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Turning a Corner with a Dubins Car

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***Abstract*— We study the problem of computing shortest collision-free Dubins paths when turning a corner. We present a sufficient condition for a closed-form solution. Specifically, consider S as the set consisting of paths of the form RSRSR, RSRSL, LSRSR and LSRSL that pass through the interior corner, where sub-paths RSR, RSL, and LSR are elementary Dubins paths composed of segments which are either straight (S) or turning left (L) or right (R). We find the closed-form optimal path around a corner when S is nonempty. Our solution can be used in an efficient path planner, for example, when navigating corridors. It can also be used as a subroutine for planners such as RRTs.**

I. Introduction

Many mobile robots have movement constraints. Some have acceleration or velocity limits, while others may move only in the direction that they are currently oriented. One of the simplest and most useful models in mobile robotics is the *Dubins car*, a point-sized robot that travels forward with a constant velocity and may only turn right or left with a minimum turning radius of one.

Path planning for a Dubins car is a well-established topic. Lester Dubins showed in 1957 that the shortest route for a Dubins car from one configuration to another on the open plane is one of six basic types [1]. When obstacles are introduced to the environment, however, closed-form solutions are not generally possible (in fact, Dubins path planning with arbitrary obstacles is NP-hard [2], [3]). Much work has been done to develop algorithms that find or approximate shortest Dubins paths in environments with polygonal obstacles [4], [5], or curvature-constrained convex obstacles [6]. Agarwal et al. found an efficient algorithm to calculate shortest Dubins paths within a convex environment [7]. For large or dynamic environments, Dubins planning can be important in more general motion planning algorithms, such as Model Predictive Control (MPC). Finding Dubins paths may be a subroutine in a Rapidly-exploring Random Tree (RRT), for example, in which case computational efficiency is crucial. Low-fidelity models for more complex systems (see [8]), such as for use in autonomous vehicles, may use a Dubins car model when planning safe trajectories. In these cases, closed-form solutions to very simple Dubins planning problems can be useful.

We address one of the most basic situations in path planning: turning around a corner. Toward this goal we present a closed-form optimal path if a set S containing feasible paths of types RSRSR, RSRSL, LSRSR and RSRSR is nonempty. We provide a method of determining when S is nonempty and outline a method to analytically find the optimal path in this case.

Our paper is organized into several important sections. First, we formally define a Dubins path, describe a corner environment and introduce further geometric assumptions and terminology (section II). Next, we present novel structural results relating to four of the six original types of Dubins paths presented in Dubins' original paper (section IV). We then present our main results (section VI). Finally, we make note of how to use our results to find the shortest path in closed-form (section VII).

II. Definitions and Terminology

We define a Dubins path as a set of poses $\Pi = (x(t), y(t), \theta(t)) \subset \mathbb{R}^2 \times [0, 2\pi)$ for differentiable functions x, y and θ that satisfy $\dot{x}(t) = \cos(\theta(t))$, $\dot{y}(t) = \sin(\theta(t))$ and $\dot{\theta}(t) \le 1$ in some interval $t \in [t_i, t_f]$. We say Π is feasible in a closed environment $\Omega \subset \mathbb{R}^2$ if and only if $(x(t), y(t)) \in \Omega$ for all $t \in [t_i, t_f]$. Finally, we define the length $|\Pi|$ as $t_f - t_i$, or, equivalently in the geometric sense,

$$|\Pi| = \int_{t_i}^{t_f} \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}\, dt\, .$$

Our problem seeks the shortest path Π^*_Ω over the environment Ω depicted in Figure 1 with fixed $\Pi^*_\Omega(t_i) = p_i$ and $\Pi^*_\Omega(t_f) = p_f$, where p_i has orientation $\theta(t_i) = \frac{\pi}{2}$ and p_f has orientation $\theta(t_f) = \psi$. We say that the shortest path Π^*_Ω is *optimal* in Ω.

We now introduce some terminology.

- *Elementary Dubins paths: RSR, RSL, RLR, LSR, LSL, LRL*. These are the six classes of paths found by Dubins to be shortest for connecting two poses on the plane. We refer to these paths as elementary Dubins paths. Generally, a string of Rs, Ls and Ss refers to a Dubins path created by stringing together right-handed arcs of radius one (R), left-handed arcs of radius one (L), and straight lines (S) in the order that they appear.
- The function $LOC(p)$: For a pose $p = (x, y, \theta)$, $LOC(p) = (x, y)$.
- $\partial\Omega$: The boundary of Ω. Note that Ω is closed: $\partial\Omega \subset \Omega$.
- *inner corner*: The origin in Figure 1, where ψ is defined.
- $\partial\Omega_{out}$, $\partial\Omega_{in}$, $\partial\Omega_{end}$: these are disjoint sets satisfying $\partial\Omega_{out} \cup \partial\Omega_{in} \cup \partial\Omega_{end} = \partial\Omega \setminus V$, where V is the set of vertices on $\partial\Omega$ (i.e. the corners). $\partial\Omega_{out}$ is partitioned into two connected sets $\partial\Omega_{out,1}$ and $\partial\Omega_{out,2}$ so that $\partial\Omega_{out,1}$ is the vertical outer wall and $\partial\Omega_{out,2}$ is the tilted outer wall. Likewise, $\partial\Omega_{in}$ is partitioned into

connected sets $\partial\Omega_{in,1}$ and $\partial\Omega_{in,2}$ and $\partial\Omega_{end}$ into connected sets $\partial\Omega_{end,1}$ and $\partial\Omega_{end,2}$ so that $\partial\Omega_{in,1}$ is the vertical inner wall, $\partial\Omega_{in,2}$ is the tilted inner wall, $\partial\Omega_{end,1}$ is the wall containing p_i and $\partial\Omega_{end,2}$ is the wall containing p_f.

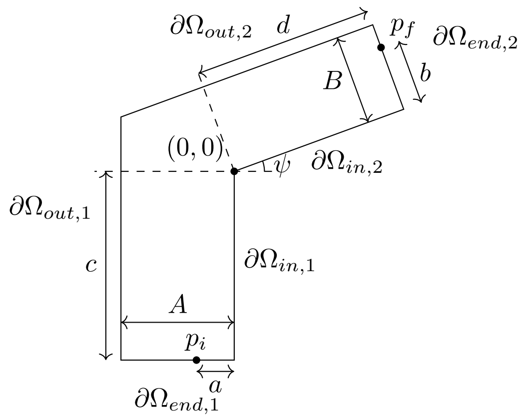

Fig. 1: We seek the shortest (or optimal) path from p_i to p_f within this corner-like environment. The angle at the inner corner, ψ, ranges from 0 to $\pi/2$.

We make the following geometric assumptions:

1) $a > 0$, $b > 0$, $A > a$ and $B > b$ (see Figure 1).
2) $d\cos\psi - b\sin\psi > 0$. This ensures that $LOC(p_f)$ lies to the right of the inner corner. We also require $\arctan\left(\frac{c}{a}\right) > \psi$, ensuring that $LOC(p_i)$ lies below the line containing $\partial\Omega_{in,2}$. This assumption is not critical to the argumentation but relieves some tedious casework.
3) The shortest distance between $\partial\Omega_{out,2}$ and $LOC(p_i)+(1,0)$ is greater than 1. Likewise the shortest distance between $\partial\Omega_{out,1}$ and $LOC(p_f)+(\sin\psi,-\cos\psi)$ is greater than 1. This is important in the proof of Lemma 9.

III. Preliminaries

We will make use of the following lemma originally developed by Jacobs and Canny and adapted from Lemma 2.2 in [7].

Lemma 1 ([4]). *Let Ω be a closed polygon environment, p_i an initial pose and p_f a final pose. Then an optimal path from p_i to p_f in Ω consists of a sequence $\Pi_1, ..., \Pi_k$ of feasible paths, where each Π_j is an elementary Dubins path from pose χ_j, such that $\chi_0 = p_i$, $\chi_k = p_f$ and, for $0 < j < k$, $LOC(\chi_j) \in \partial\Omega$.*

Equally important is the following well-known result (see [9]). For our purposes, we say *non-terminal* describes a sub-path that does not include the start or end poses p_i and p_f:

Lemma 2. *Let Π be an optimal Dubins path between the poses p_i and p_f in the environment $\mathbb{R}^2$. Then Π does not contain a non-terminal arc with arc length less than π.*

Agarwal et al. extend this result to convex polygonal environments [7]. However, our environment (Figure 1) is not convex so we must adapt Lemma 2 slightly.

Corollary 2.1. *Let Π be an optimal Dubins path between poses p_i and p_f in a polygonal, non-convex environment Ω. Then Π does not contain a non-terminal arc α with arc length less than π unless a point on $\partial\Omega$ is internally incident to α.*

When we say a point $\mathbf{p}$ on $\partial\Omega$ is *internally incident* to an arc, we mean that in a neighborhood of $\mathbf{p}$ the boundary $\partial\Omega$ is contained within the circle that extends the arc. Formally, we say that a point $\mathbf{p}$ is internally incident to a feasible arc α if and only if $\mathbf{p} \in \alpha$ and for every $\epsilon > 0$ there exists some $\mathbf{v} \in \mathbb{R}^2 \setminus \Omega$ such that $||\mathbf{p} - \mathbf{v}|| < \epsilon$ and $\mathbf{v}$ lies within the circle extending α.

We may extend Lemma 2 to Corollary 2.1 by inspecting how Lemma 2 is proved. Briefly, for any path Π containing a non-terminal arc with arc length less than π another path Π' may be constructed of shorter length. There always exists a shorter path Π' that lies entirely to one side – the concave side – of Π (see [1] and [6] for details). If there is no internally incident point on α, then a path Π' is always possible and Π is not optimal.

IV. Structural Results

Elementary Dubins paths play a crucial role in any optimal Dubins path planning problem. We find it necessary to characterize the existence of elementary Dubins paths that do not contain an arc subtending an angle π or greater. Let Π be an elementary Dubins path connecting a point $\mathbf{p}$ with orientation given by the vector $\mathbf{u}$ to a point $\mathbf{p}'$ with orientation given by $\mathbf{u}'$. Recall that all elementary Dubins paths begin and end with (possibly degenerate) arcs. Let $\mathbf{p} - \mathbf{w}$ be the center of curvature of the first arc in Π and $\mathbf{p}' - \mathbf{w}'$ be the center of curvature of the final arc.

Lemma 3. *Suppose Π is an elementary Dubins RSR path. The first R arc has length greater than or equal to π if and only if $\mathbf{p}' - \mathbf{w}'$ lies on or to the right of the line*[1] *$L(t) = \mathbf{p} - \mathbf{w} + \mathbf{u}t$. The second R arc has length less than or equal to π if and only if $\mathbf{p} - \mathbf{w}$ lies on or to the right of the line $L(t) = \mathbf{p}' - \mathbf{w}' + \mathbf{u}'t$.*

Proof. By definition an RSR path is composed of two arcs subtending the unit circles with centers $\mathbf{p}-\mathbf{w}$ and $\mathbf{p}'-\mathbf{w}'$ and a straight line, which is an external tangent to these circles. Furthermore, this tangent must lie to the left of $\mathbf{v}$, the vector pointing from $\mathbf{p}-\mathbf{w}$ to $\mathbf{p}'-\mathbf{w}'$. The external tangent always exists for two unit circles positioned arbitrarily on the plane, and it is parallel to $\mathbf{v}$ (see Figure 2). $\mathbf{v}$ then determines θ, and we observe $\theta \in [0, \pi] \iff \mathbf{v} \cdot \mathbf{w} \le 0$. This is exactly equal to the statement $\mathbf{p}' - \mathbf{w}'$ lies on or to the right of the line $L(t) = \mathbf{p} - \mathbf{w} + \mathbf{u}t$. Similar reasoning requires for the final arc $\mathbf{v} \cdot \mathbf{w}' \ge 0$, yielding the requirement that $\mathbf{p} - \mathbf{w}$ lie on or to the right of the line $L(t) = \mathbf{p}' - \mathbf{w}' + \mathbf{u}'t$. □

[1]that is, a point $\mathbf{x}$ lies to the right of a line $L(t)$ if and only if $(\mathbf{x} - L(0)) \times L'(t) > 0$.

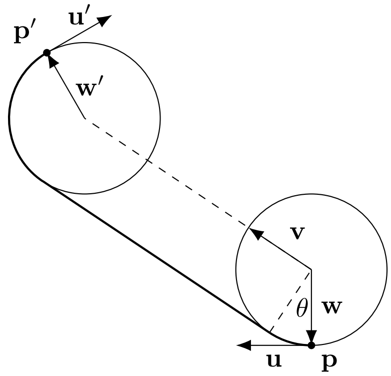

Fig. 2: RSR path with each arc shorter than π.

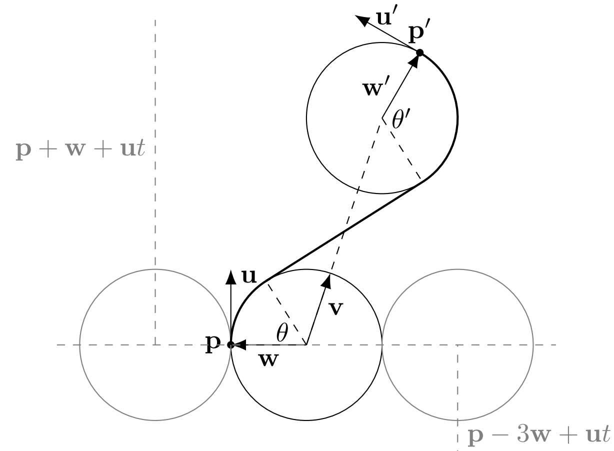

Fig. 3: An RSL path with each arc shorter than π. The parameterized lines used to determine if $\theta \in [0, \pi]$ are shown in gray.

Lemma 4. *Suppose* Π *is an elementary Dubins LSL path. Then the first L arc has length less than or equal to* π *if and only if* $\mathbf{p}' - \mathbf{w}'$ *lies on or to the left of the line* $L(t) = \mathbf{p} - \mathbf{w} + \mathbf{u}t$*, and the second L arc has length less than or equal to* π *if and only if* $\mathbf{p} - \mathbf{w}$ *lies on or to the left of the line* $L(t) = \mathbf{p}' - \mathbf{w}' + \mathbf{u}'t$*.*

Proof. An LSL path makes use of the external tangent to the right of $\mathbf{v}$. Following identically to Lemma 3, we again require $\mathbf{v} \cdot \mathbf{w} \leq 0$ and $\mathbf{v} \cdot \mathbf{w}' \geq 0$ for the first and second arcs respectively, leading directly to the claim. □

Lemma 5. *Suppose* Π *is an elementary Dubins RSL path. Let* $\mathbf{v}$ *point from* $\mathbf{p} - \mathbf{w}$ *to* $\mathbf{p}' - \mathbf{w}'$*. Then* $||(\mathbf{p} - \mathbf{w}) - (\mathbf{p}' - \mathbf{w}')|| \geq 2$*, and the first arc has length less than or equal to* π *if and only if* $\mathbf{p}' - \mathbf{w}'$ *lies on or to the right of the line*

$$L(t) = \begin{cases} \mathbf{p} + \mathbf{w} + \mathbf{u}t & \text{if } \mathbf{v} \cdot \mathbf{u} \geq 0 \\ \mathbf{p} - 3\mathbf{w} + \mathbf{u}t & \text{otherwise} \end{cases}.$$

The second arc has length less than or equal to π *if and only if* $\mathbf{p} - \mathbf{w}$ *lies on or to the left of the line*

$$L(t) = \begin{cases} \mathbf{p}' + \mathbf{w}' + \mathbf{u}'t & \text{if } \mathbf{v} \cdot \mathbf{u}' \geq 0 \\ \mathbf{p}' - 3\mathbf{w}' + \mathbf{u}'t & \text{otherwise} \end{cases}.$$

Proof. An RSL path requires the existence of an internal tangent to the unit circles centered at $\mathbf{p} - \mathbf{w}$ and $\mathbf{p}' - \mathbf{w}'$. For two unit circles on the plane, an internal tangent only exists if the circles are non-overlapping, hence the conclusion $||(\mathbf{p} - \mathbf{w}) - (\mathbf{p}' - \mathbf{w}')|| \geq 2$.

We first consider the length of the first arc, θ. Let $\mathbf{p}' - \mathbf{w}' = (r\cos t, r\sin t)$ for $(r, t) \in Q$ with $Q = [2, \infty) \times (-\infty, \infty)$. It is not difficult to verify that $\chi = (\cos\theta, \sin\theta)$ varies continuously with r and t on Q. Furthermore, $f(r, t) = ||\chi - (0, 1)||$ is monotonically decreasing as $(r\cos t, r\sin t)$ moves perpendicularly and to the right with respect to $\mathbf{u}$. Since $\theta \in [0, \pi] \iff f(r, t) \leq \sqrt{2}/2$ and $\theta = 0$ when $\mathbf{p}' - \mathbf{w}' = \mathbf{p} + \mathbf{w} + \mathbf{u}t$ for some $t \geq 0$, we conclude that $\theta \in [0, \pi]$ if and only if $\mathbf{v} \cdot \mathbf{u} \geq 0$ and $\mathbf{p}' - \mathbf{w}'$ lies on or to the right of the line $L(t) = \mathbf{p} + \mathbf{w} + \mathbf{u}t$. Similarly, $\theta = 0$ when $\mathbf{p}' - \mathbf{w}' = \mathbf{p} - 3\mathbf{w} + \mathbf{u}t$ for $t \leq 0$. Therefore $\theta \in [0, \pi]$ when $\mathbf{v} \cdot \mathbf{u} \leq 0$ if and only if $\mathbf{p}' - \mathbf{w}'$ lies on or to the right of the line $L(t) = \mathbf{p} - 3\mathbf{w} + \mathbf{u}t$ (see Figure 3).

The requirements for the second arc may be derived in a similar manner to the first. We note that $\theta' = 0$ when $\mathbf{v} \cdot \mathbf{u}' \geq 0$ and $\mathbf{p} - \mathbf{w} = \mathbf{p}' + \mathbf{w}' + \mathbf{u}'t$ for some $t \leq 0$, and $\theta' = \pi$ when $\mathbf{v} \cdot \mathbf{u}' \leq 0$ and $\mathbf{p} - \mathbf{w} = \mathbf{p}' - 3\mathbf{w}' + \mathbf{u}'t$ for some $t \geq 0$. Using the same monotonicity argument as before, we conclude that $\theta' \in [0, 2\pi]$ if and only if $\mathbf{p} - \mathbf{w}$ lies on or to the left of the line $L(t) = \mathbf{p}' + \mathbf{w}' + \mathbf{u}'t$ when $\mathbf{v} \cdot \mathbf{u}' \geq 0$ and $\mathbf{p} - \mathbf{w}$ lies on or to the left of the line $L(t) = \mathbf{p}' - 3\mathbf{w}' + \mathbf{u}'t$ when $\mathbf{v} \cdot \mathbf{u}' < 0$. □

Lemma 6. *Suppose* Π *is an elementary Dubins LSR path with* $\mathbf{v}$ *defined as in Lemma 5. Then* $||(\mathbf{p} - \mathbf{w}) - (\mathbf{p}' - \mathbf{w}')|| \geq 2$*, and the first arc has length less than or equal to* π *if and only if* $\mathbf{p}' - \mathbf{w}'$ *lies on or to the left of the line*

$$L(t) = \begin{cases} \mathbf{p} + \mathbf{w} + \mathbf{u}t & \text{if } \mathbf{v} \cdot \mathbf{u} \geq 0 \\ \mathbf{p} - 3\mathbf{w} + \mathbf{u}t & \text{otherwise} \end{cases}.$$

The second arc has length less than or equal to π *if and only if* $\mathbf{p} - \mathbf{w}$ *lies on or to the right of the line*

$$L(t) = \begin{cases} \mathbf{p}' + \mathbf{w}' + \mathbf{u}'t & \text{if } \mathbf{v} \cdot \mathbf{u}' \geq 0 \\ \mathbf{p}' - 3\mathbf{w}' + \mathbf{u}'t & \text{otherwise} \end{cases}.$$

Proof. The steps required are the same as in Lemma 5. □

Lemma 7. *Between two arbitrary poses, at most one unique elementary Dubins path in the set* {*RSL*, *LSR*, *RSR*, *LSL*} *does not contain an arc of* π *or greater.*

Proof. We examine each pair of path types and conclude that they result in the same path (they are not unique) or that one has an arc of π or more. Although it was convenient to let $\mathbf{w}$ point from the center of curvature, to prevent confusion during comparison we let $\mathbf{r}$ and $\mathbf{r}'$ be the unit perpendicular vectors of $\mathbf{u}$ and $\mathbf{u}'$ pointing to the right, and $\mathbf{l}$ and $\mathbf{l}'$ be unit perpendicular vectors of $\mathbf{u}$ and $\mathbf{u}'$ pointing to the left, respectively. For brevity, we use "short" to mean not containing an arc of length greater than π.

A short RSR path requires that $\mathbf{p}'+\mathbf{r}'$ lie on or to the right of the line $\mathbf{p}+\mathbf{r}+\mathbf{u}t$ while a short LSR path requires $\mathbf{p}'+\mathbf{r}'$ lie on or to the left of the same line[2]. Clearly, both paths can only be short if $\mathbf{p}'+\mathbf{r}'$ lies directly on the line $\mathbf{p}+\mathbf{r}+\mathbf{u}t$. Then, however, both paths are the same degenerate SR path.

Identical arguments varying the line and point in question hold for comparing LSL to RSL, RSR to RSL and LSL to LSR.

We next compare RSR to LSL. A short LSL path requires that $\mathbf{p}'+\mathbf{l}'$ lie on or to the left of the line $\mathbf{p}+\mathbf{l}+\mathbf{u}t$. Since the lines $\mathbf{p}+\mathbf{l}+\mathbf{u}t$ and $\mathbf{p}+\mathbf{r}+\mathbf{u}t$ are a distance 2 apart as are $\mathbf{p}'+\mathbf{l}'$ and $\mathbf{p}'+\mathbf{r}'$, both short path conditions are satisfied only when both $\mathbf{p}'+\mathbf{r}'$ lies on the line $\mathbf{p}+\mathbf{r}+\mathbf{u}t$ and $\mathbf{p}'+\mathbf{l}'$ lies on the line $\mathbf{p}+\mathbf{l}+\mathbf{u}t$. This case is simply a degenerate S path, and both RSR and LSL paths are identical.

Finally, we consider LSR and RSL paths. Assume $\mathbf{v}\cdot\mathbf{u}\geq 0$. Then, if both the LSR and RSL paths are short, then Lemmas 6 and 5 require that both $\mathbf{p}'+\mathbf{r}'$ and $\mathbf{p}'+\mathbf{l}'$ lie on or between the lines $\mathbf{p}+\mathbf{r}+\mathbf{u}t$ and $\mathbf{p}+\mathbf{l}+\mathbf{u}t$, and that both $\mathbf{p}+\mathbf{r}$ and $\mathbf{p}+\mathbf{l}$ lie on or between the lines $\mathbf{p}'+\mathbf{l}'+\mathbf{u}'t$ and $\mathbf{p}'+\mathbf{r}'+\mathbf{u}'t$. This is only possible if $\mathbf{p}'+\mathbf{r}'$ lies on $\mathbf{p}+\mathbf{r}+\mathbf{u}t$ and $\mathbf{p}'+\mathbf{l}'$ lies on $\mathbf{p}+\mathbf{l}+\mathbf{u}t$. In this case, both the RSL and LSR paths are degenerate S paths, and are so not unique. The case where $\mathbf{v}\cdot\mathbf{u}<0$ is similar. $\square$

V. Outline

We aim to show that a large class of optimal solutions for Π^*_Ω consists of two elementary Dubins paths joined at the corner. While there are 36 possible combinations of elementary Dubins paths possible, we will show that only four may be optimal. Define S as the set of feasible paths from p_i to p_f of the form RSRSR, RSRSL, LSRSR and LSRSL such that the middle R arc passes through the inner corner. Our goal is to prove the following:

Theorem 8. *Suppose S is not empty, i.e. there exists a feasible RSRSR, RSRSL, LSRSR or LSRSL path from p_i to p_f with the middle R arc passing through the inner corner. Then either the shortest possible feasible path lies in S, or the shortest feasible path is an elementary Dubins path joining p_i to p_f.*

The proof of Theorem 8 depends heavily on the following lemma derived from Corollary 2.1. Recall that a point $\mathbf{x}\in\partial\Omega$ is internally incident to an arc α if $\mathbf{x}$ is on α and, around $\mathbf{x}$, $\partial\Omega$ lies inside the circle extending α.

Lemma 9. *Suppose S is not empty. If a path Π is not an elementary Dubins path, is optimal, and contains an arc α passing through a point on $\partial\Omega$ that is not the inner corner, then the inner corner is an internally incident point to α.*

Using Lemma 9, we show that any path passing through a boundary point that is not the inner corner either lies in S or is not optimal. Next, we show that 32 out of the 36 combinations of possible paths that pass only through the inner corner are not optimal, leaving only members of S and any elementary Dubins paths that may exist between p_i and p_f as possible optimal paths.

Algorithm 1 describes the manner of then determining the optimal path. Note that finding the shortest path in S is based on the lemmas proven in the Structural Results section with Lemma 14 and is summarized in the conclusion.

Algorithm 1 Returns the shortest Dubins path from p_i to p_f in a corner environment Ω.

Require: S is nonempty.

```
function OptimalPath(p_i, p_f, Ω, S)
    path_1 := shortest feasible elementary path p_i → p_f
    path_2 := shortest path in S
    if (path_1 exists) and |path_1| < |path_2| then
        return path_1
    return path_2
```

VI. Results

We first establish an upper length bound on paths in S, then introduce some intermediate lemmas, prove Theorem 8, and finally prove Lemma 9.

A. Upper Length Bound on Elements of S

The following lemmas give an upper bound on the length of a path in S as a function of a, b, c and d.

Lemma 10. *Let Π_{rsr} be an elementary Dubins RSR path joining p_i and the pose $p_c=(0,0,\theta)$ for $\theta\in\left[\psi,\frac{\pi}{2}\right]$. Then $|\Pi_{rsr}|\leq\sqrt{a^2+c^2}+\frac{\pi}{2}-\theta$. Similarly, an elementary Dubins RSR path Π'_{rsr} joining p_c to p_f satisfies $|\Pi'_{rsr}|\leq\sqrt{b^2+d^2}+\theta-\psi$.*

Proof. It is not difficult to show geometrically that $|\Pi_{rsr}|=\sqrt{(a+\sin\theta-1)^2+(c-\cos\theta)^2}+\frac{\pi}{2}-\theta$. An arc greater than or equal to π at p_i or p_c is not feasible by Lemma 14, so we may apply Lemma 3, which results in the inequalities $\sin\theta\geq -a+1$ and $\tan(\theta)(a+\sin\theta-1)\leq c-\cos\theta$. Then $a+\sin\theta-1\geq 0$ so $|a+\sin\theta-1|\leq|a|\implies(a+\sin\theta-1)^2\leq a^2$. Again, since $\tan\theta\geq 0$ and $a+\sin\theta-1\geq 0$, $c-\cos\theta\geq 0$ and $|c-\cos\theta|\leq|c|\implies(c-\cos\theta)^2\leq c^2$. Then

$$|\Pi_{rsr}|=\sqrt{(a+\sin\theta-1)^2+(c-\cos\theta)^2}+\frac{\pi}{2}-\theta \leq\sqrt{a^2+c^2}+\frac{\pi}{2}-\theta\,.$$

The second half of the lemma may be verified in a similar manner to the first. $\square$

Lemma 11. *Let Π_{lsr} be an elementary Dubins LSR path joining p_i and the pose $p_c=(0,0,\theta)$ for $\theta\in\left[\psi,\frac{\pi}{2}\right]$. Then $|\Pi_{lsr}|\leq\sqrt{a^2+c^2}+\frac{\pi}{2}-\theta+\frac{(\sqrt{2}\pi-4)(1-\sin\theta)}{2\sqrt{2}}$. Likewise, for an elementary Dubins LSR path Π'_{rsl} joining p_c to p_f, $|\Pi'_{lsr}|\leq\sqrt{b^2+d^2}+\theta-\psi+\frac{(\sqrt{2}\pi-4)(1-\sin(\pi/2+\psi-\theta))}{2\sqrt{2}}$.*

Proof. We separate Π_{lsr} into two components, an arc Π_2 of angle $\pi/2-\theta$ that terminates at the inner corner and the

[2] When $\mathbf{v}\cdot\mathbf{u}<0$, $\mathbf{p}'+\mathbf{r}'$ must lie on or to the left of the line $\mathbf{p}-3\mathbf{r}+\mathbf{u}t$, which is strictly to the left of line given. In this case, both RSR and LSR paths may not be short.

remaining LSR path Π_1 that terminates with an orientation of $\pi/2$ (see Figure 4). Clearly, Π_2 has length $\frac{\pi}{2} - \theta$. To bound the length of Π_1, we replace the LSR fragment with two arcs (Π_1'). Figure 4 shows this substitution. The length of Π_1' must be longer than Π_1 since Π_2' is a valid Dubins path itself – if it were shorter, then the original elementary Dubins path would not be optimal.

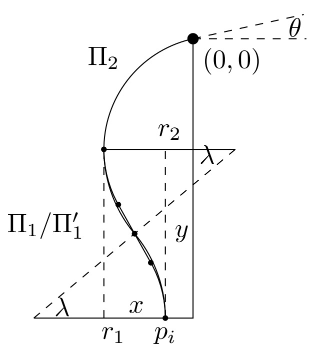

Fig. 4: Replacing a LSR path with a longer LRR path.

Let the first arc of Π_1' have radius r_1 and the second arc have radius r_2. The arcs are tangent where they meet, thus they each sweep out an equal angle λ. Let $x = 1 - a - \sin\theta$ be the width of the bounding box of Π_1' and $y = c - \cos\theta$ the height. Then $y = (r_1 + r_2)\sin\lambda$ and $x = (r_1 + r_2)(1 - \cos\lambda)$. Assuming λ lies in the first quadrant, we find[3] $\lambda = \arctan\left(\frac{2xy}{y^2 - x^2}\right)$ and $r_1 + r_2 = \frac{x^2+y^2}{2x}$. The length, then, of Π_1' is $(r_1+r_2)\lambda = \frac{x^2+y^2}{2x}\arctan\left(\frac{2xy}{y^2-x^2}\right)$, which gives an upper bound on the length of Π_1. We now consider the difference between the length of Π_1' and the shorter distance $\sqrt{x^2+y^2}$ with an error function f[4]:

$$f(x, y) = \frac{x^2 + y^2}{2x}\arctan\left(\frac{2xy}{y^2 - x^2}\right) - \sqrt{x^2 + y^2}\,.$$

Considering that x and y are positive and $y > x$, a change of variables $x = r\cos t$ and $y = r\sin t$ yields $f(r, t) = r\left(-1 - \left(t - \frac{\pi}{2}\right)\sec(t)\right)$. When $r > 0$ and $t \in (\pi/4, \pi/2)$, f is continuous with negative partial derivative equal to $\frac{\partial}{\partial t} f(r,t) = \frac{1}{2} r\sec(t)(\tan(t)(\pi - 2t) - 2)$. Hence f is largest when t is smallest for constant r. This occurs in the limit as $t \to \pi/4$:

$$\begin{aligned}\lim_{t\to\pi/4^+} f(r,t) &= r\left(-1 - \lim_{t\to\pi/4^+}(t - \pi/2)\sec(t)\right)\\ &= r\left(-1 + \frac{\pi}{2\sqrt{2}}\right).\end{aligned}$$

However x satisfies $0 \le x \le 1 - \sin\theta$. Hence, as Figure 5 shows, the maximum value of f subject to $y > x$ and $0 \le x \le 1 - \sin\theta$ lies on the line[5] $x = 1 - \sin\theta$. Since $\frac{\partial}{\partial y} f(x,y) = -1 - y\left((x^2+y^2)^{-1/2} + \frac{1}{x}\arctan\left(\frac{2xy}{y^2-x^2}\right)\right) <$

[3] $x \le 1 \implies y > 1$, so y is always greater than x. Hence $\lambda \in (0, \pi/2)$ and the following arctan function is admissable.

[4] f is always positive since the shortest distance between two points on the Euclidean plane is a straight line.

[5] It is also important to note that y is bounded by c, a finite number.

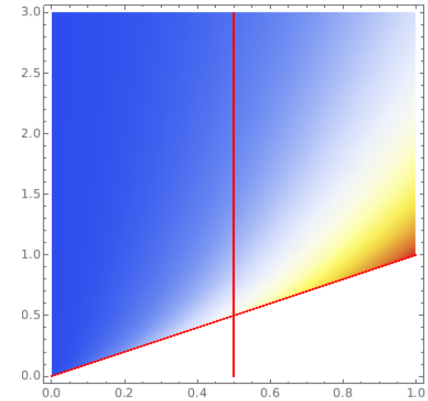

Fig. 5: A plot of $f(x, y)$ with $y = x$ and $x = 1 - \sin\theta$ overlaid for $\theta = \pi/6$. Warmer colors indicate greater values of f. The maximum of f such that $y > x$ and $x \le 1 - \sin\theta$ occurs (in the limit) somewhere on the red lines. In fact, f reaches a maximum at $(1 - \sin\theta, 1 - \sin\theta)$.

0 when $y > x$ and $x > 0$, the maximum of f must occur, in the limit, at $(1 - \sin\theta, 1 - \sin\theta)$. Returning to our polar parameterized f, we find

$$\lim_{t\to\pi/4^+} f(\sqrt{2}(1 - \sin\theta), t) = \frac{(\sqrt{2}\pi - 4)(1 - \sin\theta)}{2\sqrt{2}}\,.$$

It follows that the length of Π_1' is bounded $\frac{x^2+y^2}{2x}\arctan\left(\frac{2x}{y^2-x^2}\right) = \sqrt{x^2+y^2} + f(x,y) < \sqrt{x^2+y^2} + \frac{(\sqrt{2}\pi-4)(1-\sin\theta)}{2\sqrt{2}}$. Furthermore, $\sqrt{x^2+y^2} = \sqrt{(1 - a - \sin\theta)^2 + (c - \cos\theta)^2}$. By Lemma 6 (since a semicircle at p_i or p_c would render the path infeasible by Lemma 14), $1 - a - \sin\theta > 0$. Since $y > x$, $c - \cos\theta > 0$. Hence $\sqrt{x^2+y^2} < \sqrt{a^2+c^2}$. The length of Π_1, less than the length of Π_1', is then bounded by $\sqrt{a^2+c^2} + \frac{(\sqrt{2}\pi-4)(1-\sin\theta)}{2\sqrt{2}}$, which, when added to the length of Π_2, results in the first part of the claim. The second part follows similarly to the first. ☐

Lemma 12. *Let* $\Pi \in S$. $|\Pi| \le \sqrt{a^2+c^2} + \sqrt{b^2+d^2} + 1.73$.

Proof. If Π is an RSRSR path with orientation θ at the inner corner, then by Lemma 10 its length is bounded $|\Pi| \le \sqrt{a^2+c^2} + \frac{\pi}{2} - \theta + \sqrt{b^2+d^2} + \theta - \psi = \sqrt{a^2+c^2} + \sqrt{b^2+d^2} + \frac{\pi}{2} - \psi$. ψ is non-negative, so $\frac{\pi}{2} - \psi \le \frac{\pi}{2} < 1.73$. If Π is an RSRSL path, then its length is bounded by Lemmas 10 and 11: $|\Pi| \le \sqrt{a^2+c^2} + \frac{\pi}{2} - \theta + \sqrt{b^2+d^2} + \theta - \psi + \frac{(\sqrt{2}\pi-4)(1-\sin(\pi/2+\psi-\theta))}{2\sqrt{2}} \le \sqrt{a^2+c^2} + \sqrt{b^2+d^2} + \frac{\pi}{2} - \psi + \frac{\sqrt{2}\pi-4}{2\sqrt{2}}$. Again, since $\psi \ge 0$, $\frac{\pi}{2} - \psi + \frac{\sqrt{2}\pi-4}{2\sqrt{2}} \le \frac{\pi}{2} + \frac{\sqrt{2}\pi-4}{2\sqrt{2}} < 1.73$. The RSRSL case follows identically to the LSRSR case. Finally, when Π is an LSRSL path, Lemma 11 gives $|\Pi| \le \sqrt{a^2+c^2} + \sqrt{b^2+d^2} + \frac{\pi}{2} - \psi + \frac{(\sqrt{2}\pi-4)(2-\sin\theta-\sin(\pi/2+\psi-\theta))}{2\sqrt{2}}$. Given the restrictions $\psi \in [0, \pi/2]$ and $\theta \in \left[\psi, \frac{\pi}{2}\right]$, it can be found that $\frac{\pi}{2} - \psi + \frac{(\sqrt{2}\pi-4)(2-\sin\theta-\sin(\pi/2+\psi-\theta))}{2\sqrt{2}} < 1.73$. ☐

B. Intermediate Results.

Lemma 13. *Let a path Π contain a pose p with $LOC(p) \in (\partial\Omega \setminus \{(0,0)\})$. If $LOC(p)$ is a corner of $\partial\Omega$ or the orientation of Π at p does not match the orientation of $\partial\Omega$ at $LOC(p)$, then Π is not feasible.*

Proof. Suppose $LOC(p)$ lies on a corner of $\partial\Omega$ that is not the inner corner. Since Π is differentiable, Π is locally linear at p. At a corner not equal to the inner corner, no straight line with center $LOC(p)$ is fully contained within Ω, hence Π is not feasible.

If $LOC(p)$ lies on a line segment (non-corner) of $\partial\Omega$, then both $\partial\Omega$ and Π are locally linear at $LOC(p)$. If the orientation of Π at p is not equal to the orientation of $\partial\Omega$ at p, Π crosses $\partial\Omega$. Then Π is not feasible. □

Lemma 14. *A path Π from p_i to p_f is not feasible if it*

1) *begins with an arc of arc length π or greater*
2) *ends with an arc of arc length π or greater*
3) *contains an arc α of arc length π or greater with the inner corner internally incident to α.*

Proof. A semicircle following p_i would result in a pose $p_1 = (x, y, \frac{3\pi}{2})$ with $(x, y) \in \partial\Omega_{end,1}$. Then Π is not feasible by Lemma 13. Likewise, if a semicircle preceded p_f then Π would contain a pose $p_2 = (x, y, \psi + \pi)$ with $(x, y) \in \partial\Omega_{end,2}$. Again by Lemma 13 Π is not feasible.

Figure 6 illustrates the difficulty if the inner corner is internally incident to a semicircle α. α is guaranteed to cross $\partial\Omega$ if $\psi > 0$. If $\psi = 0$, then α will start and terminate on $\partial\Omega_{in}$. However, this arc would also not be feasible since not both the endpoints of α may match the respective orientations of $\partial\Omega_{in}$, which differ by at most $\pi/2$. Hence Π is not feasible in either case. □

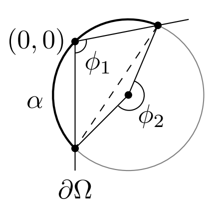

Fig. 6: α has arc length at least π. Π is only feasible if α lies within Ω. However, then ϕ_2 must be less than π, a contradiction because $\phi_1 = \pi/2 + \psi \geq \pi/2 \implies \phi_2 \geq \pi$.

Lemma 15. *If S is not empty, then the line segment connecting $LOC(p_i)$ to $LOC(p_f)$ intersects $\partial\Omega_{in}$ or passes through the inner corner.*

Proof. Let $\Pi \in S$ be a feasible path and let $L(t) = (\cos\theta, \sin\theta)t$, where θ is the orientation of Π at the inner corner (see Figure 7). We show that $LOC(p_i)$ and $LOC(p_f)$ lie on or to the right of L (where *right of* is defined as in Lemma 3). Suppose an RSR path spans p_i and the inner corner in Π. By Lemma 14, neither R arc in the RSR path has arc length greater than π. Then Lemma 3 implies that $LOC(p_i) + (1, 0)$ lies on or to the right of $L(t) + (\sin\theta, -\cos\theta)$, which in turn implies that $LOC(p_i)$ lies on or to the right of L. Similar reasoning establishes that $LOC(p_f)$ lies on or to the right of L if an RSR path spans the inner corner and p_f in Π. If an LSR path spans p_i and the inner corner, then Lemma 6 requires that $LOC(p_i)$ lie on or to the right of the leftmost part of the R arc of the LSR path. Since $\theta \in [0, \pi/2]$, $LOC(p_i)$ lies on or to the right of L. Again, similar reasoning establishes that $LOC(p_f)$ lies on or to the right of L if the second Dubins path is RSL. Therefore the line between $LOC(p_i)$ and $LOC(p_f)$ does not pass to the left of L. Since L passes through the inner corner, the line segment between $LOC(p_i)$ and $LOC(p_f)$ passes through some portion of $\partial\Omega_{in}$ or the inner corner. □

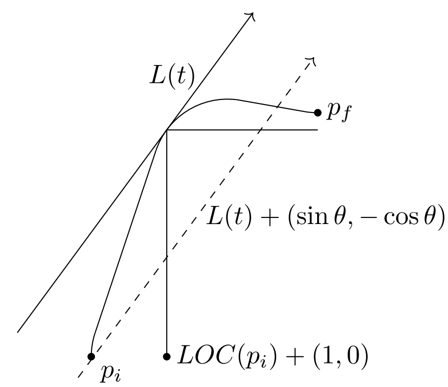

Fig. 7: A straight line through $LOC(p_i)$ and $LOC(p_f)$ (not shown) passes through $\partial\Omega_{in}$ if S is not empty.

Lemma 16. *Let S be nonempty and Π be a feasible path from p_i to p_f. Let $\mathbf{u_1}, ..., \mathbf{u_n}$ be a set of points on Π such that the length of Π between the points $\mathbf{u_1}$ and $\mathbf{u_n}$ is $|\Pi_{sub}|$ and $\sum_{i=1}^{n-1} ||\mathbf{u_i} - \mathbf{u_{i+1}}|| = L$, where $||...||$ is the Euclidean norm. Furthermore, let each line segment from $\mathbf{u_k}$ to $\mathbf{u_{k+1}}$ lie entirely within Ω. If $L < |\Pi_{sub}| - 1.73$, then Π is not optimal.*

Proof. Form a (non-Dubins) path Γ from $LOC(p_i)$ to $LOC(p_f)$ consisting of the part of Π between $LOC(p_i)$ and $\mathbf{u_1}$, the line segments connecting $\mathbf{u_k}$ to $\mathbf{u_{k+1}}$ and the part of Π between $\mathbf{u_n}$ and $\mathbf{p_f}$. Γ has length $|\Pi| - (|\Pi_{sub}| - L)$ and is feasible by the premise. However, the shortest non-Dubins but feasible path from $LOC(p_i)$ to $LOC(p_f)$ consists of a line segment from $LOC(p_i)$ to the inner corner followed by a line segment from the inner corner to $LOC(p_f)$ (recall that if S is nonempty then a straight line from $LOC(p_i)$ to $LOC(p_f)$ intersects $\partial\Omega$ by Lemma 15). Hence Γ has length greater than $\sqrt{a^2 + c^2} + \sqrt{b^2 + d^2}$. Thus we have $|\Gamma| = |\Pi| - (|\Pi_{sub}| - L) > \sqrt{a^2 + c^2} + \sqrt{b^2 + d^2} \implies |\Pi| > \sqrt{a^2 + c^2} + \sqrt{b^2 + d^2} + |\Pi_{sub}| - L$. If $|\Pi_{sub}| - L > 1.73$, then $|\Pi| > \sqrt{a^2 + c^2} + \sqrt{b^2 + d^2} + 1.73$. Since S is nonempty, there exists a shorter path by Lemma 12. Therefore Π is not optimal. □

C. Proof of Theorem 8

Suppose S is not empty. Then either the shortest possible feasible path lies in S, or the shortest feasible path is an elementary Dubins path joining p_i to p_f.

Proof. Let $\mathcal{L}$ be the set of feasible paths in Ω from p_i to p_f. By Theorem 1 from [4] (adapted from [1]), $\mathcal{L}$ contains a shortest path.

Partition $\mathcal{L}$ into three sets: $\mathcal{L}_c$ which contains paths passing through the inner corner, $\mathcal{L}_e$ which contains elementary Dubins paths between p_i and p_f and paths that do not touch any point on $\partial\Omega$ between p_i and p_f, and $\mathcal{L}_w$ which contains paths that (a) pass through some boundary point but not the inner corner and (b) are not elementary Dubins paths.

By Lemma 9, all paths in $\mathcal{L}_w$ are not optimal.

By Lemma 1, only paths in $\mathcal{L}_e$ that are elementary Dubins paths are optimal. Since Ω does not allow a semicircle immediately after p_i or before p_f (Lemma 14), Lemma 7 asserts that there are at most 3 possible optimal paths in $\mathcal{L}_e$.

We now consider by cases the paths in $\mathcal{L}_c$ that also pass through another boundary point.

Case 1: Paths that pass through the inner corner and $\partial\Omega_{end}$.

A path passing through $\partial\Omega_{end}$ must contain an arc α with a tangent point on $\partial\Omega_{end}$; otherwise the path would never leave $\partial\Omega_{end}$. Lemma 9 establishes that the inner corner is internally incident to α in an optimal path. There is only two possible arcs that satisfy this criterion, shown in Figure 8, and neither are a part of a feasible path (Lemma 14). Therefore, by Lemma 9, if S is nonempty no optimal path may pass through both the inner corner and $\partial\Omega_{end}$.

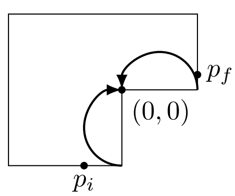

Fig. 8: The two possible arcs tangent to $\partial\Omega_{end}$ that have the inner corner internally incident. Since $\partial\Omega_{in}$ is perpendicular to $\partial\Omega_{out}$ where they meet, these arcs must be semicircles. Both are not a part of any feasible path since they begin in a corner. Note that this argument applies regardless of the direction of the arcs.

Case 2: Paths that pass through the inner corner and $\partial\Omega_{in}$.

A path passing through $\partial\Omega_{in}$ must arrive at the inner wall at some point $\mathbf{p}_{\text{arrive}}$ and depart the inner wall at some point $\mathbf{p}_{\text{depart}}$. Not both of $\mathbf{p}_{\text{arrive}}$ and $\mathbf{p}_{\text{depart}}$ may be the inner corner, otherwise the path would not have passed through $\partial\Omega_{in}$. Furthermore, the arrival or departure point contained in $\partial\Omega_{in}$ must be a part of an arc, otherwise the path would continue along the inner walls indefinitely. Therefore, by Lemma 9 the path is only optimal if the inner corner is internally incident to an arc tangent to $\partial\Omega_{in}$. There is no possible arc that meets this criterion, hence a path passing through both the inner corner and the inner walls is not optimal.

Case 3: Paths that pass through the inner corner and $\partial\Omega_{out}$.

As in the case with $\partial\Omega_{in}$, a path passing through both the inner corner and the outer walls must contain an arc α with point $\mathbf{p}$ on $\partial\Omega_{out}$. By Lemma 9, this path is only optimal if the inner corner is internally incident to α. Figure 9 shows the two arcs in which this is possible[6]. Let Π_{out} be an optimal path passing through $\mathbf{p}$ with the inner corner internally incident to an arc α. To be feasible at the inner corner, Π_{out} must have orientation $\theta_c \in \left[\psi, \frac{\pi}{2}\right]$. At $\mathbf{p}$, Π_{out} has orientation either $\theta_p = \pi/2$ or $\theta_p = \psi$ since it is tangent to $\partial\Omega_{out}$. Therefore the arc length $|\alpha| \leq \pi/2 - \psi \leq \pi/2$.

α spans the width of the hallway. Therefore bounding the length of α also bounds the width of the hallway. In this case, if $\mathbf{p} \in \partial\Omega_{out,1}$, ($\mathbf{p_1}$ in Figure 9), then $A \leq 1$. Due to this width restriction, prior to $\mathbf{p}$ Π_{out} cannot feasibly complete any arc with length greater than or equal to π. Before $\mathbf{p}$, Π_{out} is constrained to a convex environment, so by Corollary 2.1 Π_{out} must not contain any non-terminal arc before $\mathbf{p}$ distinct from α. Between p_i and $\mathbf{p}$, then, Π_{out} must be of the form RSR or LSR. By Lemmas 14 and 3, it cannot be an RSR path, leaving LSR as the only possibility. Then the subset of Π_{out} between p_i and the inner corner is itself an elementary Dubins LSR path.

Likewise, if Π_{out} contains $\mathbf{p_2}$ in Figure 9, then $B \leq 1$ and the subset of Π_{out} between the inner corner and p_f must be an elementary Dubins RSL path. Because an optimal path does not contain any point on the end or inner walls, Π_{out}, if it exists, must be the composition of two elementary Dubins paths joined at the inner corner.

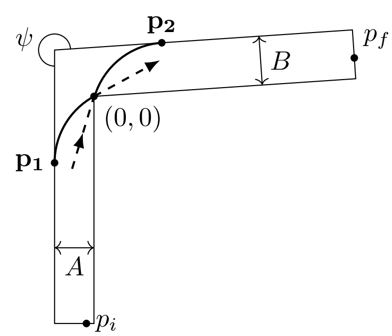

Fig. 9: The two possible arc that are tangent to $\partial\Omega_{out}$ and have the inner corner internally incident. $\mathbf{p}$ may be either of $\mathbf{p_1}$ or $\mathbf{p_2}$. Note that in the case where Π travels along the arcs in the opposite direction, say approaching $\mathbf{p_2}$ from the right, then Π would be infeasible since the hallway is too narrow to turn around in.

Hence the optimal path lies in the union of S, the set of paths that are composed of two elementary Dubins paths joined at the inner corner, and any feasible elementary Dubins paths that may exist.

[6]To be clear, at most two feasible arcs are tangent to $\partial\Omega_{out}$ with the inner corner internally incident. It is certainly possible for only one or no feasible arcs to satisfy this condition.

We now rule out many of the paths that are the composition of two elementary Dubins paths.

We first consider paths of the form (**R)(L**), where asterisks indicate any line or arc, and parentheses denote elementary Dubins paths.

Consider a path of the form (**R)(LSR). At the corner, this path begins a left arc. By Corollary 2.1, this left arc has arc length greater than π, otherwise the path is not optimal. If this is the case, however, then the remaining S and R sub-path can never reach p_f, as illustrated by Figure 10. This argument suffices for paths of the form (**R)(LSL) as well.

Next we rule out paths of the form (**R)(LRL). These paths contains two distinct arcs that, for an optimal path, must be at least semicircles by Corollary 2.1. Construct a (non Dubins) path that, instead of following the semicircles, cuts across the diameters. It follows using Lemma 16 that an (**R)(LRL) path is non-optimal.

LSL, LSR and LRL are the only three elementary Dubins paths that begin with a left arc. Therefore we have ruled out all paths of type (**R)(L**). By symmetric arguments we may also rule out all paths of type (**L)(R**).

Next, we consider paths of the type (**L)(L**). The inner corner is not internally incident to the L arc, so Corollary 2.1 asserts that the L arc has arc length greater than or equal to π, otherwise the path is not optimal. If either elementary Dubins path contained a non-terminal R arc (that is, one of the elementary Dubins paths is an LRL path), then the path contains two semicircles and we may apply the previous argument for paths of type (**R)(LRL). Furthermore, a LSL path between p_i and the inner corner or between the inner corner and p_f is not feasible by Lemma 14 in conjunction with Lemma 4. This leaves (RSL)(LSR) as the last possibility. In this final case, construct a line L through the endpoints of the L arc. It can be shown that p_i and p_f must lie on the opposite side of L as the inner corner. This, however, is a contradiction to Lemma 15 since S is not empty. Hence the path type (RSL)(LSR) is not optimal.

Since path types (**R)(L**), (**L)(R**) and (**L)(L**) are not optimal, we are left with paths of type (**R)(R**). Consider paths of type (RLR)(R**). Let the first R arc have arc length θ_1, the first L arc (the middle arc) have arc length θ_2 and the second R arc have length θ_3 (see Figure 11). By Corollary 2.1, $\theta_2 \geq \pi$. The arcs must be situated (by nature of Ω) such that $LOC(p_i)$ has lower y coordinate than the lowest point on the middle arc and the inner corner has higher x coordinate than the right-most point on the middle arc. These restrictions imply that $\theta_3 \geq \frac{2\pi}{3}$ and $\theta_1 \geq \frac{2\pi}{3}$, respectively. Hence the length of the whole RLR part is at least $\pi + \frac{4\pi}{3} = \frac{7\pi}{3}$. For an RLR path to be possible, the centers of curvature for the first arc and the third arc must be at most a distance of four apart. If θ_c is the orientation of the path at the inner corner, then we have $\sqrt{(a-1-\sin\theta_c)^2 + (c-\cos\theta_c)^2} \leq 4$, which in turn implies that $\sqrt{a^2+c^2} \leq 5$. Using $u_1 = LOC(p_i)$ and $u_2 = (0,0)$, we have by Lemma 16 that an (RLR)(R**) path is not optimal. By symmetric arguments, we also have that an (**R)(RLR) path is also not optimal.

We have proven that paths which are the composition of two elementary Dubins paths joined at the inner corner may only be optimal if they are of the form (RSR)(RSR), (LSR)(RSR), (RSR)(RSL) or (LSR)(RSL), which is precisely the elements of S. Hence we have proven that the optimal path lies in the union of S and any feasible elemntary Dubins paths that may exist between p_i and p_f. $\square$

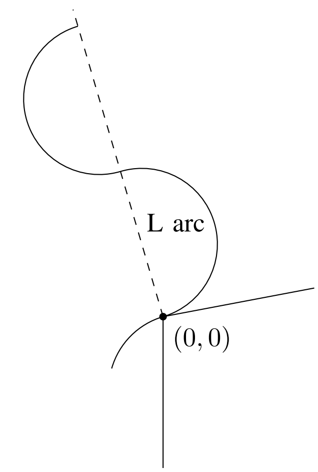

Fig. 10: After the L arc, a straight line and a curve sub-path cannot result in a final pose that lies to the right of the dotted line. Since p_f lies to the right of the dotted line (since it must be at least as far right as the rightmost part of the L arc), the path type (**R)(LSR) is not possible.

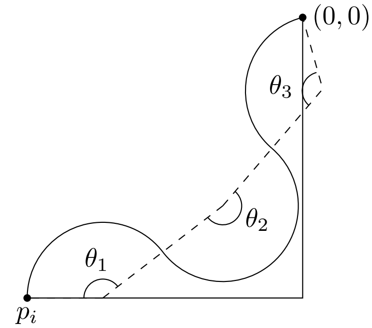

Fig. 11: An RLR path joining p_i and the inner corner.

D. Proof of Lemma 9

Suppose S is not empty. If a path Π is not an elementary Dubins path, is optimal, and contains an arc α passing through a point on $\partial\Omega$ that is not the inner corner, then the inner corner is an internally incident point to α.

Proof. Suppose S is not empty and Π is a feasible non-elementary Dubins path containing an arc α passing through a point $\mathbf{p} \in \partial\Omega - \{(0,0)\}$ such the inner corner in not internally incident to α. We aim to show that Π is not optimal.

First, we note that α is non-terminal. By lemma 13, the point $\mathbf{p}$ must match the orientation of $\partial\Omega$ at $\mathbf{p}$. Hence we must consider whether a feasible arc may start at p_i and also be tangent to $\partial\Omega$, or whether a feasible arc may start tangent to $\partial\Omega$ and terminate at p_f. Because the case with the arc terminating at p_f is geometrically symmetric to the one starting at p_i, we consider only the first case. Several such arcs are possible as illustrated in Figure 12. All immediately lead to infeasible paths except the right handed arc from p_i to $\partial\Omega_{out,2}$. Because of the assumption that $||\partial\Omega_{out} - (LOC(p_i) + (1,0))|| > 1$, however, this arc is not possible.

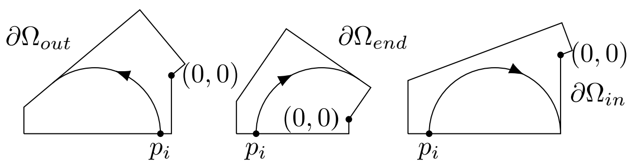

Fig. 12: Several different ways of achieving terminal arcs tangent to $\partial\Omega$. None can be a part of a feasible path if $||\partial\Omega_{out} - (LOC(p_i) + (1,0))|| > 1$.

Next, we observe that the inner corner is the only point on $\partial\Omega$ that can feasibly be an internally incident point of an arc in Π. If the inner corner is not internally incident to Π, then by Lemma 2.1 α must have length π or more. α does not pass through the inner corner, so we consider individually the cases where α passes through $\partial\Omega_{in}$, $\partial\Omega_{end}$ and $\partial\Omega_{out}$.

Case 1: $\mathbf{p} \in \partial\Omega_{in}$.

The situation where $\mathbf{p} \in \partial\Omega_{in,2}$ is geometrically symmetric to when $\mathbf{p} \in \partial\Omega_{in,1}$, so we assume that $\mathbf{p} \in \partial\Omega_{in}$. Let the terminal orientation of α be $\phi + \pi/2$, as illustrated in Figure 13. Since p_i has orientation $\pi/2$ and α begins with an orientation $\frac{3\pi}{2} - \phi$, Π must travel at least $\frac{3\pi}{2} - \phi - \frac{\pi}{2} = \pi - \phi$ before reaching α. Likewise, the orientation of p_f is ψ, so Π must travel at least $\left(\phi + \frac{\pi}{2}\right) - \psi = \frac{\pi}{2} - \psi + \phi \geq \phi$ after α. As will be shown, we further split this case into two sub-cases, one where $\min(\phi, \pi - \phi) \geq k$ for a constant $k \approx .151$ and the other where $\min(\phi, \pi - \phi) < k$.

Let I_1 be the reachable set preceding α of length[7] $\pi - \phi$ and I_2 be the reachable set following α of length ϕ. The reachable sets in question for Dubins vehicles are bounded by circle involutes and epicycloids ([10] gives a full construction).

Let $\mathbf{y}$ be the endpoint of the boundary involute of the smaller of I_1 and I_2 not on the circle that α subtends, and let $\mathbf{x}$ be a point on the involute boundary of the larger of I_1 and I_2 such that the boundary at $\mathbf{x}$ is perpendicular to $\mathbf{x} - \mathbf{y}$ (see Figure 13). Let O be the circle with diameter equal to the line segment connecting $\mathbf{x}$ and $\mathbf{y}$. Using curvature considerations, it can be shown that I_1 and I_2 lie entirely within O. Hence the distance between a point in I_1 to a point in I_2 is at most $||\mathbf{x} - \mathbf{y}||$.

Π begins at p_i, travels to some pose p_1 in I_1, continues for a length 2π before reaching some pose p_2 in I_2, then finally travels to p_f. I_1 and I_2 were constructed so that both lie to the left of $\mathbf{p}$, hence a straight line between $LOC(p_1)$ and $LOC(p_2)$ does not intersect $\partial\Omega$. Setting $u_1 = LOC(p_1)$ and $u_2 = LOC(p_2)$ and using Lemma 16, we conclude that

[7]A reachable set $W \subset \mathbb{R}^2$ of length L from a pose u is the set of all points that may be reached from u with a Dubins path of length L.

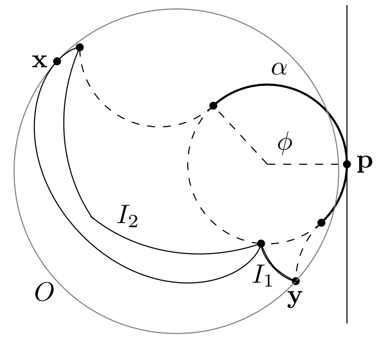

Fig. 13: An analysis of Π around $\mathbf{p}$, a point on $\partial\Omega_{in}$. Π must first travel to a point in the region I_1, then travel $\pi - \phi$ to the arc α, then complete a semicircle on the arc α and finally travel ϕ to a point within I_2.

Π is not optimal if $||LOC(p_1) - LOC(p_2)|| \leq ||\mathbf{x} - \mathbf{y}|| < 2\pi - 1.73$.

Case 1.a: $\min(\phi, \pi - \phi) \geq k$ ($k \approx .151$).

$\min(\phi, \pi - \phi) \geq k$ if and only if $||\mathbf{x} - \mathbf{y}|| < 2\pi - 1.73$, by computing the distance $||\mathbf{x} - \mathbf{y}||$ as a function of ϕ. Therefore by Lemma 16 Π is not optimal.

Case 1.b: $\min(\phi, \pi - \phi) < k$.

If $\min(\phi, \pi - \phi) < k$, then $||\mathbf{x} - \mathbf{y}|| \geq 2\pi - 1.73$. We again separate this case into two subcases. First we consider $\phi < k$. In this subcase, illustrated in Figure 14, the shortest Dubins path from $\partial\Omega_{end,1}$ to α_i (the initial pose of α) consists of a SR curve with the straight section vertical with some x coordinate x_0 and length l. Since $\phi < k$, l is guaranteed to be positive (indeed l is positive for any $\phi < \frac{\pi}{6}$). The total length of Π from p_i to $\mathbf{p}$ is then greater than $l + 2(\pi - \phi)$. If the x coordinate of $LOC(p_i)$ is greater than or equal to x_0, then we may again construct a line segment path path spanning p_i, $\mathbf{p}$, the inner corner and p_f (Π'_s). The length of Π'_s between p_i and $\mathbf{p}$ is less than $\sqrt{x_0^2 + (l + 2\sin\phi)^2}$ while the length of Π between p_i and p is at least $l + 2\pi - 2\phi$. Since $\sqrt{x_0^2 + (l + 2\sin\phi)^2} < l + 4$, Π is greater than Π'_s by at least $l + 2\pi - 2\phi - l - 4 > 1.73$. We apply Lemma 16 with $u_1 = p_i$ and $u_2 = p$ and conclude that Π is not optimal. If the x coordinate of $LOC(p_i)$ is less than x_0, then the shortest Dubins path between p_i and α_i is a RSR path. However, then second R arc has length less than π. Then Π is not optimal by Corollary 2.1, since this arc is non-terminal and cannot have an internally incident point on $\partial\Omega$.

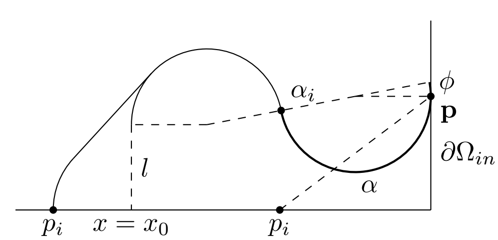

Fig. 14: Subcase 1: $\phi < k$ for various possible p_i.

Similar arguments show that Π is not optimal in the second subcase, when $\pi - \phi < k$.

Case 2: $\mathbf{p} \in \partial\Omega_{end}$. A similar argument suffices to show Π is not optimal when $\mathbf{p}$ lies on $\partial\Omega_{end}$. Briefly, for $\mathbf{p} \in \partial\Omega_{end,1}$, we note that $\mathbf{p}$ must lie to the right of p_i and Π at $\mathbf{p}$ must have orientation 0. We parameterize the initial orientation angle of α with an angle $\phi_2 \in [-\pi, 0]$. If $\phi_2 < -\frac{\pi}{6}$ we may apply an identical argument for when $\phi < k$ in case 1.b. If $\phi_2 > -\frac{\pi}{6}$, we apply Lemma 16 for $\mathbf{u_1} = LOC(p_i)$ and $\mathbf{u_2} = LOC(\alpha_f)$ for terminal pose α_f on arc α. The case where $\mathbf{p} \in \partial\Omega_{end,2}$ is very similar.

Case 3: $\mathbf{p} \in \partial\Omega_{out}$. Finally, we consider the case when p lies on $\partial\Omega_{out}$. As with case 1, the situation where $\mathbf{p} \in \partial\Omega_{out,2}$ is symmetric to the situation where $\mathbf{p} \in \partial\Omega_{out,1}$. Therefore we assume $\mathbf{p} \in \partial\Omega_{out,1}$. First, we may conclude that if Π is optimal it is *forward facing* through $\mathbf{p}$, that is, proceeds in the positive y direction. If not, then Π must cross itself at some point u. The minimum length for a loop in a Dubins path is greater than π. Using Lemma 16 with $u_1 = u_2 = u$, we conclude that Π is not optimal. Secondly, under the assumption that Π is optimal we may also conclude that Π does not visit $\partial\Omega_{out}$, leave, and revisit the same line segment; else Π would cross itself and not be optimal as before, or Π could be trivially shortened by following $\partial\Omega_{out}$.

If Π does not cross itself, there are then three possibilities: Π touches only $\partial\Omega_{out,1}$ and only once, Π touches only $\partial\Omega_{out,2}$ and only once, or Π touches both segments of $\partial\Omega_{out}$ once. A similar involute analysis to the one in case 1 may be used to show that RLR and LRL paths between p_i and $\mathbf{p}$, between $\mathbf{p}$ and p_f, or between two points on $\partial\Omega_{out}$ result in non-optimal paths. By Lemma 14, Π does not begin or end with a semicircle. Using Lemmas 3, 4, 6 and 5, we may deduce that if Π passes through a point $\mathbf{p} \in \partial\Omega_{out,1}$ then an LSR path spans p_i and $\mathbf{p}$. Likewise, if Π includes a point $\mathbf{p} \in \partial\Omega_{out,2}$, then an RSL path spans $\mathbf{p}$ and p_f. Finally, if Π includes a point p_1 on the vertical segment of $\partial\Omega$ and a point p_2 on the horizontal segment of $\partial\Omega$ then an RSR path must join p_1 and p_2 by the forward facing condition imposed earlier.

We address each combination of paths passing through $\partial\Omega_{out}$. If Π passes through both $\partial\Omega_{out,1}$ and $\partial\Omega_{out,2}$, then it has been established that the path type is LSR-RSL. If Π passes through only $\partial\Omega_{out,1}$, then an elementary Dubins path must join $\partial\Omega_{out,1}$ to p_f. Since the path must start with a R arc and an RLR or LRL elementary path is not optimal as previously discussed, this subpath must be of type RSR or RSL. Likewise if Π passes through only $\partial\Omega_{out,2}$, Π must have path type RSR-RSL or LSR-RSL. Since the arguments for RSR-RSL and LSR-RSR are the same, we only include the case for LSR-RSR.

Case 3.a Π is of the form LSR-RSR and passes through the vertical segment of $\partial\Omega_{out}$.

Recall that α is the first R arc, the arc that passes through $\partial\Omega_{out}$. α must terminate at a pose with orientation ϕ greater than zero; otherwise the RSR elementary Dubins path following would contain a semicircle and be trivially non-optimal. By Corollary 2.1, α has arc length greater than π. Let the first S path have a length L, and let $\mathbf{u}$ be an intermediate point

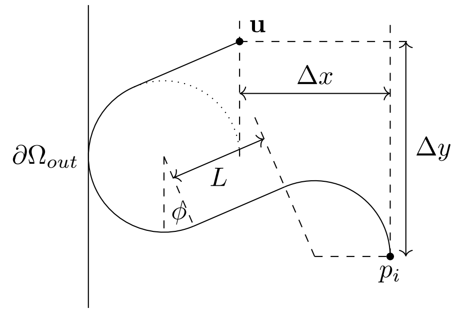

Fig. 15: The first LSR part of an LSRSR path

on the RSR path that is a distance two from the outer wall (see Figure 15). We may geometrically determine bounds on the values of Δx and Δy from L and ϕ. These are $\Delta x = 3\sin\phi + L\cos\phi$ and $\Delta y \leq 3\cos\phi + (1-L)\sin\phi$. We also bound the length of Π during this interval. The length is greater than $\frac{3\pi}{2} + \phi + L + 1$. We note that a straight line from p_i to $\mathbf{u}$ is feasible, so using Lemma 16 with $\mathbf{u_1} = p_i$ and $\mathbf{u_2} = \mathbf{u}$ we conclude that Π is not optimal.

Case 3.b Π is of the form LSR-RSL (passing through one or two points on $\partial\Omega_{out}$).

Suppose one of the straight sections in Π has length greater than zero. Then there exists a trivial feasible perturbation (Figure 16) that shortens Π, so Π is not optimal. If the straight section in Π both have length zero, then Π is an elementary Dubins LRL path and this lemma does not apply. □

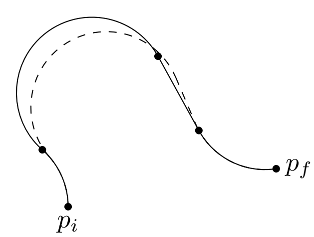

Fig. 16: Shortening an LSR-RSL path with non-degenerate straight section. Decrease the length of the first L arc by some small δ and compensate with the line before the last L arc. Since the inner corner is not internally incident to the middle R arc, this perturbation is feasible.

VII. Conclusion

We have shown that S, if it is nonempty, contains the shortest or optimal Dubins path from p_i to p_f in a corner environment Ω. Paths in S are made of elementary Dubins paths of type RSR, RSL and LSR. By Lemma 14, no arc in S may have arc length greater than π. By Lemma 7, then, a path in S is uniquely determined by $p_c = (0, 0, \theta_c)$, the pose at the inner corner. This immediately implies that the angle θ_c is a direct way to parameterize S. Lemmas 3, 4, 5 and 6 provide an algebraic method of determining existence of path types in S and the angles $\theta_{\min}$ and $\theta_{\max}$ for which $\theta_c \in [\theta_{\min}, \theta_{\max}]$ results in a feasible path in S. The length of every path type in S may be found as a function of a,

b, c, d, A, B, ψ and θ_c using relatively simple geometry. If $S(\theta_c)$ indicates the path in S that passes through the inner corner with orientation θ_c (if it exists), then $\theta_c \mapsto |S(\theta_c)|$ is a piecewise function that may be minimized with respect to θ_c using analytical methods. An implementation and online demo is provided at `sites.google.com/umn.edu/rsnlabdemos/home/dubins-2018`.

In order to obtain a complete solution, we need to characterize instances where no solution exists. In at least one case, when a and B are small, A is large and c around 2.5, another class of optimal solutions exists which needs to be analyzed. We are currently working on this problem. Our future work includes extending our solution to a sequence of turns.

VIII. Acknowledgements

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