Taylor Series Expansion for Solutions of the Korteweg- de Vries Equation with respect to Their Initial Values

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Abstract

The initial value problem for the KdV equation

$$\partial_t u + u\partial_x u + \partial_x^3 u = 0, \qquad u(x,0) = \phi(x)$$

establishes a nonlinear map K from $H^s(R)$ to $C([-T,T]; H^s(R))$. It has been known for many years that this map K is continuous [2] , [17] and is proved recently being Lipschitz continuous [23].

In this paper it is shown that the nonlinear map K is infinitely many times Frechet differentiable from $H^s(R)$ to $C([-T,T]; H^s(R))$. Furthermore, it is proved that K has a Taylor series expansion at any given $\phi \in H^s(R)$, i.e.

$$K(\phi + h) = \sum_{n=0}^{\infty} \frac{K^{(n)}(\phi)[h^n]}{n!}$$

where $K^{(n)}(\phi)$, the n-th derivative of K at ϕ, is a n linear map from the n-fold space of $H^s(R)$ to $C([-T,T]; H^s(R))$ and the series converges in the space $C([-T,T]; H^s(R))$ uniformly for $\|h\|_s \leq \delta$ with some $\delta > 0$. Each term $y_n = K^{(n)}(\phi)[h^n]$ in the series solves a linearized KdV equation. Thus any "small" perturbation $K(\phi + h)$ of $K(\phi)$ would be obtained by solving a series of linear problems.

In contrast the corresponding map K_p for periodic solutions of the initial value problem of the KdV equation is only known to be continuous from $H^s(S)$ to $C([-T,T]; H^s(S))$ where S is the unit length circle. This is due to the lack of smoothing effects for periodic solutions of the KdV equation. It is shown in the paper that K_p is Lipschitz continuous from $H^{s+1}(S)$ to $C([-T,T]; H^s(S))$ and is n times Frechet differentiable from $H^{s+n+1}(S)$ to $C([-T,T]; H^s(S))$ for any $n \geq 1$.

The method developed in this paper applies to other nonlinear dispersive wave equations.

1 Introduction

The Korteweg-de Vries (KdV) equation

$$\partial_t u + u\partial_x u + \partial_x^3 u = 0$$

in one space dimension and time was originally derived in 1895 ([27]) as an approximate model for planar, uni-directional, irrotational waves propagating on the surface of shallow water. Its range of applications has broadened considerably since 1960 when a new application of the KdV equation was founded in the study of collision-free hydromagnetic waves by Gardner and Morikawa ([10]) and it now includes many physical situations which feature wave equation wherein a balance is struck between the weak effects of nonlinearity and dispersion (cf. [29], [30] and [36]). In particular, the discovery of solitons and the inverse scattering method aroused great interests in the KdV equation.

As for the initial value problem (IVP) of the KdV equation

$$\begin{cases} \partial_t u + u\partial_x u + \partial_x^3 u = 0, \qquad x,\ t \in R \\ \\ u(x,0) = \psi(x), \end{cases} \tag{1.1}$$

a large amount of work has been devoted to the existence and uniqueness problems of its solutions in various function spaces. (cf. [4], [9], [11], [12], [14], [17], [28], [42] and [44]). In particular, it was established in the work of Bona and Smith [2] , Bona and Scott [1], Saut and Temam [35] and Kato [17] that the IVP (1.1) is locally (resp. globally) well-posed in the classical Sobolev space $H^s(R)$ with $s > 3/2$ (resp. $s \geq 2$). Recently, based on a careful analysis of the smoothing properties of the associated linear problem and the associated maximal operator, Kenig, Ponce and Vega [22] showed that the IVP (1.1) is locally well posed in $H^s(R)$ for $s > 3/4$ and globally well posed in $H^s(R)$ for $s \geq 1$ which answered the question left open by Saut and Temam [35], and by Kato [17].

Let $s > 0$ and $T > 0$ be given. For $w: \ R \times [-T,T] \to R$, define

$$\lambda_1(T,w) = \sup_{[-T,T]} \|w(t)\|_s, \tag{1.2}$$

$$\lambda_2(T,w) = \sup_x \int_{-T}^{T} |D^s \partial_x w(x,t)|^2 dt, \tag{1.3}$$

$$\lambda_3(T,w,l) = \left(\int_{-T}^{T} \|J^l \partial_x w(t)\|_\infty^4 dt\right)^{\frac{1}{4}} \tag{1.4}$$

with $l \in [0, s-3/4]$,

$$\lambda_4(T,w,r) = (1+T)^{-\rho} \left(\int_R \sup_{[-T,T]} |J^r w(x,t)|^2\right)^{\frac{1}{2}} \tag{1.5}$$

with $r \in [0, s-3/4)$ and ρ a fixed number larger than $3/4$,

$$\Lambda^s_{l,r}(T;w) = max\left\{\lambda_1(T,w), \lambda_2(T,w), \lambda_3(T,w,l), \lambda_4(T,w,r)\right\}, \tag{1.6}$$

and

$$X^{T,s}_{l,r} = \left\{w \in C([-T,T]; H^s(R)) \mid \Lambda^s_{l,r}(T;w) < +\infty\right\} \tag{1.7}$$

for any $(l,r) \in [0, s-\frac{3}{4}] \times [0, s-\frac{3}{4})$.

$X^{T,s}_{l,r}$ is a Banach space equipped with the norm

$$\|w\|_{X^{T,s}_{l,r}} := \Lambda^s_{l,r}(T;w)$$

and is obviously a subspace of $C([-T,T]; H^s(R))$ with stronger topology.

According to Kenig, Ponce and Vega [22], for $s > 3/4$ and any $\psi \in H^s(R)$, there is a $T > 0$ such that the IVP (1.1) defines a nonlinear map K from a neighborhood U of ψ in $H^s(R)$ into the space $X^{T,s}_{l,r}$ for any $(l,r) \in [0, s-\frac{3}{4}] \times [0, s-\frac{3}{4})$, i.e.

$$K(\phi) := u, \qquad \text{for any } \phi \in U \tag{1.8}$$

where u is the solution of (1.1) corresponding the initial value ϕ. If $s \geq 1$, then the map K is well defined from $H^s(R)$ to $X^{T,s}_{l,r}$ for any arbitrarily given $T > 0$.

Considering K as a map from $H^s(R)$ to $C([-T,T]; H^s(R))$, Bona and Smith [2] first proved that it is continuous. Later Kato [15] established the existence and uniqueness of the solution to the IVP (1.1) by applying his theory for abstract quasi-linear evolution equations. As a consequence, the continuity of the map K follows automatically. In [35], Saut and Temam showed that K is locally Hölder continuous with exponent $1/2$ while considering K as a map from $H^{s+1/2}(R)$ to $L^\infty([-T,T]; H^s(R))$ ($s \geq 2$). Recently, Kenig, Ponce and Vega [23] showed that the map K is Lipschitz continuous from $H^s(R)$ to $X^{T,s}_{l,r}$, and hence, in particular,is Lipschitz continuous from $H^s(R)$ to $C([-T,T]; H^s(R))$.

In this paper, we shall show that K, as a nonlinear map from $H^s(R)$ to $X^{T,s}_{l,r}$, is a C^∞ map, i.e. for any $n\geq 1$, it is n times Frechet differentiable at any $\psi\in H^s(R)$. Its n-th derivative $K^{(n)}(\psi)$ at ψ, a n-linear map from the n-fold product space $(H^s(R))^n$ into $X^{T,s}_{l,r}$, may be constructed by solving a system of inhomogeneous linearized KdV equations. More precisely, for any $n\geq 1$ and $h_k\in H^s(R)$ $(k=1,2,...,n)$, let

$$w^{(n)}_{[1,...,n]} := K^{(n)}(\psi)[h_1,...,h_n],$$

then $w^{(k)}_{[1,...,k]}$ $(k=1,2,...,n)$ solve

$$\left\{\begin{array}{l} \partial_t w^{(1)}_{[1]} + \partial_x(u w^{(1)}_{[1]}) + \partial_x^3 w^{(1)}_{[1]} = 0 \\ \\ w^{(1)}_{[1]}(x,0) = h_1(x) \end{array}\right. \tag{1.9}$$

and

$$\left\{\begin{array}{l} \partial_t w^{(k)}_{[1,...,k]} + \partial_x(u w^{(k)}_{[1,...,k]}) + \partial_x^3 w^{(k)}_{[1,...,k]} = -\frac{1}{2}\partial_x(G_k) \\ \\ w^{(k)}_{[1,...,k]}(x,0) = 0 \end{array}\right. \tag{1.10}$$

for $k=2,3,...,n$ where $u=K(\psi)$ is the solution of the IVP (1.1) and G_k is a polynomial of $w^{(j)}_{[i_1,i_2,...,i_j]}$, $1\leq i_1,...,i_j\leq k$, for $j=1,2,...,k-1$ (see section 3 for the detail of the structure of G_k).

If we let $h_1=h_2=...=h_n=h$ and denote $y_n=K^{(n)}(\psi)[h^n]$, which is a homogeneous polynomial of degree of n from $H^s(R)$ to $X^{T,s}_{l,r}$, then

$$\left\{\begin{array}{l} \partial_t y_1 + \partial_x(u y_1) + \partial_x^3 y_1 = 0 \\ \\ y_1(x,0) = h(x) \end{array}\right. \tag{1.11}$$

for $n=1$ and

$$\left\{\begin{array}{l} \partial_t y_n + \partial_x(u y_n) + \partial_x^3 y_n = -\frac{1}{2}\partial_x(\sum_{k=1}^{n-1}\left(\begin{array}{c} k \\ n \end{array}\right) y_k y_{n-k}) \\ \\ y_k(x,0) = 0 \end{array}\right. \tag{1.12}$$

for $n\geq 2$, where

$$\left(\begin{array}{c} k \\ n \end{array}\right) = \frac{n!}{k!(n-k)!}.$$

We may define the *n-th Taylor polynomial* P_n of K at $\psi \in H^s(R)$ as

$$\begin{aligned} P_n(\psi)[h] &:= K(\psi) + \sum_{k=1}^{n} \frac{K^{(n)}(\psi)}{k!}[h^k] \\ &= u + \sum_{k=1}^{n} \frac{y_k}{k!} \end{aligned}$$

for any $h \in H^s(R)$.

Let

$$z_n = K(\psi + h) - P_n(\psi)[h]$$

which is the *n-th Taylor remainder* of K at ψ. We shall see that z_n solves

$$\begin{cases} \partial_t z_0 + \frac{1}{2}\partial_x((u+v)z_0) + \partial_x^3 z_0 = 0 \\ \\ z_0(x,0) = 0 \end{cases} \tag{1.13}$$

for $n = 0$ and

$$\begin{cases} \partial_t z_n + \frac{1}{2}\partial_x((u+v)z_n) + \partial_x^3 z_n = -\frac{1}{2}\partial_x \left(\sum_{k=0}^{n-1} \frac{1}{(n-k)!} z_k y_{n-k} \right) \\ \\ z_k(x,0) = 0 \end{cases} \tag{1.14}$$

for $n \geq 1$, where $v = K(\psi + h)$, $u = K(\psi)$ and y_k $(k = 1, 2, ..., n)$ are solutions of (1.11) and (1.12).

By establishing the accurate estimates of y_n and z_n, we are able to show that for any $\psi \in H^s(R)$, there is a $\delta > 0$ such that if $h \in H^s(R)$ with $\|h\|_s \leq \delta$, then

$$K(\psi + h) = \sum_{n=0}^{\infty} \frac{K^{(n)}(\psi)}{n!}[h^n], \tag{1.15}$$

the series converging uniformly about h with $\|h\|_s \leq \delta$ in the space $X_{l,r}^{T,s}$. Hence the map K is an analytic map from $H^s(R)$ to $X_{l,r}^{T,s}$. Especially, it is analytic from $H^s(R)$ to $C([-T,T]; H^s(R))$.

Note that each term in the Taylor series (1.15) is a solution of linear problem (1.11)-(1.12). Therefore, any "small" perturbation of a given solution of the KdV equation can be reconstructed by solving a series of linear problems

Finally we should mention that our method can be used to study differentiability of solutions of the Benjamin-Ono equation

$$\begin{cases} \partial_t u + u\partial_x u + H\partial_x^2 u = 0, \quad x,\, t \in R \\ \\ u(x,0) = \phi(x) \end{cases} \tag{1.16}$$

where H is the Hilbert transform, and the generalized Korteweg- de Vries equation

$$\begin{cases} \partial_t u + \partial_x(a(u)) + \partial_x^3 u = 0, \qquad x,\, t \in R \\ \\ u(x,0) = \phi(x) \end{cases} \tag{1.17}$$

where $a(.) \in C^\infty(R)$, with respect to their initial values. More precisely, we are able to show that the nonlinear map B established by the IVP (1.16) is an analytic map from $H^s(R)$ to $C([-T.T]; H^s(R))$ and the nonlinear map G established by the IVP (1.17) is a C^∞ map from $H^s(R)$ to $C([-T,T]; H^s(R))$. The detail of the results concerning (1.16) and (1.17) and their proof will be given in our subsequent papers [47] and [48].

The paper is organized as follows.

—- In section 2, we shall consider the IVP for the following linear equation

$$\begin{cases} \partial_t u + \partial_x(a(x,t)u) + \partial_x^3 u = f(x,t) \\ \\ u(x,0) = \psi(x) \end{cases} \tag{1.18}$$

for $x \in R$ and $t \in R$.

For given $s > 3/4$ and $T > 0$, we shall show that if $a \in X_{0,0}^{T,s}$, then for any $\psi \in H^s(R)$ and $f \in L^1([-T,T]; H^s(R))$, the IVP (1.18) has a unique solution $u \in X_{l,r}^{T,s}$ and

$$\|u\|_{X_{l,r}^{T,s}} \leq c\left(\|\psi\|_s + \int_{-T}^{T} \|f(.,t)\|_s dt\right) \tag{1.19}$$

for any $(l,r) \in [0, s-\frac{3}{4}] \times [0, s-\frac{3}{4})$ where c only depends on $\|a\|_{X_{0,0}^{T,s}}$. This would be a key estimate to prove differentiability of the map K and the convergence of the Taylor series (1.15).

— In section 3, we prove the differentiability of the map K and show that the map $K(\psi)$ can be expanded as a Taylor series in a neighborhood of any give $\phi \in H^s(R)$.

— In section 4, we consider the IVP for the periodic KdV equation

$$\begin{cases} \partial_t u + u\partial_x u + \partial_x^3 u = 0, & x \in S, \quad t \in R \\ u(x,0) = \psi \end{cases} \tag{1.20}$$

where S denotes a unit length circle.

The corresponding map K_p established by (1.20) is only known to be continuous from $H^s(S)$ to $C([-T,T]; H^s(S))$ (see [2], [17]). This is due to the lack of smoothing effects for periodic solutions of the KdV equation. On the other hand, Saut and Temam [35] showed that K_p is locally Hölder continuous with exponent 1/2 from $H^{s+1/2}(S)$ to $L^\infty([-T,T]; H^s(S))$. We shall show that K_p is Lipschitz continuous from $H^{s+1}(S)$ to $C([-T,T]; H^s(S))$ and is n times Frechet differentiable from $H^{s+n+1}(S)$ to $C([-T,T]; H^s(S))$ for any $n \geq 1$.

Notations:

– The norm in $L^2(R)$ or $L^2(S)$ will be denoted by $\|.\|$ and the norm in $H^s(R)$ or $H^s(S)$ will be denoted by $\|.\|_s$. The notation $\|.\|_\infty$ is used to denote the norm in $L^\infty(R)$ or $L^\infty(S)$.

– $D^s = (-\partial_x^2)^{s/2}$ and $J^s = (1-\partial_x^2)^{s/2}$ denote the Riesz and the Bessell potential of order s respectively.

– $[A,B] = AB - BA$, where A, B are operators. Thus $[J^s; f]g = J^s(fg) - fJ^s g$ in which f is regarded as a multiplication operator.

– $H^\infty(R) := \cap_{s>0} H^s(R)$

– For $1 \leq p$, $q \leq \infty$ and $f : R \times [-T,T] \to R$,

$$\|f\|_{L_T^q L_x^p} = \left(\int_{-T}^{T} \left(\int_{-\infty}^{\infty} |f(x,t)|^p dx \right)^{\frac{q}{p}} dt \right)^{\frac{1}{q}}$$

and

$$\|f\|_{L_x^p L_T^q} = \left(\int_{-\infty}^{\infty} \left(\int_{-T}^{T} |f(x,t)|^q dt \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}}.$$

2 Linear Estimates

We use $\{W(t)\}_{-\infty}^{+\infty}$ to denote the unitary group which defines the solution of the IVP associated to

$$\begin{cases} \partial_t v + \partial_x^3 v = 0, & \text{for } x,\, t \in R \\ \\ v(x,0) = v_0(x) \end{cases} \tag{2.1}$$

where

$$v(t) = W(t)v_0 = S_t * v_0$$

with $S_t(.)$ defined by the oscillatory integral

$$S_t(x) = c\int_{-\infty}^{+\infty} e^{ix\xi}e^{it\xi^3}d\xi.$$

Then the solution of the inhomogeneous equation

$$\begin{cases} \partial_t v + \partial_x^3 v = f(x,t) \\ \\ v(x,0) = 0 \end{cases} \tag{2.2}$$

for $x \in R$, $t \in R$ is expressed as

$$v(t) = \int_0^t W(t-\tau)f(.,\tau)d\tau.$$

Lemma 2.1 (Kenig, Ponce and Vega) *([22]) For any $s \geq 0$,*

$$\left(\sup_x \int_{-\infty}^{\infty} |D^s\partial_x W(t)v_0|^2 dt\right)^{1/2} \leq c\|v_0\|_s, \tag{2.3}$$

and

$$\left(\int_{-\infty}^{\infty} \|D^{s+1/4}W(t)v_0\|_\infty^4 dt\right)^{1/4} \leq c\|v_0\|_s. \tag{2.4}$$

In addition, for any $s > 3/4$ and $T > 0$,

$$\left(\int_{-\infty}^{+\infty} \sup_{[-T,T]} |J^l W(t)v_0|^2(x)dx\right)^{1/2} \leq c(1+T)^\rho\|v_0\|_s \tag{2.5}$$

where $l \in [0, s-3/4)$ and ρ is a fixed constant larger than $3/4$.

Remark 2.1 *(2.3) is a stronger version of local smoothing effect of Kato type and (2.4) is the global smoothing effect of Strichartz type [41] present in solutions of (2.1). The estimate (2.5), which gives a bound for the associated maximal function* $\sup_{[-T,T]} |W(t).|$*, is due to Vega [46].*

Lemma 2.2 *For any* $s \geq 0$ *and* $T > 0$*,*

$$\|W(t)v_0\|_s = \|v_0\|_s, \qquad t \in [-T,T] \tag{2.6}$$

and

$$\sup_{[-T,T]} \| \int_0^t W(t-\tau) f(.,\tau) d\tau \|_s \leq \int_{-T}^T \|f(.,\tau)\|_s d\tau. \tag{2.7}$$

Proof: (2.6) and (2.7) follow easily from Kato ([17], Lemma 3.1).

Lemma 2.3 *For any* $s \geq 0$ *and* $T > 0$*,*

$$\|D_x^s \partial_x \int_0^t W(t-\tau) f(.,\tau) d\tau \|_{L_x^\infty L_T^2} \leq c \|f\|_{L^1([-T,T];H^s(R))} \tag{2.8}$$

and

$$\|D^{s+\frac{1}{4}} \int_0^t W(t-\tau) f(.,\tau) d\tau \|_{L_x^\infty L_T^4} \leq c \int_{-T}^T \|f(.,\tau)\|_s d\tau. \tag{2.9}$$

If $s > 3/4$*, then*

$$\|J^l \int_0^t W(t-\tau) f(.,\tau) d\tau \|_{L_x^2 L_T^\infty} \leq c(1+T)^\rho \int_{-T}^T \|f(.,\tau)\|_s d\tau \tag{2.10}$$

where $l \in [0, s-3/4)$ *and* ρ *is a fixed constant larger than* $3/4$*.*

Proof: (2.8) follows from (2.3) by using the Minkowski's integral inequality.

$$\begin{aligned}
\|D_x^s \partial_x \int_0^t W(t-\tau) f(.,\tau) d\tau \|_{L_x^\infty L_T^2} &\leq \| \int_0^t |D_x^s \partial_x W(t-\tau) f(.,\tau)| d\tau \|_{L_x^\infty L_T^2} \\
&\leq \sup_x \left(\int_{-T}^T \left(\int_{-T}^T |D_x^s \partial_x W(t-\tau) f(.,\tau)| d\tau \right)^2 dt \right)^{1/2} \\
&\leq \sup_x \int_{-T}^T \left(\int_{-T}^T |D_x^s \partial_x W(t-\tau) f(.,\tau)|^2 dt \right)^{1/2} d\tau \\
&\leq \int_{-T}^T \sup_x \left(\int_{-T}^T |D_x^s \partial_x W(t-\tau) f(.,\tau)|^2 dt \right)^{1/2} d\tau \\
&\leq c \int_{-T}^T \|f(.,\tau)\|_s d\tau \qquad \text{(by (2.3))}.
\end{aligned}$$

(2.9) and (2.10) follow similarly from (2.5) and (2.4) respectively. The proof is completed. □

Lemma 2.4 *Let $s > 1/2$ and $T > 0$ be given and $X_{0,0}^{T,s}$ denote the Banach space defined by (1.6) and (1.7). Then*

$$\int_{-T}^{T} \|\partial_x(uv)\|_s dt \leq cT^{1/2}(1+T)^\rho \|u\|_{X_{0,0}^{T,s}} \|v\|_{X_{0,0}^{T,s}} \tag{2.11}$$

for any $u,\ v \in X_{0,0}^{T,s}$ where c is a constant independent of u and v.

Proof:

$$\begin{aligned}
\|\partial_x(uv)\|_s &= \|J^s(v\partial_x u + u\partial_x v)\| \\
&\leq \|J^s(v\partial_x u)\| + \|J^s(u\partial_x v)\|.
\end{aligned}$$

Applying the commutator estimates due to Kato and Ponce [18],

$$\begin{aligned}
\|J^s(v\partial_x u)\| &= \|vD^s\partial_x u + v(J^s - D^s)\partial_x u + [J^s; v]\partial_x\| \\
&\leq \|vD^s\partial_x u\| + \|v(J^s - D^s)\partial_x u\| + \|[J^s; v]\partial_x u\| \\
&\leq \|vD^s\partial_x u\| + \|v\|_\infty \|u\|_s + c\left\{\|\partial_x v\|_\infty \|u\|_s + \|v\|_s \|\partial_x u\|_\infty\right\}.
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
\int_{-T}^{T} \|vD^s\partial_x u\| d\tau &\leq (2T)^{1/2}\left(\int_{-T}^{T} \|vD^s\partial_x u\|^2 d\tau\right)^{1/2} \\
&\leq (2T)^{1/2}\left(\int_R \sup_{[-T,T]} |v(x,t)|^2 dx\right)^{1/2}\left(\sup_x \int_{-T}^{T} |D^s\partial_x u|^2 d\tau\right)^{1/2} \\
&\leq (2T)^{1/2}(1+T)^\rho \Lambda_{0,0}^s(T; v)\Lambda_{0,0}^s(T; u),
\end{aligned}$$

$$\begin{aligned}
\int_{-T}^{T} \|v\|_\infty \|u\|_s d\tau &\leq \sup_{[-T,T]} \|v\|_s \int_{-T}^{T} \|u\|_s d\tau \\
&\leq 2T \sup_{[-T,T]} \|v\|_s \sup_{[-T,T]} \|u\|_s \\
&\leq 2T\Lambda_{0,0}^s(T; u)\Lambda_{0,0}^s(T; v),
\end{aligned}$$

$$
\begin{aligned}
\int_{-T}^{T} \|\partial_x v\| \|u\|_s d\tau &\leq \sup_{[-T,T]} \|u\|_s \int_{-T}^{T} \|\partial_x v\|_\infty d\tau \\
&\leq (2T)^{3/4} \left(\int_{-T}^{T} \|\partial_x v\|_\infty^4 d\tau \right)^{1/4} \sup_{[-T,T]} \|u\|_s \\
&\leq (2T)^{3/4} \Lambda_{0,0}^s(T;u) \Lambda_{0,0}^s(T;v),
\end{aligned}
$$

and similarly,

$$
\int_{-T}^{T} \|\partial_x u\|_\infty \|v\|_s d\tau \leq (2T)^{3/4} \Lambda_{0,0}^s(T;u) \Lambda_{0,0}^s(T;v).
$$

Thus

$$
\int_{-T}^{T} \|J^s(v\partial_x u)\| d\tau \leq cT^{1/2}(1+T)^\rho \Lambda_{0,0}^s(T;u) \Lambda_{0,0}^s(T;v)
$$

for some $c > 0$ independent of u, v and T. The same argument leads to

$$
\int_{-T}^{T} \|J^s(u\partial_x v\| d\tau \leq cT^{1/2}(1+T)^\rho \Lambda_{0,0}^s(T;u) \Lambda_{0,0}^s(T;v).
$$

Therefore

$$
\int_{-T}^{T} \|\partial_x(uv)\|_s d\tau \leq cT^{1/2}(1+T)^\rho \Lambda_{0,0}^s(T;u) \Lambda_{0,0}^s(T;v).
$$

The proof is completed. □

Now we consider the IVP of the following linear equation:

$$
\begin{cases}
\partial_t u + \partial_x(au) + \partial_x^3 u = f(x,t) \\
u(x,0) = \psi(x)
\end{cases}
\tag{2.12}
$$

for $x,\ t \in R$ where $a = a(x,t)$

Theorem 2.1 *Let $s > 3/4$, $T > 0$ and $a \in X_{0,0}^{T,s}$ be given. Then for any $f \in L^1([-T,T]; H^s(R))$ and $\psi \in H^s(R)$, there exists a unique solution $u \in X_{l,r}^{T,s}$ to the IVP (2.12) and*

$$
\|u\|_{X_{l,r}^{T,s}} \leq c \left(\|\psi\|_s + \int_{-T}^{T} \|f(.,\tau)\|_s d\tau \right)
\tag{2.13}
$$

for any $(l,r) \in [0, s-\frac{3}{4}] \times [0, s-\frac{3}{4})$ where the constant $c = c(\|a\|_{X_{0,0}^{T,s}})$ depends on $\|a\|_{X_{0,0}^{T,s}}$ continuously.

Proof: We use the contraction principle argument that Kenig, Ponce and Vega used in [23].

For any given $\psi \in H^s(R)$ and $f \in L^1([-T,T];H^s(R))$, denote by $u = \Phi(v)$, the solution of the IVP for the following linear inhomogeneous equation

$$\begin{cases} \partial_t u + \partial_x^3 u = f - \partial_x(av) \\ \\ u(x,0) = \psi \end{cases} \tag{2.14}$$

where

$$v \in S_b^T = \left\{ w \in X_{0,0}^{T,s} \mid \Lambda_{0,0}^s(T;w) < b \right\}$$

for some $b > 0$ to be determined.

We shall show that there exists a $b = b(\|\psi\|_s, \|f\|_{L^1([-T,T];H^s(R))}) > 0$ and a $T^* > 0$ such that $u = \Phi(v) \in S_b^{T^*}$ if $u \in S_b^{T^*}$ and

$$\Phi : \quad S_b^{T^*} \to S_b^{T^*}$$

is a contraction.

Consider the integral equation form of the IVP (2.14),

$$u(t) = W(t)\psi + \int_0^t W(t-\tau) f(.,\tau) d\tau - \int_0^t W(t-\tau)\partial_x(av) d\tau. \tag{2.15}$$

Applying (2.3) –(2.10) to (2.15) lead to

$$\begin{aligned} \Lambda_{l,r}^s(t;u) &\leq c\left(\|\psi\|_s + \int_{-t}^t \|f\|_s d\tau\right) + \int_{-t}^t \|\partial_x(av)\|_s d\tau \\ &\leq c\left(\|\psi\|_s + \int_{-T}^T \|f\|_s d\tau\right) + ct^{1/2}(1+t)^\rho \Lambda_{0,0}^s(T;a)\Lambda_{0,0}^s(t;v) \end{aligned} \tag{2.16}$$

for any $(l,r) \in [0, s-\frac{3}{4}] \times [0, s-\frac{3}{4})$. In particular,

$$\Lambda_{0,0}^s(t;u) \leq c\left(\|\psi\|_s + \int_{-T}^T \|f\|_s d\tau\right) + ct^{1/2}(1+t)^\rho \Lambda_{0,0}^s(T;a)\Lambda_{0,0}^s(t;v).$$

If we choose

$$b = 2c\left(\|\psi\|_s + \int_{-T}^T \|f\|_s d\tau\right)$$

and $T^* \leq T$ such that

$$c\sqrt{T^*}(1+T^*)^\rho \Lambda_{0,0}^s(T;a) = \frac{1}{2} \tag{2.17}$$

then

$$\Lambda_{0,0}^s(T^*;u) \leq b \tag{2.18}$$

Thus, Φ is a map from $S_b^{T^*}$ to $S_b^{T^*}$.

For any v_1, $v_2 \in S_b^{T^*}$, let $w = \Phi(v_1) - \Phi(v_2)$. Then

$$w(t) = -\int_0^t W(t-\tau)\partial_x(a(v_1-v_2))d\tau.$$

Hence

$$\begin{aligned}\Lambda_{0,0}^s(T^*;w) &\leq c\int_{-T^*}^{T^*} \|\partial_x(a(v_1-v_2))\|_s d\tau \\ &\leq c\sqrt{T^*}(1+T^*)^\rho \Lambda_{0,0}^s(T;a)\Lambda_{0,0}^s(T^*;v_1-v_2)\end{aligned}$$

and by (2.17)

$$\Lambda_{0,0}^s(T^*;w) \leq \frac{1}{2}\Lambda_{0,0}^s(T^*;v_1-v_2) \tag{2.19}$$

Consequently, by (2.18) and (2.19), we have that there exists a unique $u \in S_b^{T^*}$ with

$$\Phi(u) = u$$

i.e.,

$$u(t) = W(t)\psi - \int_0^t W(t-\tau)\partial_x(au)(\tau)d\tau + \int_0^t W(t-\tau)f(.,\tau)d\tau \tag{2.20}$$

for $-T^* \leq t \leq T^* \leq T$.

Then it follows from (2.16) and (2.17) that

$$\Lambda_{l,r}^s(T^*;u) \leq c\left(\|\psi\|_s + \int_{-T}^{T} \|f\|_s d\tau\right) \tag{2.21}$$

for any $(l,r) \in [0, s-\frac{3}{4}] \times [0, s-\frac{3}{4})$. Note that T^* determined by (2.17) only depends on $\Lambda_{0,0}^s(T;a)$ and not depends on ψ and f. Thus a standard argument shows that T^* can be extended to $T^* = T$. That is to say, (2.20) and (2.21) are true for $-T \leq t \leq T$. The proof is completed. $\square$

3 Taylor series expansion

Let X, Y be two Banach spaces. An n-linear map from X to Y is a map from the n-fold product space X^n into Y such that

$$x_k \rightarrow f(x_1, ..., x_n)$$

is linear for each fixed $(x_1, x_2, ..., x_{k-1}, x_{k+1}, ..., x_n)$, for $k = 1, 2, ..., n$. A homogeneous polynomial of degree n from X to Y is a map of the form $x \rightarrow f(x, ..., x)$, for some n-linea map f.

A map $f: X \rightarrow Y$ is Frechet differentiable at a point $x_0 \in X$ if there exists a continuous linear map $f'(x_0): X \rightarrow Y$ so that

$$\|f(x) - f(x_0) - f'(x_0)(x - x_0)\|_Y = o(\|x - x_0\|_X)$$

as $x \rightarrow x_0$ where $f'(x_0) \in \mathcal{L}(X, Y)$ is called the Frechet derivative of f at x_0. f is said twice differentiable at x_0 if f is differentiable at each point in a neighborhood of x_0 and $x \rightarrow f'(x) \in \mathcal{L}(X, Y)$ is differentiable at x_0; and so on.

A map $f: U \subset X \rightarrow Y$, U open, is analytic in U if f is infinitely often differentiable at each point of U and if, for each $x \in U$, there exists $\delta = \delta(x) > 0$ so that whenever $\|h\|_X \leq \delta$,

$$f(x + h) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(x)[h^k],$$

the series converging in Y-norm uniformly in $\|h\|_X \leq \delta$.

Consider the IVP for the KdV equation

$$\begin{cases} \partial_t u + u\partial_x u + \partial_x^3 u = 0, & x,\ t \in R \\ u(x, 0) = \psi(x) \end{cases} \tag{3.1}$$

According to Kenig, Ponce and Vega ([22], Theorem 1.1), for any $\phi \in H^s(R)$, $s > 3/4$, there is a $T > 0$ and a neighborhood U of ϕ in $H^s(R)$ such that (3.1) defines a nonlinear map $K: U \rightarrow X_{l,r}^{T,s}$

$$u := K(\psi)$$

for any $\psi \in U$ where u is the solution of (3.1) and $(l, r) \in [0, s - \frac{3}{4}] \times [0, s - \frac{3}{4})$.

We shall show in this section that K is not only initely many Frechet differentiable, but also analytic in U.

First of all,if the map K is assumed n times differentiable, then its n-th order derivative $K^{(n)}(\psi)$ at $\psi \in U$ is a symmetric n-linear map from $H^s(R)$ to $X_{l,r}^{T,s}$ and for any $h_1, ..., h_n \in H^s(R)$,

$$K^{(n)}(\phi)[h_1, ..., h_n] = \left\{ \frac{\partial^n}{\partial \xi_1 ... \partial \xi_n} K(\psi + \sum_{k=1}^{n} \xi_k h_k) \right\}_{0,...,0}.$$

Similarly, the homogeneous polynomial $K^{(n)}(\psi)[h^n]$ of degree n induced by $K^{(n)}(\psi)$ is given by

$$K^{(n)}(\psi)[h^n] = \left\{ \frac{d^n}{d\xi^n} K(\psi + \xi h) \right\}_{\xi=0}$$

for any $h \in H^s(R)$.

If we denote by

$$w_{[1,...,n]}^{(n)} = K^{(n)}(\psi)[h_1, ..., h_n] \quad \text{and } y_n = K^{(n)}(\psi)[h^n],$$

then it is easy to see that $w_{[1,...,n]}^{(n)}$ solves

$$\begin{cases} \partial_t w_{[1]}^{(1)} + \partial_x (u w_{[1]}^{(1)}) + \partial_x^3 w_{[1]}^{(1)} = 0 \\ \\ w_{[1]}^{(1)}(x, 0) = h_1 \end{cases} \tag{3.2}$$

for $n = 1$ where $u = K(\psi)$ and

$$\begin{cases} \partial_t w_{[1,...,n]}^{(n)} + \partial_x (u w_{[1,...,n]}^{(n)}) + \partial_x^3 w_{[1,...,n]}^{(n)} = -\frac{1}{2} \partial_x (\sum_{j=1}^{n-1} \sum_{1 \leq i_1 < ... < i_j \leq n} w_{[i_1,...,i_j]}^{(j)} w_{[\sigma_n(i_1,...,i_j)]}^{(n-j)}) \\ \\ w_{[1,...,n]}^{(n)}(x, 0) = 0 \end{cases} \tag{3.3}$$

for $n \geq 2$, where $\sigma_n(i_1, ..., i_j) = (k_1, ..., k_{n-j}) \leq n$ with

$$1 \leq k_1 < ... < k_{n-j} \text{ and } \{k_1, ..., k_{n-j}\} \cap \{i_1, ..., i_j\} = \emptyset.$$

As for y_n, it solves

$$\begin{cases} \partial_t y_1 + \partial_x (u y_1) + \partial_x^3 y_1 = 0 \\ \\ y_1(x, 0) = h \end{cases} \tag{3.4}$$

for $n=1$ and

$$\left\{\begin{array}{l}\partial_t y_n + \partial_x(u y_n) + \partial_x^3 y_n = -\frac{1}{2}\partial_x(\sum_{k=1}^{n-1}\left(\begin{array}{c}k\\ n\end{array}\right) y_k y_{n-k}) \\ \\ y_n(x,0)=0\end{array}\right. . \tag{3.5}$$

for $n \geq 2$.

On the other hand, we have

Proposition 3.1 *For any $u \in X_{0,0}^{T,s}$, (3.2) and (3.3) define a symmetric n- linear map $\mathcal{K}^{(n)}(u)$ from $H^s(R)$ to $X_{l,r}^{T,s}$ for any $(l,r) \in [0, s-\frac{3}{4}] \times [0, s-\frac{3}{4})$ and In addition,*

$$\|\mathcal{K}^{(n)}(u)[h_1,...,h_n]\|_{X_{l,r}^{T,s}} \leq c(n, \|u\|_{X_{0,0}^{T,s}}) \prod_{k=1}^{n} \|h_k\|_s \tag{3.6}$$

for any $h_1,...,h_n \in H^s(R)$ where $c(n, \|u\|_{X_{0,0}^{T,s}})$ continuously depends on $\|u\|_{X_{0,0}^{T,s}}$.

Proof: The existence of the n -linear map $\mathcal{K}^{(n)}$ follows from Theorem 2.1 and Lemma 2.4 easily. We just verify (3.6).

(3.6) is true for $n=1$ by applying (2.13) to (3.2). Assume that (3.6) is true for $n \leq m$. Then applying Theorem 2.1 and Lemma 2.4 to (3.3) with $n=m+1$, we have

$$\begin{aligned}
\|w_{[1,...,m+1]}^{(m+1)}\|_{X_{l,r}^{T,s}} &\leq c\sum_{j=1}^{m} \sum_{1\leq i_1<...<i_j\leq m+1} \int_{-T}^{T} \|\partial_x(w_{[i_1,...,i_j]}^{(j)} w_{[\sigma_{m+1}(i_1,...,i_j)]}^{(m+1-j)})\|_s d\tau \\
&\leq c\sum_{j=1}^{m} \sum_{1\leq i_1<...<i_j\leq m+1} \Lambda_{0,0}^s(T; w_{[i_1,...,i_j]}^{(j)})\Lambda_{0,0}^s(T; w_{[\sigma_{m+1}(i_1,...,i_j)]}^{(m+1-j)}) \\
&\leq c\sum_{j=1}^{m} \sum_{1\leq i_1<...<i_j\leq m+1} c(j, \|u\|_{X_{0,0}^{T,s}}) c(m+1-j, \|u\|_{X_{0,0}^{T,s}}) \prod_{k=1}^{m+1} \|h_k\|_s \\
&:= c(m+1, \|u\|_{X_{0,0}^{T,s}}) \prod_{k=1}^{m+1} \|h_k\|_s.
\end{aligned}$$

Thus, (3.6) is proved by induction. The proof is completed. □

Corollary 3.1 *For any $u \in X_{l,r}^{T,s}$, (3.4) and (3.5) define a homogeneous polynomial $\mathcal{K}^{(n)}(u)[h^n]$ of degree n which is induced from $\mathcal{K}^{(n)}(u)$ and*

$$\|y_n\|_{X_{l,r}^{T,s}} = \|\mathcal{K}^{(n)}(u)[h^n]\|_{X_{l,r}^{T,s}} \leq c(n, \|u\|_{X_{0,0}^{T,s}})\|h\|_s^n \tag{3.7}$$

for any $h \in H^s(R)$.

Now, formally, we may define the n-th Taylor polynomial $P_n(h)$ of K at $\psi \in U \subset H^s(R)$ as

$$\begin{aligned} P_n(h) &= K(\psi) + \sum_{k=1}^{n} \frac{1}{k!}\mathcal{K}^{(k)}(u)[h^k] \\ &= u + \sum_{k=1}^{n} \frac{1}{k!} y_k \end{aligned}$$

where $u = K(\psi)$ and y_n is the solution of (3.4) and (3.5).

Proposition 3.2 *Let z_n denote the n-th Taylor remainder of K at ψ, i.e.*

$$z_n = K(\psi + h) - P_n(h),$$

then

$$\begin{cases} \partial_t z_0 + \frac{1}{2}\partial_x((v+u)z_0) + \partial_x^3 z_0 = 0 \\ \\ z_0(x,0) = h(x) \end{cases} \tag{3.8}$$

for $n = 0$ and

$$\begin{cases} \partial_t z_n + \frac{1}{2}\partial_x((u+v)z_n) + \partial_x^3 z_n = -\frac{1}{2}\partial_x(\sum_{k=0}^{n-1} \frac{1}{(n-k)!} z_k y_{n-k}) \\ \\ z_n(x,0) = 0 \end{cases} \tag{3.9}$$

for $n \geq 1$ where $v = K(\psi + h)$ and $u = K(\psi)$.

Proof: (3.8) follows from direct computation. We prove (3.9) by induction. It is easy to see that (3.9) is true for $n = 1$. Assume that (3.9) is true for $n = m$, i.e.

$$\begin{cases} \partial_t z_m + \frac{1}{2}\partial_x((u+v)z_m) + \partial_x^3 z_m = -\frac{1}{2}\partial_x(\sum_{k=0}^{m-1} \frac{1}{(m-k)!} z_k y_{m-k}) \\ \\ z_m(x,0) = 0 \end{cases} .$$

Let

$$z_{m+1} = z_m - \frac{1}{(m+1)!} y_{m+1}$$

where y_{m+1} is the solution of (3.5) with $n = m+1$.

Then

$$\begin{cases} \partial_t z_{m+1} + \frac{1}{2}\partial_x((u+v)z_{m+1}) + \partial_x^3 z_{m+1} = -\frac{1}{2}\partial_x(H) \\ \\ z_{m+1}(x,0) = 0 \end{cases}$$

where

$$\begin{aligned} H \quad = \quad & \frac{1}{(m+1)!} z_0 y_{m+1} + \sum_{k=0}^{m-1} \frac{1}{(m-k)!} z_k y_{m-k} - \\ & - \frac{1}{(m+1)!} \sum_{k=0}^{m} \begin{pmatrix} k \\ m+1 \end{pmatrix} y_k y_{m+1-k}. \end{aligned}$$

The direct computation shows that

$$\begin{aligned} H \quad = \quad & \frac{1}{(m+1)!} z_0 y_{m+1} + \sum_{k=0}^{m-1} \frac{1}{(m-k)!}(z_k - \frac{1}{(k+1)!} y_{k+1}) y_{m-k} + \\ + \quad & \sum_{k=0}^{m-1} \frac{1}{(m-k)!} \frac{1}{(k+1)!} y_{k+1} y_{m-k} - \frac{1}{(m+1)!} \sum_{k=1}^{m} \begin{pmatrix} k \\ m+1 \end{pmatrix} y_k y_{m+1-k} \\ = \quad & \frac{1}{(m+1)!} z_0 z_{m+1} + \sum_{k=0}^{m-1} \frac{1}{(m-k)!} z_{k+1} y_{m-k} + \\ + \quad & \sum_{k=1}^{m} \frac{1}{(m+1-k)!k!} y_k y_{m+1-k} - \frac{1}{(m+1)!} \sum_{k=1}^{m} \begin{pmatrix} k \\ m+1 \end{pmatrix} y_k y_{m+1-k} \\ = \quad & \sum_{k=0}^{m} \frac{1}{(m+1-k)!} z_k y_{m+1-k} \end{aligned}$$

Therefore

$$\begin{cases} \partial_t z_{m+1} + \frac{1}{2}\partial_x((u+v)z_{m+1}) + \partial_x^3 z_{m+1} = -\frac{1}{2}\partial_x(\sum_{k=0}^{m} \frac{1}{(m+1-k)!} z_k y_{m+1-k}) \\ \\ z_{m+1}(x,0) = 0 \end{cases}$$

which is (3.9) for $n = m+1$. The proof is completed. □

Theorem 3.1 *Let $s > \frac{3}{4}$. Then, for any $\phi^* \in H^s(R)$, there exist a $T > 0$ and a neighborhood U of ϕ^* in $H^s(R)$ such that the nonlinear map K defined by the IVP (1.1) is infinitely many time Frechet differentiable in U from $H^s(R)$ to $X_{l,r}^{T,s}$. Its n-th derivative $K^{(n)}$ at $\psi \in U$ is given by*

$$K^{(n)}(\psi)[h_1, ..., h_n] = \mathcal{K}^{(n)}(u)[h_1, ..., h_n]$$

for any $h_1, ..., h_n \in H^s(R)$ where $\mathcal{K}^{(n)}(u)$ is defined by (3.2) and (3.3) with $u = K(\psi)$.

Proof: We only need to prove that for any $\psi \in U$,

$$K(\phi + h) = \sum_{k=0}^{n} \frac{1}{k!}\mathcal{K}^{(k)}(\phi)[h^k] + o(\|h\|_s^n)$$

as $h \to 0$ in $H^s(R)$ uniformly for $\|\phi - \psi\|_s \leq \|h\|_s$ by the Converse Taylor Theorem (see [8]).

Let

$$v = K(\phi + h), \qquad u = K(\phi), \qquad y^{(k)} = \mathcal{K}^{(k)}(u)[h^k]$$

for $1 \leq k \leq n$ and

$$z_0 = v - u, \qquad z_1 = z_0 - y_0, \qquad z_n = z_{n-1} - \frac{1}{n!}y_n.$$

Then, by Proposition 3.2,

$$\begin{cases} \partial_t z_0 + \frac{1}{2}\partial_x((u+v)z_0) + \partial_x^3 z_0 = 0 \\ z_0(x,0) = h \end{cases} \tag{3.10}$$

for $n = 0$,

$$\begin{cases} \partial_t z_n + \frac{1}{2}\partial_x((u+v)z_n) + \partial_x^3 z_n - \frac{1}{2}\partial_x(\sum_{k=0}^{n-1} \frac{1}{(n-k)!} z_k y_{n-k}) \\ z_n(x,0) = 0 \end{cases} \tag{3.11}$$

for $n \geq 1$.

Choose $\delta_1 > 0$ such that

$$S_{\delta_1}(\psi) = \{\phi \in H^s(R),\ \|\phi - \psi\|_s \leq \delta_1\} \subset U.$$

Note that $K(\phi)$ is bounded on $S_{\delta_1}(\psi)$ since K is continuous.

By Corollary 3.1,

$$\|y_k\|_{X^{T,s}_{l,r}} \le c(k, \|u\|_{X^{T,s}_{0,0}})\|h\|_s^k, \qquad k = 1, 2, ..., n$$

where $u = K(\phi)$ and $c(k, \|u\|_{X^{T,s}_{0,0}})$ is uniformly bounded on $S_{\delta_1}(\psi)$. It suffices to prove that

$$\|z_n\|_{X^{T,s}_{l,r}} \le \beta(n)\|h\|_s^{n+1} \tag{3.12}$$

where $\beta(n)$ is uniformly bounded for $\phi \in S_{\delta_1}(\psi)$. We verify this by induction. It is easy to obtain (3.12) for $n = 0$ by using Theorem 2.1 to (3.10). Suppose (3.12) is true for $n \le m$. Then applying Theorem 2.1 to (3.11) with $n = m+1$, we have that

$$\begin{aligned}
\|z_{m+1}\|_{X^{T,s}_{l,r}} &\le c\sum_{k=0}^{m} \frac{1}{(m+1-k)!}\int_{-T}^{T} \|\partial_x(z_k y_{m+1-k})\|_s \\
&\le c\sum_{k=0}^{m} \frac{1}{(m+1-k)!}\Lambda^s_{0,0}(T; z_k)\Lambda^s_{0,0}(T; y_{m+1-k}) \\
&\le c\sum_{k=0}^{m} \frac{1}{(m+1-k)!}\beta(k)\|h\|_s^{k+1} c(m+1-k, \|u\|_{X^{T,s}_{0,0}})\|h\|_s^{m+1-k} \\
&:= \beta(m+1)\|h\|_s^{m+2}.
\end{aligned}$$

The proof is completed. □

Corollary 3.2 (Taylor's Formula) *For any $\phi \in U$ and $h \in H^s(R)$ satisfying*

$$\phi + \xi h \in U, \qquad \textit{for any } \xi \in (0,1),$$

$$K(\phi + h) = \sum_{j=0}^{n-1} \frac{1}{j!}K^{(j)}(\phi)[h^j] + \int_0^1 \frac{(1-\xi)^{n-1}}{n!}K^{(n)}(\phi + \xi h)[h^n]d\xi$$

with any $n \ge 1$.

Proof: cf. Theorem 8.14.3 in [7].

Next we show that the map K is analytic from U to $X^{T,s}_{l,r}$ with $(l,r) \in [0, s-\frac{3}{4}] \times [0, s-\frac{3}{4})$, i.e. for any $\phi \in U$, there is a $\delta > 0$ such that

$$K(\phi + h) = K(\phi) + \sum_{k=1}^{\infty} K^{(k)}(\phi)[h^k]$$

for any $h \in H^s(R)$ with $\|h\|_s \le \delta$, and the series is uniformly convergent in $X^{T,s}_{l,r}$ for $\|h\|_s \le \delta$.

We begin with improving the estimate (3.7).

Proposition 3.3 *Let $u \in X^{T,s}_{0,0}$ be given and y_n is the solution of (3.4)-(3.5). Then there is a constant $c > 0$ only depending on $\|u\|_{X^{T,s}_{0,0}}$ and a sequence $\alpha(n)$ defined as*

$$\begin{cases} \alpha(1) = 1 \\ \\ \alpha(n) = \sum_{k=1}^{n-1} \alpha(k)\alpha(n-k), \qquad \textit{for } n \ge 2 \end{cases} \tag{3.13}$$

such that

$$\|y_n\|_{X^{T,s}_{l,r}} \le c^n \alpha(n) n! \|h\|_s^n. \tag{3.14}$$

for any given $(l, r) \in [0, s - \frac{3}{4}] \times [0, s - \frac{3}{4})$.

Proof: Let $q_n = \frac{y_n}{n!}$. It solves

$$\begin{cases} \partial_t q_1 + \partial_x(u q_1) + \partial_x^3 q_1 = 0 \\ \\ q_1(x, 0) = h \end{cases} \tag{3.15}$$

for $n \ge 1$, and

$$\begin{cases} \partial_t q_n + \partial_x(u q_n) + \partial_x^3 q_n = -\frac{1}{2}\partial_x(\sum_{k=1}^{n-1} q_k q_{n-k}) \\ \\ q_n(x, 0) = 0 \end{cases} \tag{3.16}$$

for $n \ge 2$. Applying Theorem 2.1 to (3.15) yields

$$\|q_1\|_{X^{T,s}_{l,r}} \le c_1 \|h\|_s \tag{3.17}$$

for some $c_1 > 0$ depending only on $\|u\|_{X^{T,s}_{0,0}}$. Then, by applying Theorem 2.1 and Lemma 2.4 to (3.16) with $n = 2$, we have

$$\begin{aligned} \|q_2\|_{X^{T,s}_{l,r}} &\le c_1 \frac{1}{2} \int_{-T}^{T} \|\partial_x(q_1^2)\|_s d\tau \\ &\le \frac{c_1 c_2}{2} \Lambda^s_{0,0}(T, q_1)^2 \\ &\le \frac{c_1^3 c_2}{2} \|h\|_s^2 \quad \text{by (3.17)} \end{aligned}$$

for some $c_2 > 0$ independent of u and h. If we assume that

$$\|q_m\|_{X_{l,r}^{T,s}} \leq \frac{c_1^{2m-1}c_2^{m-1}}{2^{m-1}}\alpha(m)\|h\|_s^m$$

for $1 \leq m \leq N$, then, similarly,applying Theorem 2.1 and Lemma 2.4 to (3.16) with $n = N+1$,

$$\begin{aligned}
\|q_{N+1}\|_{X_{l,r}^{T,s}} &\leq \frac{c_1}{2}\sum_{k=1}^{N}\int_{-T}^{T}\|\partial_x(q_k q_{N+1-k})\|d\tau \\
&\leq \frac{c_1 c_2}{2}\sum_{k=1}^{N}\Lambda_{0,0}^s(T,q_k)\Lambda_{0,0}^s(T,q_{N+1-k}) \\
&\leq \frac{c_1 c_2}{2}\frac{c_1^{2m}c_2^{N-1}}{2^N}\|h\|_s^{N+1}\sum_{k=1}^{N}\alpha(k)\alpha(N+1-k) \\
&= \frac{c_1^{2(N+1)-1}c_2^N}{2^m}\|h\|_s^{N+1}\alpha(N+1).
\end{aligned}$$

Thus we have proved by induction that

$$\|q_n\|_{X_{l,r}^{T,s}} \leq \frac{c_1^{2n-1}c_2^n}{2^n}\alpha(n)\|h\|_s^n,$$

which is (3.14) by choosing $c^n = \frac{c_1^{2n+1}c_2^n}{2^n}$. The proof is completed, □

Proposition 3.4 *For the sequence $\alpha(n)$ defined by (3.13), we have*

$$\alpha(n) = \frac{2^{n-1}(2n-3)!!}{n!} \tag{3.18}$$

where

$$(2n-3)!! = (2n-3)(2n-3)....(3)(1).$$

Thus

$$\alpha(n) \leq \frac{4^{n-1}}{n}, \qquad \text{for any } n \geq 2$$

or more precisely, by Sterling's formula

$$\alpha(n) \sim \frac{2^{2n-1}}{\sqrt{\pi n}(2n-1)} \qquad \text{as } n \to \infty.$$

Proof: Let

$$\beta_1 = 1 \text{ and } \beta_n = \frac{2^{n-1}(2n-3)!!}{n!}$$

for $n \geq 2$. Define

$$y(x) = \sum_{n=2}^{\infty} \beta_n x^n$$

which is convergent for any $|x| < 1/4$. It is easy to check that

$$y(x) = \frac{1}{2} - x - \frac{(1-4x)^{1/2}}{2}$$

and

$$y(x) = (x + y(x))^2$$

for any $|x| < \frac{1}{4}$ Moreover

$$\begin{aligned} (x + y(x))^2 &= (\sum_{n=1}^{\infty} \beta_n x^n)^2 \\ &= \sum_{n=2}^{\infty} (\sum_{k=1}^{n-1} \beta_k \beta_{n-k}) x^n \end{aligned}$$

for any $|x| < 1/4$, i.e.

$$\sum_{n=2}^{\infty} \beta_n x^n = \sum_{n=2}^{\infty} (\sum_{k=1}^{n-1} \beta_k \beta_{n-k}) x^n,$$

which implies that

$$\beta_n = \sum_{k=1}^{n-1} \beta_k \beta_{n-k}$$

for $n = 2, 3, \ldots$. Thus

$$\alpha(n) = \beta_n = \frac{2^{n-1}(2n-3)!!}{n!}$$

for $n = 2, 3, \ldots$. The proof is completed. □

Corollary 3.3 *Under the assumption of Proposition 3.4,*

$$\|y_n\|_{X_{l,r}^{T,s}} \leq c^n n! \|h\|_s^n \tag{3.19}$$

for $n \geq 1$ *where* c *is a constant only depending on* $\|u\|_{X_{0,0}^{T,s}}$.

Theorem 3.2 *Let $s > 3/4$ be given. For any $\psi^* \in H^s(R)$, there is a $T > 0$ and a neighborhood U of ψ^* in $H^s(R)$ such that the IVP (1.1) defines an analytic nonlinear map K from U to $X^{T,s}_{l,r}$ for any $(l,r) \in [0, s-\frac{3}{4}] \times [0, s-\frac{3}{4})$, i.e., for any $\phi \in U$, there is a $\delta > 0$ such that if $h \in H^s(R)$ with $\|h\|_s \leq \delta$, then*

$$K(\phi + h) = K(\phi) + \sum_{n=1}^{\infty} \frac{1}{n!} K^{(n)}(\phi)[h^n] \tag{3.20}$$

where the series (3.20) converges uniformly for $\|h\|_s \leq \delta$ in the space $X^{T,s}_{l,r}$. If we let $y_k = K^{(k)}(\phi)[h^k]$, $u = K(\phi)$, then

$$\begin{cases} \partial_t y_1 + \partial_x(u y_1) + \partial_x^3 y_1 = 0 \\ y_1(x,0) = h(x) \end{cases} \tag{3.21}$$

for $n = 1$ and

$$\begin{cases} \partial_t y_n + \partial_x(u y_n) + \partial_x^3 y_n = -\frac{1}{2}\partial_x(\sum_{k=1}^{n-1} \begin{pmatrix} k \\ n \end{pmatrix} y_k y_{n-k}) \\ y_n(x,0) = 0 \end{cases} . \tag{3.22}$$

for $n \geq 2$.

Proof: Denote by $v = K(\phi + h)$ with $\|h\|_s \leq 1$, $z_0 = v - u$ and

$$z_n = z_{n-1} - \frac{1}{n!} y_n \text{ for } n \geq 1.$$

By Proposition 3.2,

$$\begin{cases} \partial_t z_0 + \frac{1}{2}\partial_x((u+v)z_0) + \partial_x^3 z_0 = 0 \\ z_0(x,0) = h \end{cases} \tag{3.23}$$

for $n = 0$ and

$$\begin{cases} \partial_t z_n + \frac{1}{2}\partial_x((u+v)z_n) + \partial_x^3 z_n = \frac{1}{2}\partial_x(\sum_{k=0}^{n-1} \frac{1}{(n-k)!} z_k y_{n-k}) \\ z_n(x,0) = 0 \end{cases} \tag{3.24}$$

for $n \geq 1$.

It is easy to see by applying Theorem 2.1 to (3.23) that

$$\|z_0\|_{X_{l,r}^{T,s}} \leq c\|h\|_s$$

Similarly, applying Theorem 2.1 and Lemma 2.4 to (3.24) with $n = 1$ yields

$$\begin{aligned} \|z_1\|_{X_{l,r}^{T,s}} &\leq c\Lambda_{0,0}^s(T, z_0)\Lambda_{0,0}^s(T, y_1) \\ &\leq c^3\|h\|_s^2 \end{aligned}$$

We assume that for $2 \leq k \leq m$,

$$\|z_k\|_{X_{l,r}^{T,s}} \leq 2^{(k-1)}c^{2k+1}\|h\|_s^{k+1} \tag{3.25}$$

Then, applying Theorem 2.1 and Lemma 2.4 to (3.24) with $n = m + 1$, we have

$$\begin{aligned} \|z_{m+1}\|_{X_{l,r}^{T,s}} &\leq c\sum_{k=0}^{m} \Lambda_{0,0}^s(T, z_k)\Lambda_{0,0}^s(T, y_{m+1-k})\frac{1}{(m+1-k)!} \\ &\leq c^2\|h\|_s c^{m+1}\|h\|_s^{m+1} + c\sum_{k=1}^{m} 2^{k-1}c^{2k+1}c^{m+1-k}\|h\|_s^{k+1}\|h\|_s^{m+1-k} \\ &\leq c^{2(m+1)+1}\|h\|_s^{m+1+1}(1 + \sum_{k=1}^{m} 2^{k-1}) \\ &= c^{2(m+1)+1}2^{m+1-1}\|h\|_s^{m+1+1} \end{aligned}$$

Thus we prove by induction that

$$\|z_n\|_{X_{l,r}^{T,s}} \leq c^{2n+1}2^{n-1}\|h\|_s^{n+1}$$

for $n \geq 1$ where the constant $c > 0$ only depends on $u = K(\phi)$, and therefore only on $\phi \in H^s(R)$.

If we choose $\delta = \frac{1}{4c^2}$, then

$$\begin{aligned} \|K(\psi + h) - K(\psi) - \sum_{k=1}^{n} K^{(k)}(\psi)[h^k]\|_{X_{l,r}^{T,s}} &= \|z_n\|_{X_{l,r}^{T,s}} \\ &\leq c^{(2n+1)}2^{n-1}\|h\|_s^{n+1} \\ &\leq (\frac{1}{2})^n \end{aligned}$$

for any h with $\|h\|_s < \delta$. This implies that the series (3.20) is uniformly convergent in $X_{l,r}^{T,s}$ for any $h \in H^s(R)$ with $\|h\|_s \leq \delta$. The proof is completed. $\square$.

Corollary 3.4 *For given $T > 0$ and $s \geq 1$, the map K defined by the IVP (1.1) is analytic from $H^s(R)$ to $C([-T,T]; H^s(R))$.*

4 Periodic KdV equation

In this section, we consider the IVP for the periodic KdV equation

$$\begin{cases} \partial_t u + u\partial_x u + \partial_x^3 u = 0, & x \in S,\ t \in R \\ u(x,0) = \psi(x) \end{cases} \tag{4.1}$$

where S is a unit circle. Let $H^s(S)$ denote the real Sobolev space of order s ($s \geq 0$) which may be charaterised as the space of real periodic functions

$$v(x) = \sum_{-\infty}^{+\infty} v_k exp(2i\pi kx)$$

such that

$$\left\{ \sum_{-\infty}^{+\infty} (1+k^2)^s |v_k|^2 \right\}^{1/2} < +\infty. \tag{4.2}$$

The left-hand side of (4.2) is a Hilbert norm for $H^s(S)$ denoted also $\|v\|_s$.

It is well-known (cf. [2] and [35]) that for any $T > 0$ and $\psi \in H^s(S)$ ($s \geq 2$), there is a unique solution $u \in C([-T,T]; H^s(S))$ to (4.1). Therefore (4.1) defines a nonlinear map K_p from $H^s(S)$ to $C([-T,T]; H^s(S))$ for any $T > 0$. K_p is also known being continuous ([2], [17]) for many years. Saut and Temam [35] showed that K_p is Hölder continuous with exponent 1/2 if it is considered as a map from $H^{s+1/2}(S)$ to $L^\infty(0,T; H^s(S))$.

We shall show in this section that K_p is Lipschitz continuous from $H^{s+1}(S)$ to $C([-T,T], H^s(S))$ and is n times Frechet differentiable from $H^{s+n+1}(S)$ to $C([-T,T]; H^s(S))$ for any $n \geq 1$.

The following " fractional type Leibniz formula " is due to Saut and Temam [35].

Lemma 4.1 *Let $u,\ v \in H^s(S)$, $s > 1$, $r > 1/2$. Then*

$$\|D^s(uv) - uD^s v\| \leq c(r,s)\{\|u\|_s \|v\|_r + \|u\|_{r+1}\|v\|_{s-1}\} \tag{4.3}$$

As an application of Lemma 4.1, we have

Lemma 4.2 *Let $T > 0$ and $s > 1$ be given. Then*

$$\int_{-T}^{T} \|\partial_x(uv)\|_s d\tau \leq cT \sup_{[-T,T]} \|u\|_{s+1} \sup_{[-T,T]} \|v\|_{s+1} \tag{4.4}$$

for any $u,\ v \in C([-T,T]; H^s(S))$ where c is independent of u, v and T.

Proof: It is easy to see that

$$\begin{aligned} \|\partial_x(uv)\|_s &\leq \|v\partial_x u\|_s + \|u\partial_x v\|_s \\ &\leq c\{\|D^s(v\partial_x u)\| + \|D^s(u\partial_x v)\| + \|v\partial_x u\| + \|u\partial_x v\|\}. \end{aligned}$$

By Lemma 4.1,

$$\begin{aligned} \|D^s(v\partial_x u)\| &= \|vD^s\partial_x u + D^s(v\partial_x u) - vD^s\partial_x u\| \\ &\leq \|vD^s\partial_x u\| + \|D^s(v\partial_x u) - vD^s\partial_x u\| \\ &\leq \|v\|_1\|u\|_{s+1} + c(r,s)\{\|v\|_s\|\partial_x u\|_r + \|v\|_{r+1}\|u\|_s\} \\ &\leq c\|v\|_{s+1}\|u\|_{s+1} \end{aligned}$$

for some constant $c > 0$. Similarly,

$$\|D^s(u\partial_x v)\| \leq c\|u\|_{s+1}\|v\|_{s+1}$$

Hence

$$\|\partial_x(uv)\|_s \leq c\|u\|_{s+1}\|v\|_{s+1}$$

and

$$\int_{-T}^{T} \|\partial_x(uv)\|_s d\tau \leq cT \sup_{[-T,T]} \|u\|_{s+1} \sup_{[-T,T]} \|v\|_{s+1}.$$

The proof is completed. □

Consider the following linear problem

$$\begin{cases} \partial_t u + \partial_x(a(x,t)u) + \partial_x^3 u = f, \quad x \in S,\ t \in R \\ u(x,0) = \psi(x) \end{cases} \tag{4.5}$$

The following proposition is parallel to Theorem 2.1.

Proposition 4.1 *Let $s \geq 2$, $T > 0$ and $a = a(x,t) \in C([-T.T]; H^{s+1}(S))$ be given. If $\psi \in H^s(S)$ and $f \in L^1([-T,T]; H^s(S))$, then there exists a unique solution $u \in C([-T,T]; H^s(S))$ to (4.5). Moreover,*

$$\sup_{[-T,T]} \|u(.,t)\|_s \leq \beta\{\|\psi\|_s + \int_{-T}^{T} \|f(.,\tau)\|_s d\tau\} \tag{4.6}$$

where β depends only on $\sup_{[-T,T]} \|a(.,t)\|_{s+1}$ continuously.

Proof: The existence and uniqueness of the solution u follow from the standard semigroup theory (cf. [15]). We only need to prove (4.6).

Without loss of generality, we may assume that a, ψ and f are smooth functions, and therefore the solution u is smooth. Otherwise, we replace them by smooth ϵ approximation a_ϵ, ψ_ϵ and f_ϵ, respectively, and let $\epsilon \to 0$ after the estimate is obtained.

Applying D^s to each member of the first equation of (4.5) and taking the L^2-scalar product with $D^s u$, we get

$$\frac{1}{2}\frac{d}{dt}\|D^s u\|^2 + (D^s \partial_x(au), D^s u) = (D^s f, D^s u)$$

where $(.,.)$ denotes the inner product in $L^2(S)$. Note that

$$(D^s \partial_x(au), D^s u) = (D^s(u\partial_x a), D^s u) + (D^s(a\partial_x u), D^s u)\,,$$

$$\begin{aligned}(D^s(a\partial_x u), D^s u) &= (aD^s\partial_x u, D^s u) + ([D^s; a]\partial_x u, D^s u) \\ &= -\frac{1}{2}(\partial_x a D^s u, D^s u) + ([D^s; a]\partial_x u, D^s u)\end{aligned}$$

and

$$(D^s(u\partial_x a), D^s u) = (uD^s\partial_x a, D^s u) + ([D^s; u]\partial_x a, D^s u)\,.$$

Then it follows from Lemma 4.1 that

$$\begin{aligned}|\,(D^s(a\partial_x u), D^s u)\,| &\leq \frac{1}{2}|\,(\partial_x a D^s u, D^s u)\,| + |\,([D^s; u]\partial_x u, D^s u)\,| \\ &\leq c\Big\{\|\partial_x a\|_\infty \|D^s u\|^2 + \|a\|_s \|u\|_{1+r}\|D^s u\| + \\ &+ \|\partial_x u\|_{1+r}\|u\|_{s-1}\|D^s u\|\} \\ &\leq c\Big\{\|a\|_{s+1}\|D^s u\|^2 + \|a\|_s\|u\|_s^2 + \|a\|_{s+1}\|u\|_s^2\Big\} \\ &\leq c\|a\|_{s+1}\|u\|_s^2\end{aligned}$$

and similarly,

$$|\,(D^s(u\partial_x a), D^s u)\,| \le c\|a\|_{s+1}\|u\|_s^2.$$

Thus

$$\frac{d}{dt}\|u\|_s \le c\|a\|_{s+1}\|u\|_s + \|f\|_s$$

which implies that

$$\sup_{[-T,T]}\|u(.,t)\|_s \le c(\|\psi\|_s + \int_{-T}^{T}\|f\|_s d\tau) exp\left\{\int_{-T}^{T}\|a\|_{s+1}d\tau\right\}.$$

The proof is completed. □

Theorem 4.1 *Let $s \ge 2$ and $T > 0$ be given. Then*

(i) The nonlinear map K_p defined by the IVP (4.1) is Lipschitz continuous from $H^{s+1}(S)$ to $C([-T,T]; H^s(S))$.

(ii). The map K_p is n times Frechet differentiable from $H^{s+n+1}(S)$ to $C([-T,T]; H^s(S))$ for any $n \ge 1$.

(iii). For any given $\psi \in H^{s+1+n}(S)$, let

$$w^{(n)}_{[1,...,n]} = K_p^{(n)}(\psi)[h_1,...,h_n].$$

where $h_k \in H^{s+n+1}(S)$ ($k = 1,2,...,n$), then

$$\begin{cases} \partial_t w^{(1)}_{[1]} + \partial_x(uw^{(1)}_{[1]}) + \partial_x^3 w^{(1)}_{[1]} = 0, \quad x \in S,\ t \in R \\ \\ w^{(1)}_{[1]}(x,0) = h_1(x) \end{cases} \tag{4.7}$$

for $n = 1$ and

$$\begin{cases} \partial_t w^{(n)}_{[1,...,n]} + \partial_x(uw^{(n)}_{[1,...,n]}) + \partial_x^3 w^{(n)}_{[1,...,n]} = -\frac{1}{2}\partial_x(\sum_{j=1}^{n-1}\sum_{1\le i_1<...<i_j\le n} w^{(j)}_{[i_1,...,i_j]} w^{(n-j)}_{[\sigma_n(i_1,...,i_j)]}) \\ \\ w^{(n)}_{[1,...,n]}(x,0) = 0 \end{cases} \tag{4.8}$$

with $x \in S$, $t \in R$ for $n \ge 2$.

Denote by

$$y_n = K_p^{(n)}(\psi)[h^n]$$

with $h \in H^{s+n+1}(S)$*, then*

$$\begin{cases} \partial_t y_1 + \partial_x(uy_1) + \partial_x^3 y_1 = 0, \quad x \in S,\ t \in R \\ \\ y_1(x,0) = h(x) \end{cases} \tag{4.9}$$

for $n = 1$ *and*

$$\begin{cases} \partial_t y_n + \partial_x(uy_n) + \partial_x^3 y_n = -\frac{1}{2}\partial_x(\sum_{k=1}^{n-1} \begin{pmatrix} k \\ n \end{pmatrix} y_k y_{n-k}) \\ \\ y_n(x,0) = 0 \end{cases} \tag{4.10}$$

for $n \geq 2$ *where* $u = K_p(\psi)$ *is the solution of (4.1).*

Proof: Let $v = k_p(\psi + h)$ and $z_0 = v - u$. Then

$$\begin{cases} \partial_t z_0 + \frac{1}{2}\partial_x((u+v)z_0) + \partial_x^3 z_0 = 0 \\ \\ z_0(x,0) = h(x) \end{cases} \tag{4.11}$$

Note that $u,\ v \in C([-T,T]; H^{s+1}(S))$ if $h,\ \psi \in H^{s+1}(S)$. Applying (4.6) to (4.11), we have

$$\sup_{[-T,T]} \|z_0\|_s = \sup_{[-T,T]} \|u(.,t) - v(.,t)\|_s \leq \beta\|h\|_s \leq \beta\|h\|_{s+1}$$

where $\beta = \beta(\|u+v\|_{s+1})$, which implies that K_p is Lipschitz continuous from $H^{s+1}(S)$ to $C([-T,T]; H^s(S))$.

As in section 3, if K_p is n times Frechet differentiable from $H^{s+n+1}(S)$ to $C([-T,T]; H^s(S))$, then $K_p^{(n)}(\psi)$ is an n-linear map from $H^{s+n+1}(S)$ to $C([-T,T; H^s(S))$ and

$$w^{(n)}_{[1,...,n]} = K^{(n)}(\psi)[h_1, ..., h_n]$$

solves (4.7)-(4.8). Similarly,

$$y_n = K^{(n)}(\psi)$$

solves (4.9)- (4.10). On the other hand, according to Proposition 4.1, (4.7)- (4.8) defines a continuous n-linear map from $H^{s+1+n}(S)$ to $C([-T,T]; H^s(S))$ and similarly, (4.9)-(4.10) defines a continuous homogeneous polynomial of degree n. More precisely, if $\psi,\ h \in H^{s+n+1}(S)$, then

$$y_k \in C([-T.T]; H^{s+n+1-k}(S)), \quad k = 1, 2, ..., n$$

and

$$\sup_{[-T,T]} \|y_k\|_{s+n+1-k} \le c(k, \|\psi\|_{s+n+1})\|h\|^k_{s+n+1} \tag{4.12}$$

for $k = 1, 2..., n$ where $c(k, \|\psi\|_{s+n+1})$ is bounded if ψ runs through a bounded set in $H^{s+n+1}(S)$.

By the Converse Taylor Theorem, it suffices to show that

$$K_p(\phi + h) = K_p(\phi) + \sum_{k=1}^{n} K_p^{(k)}(\phi)[h^k] + o(\|h\|^n_{n+1+s}) \tag{4.13}$$

as $h \to 0$ uniformly for $\|\phi - \psi\|_{s+n+1} \le \|h\|_{s+n+1}$ in order to prove that K_p is n times Frechet differentiable from $H^{s+n+1}(S)$ to $C([-T, T]; H^s(S))$.

Let $v = K_p(\phi+h)$ and $u = K_p(\phi)$. Both v and u are bounded in $C([-T, T]; H^{s+n+1}(S))$ if $\phi,\ h$ satisfy

$$\|\phi - \psi\|_{s+n+1} \le \|h\|_{s+n+1} < 1.$$

Denote by $z_0 = v - u$, $y_k = K_p^{(k)}(\phi)[h^k]$ and

$$z_k = z_{k-1} - \frac{1}{k!} y_k, \qquad k = 1, 2, .., n$$

Then y_n solves (4.9) for $n = 1$ and (4.10) for $n \ge 2$. As for z_k, $0 \le k \le n$, it solves (4.11) for $k = 0$ and solves

$$\begin{cases} \partial_t z_k + \frac{1}{2}\partial_x((u + v)z_k) + \partial_x^3 z_k = -\frac{1}{2}\partial_x(\sum_{j=0}^{k-1} \frac{1}{(k-j)!} z_j y_{k-j}) \\ \\ z_k(x, 0) = 0 \end{cases} \tag{4.14}$$

for $1 \le k \le n$.

Applying (4.6) to (4.11) yields

$$z_0 \in C([-T, T]; H^{s+n}(S))$$

and

$$\|z_0\|_{s+n} \le \beta(0, \|\phi\|_{s+n+1})\|h\|_{s+n+1}$$

since $h, u = K_p(\phi) \in H^{s+n+1}(S)$.

Assume that

$$z_k \in C([-T,T]; H^{s+n+1-(k+1)}(S))$$

with

$$\sup_{[-T,T]} \|z_k\|_{s+n-k} \le \beta(k, \|\phi\|_{s+n+1})\|h\|_{s+n+1}^{k+1}$$

for $1 \le k \le n-1$.

Then, by applying (4.6) to (4.14), we have

$$\begin{aligned}
\sup_{[-T,T]} \|z_n\|_s &\le c\sum_{k=0}^{n-1} \frac{1}{(n-k)!}\int_{-T}^{T} \|\partial_x(z_n y_{n+1})\|_s dt \\
&\le c\sum_{k=0}^{n-1} \frac{1}{(n-k)!} \sup_{[-T,T]} \|z_k\|_{s+1} \sup_{[-T,T]} \|y_{n-k}\|_{s+1} \\
&\le c\sum_{k=0}^{n-1} \frac{1}{(n-k)!}\beta(k, \|\phi\|_{s+n+1})c(n-k, \|\phi\|_{s+n+1})\|h\|_{s+n+1}^{n+1} \\
&:= \beta(n, \|\phi\|_{s+n+1})\|h\|_{s+n+1}^{n+1}
\end{aligned}$$

where $\beta(n, \|\phi\|_{s+n+1})$ is bounded if ϕ runs through a bounded set in $H^{s+n+1}(S)$. Thus we have proved by induction that

$$\|K_p(\phi+h) - K_p(\phi) - \sum_{k=1}^{n} \frac{1}{k!}K_p^{(k)}(\phi)[h^k]\|_s \le \beta(n, \|\phi\|_{s+n+1})\|h\|_{s+n+1}^{n+1}$$

which implies (4.13). The proof is completed. □

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