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Abstract

Let α be a permutation of $V(G)$ of a connected graph G. Define the *total relative displacement* of α in G by

$$\delta_\alpha(G) = \sum_{x,y\in V(G)} |d_G(x,y) - d_G(\alpha(x),\alpha(y))|$$

where $d_G(x,y)$ is the length of the shortest path between x and y in G. Let $\pi^*(G)$ be the maximum value of $\delta_\alpha(G)$ among all permutations of $V(G)$ and the permutation which realizes $\pi^*(G)$ is called a *chaotic mapping* of G. In this paper, we study the chaotic mappings of complete multipartite graphs. The problem will reduce to a quadratic integer programming. We characterize its optimal solution and present an algorithm running in $O(n^5 \log n)$ time where n is the total number of vertices in a complete multipartite graph.

key word. Chaotic mapping, complete multipartite graph.

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1 Introduction

Let α be a permutation of $V(G)$ of a connected graph $= G$. Define the *total relative displacement* of α in G by

$$\delta_\alpha(G) = \sum_{x,y\in V(G)} |d_G(x,y) - d_G(\alpha(x),\alpha(y))|$$

where $d_G(x,y)$ is the length of the shortest path between x and y in G. It is easy to see that a permutation α of $V(G)$ is an automorphism of G if and only the total relative displacement of α in G is zero. Let $\pi(G)$ and $\pi^*(G)$ denote respectively the smallest nonzero total relative displacement and the largest total relative displacement in G. Computing $\pi(G)$ and $\pi^*(G)$ is an important research topic in graph theory [1, 2, 3]. The exact value of $\pi(G)$ has been obtained for G being a path or complete multipartite graphs $K_{n_1,n_2,\cdots,n_t}$.

1. [1] Let G be a path with n vertices. Then the minimum total relative displacement $\pi(G) = 2n-4$.

2. [3] Let $1 \le n_1 \le n_2 \le \cdots \le n_t$, where $t \ge 2$ and $n_t \ge 2$. Then

$$\pi(K_{n_1,n_2,\cdots,n_t}) = \begin{cases} 2n_{h+1}-2 & \text{if } 1 = n_1 = \cdots = n_h < n_{h+1} \le \cdots \le n_t, \\ & \text{and } t \ge (h+1), \text{ for some } h \ge 2, \\ 2n_{k_0} & \text{if } 1 = n_1 < n_2 \text{ or } n_1 \ge 2 \text{ and} \\ & n_{k+1} = n_k + 1 \text{ for some } k, 1 \le k \le t-1, \\ & \text{ and } 2 + n_{k_0} \le n_1 + n_2, \text{ where } k_0 \\ & \text{is the smallest index for which} \\ & n_{k_0+1} = n_{k_0} + 1, \text{ and} \\ 2(n_1+n_2-2) & \text{otherwise.} \end{cases}$$

In this paper, we study how to compute $\pi^*(K_{n_1,n_2,\cdots,n_k})$. This problem can be reduced to a quadratic integer programming due to the following result.

Lemma 1.1 *[3] Let $K_{n_1,n_2,\cdots,n_t} = (X_1, X_2, \cdots, X_t)$ be a complete t-partite graph with partite sets $X_1, X_2, \cdots, X_t$. Let α be a permutation of $V(K_{n_1,n_2,\cdots,n_t})$. For each $1 \le i, j \le t$,* *define* $a_{ij} = |A_{ij}(\alpha)| = |\{x | x \in X_i \text{ and } \alpha(x) \in X_j\}|$. *Then*

$$\delta_\alpha(K_{n_1,n_2,\cdots,n_t}) = (\sum_{i=1}^{t} n_i^2) - (\sum_{1 \le i,j \le t} a_{i,j}^2). \tag{1}$$

Since $\sum_{i=1}^{t} n_i^2$ is fixed for a given complete multipartite graph, the problem of determining $\pi^*(K_{n_1,n_2,\cdots,n_k})$ is equivalent to the following quadratic integer programming

$$\begin{array}{lll} (QIP) & \text{minimize} & \sum_{1 \le i,j \le t} a_{i,j}^2 \\ & \text{subject to} & \sum_{i=1}^{t} a_{ij} = n_j \text{ for } 1 \le j \le t \\ & & \sum_{j=1}^{t} a_{ij} = n_i \text{ for } 1 \le i \le t \\ & & a_{ij} \ge 0 \text{ are integers} \end{array}$$

In this paper, we characterize optimal solution of this minimization problem and present an algorithm running in $O(n^5 \log n)$ where n is the number of vertices in a complete multipartite graph. Based on the characterization, we also give explicite value of $\pi^*(K_{n_1,n_2,\cdots,n_k})$ in some special cases.

2 Characterization

Let $A = (a_{ij})$ be a $t \times t$ nonnegative matrix. We will call $C = (a_{i_1 j_1}, a_{i_1 j_2}, a_{i_2 j_2}, a_{i_2 j_3}, a_{i_3 j_3}, \cdots a_{i_s j_s}, a_{i_s j_1})$ a *cycle* of length $2s$, $s \ge 2$, in A. A cycle C of length $2s$ is said to be *over-weight* if either $a_{i_k j_k} \ge 1$ for $1 \le k \le s$ and

$$a_{i_1 j_1} - a_{i_1 j_2} + a_{i_2 j_2} - a_{i_2 j_3} + a_{i_3 j_3} - \cdots + a_{i_s j_s} - a_{i_s j_1} > s$$

or $a_{i_k j_{k+1}} \ge 1$ for $1 \le k \le s$ (where $j_{s+1} = j_1$) and

$$-a_{i_1 j_1} + a_{i_1 j_2} - a_{i_2 j_2} + a_{i_2 j_3} - a_{i_3 j_3} + \cdots - a_{i_s j_s} + a_{i_s j_1} > s.$$

The following shows a matrix with over-weight cycle of length 2.

$$A = \begin{pmatrix} 3 & \rightarrow 1 & 1 \\ \uparrow & \downarrow & \\ 1 & \leftarrow 2 & 1 \\ & & \\ 1 & 1 & 0 \end{pmatrix}$$

It is not difficult to see that since $A = (a_{ij})$ have an over-weight cycle, $\sum a_{ij}^2 = 19$ is not of minimum value under the constraints that row sums and column sums are fixed. The next matrix $A' = (a'_{ij})$ reaches a smaller value $\sum {a'_{ij}}^2 = 17$.

$$A' = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Theorem 2.1 *$A = (a_{ij})$ is an optimal solution of the problem (QIP) if and only if no over-weight cycle exists in A.*

Proof. (Necessity) Suppose A has an over-weight cycle $C = (a_{i_1j_1}, a_{i_1j_2}, \cdots, a_{i_sj_1})$. Assume, without loss of generality, that $a_{i_kj_k} \geq 1$ for $1 \leq k \leq s$ and

$$a_{i_1j_1} - a_{i_1j_2} + a_{i_2j_2} - a_{i_2j_3} + a_{i_3j_3} - \cdots + a_{i_sj_s} - a_{i_sj_1} > s$$

Define $A' = (a'_{ij})$ where

$$a'_{ij} = \begin{cases} a_{ij} - 1 & \text{if } (i,j) = (i_k, j_k) \text{ for some } 1 \leq k \leq s \\ a_{ij} + 1 & \text{if } (i,j) = (i_k, j_{k+1}) \text{ for some } 1 \leq k \leq s \\ a_{ij} & \text{otherwise.} \end{cases}$$

Now

$$\begin{aligned} \sum a_{ij}^2 - \sum {a'_{ij}}^2 &= a_{i_1j_1}^2 + a_{i_1j_2}^2 + \cdots + a_{i_sj_1}^2 - {a'_{i_1j_1}}^2 - {a'_{i_1j_2}}^2 - \cdots - {a'_{i_sj_1}}^2 \\ &= 2(a_{i_1j_1} + a_{i_2j_2} + \cdots + a_{i_sj_s}) - 2(a_{i_1j_2} + a_{i_2j_3} + \cdots + a_{i_sj_1}) - 2s \\ &> 0. \end{aligned}$$

Therefore $\sum a_{ij}^2$ is not the minimum. Note that the row sums and column sums of A and A' are equal respectively. Hence we have the proof.

(Sufficiency) For contradiction, assume that all cycles of A are not over-weight and $\sum a_{ij}^2$ is not the minimum. Let $A^* = (a_{ij}^*)$ denote an optimal solution.

Let $\Delta_{ij} = a_{ij} - a_{ij}^*$, $1 \leq i, j \leq t$. Define a directed bipartite mutigraph G with bipartition (X, Y), where $X = \{x_1, x_2, \cdots, x_t\}$, $Y = \{y_1, y_2, \cdots, y_t\}$, x_i joins to y_j with Δ_{ij} edges if $\Delta_{ij} > 0$ and x_i joins from y_j with Δ_{ij} edges if $\Delta_{ij} < 0$. Since $\sum_{j=1}^{t} \Delta_{ij} = 0$ for $1 \leq i \leq t$ and $\sum_{i=1}^{t} \Delta_{ij} = 0$ for $1 \leq j \leq t$, the outdegree and indegree of each vertex in G are equal. Thus, each component of G has a directed Eulerian circuit and hence G can be decomposed into directed cycles $C_1, C_2, \cdots, C_m$. For each cycle C_ℓ, define

$$w(C_\ell) = \sum_{(x_i, y_j) \in C_\ell} a_{ij} - \sum_{(y_j, x_i) \in C_\ell} a_{ij}.$$

Note that $\Delta_{ij} > 0$ for $(x_j, y_j) \in C_\ell$ and that $\Delta_{ij} > 0$ implies $a_{ij} \geq 1$ since $a_{ij}^* \geq 0$. Thus, $a_{ij} \geq 1$ for $(x_j, y_j) \in C_\ell$. This means that if $w(C_\ell) > |E(C_\ell)|/2$ where $|E(C_\ell)|$ is the number of edges in cycle C_ℓ, then C_ℓ introduces an over-weight cycle in A. Since A has no over-weight cycle, we have $w(C_\ell) \leq |E(C_\ell)|/2$ for $1 \leq \ell \leq m$. Therefore,

$$\begin{aligned}
\sum_{1 \leq i,j \leq t} a_{ij}^2 - \sum_{1 \leq i,j \leq t} {a_{ij}^*}^2 &= \sum_{1 \leq i,j \leq t} a_{ij}^2 - \sum_{1 \leq i,j \leq t} (a_{ij} - \Delta_{ij})^2 \\
&= 2 \sum_{1 \leq i,j \leq t} a_{ij} \Delta_{ij} - \sum_{1 \leq i,j \leq t} \Delta_{ij}^2 \\
&= 2 \sum_{\ell=1}^{m} w(C_\ell) - \sum_{1 \leq i,j \leq t} \Delta_{ij}^2 \\
&\leq 2 \sum_{\ell=1}^{m} \frac{|E(C_i)|}{2} - \sum_{1 \leq i,j \leq t} \Delta_{ij}^2 \\
&= |E(G)| - \sum_{1 \leq i,j \leq t} \Delta_{ij}^2 \\
&= \sum_{i,j} |\Delta_{ij}| - \sum_{1 \leq i,j \leq t} \Delta_{ij}^2 \\
&\leq 0.
\end{aligned}$$

This contradicts the fact that $A^* = (a_{ij}^*)$ is an optimal solution but $A = (a_{ij})$ is not. $\square$

With the above characterization, we are able to find a chaotic mapping for certain complete multipartite graph.

Corollary 2.2 *Let $K_{m,n}$ be a complete bipartite graph and $l = \min\{\lfloor \frac{m+n}{4} \rfloor, n\}$. Then $\pi^*(K_{m,n}) = 2(m+n-2l)l$.*

Proof. Let

$$A = \begin{pmatrix} m-l & l \\ l & n-l \end{pmatrix}.$$

Then A has no over-weight cycle, by Theorem 2.1 and (1), $\pi^*(K_{m,n}) = (m^2+n^2) - [(m-l)^2 + 2l^2 + (n-l)^2] = (-4)l^2 + 2(m+n)l = 2(m+n-2l)l$. □

Corollary 2.3 *In $K_{n_1,n_2,\cdots,n_t}$, if $a_{ij} = \frac{1}{t}(n_i+n_j) - \frac{1}{t^2}(\sum_{i=1}^{t} n_i)$ is a nonnegative integer for each $1 \le i,j \le t$, then $A = (a_{ij})$ gives a chaotic mapping of $K_{n_1,n_2,\cdots,n_t}$, and $\pi^*(K_{n_1,n_2,\cdots,n_t}) = (1-\frac{2}{t})\sum_{i=1}^{t} n_i^2 + \frac{1}{t^2}(\sum_{i=1}^{t} n_i)^2$.*

Proof. Since for each i and i', $a_{ij} - a_{i'j} = \frac{n_i - n_{i'}}{t}$ and for each j and j', $a_{ij} - a_{ij'} = \frac{n_j - n_{j'}}{t}$, all cycles in A have weight zero. By Theorem 2.1, $\sum a_{ij}^2$ is minimum. Thus, A determines a chaotic mapping of $K_{n_1,n_2,\cdots,n_t}$ by mapping a_{ij} elements of the partite set X_i to the partite set X_j. Furthermore, $\pi^*(K_{n_1,n_2,\cdots,n_t}) = (\sum_{i=1}^{t} n_i^2) - (\sum_{1\le i,j\le t} a_{ij}^2)$ is easy to check. □

Example. $\pi^*(K_{3,6,9}) = (1-\frac{2}{3})(3^2+6^2+9^2) + \frac{1}{9}(3+6+9)^2 = 78$

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix}$$

and one of the chaotic mappings is as following

$$\alpha = \left(\begin{array}{ccc|cccccc|ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\ 4 & 10 & 11 & 1 & 5 & 6 & 12 & 13 & 14 & 2 & 3 & 7 & 8 & 9 & 15 & 16 & 17 & 18 \end{array}\right).$$

$(X_1 = \{1,2,3\},\ X_2 = \{4,5,6,7,8,9\},\ X_3 = \{10,11,12,13,14,15,16,17,18\}.)$

For special $t-$tuple $(n_1, n_2, \cdots, n_t)$ we can also find $\pi^*(K_{n_1,n_2,\cdots,n_t})$.

Corollary 2.4 *Let $s = (s_1, s_2, \cdots, s_t)$ be a t-tuple such that $s_i = 0$ or 1, $1 \le i \le t$, and $n_1 - s_1, n_2 - s_2, ..., n_t - s_t$ be graphical sequence of a simple graph. Then $\pi^*(K_{n_1,n_2,\cdots,n_t}) = \sum_{i=1}^{t} n_i(n_i - 1)$.*

Proof. Since $(n_1 - s_1, n_2 - s_2, ..., n_t - s_t)$ is a graphical sequence, let G be a graph with such degree sequence. Moreover,let $B(G) = [b_{ij}]$ be the adjacent matrix of G. Now, define $A = [a_{ij}]$ such that $a_{ij} = b_{ij}$ for $i \ne j$ and $a_{ii} = s_i$ for $1 \le i, j \le t$. By direct counting, we see that A has no over-weight cycles and thus by Theorem 2.1, we have an optimal solution. By (1), we can figure out $\pi^*(K_{n_1,n_2,\cdots,n_t})$ easily. □

Combining Corollary 2.3 and 2.4, the following result is easy to see.

Corollary 2.5 *Let m, t and r be nonnegative integers and $0 \le r \le t$. Then $\pi^*(K_{t(mt+r)}) = t[(t^2-1)m^2 - 2(t-1)rm + r^2]$.*

3 Algorithm

Theorem 2.1 suggests the following algorithm to compute $\pi^*(K_{n_1,n_2,..,n_t})$:

Algorithm A: Start from an initial matrix (a_{ij})

$$a_{ij} = \begin{cases} n_i & \text{if } i = j \\ 0 & \text{if } i \ne j \end{cases}$$

Carry out the following steps in each iteration:

Step 1. Check whether matrix (a_{ij}) has an over-weight cycle. If not, then stop; we obtain

$$\pi^*(K_{n_1,n_2,\cdots,n_t}) = (\sum_{i=1}^{t} n_i^2) - (\sum_{1\le i,j\le t} a_{ij}^2).$$

Otherwise, go to Step 2.

Step 2. Suppose $(a_{i_1j_1}, a_{i_1j_2}, a_{i_2j_2}, \cdots, a_{i_sj_s}, a_{i_sj_1})$ is an over-weight cycle with $a_{i_kj_k} \ge 1$ for $1 \le k \le s$ and $a_{i_1j_1} - a_{i_1j_2} + a_{i_2j_2} - \cdots + a_{i_sj_s} - a_{i_sj_1} > s$. Then set

$$\begin{aligned}
a_{i_1j_1} &\leftarrow a_{i_1j_1} - 1,\\
a_{i_1j_2} &\leftarrow a_{i_1j_2} + 1,\\
a_{i_2j_2} &\leftarrow a_{i_2j_2} - 1,\\
\cdots,\\
a_{i_sj_s} &\leftarrow a_{i_sj_s} + 1,\\
a_{i_sj_1} &\leftarrow a_{i_sj_1} - 1.
\end{aligned}$$

Go to next iteration.

Note that initially, $\sum_{1\le i,j\le t} a_{ij}^2 = \sum_{i=1}^{t} n_i^2$. In each iteration, if Step 2 is performed, then this sum decreases at least one. Therefore, the algorithm stops within $\sum_{i=1}^{t} n_i^2 = O(n^2)$ iterations where $n = \sum_{i=1}^{t} n_i$. In the following, we explain how to implement each iteration in $O(n^3 \log n)$ time. Therefore, we have

Theorem 3.1 *$\pi^*(K_{n_1,n_2,\cdots,n_t})$ can be computed in $O(n^5 \log n)$ time.*

First, we construct a directed graph H with vertex set

$$V = \{v_{ij} \mid a_{ij} \ge 1\},$$

edge set

$$E = \{(v_{ij}, v_{i'j'}) \mid i \ne i', j \ne j'\},$$

and edge weight

$$w(v_{ij}, v_{i'j'}) = \frac{1}{2}(a_{ij} + a_{i'j'}) - a_{ij'} - 1.$$

Note that $|V| \leq \sum_{i=1}^{t} n_i = n$. Therefore, H can be constructed in $O(n^2)$ time.

Lemma 3.2 *Matrix (a_{ij}) has an over-weight cycle if and only if H has a simple directed cycle with positive total weight.*

Proof. H has a directed cycle $(v_{i_1j_1}, v_{i_2j_2}, \cdots, v_{i_sj_s})$ with a positive total weight if and only if $a_{i_kj_k} \geq 1$ for $1 \leq k \leq s$ and $a_{i_1j_1} - a_{i_1j_2} + a_{i_2j_2} - \cdots + a_{i_sj_s} - a_{i_sj_1} > s$. □

Now, let P be the subgraph of H induced by all edges with positive weight and Q the subgraph of H induced by all edges with non-positive weight. Replace each negative edge-weight in Q by its absolute value. Let Q' be the resulting edge-weighted directed graph.

Check whether P contains a directed cycle. This can be done in at most $O(n^2)$ time. If yes, then we already found a directed cycle of H, with positive total weight. Otherwise, P is acyclic.

Compute the shortest path in Q' for every pair of vertices. This can be done in $O(n^3)$ time. Let $q(v_{ij}, v_{i'j'})$ denote the length of the shortest path from v_{ij} to $v_{i'j'}$ in Q'.

Make $n+1$ disjoint copies P_0, P_1, $\cdots$, P_n of P. Denote by v_{ij}^k the copy of vertex v_{ij} in P_k. For each pair of vertices v_{ij} and v_{ij} with $q(v_{ij}, v_{i'j'}) < \infty$, add edges $(v_{ij}^{k-1}, v_{i'j'}^k)$ for $1 \leq k \leq n$ each with weight $-q(v_{ij}, v_{i'j'})$. Meanwhile, delete all edges in P_0. This results in an acyclic directed graph G with $O(n^2)$ vertices.

Lemma 3.3 *H has a simple directed cycle with positive total weight if and only if either P contains a cycle or there exist i, j, and k such that R has a directed path from v_{ij}^0 to v_{ij}^k with positive total weight.*

Proof. Suppose H has a simple directed cycle C with positive total weight. If every edge has a posive weight, then P contains a cycle. Otherwise, C can be decomposed into $2k$

paths alternatively in P and Q. Suppose v_{ij} is the starting vertex of such a path in Q. It is easy to see that we can find a path from v_{ij}^0 to v_{ij}^k in graph R, with positive total weight, corresponding to cycle C.

Conversely, if P contains a cycle or there exist i, j, and k such that R has a directed path from v_{ij}^0 to v_{ij}^k with positive total weight, then it is easy to find a directed cycle C with positive total weight in H. This cycle may not be simple. However, it can be decomposed into several simple directed cycles, at least one with positive total weight since the sum of weights of those simple directed cycles equals the positive total weight of C. $\square$

Now, for each pair of v_{ij}^0 and v_{ij}^k, compute the longest path from v_{ij}^0 to v_{ij}^k in R. Since R is acyclic, this can be done trivially by dynamic programming in $O(n^5)$ time. In fact, there are at most n v_{ij}^0's and finding all longest paths from v_{ij}^0 to other vertices needs at most $O(n^2)$ time. However, we next describe a more clever way running in $O(n^3 \log n)$ time.

Construct R' from R by adding edges (v_{ij}^{k-1}, v_{ij}^k) for all $1 \leq k \leq n$ and $1 \leq i, j \leq t$. Clearly, R' is a disjoint union of n copies of subgraph induced by vertices in $P_0 \cup P_1$. Let $g(u, v)$ denote the length of the longest path from vertex u to vertex v in R'. It is easy to see that there exist i, j, and k such that R has a directed path from v_{ij}^0 to v_{ij}^k with positive total weight if and only if $g(v_{ij}^0, v_{ij}^n) > 0$.

For every pair of vertices v_{ij}^1 and $v_{i'j'}^1$, computer the longest path from v_{ij}^1 to $v_{i'j'}^1$. This can be done in $O(n^3)$ time since P_1 is acyclic.

For every pair of vertices v_{ij}^0 and $v_{i'j'}^1$, compute the longest path from v_{ij}^0 to $v_{i'j'}^1$ by

$$g(v_{ij}^0, v_{i'j'}^1) = \max_{u \in V(P_1)} (-q(v_{ij}^0, u) + g(u, v_{i'j'}^1)).$$

where $V(P_1)$ denotes the vertex set of P_1. This can be done in $O(n^3)$ time.

Similarly, we can compute all

$$\begin{aligned} g(v_{ij}^0, v_{i'j'}^2) &= \max_{u \in V(P_1)} (g(v_{ij}^0, u) + g(u, v_{i'j'}^2)) \\ g(v_{ij}^0, v_{i'j'}^4) &= \max_{u \in V(P_2)} (g(v_{ij}^0, u) + g(u, v_{i'j'}^4)) \\ g(v_{ij}^0, v_{i'j'}^8) &= \max_{u \in V(P_4)} (g(v_{ij}^0, u) + g(u, v_{i'j'}^8)) \end{aligned}$$

$$\cdots$$

$$g(v_{ij}^0, v_{ij}^n)$$

totally in $O(n^3 \log n)$ time.

This completes the proof of Theorem 3.1.

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