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The Lion and Man Game on Polyhedral Surfaces with Boundary

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***Abstract*— We study the lion-and-man game in which a group of lions (the pursuers) try to capture a man (the evader). The players have equal speed. They can observe each other at all times. While the game is well-studied in planar domains such as polygons, very little is known about its properties in higher dimensions. In this paper, we study the lion and man game when played on the surface of a genus-zero polyhedron with boundary. We show that three lions with non-zero capture distance δ can capture the man in time $O((\frac{A}{\delta^2}+\frac{L}{\delta})^2\frac{\delta}{2})$ where A is the area of the surface, and L is the total edge length of the surface.**

I. Introduction

Many robotics applications such as tracking and search can be modeled as pursuit-evasion games. In this paper, we study a fundamental pursuit-evasion game known as the lion-and-man game. In the original version of this game, a lion tries to capture a man in a circular arena. Many variants of the lion-and-man game have been studied to model various aspects of the robotics applications. An overview of these results can be found in the survey paper by Chung et al. [1]. Briefly, assuming that the players move in turns, and they can observe each other at all times, the lion wins the game in circular arenas [2], [3] and in simply-connected polygons [4]. Moreover, in polygons with obstacles three lions are sufficient and sometimes necessary for capture [5].

The lion-and-man game has been studied in non-planar environments as well. Kopparty and Ravishankar showed that in $\mathbb{R}^d$, $d+1$ lions can capture the man if and only if the man starts inside their convex hull at the start [6]. Alexander et al. [7] study pursuit in environments with non-positive curvature (i.e. it is CAT(0)), and show that a single pursuer can eventually capture the evader by greedily moving toward it. More recently, Noori and Isler [8] showed that the class of convex terrains, which includes positive curvature examples, are still single pursuer-win. Klein and Suri [9] showed that four pursuers are sufficient to capture the man on a polyhedral surface with genus zero. In this paper, we focus on polyhedral surfaces with boundary and show that three lions suffice. A practical implication of our result is that three pursuers guarantee capture on terrains which is a special case characterized by unique height values for points in the two dimensional plane (Fig. 1(a)).

We present a pursuit strategy that is based on guarding shortest paths introduced by [10]. We show how the evader can be constrained to a region formed by two shortest paths each of which is guarded by a pursuer, along with the boundary of the surface. We use the third pursuer to split this region to two smaller regions such that one of them contains the evader. The splitting algorithm is applied iteratively. We prove that the evader will be captured in finite time by obtaining a lower-bound on how much the evader's regions shrinks at each iteration.

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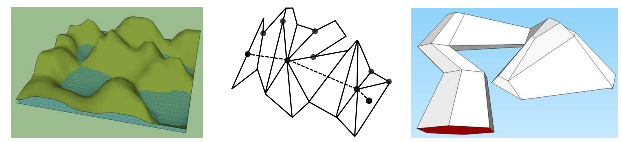


(a) (b) (c)

Fig. 1. (a) A terrain with holes. (b) An illustration of faces, edges and vertices on $\mathcal{S}$. (c) A polyhedral surface with boundary.

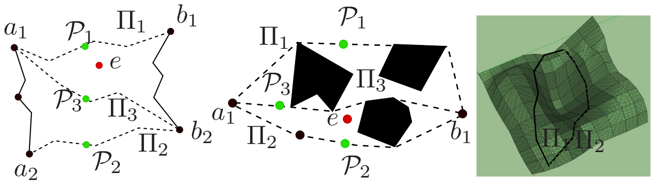
The paper is organized as follows. In Section II we present the definitions we use throughout the paper. In Section III we provide the details of the game model. Section IV presents an overview of our pursuit strategy. In Section V we discuss the required modules for our strategy. The details of the pursuit strategy are given in Section VI. In Section VII we show that the evader is captured in finite time. Concluding remarks are presented in Section VIII. Due to space limitations, we present the details of our proofs in [11].

II. Notation

The game is played on a polyhedral surface with genus zero and boundary, denoted by $\mathcal{S}$, which is defined as follows. It is a piecewise linear two-dimensional surface which is homeomorphic to a unit disk possibly with holes in it. It is represented by a set of faces f_i, a set of edges e_i, and a set of vertices v_i (Fig. 1(b)). A face f is a polygon which is bounded by a subset of edges of the polyhedron. An edge joins exactly two vertices of the polyhedron; A non-boundary edge joins exactly two faces, and the edges on the boundary are adjacent to only one face. Finally, we denote the boundary of $\mathcal{S}$ by $\partial\mathcal{S}$ (Fig. 1(c)).

In this paper, we denote the shortest path between points a and b by $\Pi(a,b)$, and its length by $d(a,b)$. We drop (a,b) if the source and destination points of Π are clear from the context. For a given path l on $\mathcal{S}$, which is not necessarily a shortest path, we use $|l|$ to denote the length of l. Given two points p and q on l we denote the sub-path of l from p to q by $l(p,q)$. For given paths l_1 and l_2 we denote their concatenation by l_1+l_2. We denote the line segment between two points p and q by pq and its length by $|pq|$.

We next present the game model.



(a) (b) (c)

Fig. 2. Shortest paths Π_1, Π_2, and Π_3 are shown with dashed lines and $\partial\mathcal{S}$ is shown with solid lines. (a) After guarding Π_3 by $\mathcal{P}_3$, the evader will be restricted to a smaller region. (b) Multiple shortest paths in polygons with obstacles are caused by obstacles. (c) On a polyhedron multiple shortest paths are caused by hills and valleys.

III. Game Model

The pursuer is denoted by $\mathcal{P}$ and the evader is denoted by e. The players take turns and each turn takes a unit time step. They move on the surface of a polyhedron which is denoted by $\mathcal{S}$. In one time step each players' path is at most one unit in length. The players can observe each other's location at all times. The pursuer captures the evader if their distance is less than the capture distance $\delta > 0$. The non-zero capture radius can model the non-zero area of robots. A collision results in capture!

IV. Overview

The main idea of our three pursuer strategy for capturing the evader on $\mathcal{S}$ is similar to the polygonal case with obstacles [5]. We start by presenting the idea for polygons. The pursuit strategy is divided into rounds. In each round, the pursuers make progress by restricting the evader to a smaller region. In the following, we refer to the subset of $\mathcal{S}$ that the evader is restricted to during the i^{th} round as the *contaminated* region and denote it by $\mathcal{S}_i$. The main idea has two components: *guarding* and *splitting*. Two pursuers are used to *guard* the boundary of the current contaminated region $\mathcal{S}_i$ in order to prevent the evader from exiting $\mathcal{S}_i$. In other words, the evader cannot cross the boundary of $\mathcal{S}_i$ without being captured by the dedicated pursuers. In Section V-A we show that a pursuer can guard a shortest path Π by locating itself on the *projection* of the evader onto Π. Therefore, we maintain the invariant that the subsets of $\partial\mathcal{S}_i$ that are guarded by the two pursuers are shortest paths. The third pursuer is used to *split* $\mathcal{S}_i$ into two smaller regions. The evader will be now restricted to one of these smaller regions which is denoted by $\mathcal{S}_{i+1}$. The evader cannot re-contaminate $\mathcal{S}_i - \mathcal{S}_{i+1}$. Furthermore, the splitting process can be continued on the new contaminated region $\mathcal{S}_{i+1}$ because as we show later in Section VI we always have a free pursuer.

The critical part of the strategy is the splitting procedure as well as showing that progress toward capture is guaranteed after each splitting step. First, let us explain the difficulties in splitting step. Let Π_1 and Π_2 be the two shortest paths on $\partial\mathcal{S}_i$ that are guarded by pursuers $\mathcal{P}_1$ and $\mathcal{P}_2$ respectively. Let a_1 and b_1 be the two endpoints of Π_1 and a_2, b_2 be the endpoints of Π_2 which are initially chosen on the boundary of $\mathcal{S}$. If these endpoints are disjoint, i.e $a_1 \neq a_2, b_2$ *and* $b_1 \neq a_2, b_2$, then it is not too difficult to see that either $\Pi(a_1, b_2)$ or $\Pi(a_2, b_1)$ can be used for splitting (Fig. 2(a)). Next, suppose $a_1 = a_2$ but $b_1 \neq b_2$. In this case, we can pick a point c along the portion of the boundary from b_1 to b_2 and use the path $\Pi(a_1, c)$ to make progress.

The remaining case $a_1 = a_2$ and $b_1 = b_2$, i.e. when there are two shortest paths between $a_1 = a_2$ and $b_1 = b_2$, is challenging on a polyhedron. In the polygonal setting, this case can be easily handled as follows. In polygons, multiple shortest paths between two points a_1, b_1 are possible only when they touch obstacles (Fig. 2(b)). The vertices on these obstacles can be used as the endpoints of the splitting path Π_3. However, on a polyhedral surface, multiple shortest paths exist because of the valleys and the hills on the surface (Fig. 2(c)). Consequently, unlike the polygonal case with obstacles, the evader is free to move in the region in between Π_1 and Π_2. Thus, finding the third splitting path is not trivial on a polyhedral surface (Fig. 2(c)).

In order to handle the multiple shortest path case on $\mathcal{S}$, we employ the capture distance δ of the pursuers. In particular, for each of the shortest paths Π_1 and Π_2 we define a *capture region* (Section V-B) which is the region around Π_1, Π_2 that the evader cannot enter without being captured by the guarding pursuers $\mathcal{P}_1, \mathcal{P}_2$. We then use the intersection points of the boundary of these two capture regions as the endpoints of the splitting path (Section VI).

The second challenge is proving that after finite time the evader will be captured. The first idea is to show that the area of the contaminated region gets smaller at the end of each round. In particular, it must be shown that $area(\mathcal{S}_i) - area(\mathcal{S}_{i+1})$ is lower bounded, i.e. $area(\mathcal{S}_i) - area(\mathcal{S}_{i+1})$ is not infinitesimal and there exists a constant number $\epsilon > 0$ such that $area(\mathcal{S}_i) - area(\mathcal{S}_{i+1}) \geq \epsilon$. It turns out that this approach is not directly applicable because of difficulties in providing lower bound on the area removed from $\mathcal{S}_i$. Instead, we partition surface of the polyhedron into *small triangles*, triangles with edges shorter than $\frac{\delta}{2}$ (Section V-B). We show that at the end of each round the pursuers claim at least one of these triangles as *cleared* for the rest of the game, i.e. the evader cannot re-contaminate the cleared triangles.

Finally, consider the result by Aigner and Fromme [10] who showed that three pursuers suffice for capture on planar graphs. It might be tempting to use this result directly since the vertices and edges of a polyhedral surface with genus zero constitute a planar graph. However, this result does not directly apply to the geometric version on the surface because the players are not restricted to stay on the vertices. Furthermore, mapping the locations to a nearest vertex may not correspond to motion along the edges.

V. Ingredients of the Pursuit Strategy

In this section, we present the components of our pursuit strategy. In Section V-A we discuss the projection of the evader onto a shortest path which is used for guarding the boundary of $\mathcal{S}_i$. In Section V-B we present the capture region of a shortest path, and finally in Section V-C we discuss the partitioning of $\mathcal{S}$ into small triangles.

A. Projection

Aigner-Fromme [10] discussed guarding of shortest paths on graphs using projections. This idea is later used for guarding shortest paths in polygonal [5] and polyhedral environments [9]. The projection of the evader onto Π is a point on Π which is closer to all the points on Π than the evader. Therefore, in order to guard Π, the pursuer can locate itself on the projection of the evader (in at most $O(D)$ steps where D is the length of the longest shortest path on $\mathcal{S}$). Afterwards, as the evader moves from e_1 to e_2 the pursuer can move to the new projection of the evader. Therefore, the evader cannot cross Π without being captured. As shown in [5], [10], shortest paths are guardable using canonical projection.

Definition 1 (Canonical Projection [5]): Given a shortest path $\Pi(a,b)$ between two points a and b, and the current evader location e, a point $p(e)$ on $\Pi(a,b)$ is called the canonical projection of e if $d(a,p(e)) = \min(d(a,e), d(a,b))$.

B. Capture Regions

We are now ready to present the capture region of a shortest path Π which is the region around Π protected by the pursuer that is guarding Π.

Definition 2 (Capture region): For a given shortest path Π, its *capture region*, denoted by $C(\Pi)$, is the set of points q on $\mathcal{S}$ such that:

$$C(\Pi) = \{q \in \mathcal{S} : \exists p \in \Pi, \quad d(p,q) \leq \frac{\delta}{2}\}.$$

The following property of the capture region plays a crucial role in our strategy.

Lemma 1: Let Π be a shortest path that is being guarded by a pursuer located on the canonical projection of the evader. Then, the evader cannot enter the capture region of Π without being captured [11].

Remark 1: The capture region of a shortest path is not necessarily a polyhedral subset of $\mathcal{S}$ because the boundary of the capture region can be curved e.g. it can contain circular arcs. In this paper, we assume that we have an oracle that computes the capture region of a given shortest path.

C. Triangulation into Small Triangles

In order to show that the evader will be captured in finite time we will partition $\mathcal{S}$ into small triangles. Later (in Section VII) we prove that after each round the pursuers remove at least one of the small triangles from the contaminated region. In this section, we present the triangulation algorithm of the faces into $\frac{\delta}{2}$-small triangles where δ is the capture radius.

Definition 3 (Small Triangles): A triangle is α-small if all of its edges are shorter than α.

We start by triangulating each face on $\mathcal{S}$ into triangles that are not necessarily $\frac{\delta}{2}$-small. This can be done in time $O(n \log n)$ where n is the number of vertices on the face [12]. We show that each of these triangles can be further partitioned into small triangles. Suppose that $\triangle abc$ is a triangle which is not small.

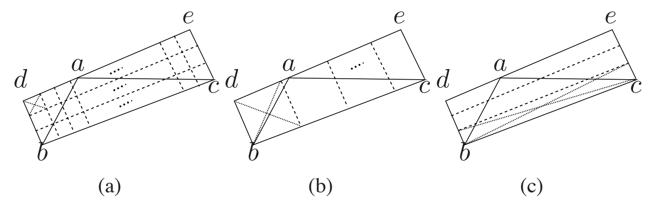


Fig. 3. Triangulation of $\triangle abc$ into small triangles. The three cases discussed in the text are shown from left to right.

In the first step, we find the bounding rectangle of $\triangle abc$ as follows. In any triangle we have at least two vertices whose angles are less than $\frac{\pi}{2}$. Let b and c be those vertices in $\triangle abc$. Then bc is one edge of the bounding rectangle (Fig. 3). The length of the other edge of the rectangle, which is perpendicular to bc, is the same as the height of the third vertex a with respect to the edge bc (Fig. 3).

Let us denote the vertices of the bounding rectangle by b, c, d, e as shown in Fig. 3(a). There are three cases whether the edges of the rectangle are shorter than $\frac{\delta}{2}$ or not.

Case (1): Both of the edges bc and bd are longer than $\frac{\delta}{2}$ (Fig. 3(a)). In this case, we put a grid of squares of sides $\frac{\delta}{2}$ on the rectangle. Since both bc and bd are longer than $\frac{\delta}{2}$ we have $O(\frac{bc}{\delta}.\frac{bd}{\delta})$ square cells. Notice that $O(\frac{bc}{\delta}.\frac{bd}{\delta}) = O(\frac{A_i}{\delta^2})$ where A_i is the area of $\triangle abc$.

We next partition each square cell into four small triangles using its diagonals. In the final step, we consider only the small triangles that are covered by $\triangle abc$. Some of these small triangles may be cut into quadrilaterals as a result of intersection with edges of $\triangle abc$. We can easily partition these quadrilaterals into small triangles by adding diagonals. Therefore, the number of small triangles on $\triangle abc$ is $O(\frac{A_i}{\delta^2})$ where A_i is the area of $\triangle abc$.

Case (2): Exactly one of the edges bc and bd is longer than $\frac{\delta}{2}$ (Fig. 3(b)). In this case, we create a grid of single row with width equal to the longer edge. Using this grid, we can partition $\triangle abc$ into $O(\frac{L_i}{\delta})$ small triangles where L_i denotes the length of the longest edge of $\triangle abc$.

Case (3): Both of the edges bc and bd are shorter than $\frac{\delta}{2}$ (Fig. 3(c)). Similar to previous cases, we can partition $\triangle abc$ into $O(1)$ small triangles (Fig. 3(c)).

We apply the aforementioned partitioning algorithm to each triangular face on $\mathcal{S}$ which is not small.

Lemma 2: Using the triangulation algorithm above, the polyhedral surface $\mathcal{S}$ is partitioned into $O(\frac{A}{\delta^2} + \frac{L}{\delta})$ small triangles where A is the area of $\mathcal{S}$, L is the sum of the length of the edges on $\mathcal{S}$, and δ is the capture distance [11].

VI. Three Pursuer Strategy

We are now ready to present the details of our pursuit strategy. The pursuit strategy is as follows. Initially (first round $i = 1$), two distinct edges on $\partial\mathcal{S}$ are chosen as Π_1 and Π_2. The two pursuers $\mathcal{P}_1$ and $\mathcal{P}_2$ guard Π_1 and Π_2 by locating themselves on the canonical projection of the evader.

Definition 4 (The contaminated region $\mathcal{S}_i$): The contaminated region $\mathcal{S}_i$ is the region bounded by shortest paths Π_1

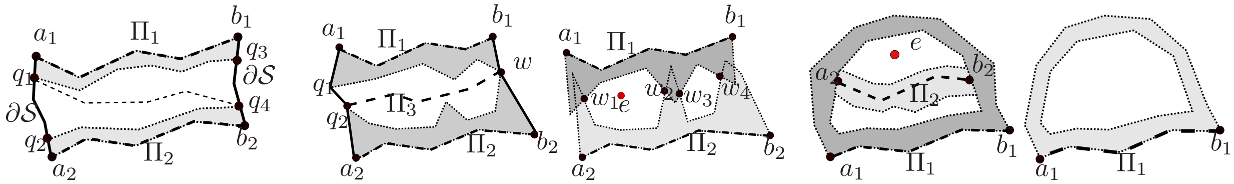


Fig. 4. Possible configurations of the contaminated region $\mathcal{S}_i$ throughout the game are shown. The shortest paths $\Pi_1(a_1, b_1)$ and $\Pi_2(a_2, b_2)$ are shown in dashed lines, the boundary $\partial\mathcal{S}$ is shown in solid lines, and the boundary of the capture regions are shown in dots. The capture regions are shaded in gray. (a) Case 1-a, (b) Case 1-b, (c,d) Case 2, (e) Case 3 of Lemma 6 are illustrated.

and Π_2 excluding the capture regions $C(\Pi_1)$ and $C(\Pi_2)$. The evader cannot exit $\mathcal{S}_i$ without being captured.

Therefore, initially $\mathcal{S}_1 = \mathcal{S} - (C(\Pi_1) \cup C(\Pi_2))$ which is bounded by $\partial C(\Pi_1)$, $\partial C(\Pi_2)$ and $\partial\mathcal{S}$. We show that the capture regions have the following properties [11]:

Lemma 3: Given a shortest path Π, its capture region $C(\Pi)$ is connected [11].

Lemma 4: Given two shortest paths Π_1 and Π_2, their capture regions $C_1 = C(\Pi_1)$ and $C_2 = C(\Pi_2)$ are pseudo-disks [11]. This property is formulated as follows. Let C be one of the connected components of $C_1 \cap C_2$. Then in the neighborhood of C at least one of the sets $C_1 - C_2$ and $C_2 - C_1$ is connected [12].

In order to show that we can free a pursuer at each iteration, we will need the following technical lemma.

Lemma 5: Let R be the region that the evader is constrained to after we choose the splitting path Π_3 for $\mathcal{P}_3$ in iteration j. This region is bounded by $\partial\mathcal{S}$ along with the capture regions $\partial C(\Pi_i)$ where $C(\Pi_i)$ is the region guarded by pursuer $\mathcal{P}_i$ ($i \leq 3$). Let $l_i = \partial R \cap \partial C(\Pi_i)$. At most one l_i is disconnected.

In Lemma 5, l_i is the contribution of $\mathcal{P}_i$ for keeping the evader contained in R. The lemma limits the interaction (i.e. number of crossings) between l_is. Its proof is presented in [11].

By exploiting the properties discussed above, we show that the pursuers can maintain the following two invariants:

Invariant on the shortest paths: The shortest paths Π_1 and Π_2 are selected such that they do not intersect each other.

Invariant on the structure of the contaminated region: At each round i, $\mathcal{S}_i$ is guarded by at most two pursuers $\mathcal{P}_1, \mathcal{P}_2$. Since by Lemma 5 we encounter $\partial C(\Pi_1)$ and $\partial C(\Pi_2)$ at most once as we traverse the boundary of $\mathcal{S}_i$, we can enumerate the possible configurations of $\partial\mathcal{S}_i$ as follows. (See also Fig. 4).

Case (1): $\partial\mathcal{S}_i$ is composed of the boundary of the capture regions of $\mathcal{P}_1$ and $\mathcal{P}_2$, and also the boundary of the surface, i.e. $\partial C(\Pi_1)$, $\partial C(\Pi_2)$ and $\partial\mathcal{S}$.

Case (2): $\partial\mathcal{S}_i$ is composed of only the boundary of the capture regions of $\mathcal{P}_1$ and $\mathcal{P}_2$, i.e. $\partial C(\Pi_1)$ and $\partial C(\Pi_2)$.

Case (3): $\partial\mathcal{S}_i$ is composed of the boundary of the capture region of exactly one of the pursuers e.g. only $\partial C(\Pi_1)$.

We now present interesting examples for the cases we presented above. An abstract illustration of case (2) is shown in Fig. 4(d) and a specific example of this configuration is presented in Fig. 5. The example in Fig. 5 is the following. Imagine a cone and cut it at height $\frac{\delta}{2}$ from its base and then mount a half-sphere on top of it (Fig. 5(a)). The endpoints of Π_1 are chosen as the antipodal points on the base circle as well as the endpoints of Π_2 that are antipodal on the cutting circle (Fig. 5(a)). The perimeter of the base circle, the circle that a_1, b_1 lie on it, is chosen small enough such that $C(\Pi_1)$ includes the whole base circle and also the portion of the surface up to height $\frac{\delta}{2}$. Now, observe that if a_2, b_2 are chosen as antipodal points on $\partial\mathcal{S}_i$, there would be infinite number of shortest paths between them (because of the half-sphere). With the choice of the evader location shown in Fig. 5(b), which provides a top view of the environment, we will have the configuration illustrated in Fig. 4(d).

A similar example for case (3) is shown in Fig. 5(c) and Fig. 5(d). The example is a cone, and the endpoints of Π_1 are the antipodal points on the base circle of the cone. The perimeter of the base circle is chosen small such that the base circle is completely inside $C(\Pi_1)$.

Remark 2: Notice that for simplicity the examples presented above are not polyhedral. It is not difficult to see that these examples can be approximated by polyhedral surfaces such that the argument above is still valid.

By maintaining the invariant on the structure of $\mathcal{S}_i$ we guarantee that guarding the boundary of $\mathcal{S}_i$ requires at most two pursuers. Consequently, at the end of each round at least one pursuer is free (we prove this later in Lemma 6). The free pursuer $\mathcal{P}_3$ is used for splitting the contaminated region as follows. A third shortest path is found inside the closure of $\mathcal{S}_i$ such that its endpoints are two distinct points on $\partial\mathcal{S}_i$. Let Π_3 be this third path (we will show how such Π_3 can be chosen in Section VI-A). The free pursuer $\mathcal{P}_3$ locates itself on the canonical projection of the evader onto Π_3. After placing $\mathcal{P}_3$ on guard position at Π_3, the evader cannot cross Π_3. The path Π_3 divides $\mathcal{S}_i$ into two smaller subsets. The new contaminated region $\mathcal{S}_{i+1}$ is the subset of $\mathcal{S}_i$ that contains the evader defined by either Π_3, Π_1 or Π_3, Π_2. Therefore, one of the pursuers $\mathcal{P}_1$ or $\mathcal{P}_2$ is free. Thus, we can repeat splitting $\mathcal{S}_{i+1}$ using the free pursuer.

We show that our pursuit strategy guarantees capture in two parts. In the first part (Section VI-A), we show that we can maintain the invariant on the structure of the contaminated region $\mathcal{S}_i$ and thus we can continue shrinking the contaminated region by using the free pursuer to split $\mathcal{S}_i$. In the second part (Section VII), we show that $\mathcal{S}_i$ is shrunken to $\mathcal{S}_{i+1}$ such that progress is ensured and finite time capture is achieved as follows. After each round, the pursuers *mark*

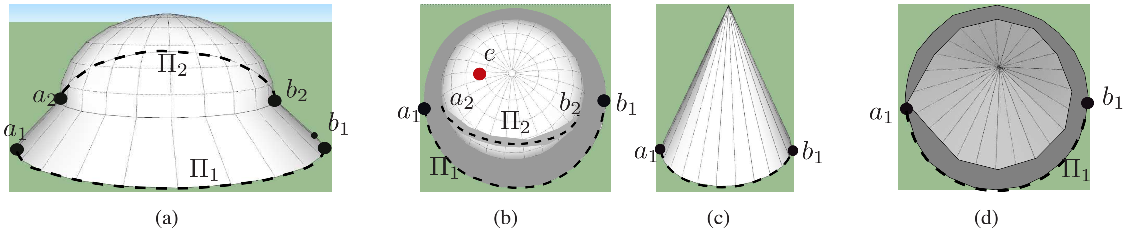


Fig. 5. Here, the shortest paths $\Pi_1(a_1, b_1)$ and $\Pi_2(a_2, b_2)$ are shown in dashed lines, and their capture regions are shaded in dark gray. (a, b) Side view and top view of an example for case 2 are shown respectively. (c,d) Side view and top view of an example for case 3 are shown.

at least one small face as *cleared*. We show that each small face is marked exactly once and thus the number of rounds is bounded by the number of small faces.

A. The Structure of $\mathcal{S}_i$ and Selection of Π_3

In this section, we show that the pursuers can maintain the invariants that we presented at the begining of Section VI. In particular, we show that the three cases for possible boundary configurations are the only possibilities for the structure of $\mathcal{S}_i$. In addition, we discuss the selection of the endpoints of the splitting path Π_3 for each of these cases. Let us denote the endpoints of Π_3 (to be determined) by a_3 and b_3. Before proceeding let us first present the following definition:

Definition 5 (Connected Component): The region $\mathcal{S}_i$ is composed of one or more *connected components*. See Fig. 4(c) for an illustration. We sometimes refer to these subsets of $\mathcal{S}_i$ as *components*.

Observe that the evader cannot move from one component to another without being caught. This is because in order to commute between components the evader has to cross the capture regions which will result in capture by the guarding pursuers $\mathcal{P}_1$ or $\mathcal{P}_2$ (Lemma 1).

We now show that in each of the three cases we can select a shortest path Π_3 inside the closure of $\mathcal{S}_i$ such that in the next round $\mathcal{S}_{i+1}$ also has a structure which is characterized by one of the three cases. This implies that we always have a free pursuer since the boundary of the contaminated region can be guarded by at most two pursuers. Therefore, we can split $\mathcal{S}_i$ using the free pursuer.

Lemma 6 (Invariant on the structure of $\mathcal{S}_i$): At each round i, we need at most two pursuers to guard the boundary of the contaminated region $\mathcal{S}_i$.

Proof: We prove the claim by induction on the round number i. In particular, we assume that the structure of $\mathcal{S}_i$ is in one of the three cases we discussed. In a constructive approach, we show that for each of the three cases we can select the endpoints of Π_3 such that $\mathcal{S}_{i+1}$ also falls into one of the three cases. The lemma follows from the fact that in each of the three cases we need at most two pursuers to guard the boundary of the contaminated region.

For the induction basis, recall that initially Π_1 and Π_2 are chosen as two edges on the boundary of $\mathcal{S}$. According to Lemma 5, the boundary of $\mathcal{S} - C(\Pi_1) \cup C(\Pi_2)$ is composed of subsets of $\partial\mathcal{S}$ and connected sub-paths of $\partial C(\Pi_i), i \leq 2$. Therefore, in the first round, $\mathcal{S}_1$ is covered by the three cases.

Next, for the inductive step, assume that $\mathcal{S}_i$ is in one of the cases we presented. In the following, we show that the third path Π_3 can be chosen such that $\mathcal{S}_{i+1}$ remains in one of the three cases. We present the correctness proof only for case (1). Similar proof is applicable to the remaining cases.

Case (1-a): In the first sub-case, the boundary of $\mathcal{S}_i$ from both a_1 to a_2 and also from b_1 to b_2 are subsets of $\partial\mathcal{S}$ (Fig. 4(a)). Let $q_i, 1 \leq i \leq 4$ be the intersection points between boundary of the capture regions and $\partial\mathcal{S}$ as shown in Fig. 4(a). Then, $\{a_3, b_3\}$ is selected from one of the following pairs: $\{q_1, q_4\}$ or $\{q_2, q_3\}$.

Suppose that we pick q_1, q_4 as the endpoints of Π_3. Without loss of generality suppose that the evader is in between Π_1 and Π_3. The subset of $\mathcal{S}_i$ which is determined by Π_3 and Π_1 defines $\mathcal{S}_{i+1}$. First, observe that by construction all the points on $\partial\mathcal{S}_{i+1}$ are on $\partial C(\Pi_1)$, $\partial C(\Pi_3)$ or $\partial\mathcal{S}$. Next, notice that $q_1 \in C(\Pi_1)$. Thereore, there exists a point $p \in \Pi_1$ such that $d(q_1, p) \leq \frac{\delta}{2}$. Thus, $p \in C(\Pi_3)$ (because there exists the point $q_1 \in \Pi_3$ such that $d(q_1, p) \leq \frac{\delta}{2}$). Thus, $C(\Pi_1) \cap C(\Pi_3) \neq \emptyset$. Consequently, $\mathcal{S}_{i+1}$ is completely guarded by $\mathcal{P}_1$ and $\mathcal{P}_3$: $\mathcal{P}_2$ is not required anymore to confine the evader inside $\mathcal{S}_{i+1}$. Finally, $C(\Pi_1)$ and $C(\Pi_3)$ are pseudo-disks (Lemma 4) and $\partial\mathcal{S}_{i+1}$ includes connected sub-paths of their boundary. Hence, $\mathcal{S}_{i+1}$ will again fall into case (1-b) or case (2) (case (3) is possible if the evader is inside a connected component of a single pursuer).

In the following, we only present the selection of the endpoints of Π_3.

Case (1-b): Let us denote the intersection point between the capture regions by w as shown in Fig. 4(b). In this case, we select $\{a_3, b_3\}$ to be either $\{q_1, w\}$ or $\{q_2, w\}$.

Case (2): Consider the intersection points between $\partial C(\Pi_1)$ and $\partial C(\Pi_2)$. Let $\{w_1, w_2, \cdots, w_k\}$ denote these intersection points. Let w_i, w_{i+1} be the points which determine the component that contains the evader. Then, we select $\{a_3, b_3\}$ to be the pair $\{w_i, w_{i+1}\}$.

Case (3): The endpoints a_3, b_3 are chosen as two arbitrary distinct points on $\partial\mathcal{S}_i$ ($\partial C(\Pi_1)$ in Fig. 4(e)). ∎

In the next section, we show that there can be a finite number of rounds before the evader is captured.

VII. Making Progress

We now prove that the evader will be captured after finite time.

Theorem 1 (Capture in finite time): For a given polyhedral surface with boundary $\mathcal{S}$, three pursuers with non-zero capture radius δ can capture the evader in time $O((\frac{A}{\delta^2}+\frac{L}{\delta})^2\frac{\delta}{2})$ where A denotes the area of the surface, and L is the sum of the length of the edges on $\mathcal{S}$.

Proof: Before proceeding let us denote the shortest path that splits the contaminated region $\mathcal{S}_i$ at round i by Π_3^i. We start by marking all faces of the polyhedron as *contaminated*. Recall that we partitioned $\mathcal{S}$ into small triangles. Thus, each face is a $\frac{\delta}{2}$-small triangle. In each round i, we mark the small triangles that are touched by the splitting path Π_3^i as *clear*. We show that a small triangle cannot be marked clear more than once. Therefore, the number of rounds is less than the number of small triangles on $\mathcal{S}$.

We now present the details of our proof. In each round, we mark the small triangles that are touched by Π_3^i. A small triangle f is touched by Π_3^i if $\Pi_3^i \cap f \neq \emptyset$. Moreover, the splitting path Π_3^i is computed inside the closure of $\mathcal{S}_i$. Hence, $\Pi_3^i \in cl(\mathcal{S}_i)$ where $cl(\mathcal{S}_i)$ denotes the closure of $\mathcal{S}_i$. In summary, for a small triangle f:

$$f \text{ is marked clear at } i^{th} \text{ round if } \Pi_3^i \cap f \cap cl(\mathcal{S}_i) \neq \emptyset \quad (1)$$

Clearly, Π_3^i is touching at least one face f. This is because the endpoints of Π_3^i are distinct (Otherwise $\mathcal{S}_i = \emptyset$ which means that the evader is already captured.). Therefore, at least one small triangle is marked clear in each round.

Next, we show that each small triangle is marked clear at most once. Assume the contrary and suppose that a small triangle f is marked twice: at rounds i and j where $i < j$. The main idea is the following. At each round i, we compute the splitting path Π_3^i inside $cl(\mathcal{S}_i)$, and then we remove the capture region of Π_3^i from $\mathcal{S}_i$ in order to obtain the new contaminated region $\mathcal{S}_{i+1}$. Since f is marked at the i^{th} round, it is being touched by Π_3^i. Shortly, we show that since f is touched by Π_3^i the capture region of Π_3^i contains f. Therefore, f does not appear in the contaminated region $\mathcal{S}_{i+1}$ and also in the future regions $\mathcal{S}_j, j > i$. The splitting path Π_3^j is computed inside $cl(\mathcal{S}_j)$. Hence, the small triangle f cannot be marked by Π_3^j (because Π_3^j marks the faces that it is touching in $\mathcal{S}_j$ and $\mathcal{S}_j$ does not contain f). The formal proof is as follows.

First we show that a small triangle f that is marked at the i^{th} round is completely inside the capture region of Π_3^i ($f \subseteq C(\Pi_3^i)$). This is because f is a $\frac{\delta}{2}$-small triangle and the distance between any pair of points inside a $\frac{\delta}{2}$-small triangle is at most $\frac{\delta}{2}$ (Definition 2).

The region $\mathcal{S}_{i+1}$ is obtained by removing the capture region $C(\Pi_3^i)$ from $\mathcal{S}_i$ (Definition 4). Thus, we have $\mathcal{S}_{i+1} \cap C(\Pi_3^i) = \emptyset$. Together with the observation that $f \subseteq C(\Pi_3^i)$ we must have $f \cap \mathcal{S}_{i+1} = \emptyset$. According to our contrary assumption, f is marked at the j^{th} round as well. Thus, according to (1) we must have $f \cap \mathcal{S}_j \neq \emptyset$.

Now observe that $\mathcal{S}_{k+1} \subset \mathcal{S}_k$ for all k since $\mathcal{S}_{k+1}$ is obtained by removing $C(\Pi_3^i)$ from $\mathcal{S}_k$. Therefore, we have $\mathcal{S}_j \subset \mathcal{S}_i$ for $j > i$. Thus, $\mathcal{S}_j \subseteq \mathcal{S}_{i+1}$. Using the properties of sets we have $f \cap \mathcal{S}_j \subseteq \mathcal{S}_j \subseteq \mathcal{S}_{i+1}$. Therefore, $f \cap \mathcal{S}_j \subseteq \mathcal{S}_{i+1}$. Moreover, $f \cap \mathcal{S}_j \subseteq f$. Consequently, $f \cap \mathcal{S}_j \subseteq f \cap \mathcal{S}_{i+1}$. But, $f \cap \mathcal{S}_{i+1} = \emptyset$ and $f \cap \mathcal{S}_j \neq \emptyset$ which is a contradiction. Hence, each small triangle is marked at most once. Therefore, the number of rounds (N) is bounded by the number of small triangles on $\mathcal{S}$ which is $O(\frac{A}{\delta^2}+\frac{L}{\delta})$ (Lemma 2).

Finally, let us compute the capture time. Each round i, takes $O(D)$ steps where D is the length of the longest shortest path inside the polyhedral surface. This time is required for initializing the pursuers on the projection of the evader. It is not too difficult to show that $D = O(N\frac{\delta}{2})$ where N is the number of $\frac{\delta}{2}$-small triangles. Also, there are $O(N)$ rounds. Thus, the capture time is $O(N^2\frac{\delta}{2})$. According to Lemma 2 we have $N = O(\frac{A}{\delta^2}+\frac{L}{\delta})$ where A is the area of $\mathcal{S}$, and L is the sum of the length of the edges on $\mathcal{S}$. Therefore, the capture time is $O((\frac{A}{\delta^2}+\frac{L}{\delta})^2\frac{\delta}{2})$. ■

VIII. Conclusion

In this paper, we studied the lion and man game on the surface of a polyhedron with boundary, which is homeomorphic to a disk (that can contain holes). We show the existence of a capture strategy with three lions when the capture radius is non-zero. We leave the computational aspects of the strategy for our future work. In particular, computing the capture region of a shortest path is a challenging problem. We believe that our proposed strategy applies to polyhedral surfaces that are homeomorphic to a sphere as well. In general, determining the class of polyhedral surfaces such that the evader will be captured with less than three pursuers is open.

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