THE REAL POSITIVE DEFINITE COMPLETION PROBLEM FOR A SIMPLE CYCLE*

WAYNE BARRETT**, CHARLES R. JOHNSON† AND PABLO TARAZAGA‡

Abstract. We consider the question of whether a real partial positive definite matrix (in which the specified off-diagonal entries consist of a full n cycle) has a positive definite completion. This lies in contrast to the previously studied chordal case. We give two solutions. In one, we describe about $\frac{n}{2}$ independent conditions on angles associated with a normalization of the data that are necessary and sufficient. The second is more computational and allows presentation of all positive definite completions, as well as answering the existence question.

1. Introduction. A *real partial matrix* A is one in which some entries are specified real numbers and the remainder are unspecified, i.e., free variables over the real numbers. We say that A is *partial symmetric* if A is square, a_{ji} is specified whenever a_{ij} is, and $a_{ji} = a_{ij}$. We shall assume throughout that the diagonal entries of A are specified. An example is

$$A = \begin{bmatrix} 5 & 2 & ? \\ 2 & 1 & 2 \\ ? & 2 & 3 \end{bmatrix} \tag{1}$$

in which the ?'s indicate unspecified entries.

A *partial positive definite matrix* is a partial symmetric matrix each of whose specified principal submatrices is positive definite. (By a specified portion of a partial matrix we always mean one composed entirely of specified entries.) *Partial positive semidefinite* matrices are defined similarly. The matrix above is not partial positive definite, but is if the 2,2 entry is replaced, for example, by 2. A completion of a partial matrix is a specification of the unspecified entries resulting in a conventional matrix, and the positive definite completion problem is to determine if a positive definite completion exists or to find a completion of a partial positive definite matrix that is positive definite. For example, $\begin{bmatrix} 5 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ is a positive definite completion of $\begin{bmatrix} 5 & 2 & ? \\ 2 & 2 & 2 \\ ? & 2 & 3 \end{bmatrix}$.

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**Dept. of Mathematics, Brigham Young University, Provo, Utah 84602.

†Dept. of Mathematics, The College of William and Mary, Williamsburg, Virginia 23185. The work of this author was supported in part by National Science Foundation grant DMS 90-00839 and office of Naval Research Contract N00014-90-J-1739.

‡Dept. of Mathematics, University of Puerto Rico, Mayaguez, Puerto Rico 00709. This author was partially supported by NSF grants RI-8905080 and EPSCoR of Puerto Rico.

For convenience, we consider the positive definite and positive semidefinite completion problems interchangeably. As observed in [GJSW], they are equivalent.

Not all partial positive semidefinite matrices have a positive semidefinite completion. An example is:

$$B = \begin{bmatrix} 1 & 1 & ? & -1 \\ 1 & 1 & 1 & ? \\ ? & 1 & 1 & 1 \\ -1 & ? & 1 & 1 \end{bmatrix} \tag{2}$$

Let x be the 1,3 entry. In order that B have a positive semidefinite completion we must choose x so that the minors

$$\begin{vmatrix} 1 & 1 & x \\ 1 & 1 & 1 \\ x & 1 & 1 \end{vmatrix}, \qquad \begin{vmatrix} 1 & x & -1 \\ x & 1 & 1 \\ -1 & 1 & 1 \end{vmatrix}$$

obtained by deleting the last row and column, and by deleting the second row and column are nonnegative. But the first equals $-|x-1|^2$ which forces $x = 1$, while the second equals $-|x+1|^2$ which forces $x = -1$. This conflict precludes the possibility of a positive definite completion.

The existence of a positive definite completion of a partial positive definite matrix A depends on the pattern of specified entries of A and can be most easily described in terms of its undirected graph.

Definition 1. *Let A be a partial positive definite n-by-n matrix. The undirected graph $G = (N, E)$ of A has node set $N = \{1, 2, \ldots, n\}$ and an edge $\{i, j\} \in E$, $i \neq j$, if and only if a_{ij} is specified.*

For example, the undirected graphs of A and B given by (1) and (2) above are

and

(3)

respectively.

We recall that the graph G is *connected* if there is a path between any two vertices in N, and that G is *chordal* if it has no minimal simple circuit of four or more edges.

A key result is

THEOREM 1 [GJSW]. *Every partial positive definite (semidefinite) matrix with graph G has a positive definite (semidefinite) completion if and only if G is chordal.*

The graph G in (3) is the simplest non-chordal graph and the matrix B illustrates that not all partial positive semidefinite matrices with pattern G have a positive semidefinite completion.

Our purpose in this paper is to give two solutions to the positive definite completion problem in the case that G is the simple cycle

$$C_n = (N, E), \quad E = \{\{1,2\},\{2,3\},\ldots,\{n-1,n\},\{n,1\}\}.$$

Thus, we are given

$$A = \begin{bmatrix} a_1 & b_1 & & & & b_n \\ b_1 & a_2 & b_2 & & ? & \\ & b_2 & a_3 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & ? & & \ddots & \ddots & b_{n-1} \\ b_n & & & & b_{n-1} & a_n \end{bmatrix} \tag{4}$$

satisfying

$$\left.\begin{aligned} a_i &> 0 \\ a_i a_{i+1} - b_i^2 &> 0 \end{aligned}\right\} \qquad i = 1, \ldots, n \tag{5}$$

where a_{n+1} denotes a_1.

Letting $D = \operatorname{diag}(1/\sqrt{a_1}, 1/\sqrt{a_2}, \ldots, 1/\sqrt{a_n})$, the congruence $C = DAD$ gives

$$C = \begin{bmatrix} 1 & \frac{b_1}{\sqrt{a_1a_2}} & & & \frac{b_n}{\sqrt{a_1a_n}} \\ \frac{b_1}{\sqrt{a_1a_2}} & 1 & \frac{b_2}{\sqrt{a_2a_3}} & ? & \\ & \frac{b_2}{\sqrt{a_2a_3}} & 1 & \ddots & \\ & ? & & \ddots & \frac{b_{n-1}}{\sqrt{a_{n-1}a_n}} \\ \frac{b_n}{\sqrt{a_1a_n}} & & & \frac{b_{n-1}}{\sqrt{a_{n-1}a_n}} & 1 \end{bmatrix}$$

in which all off-diagonal entries have modulus less than one. Since A has a positive definite completion if and only if C has, it suffices to consider the question:

When does the matrix

$$(6) \qquad C = \begin{bmatrix} 1 & c_1 & & & c_n \\ c_1 & 1 & c_2 & ? & \\ & c_2 & 1 & & \\ & ? & & \ddots & c_{n-1} \\ c_n & & & c_{n-1} & 1 \end{bmatrix}, \qquad |c_i| < 1, \;\; i \in N$$

admit a positive definite completion.

It is helpful to first consider the case $n = 4$ separately. Then

$$(7) \qquad C = \begin{bmatrix} 1 & c_1 & ? & c_4 \\ c_1 & 1 & c_2 & ? \\ ? & c_2 & 1 & c_3 \\ c_4 & ? & c_3 & 1 \end{bmatrix}.$$

and its graph G is

If the 1,3 and 3,1 entries of C can be chosen so that C is still partial positive definite, G becomes the chordal graph

It then follows from theorem 1 that the 2,4 and 4,2 entries can be chosen to make C positive definite. Therefore, C has a positive definite completion if and only if there exists a real x for which

$$\begin{vmatrix} 1 & c_1 & x \\ c_1 & 1 & c_2 \\ x & c_2 & 1 \end{vmatrix} > 0 \quad \text{and} \quad \begin{vmatrix} 1 & x & c_4 \\ x & 1 & c_3 \\ c_4 & c_3 & 1 \end{vmatrix} > 0.$$

These inequalities may be rewritten

$$(1 - c_1^2)(1 - c_2^2) - (c_1 c_2 - x)^2 > 0, \text{ and } (1 - c_3^2)(1 - c_4^2) - (c_3 c_4 - x)^2 > 0,$$

which in turn may be written

$$|x - c_1c_2| < \sqrt{(1-c_1^2)(1-c_2^2)}, \qquad |x - c_3c_4| < \sqrt{(1-c_3^2)(1-c_4^2)}. \tag{8}$$

There is an x satisfying the inequalities (8) if and only if

$$|c_1c_2 - c_3c_4| < \sqrt{(1-c_1^2)(1-c_2^2)} + \sqrt{(1-c_3^2)(1-c_4^2)}. \tag{9}$$

Squaring and simplifying gives

$$\sum_{i=1}^{4} c_i^2 < 2\left[1 + c_1c_2c_3c_4 + \left(\prod_{i=1}^{4}(1-c_i^2)\right)^{\frac{1}{2}}\right]. \tag{10}$$

We record this as:

PROPOSITION 1. *This matrix C given by (7) has a positive definite completion if and only if the inequality (10) holds.*

2. Parameterization of completable cycles. Fiedler [F] has already given necessary and sufficient conditions for the matrix C given by (6) to have a positive definite completion. However, his result is not phrased in the language of partial positive definite matrices and completion problems, and to the best of our knowledge predates other work on matrix completion problems. It is geometrical in nature and is obtained by viewing the matrix C as a Gram matrix of n unit vectors. Our purpose is to give an alternate view and a combinatorial extension to his results and to discuss its implications.

We express each c_i in (6) as $c_i = \cos\theta_i$, $\theta_i \in (0,\pi)$, $i = 1,\ldots,n$, or $\theta_i \in [0,\pi]$, $i = 1,\ldots,n$ when considering the positive semidefinite completion problem. Note that the θ_i are uniquely determined. This parameterization of the c_i enables one to state criteria for existence of a positive definite completion of C in terms of linear inequalities on the θ_i. We will need the following result concerning 3-by-3 positive semidefinite matrices, which is of independent interest.

PROPOSITION 2. *Let $0 \le \alpha, \beta, \gamma \le \pi$. Then the matrix*

$$C = \begin{bmatrix} 1 & \cos\alpha & \cos\gamma \\ \cos\alpha & 1 & \cos\beta \\ \cos\gamma & \cos\beta & 1 \end{bmatrix}$$

is positive semidefinite if and only if

$$\alpha \le \beta + \gamma, \quad \beta \le \alpha + \gamma, \quad \gamma \le \alpha + \beta, \text{ and } \alpha + \beta + \gamma \le 2\pi.$$

Furthermore, C is singular if and only if one of these inequalities is an equality.

Proof. We have the following chain of equivalences:

$$
\begin{aligned}
& C \text{ is positive semidefinite} \\
\iff & \\
& \det C \geq 0 \\
\iff & \\
& 1 + 2\cos\alpha\cos\beta\cos\gamma - \cos^2\alpha - \cos^2\beta - \cos^2\gamma \geq 0 \\
\iff & \\
& 1 - \cos^2\beta - \cos^2\gamma + \cos^2\beta\cos^2\gamma \geq \cos^2\alpha - 2\cos\alpha\cos\beta\cos\gamma + \cos^2\beta\cos^2\gamma \\
\iff & \\
& (1 - \cos^2\beta)(1 - \cos^2\gamma) \geq (\cos\alpha - \cos\beta\cos\gamma)^2 \\
\iff & \\
& \sin\beta\sin\gamma \geq |\cos\alpha - \cos\beta\cos\gamma| \\
\iff & \\
& -\sin\beta\sin\gamma \leq \cos\alpha - \cos\beta\cos\gamma \leq \sin\beta\sin\gamma \\
\iff & \\
& \cos\beta\cos\gamma - \sin\beta\sin\gamma \leq \cos\alpha \leq \cos\beta\cos\gamma + \sin\beta\sin\gamma \\
\iff & \\
& \cos(\beta+\gamma) \leq \cos\alpha \leq \cos(\beta-\gamma) \\
\iff & \\
(11) \quad & \cos(\beta+\gamma) \leq \cos\alpha \leq \cos|\beta-\gamma|
\end{aligned}
$$

Now $0 \leq \beta+\gamma \leq 2\pi$ and $0 \leq \alpha,\ |\beta-\gamma| \leq \pi$. Since $\cos x$ is decreasing on $[0,\pi]$ and increasing on $[\pi, 2\pi]$, (11) holds if and only if

$$|\beta-\gamma| \leq \alpha \leq \beta+\gamma \leq 2\pi - \alpha \tag{12}$$

which is equivalent to the four inequalities:

$$-\alpha \leq \beta-\gamma, \quad \beta-\gamma \leq \alpha, \quad \alpha \leq \beta+\gamma, \text{ and } \beta+\gamma \leq 2\pi-\alpha$$

or in other words

$$\alpha \leq \beta+\gamma, \quad \beta \leq \alpha+\gamma, \quad \gamma \leq \alpha+\beta, \quad \text{ and } \alpha+\beta+\gamma \leq 2\pi. \tag{13}$$

Finally, C is singular if and only if one of the inequalities in (12) is an equality, i.e., if and only if one of the equalities in (13) is an equality. □

The forward implication in Proposition 2 is intuitively clear if one thinks of C as the Gram matrix of three unit vectors u_1, u_2, u_3 in R^3, and α, β, γ as the angles between the pairs of vectors $\{u_1, u_2\}$, $\{u_2, u_3\}$, and $\{u_1, u_3\}$ respectively. Then C is singular if and only if u_1, u_2 and u_3 are coplanar, that is if and only if one of the inequalities in (13) is an equality.

Proposition 2 makes it trivial to construct singular 3-by-3 positive semidefinite matrices. For instance, $\alpha = \beta = \gamma = \frac{2\pi}{3}$ gives

$$\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}$$

and $\alpha = \frac{\pi}{6}$, $\beta = \frac{\pi}{3}$, $\gamma = \frac{\pi}{2}$ gives

$$\begin{bmatrix} 1 & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & 1 & \frac{1}{2} \\ 0 & \frac{1}{2} & 1 \end{bmatrix}.$$

We also note that proposition 2 parameterizes all real positive semidefinite matrices up to diagonal congruence. We know of no result analogous to proposition 2 for complex positive semidefinite matrices, which is the reason we have restricted ourselves to real matrices.

We now give criteria for a partial positive semidefinite matrix whose graph is a simple cycle to have a positive semidefinite completion. The corresponding results for positive definite matrices are obtained by replacing all inequalities by strict inequalities. We begin with $n = 4$.

THEOREM 2. *Let $0 \le \theta_1, \theta_2, \theta_3, \theta_4 \le \pi$. Then the matrix*

$$C = \begin{bmatrix} 1 & \cos\theta_1 & ? & \cos\theta_4 \\ \cos\theta_1 & 1 & \cos\theta_2 & ? \\ ? & \cos\theta_2 & 1 & \cos\theta_3 \\ \cos\theta_4 & ? & \cos\theta_3 & 1 \end{bmatrix}$$

has a positive semidefinite completion if and only if

$$(14) \qquad 2\theta_k \le \sum_{i=1}^{4} \theta_i \le 2\pi + 2\theta_k \quad k = 1, \ldots, 4$$

Proof. As in the argument preceding proposition 1, C has a positive semidefinite completion if and only if there is a $\phi \in [0, \pi]$ such that when $\cos\phi$ is substituted for the 1,3 and 3,1 entries, the principal submatrices

$$\begin{bmatrix} 1 & \cos\theta_1 & \cos\phi \\ \cos\theta_1 & 1 & \cos\theta_2 \\ \cos\phi & \cos\theta_2 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & \cos\phi & \cos\theta_4 \\ \cos\phi & 1 & \cos\theta_3 \\ \cos\theta_4 & \cos\theta_3 & 1 \end{bmatrix}$$

are positive semidefinite. By proposition 2 this occurs if and only if

$$\begin{aligned} \theta_1 &\le \theta_2 + \phi & \phi &\le \theta_3 + \theta_4 \\ \theta_2 &\le \theta_1 + \phi & \theta_3 &\le \phi + \theta_4 \\ \phi &\le \theta_1 + \theta_2 & \theta_4 &\le \phi + \theta_3 \\ \theta_1 + \theta_2 + \phi &\le 2\pi & \phi + \theta_3 + \theta_4 &\le 2\pi \end{aligned}$$

or equivalently

$$\left.\begin{array}{c} \theta_1 - \theta_2 \\ \theta_2 - \theta_1 \\ \theta_3 - \theta_4 \\ \theta_4 - \theta_3 \end{array}\right\} \le \phi \le \left\{\begin{array}{l} \theta_1 + \theta_2 \\ \theta_3 + \theta_4 \\ 2\pi - \theta_1 - \theta_2 \\ 2\pi - \theta_3 - \theta_4 \end{array}\right.$$

which is equivalent to the following 16 inequalities.

$$\begin{aligned} \theta_1 - \theta_2 &\le \theta_1 + \theta_2 & \theta_1 - \theta_2 &\le 2\pi - \theta_1 - \theta_2 \\ \theta_2 - \theta_1 &\le \theta_1 + \theta_2 & \theta_2 - \theta_1 &\le 2\pi - \theta_1 - \theta_2 \\ \theta_3 - \theta_4 &\le \theta_1 + \theta_2 & \theta_3 - \theta_4 &\le 2\pi - \theta_1 - \theta_2 \\ \theta_4 - \theta_3 &\le \theta_1 + \theta_2 & \theta_4 - \theta_3 &\le 2\pi - \theta_1 - \theta_2 \\ \theta_1 - \theta_2 &\le \theta_3 + \theta_4 & \theta_1 - \theta_2 &\le 2\pi - \theta_3 - \theta_4 \\ \theta_2 - \theta_1 &\le \theta_3 + \theta_4 & \theta_2 - \theta_1 &\le 2\pi - \theta_3 - \theta_4 \\ \theta_3 - \theta_4 &\le \theta_3 + \theta_4 & \theta_3 - \theta_4 &\le 2\pi - \theta_3 - \theta_4 \\ \theta_4 - \theta_3 &\le \theta_3 + \theta_4 & \theta_4 - \theta_3 &\le 2\pi - \theta_3 - \theta_4 \end{aligned}$$

Eight of these inequalities hold automatically. The others are:

$$\begin{aligned} \theta_3 &\le \theta_1 + \theta_2 + \theta_4 \\ \theta_4 &\le \theta_1 + \theta_2 + \theta_3 \\ \theta_1 &\le \theta_2 + \theta_3 + \theta_4 \\ \theta_2 &\le \theta_1 + \theta_3 + \theta_4 \\ \theta_1 + \theta_2 + \theta_3 &\le 2\pi + \theta_4 \\ \theta_1 + \theta_2 + \theta_4 &\le 2\pi + \theta_3 \\ \theta_1 + \theta_3 + \theta_4 &\le 2\pi + \theta_2 \\ \theta_2 + \theta_3 + \theta_4 &\le 2\pi + \theta_1 \end{aligned} \tag{15}$$

Therefore C has a positive semidefinite completion if and only if the inequalities (14) hold. $\square$

For example,

$$\begin{bmatrix} 1 & -1 & ? & -1 \\ -1 & 1 & -1 & ? \\ ? & -1 & 1 & -1 \\ -1 & ? & -1 & 1 \end{bmatrix} \qquad (\theta_1 = \theta_2 = \theta_3 = \theta_4 = \pi)$$

has a positive semidefinite completion, but

$$\begin{bmatrix} 1 & 1 & ? & -1 \\ 1 & 1 & 1 & ? \\ ? & 1 & 1 & 1 \\ -1 & ? & 1 & 1 \end{bmatrix} \qquad (\theta_1 = \theta_2 = \theta_3 = 0, \theta_4 = \pi)$$

does not. Theorem 2 seems much more understandable than the equivalent result, proposition 1.

We note that proposition 2 ($n = 3$) and theorem 1 ($n = 4$) are similar in form although only the latter is concerned with a completion problem. Apparently, there will be 2^{n-1} linear constraints on the θ_i in general, and if the inequalities are presented so that the coefficient of each θ_i is 1, there will be an odd number of θ_i on the left hand side of each inequality. The following gives necessary and sufficient conditions for general n.

THEOREM 3. *Let $n \geq 4$, let $N = \{1, 2, \ldots, n\}$, and $0 \leq \theta_1, \theta_2, \ldots, \theta_n \leq \pi$. Then the matrix*

$$C = \begin{bmatrix} 1 & \cos\theta_1 & & & \cos\theta_n \\ \cos\theta_1 & 1 & \cos\theta_2 & ? & \\ & \cos\theta_2 & 1 & & \\ & ? & & \ddots & \cos\theta_{n-1} \\ \cos\theta_n & & & \cos\theta_{n-1} & 1 \end{bmatrix}$$

has a positive semidefinite completion if and only if for each $S \subseteq N$ with $|S|$ odd,

$$\sum_{i \in S} \theta_i \leq (|S| - 1)\pi + \sum_{i \in S^c} \theta_i. \tag{16}$$

Proof. The proof is by induction on n. If $n = 4$, the inequalities (16) agree with (15) and the result follows by theorem 2. Suppose that the theorem holds for $n-1$. We examine the situation which occurs when $\cos\phi$ is substituted for the $1, n-1$ and $n-1, 1$ entries. The graph is then the simple cycle C_n with the edge $\{1, n-1\}$ added.

(17)

Now $(1, 2, 3, \ldots, n-1, 1)$ is a cycle in this graph and $C[\{1, n-1, n\}]$ is a fully specified principal submatrix. Thus, in order that the original partial matrix C have a positive semidefinite completion, it is necessary (by the inheritance property for principal submatrices) that there exist a $\phi \in [0, 2\pi]$ such that

$$B = \begin{bmatrix} 1 & \cos\theta_1 & & & \cos\phi \\ \cos\theta_1 & 1 & & ? & \\ & \ddots & \ddots & & \\ & ? & \ddots & \ddots & \cos\theta_{n-2} \\ \cos\phi & & & \cos\theta_{n-2} & 1 \end{bmatrix}$$

has a positive semidefinite completion, and

$$E = \begin{bmatrix} 1 & \cos\phi & \cos\theta_n \\ \cos\phi & 1 & \cos\theta_{n-1} \\ \cos\theta_n & \cos\theta_{n-1} & 1 \end{bmatrix}$$

is positive semidefinite. Conversely, if B has a positive semidefinite completion $\widetilde{B}$ and E is positive semidefinite, substitution of the entries from the completion $\widetilde{B}$ into their corresponding positions in C gives a partial positive semidefinite matrix $\widetilde{C}$ whose graph is obtained from (17) by adding all missing edges with vertices in $\{1, \ldots, n-1\}$. It is easy to see that this is a chordal graph and so by theorem 1, $\widetilde{C}$, and hence C, has a positive semidefinite completion.

Thus, the matrix C has a positive semidefinite completion if and only if B has a positive semidefinite completion and E is positive semidefinite.

By the induction hypothesis and theorem 2, these occur if and only if

$$\sum_{i \in S} \theta_i \leq (|S|-1)\pi + \sum_{i \in S^c} \theta_i + \phi \tag{18}$$
$$\text{for every } S \subseteq \{1, \ldots, n-2\}, |S| \text{ odd },$$

$$\sum_{i \in T} \theta_i + \phi \leq |T|\pi + \sum_{i \in T^c} \theta_i \tag{19}$$
$$\text{for every } T \subseteq \{1, \ldots, n-2\}, |T| \text{ even },$$

$$\phi \leq \theta_{n-1} + \theta_n \tag{20}$$
$$\theta_{n-1} \leq \phi + \theta_n \tag{21}$$
$$\theta_n \leq \phi + \theta_{n-1} \tag{22}$$
$$\phi + \theta_{n-1} + \theta_n \leq 2\pi \tag{23}$$

Equivalently,

$$\phi \geq \sum_{i \in S} \theta_i - \sum_{i \in S^c} \theta_i - (|S|-1)\pi \tag{24}$$
$$\text{for every } S \subseteq \{1, \ldots, n-2\}, |S| \text{ odd}$$

$$\phi \geq \theta_{n-1} - \theta_n \tag{25}$$
$$\phi \geq \theta_n - \theta_{n-1} \tag{26}$$

$$\phi \leq \sum_{i \in T^c} \theta_i - \sum_{i \in T} \theta_i + |T|\pi \tag{27}$$
$$\text{for every } T \subseteq \{1, \ldots, n-2\}, |T| \text{ even },$$

$$\phi \leq \theta_{n-1} + \theta_n \tag{28}$$
$$\phi \leq 2\pi - \theta_{n-1} - \theta_n \tag{29}$$

Equations (24)–(29) are equivalent to the 9 inequalities

$$\begin{aligned}&\text{For every } S \subseteq \{1,\ldots,n-2\}, |S| \text{ odd}\\&\text{and for every } T \subseteq \{1,\ldots,n-2\}, |T| \text{ even}\end{aligned}$$

$$\sum_{i\in S}\theta_i - \sum_{i\in S^c}\theta_i - (|S|-1)\pi \le \sum_{i\in T^c}\theta_i - \sum_{i\in T}\theta_i + |T|\pi \tag{30}$$

$$\sum_{i\in S}\theta_i - \sum_{i\in S^c}\theta_i - (|S|-1)\pi \le \theta_{n-1} + \theta_n \tag{31}$$

$$\sum_{i\in S}\theta_i - \sum_{i\in S^c}\theta_i - (|S|-1)\pi \le 2\pi - \theta_{n-1} - \theta_n \tag{32}$$

$$\theta_{n-1} - \theta_n \le \sum_{i\in T^c}\theta_i - \sum_{i\in T}\theta_i + |T|\pi \tag{33}$$

$$\theta_{n-1} - \theta_n \le \theta_{n-1} + \theta_n \tag{34}$$

$$\theta_{n-1} - \theta_n \le 2\pi - \theta_{n-1} - \theta_n \tag{35}$$

$$\theta_n - \theta_{n-1} \le \sum_{i\in T^c}\theta_i - \sum_{i\in T}\theta_i + |T|\pi \tag{36}$$

$$\theta_n - \theta_{n-1} \le \theta_{n-1} + \theta_n \tag{37}$$

$$\theta_n - \theta_{n-1} \le 2\pi - \theta_{n-1} - \theta_n \tag{38}$$

Inequalities (34), (35), (37) and (38) hold automatically. The inequalities (30) can be written

(39)

$$\sum_{i\in S}\theta_i + \sum_{i\in T}\theta_i \le (|S|+|T|-1)\pi + \sum_{i\in S^c}\theta_i + \sum_{i\in T^c}\theta_i$$

for every $S \subseteq \{1,\ldots,n-2\}, |S|$ odd and every $T \subseteq \{1,\ldots,n-2\}, |T|$ even .

Fix S and T. Since $|S|$ is odd and $|T|$ is even, $S \ne T$. Therefore, either $S\cap T^c \ne \phi$ or $S^c \cap T \ne \phi$. Assume $S \cap T^c \ne \phi$ and let $k \in S\cap T^c$. Then θ_k occurs in both sums, $\sum_{i\in S}\theta_i$ and $\sum_{i\in T^c}\theta_i$. Subtracting θ_k from both sides of (39),

$$\sum_{i\in S-\{k\}}\theta_i + \sum_{i\in T}\theta_i \le (|S|+|T|-1)\pi + \sum_{i\in S^c}\theta_i + \sum_{i\in T^c-\{k\}}\theta_i. \tag{40}$$

Since there are $|S|+|T|-1$ terms θ_i on the left hand side of (40) and each is bounded above by π, the inequality (40) holds automatically. If $S^c \cap T \ne \phi$, the same type of argument applies. It follows that the inequalities (39), and hence (30), hold automatically. Therefore, A has a positive semidefinite completion if and only if the inequalities (31), (32), (33) and (36) hold. These are:

$$\sum_{i\in S}\theta_i \le (|S|-1)\pi + \sum_{i\in S^c}\theta_i + \theta_{n-1} + \theta_n \qquad \text{for every } S\subseteq\{1,\dots,n-2\}, |S| \text{ odd} \tag{41}$$

$$\sum_{i\in S}\theta_i + \theta_{n-1} + \theta_n \le (|S|+1)\pi + \sum_{i\in S^c}\theta_i \qquad \text{for every } S\subseteq\{1,\dots,n-2\}, |S| \text{ odd} \tag{42}$$

$$\sum_{i\in T}\theta_i + \theta_{n-1} \le |T|\pi + \sum_{i\in T^c}\theta_i + \theta_n \qquad \text{for every } T\subseteq\{1,\dots,n-2\}, |T| \text{ even} \tag{43}$$

$$\sum_{i\in T}\theta_i + \theta_n \le |T|\pi + \sum_{i\in T^c}\theta_i + \theta_{n-1} \qquad \text{for every } T\subseteq\{1,\dots,n-2\}, |T| \text{ even} \tag{44}$$

But (41)–(44) are exactly equivalent to

$$\sum_{i\in S}\theta_i \le (|S|-1)\pi + \sum_{i\in S^c}\theta_i \qquad \text{for every } S\subseteq N, |S| \text{ odd}$$

and this completes the proof. □

Since the inequalities (16) are unchanged under an arbitrary permutation of $\theta_1,\dots,\theta_n$, we may assume that the θ_i's are relabeled so that

$$\theta_1 \ge \theta_2 \ge \cdots \ge \theta_n.$$

Then C has the form

$$C = \begin{bmatrix} 1 & \cos\theta_{i_1} & & & \cos\theta_{i_n} \\ \cos\theta_{i_1} & 1 & \cos\theta_{i_2} & ? & \\ & \cos\theta_{i_2} & 1 & & \\ & ? & & \ddots & \cos\theta_{i_{n-1}} \\ \cos\theta_{i_n} & & & \cos\theta_{i_{n-1}} & 1 \end{bmatrix}, \tag{45}$$

and

$$\sum_{i\in S}\theta_i \le (|S|-1) + \sum_{i\in S^c}\theta_i$$

holds for all S with $|S| = k$ if and only if

$$\sum_{i=1}^{k} \theta_i \leq (k-1)\pi + \sum_{i=k+1}^{n} \theta_i.$$

Thus, we have

Corollary 1. *Let $n \geq 4$ and $0 \leq \theta_n \leq \theta_{n-1} \leq \cdots \leq \theta_2 \leq \theta_1 \leq \pi$. Then the matrix C in (45) has a positive semidefinite completion if and only if for k odd, $k \in N$,*

$$\sum_{i=1}^{k} \theta_i \leq (k-1)\pi + \sum_{i=k+1}^{n} \theta_i. \tag{46}$$

Thus, the 2^{n-1} inequalities in (16) of theorem 3 can be replaced by about $\frac{n}{2}$ inequalities, although $\theta_1, \ldots, \theta_n$ are no longer completely symmetric in (46).

In certain cases it suffices that the first inequality in (46) hold. For example, if $\theta_2 \leq \frac{\pi}{2}$ then $\sum\limits_{i=1}^{k} \theta_i = \theta_1 + \sum\limits_{i=2}^{k} \theta_i \leq \pi + \sum\limits_{i=2}^{k} \frac{\pi}{2} = \frac{k+1}{2}\pi \leq (k-1)\pi$ for $k \geq 3$. In this case, C has a positive semidefinite completion if and only if $\theta_1 \leq \sum\limits_{i=2}^{n} \theta_i$. We restate this slightly for convenience.

Corollary 2. *Let $n \geq 4$, $0 \leq \theta_1, \ldots, \theta_n \leq \pi$ and assume that at most one of $\theta_1, \ldots, \theta_n$ is greater than $\frac{\pi}{2}$. Then*

$$C = \begin{bmatrix} 1 & \cos\theta_1 & & & \cos\theta_n \\ \cos\theta_1 & 1 & \cos\theta_2 & ? & \\ & \cos\theta_2 & 1 & & \\ & ? & & \ddots & \cos\theta_{n-1} \\ \cos\theta_n & & & \cos\theta_{n-1} & 1 \end{bmatrix}$$

has a positive semidefinite completion if and only if

$$2 \max_{k \in N} \theta_k \leq \sum_{k=1}^{n} \theta_k.$$

Corollary 3. *Let*

$$C = \begin{bmatrix} 1 & c_1 & & & c_n \\ c_1 & 1 & c_2 & ? & \\ & c_2 & 1 & & \\ & ? & & 1 & c_{n-1} \\ c_n & & & c_{n-1} & 1 \end{bmatrix}, \qquad |c_i| \leq 1, \quad i \in N. \tag{47}$$

If two or more of the c_i are 0, then C has a positive semidefinite completion.

Proof. Writing $c_i = \cos\theta_i$, $i = 1, \ldots, n$, we have $\theta_j = \theta_k = \frac{\pi}{2}$ for some $j, k \in N$, $j \neq k$. Let $S \subseteq N$. If $j, k \in S$, $\sum_{i \in S} \theta_i = 2\left(\frac{\pi}{2}\right) + \sum_{i \in S - \{j,k\}} \theta_i \leq \pi + (|S| - 2)\pi = (|S| - 1)\pi$. If $j \in S$ and $k \in S^c$, or vice versa, $\sum_{i \in S} \theta_i \leq \frac{\pi}{2} + (|S| - 1)\pi \leq (|S| - 1)\pi + \sum_{i \in S^c} \theta_i$. Finally, if $j, k \in S^c$, $\sum_{i \in S} \theta_i \leq |S|\pi = (|S| - 1)\pi + 2\left(\frac{\pi}{2}\right) \leq (|S| - 1)\pi + \sum_{i \in S^c} \theta_i$. The result follows from theorem 3. $\square$

Corollary 3 may also be proved directly as in [F, p. 115]. We now derive Fiedler's main result.

COROLLARY 4. *The matrix C given by (47) has a positive semidefinite completion if and only if*

$$2 \max_{k \in N} \arccos |c_k| \leq \sum_{k \in N} \arccos |c_k| \quad \textit{for } c_1 c_2 \ldots c_n > 0$$

and

$$\sum_{k \in N} \arccos |c_k| \geq \pi \quad \textit{for } c_1 c_2 \ldots c_n \leq 0.$$

Proof. If two or more of the c_k are 0, then C has a positive semidefinite completion and $\sum_{k \in N} \arccos |c_k| \geq 2\left(\frac{\pi}{2}\right) = \pi$.

So assume at most one $c_k = 0$. Since $P^T C P$ has a positive semidefinite completion for any cyclic permutation P if and only if C does, we may assume that $c_1, c_2, \ldots, c_{n-1}$ are nonzero. Let $D = \operatorname{diag}(1, \operatorname{sign} c_1, \operatorname{sign}(c_1 c_2), \ldots, \operatorname{sign}(c_1 \ldots c_{n-1}))$. Then C has a positive semidefinite completion if and only if

$$DCD = \begin{bmatrix} 1 & |c_1| & & & \varepsilon|c_n| \\ |c_1| & 1 & |c_2| & ? & \\ & |c_2| & 1 & & \\ & ? & & \ddots & |c_{n-1}| \\ \varepsilon|c_n| & & & |c_{n-1}| & 1 \end{bmatrix},$$

$\varepsilon = \operatorname{sign}(c_1 c_2 \ldots c_n)$, has one. Take $\varepsilon = 0$ if $c_n = 0$.

If $\varepsilon > 0$, by corollary 2, C has a positive semidefinite completion if and only if $2\max_{k \in N} \arccos |c_k| \leq \sum_{k \in N} \arccos |c_k|$. If $\varepsilon \leq 0$, then $\arccos(-|c_n|) = \pi - \arccos |c_n|$ and, again by corollary 2, C has a positive semidefinite completion if and only if $\pi - \arccos |c_n| \leq \sum_{k=1}^{n-1} \arccos |c_k|$, or $\pi \leq \sum_{k \in N} \arccos |c_k|$. This completes the proof. $\square$

Of course, Corollary 4 gives a complete characterization for the positive semidefinite completion problem in the case that the undirected graph is a simple cycle, as does theorem

3. However, we believe that theorem 3 provides additional insight not available from corollary 4. It is also difficult to see how theorem 3 might be proved from corollary 4.

We give two more corollaries to illustrate the utility of theorem 3 and its corollaries.

COROLLARY 5. *Let $n \geq 4$ and $|c| \leq 1$. Then*

$$C = \begin{bmatrix} 1 & c & & & c \\ c & 1 & c & ? & \\ & c & 1 & \ddots & \\ & ? & \ddots & \ddots & c \\ c & & & c & 1 \end{bmatrix}$$

has a positive semidefinite completion for all $c \in [-1,1]$ if n is even and for $c \geq \cos\left(\frac{n-1}{n}\pi\right)$ if n is odd.

Proof. Write $c = \cos\theta$. By corollary 1, C has a positive semidefinite completion if and only if $k\theta \leq (k-1)\pi + (n-k)\theta$ for k odd, $k \in N$. Equivalently, $(2k-n)\theta \leq (k-1)\pi$, k odd, $k \in N$, or $\theta \leq \frac{k-1}{2k-n}\pi$ for k odd, $k \in N$, and $2k > n$. Since $(k-1)/(2k-n)$ is decreasing, this is

$$\theta \leq \begin{cases} \frac{n-1}{n}\pi & \text{for } n \text{ odd} \\ \pi & \text{for } n \text{ even} \end{cases}$$

Thus C has a positive semidefinite completion if and only if $c \in [-1,1]$ for n even and $c \geq \cos\left(\frac{n-1}{n}\pi\right)$ for n odd. $\square$

In particular, the partial matrix

$$\begin{bmatrix} 1 & -1 & & & -1 \\ -1 & 1 & -1 & ? & \\ & -1 & 1 & \ddots & \\ & ? & \ddots & \ddots & -1 \\ -1 & & & -1 & 1 \end{bmatrix}$$

has a positive semidefinite completion, namely,

$$\begin{bmatrix} 1 & -1 & 1 & \dots & -1 \\ -1 & 1 & \ddots & \ddots & \vdots \\ 1 & \ddots & \ddots & \ddots & 1 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ -1 & \dots & 1 & -1 & 1 \end{bmatrix}$$

if n is even and no positive semidefinite completion if n is odd.

We end with the case of a partial positive semidefinite Toeplitz matrix.

COROLLARY 6. *Let $n \geq 4$ and $\theta, \phi \in [0, \pi]$. Then*

$$C = \begin{bmatrix} 1 & \cos\theta & & & \cos\phi \\ \cos\theta & 1 & \cos\theta & ? & \\ & \cos\theta & 1 & \ddots & \\ & ? & \ddots & \ddots & \cos\theta \\ \cos\phi & & & \cos\theta & 1 \end{bmatrix}$$

has a positive semidefinite completion if and only if

$$\phi \leq (n-1)\theta \leq (n-2)\pi + \phi \quad \textit{for } n \textit{ even}$$

and

$$\phi \leq (n-1)\theta \leq (n-1)\pi - \phi \quad \textit{for } n \textit{ odd} \; .$$

Proof. Note that (16) holds automatically if θ occurs on both sides of the inequality. Thus, we need only consider the cases in which S has minimal or maximal cardinality. If $|S| = 1$, the required inequality is $\phi \leq (n-1)\theta$. If n is even and $|S| = n-1$, we need $(n-1)\theta \leq (n-2)\pi + \phi$, while if n is odd we require $(n-1)\theta + \phi \leq (n-1)\pi$. This completes the proof. □

3. An alternative solution. As we saw in the introduction, the partial matrix C corresponds to the graph C_n. Even though the way we labeled the nodes in C_n is natural, it is not the only way. We relabel the nodes to produce an alternate pattern for the matrix associated with a simple cycle.

Given the cycle, if n is even we can label the nodes as follows:

For n odd the labeling is

Now the associated matrix is

$$
(48) \qquad D = \begin{pmatrix}
1 & d_1 & d_2 & & & & & & \\
d_1 & 1 & ? & d_3 & & & & & \\
d_2 & ? & 1 & ? & & & ? & & \\
 & d_3 & ? & 1 & & & & & \\
 & & & & \ddots & & & & \\
 & & ? & & & 1 & ? & d_{n-2} & \\
 & & & & & ? & 1 & ? & d_{n-1} \\
 & & & & & d_{n-2} & ? & 1 & d_n \\
 & & & & & & d_{n-1} & d_n & 1
\end{pmatrix}, |d_i| < 1, i \in N
$$

and its pentadiagonal structure is clear. The unspecified entries inside the pentadiagonal structure are the only ones that concern us; call them x_i, $i = 1, \ldots, n-3$.

We use Theorem 1 [GJSW] to decide if D has a positive definite completion and to see how to choose the x_i, $i = 1, \ldots, n-3$ in this event.

First consider the case $n = 5$, which illustrates all important features of the problem. The corresponding matrix is

$$
(49) \qquad D_5 = \begin{pmatrix}
1 & d_1 & d_2 & ? & ? \\
d_1 & 1 & x_1 & d_3 & ? \\
d_2 & x_1 & 1 & x_2 & d_4 \\
? & d_3 & x_2 & 1 & d_5 \\
? & ? & d_4 & d_5 & 1
\end{pmatrix}
$$

We proceed as we did to obtain Proposition 1.

From Theorem 1 [GJSW] this matrix D_5 has a positive definite completion if and only if there exist x_1 and x_2 such that $|x_1| < 1, |x_2| < 1$ and such that the three 3-by-3 contiguous principal minors are positive. For then, the matrix is partial positive definite and its associated graph is chordal.

We can transform the conditions given in the statement above by computing the 3-by-3 determinants. The first one is given by

$$\det \begin{bmatrix} 1 & d_1 & d_2 \\ d_1 & 1 & x_1 \\ d_2 & x_1 & 1 \end{bmatrix} = \det \begin{bmatrix} 1 & d_1 & x_1 \\ d_1 & 1 & d_2 \\ x_1 & d_2 & 1 \end{bmatrix}$$

and from (8) we obtain the inequality

$$|x_1 - d_1 d_2| < \sqrt{(1-d_1^2)(1-d_2^2)}. \tag{50}$$

We will denote by I_1 the set of solutions of (50).

The second determinant down the diagonal is

$$\det \begin{vmatrix} 1 & x_1 & d_3 \\ x_1 & 1 & x_2 \\ d_3 & x_2 & 1 \end{vmatrix} = 1 + 2d_3 x_1 x_2 - x_1^2 - x_2^2 - d_3^2$$

and the associated inequality is

$$x_1^2 + x_2^2 - 2d_3 x_1 x_2 < 1 - d_3^2,$$

which may be written as the quadratic form

$$(x_1, x_2) \begin{pmatrix} 1 & -d_3 \\ -d_3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} < 1 - d_3^2. \tag{51}$$

The last determinant is

$$\det \begin{vmatrix} 1 & x_2 & d_4 \\ x_2 & 1 & d_5 \\ d_4 & d_5 & 1 \end{vmatrix} = \det \begin{vmatrix} 1 & d_4 & x_2 \\ d_4 & 1 & d_5 \\ x_2 & d_5 & 1 \end{vmatrix}$$

and again from (8) we obtain the inequality

$$|x_2 - d_4 d_5| < \sqrt{(1-d_4^2)(1-d_5^2)}. \tag{52}$$

and we denote by I_2 the interval of solutions.

Now we can rewrite our statement as the following result.

Proposition 2. *The matrix D_5 given by (49) has a positive definite completion if and only if there exists a solution for the quadratic system*

$$
(53)\qquad
\begin{cases}
|x_1 - d_1 d_2| < \sqrt{(1-d_1^2)(1-d_2^2)} \\[2ex]
(x_1, x_2)\begin{pmatrix} 1 & -d_3 \\ -d_3 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} < 1 - d_3^2 \\[2ex]
|x_2 - d_4 d_5| < \sqrt{(1-d_4^2)(1-d_5^2)}.
\end{cases}
$$

At this point, we make some observation about the inequalities that remain true for the general case. Clearly the first and the last inequalities are similar and the solution sets are intervals contained in [-1,1], in other words

$$
\begin{aligned}
-1 \leq d_1 d_2 - \sqrt{(1-d_1^2)(1-d_2^2)} < x_1 < d_1 d_2 + \sqrt{(1-d_1^2)(1-d_2^2)} \leq 1 \\
-1 \leq d_4 d_4 - \sqrt{(1-d_4^2)(1-d_5^2)} < x_2 < d_4 d_5 + \sqrt{(1-d_4^2)(1-d_5^2)} \leq 1
\end{aligned}
$$

The quadratic inequality in the middle has an associated quadratic form. For the corresponding matrix $1+d_3$ and $1-d_3$ are the eigenvalues and the eigenvectors are $(1/\sqrt{2}, 1/\sqrt{2})^T$ and $(1/\sqrt{2}, -1/\sqrt{2})^T$.

Therefore the admissible solutions to the inequality are those (x_1, x_2) lying inside an ellipse:

The equation of this ellipse is

$$x_1^2 + x_2^2 - 2d_3 x_1 x_2 = 1 - d_3^2.$$

We can solve, for example, for x_1 the quadratic equation

$$x_1^2 - (2d_3x_2)x_1 + (x_2^2 + d_3^2 - 1) = 0$$

obtaining

$$x_1 = d_3x_2 \pm \sqrt{(1-d_3^2)(1-x_2^2)}.$$

For $x_2 = \pm 1$ we obtain the tangency points, $(d_3, 1)$ and $(-d_3, -1)$, between the ellipse and the lines $x_2 = \pm 1$.

A similar analysis gives us the other tangency points $(1, d_3)$ and $(-1, -d_3)$ between the ellipse and the lines $x_1 = \pm 1$.

Observe that in the picture we assumed $d_3 > 0$; for $d_3 = 0$ we obtain a unit circle and for $d_3 < 0$ the result is an ellipse rotated 90 degrees.

Finally note that in the general case each variable only appears in two inequalities. This fact suggests an easy algorithm to decide if the system has a solution and to compute solutions.

To solve the system means to find $x_1 \in I_1$ and $x_2 \in I_2$ such that the point (x_1, x_2) is inside the ellipse. We describe an algorithm that can be extended to the general case. Given an interval $I \subseteq [-1, 1]$ we define ϕ_3 as follows

$$\phi_3(I) = \{x_2 \ : \ \text{there exist } x_1 \in I \ \text{ such that } \ (x_1, x_2) \ \text{ satisfies (51)}\}.$$

It is easy to verify that for any interval $I \subseteq [-1, 1]$, $\phi_3(I)$ is a nonempty interval. Moreover for $\epsilon > 0$ small enough $\phi_3([-\epsilon, \epsilon]) \supset [-\epsilon, \epsilon]$.

Even though we will not do it here, it is not a difficult task to give an explicit definition of ϕ_3.

Applying this solvability condition to the system (53) we have an alternate version of Proposition 2.

PROPOSITION 3. *The matrix D_5 has a positive definite completion if and only if*

$$\phi_3(I_1) \cap I_2 \neq \emptyset.$$

Proof. We have that $x_2 \in \phi_3(I_1) \cap I_2$ if and only if $x_2 \in I_2$ and there exists $x_1 \in I_1$ such that (x_1, x_2) satisfies (51) if and only if (x_1, x_2) is a solution of system (53). □

Consider an example. We will denote by $\hat{D}_5$ the matrix

$$\hat{D}_5 = \begin{pmatrix} 1 & .5 & .5 & ? & ? \\ .5 & 1 & x_1 & .5 & ? \\ .5 & x_1 & 1 & x_2 & .5 \\ ? & .5 & x_2 & 1 & .5 \\ ? & ? & .5 & .5 & 1 \end{pmatrix}$$

For this example the system (53) becomes

$$
(54) \qquad \begin{cases} |x_1 - .25| < .75 \\ x_1^2 + x_2^2 - x_1 x_2 < .75 \\ |x_2 - .25| < .75 \end{cases}
$$

For every $x_1 \in (-1, 1)$ we define

$$
e(x_1) = \{x_2 \; : \; (x_1, x_2) \text{ satisfies the second inequality of } (54)\}.
$$

Observe that for this example $I_1 = I_2 = (-.5, 1)$. It is easy to see that for each $x_1 \in (-.5, 1)$, $e(x_1) \cap (-.5, 1) \neq \emptyset$ which clearly implies

$$
\phi_3(-.5, 1) \cap (-.5, 1) \neq \emptyset,
$$

which says that there exists a positive definite completion.

Also, we may easily display the set of all solutions, i. e., every pair (x_1, x_2) that allows a positive definite completion.

Given $x_1 \in (-.5, 1)$ and $x_2 \in e(x_1) \cap (-.5, 1)$, then (x_1, x_2) allows a positive definite completion.

In order to see the relation with Theorem 3, we can rewrite our matrix $\hat{D}_5$ to generate $\hat{C}_5$ defined by

$$
\hat{C}_5 = \begin{pmatrix} 1 & .5 & ? & ? & .5 \\ .5 & 1 & .5 & ? & ? \\ ? & .5 & 1 & .5 & ? \\ ? & ? & .5 & 1 & .5 \\ .5 & ? & ? & .5 & 1 \end{pmatrix}.
$$

Because $\cos(\frac{\pi}{3}) = .5$, we have $\theta_1 = \theta_2 = \ldots \theta_5 = \frac{\pi}{3}$. For our example $N = \{1, 2, \ldots, 5\}$ and given $S \subseteq N$ we have to consider S such that $|S|$ is equal to 1,3 or 5.

Since every θ_i is equal the set of inequalities (16) is very simple.

$$
\begin{aligned}
|S| = 1: &\quad \frac{\pi}{3} < \frac{4\pi}{3} \\
|S| = 3: &\quad \frac{3\pi}{3} < 2\pi + \frac{2\pi}{3} \\
|S| = 5: &\quad \frac{5\pi}{3} < 4\pi
\end{aligned}
$$

The existence of a positive definite completion is assured, but specific information about the solution set was readily obtained in the alternate approach.

Now we have all the elements to make a straightforward generalization about a positive definite completion of the matrix D, but first we will generalize the notation. The unspecified entries in the position $(2,3), (3,4), \ldots, (n-2, n-1)$ will be denoted by $x_1, x_2, \ldots, x_{n-3}$

Now we have $n-2$ contiguous principal minors of size 3×3 that have to be positive. The first and the last one yield linear inequalities, and those in the middle yield quadratic inequalities like the second one given in system (53); in other words we obtain

$$(54) \qquad (x_i, x_{i+1}) \begin{pmatrix} 1 & -d_{i+2} \\ -d_{i+2} & 1 \end{pmatrix} \begin{pmatrix} x_i \\ x_{i+1} \end{pmatrix} < 1 - d_{i+2}^2 \quad i = 1, \dots, n-4.$$

Then these positive minors are equivalent to the following system

$$(55) \qquad \begin{cases} |x_1 - d_1 d_2| < \sqrt{(1-d_1^2)(1-d_2^2)} \\ \\ [x_i, x_{i+1}] \begin{bmatrix} 1 & -d_{i+2} \\ -d_{i+2} & 1 \end{bmatrix} \begin{bmatrix} x_i \\ x_{i+1} \end{bmatrix} < 1 - d_{i+2}^2 \quad i = 1, \dots, n-4. \\ \\ |x_{n-3} - d_{n-1} d_n| < \sqrt{(1-d_{n-1}^2)(1-d_n^2)}. \end{cases}$$

Associated with each quadratic inequality we have a ϕ-function defined as follows.

For every $I \subseteq [-1,1]$

$$\phi_{i+2} = \{x_{i+1} \ : \ \text{there exist } x_i \in I \text{ such that } (x_i, x_{i+1}) \text{ satisfies (54)}\}$$

for every $i = 1, \dots, n-4$. Because every ϕ_{i+2}, $i = 1, \dots, n-4$ satisfies the properties pointed out after the definition of ϕ_3, we can apply them sequentially since

$$\phi_{i+2}(I) \qquad i = 1, \dots, n-4$$

is always an interval contained in [-1,1]. We now take I_2 to be the interval defined by the third equation in (55).

We can now establish the main result of this section.

THEOREM 4. *Given the matrix D, the following statements are equivalent.*

i) D has a positive definite completion.

ii) The system (55) has a solution.

iii) $\phi_{n-2}(\dots(\phi_3(I_1))) \cap I_2 \neq \emptyset$.

The proof of this theorem follows the proofs of propositions 2 and 3.

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