianides, 2008; Stylianides, Stylianides, & Weber, 2017). While proof has historically been viewed as a topic in a secondary geometry course (e.g., Knuth, 2002a, 2002b; Manaster, 1998;

Wu, 1996), these standards documents and further research (e.g., G. J. Stylianides, 2008) have identified the opportunity to use proof as a tool to study and learn mathematics at all levels. However, questions remain as to whether this emphasis is being executed in the way the research is advocating for. In this section, I will detail the various views, roles, and perspectives of proof in mathematics and mathematics education, followed by a summary of research related to student and teacher conceptions of proof.

Defining proof. Researchers within the field of mathematics education have defined proof in a variety of different ways (e.g., Harel & Sowder, 2007; Healy & Hoyles, 2001; Knuth, 2002b; A. J. Stylianides, 2007). Prior to engaging in this study, it was important to consider the range of definitions that have been commonly referred to in the literature in order to better understand the different views of proof held by mathematics educators in the United States. For some, proof is defined from a purely mathematical standpoint, while for others, proof is defined by taking into account the additional roles that proof can play in education. Examples of these views are outlined below.

Related to a view of proof from a mathematical standpoint, Knuth (2002b) viewed proof as "a deductive argument that shows why a statement is true by utilizing other mathematical results and/or insight into the mathematical structure involved in a statement" (p. 86). Griffiths (2000) defined proof as "a formal and logical line of reasoning that begins with a set of axioms and moves through logical steps to a conclusion" (p. 2). Lastly, Hersh (1993) succinctly described a mathematician's view of proof as a "convincing argument, as judged by qualified judges" (p. 389).

Other researchers have considered the role of conviction when defining proof, and how the nature of such conviction may be subjective. Harel and Sowder (2007) state that proof "is what establishes truth for a person or community" (p. 806). Balacheff (1988) defined proof as "an explanation which is accepted by a community at a given time" (p. 285). These views imply that what is accepted as proof in one community may not be convincing enough for another community. While Hersh's (1993) definition above also pertains to proof reaching a certain level of conviction, he differs from these more context-dependent views by stating that this conviction must be judged "by qualified judges" (p. 389), although he does not specify a definition of "qualified."

A. J. Stylianides (2007) took these views of proof and developed a definition that he viewed as appropriate for mathematics education across grade levels.

> *Proof* is a *mathematical argument*, a connected sequence of assertions for or against a mathematical claim, with the following characteristics:
>
> 1) It uses statements accepted by the classroom community (*set of accepted statements*) that are true and available without further justification;
>
> 2) It employs forms of reasoning (*modes of argumentation*) that are valid and known to, or within the conceptual reach of the classroom community; and
>
> 3) It is communicated with forms of expression (*modes of argument representation*) that are appropriate and known to, or within the conceptual reach of the classroom community (A.J. Stylianides, 2007, p. 291, emphasis in original)

While the A. J. Stylianides (2007) definition of proof has been used quite often in mathematics education research (e.g., Bieda, 2010; Lesseig, 2011; A. J. Stylianides & Ball, 2008) and accurately captures the roles that proof often plays in a classroom, in this paper I aim to use a definition of proof that is not as dependent on the classroom context. It is important to define proof in a way that is accurate and relevant for mathematicians, while also taking into account the key roles proof can play in education as outlined by the Stylianides definition. Therefore, in this paper I view proof “as careful, critical, reasoning that leads to reliable conclusions and deeper understanding (Hersh, 2009)” (Lesseig, 2011, p. 8). A reliable conclusion is considered to be a conclusion in which it is established that there are no exceptions in the given domain and the logic structure used to form such a conclusion is mathematically sound. In this way, a reliable conclusion affirms that a mathematical statement is true. Additionally, when the reasoning leads to deeper understanding, the proof is then also used as a tool to demonstrate *why* a mathematical statement is true. This element is particularly important when considering how proof can be used as a tool in mathematics education to develop mathematical understanding.

A. J. Stylianides (2007) takes the time to differentiate the process of *proving* from the product of a *proof* itself. He clarifies that the process and the product are distinct and that one can engage with the proving process without necessarily producing a proof. With colleagues, he defines proving broadly as “the activity of searching for a proof” (Stylianides, Stylianides, & Weber, 2017, p. 239). Lesseig (2011) provides the following detailed definition:

> An activity associated with the search for a proof that includes the formulation of arguments and proofs as well as empirical explorations to generate conjectures and develop possible ideas for the formulation of arguments. (p. 9)

I agree with Stylianides that it is important to distinguish the process from the product. Both the process and the product are important pieces in mathematics (Schoenfeld, 1994). Throughout this paper, I follow Stylianides and use “proof” when referring to the product and “proving” when referring to the proving process. I take up Lesseig’s definition of proving in this study.

Burton (1984) identified four phases within the proving process: (1) specializing (or exploring), (2) conjecturing, (3) generalizing, and (4) convincing (or justifying). This proving process, when completed fully, leads to a proof which fits my definition above. The process of exploring and forming conjectures is tied to careful and critical reasoning. Patterns are noticed and the person engaging in the proving process is using reasoning with these patterns to form a conjecture. Additionally, as the proving process moves towards a generalization, the careful and critical reasoning continues to ensure that all cases are accounted for and that any exceptions have been considered, which ultimately allows the proof to produce a reliable conclusion. Lastly, the proving process, and the proof itself, has provided an opportunity to develop a deeper understanding of the underlying mathematics at play. By engaging in the proving process, a person has not only determined that something is true, but *when* it is true and *why* it is true. To reiterate, it is important to note that one can engage in phases of the proving process without producing a proof.

Now that I have outlined the various views of proof that are held from both a pure mathematical perspective and from a mathematics education perspective, I will describe the various roles and perspectives held around proof in mathematics and mathematics education.

Role of proof within mathematics. Mathematicians view proof as a central construct in mathematical thinking and indispensable to their work (e.g., Hanna, 1990; Kitcher, 1984). Traditionally, the function or role of proof within mathematics is viewed in terms of verification by many mathematicians (e.g., Alibert, 1988; Hanna, 1983; Kline, 1973; Volmink, 1990; Wilder, 1944). Proof is used to remove any doubt and convince others that no exceptions remain to the statement under consideration (Lakatos, 1976). Hersh (1993) also asserts that the purpose of proof within mathematics is to convince. For many mathematicians, convincing and verifying are processes and functions that go hand-in-hand with proof (e.g., Hanna, 1989; Volmink, 1990).

This notion is challenged by de Villiers (1990, 2012), stating that proof is not necessarily a prerequisite for conviction. Instead, he posits that conviction is more often a prerequisite for proof. Mathematicians often begin the proving process because they themselves are already convinced. It is often the inductive process which confirms something for a mathematician, while the deductive process is used to reverse the doubts of others. He argues that verification is simply one purpose of proof in mathematics, perhaps even the least important role. Additional roles include: (1) explanation, (2) systematization, (3) discovery, and (4) communication. While there is much less consensus around the importance of these roles within the mathematics community when compared to the role of validity, these additional roles are very relevant to the role of proof and the proving process in mathematics education, which is outlined in the next section.

Perspectives of proof within mathematics education. While there is a range of views on the role of proof and proving in mathematics, this range of views becomes even wider when moved to the context of mathematics education. The purposes that proof and proving serve in mathematics remain when in the context of mathematics education, but additional purposes come out. Stylianides, Stylianides, and Weber (2017) outlined these additional purposes and categorized them using three research perspectives: Proving as problem-solving, proving as convincing, and proving as a socially embedded activity.

The perspective of proving as problem-solving focuses on the proving process rather than the proof itself (Stylianides, Stylianides, & Weber, 2017). This focus emphasizes student engagement in reasoning, justification, and argumentation and the proving process can serve as a vehicle to develop those important skills. The development of a formal proof, in this case, is not a requirement for developing those skills. This perspective may serve as one potential explanation for why students at the secondary level and beyond have difficulty writing proofs (Healy & Hoyles, 2000; Knuth, Choppin, & Bieda, 2009; Senk, 1989), as the emphasis on problem solving may detract from an emphasis on what the proof should look like. Additionally, while this perspective puts the emphasis on the development of reasoning skills that can lead to informal arguments, students often have difficulty transitioning these informal arguments to a proof, which is considered a more formal argument (Duval, 2007). Research from this perspective has found that students and teachers may have difficulty distinguishing between proofs and invalid arguments (Alcock & Weber, 2005; Knuth, 2002a; Selden & Selden, 2003). In this scenario, it is possible that the strong emphasis on the development of reasoning and justification skills may lead teachers and students to consider informal arguments to be just as educationally appropriate as formal proofs. In this sense, what one may consider proof in this scenario may not always meet the standard that fits my definition of proof.

The proving as convincing perspective is not unlike the convincing perspective that many hold within the field of mathematics (e.g., Hanna, 1989; Volmink, 1990). A key difference when moving into mathematics education is the range of standards of conviction. While the goal of this perspective is to apply the same standard of conviction to mathematics education as is held in mathematics, this does not often happen in practice. Students and teachers are often convinced by empirical arguments and do not necessarily require a proof to be convinced. Morris (2007) found that 41% of the teacher candidates in her study were convinced through empirical arguments that a generalization over an infinite set was true. The opposite effect can also happen, as the same study found that only 62% of participants were convinced of the same assertion when presented with a valid argument that established that generalization with absolute certainty. One possible explanation is that these valid proofs can be more difficult to follow than empirical examples, so the participants may not have fully understood the conclusion of the proof. Lastly, this perspective also places emphasis on the product, meaning that the focus is on the students' interpretation of what an acceptable proof looks like rather than a focus on the reasoning behind it (Stylianides, Stylianides, & Weber, 2017). This may result in students or teachers accepting a proof based on what it looks like rather than the logic structure behind it, as Knuth (2002a) found many participants were convinced that an argument by induction was valid because they knew induction was a valid method rather than because they understood how the method was used in the particular argument at play.

The perspective of proof as a socially embedded activity places the emphasis on a broader mathematical activity that proof is embedded in (Stylianides, Stylianides, & Weber, 2017). In this case, there is less emphasis on proof and proving itself, meaning that conviction and understanding can be underemphasized. The role of proof within a classroom may be socially negotiated (Alibert & Thomas, 1991; Fukawa-Connelly, 2012). For example, what an acceptable proof looks like may be socially negotiated, such as an understanding that proofs should be in a two-column format. This may lead students to reject valid proofs that appear in other formats. Stylianides, Stylianides, and Weber (2017) note that the body of research around this perspective is still developing, and thus does not have common, widely used constructs. More work is left to be done in this area, but the impact of these socially-negotiated norms cannot be ignored.

In addition to these three perspectives outlined by Stylianides, Stylianides, and Weber (2017), proof can also serve as a powerful tool for explanation of mathematical ideas. While Hersh (1993) agrees that the purpose of proof in general mathematics is to convince others that a mathematical statement is true, he also posits that "students are all too easily convinced" (p. 396). Hersh insists that teachers should keep the focus on explaining rather than convincing, diving deeper into concepts that students already know are true and learn *why* they are true. Hersh argues that proofs are often presented to students in a rigorous and inarguable fashion, putting so much emphasis on demonstrating that a theorem is airtight, when instead he posits that students would rather learn the critical details behind the meaning of the theorem. Schoenfeld (1994) also posits that teachers often ask students to prove concepts that may be considered trivial or obvious. At the very least, students are already aware that these concepts have been proven before. By approaching proof in this manner, I argue that teachers take the curiosity aspect away from students.

While I acknowledge that proof holds a variety of purposes in mathematics education, I agree with Hersh (1993) that proof can serve a valuable role in explaining mathematics in education. I take the stance that proof can help to build relational understanding (Skemp, 1978), as Hanna (2018) views the explanatory power of proofs to be in connecting mathematical concepts to already established knowledge. Teachers can use proof as a teaching tool for helping students understand the “why” behind mathematical relationships (Reid & Vargas, 2017). This role of proof exhibits how it can be used as a powerful tool at all levels of mathematics. However, regardless of whatever role one may feel that proof should play in education, the existence of these multiple roles establishes the importance of proof in mathematics education. This importance warrants further research related to teacher knowledge of proof, as evidence suggests that students and teachers have struggled with proof (e.g., Knuth, 2002a, 2002b; Knuth, Choppin, & Bieda, 2009; Healy & Hoyles, 2000; Senk, 1989). The next two sections summarize existing literature related to teachers’ and students’ understandings of proof.

Teachers and proof. Despite the call from educational research (e.g., Ball & Bass, 2003; Hanna, 1990; Hersh, 1993; A. J. Stylianides, Bieda, & Morselli, 2016; Stylianides, Stylianides, & Weber, 2017) for proof to be used as a tool to learn mathematics at a deep, conceptual level throughout all grades, numerous studies suggest that many teachers still struggle with proof and have difficulty in identifying what counts as proof (e.g., Bieda, 2010; Harel, 2001; Morris, 2002, 2007; Simon & Blume, 1996; Sears, 2012). One particular study by Knuth (2002b) explored teachers’ conceptions about the nature and role of proof in school mathematics. He interviewed 17 experienced secondary mathematics teachers and found that they all viewed proof similarly, describing their collective views as “an argument that conclusively demonstrates the truth of a statement” (p. 71), keeping it in line with the verification purpose commonly agreed upon in the field of mathematics. However, when specifically discussing proof in school mathematics, much more disagreement existed between the teachers. While they all agreed that informal proofs, which were described as explanations or justifications, may have a role in school mathematics, many viewed formal proof as only having a place with students in high level mathematics courses. Additionally, all but three did not consider proof to be a central idea throughout secondary mathematics, despite calls to incorporate proof as early as the elementary grades. These findings were primarily due to the teachers’ idea that proof is a difficult concept for students.

This idea of proof as a difficult concept likely influences teachers’ decisions to limit proof to a topic in a secondary geometry course. Knuth (2002b) asked teachers if they included proof in other courses and found that many did not. Knuth pressed further and found some examples of teachers justifying theorems or formulas in other courses such as algebra. However, the teachers classified these examples more as derivations and one teacher stated:

> I probably would have avoided using proof because ... my experience with the kids is that they would shut down when you use the word proof. They’re gone. Shades down. (Knuth, 2002b, p. 76)

Quotes such as this were common among Knuth’s teachers, implying that they have low expectations regarding their students’ ability to handle proofs.

Students and proof. Despite the low expectations some teachers have around students and proof, research suggests that students are capable of engaging in at least some aspects of the proving process, even at the elementary level (e.g., Ball & Bass, 2003; Lampert, 2001; Reid, 2002; A. J. Stylianides, 2007). One particular study by Maher and Martino (1996) followed the development of a child's reasoning and argumentation skills from first through fifth grade. One task was the Towers Problem, which the student worked on starting in third grade. The student and her classmates were tasked with determining how many different towers of four (and later five) cubes tall could be made selecting from red and blue cubes. Throughout the three years of engaging with this problem, the researchers saw a progression from trial-and-error and guess-and-check strategies to pictorial representations of a generic example, and later to a formal proof by cases. The researchers note that this is likely due to the mathematical learning environment that was prevalent during these five years, in which exploration was encouraged and valued on a consistent basis. This research, and others (e.g., Reid, 2002; A. J. Stylianides, 2007), suggests that students can be capable of engaging in the proving process at a young age as long as they are in an environment in which opportunities are given to explore and their less formal justifications are validated. Teachers require the knowledge and resources necessary to create these types of environments.

Inaccurate representations of proof and the proving process creates a roadblock to students understanding what proof is and what purpose it serves. Herbst and Brach (2006) found that when students were given a task in an open-ended format explaining a general case of a mathematical property, they did not consider themselves to be doing a proof, but when given the same mathematical concept in the format of a two-column proof, they felt otherwise. This implies that the students felt that a proof must look a certain way and that they lacked an appreciation for what a general case can offer to learning and understanding mathematical relationships.

The students in the Herbst and Brach (2006) study above not only have a limited idea of what a proof should look like, but they are also missing additional elements of the proving process. Students need opportunities to explore and make conjectures based on patterns and previous knowledge. These opportunities are part of the proving process as well and gives the student ownership in the process because it is something they discovered. This is also related to the discovery purpose described by de Villiers (1990, 2012).

As another example of students having a limited idea of proof, Hersh (2009) asked his upper-division undergraduate mathematics students to describe this distinction. He found that students viewed reasoning as something that occurred as early as elementary level mathematics, but proof was not something that they engaged with until 10^{th} grade at the earliest. Hersh's students expressed experience with proof at a young age and other research has given examples of elementary students engaging in reasoning and oftentimes reasoning which led to reliable conclusions and deeper understandings (e.g., Maher & Martino, 1996; Reid, 2002; A. J. Stylianides, 2007). In this sense, students engage in the proving process at a young age and it should be recognized as such.

G. J. Stylianides (2008, 2010) developed the reasoning-and-proving framework as a tool for teachers to engage students in the proving process. This framework includes three components (Figure 2.1). The mathematical component considers generalization and argumentation. Teachers should provide opportunities for students to engage with patterns. As students observe regularity within a pattern, they can be better prepared to make a conjecture. Looking for patterns and making conjectures support students in the process of developing a generalization. At that point, students are prepared to make an argument. It is important to point out that these arguments can take the form of proofs or non-proofs. Empirical arguments are still appropriate ways for students to engage in the proving process and make sense of mathematics. In summary, beginning with tasks that allows students to look at patterns provides a scaffold to support students as they move along the continuum towards a formal proof.

	Reasoning-and-proving			
Mathematical Component	Making Mathematical Generalizations		Providing Support to Mathematical Claims	
	Identifying a Pattern	Making a Conjecture	Providing a Proof	Providing a Non-proof Argument
	• Plausible Pattern • Definite Pattern	• Conjecture	• Generic Example • Demonstration	• Empirical Argument • Rationale
Psychological Component	What is the solver's perception of the mathematical nature of a pattern / conjecture / proof / non-proof argument?			
Pedagogical Component	How does the mathematical nature of a pattern / conjecture / proof / non-proof argument compare with the solver's perception of this nature? How can the mathematical nature of a pattern / conjecture / proof / non-proof argument become transparent to the solver?			

Figure 2.1. Reasoning-and-proving framework.
Note. Reprinted from "An analytic framework of reasoning-and-proving"; by G. J. Stylianides, 2008, *For the Learning of Mathematics, 28*(1), p. 10.

While the mathematical component considers patterns, conjectures, proofs, and non-proof arguments, the psychological component focuses on the students' conception of those four considerations. It is the teacher's role to question the learner to try and uncover any misunderstandings the student may hold and diagnose how where they are at on the continuum and how they are thinking about the task. The pedagogical component builds on the other two components. After the teacher has considered the students' conceptions around the task, they can make moves to bridge students' knowledge. For example, a student may be in the non-proof argument phase where they are satisfied with an empirical argument. In this case, the teacher can provide a portion of the task where the initial pattern fails. As a result, the student would have a motivation to consider more than just a few cases. When considering what knowledge and tools teachers require to support students in the area of proof, these pedagogical components could be taken into account as a way to assist students in progressing along the mathematical components in the continuum.

The research above outlined how students and teachers have struggled with proof, and also summarized the reasoning-and-proving framework as a tool to engage students in proof and move them along the reasoning-and-proving continuum. I now move to a discussion of the type of knowledge that teachers must have to experience more success with proof and to support their students in proof.

Mathematical Knowledge for Teaching (MKT)

The term pedagogical content knowledge (PCK) was first coined by Lee Shulman (1986) when describing the knowledge that all teachers must possess to be successful in the profession. While Shulman expects teachers to hold subject matter knowledge, it may be necessary for that knowledge to be even deeper than the subject matter knowledge held by other experts in a teacher's particular field:

> The teacher need not only understand *that* something is so; the teacher must further understand *why* it is so, on what grounds its warrant can be asserted, and under what circumstances our belief in its justification can be weakened or even denied. (Shulman, 1986, p. 9, emphasis in original)

Additionally, pedagogical content knowledge is the content knowledge that is specifically needed for teaching. This can include ways of representing the subject to make it comprehensible to others. This may include helpful illustrations, analogies, or examples. Pedagogical content knowledge also includes an understanding of what makes learning easy or difficult with certain topics or subjects. This may include knowledge of common errors or preconceptions built on flawed reasoning. Teachers must have the knowledge to diagnose these errors or preconceptions and also have strategies on hand to treat them.

Building on Shulman's concept of PCK, Ball, Hill, and colleagues (Ball, Hill, & Bass, 2005; Hill, Ball, & Schilling, 2008) worked to develop a framework specifically for mathematics. This framework has come to be called mathematical knowledge for teaching (MKT). The subdomains within this framework are subject matter knowledge and pedagogical content knowledge. The distinctions here elaborate on Shulman's ideas, while at the same time being similar to his categories of subject matter knowledge and pedagogical content knowledge Figure 2.2 displays the MKT framework.

Within subject matter knowledge, common content knowledge (CCK) is the domain which uses knowledge of mathematics within teaching in ways similar to how it is used in other professions that also use mathematics. Ball and colleagues (Ball, Hill, & Bass, 2005; Hill, Ball, & Schilling, 2008) state that specialized content knowledge (SCK) is the knowledge required to understand unusual solution methods to problems and ways to represent mathematical ideas. Lastly, teachers should hold knowledge at the mathematical horizon. This is knowledge of the mathematics that follows or could follow the mathematics being taught. It is important to note that none of this knowledge requires knowledge of students or teaching.

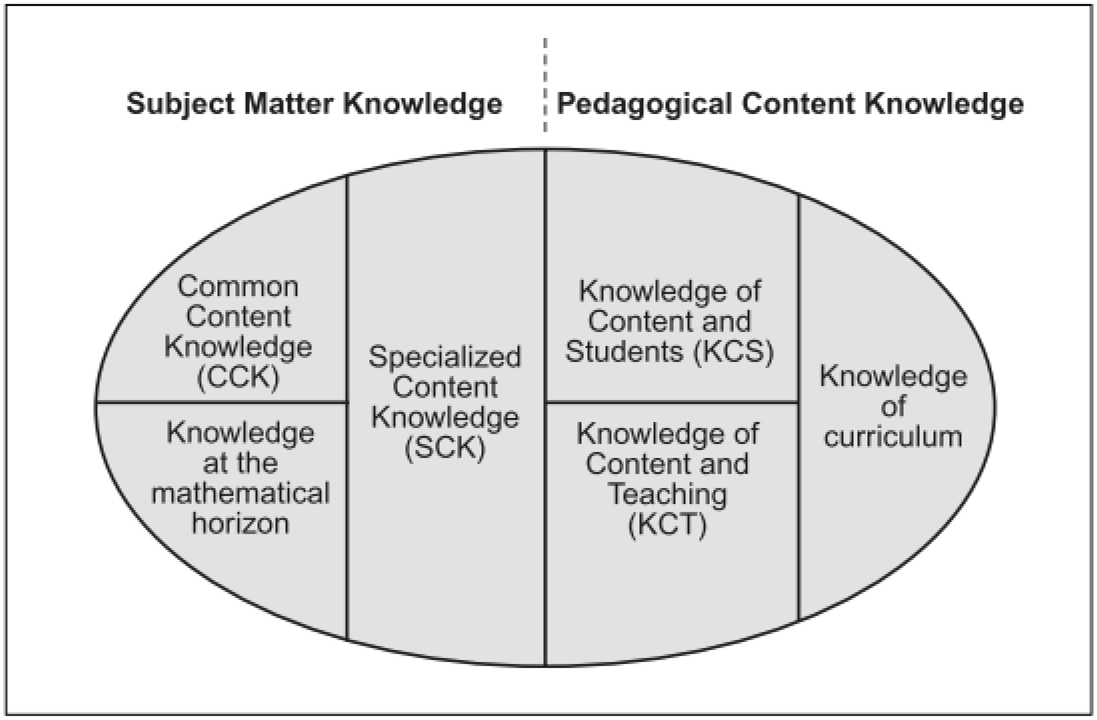

Figure 2.2. Mathematical knowledge for teaching framework.
Note. Reprinted from "Unpacking Pedagogical Content Knowledge: Conceptualizing and Measuring Teachers' Topic-Specific Knowledge of Students"; by Hill, H.C., Ball, D.L., & Schilling, S.G., 2008, *Journal for Research in Mathematics Education, 39*(4), p. 377.

Within pedagogical content knowledge, there is a distinction between knowledge of content and teaching and knowledge of content and students. Knowledge of content and students (KCS) is focused on a teacher's understanding of how their students learn. Teachers should be able to anticipate what students will struggle with and what will be accessible to them. Within knowledge of content and teaching (KCT), teachers must have knowledge of the mathematics but must also know how to use that knowledge to build on student thinking or present mathematical ideas in ways that are relevant and accessible to the student. Lastly, knowledge of curriculum is similar to Shulman's curricular knowledge, meaning that teachers should be aware of and be able to use the various tools and resources available to them to teach mathematics in a variety of ways.

It must be noted that the MKT framework is not considered a universal framework. It has faced criticism related to how well it encompasses the mathematical knowledge that secondary mathematics teachers must hold. Speer, King, and Howell (2014) specifically critique the distinction between CCK and SCK. Ball and colleagues defined common content knowledge as mathematical knowledge that "any well-educated adult should have" (Ball, Hill, & Bass, 2005, p. 22) and specialized content knowledge as mathematical knowledge that only teachers need to know. In studies attempting to identify this distinction, research has been conducted primarily in elementary settings (e.g., Ball, Thames, & Phelps 2008). Speer, King, and Howell argue that identifying the distinction between CCK and SCK in this manner is not appropriate when considering MKT at the secondary level.

A secondary mathematics teacher will require mathematical knowledge that is beyond the knowledge held by an average mathematically literate citizen (Speer, King, & Howell, 2014). As one example from the research, recognizing the mathematical accuracy of a definition may be considered part of SCK for an elementary teacher but for a secondary mathematics teacher, who has more mathematics education, this would be considered part of CCK. While there may be a distinction between CCK and SCK, using the definitions from Ball, Hill, and Bass (2005) makes it difficult to identify this distinction for a secondary teacher. Instead, it may be more appropriate to view the distinction between CCK and SCK through the specific lens of mathematical content knowledge required for the work of teaching. For example, Ball and colleagues later defined CCK as "mathematical knowledge and skill used in settings other than teaching" (Ball, Thames, & Phelps. 2008, p. 399). SCK would then be the mathematical skill and knowledge unique to teaching. This may be a more helpful distinction for the secondary mathematics teaching community; however, the lack of research related to identifying this distinction outside of elementary settings still makes it difficult to define the line between CCK and SCK.

While I acknowledge that the lines between CCK and SCK are not easy to define for many areas of mathematics, I take the stance that CCK and SCK are distinct bases of knowledge for secondary mathematics teachers. As such, I move forward with adapting the MKT model described in this section (Ball, Hill, & Bass, 2005; Hill, Ball, & Schilling, 2008) for this study. While the distinctions between CCK and SCK are not always easy to define and more work should be done in this area, I firmly believe that there is subject matter knowledge that mathematics teachers must hold that is not necessarily required by professional mathematicians (e.g., Dick, 2017; Pino-Fan, Godino, Font, & Castro, 2013). For example, a mathematician may be satisfied with coming to a solution using one method, but it would be helpful to the work of teaching if an educator could come to that same solution from using multiple pathways. As such, SCK may not simply be mathematical knowledge “beyond” that held by the average adult but may be mathematical knowledge specific for teaching and beyond or distinct from that required by professional mathematicians. Viewing the distinction in this manner is consistent with the view taken in the MKT-P framework (Lesseig, 2011) discussed below and is the view most appropriate for my study.

Mathematical Knowledge for Teaching Proof (MKT-P)

While there is a substantial research base around MKT (e.g., Hill et al., 2008; Hill, Rowan, & Ball, 2005; Morris, Hiebert, & Spitzer, 2009; Silverman & Thompson, 2008), there has been limited research around what this knowledge may consist of in the specific domain of proof. This field was first explored in detail by A. J. Stylianides and Ball (2008) and then built on in the work of Kristin Lesseig (2011, 2016). Since that time, other researchers have also built frameworks related to MKT-P (e.g., Steele & Rogers, 2012). In this section, I will outline the mathematical knowledge for teaching proof framework[1] developed by Lesseig. I have chosen to use Lesseig's framework because, as with the original MKT framework (Ball, Hill, & Bass, 2005; Hill, Ball, & Schilling, 2008), this framework is split into subject matter knowledge and pedagogical content knowledge. This contrasts with the Steele and Rogers (2012) framework, which lists components of proof subject matter knowledge and not pedagogical content knowledge, and also does not draw a distinction between CCK and SCK within subject matter knowledge.

Subject matter knowledge. According to the MKT framework, subject matter knowledge has three domains: (1) common content knowledge (CCK), (2) specialized content knowledge (SCK), and (3) knowledge at the mathematical horizon. The MKT-P framework developed by Lesseig (2011) did not explore the domain of knowledge at the mathematical horizon. While this is an area for consideration in research around MKT-P, this study directly uses Lesseig's framework and will not consider this domain. Figure 2.3 summarizes the domains of subject matter knowledge for teaching proof and I follow with a detailed summary related to CCK and SCK for proof.

[1] Some elements of the Lesseig (2011) framework may be more explicitly related to the process of proving rather than the construction of a proof itself. As such, it could be considered more broadly to be a framework around mathematical knowledge for teaching proof and proving. To remain true to the title given by Lesseig, I refer to the framework in this paper as mathematical knowledge for teaching proof, even when some elements of the framework are more related to proving.

Subject Matter Knowledge for Teaching Proof	
Common Content Knowledge for Proof	**Specialized Content Knowledge for Proof**
Ability to construct valid proof • Understand and use stated assumptions, definitions & previously established results • Build a logical progression of statements • Analyze situations by cases • Use counterexamples *Essential proof understandings* • A theorem has no exceptions • A proof must be general • The validity of a proof depends on its logic structure *Functions of proof* • To establish the validity of a statement • To provide insight into why the statement is true • To build mathematical understanding • To communicate math reasoning	*Explicit understanding of proof components* • Accepted statements o Range of useful definitions or theorems o Role of language and defined terms • Modes of representation o Variety of visual, symbolic, methods to provide a general argument • Modes of argumentation o Recognize when methods such as proof by exhaustion, counter-example are sufficient o Distinguish between empirical and deductive arguments *Knowledge of situations for proving* • Knowledge of proving tasks o Infinite vs. infinite number of cases o Existence vs. universal proof o To refute vs. verify a claim • Knowledge of relationship between proving tasks and proof activity o Which tasks provoke strategies such as systematic listing, creating generic examples or counter-examples

Figure 2.3. Subject matter knowledge for teaching proof.
Note. Reprinted from “Mathematical Knowledge for Teaching Proof”; by Lesseig, K., 2011, Unpublished Doctoral Dissertation, Oregon State University, p. 26.

Common content knowledge. Through a review of literature related to proof in mathematics education, Lesseig (2011) identified three elements that would make up a teacher’s common content knowledge for proof: (1) ability to construct a valid proof; (2) essential proof understandings; and (3) functions of proof.

Ability to construct a proof. At a basic level, teachers need to know how to do something before they can teach somebody else how to do it and the ability to construct a proof is no exception. While the “look” of a mathematical proof may be an area in which disagreement exists, there is a general consensus among mathematics educators that proof involves “the validation of propositions by application of specified rules, as of induction or deduction, to assumptions, axioms and sequentially derived conclusions” (Lesseig, 2011, pp. 18-19). Teachers should therefore have a basic understanding of what is required in a proof and how to validate claims with the axioms or assumptions at their disposal. Further expectations for students are also laid out in the Common Core State Standards for Mathematics (CCSSM) in which it is stated that “mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments” (NGA & CCSSO, 2010, p. 6). With these expectations in place for students, Lesseig argues that it makes sense that these same understandings must be present in a teacher’s common content knowledge.

Essential proof understandings. Lesseig (2011) identifies three critical proof understandings for teachers: (1) a theorem has no exceptions; (2) a proof must be general; (3) the validity of a proof depends on its logic structure. To summarize, there is an expected certainty that should result from a proof and any theorem produced by a proof should cover every case. This certainty is unique to the field of mathematics (Schoenfeld, 1994). This is accomplished through the generality of the argument, Lesseig’s second element. Lastly, the third element, related to logic structure, states that the logic underlying a proof serves as the authority of a mathematical concept, not widely agreed upon or assumed facts. A teacher’s understanding of proof in this nature is similar to Shulman’s assertion that “the teacher need not only understand *that* something is so; the teacher must further understand *why* it is so” (1986, p. 9, emphasis in original). When teachers understand the nature of a proof through its logic structure, they have gone beyond the facts of a particular concept and have come to an understanding of how that concept was established.

Functions of proof. Lesseig (2011) identifies four functions of proof: (1) to establish the validity of a statement; (2) to provide insight into why the statement is true; (3) to build mathematical understanding; and (4) to communicate mathematical reasoning. Establishing the validity of a statement is a primary function of proof in the field of mathematics (e.g., Alibert, 1988; Hanna, 1983; Kline, 1973; Volmink, 1990; Wilder, 1944). While this is certainly a function that teachers should have an understanding of, and Lesseig recognizes it as such, there remain additional functions that are essential in mathematics education. While recognizing the role of proof within mathematics is to convince, Hersh (1993) claims that proof should primarily be used as a tool to explain in mathematics education and de Villiers (1990) lists explanation as a function in mathematics. This relates to the next elements identified by Lesseig in this domain: to provide insight into why a statement is true and to build mathematical understanding. Additionally, using proof as a tool to build mathematical understanding allows students the opportunity to construct mathematical knowledge rather than receive mathematical knowledge as a collection of facts and procedures, promoting Skemp's (1978) concept of relational understanding. Lastly, when students build these arguments, they engage in Lesseig's fourth function of proof, which is to communicate mathematical reasoning. This is also a function specified by de Villiers.

Specialized content knowledge. In working to identify specialized content knowledge for teaching proof, Lesseig (2011) relied on the work of A. J. Stylianides and

Ball (2008) to identify two key elements: (1) explicit understanding of proof components; and (2) knowledge of situations for proving.

Explicit understanding of proof components. A. J. Stylianides and Ball (2008) proposed two categories of teachers' specialized content knowledge for proof, the first of which was the logico-linguistic aspects of proof. Lesseig (2011) connected this to the proof components identified by A. J. Stylianides (2007): (1) accepted statements; (2) modes of representation; and (3) modes of argumentation. Teachers need this knowledge of the connection between definitions and properties and how they can be interchangeable in certain circumstances. Regarding modes of representation and argumentation, teachers must not limit themselves to one particular format such as a two-column proof. It is also critical that teachers recognize as proof justifications of algorithms such as the quadratic formula and that pictorial models of generic examples, such as those by the elementary student in the Maher and Martino (1996) study, are also appropriate modes of argumentation, even if they sometimes fall short of a formal proof. A mathematician will often choose the most clear and direct path to prove a statement but a teacher should choose a path that is accessible for their students' developmental levels. Additionally, knowing more than one way to prove a concept provides multiple ways to make a proof accessible to students based on their learning style.

Knowledge of situations for proving. A. J. Stylianides and Ball (2008) highlighted the importance of teachers having knowledge of the various proving tasks available to them, and also the knowledge of how those tasks are related to the proving activities and strategies themselves, such as systematic listing, the use of generic examples, and the use of counter-examples. For example, common content knowledge would be knowing the specific counter-example to use to disprove a statement. However, a teacher may recognize the specific counter-example as difficult to identify and would consider what tasks to use to support students in identifying the counter-example. As teachers select tasks, they are to consider two criteria. The first consideration involves how many cases are required to complete the proving task. This could range from a single case to an infinite number of cases. Additionally, teachers should consider whether the purpose of the task is to verify or refute a statement. The type of task that is selected should impact the strategies that are used to complete the task, further emphasizing that teachers should not limit themselves to one particular structure or format.

Pedagogical content knowledge. According to the MKT framework, pedagogical content knowledge has three domains: (1) knowledge of content and students (KCS), (2) knowledge of content and teaching (KCT), and (3) knowledge of curriculum. The MKT-P framework developed by Lesseig (2011) did not explore the domain of knowledge of curriculum. While this is an area for consideration in research around MKT-P, this study directly uses Lesseig's framework and will not consider this domain. Figure 2.4 summarizes the domains of pedagogical content knowledge of proof and I follow with a detailed summary related to KCS and KCT for proof.

Knowledge of content and students. Lesseig (2011) uses Harel and Sowder's (2007) proof scheme taxonomy as a primary foundation of the knowledge that teachers need regarding how students interact with the content of proof. Additionally, Lesseig details the developmental aspects of proof.

Pedagogical Content Knowledge of Proof	
Knowledge of Content and Students	**Knowledge of Content and Teaching**
Explicit knowledge of proof schemes taxonomy • Characteristics of external, empirical and deductive proof schemes • Students tendency to rely on authority or empirical examples • Progression from inductive to deductive proof *Developmental aspects of proof* • Definitions & statements available to students • Representations within students conceptual reach • Forms of argumentation appropriate for students' level • Relationship between mathematical and everyday use of terms	*Relationship between instruction and proof schemes* • Methods of answering questions • Methods of responding to student ideas • Methods of using examples and lecturing that promote authoritarian or empirical proof schemes *Questioning strategies* • To elicit justification beyond procedures • To encourage thinking about general case *Pivotal examples or counter-examples* • To extend, bridge or scaffold thinking • To focus on key proof ideas *Knowledge of proof connections* • Linking visual, symbolic or verbal proofs of same concept or theorem • Comparing proofs in terms of accepted definitions and argument structure • Lifting general argument from numerical example or specific diagram

Figure 2.4. Pedagogical content knowledge of proof.
Note. Reprinted from "Mathematical Knowledge for Teaching Proof"; by Lesseig, K., 2011, Unpublished Doctoral Dissertation, Oregon State University, p. 34.

Explicit knowledge of proof schemes taxonomy. Harel and Sowder (2007) distinguish three proof schemes[2]: externally based, empirical, and analytical. External proof schemes are those in which minimal mathematics is used to convince an individual, a common example being when one is convinced because of the authority of the person

[2] Harel & Sowder (2007) define proof differently than I do. In particular, they define proof as something that "establishes truth for a person or a community" (p. 806). Further, they define a person's (or community's) proof scheme to consist of "what constitutes ascertaining or persuading for that person (or community)" (p. 809). From my perspective, a person or community may be convinced of something without the argument reaching the level of certainty as required in my definition of proof. As such, I consider their idea of a proof scheme to be more appropriately considered as a way someone may be convinced. However, I use the term "proof scheme" in the same way as Harel and Sowder in order to accurately represent their work.

stating a mathematical fact or property (such as a teacher). Empirical proof schemes are often based on inductive reasoning, in which a number of examples is enough to convince someone. Lesseig (2011) notes that these first two proof schemes are very common with students and that a teacher's PCK of proof must include this knowledge of students. Notably, while these are common with students, it is critical for a teacher to understand that these are not consistent with mathematical standards of proof. Inductive reasoning is a valuable tool in mathematics as it is often the basis for forming the conjectures which lead to a mathematical proof. However, it lacks the authority to make claims for all possibilities for a certain case. As such, while inductive reasoning is part of the proving process, it is not sufficient to count as proof. Teachers need to have the tools to assist students in progressing from inductive reasoning to Harel and Sowder's third proof scheme, which is analytical. An analytic proof scheme relies on deductive reasoning and is therefore accepted as proof within the mathematics community.

Developmental aspects of proofs. Lesseig (2011) notes that teachers need to be aware of where students are at developmentally when considering proof tasks and acceptable modes of argumentation and representation. This is related to where students may land on the reasoning-and-proving continuum (G. J. Stylianides, 2008) and the accepted statements, modes of representation, and modes of argumentation accepted by the classroom community noted by the A. J. Stylianides (2007) definition of proof. For example, certain statements or definitions are accepted at different developmental stages for students, such as the notion that the measurements of the three angles in a triangle sum to 180 degrees. This is true only in Euclidean geometry where the parallel postulate is accepted as true. However, it may not be appropriate to consider this detail at an early developmental level, while it is a much more critical detail when considering relationships in more advanced mathematics. Additionally, students may be more fluent in using a certain mathematical notation over another that may be more accepted by a mathematician.

Knowledge of content and teaching. Through a review of literature related to proof in mathematics education, Lesseig (2011) identified four elements that would make up a teacher’s knowledge of content and teaching for proof: (1) relationship between instruction and proof schemes; (2) questioning strategies; (3) pivotal examples or counter-examples; and (4) knowledge of proof connections. Additionally, these elements can be connected to the pedagogical components that should be considered in supporting students along the continuum of the reasoning-and-proving framework (G. J. Stylianides, 2008), in which teachers can use this knowledge to support students in moving from an empirical argument to a generic argument.

Relationship between instruction and proof schemes. In relation to Harel and Sowder’s (2007) three proof schemes, teachers must be aware of how certain instructional practices may promote one proof scheme more than others. These instructional practices may be related to how teachers respond to questions or student ideas. For example, when students present a mathematical idea, telling a student if they are right or wrong would promote an externally based proof scheme, establishing the teacher as a mathematical authority. Conversely, asking that student if they could provide reasoning or encouraging them to confirm the results for themselves would promote empirical or analytical proof schemes, establishing the mathematics itself as the authority and empowering the student.

Questioning strategies. Similarly, a teacher can promote certain proof schemes or modes of reasoning with the questions they ask. Lesseig (2011) draws from Martino and Maher (1999) to posit that knowledge of content and teaching should include an understanding of how questions can serve to extend or broaden thinking, focus students on key aspects of an argument, or stimulate further thought about a problem. Questions can also be used to lead students to think about and value the general case, moving to a more analytical and deductive approach from the empirical and inductive approach focused on specific examples.

Pivotal examples or counter-examples. Teachers need access to a range of examples and counter-examples when students bring ideas forward. The student ideas must be built on and the examples and counter-examples can serve as a tool to take student thinking and extend or bridge it to more generalized mathematical ideas. Students may also make conjectures that cover many but not quite all cases. For example, a student may conjecture that all prime numbers must be odd. The pivotal counter-example for this statement is the number 2. Through proper questioning strategies, the teacher can scaffold an opportunity for the student to discover this counter-example on their own. Moments like this allow a student to rule out certain cases on and work towards a general conclusion if one is possible.

Knowledge of proof connections. As noted above, teachers should work to progress students from empirical proof schemes to more deductive proof schemes. Connections between the various proof schemes are critical. For example, if students are still most often using empirical proof schemes and using specific examples to make claims, teachers can scaffold an opportunity to recognize the general case within the examples. Additionally, when in the analytical phase, students may propose different proofs for the same concept. Teachers must draw connections between these proofs and recognize the validity in taking different approaches or starting with slightly different assumptions. Students may also approach proofs from a visual, symbolic, or verbal perspective and connections should be made between these different representations.

Opportunities to Develop MKT-P in Teacher Preparation

The sections thus far have built towards establishing the importance of MKT-P. Given this importance, opportunities to develop MKT-P are relevant. Teacher preparation has the opportunity to serve as a primary setting to develop MKT-P, particularly before teachers start working with students full-time. In this section, I will start by outlining a brief history of mathematics teacher preparation and will identify relevant research to situate my exploration related to opportunities to develop MKT-P in teacher preparation.

Graham, Li, and Buck (2000) outlined some relevant history related to mathematics teacher preparation and noted that the nature of mathematics teacher education programs has varied considerably across the United States. From the beginning of mathematics teacher preparation in the early 1800's until the late 1980's, having a solid background in mathematics was considered sufficient for secondary mathematics teachers. It was during the reform movements in the late nineteen eighties that expectations were broadened by educational and governmental leaders to include aspects such as "knowing students as learners of mathematics" and "knowing mathematical pedagogy," tenets that appeared to begin incorporating Shulman's (1986) notion of PCK.

Graham, Li, and Buck (2000) used a survey of mathematics teacher educators to gather information related to the nature of teacher preparation programs. The results of the survey suggested that mathematics methods courses were mostly reformed-oriented, meaning they were incorporating opportunities to develop PCK and representing mathematics education with complex tasks that provided multiple pathways to a solution. This was in line with expectations of what teachers need to know to teach mathematics beyond the solid background in mathematics content. The results also suggested the opposite for content courses, in that most of the content courses were considered traditional-oriented, meaning the class sessions were often lecture-based and focused on mastering a single procedure to solve a mathematics problem. This did not necessarily model and incorporate best practices for mathematics teachers.

Further research has explored opportunities to make mathematics content courses relevant to mathematics teacher preparation, particularly if the content is beyond the scope of what one is expected to teach (e.g., Baldinger, 2018; Wasserman, 2018; Wasserman, Fukawa-Connelly, Villaneuva, Mejia-Ramos, & Weber, 2016). Connections from these upper level content courses to secondary mathematics are possible and have been observed in a few cases. However, these connections must be made explicit and they often are not.

Wasserman (2018) distinguished between local and nonlocal mathematical content, where local mathematical content is within the scope of content that teachers will engage in during instruction and nonlocal content is outside of the scope, often more advanced. He identified difficulties in the consideration of nonlocal mathematics as it relates to developing MKT. In Wasserman's best-case scenario, advanced content courses make connections to secondary content and are taught in ways that help learners learn. Even in this best-case scenario, these courses are typically organized around content ideas and the connections are often more implicit than explicit. Wasserman's conclusions suggest that connections to teaching are often not explicit enough in content courses, reinforcing the need for methods courses for opportunities to develop MKT across all domains, not just common content knowledge. While there have been other studies exploring the development of MKT in teacher preparation (e.g., Chapman, 2007; Charalambous, 2008; Holm & Kajander, 2012; Van den Kieboom, 2013), that has not been the case with MKT-P.

Teacher candidates may develop knowledge related to proof in their undergraduate content courses (e.g., Larsen & Zandieh, 2007; Mejia-Ramos, Fuller, Weber, Rhoads, & Samkoff, 2011). However, connections to teaching are not always explicit in these courses (e.g., Graham, Li, & Buck, 2000; Tatto, 2018; Wasserman, 2018). Specifically in the area of proof, methods courses have the potential to play a key role in providing opportunities for teacher candidates to unpack their knowledge of proof that they developed in their content courses in a way that is suitable for teaching. For example, as teacher candidates engage in proof in an undergraduate content course, this is likely done with an expectation of a certain amount of rigor that may not be considered by secondary teachers to be necessary for their students (e.g., Larsen & Zandieh, 2007; Weber, 2010). Many undergraduate mathematics instructors agree on the most important characteristics of proof that they expect to see from their undergraduate students (Moore, 2016). However, Baldinger and Lai (2019) found that an individual may have a different perspective on whether a proof is valid based on the context they are in, whether that be a secondary teacher or undergraduate student. As such, while teacher candidates may have an understanding of what is expected for a proof in an undergraduate mathematics course, there must be a space the unpack the expectations they should hold for secondary students.

Research Questions and Motivation

The literature above outlines much of the work around the development of MKT. Additionally, I have outlined some of the current work around mathematics teacher preparation. There has been work related to the development of MKT during teacher preparation as well (e.g., Chapman, 2007; Charalambous, 2008; Holm & Kajander, 2012; Van den Kieboom, 2013). However, there has yet to be a study which explores the development of MKT-P during teacher preparation.

It is likely that pre-service teachers engage in proof quite often while enrolled in their content courses (e.g., Larsen & Zandieh, 2007; Mejia-Ramos, Fuller, Weber, Rhoads, & Samkoff, 2011). However, the results from Graham, Li, and Buck (2000) and others (e.g., Tatto, 2018; Wasserman, 2018) suggests that there are doubts related to how well these content courses are connected to teaching, and therefore may only be connected to CCK. Even when good connections are present, the methods courses have different goals that are more directly tied to multiple elements of MKT. These goals are worth exploring in relation to proof. Specifically, the opportunities for this development seem best situated within a methods course, given that this is the space to connect content to teaching and students. This study attempts to gather evidence related the development of MKT-P in secondary mathematics methods courses and addresses the following research questions:

1. In what ways are secondary mathematics methods courses providing opportunities intended to support the development of MKT-P?

2. In the context of secondary mathematics methods courses, what factors (teaching context, personal/professional background, etc.) are associated with providing opportunities intended to support the development of MKT-P?
3. In the context of secondary mathematics methods courses, what are some instructional techniques that teacher educators use to support teacher candidate development of MKT-P?

With these research questions at the forefront, the next chapter will outline the methods used in my study.

Chapter 3: Methods

This study used a Mixed Methods Research (MMR) methodology to answer the research questions. Mixed Methods, considered the "third research paradigm", attempts to challenge the dichotomy that is often positioned by researchers situated in qualitative and quantitative research paradigms. However, MMR is a growing methodology and has become quite popular in the mathematics education research community (Buchholtz, 2019). Advocates for MMR take the stance that the combination or integration of qualitative and quantitative methods can compensate for the weaknesses embedded in each and build on the strengths of both approaches (Johnson and Christensen, 2017).

The data in this study came from two different sources in two distinct phases. The first phase consisted of a survey distributed to mathematics teacher educators. This data was analyzed using both qualitative and quantitative methods. This analysis influenced sampling selections for the second phase, which consisted of interviews with five teacher educators who completed the survey. The interviews were analyzed qualitatively to bring further clarity to and explain the results of the survey, while providing further details on what strategies teacher educators may implement to provide opportunities to develop MKT-P. Additionally, these interviews spurred further analysis within the survey data to explore themes that emerged in the interviews. As such, the MMR analysis process was iterative. In the following sections, I give further details on the survey design, followed by details related to the collection and analysis process.

Survey Design

The survey was designed to contain five sections: (1) a section to capture course information; (2) a section to capture the participant's view of proof; (3) a section to capture how proof is or is not addressed through the participant's teacher education program; (4) a section to provide further background information on the participant; and (5) a section asking for follow-up and demographic information. Throughout the survey design, survey consultants provided feedback that was intended to maximize participation and validity. Additionally, "think-alouds" (Ericsson & Simon, 1980) were conducted in which three volunteers piloted the survey and expressed as many thoughts out load as possible. These thoughts were recorded and transcribed. The volunteers all had some experience in mathematics teacher education and did not participate in the final survey for various reasons. Feedback from the survey consultants and those who participated in "think-alouds" led the survey through multiple drafts towards its final design. I describe the five sections of the survey in detail below and the full survey is provided in the appendix.

Course information. The survey began by asking participants for information related to the course(s) they are teaching or have taught in order to categorize them for eligibility for analysis. An additional purpose of asking these questions first is that it requires participants to identify the course(s) they are currently teaching or have taught most recently. As the survey progressed to the more critical questions associated with proof, it was important that participants answered these questions through the lens of a specific course or set of courses.

Through these initial questions, participants were asked if they were teaching a secondary mathematics methods course during the current academic year. If indicated as such, they were asked to provide the name of the course(s) they are assigned to teach during that year. If they were not teaching a secondary mathematics methods course during the current academic year, they were asked to provide the name of the course they most recently taught and the academic year in which it was taught. In either scenario, participants were then asked to identify the grade levels which the teacher candidate enrolled in the course(s) are prepared to teach. Participants were also asked to name the state or territory where their university is located. Both pieces of information are related to the educational standards that the mathematics teacher candidates will be responsible for following preparation.

View of proof. Prior to engaging with items related to proof in their course(s), participants responded to two items designed to capture their view related to the purpose of proof and what counts as proof. The purpose of these items was to identify to what extent each participant's view was encompassing or formal. Additionally, the literature outlined in the previous chapter details how the view of proof in mathematics and mathematics education is not in agreement at times. As such, the design of these items was done in consultation with professionals in mathematics and mathematics education to accurately capture the range of disagreements related to proof that can be held in both fields.

First, participants were asked to what extent they agreed or disagreed with the five purposes of proof given by de Villiers (1990, 2012): (1) verification, (2) explanation, (3) systematization, (4) discovery, and (5) communication. The second item (referred to as the "potential examples of proof item") gave four classroom activities that may occur at the secondary level. Participants were asked to what extent they agreed or disagreed that each of these activities would be considered proof at the secondary level. These activities all require students to use reasoning skills and engage in justifying. These activities, and the details related to their selection, are described below.

The first activity was completing the square of the general form of the equation $ax^2 + bx + c = 0$ to verify that the quadratic formula identifies the zeros of any quadratic function. This item was selected due to its common presence in secondary algebra courses but often outside the context of proof. Additionally, it may challenge participants who view proof as strictly geometric.

The second activity was a student explaining that "no triangle exists with side lengths 1, 2, and 10 because the long side is too long to allow the shorter sides to meet." This is a geometric example of justifying the triangle inequality theorem. The potential disagreements on this item lie in that it is only one specific example and that it is never stated that the student produces something on paper as a proof.

The third activity was tearing corners off of paper triangles and rearranging them to demonstrate that the measures of the interior angles of a triangle sum to 180 degrees. This is another common activity used in classrooms to justify the triangle sum theorem. Participants may disagree with this item because it is still a finite number of triangles and there is no indication that students would produce a proof on paper. Additionally, there is no use of definitions, axioms, or other theorems or postulates in this activity.

The fourth activity was using a counterexample of 2 to refute the claim that all prime numbers are odd. In this case, a student is using a counterexample properly but is disproving something rather than proving something. However, one may agree that this counts as proof because one counterexample is sufficient to make a valid mathematical claim that something is not true.

Proof in their course(s). The main purpose of the survey was to capture how secondary mathematics methods instructors approach proof in their courses. After providing the course information, participants were asked if the teaching and learning of proof is a topic addressed in their course(s). If so, participants were asked to classify the amount of time dedicated to proof as either: (a) a single class session; (b) a sequence of class sessions; or (c) embedded throughout the course.

Participants who indicated that proof is addressed were then asked to describe their learning goals for teacher candidates around the teaching and learning of proof in their methods course(s). Participants were also asked if there were specific tasks or assignments related to proof that teacher candidates must complete and, if so, were then asked to describe them. The response to both of these items were provided in an open-ended text box allowing for multiple sentences to describe their learning goals or possible task/assignment. These open-ended responses were limited throughout the survey but deemed necessary in this case in order to allow participants to describe these items in ways that are natural to them. It was important to not limit their descriptions to a set of specific choices in order to capture their approach towards proof as authentically as possible.

If a participant indicated that they do not address proof in their course(s), they were given the opportunity to provide a reason. The reasons to choose from included an expectation that proof is a focus in another methods course taught by a different instructor or due to proof being learned about sufficiently thought mathematics content courses. Additionally, participants could indicate that time constraints within the semester may be a factor. In addition to these reasons, participants were provided a text box for "other."

All participants were then directed to a matrix with various elements of the MKT-P framework (Lesseig, 2011) listed. Unlike the open-ended items described above, this item was designed to intentionally capture specific elements of the framework to protect against cases where participants' courses provide opportunities intended to develop certain elements of the framework, but their descriptions did not provide adequate information to capture those elements. Additionally, as mentioned above, there may have been cases where proof (or certain elements of proof) were not addressed in the participant's course(s), but this is because it was expected by the participant to be addressed in other courses, such as a methods course taught by another instructor or in a mathematics content course. It is also possible that certain elements were addressed in more than one course. It is for these reasons that each element allowed a participant to select multiple options: (a) addressed in my methods course(s); (b) I expect that this is addressed in other methods courses; and (c) I expect that this is addressed in mathematics content courses. These survey items are referred to throughout the paper as the "MKT-P items." The MKT-P items included eight potential methods class learning goals, with two activities intended to represent each of the four domains of MKT-P (KCS, KCT, CCK, SCK). I will describe the selection of these learning goals below.

For CCK, participants were presented items related to learning goals associated with writing proofs and the purposes of proof. In particular, the items were "teacher candidates learn how to write proofs" and "teacher candidates learn about the various purposes of proof." For SCK, the focus was on the various types of proofs teachers may require expertise in. These items stated that “teacher candidates learn about the variety of visual and symbolic methods to provide a general argument” and “teacher candidates learn when certain types of proofs are relevant for certain situations.” The KCT items focused on questioning strategies and pivotal examples and counter-examples. They were presented as “teacher candidates learn questioning strategies that will encourage secondary students to move from specific examples to general cases” and “teacher candidates learn of pivotal examples and counter-examples that can be used to extend, bridge, or scaffold secondary student thinking toward the development of a mathematical proof.” Lastly, the KCS items focused on the types of proofs students might produce and how these may relate to their learning level. These items stated that “teacher candidates learn about forms of argumentation at various secondary student learning levels” and “teacher candidates learn how to categorize student proof productions using the proof schemes of external, empirical, and deductive”

Participant background information. Research has shown that educational level, educational background, and departmental/university status can have an impact on a mathematics teacher educator’s practice (Graham, Li, & Buck, 2000). It is for this reason that participants were asked to provide various background information related to these variables. Each participant selected the various degrees they have earned (e.g., PhD, EdD, MA, etc.) and also selected the field which best describes each degree (mathematics, education, other). Additional information provided in this section included items such as years of experience, current professional title, and departmental assignment (education, mathematics, or joint).

Follow-up and demographics. Participants were asked if they would be willing to follow-up with the researcher via email if further information would help this study. If so, participants provided their name and email address in order to be contacted if they were selected for an interview. This information was hidden until the completion of analysis and was only used to contact those selected for an interview.

Throughout the survey design phase, there were multiple occasions where the issue of gender and race/ethnicity was raised by consultants and those who participated in "think-alouds." Upon further review of existing literature, gender and race/ethnicity was never discussed through Lesseig's (2011, 2016) MKT-P framework. In other reviewed studies around MKT, gender and race/ethnicity was either not discussed or considered infeasible to account for in a quantitative analysis. For example, Mohr's (2006) study assessed preservice middle school mathematics teachers' level of MKT and considered inclusion of gender and ethnicity as part of the assessment. However, this was eventually not included because of the feasibility of the study. Schilling and Hill (2007), considered two of the seminal developers of the MKT framework, also attempted to assess levels of MKT and never discussed race, ethnicity, or gender. Ball, Hill, and Bass (2005) studied the relationship between a teacher's level of MKT and the race of their students, but did not focus on the race of the teacher.

While the current literature around MKT and MKT-P gives no indication that gender or race/ethnicity play a specific role in the development of preservice teachers' MKT-P, these variables are all too often ignored in mathematics education research (e.g., Martin, 2013). When participating in surveys, a participant's gender or race may influence the answers they provide (Dillman, Smyth, and Christian, 2014). Additionally, education is no exception to the common problem of White males occupying positions of advantage within the educational system (Riegle-Crumb, Kyte, & Morton, 2018).

Throughout the survey design process, particular attention was paid to how this information could be incorporated in a way that allows all participants to be as comfortable as possible. For example, rather than using the terms gender, race, and ethnicity, participants were asked to select what best describes them and were allowed to select multiple options. Additionally, a “prefer not to answer” option was provided to offer a chance for participants concerned about providing a certain level of identifying information.

Data Collection

Sampling and participants. The initial sample for the survey included instructors of secondary mathematics methods courses. Instructors must have taught a course within the last three academic years to be eligible for analysis. This restriction was put in place to keep the sample focused on teacher educators who have likely been engaged with the most current standards and research associated with proof. Additionally, as teacher educators spend more time removed from a course, their memory may be less reliable as they attempt to recall their approaches towards proof in the course. The survey was initially distributed through the Association of Mathematics Teacher Educators (AMTE) membership roster. While not everyone on this roster is a secondary mathematics methods instructor, there were safeguards within the survey to exclude non-secondary mathematics methods instructors from analysis. The sample is considered a convenience sample.

The survey yielded 110 responses. Nine participants indicated that they have never taught a secondary methods course. Thirty-one participants began the survey but ended it after fewer than five questions. These participants were removed from the data set and this left me with 70 participants. Additionally, four of these 70 participants ended the survey prior to the MKT-P items which were required to be completed for quantitative analysis. Forty out of 70 participants indicated that the teaching and learning of proof is a topic that is addressed in their methods course(s). These 40 participants were then asked to describe their learning goals for teacher candidates around the teaching and learning of proof in their methods course. Thirty-seven of these participants provided a response to this prompt. Therefore, this study had a sub-sample size of 37 for much of the qualitative analysis and a sub-sample size of 66 for much of the quantitative analysis. The sample demographics related to gender and race/ethnicity are displayed in Tables 3.1 and 3.2, respectively.

Table 3.1. Survey demographics related to gender.

Response	Count
Female	38
Male	23
Nonbinary	2
No answer	3
Total	66

Table 3.2. Survey demographics related to race/ethnicity.

Response	Count
White	54
White AND Hispanic or Latino	3
Middle Eastern	2
Black or African American	1
White AND Native American or American Indian	1
European American	1
No answer	4
Total	66

Note. Participants were allowed to make multiple selections. The capitalized AND is used to denote that a participant selected multiple races/ethnicities.

Throughout the survey phase, participants only provided their names if consenting for further follow-up. Even in these cases, this information was only accessed upon completion of analysis as I began the selection process of interview participants. No identifiable information is included in the presentation of results.

Survey implementation. The survey was primarily distributed by email to mathematics teacher educators through the AMTE membership list. Some follow-up distribution also took place through the form of social media. Participants were asked to consent to the research prior to beginning the survey. The survey was administered through Qualtrics. Likewise, this is where responses were stored. The distribution email requested that participants complete the survey within two weeks of receipt.

Interviewee selection. Of the 70 participants in the sample, 44 provided consent to a follow-up interview. With those 44 candidates, I began by considering their responses to the MKT-P items (survey items 17 and 18). These items each contained one learning goal that was associated with each domain of MKT-P. The initial intent was to select interviewees who checked at least one of the items for each domain. In all, 15 participants fit this criterion. This selection allowed me to gather specific information related to each domain from the interviewees.

Additionally, the potential examples of proof item (survey item 10) showed that participants disagreed on what activities may count as proof on the secondary level. The responses to these items served as another criterion when considering interview candidates, as it had the potential to bring a wide range of views on what counts as proof. Four participants (Nathan, Kelly, Mara, Cassie) were selected based on the fact that they provided a range of views related to the activities on the potential examples of proof item.

In particular, there were strong stances taken in each direction by at least one of these four on each item. In addition, a fifth interviewee (Karen) was selected because she was the only participant to strongly disagree with all four potential examples of proof. Karen was not on the original list of 15 participants because she did not provide evidence that she addressed either of the learning goals associated with CCK. However, the nature of her response to the potential examples of proof item would bring a unique perspective to the interview phase. The names of these interviewees are all pseudonyms.

Interview structure. The interviews were designed to provide further clarity on select participants' opportunities intended to support the development of MKT-P within their course(s). It was also intended to provide a picture of what these opportunities look like in practice. There was also a significant amount of time dedicated to unpacking their responses on the potential examples of proof item. In this sense, each interview was conducted in a unique manner that took into account responses from the survey. Interviews were conducted over the phone or by video chat. One circumstance (Kelly) allowed for a face-to-face interview at a conference. All interviews were recorded and transcribed for analysis.

Data Analysis

The data analysis began with an initial qualitative analysis of the survey data to select interviewees as described in the previous section. Quantitative data analysis commenced with survey data while interviews were being conducted. Following completion of the interviews, the interviews were analyzed qualitatively. During this process, I also moved back for further qualitative analysis of the survey data based on themes identified in the interviews. As such, the data analysis was conducted in three phases and I engaged with these phases in an iterative approach. I outline the details of these three phases below.

Qualitative survey data analysis. The two open-ended response items in the survey were qualitatively coded to identify elements of the MKT-P framework. These open-ended response items asked the participant to describe their learning goals for their teacher candidates related to proof and to describe a specific task or assignment related to proof, if applicable. The goal of this coding was to primarily identify which of the four domains of MKT-P (KCS, KCT, CCK, SCK) were present in their response. For example, if a participant's response included a statement related to teacher candidates engaging in various forms of argumentation at different student learning levels, this was coded as KCS for proof as it is one of Lesseig's (2011) developmental aspects of proof. A participant may also make note of teacher candidates valuing the role of general cases or general arguments. This would be coded as KCT for proof as it is one of Lesseig's "knowledge of proof connections." Additionally, many responses contained learning goals that were not related to specific items in Lesseig's MKT-P framework. These were still coded as CCK, SCK, KCT, or KCS based off of my understanding of the broad MKT framework (Ball, Hill, & Bass, 2005; Hill, Ball, & Schilling, 2008). For example, one response stated that "PSTs should be familiar with different ways to write a proof." This was coded as CCK due to it being related to a skill that is also used directly in the field of mathematics. Table 3.3 provides an overview of the key phrases used to code in this phase.

Table 3.3. Summary of keywords to code for the various domains of MKT-P.

Code	Key phrase	Example
CCK	Purpose of proof What is proof Do a proof Complete [a math task] Write a proof	Discussing with students what proof is for (in real life, logical thinking).
SCK	Write a proof using multiple methods Connect between different proofs Expand conception of proof beyond… Proving techniques	Understanding different types of proving techniques and connecting between them.
KCT	Teaching Questioning strategies Lesson plan Assessment Ways to connect…	Ways to connect conjecturing, reasoning, and proving as a sequence.
KCS	Different learning levels Student work Make sense of what students did… Develop students…	PSTs examine sample student work with an eye toward methods for moving students along the continuum in the reasoning-and-proving framework.

Note. This list of key phrases is not meant to be exhaustive, but rather to share the notable words that guided my coding. Additionally, not every key phrase was used exactly as specified in this table, as interpreting the context of the learning goals may cause a response to be coded in a different domain than what the key phrases specify.

The prompt related to learning goals did not specify nor make mention of any relations to the MKT-P framework. The prompt was designed to be open-ended and to capture whatever comes to mind first when a teacher educator thinks of their learning goals. Due to this, it may have been possible that a participant made mention of some elements of the framework but not others, even if other elements were to be considered learning goals upon more specific prompting. This is not taken as evidence that the teacher educator does not incorporate these elements of MKT-P in their instruction. The purpose is to capture some specific examples of opportunities intended to support the development of MKT-P in their learning goals and instructional practice. These codes were later compared with responses on the MKT-P items, which lists specific examples from each element of the framework.

Quantitative survey data analysis. Much of the quantitative analysis took place when considering responses to the MKT-P items (survey items 17 and 18). I looked for evidence or lack of evidence for the four constructs of MKT-P, as well as a combination of some of these constructs. For these tests, I only considered evidence to be present if a participant selected “Addressed in my methods course.” The other boxes were in place as explanatory data to provide details on why a teacher educator may not cover a certain topic. However, this data would not be appropriate to quantitatively analyze due to much of the other data throughout the survey being associated with the teacher educator taking the survey. I understand that, for example, having no evidence of opportunities to develop CCK is not evidence that CCK is not addressed. Therefore, any claims made are not to the effect of “this factor is associated with addressing CCK for proof” but instead the claims are “this factor is associated with evidence of providing opportunities intended to support the development of CCK for proof.”

The focus of the quantitative data analysis was to identify factors associated with opportunities intended to support the development of each of the four domains of MKT-P. These factors were captured on the survey and included information related to: departmental assignment, the degrees they have earned, the grade levels for which their teacher candidates will be certified to teach, their years of experience teaching secondary methods courses, their years of experience teaching secondary mathematics, their gender, and their race/ethnicity. When appropriate, Chi-square tests of independence were conducted to determine if any of these factors were associated with evidence related to one of the four domains of MKT-P. When necessary, a Yates correction was applied. For the purposes of analysis, showing evidence in one example of a domain was sufficient to be counted for running a Chi-square test. For example, if a participant selected both learning goals related to CCK, they would be counted the same as someone who only selected one of the learning goals related to CCK.

Interview data analysis. The interviews were coded in a similar manner to the open-ended survey items, looking to identify specific elements of the framework in responses. Since most interviewees were chosen based on an indication that their methods course(s) provides opportunities to develop multiple elements of the MKT-P framework, the interviews were in place to gather further details on how these opportunities were present in their practice. All specific examples of these opportunities were flagged and an analytic memo was written to summarize these examples for each interview. Additionally, each interviewee discussed their view of proof extensively based on their responses in the potential examples of proof item (survey item 10). This view, and how it influences their teaching practice, was also captured in each memo. In summary, each memo contained details related to their view of proof, how their view of proof influenced their practice, specific mentions of domains in the MKT-P framework, and particular activities they shared. The memos for each of the five interview were compared for similarities with a final analytic memo written to capture these similarities. These similarities may include common goals for assignments, identical or nearly identical tasks, and common influences on their approach. A summary of data collection and analysis is provided in Table 3.4.

Table 3.4. Summary of data collection and analysis.

Research Question	Data Collection Phase	Data Analysis
In what ways are secondary mathematics methods courses providing opportunities intended to support the development of MKT-P?	Survey (items 13, 15) Interviews	Coding of responses Coding of interview
In the context of secondary mathematics methods courses, what factors (teaching context, personal/professional background, etc.) are associated with providing opportunities intended to support the development of MKT-P?	Survey	Chi-Square Test
In the context of secondary mathematics methods courses, what are some instructional techniques that teacher educators use to support teacher candidate development of MKT-P?	Interview Survey (item 15)	Coding of interview Coding of responses

Chapter 4: Survey Findings

Qualitative Survey Results

Forty out of 70 participants indicated that the teaching and learning of proof is a topic that is addressed in their methods course(s). These 40 participants were then asked to describe their learning goals for teacher candidates around the teaching and learning of proof in their methods course(s). Thirty-seven of these participants provided a response to this prompt and most of these responses contained evidence related to the mathematical knowledge for teaching proof framework. This evidence is outlined in the following sections by breaking it up into the four domains of MKT-P: Common content knowledge, specialized content knowledge, knowledge for content and teaching, and knowledge for content and students.

Additionally, of these 37 participants, 25 indicated that they had specific tasks or assignments related to proof that their teacher candidates must complete. When asked to briefly describe one such task or assignment, 20 participants provided a description. Most of these descriptions were mathematical tasks designed to build reasoning and justification skills. In many cases, the description of the tasks did not make explicit connections to students or teaching. As such, there was a significant amount of evidence of CCK throughout these responses.

In the following sections, I outline the evidence of learning goals associated with the four domains of MKT-P: CCK, SCK, KCT, and KCS. Additionally, I outline the various tasks and assignments that are associated with each domain.

Common content knowledge. Twenty-five of the 37 responses outlined learning goals which had some relation to common content knowledge. While these numbers may initially indicate that a majority of participants focused on CCK in their response, it should be noted that 15 of these responses also contained some evidence of one of the other three domains of MKT-P. The other 10 responses only contained evidence of CCK. Within these responses, a variety of themes emerged. Some of the notable themes included: the nature/role of proof, a focus on proof writing, generalized cases, and using proof for communication.

The nature/role of proof. Eight participants indicated that understanding the nature and/or role of proof was a learning goal for their teacher candidates. For some of these participants, the description in this area was broad, with responses such as "PSTs should understand the role of proof" or that they should "understand the nature of proof." Some were more specific in relation to what specific role proof may play in mathematics. One response stated that "proof is about more than verification, it is about understanding why something works and how that how [*sic*] it connects to other ideas." Another response stated that their teacher candidates will learn different purposes of proof. This response is notable in that it specified that there are multiple different purposes, and gave examples of explanation and verification. When examining these learning goals, those that were specific did not specify any purposes or role of proof outside of the field of mathematics.

Focus on proof writing. Three participants indicated that they place an emphasis on their teacher candidates engaging in the process of writing a proof. One response did not indicate any other goals, simply stating that "teacher candidates will demonstrate their ability to teach proof writing in high school Geometry class." Another participant held this goal with more of a focus on the future success of their teacher candidates in further undergraduate coursework, saying "one of my goals is to make them better proof writers since they are mathematics majors and will see proof again during the [*sic*] undergraduate degree."

Generalized cases. Three participants discussed the importance of proofs being generalized cases. One of these responses kept the focus on the shortcomings of specific cases, saying that "just checking examples (e.g., Geogebra) does not constitute proof." This participant clarified that checking examples can be very helpful in the process of making conjectures. In the second case, the focus was on how valid proofs hold for all cases, saying that their teacher candidates should "understand that proving a relationship means demonstrating that the relationship is true in general for the defined context." The third response was focused on teacher candidates helping students create "proofs for generalizations that arise in exploring problem situations."

Proof for communication. Communication was viewed as part of learning goals for three participants. For one, "proof is about communicating a reasoned argument in whatever form accomplishes your purposes," while also making a point to de-emphasize specific forms. Other responses included "focusing on proof as communication in mathematics" and a goal to "connect proof to communication."

Tasks/assignments related to CCK. Eleven of the described tasks/assignments gave evidence that they were intended for the development of common content knowledge. While these were tasks that were assigned to teacher candidates, I inferred that many of these tasks were presented as model tasks for the teacher candidates to use with their own future secondary students. Figure 4.1 provides a description of one such task, called the locker problem. After solving the locker problem, the teacher candidates are to determine which doors would be open for an infinite number of lockers and must provide a justification for why it is true. The move towards an infinite number of lockers requires the teacher candidates to generalize, which is part of the proving process. I considered a task like this to provide an opportunity to develop common content knowledge for proof because it requires teacher candidates to directly engage in the proving process themselves.

> Students at an elementary school tried an experiment. When recess was over, each student walked into the school one at a time. The first student opened all the first 100 locker doors. The second student closed all the locker doors with even numbers. The third student changed all the locker doors with numbers that were multiples of three. (Change means closing lockers that were open and opening lockers that were closed.) The fourth student changed the position of all locker doors numbered with multiples of four; the fifth student changed the position of the lockers that were multiples of five, and so on. After 100 students had entered the school, which locker doors were open?

Figure 4.1. A description of the locker problem provided on the survey.

Some responses contained specific examples of tasks such as the locker problem, while others were more general, containing phrases such as “students are asked to solve a challenging problem (like the surfer and spotter task) and justify their solution outside of class time.” Another participant wrote that they ask their teacher candidates “to come up with an algorithm for something and to give justify [*sic*] why their algorithm was efficient.” Only three of these 11 responses made connections to other domains of MKT-P. It is entirely possible that the other tasks described also provided opportunities to draw these connections and the response simply did not provide these details.

Specialized content knowledge. Eleven of the responses outlined learning goals which had some relation to specialized content knowledge. Within these responses, a variety of themes emerged. Most of these learning goals were related to exposing teacher candidates to a variety of types of proofs and methods for completing proofs, seeming to acknowledge that mastering proof when holding a narrow view of proof is not sufficient for teachers.

Different types of proofs. Seven participants listed goals related to different types of proofs. For some, this goal was stated broadly, such as "I want them to understand different types of proof." For others, they were more specific, in that it could have been related to a certain format or structure. Specifically, a number of participants wanted to move beyond what some called a "rigid" idea of proof having to be in a two-column structure. One response indicated that they want their teacher candidates to "see valid proofs as more than only formats such as two-column proofs," while another referred to pointing out the "limitations of two-column proofs." A third participant said they wanted to "deepen" their teacher candidates' "understanding of proof with a shift away from thinking of proof procedurally" with two-column proofs provided as an example of a procedurally-oriented proof. Some participants gave specific examples of alternative types, such as paragraph proofs and flow chart proofs, implying a focus on proof as a product. Alternatively, others seemed to be intentional about keeping the idea of what counts as proof as broad, implying that proof can be done visually or that what counts as proof is "socially negotiated."

Different approaches to proof. Four participants discussed goals around familiarizing their teacher candidates with a variety of techniques and approaches to proof, bringing the focus to proof as a process. One participant wanted their teacher candidates to understand different types of proving techniques, specifically listing "using visuals (e.g., interactive online applets) in geometry, presenting counter examples, direct and indirect proofs with numbers." Another listed further "various forms of proving" as "synthetic, analytical, [and] transformational." Two others referred in general to "different approaches" related to proof.

Tasks/assignments related to SCK. Five of the tasks/assignments gave evidence that they were intended to support the development of specialized content knowledge. Most of these tasks involved exploring how to prove a concept using multiple techniques or formats, and drawing connections between them. One participant described how their teacher candidates prove that the product of an even number and an odd number is always even by using pictures. The teacher candidates then prove it using an argument suitable for an abstract algebra class and draw connections to their pictorial proof. Another response considered informal and formal methods of proving that the measures of the interior angles of a triangle sum to 180 degrees. Another response did not specify a specific concept but outlined the various methods to be used.

> Students are first given the task to prove themselves using any method they prefer and then given completed proofs that use different approaches (e.g., Euclidean geometry vs. algebra) and also different proof formats (two column, paragraph, flow chart).

Knowledge of content and teaching. Nine of the responses outlined learning goals which had some relation to knowledge of content and teaching. One of these responses stated that they wanted their teacher candidates to be "able to lead explorations of proofs in their own classes" and another stated that teacher candidates are shown ways to use various proofs "in the classroom." Among the other responses, themes around proof connections, proof for explanation, and supporting students emerged.

Proof connections. Two participants expressed that their goal was to have their teacher candidates draw connections when discussing proof. One specifically wrote that they explored "ways to connect conjecturing, reasoning, and proving as a sequence." The other participant wrote about drawing connections between the multiple formats of proof that "fall at different points on the continuum of the reasoning-and-proving framework." The teacher candidates in this methods course work with the reasoning-and-proving framework developed by G. J. Stylianides (2010).

Proof for explanation. A number of participants referred to explanation as a specific purpose of proof that they emphasize in their courses. A proof that explains can be powerful in deepening a student's understanding of other mathematical concepts. One response noted that "future teachers need to be comfortable with proof in general to answer the 'why' and 'how' questions that students will have." Another response gave specific examples of proofs that teacher candidates "should know and be able to walk students through" in order to explain that 2 is irrational and for justifying the Pythagorean Theorem. It was noted through analysis that while explanation serves as a function or role of proof for these participants, this role is different than those discussed above in the area of common content knowledge. This is a role of proof specific to the work of teaching, as these participants are encouraging their teacher candidates to use proof as a tool to explain other mathematical concepts, such as the Pythagorean Theorem.

Supporting students. Three participants wrote about developing ways to support students in the work of proving. Sometimes these ways came in the form of using specific teaching strategies. One wrote that "PSTs identify ways that teaching practices can support student learning of proof and proving." Another wrote that they wanted their teacher candidates to have "developed some specific ways of supporting students who are learning to prove." One response was more specific in their description of teaching strategies, writing that "teacher candidates must be prepared to coach students in the creation of proofs for generalizations that arise in exploring problem situations," implying that students need support from their teachers to lift a general case from multiple examples. This coaching may come in the form of questioning strategies, as Lesseig (2011) notes that certain questioning strategies can encourage thinking about the general case.

Additionally, some participants held a goal of their teacher candidates supporting students in the work of proving through careful planning and selection of tasks. One participant elaborated on this idea in detail.

> I would like them to have a better understanding of how to choose problems/contexts that are likely to engage students in authentic mathematical activity and how to modify existing contexts/problems to make them more likely to engage students in proving.

This response emphasizes that there exist certain tasks that are likely to support students in the work of proving more than others. Another participant specifically wrote that their teacher candidates write lesson plans related to proof.

Tasks/assignments related to KCT. Five of the tasks/assignments gave evidence that they were intended for the development of knowledge for content and teaching. One of these was specifically related to assessment in the understanding of proof.

> We read an article about authentic proof and there are problems within the article so we discuss those problems and how they can be used as scaffolds in assessing students understanding of proofs.

The other four responses were related to the careful design of instructional tasks for students to engage in. For example, one teacher educator asks their teacher candidates to "choose a problem/context that (as currently written) does not engage students in proving and modify it into one that is likely to do so." Another involved having teacher candidates plan lessons designed for students to prove a mathematical relationship using a dynamic geometry software called Geogebra. Other responses related to a focus on instructional sequencing, to consider how to sequence a set of activities to move from informal to formal justifications.

Knowledge of content and students. Six of the responses outlined learning goals which had some relation to knowledge of content and teaching. All six of these responses also made connections to other domains of MKT-P. Two themes emerged when analyzing these responses: examining student work and different student learning levels.

Examining student work. Two responses stated that their teacher candidates examined student work related to proof. One of these responses said that their teacher candidates examined this work "with an eye toward methods for moving students along the continuum in the reasoning and proving framework." This is the same participant who used the reasoning-and-proving framework developed by G. J. Stylianides (2010). The other response had stated that their teacher candidates examined student work to identify "when an argument provides sufficient justification and when it does not." Both of these responses indicated a belief that students often develop arguments or justifications that may fall short of what would be considered proof. Examining this student work gives the teacher candidates opportunities to interpret the missing pieces that may be necessary to satisfy sufficient conditions to be considered proof.

Different student learning levels. Four participants discussed proof at different student learning levels. Two of these participants kept the focus on how teachers should adjust their approach to proof based on the learning level of their students, with one saying that their teacher candidates should feel comfortable approaching proof "at a variety of learning levels." The third participant focused on what would be an acceptable proof at different ages.

> We discuss how proofs might be validated at middle and high school levels, and how the criteria for proof is context dependent. What counts as a proof in middle school might not count at the University level.

Lastly, the fourth participant noted how a "proof" can look different at different levels. Examples of proof at different learning levels given by this participant included: providing examples, informal explanations, and formal deductive structures.

Tasks/assignments related to KCS. Three responses provided evidence that their task/assignment was intended for the development of knowledge of content and students. In two of these responses, the description was brief and involved looking at student work to "make sense of what the students did." The third response indicated an extensive use of a textbook (Arbaugh, Smith, Boyle, Stylianides, & Steele, 2019) which contains many tasks relating to reasoning and proof. The response stated that their teacher candidates complete every task in that textbook and specified that "the idea is for my students to see proof from the perspective of a high school student."

Summary of learning goals, tasks, and assignments. A notable finding among these responses is the high number of participants including goals around common content knowledge, especially compared to the significantly lower number of responses including goals around pedagogical content knowledge, particularly knowledge of content and students. Of the 37 participants who wrote a response to this prompt, 28 did not specify goals related to KCT and 31 did not specify goals to KCS. Similar findings are reflected in the tasks and assignments participants provided. That is not to say that these teacher educators do not have goals relating to these domains. These responses are simply a brief snapshot of how they would briefly describe their goals. Additionally, many participants who provided goals, tasks, and assignments related to CCK and SCK may provide opportunities to connect these types of goals more explicitly to teaching and students. However, further research is needed to identify learning goals, tasks, and assignments related to KCT and KCS for proof. The data here does not paint a vivid picture of how this is addressed in methods courses.

Another noteworthy finding from these responses is that some participants kept a focus on justification more so than the production of a proof. The juxtaposition of proof versus justification raises interesting questions. Two participants who indicated that proof is addressed in their methods course admitted that they do not explicitly address proof but instead focus on justification or argumentation. One response said:

> I would say that I emphasize justification rather than proof. I emphasize to my teacher candidates that they should expect that students should engage in mathematical justification and should understand why formulas/theorems are true.

The other response stated that “I do not use proof, but askt [*sic*] students to construct and critique arguments.” Both of these responses are direct in saying that they emphasize arguments and justification over proof, yet they still consider themselves to address the teaching and learning of proof in their methods course(s). This may imply that these participants view justification or argumentation as a proper substitute for proof.

Quantitative Survey Results

The MKT-P items (survey items 17 and 18) from the survey described various learning objectives for teacher candidates. On each item, there was a learning objective tied to CCK, SCK, KCS, and KCT. Each learning objective was linked to a specific element in the MKT-P framework (Lesseig, 2011). Sixty-six of 70 participants engaged in these items, while the other four participants ended the survey prior to completing these items. It is important to note the difference in sample size from the section above. This is because participants did not engage with the items analyzed above if they indicated that the teaching and learning of proof is not addressed in their methods course(s). On the other hand, all participants were asked if each of the learning objectives on the MKT-P items were addressed in their methods course(s), while also being given the opportunity to clarify if they expect any learning objective to be addressed in other courses, such as another methods course or content courses. In the following sections, I outline findings related to evidence of CCK, SCK, KCS, and KCT in each participant’s course(s). Only one of the two learning objectives related to a specific domain needed to be checked in order to provide evidence of that domain being addressed in the methods course(s). These findings are not meant to imply that any participant does not address a specific domain of the MKT-P framework. For example, a participant may not indicate that either of the learning objectives associated with CCK are addressed in their course(s), but they may address other learning objectives associated with CCK that were not given as options on the survey.

Participants providing evidence of MKT-P in their methods course(s). Table 4.1 below outlines results of the 66 participants who engaged with the MKT-P items (survey items 17 and 18).

Table 4.1. Summary of evidence related to opportunities intended to support the development of MKT-P.

n = 66	Participants indicate evidence of addressing MKT-P domain in methods class	Participants do not indicate evidence of addressing MKT-P domain in methods class	
MKT-P domain		Participants expect that the MKT-P domain is addressed elsewhere	Participants do not expect that the MKT-P domain is addressed at all
CCK	31	33	2
SCK	50	12	4
KCS	42	17	7
KCT	55	6	5
All Four	23	33*	10

**Note*. To be counted in this cell, one would need to check at least one example related to each of the domains that that they did not provide evidence for in their own methods course(s).

The highest amount of evidence for methods courses was related to KCT, where 55 out of 66 participants indicated that they provided opportunities intended to support the development of KCT for proof in their methods course(s). In contrast, only 31 participants indicated such evidence with CCK. Also notable among the CCK evidence is the large number of participants who indicated it was addressed in other courses. As such, while CCK had the least amount of evidence for each participant’s course(s), it also had the largest amount of evidence that it is addressed at some point within the coursework in each participant’s teacher preparation program, with only two participants not providing evidence that it is addressed at any point. Lastly, nearly one third (23) of the participants indicated that they provided opportunities intended to support the development of MKT-P in all four domains.

Factors associated with MKT-P. There was a variety of other information that participants provided as part of the survey. This other information included their departmental assignment, the degrees they have earned, the grade levels for which their teacher candidates will be certified to teach, their years of experience teaching secondary methods courses, their years of experience teaching secondary mathematics, their gender, and their race/ethnicity. These factors were compared with the evidence they provided on the MKT-P items described above.

There was no evidence to suggest that departmental assignment, degrees earned, gender, or years of experience was associated with addressing certain domains of MKT-P in a participant’s methods course(s). Further, because the sample was mostly White, it was not possible to determine whether race/ethnicity was or was not associated with the domains of MKT-P being addressed in a participant’s methods course(s). Notable findings related to these factors are outlined below.

Departmental assignment. The survey was designed with an anticipation that a teacher educator with a mathematics-focused background may approach the teaching and learning of proof differently than a teacher educator with an education-focused background. In particular, it was hypothesized that teacher educators with an education-focused background may be more likely to indicate that they provide opportunities to support the development of KCT or KCS than a teacher educator with a mathematics-focused background. The data in this study did not indicate that this was a significant factor. In fact, the data from teacher educators in mathematics departments was very similar to the data from teacher educators in education departments. These results are displayed in Tables 4.2-4.5 below, with accompanying Chi-square statistics.

Table 4.2. Comparison across departmental assignment related to providing evidence of opportunities intended to support the development of CCK for proof.

	Selected one or more CCK items	Selected no CCK items	Total
Mathematics	11	12	23
Education	18	16	34
Total	29	28	57

χ^2 (1, *N* = 57) = 0.144, *p* = .705

Table 4.3. Comparison across departmental assignment related to providing evidence of opportunities intended to support the development of SCK for proof.

	Selected one or more SCK items	Selected no SCK items	Total
Mathematics	18	5	23
Education	29	5	34
Total	47	10	57

χ^2 (1, *N* = 57) = 0.469, *p* = .493

Table 4.4. Comparison across departmental assignment related to providing evidence of opportunities intended to support the development of KCT for proof.

	Selected one or more KCT items	Selected no KCT items	Total
Mathematics	21	2	23
Education	28	6	34
Total	49	8	57

χ^2 (1, *N* = 57) = 0.320, *p* = .571

Note. Yates correction was applied in this case so these results should be interpreted modestly.

Table 4.5. Comparison across departmental assignment related to providing evidence of opportunities intended to support the development of KCS for proof.

	Selected one or more KCS items	Selected no KCS items	Total
Mathematics	17	6	23
Education	21	13	34
Total	38	19	57

χ^2 (1, N = 57) = 0.911, p = .340

Amount of time dedicated to proof. Participants were asked to describe the amount of time dedicated to proof in their methods course(s). Their options were: (a) a single class session, (b) a sequence of class sessions, and (c) embedded throughout the course. Participants only responded to this item if they indicated that the teaching and learning of proof is addressed in their methods course. As such, 37 participants responded to this item, with results shown in Table 4.6 below.

Table 4.6. Summary of how participants described the amount of time dedicated to proof in their methods course(s).

A single class session	A sequence of class sessions	Embedded throughout the course	Total
2	10	25	37

With only 12 participants not indicating that proof is embedded throughout the course, it is notable that half of these participants (6) did not provide evidence of addressing CCK in their course. On the other hand, of the 25 who indicated that proof is embedded throughout the course, 22 provided evidence of addressing CCK in their methods course(s). With only three in this sub-sample not providing evidence, Yates correction was applied. These results should be interpreted modestly but still appear interesting, as only 50% of those who dedicated a single session or sequence of class sessions to proof provided evidence of addressing CCK, while 88% of those who embed proof throughout the course provided such evidence. Table 4.7 presents these results.

Table 4.7. Comparison across how much time participants dedicated to proof related to providing evidence of opportunities intended to support the development of CCK for proof.

	Selected one or more CCK items	Selected no CCK items	Total
Embedded	22	3	25
Not Embedded	6	6	12
Total	28	9	37

χ^2 (1, N = 57) = 6.361, p = .012
Note. Yates correction was applied in this case so these results should be interpreted modestly.

Grade levels included in licensure. Participants were asked to indicate the grades that their teacher candidates would be licensed to teach upon completion of their program. Two participants did not indicate any grade levels. Of the remaining 64, six indicated some inclusion of elementary grades below grade 5, 12 indicated only secondary grade levels (9-12), while 46 indicated some middle school grades (5-8) but no elementary grade levels. Table 4.8 presents a summary of this sample.

Table 4.8. Summary of how participants described the grade levels that their teacher candidates would be licensed to teach upon completion of their program.

Only secondary	Some middle, no lower than grade 5	Some elementary	Total
12	46	6	64

While KCT was the domain showing the largest amount of evidence across all participants, it is notable that all six of the participants who indicated some elementary grade levels indicated evidence of KCT. This is evident in Table 4.9.

Table 4.9. Comparison across grade level preparation related to providing evidence of opportunities intended to support the development of KCT for proof; elementary included.

	Selected one or more KCT items	Selected no KCT items	Total
Only secondary	7	5	12
Some middle, no lower than grade 5	41	5	46
Some elementary	6	0	6
Total	54	10	64

The low sub-sample size of the elementary category made Chi-square tests of independence impractical. However, when comparing across the sub-groups of middle grade inclusion and only secondary, there is evidence to suggest that preparing teacher candidates to only teach secondary mathematics is associated with whether or not a teacher educator addresses SCK and KCT. This evidence is presented in Tables 4.10 and 4.11, respectively.

Table 4.10. Comparison across grade level preparation related to providing evidence of opportunities intended to support the development of SCK for proof.

	Selected one or more SCK items	Selected no SCK items	Total
Only secondary	6	6	12
Some middle, no lower than grade 5	39	7	46
Total	45	13	58

χ^2 (1, N = 58) = 6.621, p = .010

Table 4.11. Comparison across grade level preparation related to providing evidence of opportunities intended to support the development of KCT for proof.

	Selected one or more KCT items	Selected no KCT items	Total
Only secondary	7	5	12
Some middle, no lower than grade 5	41	5	46
Total	48	10	58

χ^2 (1, N = 58) = 6.326, p = .012

This data may suggest that the secondary methods course(s) preparing teacher candidates for only secondary grade levels are less likely to address elements of SCK and KCT related to proof compared to those who are also (or solely) preparing teaching candidates for middle grade levels. This was not an anticipated finding and it is unknown what would be the cause of this. One potential explanation would be that a focus on only secondary grades may cause a methods course instructor to focus more on advanced mathematical common content knowledge rather than specific knowledge related to teaching. It should also be noted that the sub-sample size for secondary grade levels was only 12, so more participants in this category would be helpful before making any wide-ranging claims.

Is the teaching and learning of proof addressed in your methods course(s)? A surprising development from the data is that there were 29 participants who indicated that the teaching and learning of proof is not addressed in their methods course(s), yet 21 of those participants indicated that they provided opportunities intended to support the development of MKT-P in their methods course(s). It should be noted that some of the learning goals associated with MKT-P on the survey items can be addressed outside the context of proof. However, the fact that so many participants indicated that the teaching and learning of proof is not addressed in their methods course(s) and also indicated that they provided opportunities intended to support the development of MKT-P in their methods course(s) is a notable finding worthy of exploration. It is clear that teacher educators feel they can provide opportunities intended to develop MKT-P without addressing the teaching and learning of proof in their methods courses. As such, I explored whether this appeared more likely to occur within one domain of MKT-P more than others, given the items on the survey. Table 4.12 below outlines this data for CCK.

Table 4.12. Comparison related to providing evidence of opportunities intended to support the development of CCK for proof across whether participants indicated that the teaching and learning of proof is addressed in their methods course(s).

	Selected one or more CCK items	Selected no CCK items	Total
Indicated proof is addressed	28	9	37
Indicated proof is not addressed	3	26	29
Total	31	35	66

The data associated with CCK for proof reflects the data one would expect to see when exploring how these results interact, as the largest two cell values fall in the main diagonal. For those that indicated that the teaching and learning of proof is addressed in their methods course(s), all but 3 provided opportunities intended to support the development of CCK for proof. Conversely, for those that indicated that the teaching and learning of proof is not addressed in their methods course(s), only 9 of 37 provided opportunities intended to support the development of CCK for proof. In this sense, the development in the data is not surprising when looking at it through a CCK lens. However, that was not the case when considering SCK, KCT, and KCS. These results are displayed in Tables 4.12-4.15 below.

Table 4.13. Comparison related to providing evidence of opportunities intended to support the development of SCK for proof across whether participants indicated that the teaching and learning of proof is addressed in their methods course(s).

	Selected one or more SCK items	Selected no SCK items	Total
Indicated proof is addressed	32	5	37
Indicated proof is not addressed	18	11	29
Total	50	16	66

Table 4.14. Comparison related to providing evidence of opportunities intended to support the development of KCT for proof across whether participants indicated that the teaching and learning of proof is addressed in their methods course(s).

	Selected one or more KCT items	Selected no KCT items	Total
Indicated proof is addressed	33	4	37
Indicated proof is not addressed	22	7	29
Total	55	11	66

Table 4.15. Comparison related to providing evidence of opportunities intended to support the development of KCS for proof across whether participants indicated that the teaching and learning of proof is addressed in their methods course(s).

	Selected one or more KCS items	Selected no KCS items	Total
Indicated proof is addressed	28	9	37
Indicated proof is not addressed	14	15	29
Total	42	24	66

Where one would expect the largest values to be present in the main diagonal as seen with CCK, the largest values are instead present in the top row with SCK and KCT. In addition, the same phenomenon is almost present with KCS. There are two possible explanations for this that are worth noting. First, the learning goals I selected for these survey items may have been easily accomplished outside of the context of proof, particularly with SCK, KCS, and KCT. The other explanation may be that when teacher educators were considering whether or not the teaching and learning of proof is addressed in their coursework, perhaps they were conceptualizing proof within only a common content knowledge framing. In this case, these results would be more of a window into a teacher educator's view of proof.

It is important to note that Chi-square tests were not conducted with this particular set of data. In most cases throughout this study, the Chi-square test was testing the assumption that the expected value of the two populations would be equal. In this case, I have sufficient reason to believe that this assumption would be inappropriate for these two populations. This is because I have reason to assume that those who do not address the teaching and learning of proof in their methods course(s) would not provide as many opportunities intended to support the development of MKT-P.

Participants providing no evidence of any domain. Eight of the participants who engaged with the MKT-P items (survey items 17 and 18) indicated that they did not address any of the learning objectives presented to them. As such, they provided no evidence of addressing any of the domains of MKT-P. Of these eight, six indicated that at least one of the learning objectives was addressed in other coursework, leaving only two who indicated that none of the learning objectives were addressed in any coursework in their mathematics teacher preparation program. There are various potential explanations for these two participants. It is likely that there are learning objectives for their teaching candidates related to proof that were not presented to them, as both of these participants indicated that proof is addressed in their course and is embedded throughout. One of these participants skipped over the open-ended items, perhaps indicating a reluctance to spend a lot of time on the survey. The other participant completed all other aspects of the survey and provided details in the open-ended items that indicated that they provide opportunities intended to support the development of MKT-P.

Summary of Survey Findings

The survey results indicated that many teacher educators provide opportunities intended to support the development of MKT-P in their methods course(s), especially in the area of KCT. This was true even though only 40 of the 70 participants indicated that the teaching and learning of proof is addressed in their methods course(s). While responses to the MKT-P items suggested that many teacher educators provided opportunities intended to support the development of MKT-P in their methods course(s) across all four domains, the details provided in the open-ended items outlined strategies that were mostly related to CCK. As such, while the survey provided many details related to the strategies that teacher educators use to address the teaching and learning of proof, connections to teaching and learning in many of these strategies were not explicit. The interview findings outlined in the next chapter explore these strategies in further detail, while also identifying ways in which a teacher educator’s view of proof is associated with their approach towards proof in their methods course(s).

Chapter 5: Interview Findings

The analysis of the five interviews brought further clarity to answering all three research questions. All interviewees discussed the opportunities that their methods courses provided intended to support their teacher candidates in instruction around proof. Additionally, specific instructional techniques and tasks revealed themselves through the interviews. Notably, many of these instructional techniques and tasks were associated with the teacher educator's views related to proof, which varied across all five participants. Their view around proof appeared to be the factor which was most associated with their approach towards proof with their teacher candidates.

Two of the interviewees stood out in unique ways when discussing their views around proof. Kelly provided the most evidence related to a very encompassing idea of proof, using broad ideas to describe what is required to count as proof. Karen provided the most evidence related to a formal idea of proof, describing very specific requirements when describing what counts as proof. The other three (Cassie, Mara, and Nathan) appeared to fall in between Kelly and Karen. They all had ideas related to what is required for something to count as proof. These requirements were more specific than Kelly's but were not as rigorous as Karen's. Additionally, each of the three emphasized one or two of these requirements more than others. Some of these requirements for proof were flagged during analysis because they were mentioned across more than one interview. In the following sections, I describe each of the five interviews, starting with the most encompassing view of proof and moving towards the most formal. I close by drawing connections across interviews.

The Case of Kelly and Her Encompassing Notion of Proof

Kelly's survey results were unique in the sense that she agreed that all four of the potential examples of proof would be considered proof. In fact, she strongly agreed with three of the four examples. The case in which she somewhat agreed was the example of tearing off corners of a triangle and rearranging them to demonstrate that the interior angle measures sum to 180 degrees. She also indicated that she considered proof to be embedded throughout her course. She was the only individual among the five interviewees who indicated this in the survey. This data led me to enter the interview with the hypothesis that Kelly's idea of proof was very encompassing.

In the interview, Kelly described that in their teacher education program, each of the three seminal courses for their teacher candidates has a specific content focus. Notably, the course which Kelly teaches has a content focus on reasoning and proof. Early in the interview, Kelly spoke about why she appreciates that focus in her course:

> I taught high school for a long time--and I don't think it's something that secondary teachers are great at--at knowing how to support students in developing reasoning. And I think it's seen as an area that's just for geometry, but it can be everywhere, and overall great. I mean, you don't have to talk about proof with just geometry students or just with high school students. You can take proof back down--you know, kindergarten students can give you reasons why something works, in an appropriate way. But they can come up with some pretty convincing arguments.

In this excerpt, Kelly alluded to an idea that her teacher candidates need support in the area of proof and that thinking of proof as just a geometry topic or a high school topic may be associated with this need for support.

Kelly expanded on this idea when discussing how her teacher candidates often say that they did not have proof experiences in high school. She felt as if a very formal idea of proof is likely the reason why they felt inexperienced in this area and her goal is to challenge this formal idea of proof.

> As we start to work through those ideas this year, I wonder how many of them are thinking about proof as this big formal argument--you know, two-column versus the informal justifications, because I think if they looked at it that way, they'd realize how much proof they'd actually done. But I don't know that they think of that as proof. I think they think of proof as being very formal. I don't think that they see visual arguments as necessarily being valid proofs. I just wonder how they'll support student thinking if they haven't had to go through those kinds of thinking experiences themselves.

This conversation led Kelly to confirm that she does consider her idea of proof to be "flexible" and that is something "that's changed over the years." She also clarified that "different proofs have different explanatory power."

Kelly's issues with two-column proofs were brought up multiple times throughout the interview.

> I think that two-column proofs are--I could probably get in trouble for saying--an obsolete mathematical device. I think it's a left-over of this Greek-Western civilization ideal of what counts as proof and it's funneling into a very narrow

> way of thinking when there are very wide ways of thinking that are just as valid. So I think it's a remnant that because so many of us have been through it in our own high school experience, [it] lingers.

She also mentioned that the kind of logic used in two-column proofs "doesn't transfer well to other subjects or other areas of life." She instead wants to help students build their thinking skills by "doing some sort of exploration, making a conjecture, testing the conjecture, and trying to verify the conjecture." This evidence implies that Kelly does not feel as if two-column proofs are productive in building those types of thinking skills. Additionally, Kelly incorporates a lot of exploratory mathematical tasks in her course and views these tasks as facilitating the development of teaching skills around reasoning and proof.

Kelly expanded on the "wet box task" during the interview. This task was one that Kelly initially described in the survey as a task related to proof that her teacher candidates are required to complete. Kelly's description of this task from the survey is provided in Figure 5.1. Within this task, students are clearly expected to explain or justify their answer. I asked Kelly how she would respond to an individual who says that this task is not a proof, but instead just a regular mathematics task requiring justification. Kelly responded by saying that "it's not a matter of proving for generalization, but it's a matter of proof for convincing somebody why your reasoning is the best." This led Kelly to liken this sort of convincing to argumentation and in turn, she feels as if argumentation and proof are in "the same box" while clarifying that "I think there's argumentation that isn't proof but I think all proof is argumentation."

Part 1: A shipper is shipping large boxes of iPhones from Shanghai. Each large box measures 36” × 18” × 16”. It is filled completely with iPhones, which are in boxes that are 3” × 4” × 6”. One box falls off the boat and submerges completely, but is pulled out of the water almost immediately. Upon talking with Apple, the shipper concludes that all iPhones that were touching the outside of the wet box will have to be returned to Apple to check if they still work. How many iPhones will have to go back? Be sure to explain how you arrived at your answer, and use words, symbols, and/or diagrams to support your explanation. Part 2: Upon hearing the bad news, Apple decides that they need to ship their iPhones in a larger box. They want to design a box that has twice the volume of the original and that minimizes the number of iPhones that would be damaged in a similar accident. What are the dimensions of a box that serves this purpose? If Apple’s box supplier charges by the square inch of surface area, how much more will the new box cost?

Figure 5.1. The “wet box task” provided by Kelly.

In summary, I inferred that Kelly’s view of proof is very encompassing, in the sense that there are many forms of argumentation that would count as proof. Notable evidence for this inference would be Kelly’s statement that “all proof is argumentation,” her view that the “wet box task” facilitates proof in the form of “convincing somebody,” and her emphasis on seeing proof as much more than two-column proofs. This encompassing view causes her tasks to be structured around exploratory mathematics and complex tasks. This view also may be a reason why Kelly did not mention some of the typical attributes of proof in the interview, such as generalizations, providing reasons for statements, or the justification of specific mathematical concepts such as theorems.

The Case of Nathan and His Emphasis on Student Agency

Nathan was similar to Kelly in the sense that it is important to him that his teacher candidates engage with justification from a more global perspective rather than a narrowing in on any formal idea of proof. However, Nathan still outlined some important criteria that implies a recognition that proof is a more specific notion than justification and there is an important place for that more specific notion. One of the criteria that came up was the importance of moving beyond a finite number of cases and constructing a more general argument. Another piece that came up more frequently was the importance of student agency. Still, Nathan's emphasis in his classroom was more on justification, as the term "proof" brings with it certain "connotations."

Nathan expressed that "often people's experience with proof through high school is this sort of two-column geometry proof and so it's a very format specific and content specific venture" and that the justification that students often engage in throughout all of their schooling is often not validated as sound mathematical reasoning in the same way that proof is. Nathan further elaborated on this tension between proof and justification that he tries to reconcile in his course.

> When I talk about proof or justification--I keep saying both, proof and justification, because I tend to broaden--my idea of broadening it is to talk about valid justifications, almost avoiding the idea of proof because it has certain connotations, and I want prospective teachers to think more globally about what counts as a valid justification, even in contexts and situations that they wouldn't typically think of as a quote "proof," or that may actually be inconsistent in some ways with the kinds of proofs they would see in an academic math paper.

Nathan later clarified that he would not necessarily equate proof and justification, but continued to allude to this "connotation" that proof possesses[3]. While not equating the two, he was also hesitant to place one above the other, saying "I don't want to say that justification is somehow lesser than or proof is somehow greater than because of that [connotation]."

While Nathan often avoids the term "proof" and instead uses "justification" quite often, he acknowledges the tension that exists when recognizing that they are not, in fact, synonymous.

> I'm admittedly playing a little fast and loose with it. I've just found it fruitful in ultimately giving it a small amount of time within a course to really dedicate to [proof]. Framing it as "let's think about valid justifications and how to kind of support that work through discussion" is where I've sort of landed in talking about that.

As such, when discussing what does or does not count as proof, Nathan admittedly focused less on the criteria others consider when labeling something as a proof, such as general cases, logic structure, and others. While acknowledging that empirical examples are not satisfactory as a valid proof, he also noted that they are an important part of the process. Instead, Nathan's focus was on the amount of student agency in the process.

On the survey, Nathan strongly disagreed with the quadratic formula example, which was described as a classroom activity in which it was verified that the quadratic formula identifies the zeros of any quadratic function by completing the square of the

[3] While Nathan did not specify what he meant by this connotation, I inferred that this was related to a tendency for students, and sometimes teachers, to avoid proof due to a perceived difficulty that is associated with it. This is not unlike the teacher in Knuth's (2002b) study that stated that their "experience with the kids is that they would shut down when you use the word proof" (p. 76).

general form of the equation $ax^2 + bx + c = 0$. At first, Nathan was surprised that he originally felt so strongly about this example. However, he noted that his skepticism came from thinking that the teacher is just "demonstrating that to the students." He admits that "it is in fact a way to show that and justify that algebraically," but that he was also thinking "that it's not the students engaging in anything."

It is this idea of proof "at the purely demonstration level" that concerned Nathan. He brought up another example often present in high school classrooms related to a proof of the Pythagorean Theorem.

> We would not necessarily expect students to immediately come to the realization that "oh maybe I can draw a square off of each side of the triangle." That's not like a tool in their toolbox, and the same with completing the square. If you just presented this situation to students, they wouldn't necessarily have that as a tool to engage [with] in the process of proof.

Nathan further elaborated how he likely was focused on the student agency rather than the line between proof and not a proof when filling out the survey.

> Maybe that highlights that my focus is really on the process and not the product. So to say that I wouldn't strongly disagree that the final product is not a proof, I don't know that I would agree with that, but in the moment, I think I was interpreting that as [whether it is] an effective way to teach students about proof of this particular thing. I would have more doubts about that.

This idea came up again when talking about the other cases that Nathan took a strong stance on while filling out the survey. Nathan strongly disagreed that tearing corners off of paper triangles and rearranging them to demonstrate that the interior angle measures of a triangle sum to 180 degrees would be considered proof. While he alluded to the fact that it is only a finite number of cases, he spent more time discussing how unlikely it is that students would take that step on their own.

> Yeah, that's a great example of where students have to probably be told, and they [often] are on a worksheet, to tear off those corners. That would not be something they would probably know to do on their own. So it might be interesting, with dedicating little to it, to at least let students puzzle for 5 seconds about how to prove this, just given a triangle before we immediately tell them that step 2 is to tear off the angles. But it would not be realistic to expect that that would be kind of a figuring out that they would have at the ready.

This idea centers on Nathan's goal of having "students engaged in the mathematical work" and this goal is a central focus in most of Nathan's discussions throughout his course.

Lastly, Nathan strongly agreed that using a counterexample of 2 to refute the claim that all prime numbers are odd would be considered proof. When reflecting, he acknowledges that this is more of an example of disproving something than proving something. However, his strong reaction was related to "acknowledging that that is something important about mathematical proof, that you need the one counterexample to refute effectively and completely." Additionally, he noted how realistically this sort of mathematical work could come directly from a student, again bringing up the theme of student agency.

In summary, Nathan focuses on "valid justifications" and the process of proving more than a traditional product of mathematical proof. This focus is due to the "connotation" around proof and his idea that many of the skills required for both are the same. Furthermore, Nathan places a high value on students engaging in the mathematical work, and this applies when engaging in justification or other modes of mathematics. As such, this theme of student agency is present throughout the entirety of Nathan's methods course.

The Case of Mara and Her Experience in Elementary Teacher Preparation

Mara's interview drew a lot on her duties within her department. She is a member of the mathematics department and teaches a variety of mathematics courses such as statistics and calculus. She is the only mathematics teacher educator in the department and is responsible for all mathematics methods courses in the K-12 teacher education program. She referenced this wide range of grade levels multiple times during the interview.

When asked on the survey to describe her learning goals for teaching candidates around the teaching and learning of proof in her methods course, Mara stated that she wants "them to understand different types of proof." When asked during the interview to expand on why this is important to her, she brought it back to her responsibilities beyond secondary teacher preparation and her experience in elementary teacher preparation.

> Why it's important to me--probably because I do K-12 math teacher education and I see little kids can prove things. You know, when we talk about even and odd numbers and how kids can draw a picture that's a proof or explain something that does prove why a number is even or odd. So I think that it's probably more important to me that my future high school teachers think of proof as not just two-column proofs. That there's lots of ways that I can prove something. And so

> visual proofs, I think, are really important and I think that's a huge stepping stone for kids in high school--that they need to be encouraged to do such proofs, not just to write formal two-column geometry proofs, which is what my future teachers think proofs are about.

Mara later clarified that she is not necessarily against two-column proofs, but that she views it as her role to challenge her teacher candidates' ideas that a two-column proof is the only type of proof. She feels that the two-column proof "pigeonholes" high school proof to geometry, and she wants them "to think about proving and justification in other courses." She often has teacher candidates push back and say that "it's not a proof if it's a picture" and that her "whole goal" is to challenge them to think about how they can "engage kids with proof in different manners."

Exposing teacher candidates to different types of proofs was also the focus of the task that Mara shared. Her teacher candidates are asked to consider the two tasks shown in Figure 5.2. The first simply asked students to make a conjecture based on empirical examples, while the second one pushes students to either prove or disprove that conjecture. Mara then asks her teacher candidates to examine a variety of student work that attempted to prove or disprove the conjecture. The teacher candidates are asked to determine which pieces of student work would count as proof or not and a variety of formats are presented. She also highlights that this is not a geometry proof, further challenging her teacher candidates' notion of what does and does not count as proof.

TASK A	TASK A'
MAKING CONJECTURES – Complete the conjecture based on the pattern you observe in the specific cases.	For problems 29 and 30, complete the conjecture based on the pattern you observe in the examples. Then explain why the conjecture is always true <u>or</u> show a case in which it is not true.
29. Conjecture: The sum of any two odd numbers is: ______	**29. Conjecture:** The sum of any two odd numbers is: ______
1 + 1 = 2 7 + 11 = 18 1 + 3 = 4 13 + 19 = 32 3 + 5 = 8 201 + 305 = 506	1 + 1 = 2 7 + 11 = 18 1 + 3 = 4 13 + 19 = 32 3 + 5 = 8 201 + 305 = 506
30. Conjecture: The product of any two odd numbers is: ______	**30. Conjecture:** The product of any two odd numbers is: ______
1 x 1 = 1 7 x 11 = 77 1 x 3 = 3 13 x 19 = 247 3 x 5 = 15 201 x 305 = 61,305	1 x 1 = 1 7 x 11 = 77 1 x 3 = 3 13 x 19 = 247 3 x 5 = 15 201 x 305 = 61,305
(McDougal Littell (204), Geometry, p. 7, #29-30)	

Figure 5.2. The proof task Mara provided during her interview.

Mara also placed value in student agency and this came up when discussing the quadratic formula example. Mara neither agreed nor disagreed with that example and elaborated on why she felt she was unable to take a stance.

> I think that one is one where I'd want to see it really written down to see how a kid had done. Are they just applying some procedures and algebraic manipulations or are they just understanding what they are doing? I probably want to see a little more explanation along the side of it, but it's probably a proof. That one I felt like I needed more context for.

Mara acknowledged that it likely would look like a perfect proof, but alluded to the same idea as Nathan in that the product is not necessarily the most important aspect of the proof, but the process is also a critical piece to consider. If the students are not engaged through the process of reasoning in this case, that would be the missing piece.

Mara also spent time discussing the importance of moving from specific examples to general cases. When discussing the triangle inequality example, Mara strongly disagreed that a student explaining that "no triangle exists with side lengths 1, 2 and 10 because the long side is too long to allow the shorter sides to meet" would be considered a valid proof. This was because it revolved around “specific side lengths” and it was “not generalized.” Mara also identified this as a common missing piece of something that is maybe a really good argument but not quite proof.

> I think it’s the jump to complete generalization that is one of the things that kids struggle with. Where they can come up with examples and reasons why these examples work. The jump to actual generalization can be really tough for kids.

Her advocacy for non-traditional proof formats was brought up when discussing her thoughts on the triangle sum example from the survey, where students tear corners off of paper triangles. Mara strongly agreed that this constituted proof, saying that it is “definitely one we do in my elementary content courses and one that we talk about as a visual proof.” When doing this in her class, she makes sure to get “a breadth of different types of triangles.” This appears to conflict with her idea that generalized cases are an important criterion, as even though this visual proof moves beyond a single case, there are still a finite number of triangles in the classroom.

Mara was asked about the perspective that this sort of activity is just a step towards a more formal proof. Her response pulled further on her experience preparing elementary teacher candidates.

> I think it’s very age dependent. But yes, for an elementary school child, I absolutely think that is a proof. When I do it in class, I make sure we’re getting a breadth of different types of triangles. We don’t need to all have isosceles triangles when we do this. I talk about how it’s important that we’re seeing different types of triangles before we do such a thing.

The survey item asked if this sort of activity would be considered proof at the secondary level. While Mara strongly agreed with this item, it is notable that her reasoning was more dependent on whether it would be considered proof at the elementary level and she did not justify her response with reference to secondary students.

In summary, Mara has an encompassing idea of what proof looks like but has criteria related to the generality of the cases and the student reasoning that is used. This is reflected in her practice by her goal of challenging her teacher candidates' ideas of what proof looks like and pushing them to validate authentic student reasoning. Relying on empirical examples is one step in the process, but her teacher candidates must develop the tools to progress from those empirical examples to general cases. She does not place as much emphasis on what that generalized argument must look like and whether it must come in the form of a formal proof. Additionally, Mara's idea of what counts as proof appears to be dependent on the age group under consideration. This condition stems from her experience not just with secondary teacher preparation, but elementary teacher preparation as well.

The Case of Cassie and Her Emphasis on General Cases

A unique aspect of Cassie's survey results was that she indicated that the teaching and learning of proof is not a topic that is addressed in her methods course, providing the reason that teacher candidates learn about proof sufficiently through other methods instructors. However, when asked about specific topics associated with the MKT-P framework, Cassie indicated that six of the eight topics on the survey were addressed in her methods course. One potential reason for this could be that Cassie does address some of these topics in her course, while at the same time not necessarily considering these topics to be directly associated with the teaching and learning of proof. Regardless of these reasons, there is evidence to suggest that Cassie does, in fact, address topics related to MKT-P in her methods course.

Cassie also provided evidence that may infer that her idea of proof is somewhat encompassing. She immediately stated that she does not "focus so much on having them do formal proofs."

> We want them to take a problem, work on it, hypothesize how this can be generalized, and then we'll talk about how they can prove that, and then how they can help their students with that proof.

This emphasis on generalization came up in response to almost every prompt, often saying that a major goal of her class is to help her teacher candidates "think about how they can get their students to generalize that and prove it, but not in like a formal proof manner." In this sense, Cassie places a great emphasis on generalization but less of an emphasis on the traditional formalities of proof.

When asked to give an example of a task intended to support her teacher candidates with proof, Cassie described a task related to generalizing a pattern.

> We do a lot with patterns and helping students find patterns. So one of the problems that we do is inverting a triangle, so they will have pennies and they have to see how many moves they need to make to invert a triangle. Essentially it's just the triangular numbers. So they will test it and say "oh well I just move these two." And then on the next triangle it's like "oh I move these three." So they think it just increases by 1 but once they get to the larger triangles, that falls apart. So we say "keep testing it with larger triangles to see if it will work." Eventually

> they realize it's just the triangular numbers…We try to get them to think about how they can generalize to larger cases. And we'll ask them "if you had a triangle that had 15 rows, how many moves do you think it would take?" And then we'll have them test that hypothesis and see if it works and stuff like that.

This type of pattern task came up multiple times throughout the interview. Often, Cassie's teacher candidates continue these patterns for a small number of cases and then she will push them to consider a case that is so far out that continuing the pattern is not realistic. It is these types of questions that are often what pushes students to consider generalized cases. Cassie feels that "posing those different follow-up questions [such as the 15th case] to the [teacher candidates] really helps them understand that and how to get them to understand proof."

In relation to the four potential examples of proof, Cassie agreed with three of the four examples in the survey. The one case where Cassie indicated disagreement was the example of a student using a counterexample of 2 to refute the claim the all prime numbers are odd. While this may be a valid way to refute that claim, Cassie felt more work would need to be done.

> It's just one counterexample, and that would prove that it doesn't work for that case, but it doesn't prove that they're all odd...We want them to focus on the generalizations...It's not a true proof for that statement.

While the task is not to prove all prime numbers are odd, Cassie again identifies generalization as the missing piece, even though generalization is not required to disprove something.

Additionally, the one case where Cassie took a strong stance was on the triangle sum theorem exploration where students tear off the corners of paper triangles. Like Mara, Cassie strongly agreed that this is proof and immediately mentioned that it was a task she has experience with.

> That is something I've actually done in our classes when I've taught high school geometry. Instead of just telling them, here's a theorem, the interior angles add up to 180 degrees, if they do that activity they think "oh, yes this does work." So that's something that I've actually used in my classroom and I'm comfortable with and I feel like definitely proves that example.

Cassie appeared to deviate from her typical criterion of generalized cases when considering this task.

In summary, Cassie places a lot of emphasis on generalized cases and less emphasis on a certain format. Cassie's reference to generalized cases came up frequently and the tasks she asks her teacher candidates to engage in often push them to consider these generalized cases. She pushes her teacher candidates to let students explore patterns and come up with conjectures and gives them opportunities to develop the tools to bridge students from specific examples to generalized cases. Cassie rarely spoke about the notion of proof without referring to a condition of generalization.

The Case of Karen and Her Formal Notion of Proof

A unique aspect of Karen's survey results was that she indicated strong disagreement with all four examples posed as potential proofs. Additionally, she did not show evidence of addressing elements of common content knowledge in her methods coursework on the MKT-P survey items. However, she indicated that she expects some of those topics to be addressed in her teacher candidates' content courses. Karen was the only participant who strongly disagreed with all four examples of potential proof and this data led me to enter the interview with the hypothesis that Karen's idea of proof was very formal.

During the interview, Karen confirmed my hypothesis by saying she is probably more along the lines of a "pure mathematician" in terms of her views of proof. She alluded to ideas that were not brought up in the other four interviews, talking about the importance of rules, the axiomatic system, providing reasons for every step, and specificity.

> I want my teacher candidates to be specific about things. I want them to be precise in the language that they use, the notation that they use, the structure that they use.

Karen has strong expectations for her teacher candidates when it comes to precision and structure, though she acknowledges that she does not expect her teacher candidates to hold their future students to the same standards, saying "I want them to have a very solid understanding of the content but that doesn't mean that they have to teach at that level or have those same expectations for students."

It was also clear that her views on proof were a major influence on her expectations for her teacher candidates and these views were accurately reflected in the survey items around the four potential examples of proof. Karen said that using 2 as a counterexample to refute the claim that all prime numbers are odd is simply a counterexample and "that discussion between proof by contradiction and counterexample is often confused." Additionally, the example of a student explaining that "no triangle exists with side lengths 1, 2 and 10 because the long side is too long to allow the shorter sides to meet" was not valid because “it’s an anecdotal statement from a student.”

Karen also discussed the example of completing the square of the general form of the equation $ax^2 + bx + c = 0$ to verify that the quadratic formula identifies the zeros of any quadratic function. The concern there is that “it’s just kind of a procedure that you’re following. It’s not a conceptual development of something.” Additionally, she often does not see reasons provided for every step when that is done in a classroom, and providing reasons is a critical aspect of proof from Karen’s perspective.

Karen also discussed the triangle sum theorem activity. Like Mara and Cassie, Karen has done this activity many times in her classes. However, she says that it is not a proof and that it “shows them the triangle sum theorem but that’s certainly not a proof of it.” She says that this sort of activity can often lead to a proof. It is also notable that while Karen does not consider this activity to be a proof, that does not mean that teachers in this case have to move students towards a proof to meet the goals of the lesson. This implies that while Karen has a formal idea of what counts as proof, there are still activities that do not count as proof that can have explanatory power. Additionally, sometimes those activities that just “show” something can be enough to meet the learning objective in a high school classroom.

When discussing tasks and instructional strategies in her course, Karen described a progression of activities designed to establish the various “rules” involved when proving something. She discussed an article from *The Mathematics Teacher* (1999) called The Rules of the Game, which is used to dive deeply into understanding characteristics of a deductive system. A portion of this task is provided in Figure 5.3.

The "Letter Game"

Undefined terms: Letters M, I, and U
Definition: x means any string of I's and U's.
Postulates:
1. If a string of letters ends in I, you may add a U at the end.
2. If you have Mx, then you may add x to get Mxx.
3. If 3 I's occur, that is, III, then you may substitute U in their place.
4. If UU occurs, you drop it.

Objective: Given one string, you are to derive or prove some other string.

Example: Given: MI
Prove: MIIU

Statement	Reason
MI	Given
MII	Rule 2
MIIU	Rule 1

Prove the following theorems:

1. Given: MIII
Prove: M

Statement	Reason

2. Given: MIIIUUIIIII
Prove: MIIU

Statement	Reason

3. Given: MI
Prove: MUI

Statement	Reason

4. Given: MI
Prove: MIUIU

Statement	Reason

5. Given: MI
Prove: MIIUIIU

Statement	Reason

6. Given: MIIIUII
Prove: MIIUIIU

Statement	Reason

From the *Mathematics Teacher*, May 1999

Figure 5.3. The "rules of the game" task provided by Karen.
Note. Reprinted from "The rules of the game"; by D. Gernes, 1999, *The Mathematics Teacher*, *92*(5), p. 428.

She then has her teacher candidates engage with a task where she has cut up the pieces to a complete proof and they have to put it back together. The next task has teacher candidates fill in a partially complete two-column proof, where some statements and reasons are missing. The progression then moves to a proof with a set of statements and missing reasons and then to a proof with a set of reasons and missing statements. These types of tasks seem to promote the format of a two-column proof, which Karen values due to her emphasis on structure and specificity.

In summary, Karen's views on proof appear more formal than the other interviewees. There are many items that would be considered proof by the other interviewees that Karen would not consider proof. However, Karen still values those sorts of activities and those activities can, at times, be just as powerful in explaining a mathematical concept as a formal proof. Due to these views, she has high expectations for her teacher candidates related to the formality and specificity in which she wants them to prove things and provides tasks that promote this formality and specificity. Despite this, she does not expect her teacher candidates to hold these same expectations for their future students.

Themes across the Five Interviews

The notion of what counts as proof was a significant factor in the selection of each interview participant. This became a major topic in each interview and it became clear that this impacted their instructional techniques and tasks in their methods coursework.

Formal vs. informal proof. The notion of a "formal proof" came up in all but one interview (Nathan). Kelly, Mara, and Cassie discussed an idea of proof that was separate from formal proofs, implying that there are formal proof and informal proofs. Additionally, Kelly and Mara alluded to two-column proofs as falling into the category of formal proofs. It is not clear from the interviews if two-column proofs are considered by these participants to be one type of formal proofs or if they considered formal proofs and two-column proofs to be synonymous.

Karen also referenced formal proofs but never did so in a way to suggest there are informal proofs. Karen discussed some examples that were similar to or identical to some of the ideas put forth as informal proofs by the other participants. However, when discussing these examples, Karen simply labeled them as “not a proof.” This implies that Karen may feel there is no such thing as an informal proof and that a proof must be formal. That would not necessarily imply that Karen feels that a proof must be done in a two-column format, although most of Karen’s discussions around proof centered on the idea of being in a two-column format.

Paper triangles and the triangle sum theorem. Another theme across the interviews surrounded the paper triangles example from the survey, particularly in how much time each interviewee spent discussing this example and the range of views that emerged. Four of the participants noted in their interview that they have students tear off the corners of paper triangles to demonstrate that the interior angle measures of a triangle sum to 180 degrees. Of those four, Cassie and Mara strongly agreed that this counted as proof at the secondary level and used the fact that they do this activity in their course as an immediate rationale for this agreement. Additionally, Mara mentioned that she does this in her elementary methods course, despite the focus of the survey item being on the secondary level. Kelly somewhat agreed that it counted as proof and her lack of a strong stance came from the fact that it is still a finite number of empirical examples. She did feel it was a really good task, which is why she does it in her class, and that it is a good step for “building a more solid argument.”

On the other hand, Karen also does this task in her class but still feels strongly that it is not considered a proof. The familiarity with the task appeared to lead Mara, Cassie, and Kelly to agree that it counted as proof but it did not have that effect for Karen. Instead, she continued to rely on her formal criteria for what counted as proof. While it may be a great activity towards building that mathematical principle, Karen feels that you still have not proven anything.

Nathan was aligned with Karen in the sense that he strongly disagreed that this counted as proof. However, he implied that he would not even necessarily consider this a good task, which strongly deviated from all four of the other participants. Nathan’s strong emphasis on student agency played a big role in this stance, as he felt that many times in this activity, students are likely only taking steps that are prescribed by the teacher. The discussion around this particular survey item illuminated the prominent themes that arose out of each interview.

The shortcomings of empirical examples. All five interviewees brought up the shortcomings of empirical examples. Kelly, whose stance on proof appeared most encompassing, felt that the example of tearing off corners of paper triangles came up slightly short of counting as proof. While she still somewhat agreed that it counted as proof, it was the only one of the four examples that she did not strongly agree with. Her reason was that even if you saw hundreds of examples, they are still all just empirical examples. Although there were a variety of reasons why Nathan strongly disagreed with this particular example, one of those reasons was that it was just one example. Karen and Mara brought up these same concerns when discussing the triangle inequality explanation around a triangle with side lengths of one, two and ten. Lastly, the prominent theme throughout the entirety of Cassie's interview surrounded moving from specific examples to general cases. Despite these five participants falling in different positions on the encompassing-to-formal spectrum, generalization appeared to be a common essential criterion.

Argumentation, justification, and proof. Terms such as argumentation and justification came up in addition to the term proof in all but one interview (Cassie). These four participants felt there was certainly some overlap between these terms but the nature of that overlap varied. When discussing the "wet box" task (Figure 5.1), Kelly noted that "it's not a matter of proving for generalization, but it's a matter of proof for convincing somebody why your reasoning is the best." In that sense, she felt that this task required students to engage in argumentation and that argumentation and proof are in the same box. She clarified this by saying that "I think there's argumentation that isn't proof but I think all proof is argumentation."

Nathan referenced justification very often in the interview and valid mathematical justifications are a prominent theme in his methods course. When discussing the role that justification plays in his course, he admitted to playing "a little fast and loose" with the terms "justification" and "proof". While he would not equate proof and justification, he views them as building skill in similar modes of mathematical thinking. Since one is not necessarily greater than the other, he finds himself using the term "justification" more often due to the "connotation" that comes with the term "proof". Mara referenced a similar idea, albeit with less of a focus throughout her interview, when she talked about pushing her teacher candidates to think “about argumentation and justification more so than just geometry proofs.”

The themes above seem to imply that Nathan, Kelly, and Mara value things such as argumentation and justification and that these are the underlying ways of thinking that they want their teacher candidates to engage in. Sometimes that may come in the form of proof, but other times it may not. Karen, on the other hand, views proof as a way of doing mathematics that is essential for her teacher candidates in and of itself. She noted that argumentation and justification “could be included in proof” but that they are not synonymous with proof. The other participants did not refer to proof in ways that implied it was an essential piece separate from argumentation and justification. This unique viewpoint of Karen’s is likely related to her more formal idea of what counts as proof, as she stated that she is probably along the lines of a “pure mathematician” compared to others.

Summary of Interview Findings

The interviews brought important context to the survey findings, particularly related to teacher educators’ views of proof and how these views are associated with how they address MKT-P, and proof broadly, in their methods course(s). These views ranged from encompassing to formal and each teacher educator brought a unique view of proof to consider. Additionally, the range of views expressed resulted in each teacher educator emphasizing different aspects of proof and proving in their methods course(s). The discussion in the next chapter centers around the importance of narrowing this range of views to come closer to a specific idea of what counts as proof at the secondary level, while outlining contributions to the Lesseig (2011) MKT-P framework and directions for further research.

Chapter 6: Discussion

In this study, I have identified the various ways in which teacher educators are providing opportunities to support the teaching and learning of proof in their method course(s). The data shows that there are many different ways that teacher educators are approaching this topic in their courses. However, a key takeaway from the data is related to the term "proof" itself. In particular, the interviews showed that a teacher educator's idea of what counts as proof is associated with the ways in which they address proof in their courses. The term itself comes with a connotation that some teacher educators work to avoid. Some teacher educators prefer to focus on ideas such as justification and reasoning due to this connotation. This may be due to an idea that these ideas are synonymous with proof or that they serve similar or equivalent learning goals in relation to proof. Other teacher educators were hesitant or even resistant to referring to those terms as if they were anything close to synonymous with proof. For these teacher educators, proof is a label that is reserved for something very specific, whether it be a process or a product.

In this chapter, the discussion centers on the term "proof" itself and the purposes behind it as a learning goal. Connections are made to the findings related to how this impacts the teaching and learning of proof in a secondary mathematics methods course. Additionally, I explore other key implications from the findings. In particular, I discuss how the findings fit with the MKT-P framework from Lesseig (2011), both in ways that match with the framework and in ways that can contribute to the framework. I close this chapter with a discussion around limitations in this study and directions for further research.

What Counts as Proof?

I opened this paper by defining proof in a way that seemed most appropriate with the MKT-P framework in mind. The findings suggest that the definition of proof is far from agreed upon among teacher educators. As such, the purpose of teaching and learning proof also varied among participants. In this section, I outline the ways in which this variation manifested itself in the findings and how it connects to implications for further research around the MKT-P framework in teacher education.

Proof versus justification and reasoning. Several participants in the survey wrote about justification or reasoning when discussing their learning goals related to proof. In some of these responses, the term "proof" was never even mentioned. In a specific response, the participant wrote that they "emphasize justifications rather than proof." Another drew more specific connections, stating that "proof is a formal presentation of the kinds of justification and reasoning that students should be doing each and every day."

To further unpack the role of proof in secondary mathematics teacher preparation, the relationships between justification, reasoning, argumentation, and proof requires further exploration. Most participants presented evidence that justification and reasoning are important skills for students to engage in and develop. However, substantially fewer participants indicated that proof was just as important as those skills. In some cases, proof was simply one example of a vehicle to develop those skills. For others, it was a more formal presentation of justification and reasoning. For others still, there was evidence that these ideas are considered synonymous. These differing views make it difficult to discern a consensus among teacher educators around learning goals related to proof in a secondary classroom.

From the interviews, Nathan served as a prominent example of a teacher educator walking this fine line between these various terms, admittedly playing “fast and loose” with these different ideas. On the survey, Nathan indicated that the teaching and learning of proof is addressed in his methods course, but findings from his interview indicated that he acknowledges that the teaching and learning of proof is not addressed directly. He still felt as if these learning goals are addressed even though he places a much greater emphasis on developing “valid justifications” in mathematics. It is possible that Nathan reconciles these seemingly contradictory statements because he feels that he is ultimately developing the same modes of mathematical thinking that proof is intended to develop. In this sense, Nathan views justification as an appropriate substitute for proof, while still acknowledging that they are not synonymous.

The process versus the product. Nathan’s discussion around justification versus proof is related to another theme from the data related to whether a teacher educator considers proof as a process or a product. In Nathan’s case, he recognizes that proof is a more formal presentation of the justifications that he wants his teacher candidates to engage in. He admittedly does not typically push his students to this formal presentation, implying that as long as his students are engaged in the process of justifying their mathematical work, he will place less emphasis on the product. In this way, the teacher candidates in Nathan’s course may engage in the proving process without completing the proving process in the form of a formal proof. This view relates to the problem-solving perspective of proving outlined by Stylianides, Stylianides, and Weber (2017).

This theme also came up in the survey in relation to the items asking participants what counted as proof. Over half of the participants agreed that the activity of tearing off corners of paper triangles and rearranging them to demonstrate that measures of the interior angles of a triangle sum to 180 degrees would count as a proof. While there was variation related to how strongly participants agreed with this example, it is notable that very few disagreed that this activity counted as proof at the secondary level. The way this item was written provided no indication that students would write down anything at all as a part of this activity. This item, as written, does not emphasize proof as a product, implying that those who agreed with this statement may also view the process of proof as being separate from any particular final product.

On the other hand, Nathan's experience with an activity such as this gives him hesitation to consider it proof as he does not feel that students are actually engaged in the justification and reasoning process. In his view, both the process *and* the product are missing, as students are often following directions by a teacher and then making an observation at the end of the activity. Karen sees a role for this sort of activity, even though she does not view it as a proof activity. Karen saw justifying and reasoning skills at play in this activity but ultimately felt that the activity needed to move to something more formal to be considered proof. In this way, Karen emphasizes a view of proof as a product. The process still includes justification and reasoning, and the proof itself requires justification and reasoning. However, for Karen, the term "proof" is reserved for a formal product.

Proof as formal or encompassing. The findings in this study imply that the view of proof as a product or a process may also be related to a view of proof as formal or encompassing. For the interviewees who indicated a more encompassing view of proof, it was ultimately more important that their teacher candidates have the opportunity to develop the mathematical thinking skills related to justification and reasoning. In Kelly's case, the idea of a formal proof is an outdated and narrow mathematical way of thinking. Kelly wanted to validate "the very wide ways of thinking" that students are capable of engaging in.

The juxtaposition of formal proof versus a more encompassing idea of proof is not foreign to the current landscape of mathematics education research. As just one example, Herbst and Brach (2006) found students struggling to identify what counts as proof when engaging in the justification of mathematical properties. Explaining these properties using general cases in an open-ended format was not considered proof by the students engaged in that work. However, doing the same explanation in a two-column format counted as proof for these students. For the students in the Herbst and Brach study, the two-column proof represented their view of formal proof. It is apparent there is no such thing as "informal" proof for these students.

The teacher educators in my study who had a more encompassing idea of proof certainly indicated that proof can be formal or informal. In many of the cases presented, these informal proofs were focused on justifying mathematical relationships regardless of the product produced on paper. Additionally, for some of these participants, these informal justifications were sufficient and there was no necessity to move towards a more formal presentation. On the other hand, those who held a more formal idea of proof felt that these informal explanations and justifications held an important role in mathematics education, but it would be inappropriate to label them as proof. In this sense, the term itself holds a certain amount of weight for these teacher educators.

The weight of the term. As discussed, Karen is one example of a teacher educator who wishes to reserve the term “proof” for modes of mathematical justification presented in a formal product. Of all of the potential examples of proof presented in the survey, Karen strongly disagreed that any of them should be considered proof at the secondary level. At the same time, Karen saw some value in using all of those potential proof activities in a secondary classroom or methods course. From Karen’s perspective, these activities engage students in justifying and can lead to a proof, but to label these as proof as is would be a misuse of the term. Karen has high expectations for her teacher candidates when it comes to proofs, expecting them to be quite formal. However, she also said that her teacher candidates do not necessarily need to hold those same expectations for their secondary students. A secondary student’s argument can be “more informal.” Taking this with Karen’s careful use of the term “proof”, I inferred that Karen feels that not every justification or argument from a secondary student needs to lead to a proof.

While Nathan does not necessarily hold a formal view of proof, he feels the term carries some weight in a different manner. In his case, proof has a certain “connotation,” which causes him to ultimately emphasize justification rather than proof. In this sense, justification serves as a bit of a substitute for proof from Nathan’s perspective. He is able to avoid the connotation, while still serving similar learning goals in his eyes. The connotation around this term is also not new to mathematics education research. Knuth’s (2002b) study found that teachers are often resistant to labeling something as a proof because it causes students to “shut down” (p. 76). Research in this study and in others has shown that the weight of the term has a tendency to invalidate wide ways of mathematical thinking, which teachers and teacher educators are resistant to.

The weight of the term may also explain a seemingly significant discrepancy in the survey data. Only 40 out of 70 participants indicated that the teaching and learning of proof was addressed in their methods course(s). When examining the MKT-P items (survey items 17 and 18), it was very surprising to see 58 participants indicate that at least one of the learning goals from Lesseig's (2011) MKT-P framework was addressed in their methods course(s). For context, this means that approximately 57% of survey participants indicated that the teaching and learning of proof was not addressed in their methods course(s), while approximately 83% of survey participants marked that at least one of the MKT-P learning goals is addressed in their methods course(s) on a later item.

This discrepancy may be explained by the fact that certain elements of the MKT-P framework can be developed in spaces outside of proof. For example, one learning goal stated that "teacher candidates learn about forms of argumentation appropriate at various secondary student learning levels." While argumentation is involved with proof, it is certainly possible to learn about forms of argumentation at different learning levels without talking about formal proofs. However, this knowledge is still necessary for teachers if they are to support students in the area of proof. The findings also imply that some teacher educators may provide opportunities to develop SCK, KCS, and KCT for proof without making direct connections to proof. This raises further questions related to the role of formal proofs in secondary education and teacher preparation.

Proof is a lot of things, but not everything is proof. The participants in this study varied widely in their expectations around what counts as proof at the secondary level. Previous research has shown that this expectation is clearer in the mathematics community and at the undergraduate level, particularly when it comes to the most important characteristics (Moore, 2016). Many teacher educators in this study appear to recognize that the rigor that is expected in a proof in an undergraduate course is not required at the same level with secondary students. For Mara, she spends less time on formal proof in her methods course knowing that her teacher candidates engage with formal proof in their content courses. However, those content courses may lack explicit connections to teaching and learning (e.g., Graham, Li, & Buck, 2000; Tatto, 2018; Wasserman, 2018). Knowing that their teacher candidates engage with proof at the undergraduate level in their content courses, it is the role of the teacher educator to draw connections to how teacher candidates can use their knowledge of proof from undergraduate content courses and identify age-appropriate ways for their secondary students to prove.

Some teacher educators are also wary of the connotation that comes with proof, and this is likely due to the rigorous method of proving that their teacher candidates experience in undergraduate content courses. This rigor often manifests itself in what has been referred to throughout this study as "formal proof." For example, there is a perception among teachers that algebraic notation is required for formal proofs at the undergraduate level but this would not be necessary to require from a secondary student if the underlying logic is correct (Baldinger & Lai, 2019). In this sense, many teachers wish to validate alternative ways to represent mathematical proofs without always requiring such formal notation.

This study finds similar results among teacher educators. For example, Karen holds high expectations for her teacher candidates when it comes to the rigor of developing a formal proof. At the same time, she recognizes that these same expectations are often not appropriate at the secondary level. Whether these expectations are bridged in her methods course is not evident in this data. Additionally, it is not clear if a less formal argument from a student would be considered a valid proof by Karen or if it is merely a valid argument that falls short of her standard of proof.

On the other hand, other participants put great emphasis on lowering the rigor and challenging the connotation of what counts as proof at the secondary level. For example, Mara challenges her teacher candidates to accept that a proof can be in the form of a picture at the secondary level. Kelly feels that teacher candidates come to her methods course with a view that visual proofs are not valid proofs. She views one of her objectives as challenging the notion that proof has to be "this big formal argument." Based on these results, I argue that teacher educators feel that teacher candidates need to develop a more encompassing view of proof for their future secondary students than the view that they experience in their undergraduate content courses. However, there is a fine line that must not be crossed in a way to hold proof as a distinct mathematical construct.

One way in which this was apparent in the data was the number of people who stated that they emphasized justification over proof. Research advocates for the development of reasoning and justification skills and the MKT-P framework incorporates those skills. In many ways, these skills overlap with proof, yet by emphasizing justification over proof it implies that these participants view these ideas as close to synonymous. Not all justification is proof and proof is still something that must be addressed at the secondary level. In an attempt to remove some weight from the term "proof" by treating proof and justification as synonymous, proof may be trivialized to a point where too much weight has been removed from the term. As such, proof can be many things at the secondary level, but it is not everything. Teachers, and thus teacher educators, must still hold a certain weight to the term, while still removing some of the weight it holds at the undergraduate level.

This phenomenon of treating proof and justification as synonymous was also apparent when participants were presented with the activity of tearing off corners of paper triangles to demonstrate that the measures of the interior angles of a triangle sum to 180 degrees. When asked if they agreed or disagreed that this activity would be considered proof at the secondary level, most participants agreed. This was surprising since this activity produces a finite number of cases. One could also argue that this activity lacks the power of explaining *why* the measures sum to 180 degrees. In her interview, Kelly discussed why she agreed that this counted as proof, alluding to the fact that students are often convinced that the triangle sum theorem holds when participating in this activity. This relates back to Hersh's (1993) notion of convincing versus explaining in mathematics education. Hersh feels that the purpose of proof in mathematics education is to explain rather than convince, because students "are all too easily convinced" (p. 396). The standard of convincing a secondary student is vastly different from the standard of convincing a mathematician.

This idea of convincing is at the heart of considering the ideas of convincing arguments and valid proofs. An argument may be quite convincing to a secondary student while coming short of being a valid proof. Additionally, teachers may be wary of labeling the proof as "invalid" due to the accompanying connotation that the reasoning is invalid (Baldinger & Lai, 2019). It is also important to consider the mathematical goal of a lesson or activity. If the goal is to develop a valid proof, it is possible to construct a convincing argument with valid reasoning while still coming short of a valid proof. Teachers require opportunities to develop tools that will validate the reasoning while still giving the student support in moving towards a valid proof.

Additionally, the MKT-P framework contains many elements that are related to justification, argumentation, and reasoning but could be addressed outside of the context of proof. One such example is having knowledge of various forms of argumentation appropriate for various students' level. This example is not directly related to proof but is knowledge that teachers must hold if they are to ultimately assist students in the development of a proof. The findings in this study suggest that many teachers may provide opportunities for teacher candidates to develop MKT-P in this space but the work of extending that to the product of a formal proof is missing for some.

To further consider ways of approaching the teaching and learning of proof in methods courses, the expectations for proof at the secondary level must become more specific. Must proof be formal? Does formal proof at the secondary level look different from the formal proof that is expected in undergraduate mathematics? When proof at the secondary level becomes more explicitly defined, the MKT-P framework can serve as a tool for teacher educators to provide opportunities to develop the skills the teachers need to support their students in the development of proof.

Contributions to the Lesseig (2011) Framework

The purpose of this study is to:

1. Identify ways that secondary mathematics methods courses provide opportunities to support the development of MKT-P.
2. Identify instructional techniques that teacher educators use to provide opportunities to support the development of MKT-P.

The data identified a variety of instructional techniques to provide these opportunities in certain areas of MKT-P and I will outline these below. In addition, there were other areas for MKT-P in which I was unable to identify instructional strategies. I will identify these for further consideration. Lastly, there were other ways that secondary mathematics methods courses provided opportunities to support the development for MKT-P in areas which are not explicitly called out in the framework. I close this section by outlining these methods as potential additions to the framework.

Areas for which instructional strategies were identified. Participants provided many examples of mathematical tasks that are intended to develop CCK for proof. In particular, many of these activities asked students to justify a mathematical relationship, which has the potential to help teacher candidates develop the ability to construct valid proofs. These activities also have the potential to develop essential proof understandings, such as a theorem having no exceptions and that a proof much be general. Lastly, the learning goals given by participants provided evidence that implied that discussions take place in methods courses related to the various functions of proof.

Much of the data provided around SCK was related to explicit understanding of proof components. In particular, a number of participants indicated that they ask their teacher candidates to prove something using a variety of methods, such as visual proofs, proofs using technology, and two-column proofs. Some of these participants made explicit mention of drawing connections between the different methods of proving a particular relationship. These are intended to support the development of knowledge around modes of representation and modes of argumentation.

Drawing connections between various proofs of the same concept or thinking can also provide the opportunity to develop KCT, as knowledge of proof connections is a criterion within that space. Also related to KCT, a number of participants ask their teacher candidates to design activities to engage students in proving. This can support the development of knowledge related to the relationship between instruction and proof schemes. One assignment in particular asked teacher candidates to "choose a problem/context that (as currently written) does not engage students in proving and modify it into one that is likely to do so." Another teacher educator provided opportunities intended to support the development of questioning strategies. This opportunity involved teacher candidates examining student work and identifying two or three questions that they would ask each student.

In the KCS space, the most common strategy was having teacher candidates examine student work, which relates to the developmental aspects of proof. Much of this examination involves making sense of what the student did and determining what work remains to reach a valid proof. Further clarification related to the selection of student work would help refine what this strategy looks like.

Areas for which we still must identify instructional strategies. The data did not provide much evidence related to how teacher educators can provide opportunities for teacher candidates to develop knowledge of situations for proving in the area of SCK. Within learning goals, one participant wrote about wanting their teacher candidates to know how to motivate proof, but did not provide specific strategies.

Within KCT, the data did not identify ways that may support teacher candidates in using pivotal examples or counter-examples that could be used to focus on key proof ideas or extend a student's thinking. Related to KCS, the data did not provide much evidence related to how teacher educators can provide opportunities to develop explicit knowledge of proof schemes taxonomy. A small number of participants referred to their incorporation of the G. J. Stylianides (2010) reasoning-and-proving continuum. However, critical work remains in identifying strategies related to how teachers can progress students from a space in which reasoning is sound to a space in which reasoning is sound *and* a valid proof is constructed.

Potential additions to the framework. Based on the responses around learning goals and tasks/assignments that were provided throughout the survey, I have identified additional learning goals for developing MKT-P that have not yet been explicitly called out in the Lesseig (2011) framework. Table 6.1 outlines various forms of CCK, SCK, KCS, and KCT that are not explicitly called out in the framework but were survey contributions that may be worthy of consideration as I considered the mathematical knowledge required for teaching proof. These are merely meant to be considerations for how this data can be used to expand the framework, acknowledging that some of these considerations may be more specific forms of elements already present in the framework.

Table 6.1. Potential additions to the mathematical knowledge for teaching proof framework.

Common Content Knowledge	Specialized Content Knowledge
• Methods of writing to present a formal proof • Using proof to deepen content understanding	• Socially negotiated aspects of what counts as proof • Affordances and constraints of various proof formats
Knowledge of Content and Students	**Knowledge of Content and Teaching**
• Progressing students along the continuum in the reasoning-and-proving framework (G. J. Stylianides, 2008, 2010).	• Ways to connect conjecturing, reasoning, and proving as a sequence • Ways to modify contexts or problems to make students more likely to engage in proving

Within CCK, multiple participants wrote about proof writing at the secondary level. While this may be related to Lesseig's (2011) knowledge related to "ability to construct a valid proof," there may be other skills at play with proof writing that should be considered beyond the broad idea of proof construction. For example, this could be related to specific details around how mathematicians write, whether in paragraph form or otherwise. Consideration of the use of symbols to write could also be considered, such as familiarizing students with the meaning behind $\therefore$ and $\subset$. While some may argue that these sorts of skills are more appropriate for undergraduate students than secondary students, the data implies that they are worthy of consideration when considering what knowledge teachers must hold. Using proof as a tool to deepen content understanding is another element of CCK to be considered. Research has shown that this is often an underappreciated role that proof can play when learning mathematics in a secondary classroom (Knuth, 2002a, 2002b). Within her framework, Lesseig lists "to build mathematical understanding" as a function of proof to be known within CCK. However, this consideration may go beyond knowing it as a function of proof and it could involve using proof to deepen a student's understanding of a particular concept, such as the quadratic formula or the triangle sum theorem.

Within SCK, considerations around the socially negotiated aspects of proof may be appropriate to address in teaching. There were multiple mentions related to expectations of what counts as proof and how this interacts with social norms and the culture of mathematics. One participant specifically wrote about building towards a "culturally accepted meaning of what constitutes a proof." Another related response included discussion around issues of power and identity in proving, which leads to the process often being "inequitable and less productive." This is knowledge related to proof outside of teaching, but is often not directly considered when writing a proof in the field of mathematics, due to the more agreed upon nature of what counts as proof in that field. These findings complement the socially negotiated perspective outlined by Stylianides, Stylianides, and Weber (2017). Another element of SCK to be considered is the affordances and constraints of various proof formats. Lesseig's (2011) framework includes knowledge of different proof formats and forms of argumentation within SCK, but having knowledge of the affordances and constraints may be another, more specific, aspect of that knowledge to be considered.

Within KCS, multiple participants discussed the G. J. Stylianides (2008, 2010) reasoning-and-proving framework. This framework identifies the mathematical, psychological, and pedagogical components of reasoning and proving, and considers the process students move through as they progress towards a formal proof. A teacher must have knowledge of where a student is at on this continuum and having strategies available to move them to the next phase where appropriate. For example, if a teacher notices that a student has identified a pattern, what strategies should they have at their disposal to assist that particular student in progressing towards a conjecture? What is missing from the student work to move it towards a proof?

While Lesseig (2011) includes knowledge of proof connections with KCT, a specific piece to be considered would be connections within the sequence of conjecturing, reasoning, and proving. Another teaching skill that came up within the survey was related to how teachers engage students in proving using particular contexts or problems. As teachers design mathematical tasks, having specific strategies to adjust or adapt those tasks to make them more likely to engage students in proving can be useful as teachers consider how to motivate proof.

Limitations

As with most survey research, limitations with the survey itself represent the primary limitations with my research. In particular, the survey is limited in space. Due to this, some examples of MKT-P may not be captured. Only two examples from each of the domains of the framework were listed within the matrix and a teacher educator may address a certain domain without necessarily doing so through either of the two examples listed. Additionally, the open-ended item related to learning goals is limited in space and may also require substantial reflection to answer, with no assurance that participants are taking the time to reflect substantially. This is another example in which certain examples of MKT-P may not be captured in the survey. Lastly, there are limits in survey distribution. While I used a distribution list from an organization that many secondary mathematics methods instructors are a part of, there is no single database of all methods instructors in the United States. Additionally, participants are taking the survey on a voluntary basis. Hence, this survey is not meant to be a representative sample of all secondary mathematics methods instructors and I do not generalize to make any claims about methods instructors as a whole.

Another significant issue with my sample is the lack of diversity. Of the 62 participants who provided information related to race and ethnicity, 54 of those identified as White. This is likely due to the lack of diversity among teacher educators as a whole. This means that the broad issue of diversity among mathematics teacher educators must be addressed. In addition, further research is needed around opportunities that nonwhite mathematics teacher educators provide which are intended to support the development of MKT-P. Variables such as race and ethnicity are all too often ignored in mathematics education research (e.g., Martin, 2013). Research has also shown that a participant's race may influence the answers they provide in survey research (Dillman, Smyth, and Christian, 2014). The lack of diversity in my sample leaves work to be done in this area.

This work attempts to explore the development of MKT-P in teacher preparation. However, the research questions specifically target secondary mathematics methods courses. The course itself is the unit of analysis, even though a lot of the focus of the survey is related to the instructor of the course. I also acknowledge that there exist other opportunities for the development of MKT-P throughout the teacher preparation process besides methods courses. For instance, I did not study fieldwork or content courses through this research in order to narrow the focus on a specific facet of the teacher preparation process. However, I fully acknowledge that these other facets could serve as key figures in the development of MKT-P.

Lastly, I am not attempting to “measure” MKT-P. I am not focused on the amount of MKT-P that is held by the teacher educator. I am also not focused on how much MKT-P is developed within teacher candidates through the methods course. Rather, my focus is on *opportunities* to develop MKT-P within the methods courses and the strategies used within these opportunities.

Future Research

Many of the tasks and assignments described in the survey appeared primarily designed for the development of CCK or SCK. In these cases, connections to teaching and student were often vague or implicit. In particular, connections to KCS were rare and in most cases included the examination of student work. Many of the tasks and assignments were tasks designed for teacher candidates to prove something or justify a solution. It is possible, and perhaps likely, that teacher educators make explicit connections to students and teaching as their teacher candidates engage in these tasks. Perhaps these connections were not provided in the survey due to time constraints. Regardless of the reason, I have much less evidence related to how teacher educators use tasks and assignments in ways that are intended to support the development of KCS and KCT in the area of proof.

As research seeks to identify further strategies related to KCS and KCT, the specificity in which data is collected will be critical. This study revealed many tasks that are appropriate for providing opportunities to develop CCK and SCK. One could present these tasks to teacher educators and ask to specify ways in which these tasks are connected to teaching and students. Knowing that many teacher educators use student work to draw connections to KCS and KCT, one could further explore what type of student work is considered by teacher educators to be most helpful to present to teacher candidates. This type of work is important as we consider how the MKT-P framework can be a tool for teacher educators.

Future directions for this particular research could include classroom observations of methods courses taught by instructors who participated in this study. In particular, many classroom activities were shared through the survey, but connections to teaching and students were not always explicit in descriptions. A classroom observation could provide details around those opportunities to connect to teaching and students. Additionally, future work could focus on the teacher candidates enrolled in the methods courses. This could include observations during their fieldwork to consider opportunities where MKT-P is on display in their practice and how this relates to the opportunities for development of MKT-P in their methods courses.

With the importance of proof within mathematics education, the remaining work within teacher education is clear. I have identified ways in which teacher educators provide opportunities intended to support certain areas of MKT-P. Moving forward, a critical piece remains coming closer to a consensus on what is expected in a proof from secondary students. Once there is more consensus, research can further identify how best to prepare secondary mathematics teachers to support students in the area of proof.

References

Alcock, L., & Weber, K. (2005). Proof validation in real analysis: Inferring and checking warrants. *The Journal of Mathematical Behavior, 24*(2), 125–134.

Alibert, D. (1988). Towards new customs in the classroom. *For the Learning of Mathematics, 8*(2), 31-35, 43.

Alibert, D., & Thomas, M. (1991). Research on mathematical proof. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 215–230). Dordrecht, The Netherlands: Kluwer.

Arbaugh, F., Smith, P., Boyle, J., Stylianides, G. J., & Steele, M. (2019). *We reason and we prove for all mathematics. Building students' critical thinking, Grades 6-12*. Thousand Oaks, CA: Sage.

Balacheff, N. (1988). A study of students' proving processes at the junior high school level. In I. Wirszup & R. Streit (Eds.), *Proceedings of the Second UCSMP International Conference on Mathematics Education* (pp. 284–297). Reston, VA: National Council of Teachers of Mathematics.

Baldinger, E. E. (2018). Learning mathematical practices to connect abstract algebra to high school algebra. In N. H. Wasserman (Ed.), *Connecting Abstract Algebra to Secondary Mathematics, for Secondary Mathematics Teachers* (pp. 211–239). New York: Springer International Publishing.

Baldinger, E. E., & Lai, Y. (2019). Pedagogical context and proof validation: The role of positioning as a teacher or student. *Journal of Mathematical Behavior, 55*, [100698] Ball, D. L., & Bass, H. (2003). Making mathematics reasonable in school. In J. Kilpatrick, W.G. Martin, & D. Schifter (Eds.), *A research companion to principles and standards for school mathematics* (pp. 27-44). Reston, VA: National Council of Teachers of Mathematics.

Ball, D. L., Hill, H. C., & Bass, H. (2005). Knowing mathematics for teaching: Who knows mathematics well enough to teach third grade, and how can we decide? *American Educator, 29*(3), 14-22.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching. What makes it special? *Journal of Teacher Education, 59*(5), 389-407.

Bieda, K. N. (2010). Enacting proof-related tasks in middle school mathematics: Challenges and opportunities. *Journal for Research in Mathematics Education, 41*, 351–382.

Buchholtz, N. (2019). Planning and conducting mixed methods studies in mathematics educational research. In G. Kaiser & N. Presmeg (Eds.), *Compendium for early career researchers in mathematics education. ICME-13 monographs* (pp. 131–152).

Burton, L. (1984). Mathematical thinking: The struggle for meaning. *Journal for Research in Mathematics Education, 15*, 35–49.

Chapman, O. (2007). Facilitating preservice teachers' development of mathematics knowledge for teaching arithmetic operations. *Journal of Mathematics Teacher Education, 10*(4-6), 341-349.

Charalambous, C. Y. (2008). Preservice teachers' mathematical knowledge for teaching and their performance in selected teaching practices: Exploring a complex relationship. Unpublished Doctoral Dissertation, University of Michigan.

de Villiers, M. (1990). The role and function of proof in mathematics. *Pythagoras*, *24*, 17-24.

de Villiers, M. (2012). An illustration of the explanatory and discovery functions of proof. *Pythagoras*, *33*(3), 1-8.

Dick, L. K. (2017). Investigating the relationship between professional noticing and specialized content knowledge. In E. O. Schack, M. H. Fisher, & J. Wihlem (Eds.). *Teacher Noticing: Bridging and Broadening Perspectives, Contexts, and Frameworks*. New York: Springer.

Dillman, D.A., Smyth, J.D., & Christian, L.M. (2014). *Internet, phone, mail, and mixed-mode surveys: The tailored design method* (4rd ed.). Hoboken, NJ: John Wiley & Sons, Inc.

Duval, R. (2007). Cognitive functioning and the understanding of mathematical processes of proof. In P. Boero (Ed.), *Theorems in school: From historic, epistemology, and cognition to classroom practice* (pp. 163–181). Rotterdam: The Netherlands: Sense.

Ellis, A. B., Bieda, K., & Knuth, E. (2012). *Developing Essential Understanding of Proof and Proving for Teaching Mathematics in Grades 9-12*. Reston, VA: National Council of Teachers of Mathematics.

Ericsson, K. A., & Simon, H. A. (1980). Verbal reports as data. *Psychological Review, 87*(3), 215–251.

Fukawa-Connelly, T. (2012). Classroom sociomathematical norms for proof presentation in undergraduate in abstract algebra. *Journal of Mathematical Behavior, 31*(3), 401–416.

Gernes, D. (1999). The rules of the game. *The Mathematics Teacher, 92*(5), 424-429.

Graham, K., Li, Y, & Curran Buck, J. (2000). Characteristics of mathematics teacher preparation programs in the United States: An exploratory study. *The Mathematics Educator, 5*(1/2), 5-31.

Griffiths, P. A. (2000). Mathematics at the turn of the millennium. *American Mathematical Monthly, 107*, 1–14.

Hanna, G. (1983). *Rigorous proof in mathematics education*. Toronto: OISE Press.

Hanna, G. (1989). More than formal proof. *For the Learning of Mathematics, 9*(1), 20-23.

Hanna, G. (1990). Some pedagogical aspects of proof. *Interchange, 21*(1), 6–13.

Hanna, G. (2018). Reflections on proof as explanation. In *Advances in Mathematics Education Research on Proof and Proving* (pp. 3-18). Springer, Cham.

Harel, G. (2001). The development of mathematical induction as a proof scheme: A model for DNR-based instruction. In S. Campbell & R. Zaskis (Eds.), *Learning and teaching number theory: Research in cognition and instruction* (pp. 185–212). Dordrecht, The Netherlands: Kluwer.

Harel, G., & Sowder, L. (2007). Toward comprehensive perspectives on the learning and teaching of proof. In F. K. Lester, Jr. (Ed.), *Second handbook of research on mathematics teaching and learning: A project of the National Council of*

Teachers of Mathematics (pp. 805–842). Charlotte, NC: Information Age Publishing.

Healy, L., & Hoyles, C. (2000). Proof conceptions in algebra. *Journal for Research in Mathematics Education, 31*, 396–428.

Herbst, P., & Brach, C. (2006). Proving and doing proofs in high school geometry classes: What is it that is going on for students? *Cognition and Instruction, 24*(1), 73–122.

Hersh, R. (1993). Proving is convincing and explaining. *Educational Studies in Mathematics*, *24*, 389–399.

Hersh, R. (2009). What I would like my students to already know about proof. In D. A. Stylianou, M. L. Blanton, & E. J. Knuth (Eds.), *Teaching and learning proof across the grades: A K−16 perspective* (pp. 17-39). New York, NY: Routledge.

Hill, H., Ball, L. D., & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers’ topic-specific knowledge of students. *Journal for Research in Mathematics Education, 39*(4), 372–400.

Hill, H. C., Blunk, M. L., Charalambos, C. Y., Lewis, J. M., Phelps, G. C., Sleep, L., & Ball, D. L. (2008). Mathematical knowledge for teaching and the mathematical quality of instruction: An exploratory study. *Cognition and Instruction, 26*, 1–81.

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal, 42*(2), 371-406.

Holm, J., & Kajander, A. (2011). “I finally get it”: Developing mathematical understanding during teacher education. *International Journal of Mathematical Education in Science and Technology, 43*(5), 563-574.

Johnson, R. B., & Christensen, L. B. (2017). *Educational research: Quantitative, qualitative, and mixed approaches* (6th ed.). Thousand Oaks, CA: Sage.

Kitcher, P. (1984). *The nature of mathematical knowledge*. New York, NY: Oxford University Press.

Kline, M. (1973). *Why Johnny can’t add: The failure of the New Math*. New York: St. Martin’s Press.

Knuth, E. J. (2002a). Secondary school mathematics teachers’ conceptions of proof. *Journal for Research in Mathematics Education*, *33*(5), 379–405.

Knuth, E. (2002b). Teachers' conceptions of proof in the context of secondary school mathematics. *Journal of Mathematics Teacher Education*, *5*(1), 61-88.

Knuth, E. J., Choppin, J., & Bieda, K. (2009). Middle school students’ productions of mathematical justification. In M. Blanton, D. Stylianou, & E. Knuth (Eds.), *Teaching and learning proof across the grades: A K–16 perspective* (pp. 153–170). New York, NY: Routledge.

Lakatos, I. (1976). *Proofs and refutations: The logic of mathematical discovery*. Cambridge: Cambridge University Press.

Lampert, M. (2001). *Teaching problems and the problems of teaching*. New Haven, CT: Yale University Press.

Larsen, S. P., & Zandieh, M. (2007). Proofs and refutations in the undergraduate mathematics classroom. *Educational Studies in Mathematics, 67*(3), 205–216.

Lesseig, K. (2011). Mathematical knowledge for teaching proof. Unpublished Doctoral Dissertation, Oregon State University.

Lesseig, K. (2016). Conjecturing, generalizing and justifying: Building theory around teacher knowledge of proving. *International Journal for Mathematics Teaching and Learning, 17*(3), 1–31.

Maher, C. & Martino, A. (1996). The development of the idea of mathematical proof: A 5-year case study. *Journal for Research in Mathematics Education*, *27*(2), 194–214.

Manaster, A. B. (1998). Some characteristics of eighth grade mathematics classes in the TIMSS videotape study. *American Mathematical Monthly, 105*, 793–805.

Martin, D. B. (2013). Race, racial projects, and mathematics education. *Journal for Research in Mathematics Education, 44*(1), 316–333.

Martino, A. M., & Maher, C. A. (1999). Teacher questioning to promote justification and generalization in mathematics: what research practice has taught us. *Journal of Mathematical Behavior, 18*, 53-78.

Mejía-Ramos, J. P., Fuller, E., Weber, K., Rhoads, K., & Samkoff, A. (2012). An assessment model for proof comprehension in undergraduate mathematics. *Educational Studies in Mathematics, 79,* 3-18.

Mohr, M. J. (2006). An assessment of middle grades preservice teachers' mathematics knowledge for teaching. Unpublished Doctoral Dissertation, Texas A&M University.

Moore, R. C. (2016). Mathematics professors’ evaluations of students’ proofs: A complex teaching practice. *International Journal of Research in Undergraduate Mathematics Education, 2*, 246-278.

Morris, A. K. (2002). Mathematical reasoning: Adults’ ability to make the inductive-deductive distinction. *Cognition and Instruction*, *20*(1), 79–118.

Morris, A. K. (2007). Factors affecting pre-service teachers’ evaluations of the validity of students’ mathematical arguments in classroom contexts. *Cognition and Instruction*, *25*(4), 479–522.

Morris, A. K., Hiebert, J., & Spitzer, S. M. (2009). Mathematical knowledge for teaching in planning and evaluating instruction: What can preservice teachers learn? *Journal for Research in Mathematics Education*, *40*(5), 491-529.

National Council of Teachers of Mathematics (NCTM). (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). *Common Core State Standards for Mathematics (CCSSM).* Washington, DC: Authors.

Pino-Fan, L., Godino, J. D., Font, V., & Castro, W. F (2013). Prospective teacher’s specialized content knowledge on derivative. In B. Ubuz, Ç. Haser & M. Mariotti (Eds.), *Proceedings of the Eighth Congress of European Research in Mathematics Education* (pp. 3195 – 3205). Antalya, Turkey: CERME.

Reid, D. A. (2002). Conjectures and refutations in grade 5 mathematics. *Journal for Research in Mathematics Education*, *33*(1), 5-29.

Reid, D., & Vargas, E. V. (2017). Proof-based teaching as a basis for understanding why. *CERME10*, (pp. 235–242).

Riegle-Crumb, C., Kyte, S. B., & Morton, K. (2018). Gender and racial/ethnic differences in educational outcomes: Examining patterns, explanations, and new directions for research. In B. Schneider (Ed.), *Handbook of the sociology of education in the 21st century* (pp. 131–152). New York, NY: Springer.

Schilling, S. G. & Hill, H. C. (2007). Assessing measures of mathematical knowledge for teaching: A validity argument approach. *Measurement: Interdisciplinary Research and Perspectives, 5*, 93–106.

Sears, R. (2012). An examination of how teachers use curriculum materials for the teaching of proof in high school geometry. Unpublished Doctoral Dissertation, University of Missouri, Columbia.

Schoenfeld, A. (1994). What do we know about mathematics curricula? *Journal of Mathematical Behavior*, *13*(1), 55–80.

Selden, A., & Selden, J. (2003). Validations of proofs written as texts: Can undergraduates tell whether an argument proves a theorem? *Journal for Research in Mathematics Education, 36*(1), 4–36.

Senk, S. L. (1989). Van Hiele levels and achievement in writing geometry proofs. *Journal for Research in Mathematics Education, 20*, 309–321.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14.

Skemp, R. R. (1978). Relational understanding and instrumental understanding. *Arithmetic Teacher*, *26*(3), 9–15.

Silverman, J., & Thompson, P. W. (2008) Toward a framework for the development of mathematical knowledge for teaching. *Journal of Mathematics Teacher Education, 11*, 499-511.

Simon, M. A., & Blume, G. W. (1996). Justification in the mathematics classroom: A study of prospective elementary teachers. *Journal of Mathematical Behavior, 15*, 3–31.

Speer, N. M., King, K. D., & Howell, H. (2014). Definitions of mathematical knowledge for teaching: Using these constructs in research on secondary and college mathematics teachers. *Journal of Mathematics Teacher Education, 18*, 105-122.

Steele, M. D., & Rogers, K. C. (2012). Relationships between mathematical knowledge for teaching and teaching practice: the case of proof. *Journal of Mathematics Teacher Education, 15*(2), 159–180

Stylianides, A. J. (2007). Proof and proving in school mathematics. *Journal for Research in Mathematics Education*, *38*(3), 289-321.

Stylianides, A. J., & Ball, D. L. (2008). Understanding and describing mathematical knowledge for teaching: knowledge about proof for engaging students in the activity of proving. *Journal of Mathematics Teacher Education, 11*, 307-332.

Stylianides, A. J., Bieda, K. N., & Morselli, F. (2016). Proof and argumentation in mathematics education research. In A. Gutiérrez, G. C. Leder, & P. Boero (Eds.), *The second handbook of research on the Psychology of Mathematics Education* (pp. 315–351). Rotterdam, The Netherlands: Sense Publishers.

Stylianides, G. J. (2008). An analytic framework of reasoning-and-proving. *For the Learning of Mathematics, 28*(1), 9-16.

Stylianides, G. J. (2010). Engaging secondary students in reasoning and proving. *Mathematics Teaching*, *219*, 39-44.

Stylianides, G. J., Stylianides, A. J., & Weber, K. (2017). Research on the teaching and learning of proof: Taking stock and moving forward. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 237-266). Reston, VA: National Council of Teachers of Mathematics

Tatto, M. T. (2018). The mathematical education of secondary teachers. In M. T. Tatto, W. M. Smith, M.C. Rodriguez, M. Reckase, & K. Bankov (Eds.), *Exploring the mathematical education of teachers using TEDS-M data* (pp. 409-450). Switzerland: Springer.

Van den Kieboom, L. A. (2013). Examining the mathematical knowledge for teaching involved in pre-service teachers' reflections. *Teaching & Teacher Education, 35*, 146–156

Volmink, J. D. (1990). The nature and role of proof in mathematics education. *Pythagoras, 23*, 7-10.

Wasserman, N. H. (2018). Knowledge of nonlocal mathematics for teaching. *Journal of Mathematical Behavior, 49*(November 2017), 116–128.

Wasserman, N. H., Fukawa-Connelly, T., Villanueva, M., Mejia-Ramos, J. P., & Weber, K. (2017). Making real analysis relevant to secondary teachers: Building up from and stepping down to practice. *PRIMUS, 27*(6), 559-578.

Weber, K. (2010). Mathematics majors' perceptions of conviction, validity, and proof. *Mathematical Thinking and Learning*, *12*, 306–336.

Wilder, R. L. (1944). The nature of mathematical proof. *American Mathematical Monthly, 51*, 309-323.

Wu, H. (1996). The role of Euclidean geometry in high school. *Journal of Mathematical Behavior, 15,* 221-237.

Appendix: Survey Questions

Q1 Are you teaching one or more secondary math methods courses during the 2019-2020 academic year?

- Yes (1)
- No (2)

If....NO to Q1

Q2 What is the official name of the secondary math methods course you most recently taught?

____

If....NO to Q1

Q3 Which grade levels are teacher candidates enrolled in the course above preparing to teach? (select all that apply)

- K-4 (1)
- 5 (2)
- 6 (3)
- 7 (4)
- 8 (5)
- 9 (6)
- 10-12 (7)

If….NO to Q1

Q4 Where is your college or university located?

▼Alabama (1) ... Other (57)

If….NO to Q1

Q5 What academic year did you most recently teach this course?

- ○ 2018-2019 (1)
- ○ 2017-2018 (2)
- ○ 2016-2017 (3)
- ○ Prior to 2016-2017 (4)
- ○ I have never taught a secondary math methods course (5)

If….YES to Q1

Q6 What is the official name of the secondary math methods course(s) that you teach?

- ○ Course 1 (1) ____
- ○ Course 2 (if applicable) (2) ____
- ○ Course 3 (if applicable) (3) ____
- ○ Course 4 (if applicable) (4) ____

If....YES to Q1

Q7 Which grade levels are teacher candidates enrolled in the course(s) above preparing to teach? (select all that apply)

- ▢ K-4 (1)
- ▢ 5 (2)
- ▢ 6 (3)
- ▢ 7 (4)
- ▢ 8 (5)
- ▢ 9 (6)
- ▢ 10-12 (7)

If....YES to Q1

Q8 Where is your college or university located?

▼Alabama (1) ... Other (57)

Q9 To what extent do you agree or disagree that each of the following is a purpose of proof?

	Strongly disagree (1)	Somewhat disagree (2)	Neither agree nor disagree (3)	Somewhat Agree (4)	Strongly Agree (5)
Verification (1)	○	○	○	○	○
Explanation (2)	○	○	○	○	○
Systematization (3)	○	○	○	○	○
Discovery (4)	○	○	○	○	○
Communication (5)	○	○	○	○	○

Q10 To what extent do you agree or disagree that the following classroom activities would be considered proof at the secondary level?

	Strongly disagree (1)	Somewhat disagree (2)	Neither agree nor disagree (3)	Somewhat agree (4)	Strongly agree (5)
Completing the square of the general form of the equation to verify that the quadratic formula identifies the zeros of any quadratic function (1)	○	○	○	○	○
A student explains "no triangle exists with side lengths 1, 2 and 10 because the long side is too long to allow the shorter sides to meet" (2)	○	○	○	○	○
Tearing corners off of paper triangles and rearranging them to demonstrate that the interior angles of a triangle sum to 180 degrees (3)	○	○	○	○	○
Using a counterexample of 2 to refute the claim that all prime numbers are odd (4)	○	○	○	○	○

Q11 Is the teaching and learning of proof a topic that is addressed in your methods course(s)?

○ Yes (1)

○ No (2)

If....YES to Q11

Q12 How would you describe the amount of time dedicated to proof in your course(s)?

○ A single class session (1)

○ A sequence of class sessions (2)

○ Embedded throughout the course(s) (3)

If....YES to Q11

Q13 Please describe your learning goals for teaching candidates around the teaching and learning of proof in your methods course(s):

____

____

____

____

____

If….YES to Q11

Q14 Are there specific tasks or assignments related to proof that your teacher candidates must complete?

◯ Yes (1)

◯ No (2)

If….YES to Q14

Q15 Briefly describe one such task/assignment.

____

____

____

____

____

If….NO to Q11

Q16 What would best describe the reason for not including proof as a topic?

◯ Teacher candidates learn about proof sufficiently through the mathematics department (1)

◯ Teacher candidates learn about proof sufficiently through other methods instructors (2)

◯ Given time constraints, priority is given to other topics in my methods course(s) (3)

◯ Other (please explain in text box (4) ____________________

Q17 The prompts below relate to areas associated with the teaching and learning of proof that may be addressed in your methods course(s) and any expectations you hold that they are addressed through mathematics content courses or in other methods courses which you do not teach. Please select any and all that apply for each area.

	Addressed in my methods course(s) (1)	I expect that this is addressed in other methods courses (2)	I expect that this is addressed in mathematics content courses (3)
Teacher candidates learn how to write proofs (1)	☐	☐	☐
Teacher candidates learn about forms of argumentation appropriate at various secondary student learning levels (2)	☐	☐	☐
Teacher candidates learn when certain types of proofs are relevant for certain situations (3)	☐	☐	☐
Teacher candidates learn of pivotal examples and counter-examples that can be used to extend, bridge, or scaffold secondary student thinking toward the development of a mathematical proof. (4)	☐	☐	☐

Q18 The prompts below relate to areas associated with the teaching and learning of proof that may be addressed in your methods course(s) and any expectations you hold that they are addressed through mathematics content courses or in other methods courses which you do not teach. Please select any and all that apply for each area.

	Addressed in my methods course(s) (1)	I expect that this is addressed in other methods courses (2)	I expect that this is addressed in mathematics content courses (3)
Teacher candidates learn how to categorize secondary student proof productions using the proof schemes of external, empirical, and deductive. (1)	☐	☐	☐
Teacher candidates learn about the various purposes of proof. (2)	☐	☐	☐
Teacher candidates learn questioning strategies that will encourage secondary students to move from specific examples to general cases. (3)	☐	☐	☐
Teacher candidates learn about the variety of visual and symbolic methods to provide a general argument. (4)	☐	☐	☐

Q19 Which degrees have you completed?

- ▢ PhD (1)
- ▢ EdD (2)
- ▢ MA/MS/MEd (3)
- ▢ BS/BA/BEd (4)
- ▢ Associates (5)
- ▢ Other (6) ____

Q20-Q25 will only display if the appropriate degree is selected here

---

Q20 Which field best describes your PhD degree?

- ◯ Mathematics (1)
- ◯ Education (2)
- ◯ Other (3) ____

Q21 Which field best describes your EdD degree?

- ◯ Mathematics (1)
- ◯ Education (2)
- ◯ Other (3) ____

Q22 Which field best describes your Master's degree?

- Mathematics (1)
- Education (2)
- Other (3) ____

Q23 Which field best describes your Bachelor's degree?

- Mathematics (1)
- Education (2)
- Other (3) ____

Q24 Which field best describes your Associates degree?

- Mathematics (1)
- Education (2)
- Other (3) ____

Q25 Which field best describes your "Other" degree?

- Mathematics (1)
- Education (2)
- Other (3) ____

Q26 How many years have you taught secondary math methods courses?

- ◯ This is my first year (1)
- ◯ 1-5 (2)
- ◯ 6-10 (3)
- ◯ 11-15 (4)
- ◯ 16-20 (5)
- ◯ >20 (6)

Q27 How many years have you taught mathematics at the secondary level?

- ◯ I have never taught secondary mathematics (1)
- ◯ 1-5 (2)
- ◯ 6-10 (3)
- ◯ 11-15 (4)
- ◯ 16-20 (5)
- ◯ >20 (6)

Q28 What is your current professional title?

- ◯ Adjunct Professor (1)
- ◯ Lecturer/Professor of Practice (2)
- ◯ Assistant Professor (3)
- ◯ Associate Professor (4)
- ◯ Professor (5)
- ◯ Professor Emeritus (6)
- ◯ Graduate Student (7)
- ◯ Other (8) ____________________

Q29 Which of the following best describes your department?

- ◯ Education (1)
- ◯ Mathematics (2)
- ◯ Joint appointment (Mathematics and Education) (3)
- ◯ Other (4) ____________________

Q30 Would you be willing to follow-up with the researcher via email if further information would help his study?

- ◯ Yes (1)
- ◯ No (2)

If….YES to Q30

Q31 Name

____

Q32 Email

____

Q33 What category best describes you?

- ○ Male (1)
- ○ Female (2)
- ○ Nonbinary (3) ____
- ○ Prefer not to answer (4)

Q34 Which categories best describe you? (select all that apply)

- ☐ White (1)
- ☐ Hispanic or Latino (2)
- ☐ Black or African American (3)
- ☐ Native American or American Indian (4)
- ☐ Asian / Pacific Islander (5)
- ☐ Other (6) ____
- ☐ Prefer not to answer (7)

End of Survey