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Exponential Modeling with Unknown Model Order Using Structured Nonlinear Total Least Norm

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Abstract: A method based on Structured Nonlinear Total Least Norm is presented for estimating the parameters of exponentially damped sinusoidal signals in noise when the model order is unknown. It is compared to two other existing methods to show its robustness in recovering correct values of parameters when the mo del order is unknown, in spite of some large errors in the measured data.

Keywords: Exponential damped signals, model order, overdetermined linear system s, Hankel matrices, singular value decomposition.

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I. Introduction

A record of a data sequence y_n, $n = 1, 2, \ldots, N$, is assumed to be composed of uniformly spaced samples of a sum of exponential damped signals y_n^* and measurement noise w_n. That is

$$\begin{aligned} y_n &= y_n^* + w_n, \\ y_n^* &= \sum_{k=1}^{K} x_k e^{s_k n\Delta t} = \sum_{k=1}^{K} (a_k e^{i\phi_k}) e^{(-d_k + i2\pi f_k)n\Delta t}, \\ & \quad n = 0, \ldots, N-1, \end{aligned} \tag{I.1}$$

where $i = \sqrt{-1}$, K is the model order, which is unknown, and Δt is the constant sampling interval. The values y_n^* are called true signals and y_n observed signals. The objective is to estimate the model order K and the parameters, which are frequencies f_k, damping factors d_k, amplitudes a_k, and phases ϕ_k, $k = 1, \ldots, K$.

Two most commonly used methods for estimating the model order and the parameters are the Kumaresan-Tufts' linear prediction (LP) method [4], [5] and the HSVD method [1], [6].

The LP method first sets up the backward linear prediction equations

$$\begin{pmatrix} \overline{y_1} & \overline{y_2} & \cdots & \overline{y_M} \\ \overline{y_2} & \overline{y_3} & \cdots & \overline{y_{M+1}} \\ \vdots & \vdots & \cdots & \vdots \\ \overline{y_L} & \overline{y_{L+1}} & \cdots & \overline{y_{N-1}} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_M \end{pmatrix} = - \begin{pmatrix} \overline{y_0} \\ \overline{y_1} \\ \vdots \\ \overline{y_{L-1}} \end{pmatrix}$$

or

$$Ab = -h$$

where b is the vector of backward prediction coefficients and $\overline{y_n}$ denotes the complex conjugate of y_n. If the data is noiseless, the prediction-error filter polynomial

$$B(z) = 1 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}$$

will have zeros at $e^{\overline{s_k}}$, $k = 1, 2, \cdots, K$ if M is chosen to satisfy the inequality $K \leq M \leq N - K$. These zeros, called signal zeros, fall outside the unit circle. If $M > K$, $B(z)$ has $M - K$ other zeros called extraneous zeros which will always fall within the unit circle. In the presence of noise in the data, the above procedure is shown experimentally still valid to estimate s_k's. However, A will have full rank and $M - K$ columns of A tend to be closely dependent. To alleviate this ill-conditioning problem, the SVD of A is computed and truncated by setting the smaller singular values of A to zero [4].

The HSVD method arranges the data y_n into a Hankel matrix $A \in \mathbf{C}^{L\times M}$, $L \geq M > K$, $N = L + M - 1$,

$$\begin{pmatrix} y_0 & y_1 & \cdots & y_{M-1} \\ y_1 & y_2 & \cdots & y_M \\ \vdots & \vdots & \cdots & \vdots \\ y_{L-1} & y_L & \cdots & y_{N-1} \end{pmatrix}.$$

The SVD of A is computed and truncated by setting the smaller singular values of A to zero, similar to the procedure of LP. Suppose the rank K' truncated SVD of A is $U_{K'}\Sigma_{K'}V_{K'}^H$, where H denotes complex conjugate. A matrix

$$Z' = (I + uu^H/(1 - u^H u))(U_{K'})^{\downarrow H}(U_{K'})^{\uparrow}$$

is then obtained, where u is the bottom row of $U_{K'}$, I is the $K' \times K'$ identity matrix, and $(U_{K'})^{\uparrow}$, $(U_{K'})^{\downarrow}$ are matrices derived from $U_{K'}$ by deleting the first row and last row, respectively. The estimates of the frequencies and damping factors are derived by diagonalizing Z'. Finally, the amplitudes and phases are estimated by linear least squares fitting of (I.1). See [1], [6] for more details.

The LP and HSVD methods constitute the so called black-box methods: they require no prior knowledge. However, it is known that when some of the noise components w_n are large, i.e., the SNR is small, these methods may break down. The SNR at which this takes place is called breakdown SNR. In addition, the gap between small and large singular values of A becomes unclear, which makes it very difficult to correctly estimate the model order. The situation can be improved by imposing prior knowledge on the model parameters, which causes the Cramer-Rao (CR) bounds to decrease and lowers the threshold SNR. For more details see [2]

Recently, a new algorithm, called Structured Nonlinear Total Least Norm (SNTLN), has been developed [8], [9], [10]. A theoretical justification and computational testing of the SNTLN algorithm confirm that it is an efficient method for solving structured overdetermined systems by minimizing structured perturbations in the discrete L_p norm, where $p = 1, 2,$ *or* ∞ [9], [10], [11]. In the next section, we will discuss how SNTLN can be applied to estimation of the model order and the parameters in (I.1). Then in Section III, the computational results, which compare the performance of the SNTLN based methods with that of the HSVD and the LP methods, are summarized, and the advantage of the SNTLN methods discussed.

II. Formulation of the SNTLN based method

Since the model order K is unknown, to apply SNTLN, we first choose a prediction order M, where M should be large enough such that $M \geq K$. We assume that the correct signal y_n^* is represented by (I.1) with M possible terms (rather than K), but with $M-K$ of the amplitudes $a_k = 0$. The model order is then determined by identifying the zero amplitudes in the computed signal. Eqn. (I.1) can be rewritten as

$$A(\alpha)x \approx y, \tag{II.2}$$

where

$$\begin{aligned} &y = (y_0, \ldots, y_{N-1})^T, \quad && \alpha = (\alpha_1, \ldots, \alpha_M)^T, \\ &\alpha_k = -d_k + i2\pi f_k, && A(\alpha)_{jk} = e^{\alpha_k j \Delta t}, \\ &x = (x_1, \ldots, x_M)^T, && \\ &k = 1, \ldots, M, && j = 0, \ldots, N-1, \end{aligned}$$

M is the prediction order, and $N > M \geq K$.

To solve this problem, we minimize the following:

$$\min_{\alpha,x} \left\| \begin{array}{c} r(\alpha, x) \\ D(\alpha - \hat{\alpha}) \end{array} \right\|_p, \quad p = 1, 2, \infty, \tag{II.3}$$

where $r(\alpha, x) = y - A(\alpha)x$, $\hat{\alpha}$ is an initial estimate of the parameter vector α, and D is a diagonal matrix of positive weights. We compute the minimum solution to (II.3) iteratively by linearizing $r(\alpha, x)$:

$$r(\alpha + \Delta\alpha, x + \Delta x) = r(\alpha, x) - A(\alpha)\Delta x - J(\alpha, x)\Delta\alpha, \tag{II.4}$$

where $\Delta\alpha$ and Δx represent small changes in the elements of α and x respectively, and $J(\alpha, x)$ is the Jacobian, with respect to α, of $A(\alpha)x$. Let $A(\alpha)_k$ represent the k^{th} column of $A(\alpha)$. Then

$$J(\alpha, x) = \nabla_\alpha (A(\alpha)x) = \sum_{k=1}^{M} x_k \nabla_\alpha (A(\alpha)_k). \tag{II.5}$$

The SNTLN method is summarized in Algorithm SNTLN.

An initial estimate of $\hat{\alpha}$ is needed to start the iterative procedure. We will show how this value can be obtained, in Section III. The values of the diagonal weight matrix D are chosen according to the error levels in the data and how good the initial estimate of $\hat{\alpha}$ is, and they can be used to fine tune the algorithm. The computational method by which Step 2a is carried out depends on the value of p. For $p = 2$, the least squares problem is solved. For $p = 1$ or ∞, Step 2a is

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Algorithm SNTLN

Input – Matrices $A(\alpha)$, $\nabla_\alpha a_k$, $1 \le k \le M$, D, vector b, initial estimate $\hat{\alpha}$, and tolerance tol.

Output – α, residual vector r, and vector x

1. Set $\alpha = \hat{\alpha}$, compute x from (II.3) with $A = A(\alpha)$, $J(\alpha, x)$, and set $r = b - A(\alpha)x$.

2. **repeat**

(a) $\underset{\Delta x, \Delta\alpha}{\text{minimize}} \left\| G \begin{pmatrix} \Delta x \\ \Delta\alpha \end{pmatrix} + \begin{pmatrix} -r \\ D(\alpha - \hat{\alpha}) \end{pmatrix} \right\|_p ,$

where $G = \begin{pmatrix} A(\alpha) & J(\alpha, x) \\ 0 & D \end{pmatrix}.$

(b) Set $x := x + \Delta x$, $\alpha := \alpha + \Delta\alpha$.

(c) Compute $J(\alpha, x)$, $A(\alpha)$, $r = b - A(\alpha)x$.

until ($\|\Delta x\| \le tol$ and $\|\Delta\alpha\| \le tol$)

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solved as a linear program. To further improve robustness of the SNTLN1 algorithm, bounds can be imposed on α, which can easily be imposed as constraints. It has been shown that this algorithm will always converge [11].

The following theorem explains how the SNTLN algorithm for exponential modeling with unknown model order works.

Theorem 1: Suppose x^* is the solution to the following zero residual overdetermined system

$$Ax = b, \tag{II.6}$$

where A is an $m \times n$, $m > n$, matrix. Now consider the following two systems

$$A_1 y = b, \tag{II.7}$$

where $m \times l$ matrix A_1, $l < n$, is a submatrix of A, and

$$Bz = b, \tag{II.8}$$

where B is an $m \times q$, $m > q$, matrix, and has A as its submatrix. If A and B have full rank then the following results hold:

(a) Let y^* denote the vector with $n - l$ components consisting of elements of x^* that correspond to columns of A not in A_1. The Eqn. (II.7) has a solution only if $y^* = 0$;

(b) The elements z_j^*, $j = 1, \ldots, q$, of the solution to (II.8) are given as follows:

$$z_j^* = \begin{cases} x_i^* & \text{if the } j^{th} \text{ column of } B \text{ is} \\ & \text{the } i^{th} \text{ column of } A, \\ 0 & \text{if the } j^{th} \text{ column of } B \text{ is not} \\ & \text{a column of } A, \end{cases}$$

where x_i^* is the i^{th} element of x^*.

Proof: Without loss of generality, we assume that A_1 consists of the first l columns of A, and that the first n columns of B are given by A.

To prove (a), note that if the last $n-l$ elements of x^* are not all zeros and (II.7) has a solution then (II.6) has two distinct solutions. This contradicts the fact that A has full rank.

Similarly, since B has full rank, $z^* = \begin{pmatrix} x^* \\ 0_{m\times(q-n)} \end{pmatrix}$ is the only solution to (II.8). ■

In Eqn. (II.2), the matrix $A(\alpha)$ has Vandermonde-like structure except that the first row is not all 1's. It has full rank if the elements of α are distinct.

Theorem (1) shows that if there is no noise in the data, after solving (II.2) the x_k corresponding to the non existent frequency f_k and damping factor d_k will be zero. If there is noise in the data, being applied to (II.2), the SNTLN algorithm tries to recover the correct values of x_k. This means some of the x_k are close to zero. We can then determine the value of the model order K from the number of nonzero elements of x, and select the damping factors d_k and frequencies f_k correspondingly.

Both theoretical and computational results show that 1-norm SNTLN is able to obtain accurate approximations with data containing a relatively small number of large errors, which we call "outliers" [10]. In contrast, our results show that HSVD and LP have difficulty in estimating the model order correctly, and parameters accurately, when some of the errors are large. The SNTLN based methods can correctly estimate the model order, and the errors in parameter estimates are much smaller.

III. Computational Results

We now present the test results using 1-norm SNTLN (SNTLN1), 2-norm SNTLN (SNTLN2), the HSVD and the LP methods. To construct the test problems, we assume that the correct parameter vector α_c and amplitude vector x_c are known. This information is of course not used in the algorithms tested, but allows us to investigate the ability of the algorithms to recover the

correct α and x, in spite of errors in the data. A total of six test cases were considered.

In the first case, we used the following values: exact model order $K = 4$, the prediction order $M = 7$, $N = 128$, $\Delta t = 0.0004$. The measurement noise w_n consisted of a small random (uniformly distributed) error with magnitude $\leq 10^{-6}$ for each n, and no large errors were added. For SNTLN1, bounds were imposed on $d = (d_1, \ldots, d_M)^T$ and $f = (f_1, \ldots, f_M)^T$:

$$\begin{array}{l} d_{lk} \leq d_k \leq d_{uk} \\ f_{lk} \leq f_k \leq f_{uk} \end{array} \qquad k = 1, \ldots, 7,$$

where

$$\begin{aligned} d_l &= (40, 40, 130, 100, 160, 190, 240), \\ d_u &= (65, 65, 160, 130, 190, 220, 280), \\ f_l &= (8, 18, 32, 120, 370, 530, 790), \\ f_u &= (13, 28, 42, 140, 400, 560, 825). \end{aligned}$$

A total of 10 test problems were constructed. For each problem, the correct values x_c, d_c, and f_c were generated by selecting random values within the bounds. For $x_{ck} = x_{Rk} + \sqrt{-1}x_{Ik}$, $k \leq K$, each x_{Rk} and x_{Ik} were randomly selected in the range $[-10, 10]$, and x_{ck} was set to zero for $k > K$. These test problems were chosen so as to be similar to the NMR signal data problem described in [3]. For SNTLN1 and SNTLN2, we used the following initial values:

$$\begin{aligned} d_{init} &= \tfrac{1}{2}[d_l + d_u] \\ f_{init} &= \tfrac{1}{2}[f_l + f_u] \end{aligned}$$

for each problem. For SNTLN, the tolerance value used to truncate x and estimate the model order was set to be 0.02 and for HSVD and LP, the tolerance value used to truncate the singular values of Hankel matrices formed by the signals was set to be 2.0×10^{-5}. Those tolerance values were chosen based on the knowledge of the approximate size of errors in the signals, as well as the approximate magnitude of x and y. We measured the performance of each method on a test problem in terms of the relative errors:

$$\begin{aligned} RE_d &= \|d_A - d_c\|/\|d_c\|, \\ RE_f &= \|f_A - f_c\|/\|f_c\|, \\ RE_x &= \|x_A - x_c\|/\|x_c\|, \end{aligned} \tag{III.9}$$

where d_A, f_A, and x_A are parameter vectors estimated by a method. The results are summarized in Table I. For each method, the average of relative errors over 10 different correct signals are

shown. In this case, all the methods can correctly estimate the model order. However, we see that SNTLN based methods performed much better.

In the second case, we used the same data as in the first case except that in addition to the small random error, for 25 randomly selected n, an outlier $\delta = 0.01$ were added to the measurement noise w_n. Also the tolerance value used to truncate the singular values of Hankel matrices formed by the signals was set to be 0.05, while for SNTLN the tolerance value to truncate x remained the same. See Table II for results. Since in this case, both the HSVD and the LP methods had difficulty in estimating the correct model order, for the purpose of comparison, only those test problems for which those two methods happened to correctly estimate the model order were chosen. SNTLN based methods still estimated the correct model order easily.

In the third and fourth cases, the test problems were constructed in the same way as in the second test except that the actual model order K was set to be 5 and 6, respectively. The results are shown in Table III and Table IV.

TABLE I

AVERAGE RELATIVE ERRORS IN f, d, AND x FOR 10 RUNS. MAGNITUDE OF SMALL RANDOM ERRORS IS 10^{-6}. NO LARGE ERRORS ($\delta = 0$). ACTUAL MODEL ORDER $K = 4$.

	SNTLN1	SNTLN2	HSVD	LP
RE_d	2.52e-6	1.36e-6	4.28e-4	3.13e-4
RE_f	1.38e-7	4.87e-7	1.61e-4	6.19e-5
RE_x	8.48e-6	2.63e-6	2.46e-3	5.38e-4

TABLE II

AVERAGE RELATIVE ERRORS IN f, d, AND x FOR 10 RUNS. MAGNITUDE OF SMALL RANDOM ERRORS IS 10^{-6}. $\delta = 0.01$. ACTUAL MODEL ORDER $K = 4$.

	SNTLN1	SNTLN2	HSVD	LP
RE_d	2.71e-6	6.17e-3	2.13e+0	2.69e+0
RE_f	1.54e-7	1.28e-3	9.10e+0	4.85e+0
RE_x	1.22e-5	1.53e-2	8.77e-1	1.08e+0

TABLE III

AVERAGE RELATIVE ERRORS IN f, d, AND x FOR 10 RUNS. MAGNITUDE OF SMALL RANDOM ERRORS IS 10^{-6}. $\delta = 0.01$. ACTUAL MODEL ORDER $K = 5$.

	SNTLN1	SNTLN2	HSVD	LP
RE_d	2.13e-6	5.01e-4	3.29e-1	1.70e+0
RE_f	2.33e-7	8.59e-5	3.00e+0	1.86e+0
RE_x	3.41e-5	2.71e-3	1.12e+0	1.21e+0

TABLE IV

AVERAGE RELATIVE ERRORS IN f, d, AND x FOR 10 RUNS. MAGNITUDE OF SMALL RANDOM ERRORS IS 10^{-6}. $\delta = 0.01$. ACTUAL MODEL ORDER $K = 6$.

	SNTLN1	SNTLN2	HSVD	LP
RE_d	2.38e-6	2.94e-3	2.57e-1	1.39e+0
RE_f	1.72e-7	1.45e-4	1.86e+0	1.22e+0
RE_x	2.67e-5	5.32e-3	8.43e-1	9.39e-1

The fifth and sixth cases are similar to the first and second cases, respectively, except exact model order $K = 4$, the prediction order $M = 6$ and the bounds were changed as follows

$$\begin{aligned} d_l &= (28.5, 12, 24, 28.5, 12, 24), \\ d_u &= (33, 18, 28.5, 33, 18, 28.5), \\ f_l &= (40, 70, 100, 220, 320, 550), \\ f_u &= (60, 80, 140, 270, 380, 600). \end{aligned}$$

The results are shown in Table V and Table VI.

The remarkable ability of SNTLN1 to obtain the correct parameter and amplitude vectors in spite of large errors in the data and initial estimates, is clearly shown. This is due both to the use of the L_1 norm and the imposed bounds. In fact, the errors in the parameters estimated by SNTLN1 are determined by the small ($< 10^{-6}$) data errors, and are essentially independent of the 25 outliers. More test results also show that the SNTLN1 solution is changed only slightly by a change in the size of outliers, δ. In contrast, the errors in parameters estimated by other methods are determined primarily by the largest data errors, and will increase in proportion to δ,

TABLE V

AVERAGE RELATIVE ERRORS IN f, d, AND x FOR 10 RUNS. MAGNITUDE OF SMALL RANDOM ERRORS IS 10^{-6}. NO LARGE ERRORS ($\delta = 0$). ACTUAL MODEL ORDER $K = 4$.

	SNTLN1	SNTLN2	HSVD	LP
RE_d	1.06e-4	1.95e-6	1.06e-3	1.00e-3
RE_f	2.29e-6	1.19e-8	3.21e-5	8.14e-6
RE_x	4.96e-5	5.04e-7	4.46e-4	1.89e-4

TABLE VI

AVERAGE RELATIVE ERRORS IN f, d, AND x FOR 10 RUNS. MAGNITUDE OF SMALL RANDOM ERRORS IS 10^{-6}. $\delta = 0.01$. ACTUAL MODEL ORDER $K = 4$.

	SNTLN1	SNTLN2	HSVD	LP
RE_d	2.59e-7	2.98e-3	1.40e+1	1.05e+1
RE_f	9.72e-9	8.90e-5	4.21e+0	2.66e+0
RE_x	1.74e-7	2.45e-3	1.12e+0	6.07e-1

as δ is increased. While SNTLN2 is not as robust as SNTLN1, it performed significantly better than the HSVD and the LP.

To illustrate the difficulty encountered by HSVD and LP in estimating the correct model order, we show in the following the singular values of one of the Hankel matrices used in the second test case:

$$\begin{pmatrix} 1.4825e+2 \\ 1.9843e+1 \\ 8.7398e-1 \\ 5.3410e-2 \\ 4.7958e-2 \\ 4.6210e-2 \\ 4.4174e-2 \end{pmatrix}.$$

The gap which should occur between the 4^{th} and 5^{th} singular values is not identified by this

result. Now look at the x value obtained by SNTLN1 in the same test run:

$$\begin{pmatrix} 6.92e+0 & - & 3.84e+0i \\ -4.75e+0 & - & 8.07e+0i \\ -1.88e+0 & + & 4.13e+0i \\ 8.67e+0 & - & 2.18e+0i \\ 3.44e-5 & - & 7.11e-6i \\ 7.73e-6 & + & 1.05e-5i \\ -6.63e-6 & - & 2.63e-6i \end{pmatrix}.$$

We note that there is a well defined gap between the 4^{th} and 5^{th} elements. This is true for all test problems. SNTLN2 also did well in identifying the gap.

TABLE VII

VALUES OF d AND f WHEN THE MODEL ORDER IS *overestimated*. COMPARISON OF SNTLN2, SNTLN1, HSVD, AND LP.

d_c	151	116	61.4	57.4	
f_c	37.1	127	12.5	21.3	
d_{SNTLN2}	151	116	61.4	57.4	260
f_{SNTLN2}	37.1	127	12.5	21.3	804
d_{SNTLN1}	151	116	61.4	57.4	190
f_{SNTLN1}	37.1	127	12.5	21.3	376
d_{HSVD}	151	116	61.4	57.4	559
f_{HSVD}	37.1	127	12.5	21.3	1208
d_{LP}	151	116	61.4	57.4	-515
f_{LP}	37.1	127	12.5	21.3	-813

In Table VII and Table VIII, we show the computed parameters when the model order was estimated incorrectly in some of our tests. In the SNTLN based method, it is easy to identify the correct parameters even when the model order is overestimated, since we can discard the damping factors and frequencies that correspond to zero amplitudes. Thus for the SNTLN based method the amplitude was essentially zero for the d and f in the 5^{th} column of Table VII. It is interesting to notice that even when the model order was underestimated, the parameters

TABLE VIII

VALUES OF d AND f WHEN THE MODEL ORDER IS *underestimated*. COMPARISON OF SNTLN2, SNTLN1, HSVD, AND LP.

d_c	155	52.1	41.1	108
f_c	37.7	26.0	9.65	134
d_{SNTLN2}	155	52.1	41.1	
f_{SNTLN2}	37.7	26.0	9.65	
d_{SNTLN1}	155	52.1	41.1	
f_{SNTLN1}	37.7	26.0	9.65	
d_{HSVD}	110	101	19.6	
f_{HSVD}	37.3	134	11.2	
d_{LP}	100	90.5	18.4	
f_{LP}	134	37.2	11.3	

computed by the SNTLN based method make a subset of the correct set of parameters. In contrast, the HSVD and the LP methods produced completely wrong parameters when the model order was underestimated.

In summary, we have introduced a new, SNTLN based method for estimating the model order and parameters in exponential modeling. Then the SNTLN based method was compared to the HSVD and the LP methods, and its superior capability in estimating the correct model order, and the accurate parameters is shown. Moreover, we have shown the robustness of SNTLN with L_1 norm in recovering correct values of parameters, in spite of some large errors in the measured data. Often in real applications, estimated bounds on the parameters can be easily obtained. To fully take advantage of the prior knowledge, for the 1-norm SNTLN based method these bounds can be imposed as constraints, further improving the robustness.

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