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Transmission Fault-Tolerance of Iterated Line Digraphs

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Abstract

Many interconnection networks can be constructed with line digraph iterations. In this paper, we will establish a general result on super line-connectivity based on the line digraph iteration which improves and generalizes several existing results in the literature.

Key Words: line digraph iterations, super line-connectivity, interconnection networks.

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1 Introduction

When an interconnection network contains possible link-fault there are two fault-tolerance measures in the literature.

The deterministic measure is the maximum number of faulty links which, in any case, cannot disconnect the network. This measure is called the *line-connectivity*.

The probabilistic measure is the probability of the network being connected when links fail with certain probabilistic distribution. Let F be the family of all line-cuts of a digraph G. By the exclusion-inclusion principle,

$$\begin{aligned} \text{Prob } (G \text{ connected}) &= 1 - \text{Prob } (G \text{ disconnected}) \\ &= 1 - \sum_{c \in F} \text{Prob } (c) + \sum_{c_1, c_2 \in F, c_1 \neq c_2} \text{Prob } (c_1 \cup c_2) - \ldots \end{aligned}$$

When all links are independent, Prob (c) (respectively Prob $(c_1 \cup c_2)$) is a product of failure probabilities of links in c (respectively in $c_1 \cup c_2$). Therefore, if every link has the same fault probability of a small number, then Prob $(G \text{ connected})$ depends mainly on the number of the minimum line-cuts.

Consider a digraph G with line-connectivity c. If a vertex of G has c in-links (or c out-links) other than loops, then those c in-links (out-links) form a line-cut of size c. Those line-cuts are called *natural line-cuts*. A digraph G is said to have *super* line-connectivity c if its line-connectivity is c and every line-cut of size c is natural. Clearly, the super line-connected digraph, in some sense, reaches maximum fault-tolerance.

Given a degree bound d, many constructions have been found in the literature to achieve the maximum connectivity d and near-minimum diameter [16, 5], including Kautz digraphs, cyclically-modified de Bruijn digraphs, generalized cycles, etc. Do they also have super line-connectivity? This is an interesting question. Indeed, several related research works have been published in the literature [17, 3].

In this paper, we study the super line-connectivity with line digraph iterations. In fact, many interconnection networks can be constructed with line digraph iterations, such as de Bruijn digraphs [2], Kautz digraphs [12], some of generalized de Bruijn digraphs [5, 13], Imase-Itoh digraphs [10, 11, 9], large bipartite digraphs[15], and large generalized cycles[7]. We will show that the super line-connectivity can be generally established through line digraph iterations.

2 Main Results

Consider a d-regular digraph G, that is, every vertex of G has in-degree d and out-degree d. Suppose each vertex of G has at most one loop. A vertex with a loop is called a *loop-vertex*. A *cyclic modification* of G is a digraph obtained from G by deleting all loops and connecting all loop-vertices into a cycle.

Lemma 2.1. *Let $d \geq 2$. Suppose G is a d-regular digraph that each vertex has at most one loop. Then every cyclic modification of G has super line-connectivity d if and only if G satisfies the following conditions:*

(a) G has super line-connectivity at least $d-1$, and

(b) every line-cut of size d breaks the vertex set of G into two parts A and B such that either every part contains a loop-vertex or one of A and B is a singleton.

Proof. For sufficiency, assume G has properties (a) and (b). Consider a line-cut C of size d in a cyclic modification G^* of G. Suppose C breaks the vertex set of G^* into two parts A and B such that every link from A to B belongs to C. By (b), we have two cases.

Case 1. Both A and B contain at least one loop-vertex. Then, C must contain a link e from the cycle $G^* \setminus G$. Then, $C - \{e\}$ forms a line-cut of G. By (a), $C - \{e\}$ must be natural. Thus, either A or B contains only one vertex, so C must be natural in G^*.

Case 2. Either A or B contains only one vertex. Thus, C must be natural in G^*.

For necessity, we first assume that G does not have property (b). This means that there exists a line-cut C of size d which breaks the vertex set of G into two parts A and B such that $|A| \geq 2$, $|B| \geq 2$, and either A or B contains no loop-vertex. Clearly, C is also an evidence to witness that G^* has no super line-connectivity d.

Now, we assume that G does not have property (a). Suppose C is a line-cut of size $d-1$ which breaks the vertex set of G into two parts A and B such that $|A| \geq 2$, $|B| \geq 2$, and all links from A to B belong to C. We connect all loop-vertices in A into a path P_A and all loop-vertices in B into a path P_B, and then connect two path into a cycle Q. With this cycle, we can obtain a cyclic modification G^* of G such that C together with the link in the cycle Q from A to B form a line-cut of size d for G^*, which witnesses that G^* has no super line-connectivity d. ☐

We should be careful with the case $d = 1$. In fact, Lemma 2.1 does not hold for $d = 1$. For a counterexample, consider a digraph G consisting of disjoint union of two loops and a cycle of size three. The cyclic modification of G is not connected. In fact, when $d-1=0$, the condition (a) is vague.

It is worth mentioning that if G has no loop, then conditions (a) and (b) are equivalent to the fact that G has super line-connectivity d. In fact, it follows from (a) that G has line-connectivity d. It then follows from (b) that every line-cut of size d is natural. Hence, G has super line-connectivity d. Conversely, if G has super line-connectivity d, then (a) and (b) hold trivially.

For any digraph $G=(V,E)$, we denote by $L(G)$ the line digraph of G defined as follows: The vertex set of $L(G)$ is E. For $(a,b),(c,d)\in E$, there exists a link in $L(G)$ from (a,b) to (c,d) if and only if $b=c$. For any natural number $k\geq 1$, recursively define $L^k(G)=L(L^{k-1}(G))$, where $L^0(G)=G$.

Theorem 2.2. *Let G be a d-regular digraph where each vertex has at most one loop. If every cyclic modification of G has super line-connectivity d, then for $k\geq 1$, every cyclic modification of $L^k(G)$ also has super line-connectivity d unless $d=2$ and G contains a loop.*

Proof. For $d=1$, since G is d-regular, G consists of disjoint union of cycles. If G has no loop, then G is a cycle since G has super line-connectivity 1. Thus, for every $k\geq 1$, $L^k(G)$ is a cycle and hence has super line-connectivity d. If G has a loop, then every cycle in G is a loop because every cyclic modification of G has super line-connectivity 1. This means that every vertex of G has a loop and so does every vertex of $L^k(G)$ for $k\geq 1$. Hence, every cyclic modification of $L^k(G)$ has super line-connectivity 1.

Next, we assume $d\geq 2$. By Lemma 2.1, it suffices to show that if G has properties (a) and (b), then $L(G)$ has properties (a) and (b). The fact that $L^k(G)$ satisfies (a) and (b) then follows by induction.

To do so, consider a minimum line-cut C of $L(G)$. Since G is d-regular, $L(G)$ is also d-regular. Hence, the line-connectivity of $L(G)$ is at most d, i.e., $|C|\leq d$. Suppose C breaks the vertex set of $L(G)$ into two parts A and B such that no link other than those in C is from A to B. Let

$$\begin{aligned}
U &= \{(u,v)\mid ((u,v),(v,w))\in C,\ \text{for some vertex } w \text{ of } G\ \}\\
W &= \{(v,w)\mid ((u,v),(v,w))\in C,\ \text{for some vertex } u \text{ of } G\ \}\\
V &= \{v\mid ((u,v),(v,w))\in C,\ \text{for some vertices } u,w \text{ of } G\ \}
\end{aligned}$$

Our plan is to first show that C is a natural line-cut of $L(G)$ of size at least $d-1$, namely $L(G)$ satisfies condition (a). We show several claims as follows.

Claim 1. *If* $|V| \geq 2$*, then* $|A| > |C|$ *and* $|B| > |C|$.

Proof. Note that for each $v \in V$, there are d out-links and d in-links at v. Each of the out-links belongs to either W or A and each of the in-links belongs to either U or B. Moreover, for each $v \in V$, there exists at least one in-link in A and at least one out-link in B. Therefore, when $|V| \geq 2$ and no loop-vertex exists in V, we have $|A| \geq 2(d+1) - |C| > |C|$ and $|B| \geq 2(d+1) - |C| > |C|$. When there exists loop-vertex in V, $L(G)$ must have a loop-vertex. Therefore, $|C| \leq d - 1$. Therefore, $|A| \geq 2d - |C| > |C|$ and $|B| \geq 2d - |C| > |C|$. □

Claim 2. $|V| = 1$.

Proof. For contradiction, suppose $|V| \geq 2$. By Claim 1, $|A| > |C|$ and $|B| > |C|$. Since $|A| > |C|$, U is a vertex-cut of $L(G)$ and hence a line-cut of G. If G has no loop, then G has super line-connectivity d. Thus, $|U| \geq d$. Note that $|U| \leq |C| \leq d$. Therefore, $|U| = |C| = d$ and hence U is a natural line-cut of G. If G has a loop, then $L(G)$ has a loop. Hence, $|U| \leq |C| \leq d - 1$. However, in this case, G has super line-connectivity $d - 1$. Therefore, $|U| = |C| = d - 1$ and U is a natural line-cut of G. Similarly, we can show that W is natural and $|W| = d$ if G has no loop and $d - 1$ if G has a loop. Hence, we have $|C| = |U| = |W|$. It follows that any two links in

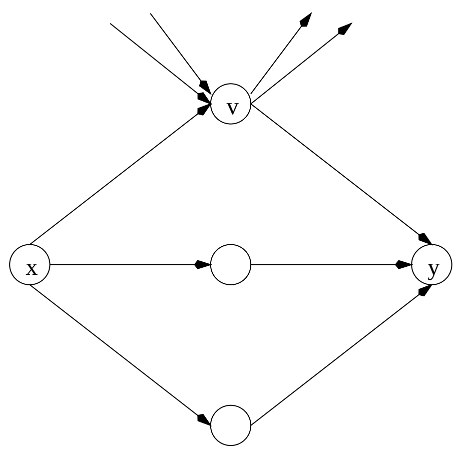

Figure 1: $|C| = |U| = |W| = |V|$.

U cannot share the same ending vertex (recall the assumption that $|V| \geq 2$). Therefore, U must consist of out-links at a vertex x and W must consist of in-links at a vertex y (Fig. 1). It also follows that $|V| = |C|$. We next show that any v in V is not a loop-vertex. In fact, for otherwise, suppose some $v \in V$ has a loop. Then the loop being in A would introduce a link $((v, v), (v, y))$ from A to B, but not in C, and the loop being in B would introduce a link $((x, v), (v, v))$ from A to B, but not in C, a contradiction.

Note that at each $v \in V$, every in-link other than (x, v) belongs to B and every out-link other than (v, y) belongs to A. Those links induce $(d-1)^2$ links in $L(G)$ from B to A. Therefore, there exist at least $|C|(d-1)^2$ links in $L(G)$ from B to A. However, as $L(G)$ is d-regular, every vertex in $L(G)$ has the same in-degree and out-degree. It follows that in $L(G)$ the number of links from B to A equals the number of links from A to B. Therefore, $|C|(d-1)^2 \leq |C|$. It follows that $d = 2$ (Fig. 2). Since $|C| = |V| \geq 2$, we must have $|C| = 2$. Thus, we may write $V = \{v_1, v_2\}$. Note that every path from v_1 to y in G, not containing link (v_1, y), must pass

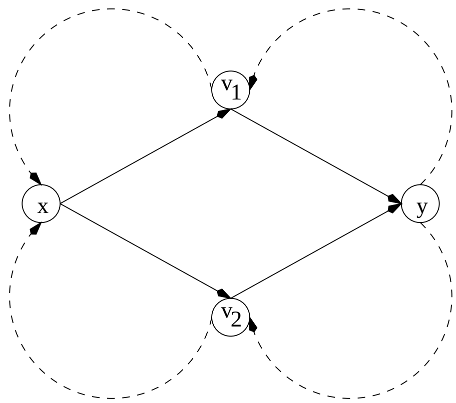

Figure 2: $d = 2$.

through vertex x and hence must contain link (x, v_2). To see this, suppose P is a path from v_1 to y not going through x. It is clear that P must go through v_2. The first link in P is an out-link of v_1 and thus it is in A. The last link in P is (v_2, y) which is in B. Hence, there must be a transition from A to B along the way. Thus, P induces a link of $L(G)$ connecting A to B which is not in C. This means that (v_1, y) and (x, v_2) form a line-cut of size $|C| = d = 2$, which is not natural. However, $|C| = d$ implies that $L(G)$ has no loop and hence G has no loop. It follows that G has super line-connectivity d, contradicting the existence of an un-natural line-cut of size d. $\square$

Claim 3. $|U| = 1$ *or* $|W| = 1$.

Proof. For contradiction, suppose $|U| \geq 2$ and $|W| \geq 2$. By Claim 2, $|V| = 1$, i.e., $V = \{v\}$. This means that for any $(u, v) \in U$ and $(v, w) \in W$, $((u, v), (v, w)) \in C$. It follows that $|U| \cdot |W| = |C|$.

Since $|U| \geq 2$ and $|W| \geq 2$, we have $|C| - |W| = (|U| - 1)|W| \geq |W|$ and $|C| - |U| = |U|(|W| - 1) \geq |U|$. Therefore, $(|C| - |U|)(|C| - |W|) \geq |U| \cdot |W| = |C|$. Note that v has at least $|C| - |U|$ in-links not in U, which must belong to B, and at least $|C| - |W|$ out-links not in W, which must belong to A. Those links at v induce at least $(|C| - |U|)(|C| - |W|)$ links in $L(G)$ from B to A. However, the number of links from B to A equals the number of links from A to B, which equals $|C|$. Therefore, $(|C| - |U|)(|C| - |W|) \leq |C|$. This means that $(|C| - |U|)(|C| - |W|) = |C|$ and all links from B to A in $L(G)$ are also located at v in G (Fig. 3). (An link of $L(G)$ is said to be *located* at a vertex of G if the link is in the form

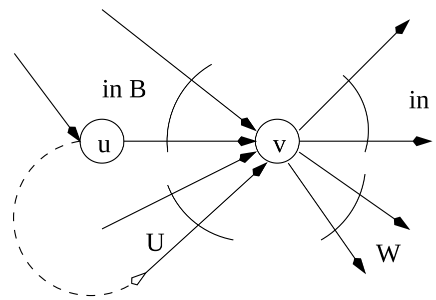

Figure 3: All links in C are located at v.

$((u, v), (v, w))$.) Now, consider a link $(u, v) \notin U$, which is not a loop at v. Such a link exists because $|C| - |U| \geq |U| \geq 2$. Then (u, v) must belong to B and all in-links at u must also belong to B. Note that the number of in-links at v other than those in U is $d - |U| \leq d - 2$ and G is at least $(d-1)$-line-connected. Therefore, after deleting all in-links at v which are not in U, the remaining digraph is still connected. Hence, there exists a path from u to v passing through a link in U. This path would induce a link in $L(G)$ from B to A, not located at v, a contradiction. ☐

Claim 4. *C is a natural line-cut of $L(G)$.*

Proof. By Claim 3, $|U| = 1$ or $|W| = 1$. It follows that if $|C| = d$, then C is natural. Next, we assume $|C| \leq d - 1$. First, we consider the case that $|U| = 1$. Note that $|W| \leq |C| \leq d - 1$. Let $V = \{v\}$. There exists at least one out-link (possibly a loop) at v not in W. Suppose (v, w) is an out-link at v not in W. Then (v, w) must belong to A.

If (v, w) is not a loop, then $(v, w) \notin U$, i.e., $A - U \neq \emptyset$. Define

$$X = \{x \mid (u, x) \in A - U \text{ for some vertex } u \text{ of } G\}$$
$$Y = \{y \mid (y, w) \in B \text{ for some vertex } w \text{ of } G\}.$$

Then any vertex z not in $X \cup Y$ must satisfy property that all in-links at z belong to B and all out-links at z belong to A. Thus, the existence of such a vertex z induces d^2 links from B to A in $L(G)$. Since the number of links from A to B equals the number of links from B to A, we have $d^2 \le |C| \le d-1$, a contradiction. Therefore, X and Y form a partition of the vertex set of G. Note that every link from X to Y belongs to U. Moreover, $X \cap Y$ must be empty, as the non-emptiness of $X \cap Y$ implies the existence of a link from A to B in $L(G)$ which is not in C. Therefore, U is a line-cut of G. Since $|U| = 1$, we have $d - 1 \le 1$ and hence $d = 2$. This falls into the case that we wanted to avoid.

If (v, w) is a loop, i.e., $(v, w) = (v, v)$, then (v, v) must belong to U. Otherwise, from (v, v) to other out-links at v would induce more links of $L(G)$ from A to B, but not in C, a contradiction. Summarizing the above arguments, we conclude that U contains only one element (v, v) which is a loop in G and all out-links at v except the loop belong to W. Therefore, C contains $d-1$ links from U to W and at (v, v) there is a loop $((v, v), (v, v))$. Hence, C is natural.

The case when $|W| = 1$ can be done similarly. $\square$

Claim 5.

$$|C| = \begin{cases} d & \textit{if } G \textit{ has no loop} \\ d-1 & \textit{otherwise}. \end{cases}$$

Proof. If G has no loop, then $L(G)$ has no loop. Thus, every natural line-cut of $L(G)$ has cardinality d. By Claim 4, $|C| = d$. If G has a loop, then this loop will induce a loop for $L(G)$. Therefore, the line-connectivity of $L(G)$ is at most $d-1$. However, since each vertex of G has at most one loop, so does each vertex of $L(G)$. Thus, every natural line-cut of $L(G)$ has cardinality at least $d-1$. By Claim 4, $|C| = d-1$. $\square$

By Claims 4 and 5, if G has no loop, then $L(G)$ has super line-connectivity d; if G has a loop, then $L(G)$ has super line-connectivity $d-1$, i.e., $L(G)$ satisfies condition (a). Thus, it remains to show that if G has a loop and $d > 2$, $L(G)$ satisfies condition (b). To do so, consider a line-cut C^* of size d in $L(G)$. Suppose C^* is not natural and C^* breaks the vertex set of $L(G)$ into two parts A^* and B^* such that no link other than those in C^* is from A^* to B^*. Let

$$\begin{aligned} U^* &= \{(u,v) \mid ((u,v),(v,w)) \in C^*, \text{ for some vertex } w \text{ of } G\ \} \\ W^* &= \{(v,w) \mid ((u,v),(v,w)) \in C^*, \text{ for some vertex } u \text{ of } G\ \} \\ V^* &= \{v \mid ((u,v),(v,w)) \in C^*, \text{ for some vertices } u, w \text{ of } G\ \} \end{aligned}$$

We show the following claims.

Claim 6. *$A^* - U^* \neq \emptyset$ and $B^* - W^* \neq \emptyset$.*

Proof. If $|V^*| \geq 2$, then by an argument similar to the proof of Claim 1, we can show that $A^* - U^* \neq \emptyset$ and $B^* - W^* \neq \emptyset$. If $|V^*| = 1$, then as C^* is not natural, it must be the case that $|U^*| \geq 2$ and $|W^*| \geq 2$. Since $|C^*| = |U^*| \cdot |W^*|$, we have $|C^*| - |U^*| \geq |U^*|$ and $|C^*| - |W^*| \geq |W^*|$. Hence, $(d-1) - |U^*| \geq |U^*| - 1 \geq 1$ and $(d-1) - |W^*| \geq |W^*| - 1 \geq 1$. Assume $V^* = \{v\}$. Then at v, there exist at least $(d-1) - |U^*|$ in-links not in U^*, which must belong to $B^* - W^*$, and there exist at least $(d-1) - |W^*|$ out-links not in W^*, which must belong to $A^* - U^*$. Therefore, $A^* - U^* \neq \emptyset$ and $B^* - W^* \neq \emptyset$. □

Claim 7. *If $A^* - U^* \neq \emptyset$ and $B^* - W^* \neq \emptyset$, then both A^* and B^* contain at least one loop-vertex.*

Proof. Define

$$\begin{aligned} X &= \{x \mid (u,x) \in A^* - U^*, \text{ for some vertex } u \text{ of } G\} \\ Y &= \{y \mid (y,w) \in B^*, \text{ for some vertex } w \text{ of } G\}. \end{aligned}$$

Similar to the proof of Claim 4, we can show that every link from X to Y belongs to U^*, that X and Y form a partition for the vertex set of G, so that U^* is a line-cut of G. Note that $|U^*| \leq |C| = d$. Since G has properties (a) and (b), we have $|U^*| \geq d-1$ and either

(u1) U^* is a natural line-cut in G, or

(u2) $|U^*| = d$ and both X and Y contain at least one loop-vertex. (When either X or Y is a singleton, case (u1) applies.)

If (u2) holds, then both A^* and B^* contain a loop-vertex. Thus, we may assume that (u1) holds.

Similarly, we may assume that W^* is a natural line-cut of G. Note that $d-1 \leq |U^*| \leq d$ and $d-1 \leq |W^*| \leq d$. Therefore, we have four cases as Figure 4 illustrated.

Case 1. $|U^*| = |W^*| = d-1$. Since $|C^*| = d$, in $L(G)$ there is exactly one element of U^* incident to two links of C^*. Meanwhile, there is exactly one element of W^* incident to two links of C^*. This happens only if $d \geq 4$. Note that at each $v \in V^*$, all in-links not in U^* must belong to B^* and all out-links not in W^* must belong to A^*. Therefore, there are totally

$$(d-4)(d-1)^2 + 2(d-2)(d-1) = (d^2 - 3d)(d-1)$$

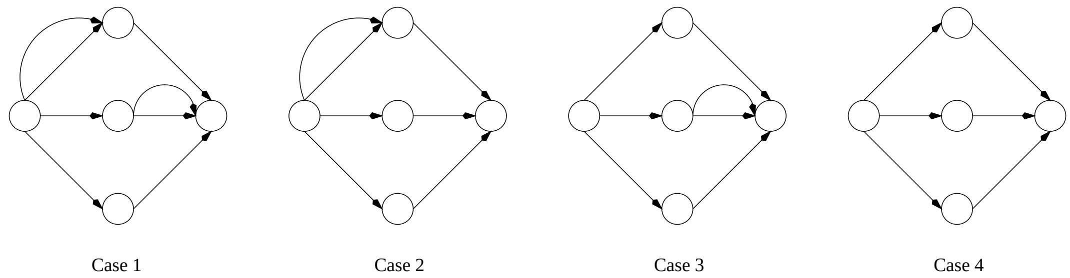

Figure 4: The proof of Claim 7.

links in $L(G)$ from B^* to A^* located at vertices in V^*. Since the number of links from A^* to B^* equals the number of links from B^* to A^*, we have

$$(d^2 - 3d)(d-1) \leq |C^*| = d.$$

This inequality cannot hold for $d \geq 4$.

Case 2. $|U^*| = d-1$ and $|W^*| = d$. In this case there is exactly one element in U^* incident to two links in C^* and none of the elements of W^* is incident to more than one link in C^*. This occurs only if $d \geq 3$ (recall that $d > 2$). In this case, there are totally $(d-2)(d-1)^2 + (d-2)(d-1)$ $(= d(d-2)(d-1))$ links in $L(G)$ from B^* to A^* located at vertices in V^*. Since the number of links from A^* to B^* equals the number of links from B^* to A^*, we have

$$d(d-2)(d-1) \leq |C^*| = d.$$

This inequality cannot hold for $d \geq 3$.

Case 3. $|U^*| = d$ and $|W^*| = d-1$. A contradiction can be found by an argument similar to that in Case 2.

Case 4. $|U^*| = |W^*| = d$. By an argument similar to the proof of Claim 2, we can find a contradiction. □

By Claims 6 and 7, $L(G)$ satisfies condition (b), completing the proof of Theorem 2.2. □

The following is a special case of Theorem 2.2, since every cyclic modification of a d-regular digraph G with super line-connectivity d is the same as G.

Corollary 2.3. *If a d-regular digraph G has super line-connectivity d, then $L^k(G)$ has super line-connectivity d for every $k \geq 1$.*

What would happen to Theorem 2.2 if $d = 2$ and G contains a loop? In this exceptional case, Theorem 2.2 does not hold. A counterexample is shown in Fig. 5. However, with certain additional condition, we can still establish the same result.

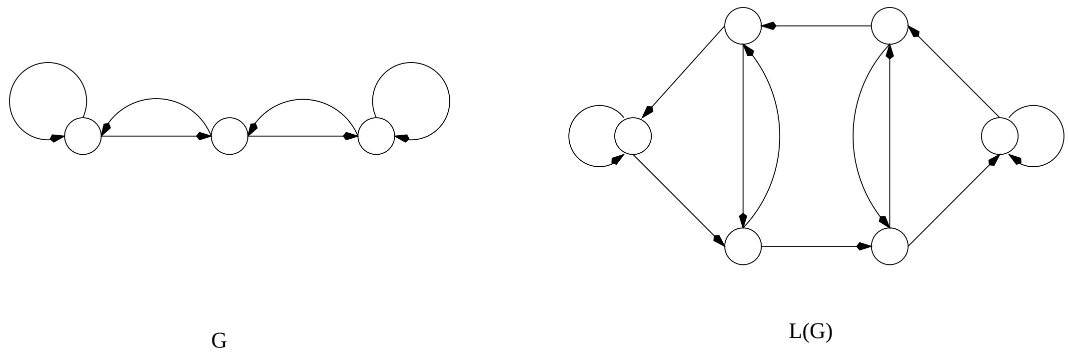

Figure 5: Exceptional case in Theorem 2.2.

Corollary 2.4. *Consider a 2-regular digraph G with some loops. Suppose G has super line-connectivity one and no path of length two is between two loop-vertices. If every cyclic modification of G has super line-connectivity two, then every cyclic modification of $L^k(G)$ ($k \geq 1$) has super line-connectivity two.*

Proof. Going over the proof of Theorem 2.2, we may find that only in the proof of Claim 4 we need to avoid the exceptional case that $d = 2$ and G contains a loop. The proof cannot proceed because in this case $|U| = |W| = 1$. Assume $U = \{(u, v)\}$ and $W = \{(v, w)\}$. Then, both U and W can be natural line-cuts of G while u and w are loop-vertices, but $v \in V$ is not. This produces a path of length two between two loop-vertices. □

3 Applications

We look at several examples in this section.

Example 3.1. The Kautz digraph $K(d, 1)$ is the complete digraph on $d + 1$ vertices without loop and in general $K(d, D) = L^{D-1}(K(d, 1))$ [12]. We claim that $K(d, 1)$ has super line-connectivity d. Consider a line-cut C of size d in $K(d, 1)$, which breaks the vertex set of $K(d, 1)$ into two parts A and B such that every link from A to B belongs to C. Note that there are $|A|(|A|-1)$ links from A to A and each vertex has d out-links. Therefore, $|A|d - |A|(|A|-1) = d$. That is, $(|A| - 1)(d - |A|) = 0$. Thus, $|A| = 1$ or $|A| = d$. Since $|A| = d$ implies $|B| = 1$, C is a natural line-cut.

Corollary 3.2. *The Kautz digraph $K(d, D)$ has super line-connectivity d.*

Example 3.3. The de Bruijn digraph $B(d, 1)$ is the complete digraph on d vertices with all loops and in general $B(d, D) = L^{D-1}(B(d, 1))$. We claim that every cyclic modification of $B(d, 1)$

has super line-connectivity d. In fact, every vertex of $B(d,1)$ has a loop and hence it has property (b). Moreover, removal all loops of $B(d,1)$ results in $K(d-1,1)$ and hence $B(d,1)$ has super line-connectivity $d-1$. By Lemma 2.1, every cyclic modification of $B(d,1)$ has super line-connectivity d for $d \geq 2$. By Theorem 2.2, every cyclic modification of $B(d,D)$ has super line-connectivity d for $d \geq 3$. For $d=2$, we may directly verify that every cyclic modification of $B(2,2)$ and $B(2,3)$ (Fig. 6) have super line-connectivity 2. Note that the distance between

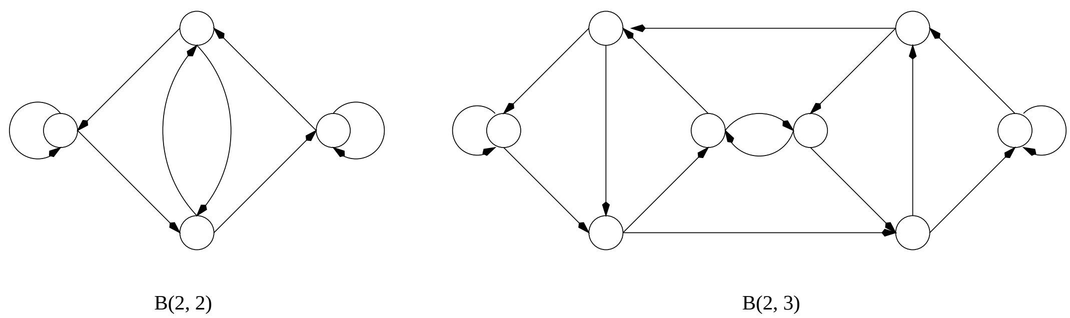

Figure 6: $B(2,2)$ and $B(2,3)$.

two loop-vertices in $B(2,D)$ is at least $D-1$. By Corollary 2.4, every cyclic modification of $B(2,D)$ for $D \geq 4$ also has super line-connectivity 2. Therefore, we have

Corollary 3.4. *Every cyclic modification of the de Bruijn digraph $B(d,D)$ has super line-connectivity d.*

Example 3.5. Fiol and Yebra [8] defined a family of bipartite digraphs $BD(d,n)$ as follows: The vertex set is $\mathbb{Z}_2 \times \mathbb{Z}_n = \{(\alpha,i) \mid \alpha \in \mathbb{Z}_2, i \in \mathbb{Z}_n\}$. There is a link from (α,i) to $(1-\alpha,(-1)^\alpha d(i+\alpha)+t)$ for every $t=0,1,\ldots,d-1$. This family of digraphs has a property that $BD(d,dn)=L(BD(d,n))$. We will show the following.

Corollary 3.6. *For $d \geq 3$ and $D \geq 1$, the bipartite digraph $BD(d,d^D)$ has super line-connectivity d.*

Proof. It is easy to see that $BD(d,d)$ is the complete bipartite digraph. For $d \geq 3$, $BD(d,d)$ has super line-connectivity d. Note that a simple digraph without loop has super line-connectivity d if and only if every line-cut of size at most d is natural. Thus, it suffices to show that every line-cut of size at most d in $BD(d,d)$ is natural. To do it, suppose $BD(d,d)$ has a line-cut C with cardinality at most d. We will prove that C is a natural line-cut and hence C must have cardinality d. Suppose C breaks the vertex set of $BD(d,d)$ into two parts A and B such that every link from A to B belongs to C. Let (P_1,P_2) be the partition of the vertex set such that every link is between P_1 and P_2. Denote $x = |A \cap P_1|$ and $y = |A \cap P_2|$. Then $d - x = |B \cap P_1|$ and $d - y = |B \cap P_2|$. Note that there are $x(d-y)$ links from $A \cap P_1$ to $B \cap P_2$ and there are $y(d-x)$ links from $A \cap P_2$ to $B \cap P_1$. Therefore

$$x(d-y) + y(d-x) \leq d.$$

We claim that one of x, y, $d-x$, and $d-y$ must be 0. For contradiction, suppose $x > 0$, $y > 0$, $d - x > 0$, and $d - y > 0$. Note that $y \geq 2$ or $d - y \geq 2$. Thus,

$$x(d-y) + y(d-x) > x + (d-x) = d,$$

a contradiction. Now, without loss of generality, assume $y = 0$. Then $d - y = d$ and $x = |A| > 0$. Note that $x(d-y) \leq d$. This implies $x = 1$. Hence, C is a natural line-cut. □

It is worth mentioning that $BD(2, 2)$ does not have super line-connectivity 2.

The bipartite digraph $BD(d, d^2 + 1) = (P_1, P_2, E)$ has the property that for each vertex $v \in P_1$ (or $v \in P_2$),

$$\{w \mid \exists u \text{ such that } (v, u), (u, w) \in E\} = P_1 - \{v\} \text{ (or } P_2 - \{v\}).$$

Now, we show that for $d \geq 3$, $BD(d, d^2 + 1)$ has super line-connectivity d. To do so, let C be a line-cut of size at most d, which breaks the vertex set into two nonempty parts A and B such that every link from A to B belongs to C. Denote $x = |A \cap P_1|$ and $y = |A \cap P_2|$. Then $(d^2 + 1) - x = |B \cap P_1|$ and $(d^2 + 1) - y = |B \cap P_2|$. Note that $x + y = |A| \geq 1$. Without loss of generality, we may assume $x \geq 1$. For each $v \in A \cap P_1$, let t_v be the number of links from $A \cap P_1$ to $B \cap P_2$. Denote $t = \min_{v \in A \cap P_1} t_v$. Suppose $v^* \in A \cap P_1$ achieves $t_{v^*} = t$. Then there are $(d - t)$ vertices in $A \cap P_2$ adjacent to v^*. From those $(d - t)$ vertices, there are at least $d(d - t) - (x - 1)$ links to $B \cap P_1$. Therefore, there are at least $xt + d(d - t) - (x - 1)$ links from A to B. This means that

$$xt + d(d-t) - (x-1) \leq d.$$

If $t \geq 1$, then $(x - 1)(t - 1) \geq 0$, i.e., $xt \geq x + t - 1$. It follows that

$$x + t - 1 + d(d-t) - (x-1) \leq d.$$

Therefore, $(d - 1)(d - t) \leq 0$. Hence, $t = d$. So, $xd - (x - 1) \leq d$, that is, $(x - 1)(d - 1) \leq 0$. This implies $x = 1$.

If $t = 0$, then $d^2 + 1 - d \leq x$. Similarly, if $(d^2 + 1) - x > 0$, then either $(d^2 + 1) - x = 1$ or $(d^2 + 1) - x \geq (d^2 + 1) - d$. This implies that x has only three possible values 1, d^2 and $d^2 + 1$. Similarly, each of y, $(d^2 + 1) - x$, and $(d^2 + 1) - y$ has four possible values 0, 1, d^2, and $(d^2 + 1)$.

Suppose $x = 1$. Then $t = d$ and hence $(d^2 + 1) - y = d^2$ or $d^2 + 1$. If $(d^2 + 1) - y = d^2$, then $y = 1$. This would imply the existence of d links from $A \cap P_2$ to $B \cap P_1$. Therefore, there are totally $2d$ links from A to B, a contradiction. This means that $(d^2 + 1) - y = d^2 + 1$. Hence, $y = 0$. Thus, C is natural.

Note that by the same argument, we can show that $y = 1$ or $(d^2+1)-x = 1$ or $(d^2+1)-y = 1$ implies that C is natural. Thus, it remains to prove that we must have $x = 1$ or $y = 1$ or $(d^2 + 1) - x = 1$ or $(d^2 + 1) - y = 1$. For contradiction, suppose none of them equals 1. Then they must equal 0 or $d^2 + 1$. That is, $x = d^2 + 1$, $(d^2 + 1) - y = 0$ (since $t = 0$), and $(d^2 + 1) - x = 0$. Hence, $|B| = (d^2 + 1) - x + (d^2 + 1) - y = 0$, a contradiction.

Corollary 3.7. *For $d \geq 3$ and $D \geq 2$, the bipartite digraph $BD(d, d^D + d^{D-2})$ has supper line-connectivity d.*

Example 3.8. Ferrero and Padró [7] studied two families of digraphs $BGC(p, d, n) = C_p \otimes B(d, n)$ and $KGC(p, d, n) = C_p \otimes K(d, n)$ where C_p is a directed cycle of length p and operation $\otimes$ is defined as follows. Let $G = (V, E)$ and $G' = (V', E')$. Then $G \otimes G'$ has vertex set $V \times V'$ and link set $\{((u, u'), (v, v')) \mid (u, v) \in E, (u', v') \in E'\}$.

With arguments similar to those in Example 3.5, we can show the following:

Corollary 3.9. *For $d \geq 3$, $BGC(p, d, d^k)$ has supper line-connectivity d.*

Corollary 3.10. *For $d \geq 3$, $KGC(p, d, d^{p+k} + d^k)$ (= $L^k(KGC(p, d, d^p + 1))$) has supper line-connectivity d.*

4 Discussion

The line digraph iteration preserves the degree, that is, the line digraph of a d-regular digraph is still d-regular. This is a very important property different from line graph iteration. This property enable the line digraph iteration to become a very useful tool to study interconnection networks. Many important properties can be preserved through line digraph iterations [8, 6, 4, 14] under certain conditions. Those conditions should be carefully established.

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