Applications of Functional-analytic Methods in Nonlocal and Dynamical System Problems

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Abstract

This paper is concerned with functional analytic techniques in problems related to dynamical systems and contains two parts. In the first part we show bifurcation of localized spike solutions from spatially constant states in systems of nonlocally coupled equations in the whole space. The main assumptions are a generic bifurcation of saddle-node or transcritical type for spatially constant profiles, and a symmetry and second moment condition on the convolution kernel. Rather than relying on center manifolds methods, we pursue a more direct approach, deriving leading order asymptotics and Newton corrections for error terms. We are able to extend well-known results for spots, spikes, and fronts in locally coupled systems on the real line, and for radially symmetric profiles in higher space dimensions.

In the second part, we revisit the classical problem of determining the asymptotic expansion of the solution near the passage of a fold point in a singularly perturbed system, where the theory of normally hyperbolic invariant manifold cannot be directly applied. While the standard remedy is the blow-up method first demonstrated by Krupa and Szmolyan, we will show how one can use a functional analytic approach to achieve the same goal.

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Chapter 1

Introduction

1.1 Geometric and functional analytic methods in dynamical systems

Dynamical systems deal primarily with systems of ordinary differential equations (ODEs). In most introductory graduate classes about ODEs, two perspectives are often presented to the students: on the one hand, we have analysis approaches. The most obvious example is the proof of the basic existence and uniqueness theorem for a general $n-$dimensional system of ODEs, where one usually sees Banach's fixed point theorem for the first time. On the other hand, we have geometric approaches that qualitatively describes the behavior of system of ODEs. One would think of phase-plane analysis, and perhaps the theory of invariant manifolds which proves to be essential in long-time behavior questions.

While both approaches are important, this paper will focus on the analytic approaches for two main reasons. First, for problems beyond the traditional ODE-setting, it is easier to recast the problem in a functional analytic framework and extend known results. Second, we found that even in problems usually dealt with by geometric approaches, it is possible to give a conceptually cleaner and easier-to-follow proof when reinterpreting the problem using functional analysis. We shall demonstrate these two points via two examples. In the first example, we show how to extend results about radially symmetric solutions bifurcating from systems of partial differential equations to nonlocally coupled systems by appropriately setting them up as a fixed point problem. In the second example, we revisit a problem from singular perturbation theory that is solved using the geometric blow-up method, and show that we can achieve the same result again by rewriting it as an appropriate fixed point problem with multiple parts and gluing parts of solutions.

1.2 Nonlocal spikes

Much of the success of modeling has been based on infinitesimal descriptions of physical laws, leading to differential and partial differential equations as models for physical processes. However, many physical interactions are inherently nonlocal, at least at a coarser modeling level, leading to nonlocal spatial coupling, as well as dependence of time evolution on history. From a dynamical systems point of view, a natural question in this context is in how far nonlocally coupled equations may behave qualitatively differently from locally coupled problems. As usual, one can approach this question from several vantage points, striving in particular to point to phenomena where nonlocality generates new phenomena, or delineate situations where nonlocal and local equations behave in analogous fashions. Our contribution here is mainly towards the latter aspect, showing that local bifurcations in nonlocal systems produce coherent structures completely analogous to local systems. We will however also comment on phenomena that are qualitatively different as a result of nonlocality.

In the first part of this paper, we will focus on a fairly simple model problem, stationary solutions to nonlocally coupled systems, solving

$$U + \mathcal{K} * U = \mathcal{N}(U; \mu), \tag{1.2.1}$$

where $U = U(x) \in \mathbb{R}^k$, $x \in \mathbb{R}^n$, $\mathcal{K} * U$ stands for matrix convolution,

$$(\mathcal{K} * U(x))_i = \sum_{j=1}^{m} \int_{\mathbb{R}^n} \mathcal{K}_{i,j}(x-y) U_j(y) dy, \qquad 1 \leq i \leq k,$$

and $\mathcal{N}(U; \mu)$ encodes nonlinear terms which depend on a parameter $\mu \sim 0$.

We are interested in the existence of spatially localized solutions $U(x) \to U_\infty$, $|x| \to$

∞ in the prototypical setting of a transcritical bifurcation of spatially constant solutions $U(x) \equiv m$.

In the remainder of the introduction for the first part, we shall first provide some background and motivation for this kind of question, Section 1.2.1, and then give precise assumptions and results in Sections 1.2.2 and 1.2.3. We collect some standard notation used throughout the first part at the end.

1.2.1 Motivation

Applications with nonlocal coupling. Nonlocal coupling has been suggested as a more appropriate modeling assumptions in fields all across the sciences. Prominent examples are nonlocal dispersal of plant seeds in ecology [1], fractional powers of the Laplacian as limits of random walks with Levy jumps [2], models for neural fields [3], nonlocal interactions in models for Bose-Einstein condensates [4], kinetic equations for interacting particles [5], shallow water-wave models [6], or material science [7]. In many of these models, one is interested in spatially localized or front-like solutions, stationary, periodic, or propagating in time, which we will here refer to collectively as coherent structures. Such solutions are usually found through a traveling-wave ansatz, thus eliminating or compactifying time dependence. In some special situations, Many of the above examples can thereby be reduced to problems of the type (1.2.1), and we will give details for some cases in Section 2.3.

Arguably, the most powerful results for existence and stability of coherent structures rely on formulating the existence problem as an ordinary differential equation, a method sometimes referred to as "spatial dynamics" [8] — for locally coupled equations, in essentially one-dimensional geometries such as the real line, cylinders $\mathbb{R} \times \Omega$, or with radial symmetry. We will briefly overview such results from our perspective here and discuss in how far they generalize to nonlocally coupled equations, before turning to our contribution in more detail in Sections 1.2.2–1.2.3.

Local coupling — results. Replacing nonlocal coupling by diffusive coupling, say $D\Delta U$, with D positive, diagonal, we can consider the elliptic system

$$\Delta U + \mathcal{N}(U; \mu) = 0, \tag{1.2.2}$$

with $x \in \mathbb{R}^n$. In the simplest case of $n = 1$, one can study the resulting ordinary differential equation

$$U_x = V, \qquad V_x = -\mathcal{N}(U;\mu),$$

using dynamical systems methods such as center-manifold reduction and normal form transformations, thus leading to nearly universal predictions for bifurcations of coherent structures. In addition to existence results, such methods also allow one to state rather general uniqueness and non-existence results. In higher space dimensions, $n > 1$, such reduction techniques are not known to be applicable, except for the context of radial symmetry, which allows one to formulate the existence problem as a dynamical system in the radial variable r,

$$U_r = V, \qquad V_r = -\frac{n-1}{r}V - \mathcal{N}(U;\mu).$$

Slightly adapted center manifold and normal form theory again leads to near-universal classifications of small-amplitude coherent structures; see [9] for a comprehensive exposition of these techniques and [10] for background on center manifolds and normal forms.

In the simplest case of a transcritical bifurcation in the nonlinearity, with suitable additional assumptions on the linear part, one finds a reduced equation on the center manifold of the form

$$\begin{aligned} u_x &= v + \mathrm{O}\left(|\mu|(|u| + |v|) + u^2 + v^2\right), \\ v_x &= \mu u - u^2 + \mathrm{O}\left(v^2 + (|u| + |v|)(\mu^2 + u^2 + v^2)\right), \end{aligned}$$

which at leading order, after scaling, reduces to

$$u_{xx} - u + u^2 = 0, \tag{1.2.3}$$

with an explicit nontrivial localized solution $u(x) \to 0$, $|x| \to \infty$. Using reversibility $x \mapsto -x$ one then readily shows persistence of this solution at higher order. Exploiting the characterization of center manifolds, one also obtains non-existence of other localized solutions and, in fact, a complete characterization of small bounded solutions.

The results in [9] extend this machinery to radially symmetric solutions in higher space dimension, leading to a leading order equation of the form

$$u_{rr} + \frac{n-1}{r}u_r - u + u^2 = 0.$$

This type of results is also available for Turing and Hopf bifurcations [9]. It is however not immediately applicable to the type of nonlocally coupled equations described above, with recent progress that we shall describe next.

Nonlocal coupling — center manifolds. Going back to (1.2.1), we can in general not find an obvious formulation as a dynamical system, with the notable exception of convolution kernels with a rational Fourier transform. Consider for instance $\mathcal{K}(x) = \frac{1}{2}\mathrm{e}^{-|x|}$, $x \in \mathbb{R}$, with Fourier transform $\hat{\mathcal{K}}(\xi) = (1+\xi^2)^{-1}$, for which we can formally write (1.2.1) as $(\,\mathrm{id}\, - \Delta)^{-1}U + \mathcal{N}(U;\mu) = 0$, which in turn is equivalent to

$$\begin{aligned} -U + W + \mathcal{N}(U;\mu) &= 0, \\ W - \Delta W &= U. \end{aligned}$$

Under suitable assumptions, equivalent to stability assumptions made in [9], one can solve the first equation for $U = \Psi(W;\mu)$ and insert into the second equation, thus obtaining a local equation for W,

$$\Delta W - W + \Psi(W;\mu) = 0,$$

which is amenable to the methods from [9].

The restriction to kernels with rational Fourier transform is clearly restrictive, excluding for instance Gaussians, and one naturally wonders if similar results hold outside of this class. The more recent results in [11] answer this question in the affirmative, for $x \in \mathbb{R}$ and $\mathcal{K}, \mathcal{K}'$ exponentially localized, such that $\hat{\mathcal{K}}$ is analytic in a strip of the complex plane $|\,\mathrm{Re}\,\xi| < \eta$.

Some smoothness of $\mathcal{K}$ appears to be necessary, as the counter example of $\mathcal{K} =$

$\frac{1}{2}(\delta_{-1}+\delta_1)$ shows, which produces a simple iteration

$$\frac{1}{2}\left(U(x+1)-2U(x)+U(x-1)\right)+\mathcal{N}(U;\mu)=0,$$

with completely uncorrelated solutions on lattices $x \in x_0+\mathbb{Z}$. A finite-dimensional reduction to an ordinary differential equation here does not seem possible.

Also, exponential localization appears to be necessary. Algebraically localized kernels typically yield algebraically localized profiles but solutions of reduced differential equations would typically converge exponentially. Nevertheless, our results remove the assumption of exponential localization, at the expense of lacking a general uniqueness argument. We still reduce to the simple ordinary differential equation (1.2.3) or its higher-dimensional analogue, with exponentially localized solutions, and find weaker far-field decay only at higher order in the bifurcation parameter μ.

1.2.2 Setup — linear nonlocal diffusive coupling and local bifurcations of spatially constant states

Within the context described in the previous section, we are now ready to state our main assumptions and results. We start with assumptions on the linear part, keeping in mind that the nonlinearity will be assumed to be of quadratic order in U, μ, then turn to assumptions on the nonlinearity, before stating our main result.

Linear diffusive coupling. Let I_k denote the identity matrix of size k and consider the linearized operator $I_k + \mathcal{K}*$.

Hypothesis (L) *We assume that $\mathcal{K}$ satisfies the following properties:*

(i) localization: *$\mathcal{K}$ has finite moments of order 2, that is,*

$$\mathcal{K}(x),\ |x|^2\mathcal{K}(x) \in L^1(\mathbb{R}^n, \mathbb{R}^{k\times k});$$

(ii) symmetry: *$\mathcal{K}(x)=\mathcal{K}(\gamma x)$ for all $x\in\mathbb{R}^n$ and all $\gamma\in\Gamma\subset\mathbf{O}(n)$, a subgroup of the orthogonal matrices with*

$$\operatorname{Fix}\Gamma=\{x \mid \gamma x = x,\ \textit{for all } \gamma\in\Gamma\}=\{0\};$$

(iii) minimal nullspace: $\ker(I_k + \int \mathcal{K}) = \operatorname{span} \boldsymbol{e}$ *for some* $0 \neq \boldsymbol{e} \in \mathbb{R}^k$*; we then choose* $\boldsymbol{e}^*$ *such that* $\ker(I_k + \int \mathcal{K}^T) = \operatorname{span} \boldsymbol{e}^*$*;*

(iv) nondegenerate second moments: *the matrix of projected second moments* S *with entries*

$$S_{ij} = \int x_i x_j \langle \boldsymbol{e}^*, \mathcal{K}(x)\boldsymbol{e}\rangle \mathrm{d}x,$$

is positive definite;

(v) invertibility for nonzero wavenumbers: $I_k + \int \mathrm{e}^{\mathrm{i}\langle \xi, x\rangle}\mathcal{K}(x)\mathrm{d}x$ *is invertible for all* $\xi \neq 0$.

The assumption on positive definiteness can be readily replaced by negative definiteness, simply multiplying the equation by -1. Note that first moments, $\int x\mathcal{K}(x)\mathrm{d}x$ vanish, since

$$\int x\mathcal{K}(x)\mathrm{d}x = \int xK(\gamma x)\mathrm{d}x = \gamma^{-1}\int y\mathcal{K}(y)\mathrm{d}y, \tag{1.2.4}$$

for all $\gamma \in \Gamma$, hence $\int x\mathcal{K}(x)\mathrm{d}x = 0$ since Γ fixes the origin, only. In fact, this is the primary reason for us to require the symmetry mentioned here. Nonvanishing first moments correspond to an effective directional transport which would need to be compensated by a drift term $c \cdot \nabla U$, say, in order to find coherent structures.

Typical examples of symmetry groups Γ are $\Gamma = \mathbf{O}(n)$, $\Gamma = \{\mathrm{id}, -\mathrm{id}\}$ or the group generated by reflections at hyperplanes, $x_j \mapsto -x_j$.

Remark 2.1 (normalizing second moments) *There exists a coordinate change* $x = T_0 y$ *such that the transformed kernel* $\tilde{\mathcal{K}}(y) := |\det T_0|\mathcal{K}(T_0 y)$ *satisfies*

$$\tilde{S}_{ij} = \int x_i x_j \langle \boldsymbol{e}^*, \tilde{\mathcal{K}}(x)\boldsymbol{e}\rangle \mathrm{d}x = 2\delta_{ij}.$$

Indeed, let λ_i*,* $i = 1, \ldots, k$ *be the eigenvalues of* S *and* P_i *be the associated spectral projections. Define*

$$T_0 = \sum_i \sqrt{2}(\lambda_i)^{-1/2}P_i, \qquad \tilde{M}(y) = \langle \boldsymbol{e}^*, \tilde{K}(x)\boldsymbol{e}\rangle, \qquad M(y) = \langle \boldsymbol{e}^*, K(x)\boldsymbol{e}\rangle.$$

Then

$$\tilde{S}_{ij} = \int x_i x_j \tilde{M}(x)\mathrm{d}x = \int (T_0^{-1}y)_i (T_0^{-1}y)_j M(y)\mathrm{d}y = \sum_{k,l} T_{0,ik}^{-1} T_{0,jl}^{-1} S_{kl} = T_0^{-1} S (T_0^{-1})^T$$

$$= \left(\sum_\ell \sqrt{2}\lambda_\ell^{-1/2} P_\ell\right)\left(\sum_m \lambda_m P_m\right)\left(\sum_k \sqrt{2}\lambda_k^{-1/2} P_k\right) = 2\sum P_\ell = 2I_k.$$

Note that the new kernel $\tilde{\mathcal{K}}$ possesses all the properties assumed for $\mathcal{K}$ in Hypothesis (L). In particular, $\tilde{K}$ is invariant under Γ. To see this, notice that $S\gamma = \gamma S$ for all γ, and conclude that $T_0\gamma = \gamma T_0$ since spectral projections commute with $\gamma \in \Gamma$, as well. As a consequence, $\tilde{K}$ is invariant.

The assumptions can also be stated in terms of the associated Fourier determinant $\mathcal{D}$,

$$\mathcal{D}(\xi) := \det(I_k + \widehat{\mathcal{K}}(\xi)).$$

One readily finds that $\mathcal{D}$ is of class $\mathscr{C}^2$ by the assumption on second moments, and

$$\mathcal{D}(0) = 0, \qquad \mathcal{D}'(0) = 0, \qquad \mathcal{D}''(0) \neq 0.$$

The characteristic function $\mathcal{D}$ was also used in [11], identifying zeros of $\mathcal{D}$ on the real axis with bounded solutions to the linear equation, and, more generally, multiplicities of zeros adding up to the dimension of a reduced center manifold as algebraic multiplicities of a center subspace. In the setup there, $\mathcal{D}$ was analytic, allowing readily for characterizing the multiplicity of roots. We here assume just enough regularity, $\mathscr{C}^2$ to make sense of a "double root" of $\mathcal{D}$.

Remark 2.2 (generalizing linear assumptions) *Most examples of nonlinear problems would involve a nontrivial pointwise linear part, say, $AU + \mathcal{K} * U$. One quickly sees that these and more general terms should be viewed as less smooth, namely Dirac-δ contributions to the matrix kernel. Whenever this principle part, say, the matrix A, is invertible, the system can be readily put into our form by applying A^{-1}. On the other hand, when this principal part possesses a kernel, our assumption of invertibility for nonzero wavenumbers would be violated asymptotically, for $\ell \to \infty$. Bifurcation solutions in such situations are not necessarily smooth.*

Yet a different interpretation would refer to the spectrum of the linear part $I_k + \mathcal{K}*$, given by the set of λ for which $I_k + \mathcal{K} * -\lambda I_k$ is not invertible, or, equivalently, the closure of the set of λ such that $\det(I_k + \widehat{\mathcal{K}}(\xi) - \lambda I_k) = 0$ for some $\xi \in \mathbb{R}$. Clearly, $\lambda = 0$ is in the spectrum, choosing $\xi = 0$. Also, $\lambda = 0$ is minimal in multiplicity in the sense that it is an eigenvalue only for $\xi = 0$, and its geometric multiplicity at $\xi = 0$ is minimal. Assuming that, in addition, $\lambda = 0$ is algebraically simple, one readily finds that the continuation of λ as an eigenvalue in ξ is quadratic, $\lambda = \langle \mathcal{S}\xi, \xi\rangle + \ldots$, with definite symmetric matrix $\mathcal{S}$. Such spectral information can in general be converted into heat decay estimates for $U_t = -U + \mathcal{K} * U$ [12].

Transcritical bifurcation for spatially constant solutions. As mentioned, we assume the presence of a simple transcritical bifurcation in the nonlinearity $\mathcal{N}$. The assumptions that follow are generic and necessary for a non-degenerate bifurcation scenario; see for example [13]. They single out relevant terms in systems of equations that lead to bifurcations as in the simple scalar example $\mathcal{N}(u;\mu) = \mu u - u^2$.

Hypothesis (TC) *We assume that* $\mathcal{N} = \mathcal{N}(U;\mu) : \mathbb{R}^k \times \mathbb{R} \to \mathbb{R}^k$ *satisfies the following conditions:*

(i) smoothness: $\mathscr{C}^K(\mathbb{R}^k \times \mathbb{R}; \mathbb{R}^k)$, $K = 1 + \ell + 2$*;*

(ii) trivial solution: $\mathcal{N}(0;\mu) = 0$ *for all* μ*;*

(iii) criticality: $D_U\mathcal{N}(0;0) = 0$*;*

(iv) generic linear unfolding:

$$\alpha := \langle D_{\mu,U}\mathcal{N}(0;0)e, e^*\rangle \neq 0; \tag{1.2.5}$$

(v) generic nonlinearity:

$$\beta := \frac{1}{2}\langle D_{U,U}\mathcal{N}(0;0)[e,e], e^*\rangle \neq 0. \tag{1.2.6}$$

Smoothness assumptions ensure that the superposition operator $U(\cdot) \mapsto \mathcal{N}(U(\cdot);\mu)$ defined by $\mathcal{N}$ is of class $\mathscr{C}^1$ as a map on $H^\ell(\mathbb{R}^n, \mathbb{R}^k)$; see for instance [14].

The choice of a transcritical setup here is for convenience and other elementary bifurcations can be treated in a similar fashion. A saddle-node bifurcation, where $\langle D_\mu \mathcal{N}(0;0), e^*\rangle \neq 0$, $\langle D_{U,U}\mathcal{N}(0;0)[e,e], e^*\rangle \neq 0$, can be transformed into a transcritical bifurcation after subtracting one of the branches, $\tilde{U} = U - U_-(\mu)$ and reparameterizing, say, $\mu = \tilde{\mu}^2$. Also, more general nonlocal dependence of $\mathcal{N}$ on U can be allowed. In such a case, Hypothesis (TC) applies to $\mathcal{N}$ acting on spatially constant solutions U.

1.2.3 Bifurcation of spikes — main result

We are now ready to state our main result. As suggested above, we would like to compare our problem to the local problem

$$\Delta u - \alpha\mu u + \beta u^2 = 0, \tag{1.2.7}$$

when $U \sim ue$. For $\alpha\mu > 0$, this equation possesses localized ground states of the form

$$u_{\mathrm{c}}(x;\mu) = -\alpha\beta^{-1}\mu u_*(\sqrt{\alpha\mu}x), \tag{1.2.8}$$

where $u_*(y)$ is the (positive) ground state to

$$\Delta u - u + u^2 = 0; \tag{1.2.9}$$

see [15] for background information on existence of such ground states and their properties.

Theorem 1 (bifurcation of spikes) *Fix $n < 6$ and $\ell > n/2$. Assume Hypotheses (L) and (TC), and recall the definition of T_0 from Remark 2.1 and α, β from* (1.2.5) *and* (1.2.6).

There then exists a constant $\mu_0 > 0$ such that for all $0 < \alpha\mu < \mu_0$, the nonlocally coupled system (1.2.1) *possesses a family of nontrivial solutions $U_* = U_*(\cdot;\mu) \in H^\ell(\mathbb{R}^n;\mathbb{R}^k)$. Moreover, $U_*(x;\mu)$ is given to leading order through*

$$U_*(x;\mu) = -\alpha\beta^{-1}\mu\,[u_*(\sqrt{\alpha\mu}\,T_0 x) + w(\sqrt{\alpha\mu}\,x;\mu)]\,e + u_\perp(x;\mu), \tag{1.2.10}$$

where

- *u_* from (1.2.8) is the (scalar) radially symmetric ground state to (1.2.9);*
- *the corrector $w(\sqrt{\alpha\mu}\, x;\mu)$ satisfies $\|w(\cdot;\mu)\|_{H^\ell} \to 0$ as $\mu \to 0$;*
- *$\langle u_\perp(x;\mu), e\rangle = 0$, and $\|u_\perp\|_{H^\ell} = \mathrm{O}(\mu^2)$.*

Moreover, $U_*(x;\mu) = U_*(\gamma x;\mu)$ for all $\gamma \in \Gamma$.

We comment briefly on the scope of this result and outline the main idea of proof.

We believe that our assumptions are to some extent sharp. The assumption $n < 6$ relates to subcriticality of the nonlinearity in the local, scalar version of our problem, $\Delta w - w + w^p = 0$. Subcriticality of the nonlinearity implies existence of ground states and minimal critical spectrum, and holds for $1 < p < \frac{n+2}{n-2}$ ([15, Lem. 13.3]), or in our case, $n < 6$ for $p = 2$. Second moments are necessary to define diffusive behavior and obtain the limiting ground state problem. One would suspect that weaker localization, $\hat{\mathcal{K}} \sim |\xi|^{2\nu}$ for $\xi \sim 0$ would lead to reduced problems based on the fractional Laplacian $(-\Delta)^\nu$, with somewhat analogous results. Symmetry of the kernel is necessary to some extent to prevent drift of the resulting spikes. In other words, we find at leading order a manifold of translates of a ground state as solutions. The symmetry condition guarantees that all these solutions persist to higher order as non-degenerate fixed points in the fixed point subspace of the action of Γ on profiles. We suspect that alternate assumptions could involve a variational structure in the problem. We will comment on possible drift and the technical ramifications in the discussion section.

1.3 Passage through fold

In the second part of this paper, we study singularly perturbed ordinary differential equations (ODEs) of the form

$$\begin{aligned} \varepsilon \dot{x} &= f(x,y;\varepsilon), \\ \dot{y} &= g(x,y;\varepsilon), \end{aligned} \qquad x \in \mathbb{R}^n, \quad y \in \mathbb{R}^m, \quad 0 < \varepsilon \ll 1, \tag{1.3.1}$$

where f and g are smooth functions.

1.3.1 Background — geometric singular perturbation theory and blow-up

The standard way of studying (1.3.1) is using methods from geometric singular perturbation theory. An brief overview of this theory involves treating (1.3.1), which is referred to as the *slow-system*, together with its equivalent counterpart, the *fast-system*:

$$\begin{aligned} x' &= f(x, y; \varepsilon), \\ y' &= \varepsilon g(x, y; \varepsilon), \end{aligned} \tag{1.3.2}$$

where, if τ denotes the (slow) time variable in (1.3.1), then $t = \tau/\varepsilon$ is the (fast) time variable used in (1.3.2). The dynamics can then be studied by setting $\varepsilon = 0$ in both systems to obtain the so-called *reduced problem*

$$\begin{aligned} 0 &= f(x, y, 0), \\ \dot{y} &= g(x, y, 0), \end{aligned} \tag{1.3.3}$$

and the *layer problem*

$$\begin{aligned} x' &= f(x, y, 0), \\ y' &= 0. \end{aligned} \tag{1.3.4}$$

The basic premise of the theory, which was first laid out by Fenichel, is that the dynamics of the reduced problem (1.3.3) happens on the *critical manifold*

$$S := \{(x, y) \mid f(x, y; 0) = 0\}.$$

One then focuses on a *normally hyperbolic submanifold* of equlibra $S_0 \subset S$ of the layer problem (1.3.4), which will perturb to a so-called "slow manifold", S_ε for $0 < \varepsilon \ll 1$, on which the dynamics of (1.3.1) is an ε-perturbation of the reduced problem (1.3.3). In addition to the existence of a slow manifold, we have the existence of stable and unstable invariant foliations with base S_0, which also persist for $\varepsilon > 0$. More details can be found in [28] or [29].

The above approach relies heavily on the notion of normal hyperbolicity, which may not be always satisfied in the problems to be studied. The most common case where normal hyperbolicity fails is the so-called *fold point*, where the critical manifold S loses its normal hyperbolicity due to a zero eigenvalue in the Jacobian $\frac{\partial f}{\partial x}$.

To overcome these difficulties, Krupa and Szmolyan proposed the method of *blow-up* to extend the reach of geometric singular perturbation theory in [30]. Roughly speaking, it is a set of coordinate transformations which desingularizes the vector field near the fold point so that information can be gained by using standard tools in dynamical system.

For fold points, their example was the following extended system

$$\begin{aligned} u' &= \mu + u^2 + f(u, \mu; \varepsilon), \\ \mu' &= \varepsilon g(u, \mu; \varepsilon), \\ \varepsilon' &= 0 \end{aligned} \tag{1.3.5}$$

where (μ, u, ε) in a sufficiently small neighborhood $\mathcal{U}$ of the origin so that the critical manifold

$$S_0 = \{(\mu, u) \mid \mu + u^2 + f(u, \mu; 0) = 0\}$$

has $(0, 0, 0)$ as the only fold point. Further, a generic condition on the nonlinearity f, g are assumed, so that they have the following expansions

$$f(u, \mu; \varepsilon) = \mathrm{O}(\varepsilon, u\mu, \mu^2, u^3), \quad g(u, \mu; \varepsilon) = 1 + \mathrm{O}(u, \mu, \varepsilon), \tag{1.3.6}$$

for $(\mu, u, \varepsilon) \in \mathcal{U}$.

Away from the fold point $(0, 0, 0)$, there exists the attracting manifold of equilibria S_a with a section sketched in Figure 1. By standard Fenichel's theory, S_a perturbs to $S_{a,\varepsilon}$ until it reaches the fold point. Thus a natural question is to track how a trajectory starting on the slow manifold $S_{a,\varepsilon}$ passes through the fold point.

For a small interval J containing $-\sqrt{\delta}$, Krupa and Szmolyan start by setting two sections $\Delta_{in} = \{(-\delta, u) : u \in J\}$ and $\Delta_{out} = \{(\mu, \sqrt{\delta}) : \mu \in \mathbb{R})\}$ with $\delta > 0$ small but fixed, which are also shown in Figure 1. To track the passage through the fold amounts to studying the transition map

$$\pi : \Delta_{in} \to \Delta_{out},$$

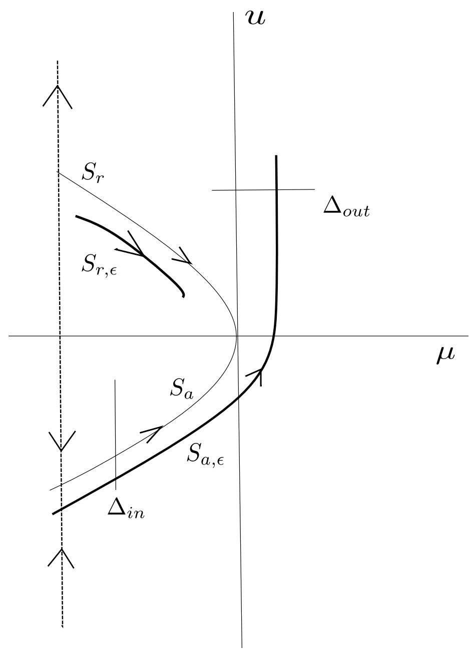

Figure 1: Critical manifold and slow manifolds, sections near the fold.

Krupa and Szmolyan then proceed by defining the blow up transformation

$$u = \bar{r}\bar{u}, \quad \mu = \bar{r}^2\bar{\mu}, \quad \varepsilon = \bar{r}^3\bar{\varepsilon},$$

which "blows up" the vector field of the extended (μ, u, ε) system near $(0,0,0)$ into a vector field on the ball $B = S^2 \times [0, \varepsilon_0^{1/3}] \ni (\bar{u}, \bar{\mu}, \bar{\varepsilon}, \bar{r})$ for some $\varepsilon_0 > 0$ small. Further directional blow-ups to obtain charts K_1, K_2, K_3 are then used to describe the flows in regions near different parts of the manifold B. After a careful and thorough analysis, they were able to prove the following results.

Theorem 2 *(Theorem 2.1 in [30])*

For ε sufficiently small, the following statements are true:

(i) The manifold $S_{a,\varepsilon}$ passes through Δ_{out} at a point $(h(\varepsilon), \sqrt{\delta})$ where $h(\varepsilon) = \mathrm{O}(\varepsilon^{2/3})$.

(ii) The transition map π is a contraction with rate $\mathrm{O}(e^{-c/\varepsilon})$, where c is a positive constant independent of ε.

In this paper, we focus on the same problem (1.3.5) with an aim to recreate the result of Theorem 2 via a different, functional-analytical approach. Instead of proceeding with the geometrically-flavored blow-up approach, we directly prove a trajectory exists with the properties claimed in Theorem 2 by dividing the time of passage into appropriate parts and setting up a corresponding ansatz in each region, closing the arguments by carefully reformulating the existence question as a fixed-point argument.

To summarize, we first make a change of variables to transform (1.3.5) into a more convenient form in Section 1.3.2. Then we introduces the different ansatz used in Sections 1.3.3 and 1.3.4. We then give a statement of our main result in Section 1.3.5.

1.3.2 Euler multiplier

Let τ denote the independent time variable in (1.3.5), since for u, μ, ε small, $g(u, \mu, \varepsilon) = 1 + \mathrm{O}(u, \mu, \varepsilon) > 0$, we can define a new time $t = t(\tau)$ by $\frac{dt}{d\tau} = g$, consequently, (1.3.5) is transformed into

$$\begin{aligned} \frac{d}{dt}u &= \mu + u^2 + \tilde{f}(u, \mu; \varepsilon), \\ \frac{d}{dt}\mu &= \varepsilon, \end{aligned} \tag{1.3.7}$$

where now $\tilde{f}$ satisfies the asymptotics

$$\tilde{f}(u, \mu, \varepsilon) = \mathrm{O}(\varepsilon, u\mu, \mu^2, \varepsilon u, \varepsilon\mu, \varepsilon u^2, u^3). \tag{1.3.8}$$

The critical manifold

$$\tilde{S}_0 = \{(\mu, u) \mid \mu + u^2 + \tilde{f}(u, \mu; 0) = 0\},$$

still has $(0, 0, 0)$ as the only fold point, and our goal is to track how a trajectory of the flow of (1.3.7) passes through the fold. Following the set up of the sections Δ_{in} and Δ_{out} in Krupa-Szmolyan, we propose the following boundary conditions

$$\mu(0) = -\delta_-, \qquad u(T) = \delta_+, \tag{1.3.9}$$

where T is also an unknown variable which marks the “time of exit” as the trajectory hits the section Δ_{out}, and $\delta_\pm$ are small positive numbers, independent of ε.

That is, we think of tracking the passage through the fold as a boundary value problem: we prove the existence of a solution (μ, u) to (1.3.7) with the boundary condition (1.3.9), and give the expansion for the components μ, in order to recover the results in (1.3.5). Our hope is that this new, different approach will prove to be more robust and flexible in new applications. In the rest of the paper, we shall drop the tilde to use $f(u, \mu, \varepsilon)$ as the nonlinearity and S_0 to denote the critical manifold.

1.3.3 The Riccati equation

First, we want to get an idea of what kind of ansatz one might use. Consider (1.3.7) with $f(u, \mu; \varepsilon) = 0$, which can then be scaled such that $\varepsilon = 1$, $\mu = s$, and

$$\frac{d}{ds}u(s) = s + u(s)^2, \tag{1.3.10}$$

which is the Ricatti equation in its simplest form. We denote any solution to (1.3.10) as u_R. It is known that there is a unique solution, denoted by $\bar{u}_R$, with the following asymptotic expansions (see [30] as well),

$$\bar{u}_R(s) = \begin{cases} (\Omega_0 - s)^{-1} + \mathrm{O}(\Omega_0 - s), \text{ as } s \to \Omega_0, \\ -(-s)^{1/2} - \frac{1}{4}(-s)^{-1} + \mathrm{O}(|s|^{-3/2}), \text{ as } s \to -\infty. \end{cases} \tag{1.3.11}$$

Here the constant $\Omega_0 \approx 2.3381$ is the smallest positive zero of

$$J_{-1/3}(2z^{3/2}/3) + J_{1/3}(2z^{3/2}/3),$$

where $J_{\pm 1/3}$ are Bessel functions of the first kind.

More generally, we consider family of solutions $u_R(s; u_0)$ of the Riccati equation (1.3.10) such that $u_R(0; u_0) = u_0$. That is, we take the initial condition u_0 as a parameter to the Riccati equation. For the special Riccati solution $\bar{u}_R$, we define

$$\bar{u}_0 := \bar{u}_R(0). \tag{1.3.12}$$

In fact, using simple phase plane analysis, we can extend the asymptotic results about the special Riccati solution $\bar{u}_R$ to the family of solutions $u_R(s; u_0)$ as follows.

Proposition 3.1 *There exists $\eta > 0$ small, so that for all initial conditions u_0 with $|u_0 - \bar{u}_0| < \eta$, there exist a $\Omega_\infty = \Omega_\infty(u_0)$ which depends smoothly on u_0, and a solution $u_R(s; u_0)$ of (1.3.10) on $[0, \Omega_\infty)$ with the following asymptotic expansion as $s \to \Omega_\infty$:*

$$u_R(s; u_0) = \frac{1}{\Omega_\infty - s} + (\Omega_\infty - s) \cdot r(\Omega_\infty - s; u_0), \tag{1.3.13}$$

where the function r is smooth in both variables and satisfies

$$r(\Omega_\infty - s; u_0) = -\frac{\Omega_\infty}{3} + \mathrm{O}(\Omega_\infty - s), \tag{1.3.14}$$

as $s \to \Omega_\infty$.

We postpone the proof of this proposition to the appendix.

1.3.4 Critical manifold

Another piece of the ansatz comes from part of the critical manifold. We expect this piece of the ansatz because away from the fold, the critical manifold has an attracting branch S_a which implies that the trajectory has to stay close to it.

Recall the critical manifold for (1.3.15) is the set of points (μ, u) in a small neighborhood of the origin which satisfies

$$\mu + u^2 + f(u, \mu; 0) = 0. \tag{1.3.15}$$

From (1.3.8) we see that $\mu = -u^2 + \mathrm{O}(u^3)$. Rescaling $\mu = -\mu_1^2$ with μ_1 positive and $u = \mu_1 u_1$ we see that u_1 satisfies

$$u_1^2 = 1 + \mathrm{O}(\mu_1),$$

and hence we obtain two branches of solutions

$$u_1 = \pm\sqrt{1} + \mathrm{O}(\mu_1),$$

and consequently $u_\pm(\mu) = \sqrt{-\mu} \cdot u_1$, where $u_\pm$ satisfy

$$\begin{aligned} u_-(\mu) &= -\sqrt{-\mu} + \mathrm{O}(|\mu|), \\ u_+(\mu) &= +\sqrt{-\mu} + \mathrm{O}(|\mu|). \end{aligned} \tag{1.3.16}$$

For our ansatz, we set

$$u_s(t) = u_-(\mu(t)), \tag{1.3.17}$$

so that

$$0 = \mu(t) + u_s^2(t) + f(u_s(t), \mu; 0).$$

Due to the simple form of (1.3.7) and (1.3.9), we have that $\mu(t) = \varepsilon t - \delta_-$, hence $u_s(\mu)$ satisfies

$$u_s(t) = -\sqrt{\delta - \varepsilon t} + \mathrm{O}(\delta - \varepsilon t). \tag{1.3.18}$$

1.3.5 Main result - summary

We now state our main result. Recall $\mathcal{U}$ is a neighborhood small enough so that (1.3.6) holds for $(\mu, u, \varepsilon) \in \mathcal{U}$.

Theorem 3 (passage through fold) *Let Ω_0 be the constant defined in* (1.3.11). *Fix $\delta_-, \delta_+ > 0$ and α with $0 < \alpha < 3/4$. There exist $\varepsilon_0 > 0$, a constant $C = C(\delta_-, \delta_+, \alpha)$, such that for all $0 < \varepsilon < \varepsilon_0$ and u_i with $|u_i - u_-(-\delta)| \le C\varepsilon^{1-\alpha/3}$, a solution of the rescaled system*

$$\begin{aligned} \frac{d}{dt}u &= \mu + u^2 + f(u, \mu; \varepsilon), \\ \frac{d}{dt}\mu &= \varepsilon. \end{aligned} \tag{1.3.19}$$

with the initial condition

$$\begin{aligned} u(0) &= u_i, \\ \mu(0) &= -\delta_-, \end{aligned} \tag{1.3.20}$$

exists for $t \in (0, T)$, where the end point T is the desired "exit time" in the sense that

$$u(T) = \delta_+. \tag{1.3.21}$$

Moreover, T has the following expansion in ε

$$T = T(\varepsilon; u_i) = \varepsilon^{-1}\delta_- + \varepsilon^{-1/3}\Omega_0 + \mathcal{H}(\varepsilon; u_i), \tag{1.3.22}$$

where the term $\mathcal{H}(\varepsilon; u_i)$ satisfies

$$\varepsilon\mathcal{H}(\varepsilon; u_i) = \mathrm{O}(\varepsilon^{1-\alpha/3}), \qquad Lip_{u_i}\varepsilon\mathcal{H}(\varepsilon; u_i) = \mathrm{O}(1). \tag{1.3.23}$$

In particular, since $\mu(t) = \varepsilon t - \delta_-$, we have the following expansion of the exit location $\mu(T)$ on the section Δ_{out} in ε:

$$\mu(T) = \varepsilon^{2/3}\Omega_0 + \varepsilon\mathcal{H}(\varepsilon; u_i) = \varepsilon^{2/3}\Omega_0 + \mathrm{O}(\varepsilon^{1-\alpha/3}). \tag{1.3.24}$$

To fully recover Theorem 2, we state the following corollary.

Corollary 3.2 *For $\alpha, \delta_-, \delta_+ > 0$, there is a compact interval $K \subset (-\infty, u_+(-\delta_-))$, independent of ε, so that for all $u_i \in K$ with $(u_i, \mu, \varepsilon) \in \mathcal{U}$, the conclusion of Theorem 3 holds. In fact, there exist a constant c independent of ε, such that the Lipschitz constant of $mathcal H(\varepsilon; u_i)$ in the expansion* (1.3.22) *satisfies*

$$Lip_{u_i}\varepsilon\mathcal{H}(\varepsilon; u_i) = \mathrm{O}(e^{-c/\varepsilon}), \tag{1.3.25}$$

for all $u_i \in K$.

Remark: Notice that $\alpha > 0$ could be chosen arbitrary small in the expansion (1.3.24). In fact, it is well-known from formal matched-asymptotics that the next order term after $\varepsilon^{2/3}\Omega_0$ is of order $\varepsilon \log(\varepsilon)$. This is also confirmed in [30] using the blow-up method.

Chapter 2

Nonlocal spikes

In this chapter we prove Theorem 1. The proof is eventually based on a rather direct contraction mapping principle. We prepare the equation by performing linear transformations singling out the neutral direction $\mathfrak{e}$, followed by scaling and solving the equation in the complement $\mathfrak{e}^{\perp}$. Finally, an ansatz substituting u_* and a suitable corrector yields a contraction mapping for the corrector in a small neighborhood of the origin.

Key to the argument is a precise formulation of the convergence of $-I_k + \mathcal{K}$ to Δ, which we accomplish by carefully preconditioning our system; see equation (2.2.2).

Outline. We perform coordinate changes and rescalings in Section 2.1, preparing for the proof through Lyapunov-Schmidt reduction and contraction mappings in Section 2.2. Section 2.3 shows some more concrete applications of our result, and we conclude with a discussion in Section 2.4.

Notation. For a vector $u = (u_1, \ldots, u_k) \in \mathbb{R}^k$, we write $|u|$ to denote its usual Euclidean norm $|u| = \sum_{i=1}^k u_i^2$. We also use the standard multi-index notation in $\mathbb{R}^n$, that is we have $\alpha = (\alpha_1, \cdots, \alpha_n)$ with $\alpha_i \in \{0, 1, \ldots\}$ and $\alpha! = \alpha_1! \cdots \alpha_n!$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$. So that $D^\alpha u = \dfrac{\partial^{|\alpha|} u}{\partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}}$.

We shall use the standard Sobolev spaces on $\mathbb{R}^n$ with values in $\mathbb{R}^k$, which are denoted by $W^{\ell,p}(\mathbb{R}^n, \mathbb{R}^k)$ or simply $W^{\ell,p}(\mathbb{R}^n)$ when $k = 1$ or even $W^{\ell,p}$ whenever it is convenient to do so and does not cause confusions. For $\ell \geq 0$ and $1 \leq p \leq \infty$

$$W^{\ell,p}(\mathbb{R}^n, \mathbb{R}^k) := \{u \in L^p(\mathbb{R}^n, \mathbb{R}^k) : \partial^\alpha u \in L^p(\mathbb{R}^n, \mathbb{R}^k), 1 \leq |\alpha| \leq \ell\},$$

with norm

$$\|u\|_{W^{\ell,p}(\mathbb{R}^n,\mathbb{R}^k)} = \begin{cases} \left(\sum_{1\leq|\alpha|\leq\ell} \|\partial^\alpha u\|_{L^p(\mathbb{R}^n;\mathbb{R}^k)}\right)^{1/p}, & 1 \leq p < \infty \\ \max_{1\leq|\alpha|\leq\ell} \|\partial^\alpha u\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^k)}, & p = \infty. \end{cases}$$

We use $H^\ell(\mathbb{R}^n, \mathbb{R}^k)$ to denote the space $W^{\ell,2}(\mathbb{R}^n, \mathbb{R}^k)$, we will also use $\mathscr{C}_b^\ell(\mathbb{R}^n, \mathbb{R}^k)$ to denote the space of $\ell-$times bounded continuously differentiable functions for $\ell = 0, 1, \ldots, \infty$.

Finally, we use the usual Fourier transform on $\mathbb{R}^n$,

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)\mathrm{e}^{-\mathrm{i}\langle\xi,x\rangle}\mathrm{d}x,$$

for a Schwartz function f, which extends by isometry to all $f \in L^2(\mathbb{R}^n, \mathbb{R}^k)$.

2.1 Normal forms and scalings

We change variables in $\mathbb{R}^k$ such that the linear part takes a simple form, Section 2.1.1, and introduce the long-wavelength scaling that exhibits the leading order asymptotics in Section 2.1.2.

2.1.1 Normal forms on the linear part

We work in coordinates $y = T_0^{-1}x$, and drop tildes for the transformed kernel. Recall that first moments vanish (1.2.4) and recall the definition of the operator $\mathcal{T} := I_k + \mathcal{K}*$ with symbol $\widehat{\mathcal{T}}(\xi) = I_k + \widehat{\mathcal{K}}(\xi)$. The next lemma isolates the center part of our equation in the first coordinate and factors a long-wavelength contribution of the convolution.

Lemma 1.1 *There exist invertible $k \times k$ matrices P, Q, and a pseudo-differential operator L with symbols $\widehat{L}(\xi), \widehat{L}^{-1}(\xi) \in L^\infty(\mathbb{R}^n, \mathbb{R}^{k\times k})$ such that*

$$\widehat{L}(\xi)P\widehat{\mathcal{T}}(\xi)Q = \operatorname{diag}\left\{\frac{|\xi|^2}{1+|\xi|^2}, I_{k-1}\right\}.$$

Moreover, in the canonical basis of $\mathbb{R}^k$, we can write L and its symbol in matrix form

$$L = \begin{pmatrix} L_{\mathrm{cc}} & L_{\mathrm{ch}} \\ L_{\mathrm{hc}} & L_{\mathrm{hh}} \end{pmatrix}, \qquad \widehat{L}(\xi) = \begin{pmatrix} \widehat{L}_{\mathrm{cc}}(\xi) & \widehat{L}_{\mathrm{ch}}(\xi) \\ \widehat{L}_{\mathrm{hc}}(\xi) & \widehat{L}_{\mathrm{hh}}(\xi) \end{pmatrix},$$

where terms with subscript cc *denote a scalar, terms with subscript* ch *a $(k-1)$ dimensional row vector, terms with subscript* hc *a $(k-1)$ dimensional column vector, and terms with subscript* hh *a $(k-1)\times(k-1)$ matrix. We then have that $\widehat{L}_{\mathrm{cc}}, \widehat{L}_{\mathrm{ch}}, \widehat{L}_{\mathrm{hh}}$ are continuous and bounded in ξ, and $\widehat{L}_{\mathrm{hc}} \in L^\infty(\mathbb{R}^n, \mathbb{R}^{k-1})$.*

Proof. We divide our construction in two steps. We first normalize the constant-coefficient problem, $\xi = 0$ in Fourier space, and expand. We then factor the leading-order Fourier multiplier.

Step 1. Since the rank of $\widehat{\mathcal{T}}(0)$ is equal to $k-1$, there exist invertible matrices P and Q such that

$$P\widehat{\mathcal{T}}(0)Q = \operatorname{diag}\{0, I_{k-1}\}.$$

By finiteness of second moment, the entries of $\widehat{\mathcal{T}}(\xi)$ are $\mathscr{C}^2$ functions in ξ. Taylor expanding near $\xi = 0$, we find

$$P\widehat{\mathcal{T}}(\xi)Q = \begin{pmatrix} \widehat{\mathcal{T}}_{\mathrm{cc}}(\xi) & \widehat{\mathcal{T}}_{\mathrm{ch}}(\xi) \\ \widehat{\mathcal{T}}_{\mathrm{hc}}(\xi) & \widehat{\mathcal{T}}_{\mathrm{hh}}(\xi) \end{pmatrix},$$

where $\widehat{\mathcal{T}}_{\mathrm{cc}} = |\xi|^2 + \mathrm{o}(|\xi|^2)$ using the normalization condition $\int x_i x_j \langle e^*, \tilde{\mathcal{K}}(x)e\rangle dx = 2\delta_{ij}$. The expansion for the other matrix elements follows similarly.

Step 2. Set $H(\xi) := \mathrm{diag}\{\frac{1+|\xi|^2}{|\xi|^2}, I_{k-1}\}$ for $\xi \neq 0$. We find

$$P\widehat{\mathcal{T}}(\xi)QH(\xi) = \begin{pmatrix} \widehat{\mathcal{T}}_{\mathrm{cc}}(\xi) & \widehat{\mathcal{T}}_{\mathrm{ch}}(\xi) \\ \widehat{\mathcal{T}}_{\mathrm{hc}}(\xi) & \widehat{\mathcal{T}}_{\mathrm{hh}}(\xi) \end{pmatrix} \begin{pmatrix} \frac{1+|\xi|^2}{|\xi|^2} & 0 \\ 0 & I_{k-1} \end{pmatrix} = \begin{pmatrix} \widehat{\mathcal{T}}_{\mathrm{cc}}(\xi)\frac{1+|\xi|^2}{|\xi|^2} & \widehat{\mathcal{T}}_{\mathrm{ch}}(\xi) \\ \widehat{\mathcal{T}}_{\mathrm{hc}}(\xi)\frac{1+|\xi|^2}{|\xi|^2} & \widehat{\mathcal{T}}_{\mathrm{hh}}(\xi) \end{pmatrix}.$$

The fact that $\widehat{\mathcal{T}}_{\mathrm{cc}}(0) = D_\xi \widehat{\mathcal{T}}_{\mathrm{cc}}(0) = 0$ and the normalization assumption on the second moment matrix together imply that $\widehat{\mathcal{T}}_{\mathrm{cc}}(\xi) = |\xi|^2 T_2(\xi)$ for some continuous function $T_2(\xi)$. On the other hand, we have $|\widehat{\mathcal{T}}_{\mathrm{hc}}(\xi)|/|\xi|^2 \leq C$ for some constant C near $\xi = 0$. Therefore, for $|\xi| \neq 0$ small,

$$P\widehat{\mathcal{T}}(\xi)QH(\xi) = \begin{pmatrix} 1 & 0 \\ \mathrm{O}(1) & I_{k-1} \end{pmatrix} + \mathrm{o}(|\xi|).$$

It follows that $P\widehat{\mathcal{T}}(\xi)QH(\xi)$ is invertible with uniform bounds on the inverse near $\xi = 0$, and its inverse is also of the form $\begin{pmatrix} 1 & 0 \\ \mathrm{O}(1) & I_{k-1} \end{pmatrix} + \mathrm{o}(|\xi|)$, for $\xi \neq 0$ small.

For ξ bounded away from the origin, it follows from Hypothesis (L)(v) that the matrix $P\widehat{\mathcal{T}}(\xi)QH(\xi)$ is invertible for each ξ. Moreover, $\widehat{\mathcal{T}}(\xi) \to I_k$ for $|\xi| \to \infty$ and $H(\xi) \to I_k$ by Riemann-Lebesgue. We therefore conclude that $P\widehat{\mathcal{T}}(\xi)QH(\xi)$ is invertible on $\mathbb{R}^n \backslash \{0\}$ with uniform bounds.

We then define the pseudo-differential operator L by setting its symbol $\widehat{L}(\xi)$ equal to $[P\widehat{\mathcal{T}}(\xi)QH(\xi)]^{-1}$. Then $\widehat{L}(\xi) \in L^\infty$ and it follows that

$$L(\xi)P\widehat{\mathcal{T}}(\xi)Q = H(\xi)^{-1}\left[P\widehat{\mathcal{T}}(\xi)Q\right]^{-1} P\widehat{\mathcal{T}}(\xi)Q = H(\xi)^{-1} = \mathrm{diag}\left\{\frac{|\xi|^2}{1+|\xi|^2}, I_{k-1}\right\},$$

as claimed.

If we write

$$L = \begin{pmatrix} L_{\mathrm{cc}} & L_{\mathrm{ch}} \\ L_{\mathrm{hc}} & L_{\mathrm{hh}} \end{pmatrix}, \qquad \widehat{L}(\xi) = \begin{pmatrix} \widehat{L}_{\mathrm{cc}}(\xi) & \widehat{L}_{\mathrm{ch}}(\xi) \\ \widehat{L}_{\mathrm{hc}}(\xi) & \widehat{L}_{\mathrm{hh}}(\xi) \end{pmatrix},$$

then it follows from the above computations that the entries $\widehat{L}_{\mathrm{cc}}, \widehat{L}_{\mathrm{ch}}, \widehat{L}_{\mathrm{hh}}$ are continuous at $\xi = 0$ and $\mathscr{C}^2$ for $\xi \neq 0$, while $\widehat{L}_{\mathrm{hc}}$ is only bounded in ξ. This verifies the last claim in the lemma and concludes the proof. ∎

Since $\widehat{L} \in L^\infty$, we know that the pseudo-differential operator L maps H^ℓ into H^ℓ and is bounded. Denote by M the pseudo-differential operator with symbol $|\xi|^2/(1+|\xi|^2) =: m(\xi)$, and set $V(y) = Q^{-1}U(y)$, with standard coordinates $V(y) = (v_\mathrm{c}(y), v_\mathrm{h}(y))^T$. Then, after precondition (1.2.1) with LP, we obtain the equivalent equation

$$\operatorname{diag}\{M, I_{n-1}\}V(y) = LP\mathcal{N}(QV(y);\mu). \tag{2.1.1}$$

In particular, the linear part of the system is block-diagonal, decoupled between invertible and diffusive components. We will next introduce rescalings to further simplify this equation.

2.1.2 Taylor expansion and rescaling

We start by expanding our bifurcation equation and then introduce appropriate scalings.

Taylor jets of the nonlinearity. We write $v_\mathrm{h} = (v_2, \ldots, v_k)^T$ in the standard coordinates of $\mathbb{R}^{k-1}$ and set $\mathcal{H}(V;\mu) := P\mathcal{N}(QV;\mu)$. Then, with respect to the standard basis in $\mathbb{R}^k$, we denote by $\mathcal{H}_\mathrm{c}(V;\mu)$ the first component of the nonlinearity $\mathcal{H}$, and by $\mathcal{H}_\mathrm{h}(V;\mu)$ the remaining $k-1$ components.

In this notation, (1.2.1) becomes

$$Mv_\mathrm{c} + L_\mathrm{cc}\mathcal{H}_\mathrm{c}(v_\mathrm{c}, v_\mathrm{h};\mu) + L_\mathrm{ch}\mathcal{H}_\mathrm{h}(v_\mathrm{c}, v_\mathrm{h};\mu) = 0, \tag{2.1.2}$$

$$v_\mathrm{h} + L_\mathrm{hc}\mathcal{H}_\mathrm{c}(v_\mathrm{c}, v_\mathrm{h};\mu) + L_\mathrm{hh}\mathcal{H}_\mathrm{h}(v_\mathrm{c}, v_\mathrm{h};\mu) = 0. \tag{2.1.3}$$

By Hypothesis (TC)(i), we may use Taylor's theorem to write $\mathcal{H}_j$ as

$$\begin{aligned}\mathcal{H}_j(v_\mathrm{c}, v_\mathrm{h};\mu) &= \Big(a^j_{101}\mu v_\mathrm{c} + a^j_{011}\mu v_\mathrm{h} + a^j_{110}v_\mathrm{c}v_\mathrm{h} + a^j_{200}v_\mathrm{c}^2 + a^j_{020}[v_\mathrm{h}, v_\mathrm{h}]\Big) + \mathcal{R}_j(v_\mathrm{c}, v_\mathrm{h};\mu)\\ &:= \mathcal{B}_j(v_\mathrm{c}, v_\mathrm{h};\mu) + \mathcal{R}_j(v_\mathrm{c}, v_\mathrm{h};\mu),\end{aligned}$$

where $j = c, h$, and with multi-index notation $\omega = (l, m, n)$, $|\omega| = 2$, $a^j_\omega = \dfrac{1}{\omega!}D^\omega\mathcal{H}_j(0,0;0)$. The remainder R_j satisfies the pointwise estimate

$$|R_j(v_\mathrm{c}, v_\mathrm{h};\mu)| = |R_j(V;\mu)| = \mathrm{O}(\mu^2|V| + \mu|V|^2 + |V|^3) \tag{2.1.4}$$

for $(V;\mu)$ close to zero.

We are in particular interested in the terms μv_{c} and v_{c}^2. In (2.1.2), the term μv_{c} is preconditioned by $L_{\mathrm{cc}}a_{101}^c + L_{\mathrm{ch}}a_{101}^h$, and the coefficient of v_{c}^2 is preconditioned by $L_{\mathrm{cc}}a_{200}^c + L_{\mathrm{ch}}a_{200}^h$. Using Hypothesis (TC), we claim that

$$\alpha = a_{101}^c = \widehat{L}_{\mathrm{cc}}(0)a_{101}^c + \widehat{L}_{\mathrm{ch}}(0)a_{101}^h, \quad \text{and} \quad \beta = a_{200}^c = \widehat{L}_{\mathrm{cc}}(0)a_{200}^c + \widehat{L}_{\mathrm{ch}}(0)a_{200}^h. \tag{2.1.5}$$

Indeed, to verify the first assertion, use the definition of L in Lemma 1.1. We find $\widehat{L}_{\mathrm{cc}}(0) = 1$ and $\widehat{L}_{\mathrm{ch}}(0) = (0,\ldots,0)$, thus verifying the second equality $a_{101}^c = \widehat{L}_{\mathrm{cc}}(0)a_{101}^c + \widehat{L}_{\mathrm{ch}}a_{101}^h$. To verify the first equality, let e_1 denote the standard coordinate vector $(1,0,\ldots,0)^T \in \mathbb{R}^k$. Then the derivative $a_{101}^c = \dfrac{\partial^2}{\partial\mu\partial v_{\mathrm{c}}}\mathcal{H}_{\mathrm{c}}(0,0;0)$ is given by

$$\begin{aligned}\langle D_{\mu,V}\mathcal{H}(0;0)e_1, e_1\rangle &= \langle D_{\mu,U}P\mathcal{N}(0;0)Qe_1, e_1\rangle \\ &= \langle PD_{\mu,U}\mathcal{N}(0;0)\boldsymbol{e}, e_1\rangle = \langle D_{\mu,U}\mathcal{N}(0;0)\boldsymbol{e}, \boldsymbol{e}^*\rangle = \alpha,\end{aligned}$$

which verifies the first equality $\alpha = a_{101}^c$. The computations for β are similar.

In the next paragraph, we shall make a series of rescalings to further simplify the equation and exhibit leading order parts.

Rescaling. Recall that throughout $\alpha\mu > 0$. Now set $\tilde{\mu} = \alpha\mu$ and write $\sqrt{\tilde{\mu}} =: \varepsilon$. We then rescale the functions $v_{\mathrm{c}}, v_{\mathrm{h}}$ to $\tilde{v}_{\mathrm{c}}, \tilde{v}_{\mathrm{h}}$ through

$$v_{\mathrm{c}}(\cdot) = \frac{-1}{\beta}\varepsilon^2\tilde{v}_{\mathrm{c}}(\varepsilon\cdot), \quad v_{\mathrm{h}}(\cdot) = \varepsilon^2\tilde{v}_{\mathrm{h}}(\varepsilon\cdot).$$

We substitute these variables into (2.1.2) and (2.1.3), divide the first equation by $(-1/\beta)\varepsilon^4$, the second by ε^2, and then obtain,

$$\varepsilon^{-2}M^{\varepsilon}\tilde{v}_{\mathrm{c}} + \sum_{j=\mathrm{c,h}} L_{\mathrm{cj}}^{\varepsilon}[\tilde{\mathcal{B}}_j(\tilde{v}_{\mathrm{c}},\tilde{v}_{\mathrm{h}}) + \varepsilon^{-4}\tilde{\mathcal{R}}_j(\tilde{v}_{\mathrm{c}},\tilde{v}_{\mathrm{h}};\varepsilon)] = 0, \tag{2.1.6}$$

$$\tilde{v}_{\mathrm{h}} + \sum_{j=\mathrm{c,h}} L_{\mathrm{hj}}^{\varepsilon}[\varepsilon^2\tilde{\mathcal{B}}_j(\tilde{v}_{\mathrm{c}},\tilde{v}_{\mathrm{h}}) + \varepsilon^{-2}\tilde{\mathcal{R}}_j(\tilde{v}_{\mathrm{c}},\tilde{v}_{\mathrm{h}};\varepsilon)] = 0. \tag{2.1.7}$$

Note that both (2.1.6) and (2.1.7) hold pointwise in $z = \sqrt{\tilde{\mu}}y$. Since y is arbitrary, they hold for all $z \in \mathbb{R}^n$. We will subsequently view them as functional equations in $\tilde{v}_{\mathrm{c}}(\cdot)$

and $\tilde{v}_{\mathrm{h}}(\cdot)$.

In Fourier space, we find symbols for the rescaled linear operators M^ε and L_j^ε, $j = \mathrm{cc}, \mathrm{ch}, \mathrm{hc}, \mathrm{hh}$, of the form $m(\varepsilon\xi)$ and $\widehat{L}_j(\varepsilon\xi)$, respectively.

The rescaled nonlinear terms $\tilde{\mathcal{B}}_j, \tilde{\mathcal{R}}_j$ for $j = \mathrm{c}, \mathrm{h}$ are defined through

$$\begin{aligned}\tilde{\mathcal{B}}_j(u,v) &= \frac{a_{101}^j}{\alpha}u + \frac{a_{011}^j}{\alpha}v + a_{110}^j uv + \frac{a_{200}^j}{-\beta}u^2 + a_{020}^j(-\beta)v^2,\\ \tilde{R}_j(u,v;\varepsilon) &= \mathcal{R}_j\left(\frac{\varepsilon^2 u}{-\beta}, \varepsilon^2 v; \frac{\varepsilon^2}{\alpha}\right).\end{aligned}$$

In particular, the coefficient of the term $\tilde{v}_{\mathrm{c}}$ now equals $a_{101}^c/\alpha = 1$, and the coefficient of $\tilde{v}_{\mathrm{c}}^2$ now equals $a_{200}^h/(-\beta) = \beta/(-\beta) = -1$, using (2.1.5). As a consequence, we have

$$\tilde{B}_{\mathrm{c}}(\tilde{v}_{\mathrm{c}}, \tilde{v}_{\mathrm{h}}) = \tilde{v}_{\mathrm{c}} - \tilde{v}_{\mathrm{c}}^2 + \mathrm{O}\left(|\tilde{v}_{\mathrm{h}}| + |\tilde{v}_{\mathrm{h}}|^2 + \tilde{v}_{\mathrm{c}}|\tilde{v}_{\mathrm{h}}|\right).$$

From the Sobolev embedding $H^\ell(\mathbb{R}^n, \mathbb{R}^k) \hookrightarrow L^\infty(\mathbb{R}^n, \mathbb{R}^k)$ with $\ell > n/2$ and the fact that $H^\ell(\mathbb{R}^n, \mathbb{R}^k)$ is a continuous multiplication algebra, we find

$$\|\tilde{\mathcal{B}}_j(u,v)\|_{H^\ell} \le C\left(\|u\|_{H^\ell} + \|v\|_{H^\ell} + \|u\|_{H^\ell}\|v\|_{H^\ell} + \|u\|_{H^\ell}^2 + \|v\|_{H^\ell}^2\right), \tag{2.1.8}$$

for any $u \in H^\ell(\mathbb{R}^n), v \in H^\ell(\mathbb{R}^n, \mathbb{R}^{k-1})$, with some constant C and $j = \mathrm{c}, \mathrm{h}$. For the remainder terms $\tilde{\mathcal{R}}_j$, we have

$$\|\tilde{\mathcal{R}}_j(u,v;\varepsilon)\|_{H^\ell} = \mathrm{O}\left(\varepsilon^6\right), \qquad \|D_u\tilde{\mathcal{R}}_j(u,v;\varepsilon)\|_{H^\ell \to H^\ell} = \mathrm{O}\left(\varepsilon^6\right), \tag{2.1.9}$$

as $\varepsilon \to 0$ from (2.1.4).

For the rescaled linear operator L^ε, using the definition of L in Lemma 1.1 and the Fourier transform characterization of H^ℓ, we find, for $\varepsilon \to 0$,

$$\|(L_{\mathrm{cc}}^\varepsilon - 1)u\|_{H^\ell} \to 0, \quad \|L_{\mathrm{ch}}^\varepsilon v\|_{H^\ell} \to 0, \quad \|(L_{\mathrm{hh}}^\varepsilon - I_{k-1})w\|_{H^\ell} \to 0, \quad \text{and } \|L_{\mathrm{hc}}^\varepsilon v\|_{H^\ell} \le C, \tag{2.1.10}$$

where $u \in H^\ell(\mathbb{R}^n)$, $v \in H^\ell(\mathbb{R}^n, \mathbb{R}^{k-1})$, $w \in H^\ell(\mathbb{R}^n, \mathbb{R}^k)$, and C is a constant independent of ε.

We will study the behavior of the term $\varepsilon^{-2}M^\varepsilon v_{\mathrm{c}}$ as $\varepsilon \to 0$ in the next section. To further ease notation, we drop tildes, and use $v_j, \mathcal{B}_j, \mathcal{R}_j$ ($j = \mathrm{c}, \mathrm{h}$) for the variables also after the rescaling.

2.2 Lyapunov-Schmidt reduction, leading-order ansatz, and corrections

We first solve (2.1.7) to obtain v_h as a function of v_c using a fixed point argument in Section 2.2.1. We then substitute this function back into (2.1.6) to obtain a scalar equation for v_c and ε, which we solve again using a fixed point argument in Section 2.2.2. For this, the crucial ingredients are estimates on the operator M^ε stated in Lemma 2.3.

2.2.1 Lyapunov Schmidt reduction

We write the left hand side of (2.1.7) as $\mathcal{G}(v_\mathrm{h}; v_\mathrm{c}, \varepsilon)$, where

$$\mathcal{G}(v; u, \varepsilon) = v + \sum_{j=c,h} L^\varepsilon_{\mathrm{hj}} \left(\varepsilon^2 \mathcal{B}_j(u, v; \varepsilon) + \varepsilon^{-2} \mathcal{R}_j(u, v; \varepsilon) \right).$$

Using estimates (2.1.8) and (2.1.9), we have $\mathcal{G} : H^\ell(\mathbb{R}^n, \mathbb{R}^{k-1}) \times H^\ell(\mathbb{R}^n) \to H^\ell(\mathbb{R}^n, \mathbb{R}^{k-1})$ for each $\varepsilon > 0$, small. Note that we are treating v_c as an additional (Banach space-valued) parameter. The following lemma accomplishes the key reduction step to a scalar equation.

Lemma 2.1 *Fix $r > 0$ not necessarily small and let B_r denote the ball centered at 0 with radius r in $H^\ell(\mathbb{R}^n)$. Then there exists $\varepsilon_0 > 0$, sufficiently small, and a map $\psi(u, \varepsilon) : B_r \times (0, \varepsilon_0) \to H^\ell(\mathbb{R}^n, \mathbb{R}^{k-1})$, such that $v = \psi(u, \varepsilon)$ solves $\mathcal{G}(v; u, \varepsilon) = 0$. Moreover, the map $u \mapsto \psi(u, \varepsilon)$ is of class $\mathscr{C}^1$ for $u \in B_r$, and we have*

$$\|\psi(u, \varepsilon)\|_{H^\ell} = \mathrm{O}(\varepsilon^2), \qquad \|D_u \psi(u, \varepsilon)\|_{H^\ell \to H^\ell} = \mathrm{O}(\varepsilon^2),$$

as $\varepsilon \to 0$, uniformly for $u \in B_r$. Here $D_u \psi(u, \varepsilon)$ denotes the derivative of ψ with respect to u at the point (u, ε).

Proof. We solve $\mathcal{G}(v; u, \varepsilon) = 0$ using a Newton iteration scheme. For $u \in B_r$ and ε_0 small, we claim the following properties for $\mathcal{G}$:

(i) $\|\mathcal{G}(0; u, \varepsilon)\|_{H^\ell} = \mathrm{O}(\varepsilon^2)$, uniformly in $u \in B_r$ and $\varepsilon < \varepsilon_0$;

(ii) $\mathcal{G}$ is smooth in v, and $D_v\mathcal{G}(0;u,\varepsilon) : H^\ell(\mathbb{R}^n,\mathbb{R}^{k-1}) \to H^\ell(\mathbb{R}^n,\mathbb{R}^{k-1})$ is bounded invertible with uniform bounds on the inverse for $\varepsilon < \varepsilon_0$ and $u \in B_r$.

For (i), since L^ε is uniformly bounded in ε, there exist a constant C such that $\|L^\varepsilon_{\mathrm{hc}}\|_{H^\ell\to H^\ell} + \|L^\varepsilon_{\mathrm{hh}}\|_{H^\ell\to H^\ell} \le C$. We then have

$$\begin{aligned}\|\mathcal{G}(0,u;\varepsilon)\|_{H^\ell} \le \varepsilon^2 C(\|\mathcal{B}_{\mathrm{c}}(u,0;\varepsilon)\|_{H^\ell} + \|\mathcal{B}_{\mathrm{h}}(u,0;\varepsilon)\|_{H^\ell})\\ + \varepsilon^{-4} C(\|\mathcal{R}_{\mathrm{c}}(u,0;\varepsilon)\|_{H^\ell} + \|\mathcal{R}_{\mathrm{h}}(u,0;\varepsilon)\|_{H^\ell}).\end{aligned}$$

Using estimates (2.1.8) and (2.1.9), we have $\|\mathcal{G}(0;u,\varepsilon)\|_{H^\ell} \le C(r)\varepsilon^2$ uniformly in $u \in B_r$ and ε small.

For (ii), we conclude the smoothness of $\mathcal{G}$ in v by the smoothness of the superposition operator and the fact that L^ε_j are bounded linear operators. We compute the Fréchet derivative of $\mathcal{G}$ and obtain

$$D_v\mathcal{G}(v;u,\varepsilon)w = w + \sum_{j=\mathrm{c,h}} L^\varepsilon_{\mathrm{h}j}\left(\varepsilon^2 D_v\mathcal{B}_j(u,v;\varepsilon) + \varepsilon^{-2} D_v\mathcal{R}_j(u,v;\varepsilon)\right) w,$$

for $w \in H^\ell(\mathbb{R}^n,\mathbb{R}^{k-1})$. Using estimate (2.1.9), we see that $D_v\mathcal{G}(0;u,\varepsilon)$ is an $\mathrm{O}(\varepsilon^2)$ perturbation of the identity as an operator on $H^\ell(\mathbb{R}^n,\mathbb{R}^{k-1})$ uniformly for $u \in B_r$. Thus, if ε_0 is small enough, then for all ε with $\varepsilon < \varepsilon_0$, we have that $D_v\mathcal{G}(0;u,\varepsilon)$ is bounded invertible with uniform bounds in ε.

Having established (i) and (ii), we fix $\delta > 0$ and $u \in B_r$. Let N_δ denote the closed ball of radius δ around 0 in $H^\ell(\mathbb{R}^n,\mathbb{R}^{k-1})$, we introduce a map $\mathcal{S}(\cdot;u,\varepsilon)$: $H^\ell(\mathbb{R}^n,\mathbb{R}^{k-1}) \to H^\ell(\mathbb{R}^n,\mathbb{R}^{k-1})$ through

$$\mathcal{S}(v;u,\varepsilon) = v - D_v\mathcal{G}(0;u,\varepsilon)^{-1}[\mathcal{G}(v;u,\varepsilon)].$$

We then find

$$\|\mathcal{S}(0;u,\varepsilon)\|_{H^\ell} \le \|D_v\mathcal{G}(0;u,\varepsilon)^{-1}\|_{H^\ell\to H^\ell}\|\mathcal{G}(0;u,\varepsilon)\|_{H^\ell} = \mathrm{O}(\varepsilon^2).$$

Also, $D_v\mathcal{S}(0;u,\varepsilon) = 0$ by definition, and $\mathcal{S}$ is smooth in v by (ii). Therefore, if δ is small and $v \in N_\delta$, it then follows that $\|D_v S(v;u,\varepsilon)\|_{H^\ell\to H^\ell} \le C\delta$ for some constant C independent of δ.

We now start our iteration with $v_0 = 0$, $v_{n+1} := \mathcal{S}(v_n; u, \varepsilon)$, $n \geq 0$. Suppose that by induction $v_k \in N_\delta$, for $1 \leq k \leq n$. Then

$$\|v_{n+1} - v_n\|_{H^\ell} \leq C\delta \|v_n - v_{n-1}\|_{H^\ell},$$

by the mean value theorem. Therefore

$$\|v_{n+1}\|_{H^\ell} \leq \frac{C}{1 - C\delta} \|v_1 - v_0\|_{H^\ell} = \frac{C}{1 - C\delta} \|\mathcal{S}(0; u, \varepsilon)\|_{H^\ell}.$$

This implies that for ε small and $u \in B_r$, we have $v_{n+1} \in N_\delta$, and that $\mathcal{S}$ is a contraction for δ sufficiently small. As in Banach's fixed point theorem, we conclude that $v_n \to v = \psi(u, \varepsilon)$ as $n \to \infty$, and that v is a fixed point of $\mathcal{S}$. Note that, from the construction, we also obtain $\|\psi(u, \varepsilon)\|_{H^\ell} = \mathrm{O}(\varepsilon^2)$, uniformly for $u \in B_r$.

To show the smooth dependence of $\psi(u, \varepsilon)$ on u, we note that $\mathcal{G}(v; u, \varepsilon)$ is also smooth in u by Hypothesis (TC). Choosing ε small, the contraction constant for $\mathcal{S}$ can be chosen uniformly for $u \in B_r$. Hence by the uniform contraction principle, e.g. [16, Thm 1.244], we conclude that ψ depends smoothly on u as well.

Finally, in order to show that $\|D_u \psi\|_{H^\ell \to H^\ell} = \mathrm{O}(\varepsilon^2)$, we differentiate the equation $0 = \mathcal{G}(\psi(u, \varepsilon); u, \varepsilon)$ in u for $u \in B_r$ to see that $D_u \psi$ satisfies the equation

$$D_v \mathcal{G}(\psi(u, \varepsilon); u, \varepsilon) D_u \psi(u, \varepsilon) + D_u \mathcal{G}(\psi(u, \varepsilon); u, \varepsilon) = 0.$$

Now, $D_u \mathcal{G}(v; u, \varepsilon)$ is of the form

$$D_u \mathcal{G}(v; u, \varepsilon) w = \sum_{j = \mathrm{c}, \mathrm{h}} L^\varepsilon_{\mathrm{hj}} (\varepsilon^2 D_u \mathcal{B}_j(u, v; \varepsilon) + \varepsilon^{-2} D_u \mathcal{R}_j(u, v; \varepsilon)) w.$$

Hence, for $u \in B_r$ and $v = \psi(u, \varepsilon) \in N_\delta$, we have $\|D_u \mathcal{G}(v; u, \varepsilon)\|_{H^\ell \to H^\ell} = \mathrm{O}(\varepsilon^2)$, again by (2.1.8) and (2.1.9).

On the other hand, $D_v \mathcal{G}(v; u, \varepsilon)$ is uniformly invertible in ε for $v = \psi(u, \varepsilon) \in N_\delta$ and $u \in B_r$ as shown above. Therefore we can write $D_u \psi(u, \varepsilon) = -[D_v \mathcal{G}]^{-1} D_u \mathcal{G}$ and conclude that

$$\|D_u \psi(u, \varepsilon)\|_{H^\ell \to H^\ell} \leq C(r, \delta) \varepsilon^2.$$

This finishes the proof. ■

Remark 2.2 *We cannot use the standard implicit function theorem directly to solve the equation* $\mathcal{G}(v;u,\varepsilon)=0$ *since the dependence of the convolution operators* $L^{\varepsilon}_{\mathrm{hc}}, L^{\varepsilon}_{\mathrm{hh}}$ *on* ε *is not well-defined at* $\varepsilon=0$.

2.2.2 Preconditioning the reduced equation and existence of spikes

We substitute $v_{\mathrm{h}}=\psi(v_{\mathrm{c}},\varepsilon)$ from Lemma 2.1 into (2.1.6) and obtain the scalar equation,

$$0=\varepsilon^{-2}M^{\varepsilon}v_{\mathrm{c}}+\sum_{j=\mathrm{c,h}}L^{\varepsilon}_{\mathrm{cj}}\left[B_j(v_{\mathrm{c}},\psi(v_{\mathrm{c}},\varepsilon))+\varepsilon^{-4}\mathcal{R}_j(v_{\mathrm{c}},\psi(v_{\mathrm{c}},\varepsilon);\varepsilon)\right]. \tag{2.2.1}$$

The key issue now is the behavior of the linear operator M^{ε} as $\varepsilon\to 0$. Recall that by construction

$$\widehat{M^{\varepsilon}v}(\xi)=m(\varepsilon\xi)\widehat{v}(\xi)=\frac{|\varepsilon\xi|^2}{1+|\varepsilon\xi|^2}\widehat{v}(\xi),$$

for any $v\in H^{\ell}(\mathbb{R}^n,\mathbb{R}^k)$. We then define a new operator $\mathcal{M}^{\varepsilon}$ through

$$\widehat{\mathcal{M}^{\varepsilon}v}(\xi)=\frac{m(\varepsilon\xi)}{|\varepsilon\xi|^2}\widehat{v}(\xi)=\frac{1}{1+|\varepsilon\xi|^2}\widehat{v}(\xi).$$

Since $1/(1+|\varepsilon\xi|^2)$ is a bounded function on $\mathbb{R}^n$, $\mathcal{M}^{\varepsilon}$ maps $H^{\ell}(\mathbb{R}^n,\mathbb{R}^k)$ into itself. For $v\in H^{\ell}(\mathbb{R}^n,\mathbb{R}^k)$, $(\mathcal{M}^{\varepsilon})^{-1}$ is defined through

$$\widehat{(\mathcal{M}^{\varepsilon})^{-1}}v(\xi)=\frac{|\varepsilon\xi|^2}{m(\varepsilon\xi)}\widehat{v}(\xi)=(1+|\varepsilon\xi|^2)\widehat{v}(\xi).$$

Moreover, we have

$$\begin{aligned}\|((\mathcal{M}^{\varepsilon})^{-1}-1)v\|_{H^{\ell-2}}&=\left\|\left(1+|\varepsilon\xi|^2-1\right)\widehat{v}(\xi)(1+|\xi|^2)^{\frac{\ell-2}{2}}\right\|_{L^2}\\&\le\sup_{\ell}\left|\frac{|\varepsilon\xi|^2}{1+|\xi|^2}\right|\|\widehat{v}(\xi)(1+\xi^2)^{\frac{\ell}{2}}\|_{L^2}\\&\le\varepsilon^2\|v\|_{H^{\ell}}.\end{aligned}$$

Therefore, considered as an operator from $H^{\ell}(\mathbb{R}^n,\mathbb{R}^k)$ to $H^{\ell-2}(\mathbb{R}^n,\mathbb{R}^k)$, $(\mathcal{M}^{\varepsilon})^{-1}$ is well-defined, and we have $\|(\mathcal{M}^{\varepsilon})^{-1}v-v\|_{H^{\ell-2}}\to 0$ as $\varepsilon\to 0$ for $v\in H^{\ell}(\mathbb{R}^n,\mathbb{R}^k)$. This simple observation is key to identifying the leading-order terms and we state it as a lemma.

Lemma 2.3 *The pseudo-differential operator* $(\mathcal{M}^\varepsilon)^{-1}$ *with symbol* $\dfrac{|\varepsilon\xi|^2}{m(\varepsilon\ell)} = 1 + |\varepsilon\xi|^2$ *is well defined as a map from* $H^\ell(\mathbb{R}^n, \mathbb{R}^k)$ *into* $H^{\ell-2}(\mathbb{R}^n, \mathbb{R}^k)$*. Moreover, it converges to the identity in the operator norm,*

$$\|(\mathcal{M}^\varepsilon)^{-1} - I\|_{H^\ell \to H^{\ell-2}} = \mathrm{O}(\varepsilon^2). \tag{2.2.2}$$

Since we are seeking solutions that inherit the symmetry of $\mathcal{K}$, we shall work with the subspace of functions in H^ℓ that are invariant under $\Gamma \subset \mathbf{O}(n)$,

$$H^\ell_\Gamma(\mathbb{R}^n, \mathbb{R}^k) := \left\{ u \in H^\ell(\mathbb{R}^n, \mathbb{R}^k) \;\middle|\; u(\cdot) = u(\gamma\cdot) \text{ for any } \gamma \in \Gamma \right\}.$$

We remark that $(\mathcal{M}^\varepsilon)^{-1}$ takes H^ℓ_Γ into $H^{\ell-2}_\Gamma$ since its symbol is radially symmetric, that is, it commutes with the full group $\mathbf{O}(n)$.

We next turn to the model equation

$$\Delta v - v + v^2 = 0. \tag{2.2.3}$$

Lemma 2.4 *Assume* $n < 6$ *and fix* $\ell \geqslant 2$*. Then* (2.2.3) *possesses a unique (up to translations) localized solution* $v_*(x)$*, which is smooth and radially symmetric. Moreover the linearization* $\mathcal{L} = -\Delta + 1 - 2v_*$ *at* v_* *is nondegenerate in the sense that* $\ker \mathcal{L} \cap L^2_\Gamma(\mathbb{R}^n) = \{0\}$*. In particular,* $\mathcal{L}$ *is bounded invertible from* $H^\ell_\Gamma(\mathbb{R}^n)$ *to* $H^{\ell-2}_\Gamma(\mathbb{R}^n)$*.*

Proof. For existence and uniqueness of the ground state v_*, we refer to [15, Lem. 13.3]. Next, by [15, Lem. 13.4], any element $\eta(x) \in \ker \mathcal{L}$ is of the form $\langle a, \nabla v_*(x)\rangle$ for some vector $a \in \mathbb{R}^n$. Now suppose $\eta \in \ker \mathcal{L} \cap L^2_\Gamma$. Then $\eta(x) = \eta(\gamma x)$ for all $\gamma \in \Gamma$. On the other hand, using radial symmetry of v_*, we have $v_*(\gamma x) = v_*(x)$ for all γ. Differentiating in x, we have $\nabla v_*(x) = \gamma \nabla v_*(\gamma x)$. Together, this gives

$$\langle a, \nabla v_*(x)\rangle = \eta(x) = \eta(\gamma x) = \langle a, \nabla v_*(\gamma x)\rangle = \langle \gamma a, \nabla v_*(x)\rangle.$$

As a consequence, $\gamma a - a$ is orthogonal to $\nabla v_*(x)$ for any $x \in \mathbb{R}^n$. However, using again the radial symmetry of v_*, we have $\nabla v_*(x) = \frac{V'(|x|)}{|x|} x$ for some scalar function $V = V(r)$ defined for $r \geq 0$, not identically constant. Since $x \in \mathbb{R}^n$ is arbitrary, we conclude that $\gamma a = a$, which implies $a \in \operatorname{Fix}\Gamma = \{0\}$. Since $-\Delta + 1$ is bounded invertible, and the multiplication operator $2v_*$ is compact as maps from H^ℓ to $H^{\ell-2}$, we conclude that $\mathcal{L}$ is Fredholm of index zero. Its restriction to the invariant subspace H^ℓ_Γ is therefore also Fredholm of index 0, hence invertible, as claimed. ■

We are now ready to set up the final fixed point iteration, using a preconditioning of the scalar reduced equation (2.2.1) with $(\mathcal{M}^\varepsilon)^{-1}$.

Proposition 2.5 *Assume $n < 6$ and $\ell > n/2$. There is $\varepsilon_1 > 0$ sufficiently small, such that for $0 < \varepsilon < \varepsilon_1$, there exist a family of solutions to (2.2.1) of the form $v_\mathrm{c}(\cdot;\varepsilon) = v_*(\cdot) + w(\cdot;\varepsilon)$. Here $w = w(\cdot,\varepsilon) \in H^\ell_\Gamma(\mathbb{R}^n,\mathbb{R})$ is a family of correctors parameterized by ε such that $\|w(\cdot,\varepsilon)\|_{H^\ell} \to 0$ as $\varepsilon \to 0$.*

Proof. We substitute the ansatz $v_\mathrm{c} = v_* + w$ into (2.2.1), where v_* is as stated in Lemma 2.4 and $w \in H^\ell_\Gamma$. We will determine an equation for w and ε and show that it can be solved using a Newton iteration scheme near $(w,\varepsilon) = (0,0)$. First, for the term $\varepsilon^{-2}M^\varepsilon v_\mathrm{c}$ with $v_\mathrm{c} \in H^\ell$, we apply Fourier transform to obtain

$$\varepsilon^{-2}m(\varepsilon\xi)\widehat{v}_\mathrm{c}(\xi) = -\frac{m(\varepsilon\xi)}{|\varepsilon\xi|^2}(-|\xi|^2)\widehat{v}_\mathrm{c}(\xi) = -\widehat{\mathcal{M}^\varepsilon\Delta v}_\mathrm{c},$$

and equation (2.2.1) becomes

$$0 = -\mathcal{M}^\varepsilon\Delta v_\mathrm{c} + \left(L^\varepsilon_\mathrm{cc} + L^\varepsilon_\mathrm{ch}a^h_{101}\right)v_\mathrm{c} + \left(L^\varepsilon_\mathrm{cc}a^c_{110} + L^\varepsilon_\mathrm{ch}a^h_{110}\right)v_\mathrm{c}^2 + \mathcal{R}(w,\psi;\varepsilon),$$

where $\mathcal{R}(w,\psi;\varepsilon)$ contains all the terms of order ε^2 and higher,

$$\mathcal{R}(w,\psi;\varepsilon) = \sum_{j=\mathrm{c,h}} L^\varepsilon_\mathrm{cj}\left[\frac{a^j_{011}}{\alpha}\psi + a^j_{110}(v_* + w)\psi + a^j_{020}(-\beta)[\psi,\psi] + \varepsilon^{-4}\mathcal{R}_j(v_* + w,\psi;\varepsilon)\right].$$

Indeed, for w with $v_* + w \in B_r$, we claim that $\mathcal{R}$ satisfies the estimate $\|\mathcal{R}\|_{H^\ell} = \mathrm{O}(\varepsilon^2)$ and $\|D_w\mathcal{R}\| = \mathrm{O}(\varepsilon^2)$. To see this, we first apply Lemma 2.1 with $r = 2\|v_*\|_{H^\ell}$ to obtain $\psi = \psi(v_* + w,\varepsilon)$ which satisfies $\|\psi(v_* + w,\varepsilon)\|_{H^\ell} = \mathrm{O}(\varepsilon^2)$.

The linear operators $L^\varepsilon_\mathrm{cc}, L^\varepsilon_\mathrm{ch}$ are uniformly bounded in ε, so that we have

$$\left\|\sum_{j=\mathrm{c,h}} L^\varepsilon_\mathrm{cj}\left(\frac{a^j_{011}}{\alpha}\psi + a^j_{110}v_\mathrm{c}\psi + a^j_{020}(-\beta)[\psi,\psi]\right)\right\|_{H^\ell} \le C(\|\psi\|_{H^\ell} + \|\psi\|^2_{H^\ell}) = \mathrm{O}(\varepsilon^2).$$

On the other hand, the remainders $\mathcal{R}_\mathrm{c}$ and $\mathcal{R}_\mathrm{h}$ satisfy $\|\mathcal{R}_\mathrm{c}\|_{H^\ell} = \mathrm{O}(\varepsilon^6)$, $\|\mathcal{R}_\mathrm{h}\|_{H^\ell} = \mathrm{O}(\varepsilon^6)$, uniformly for v_* and w such that $v_* + w \in B_r$ as $\varepsilon \to 0$ by (2.1.8) and (2.1.9). Therefore, we conclude that $\|\mathcal{R}(v_\mathrm{c}, \psi; \varepsilon)\|_{H^\ell} = \mathrm{O}(\varepsilon^2)$ for $v_\mathrm{c} = v_* + w \in B_r$. Similarly, we find that $\|D_w\mathcal{R}\| = \mathrm{O}(\varepsilon^2)$ in the operator norm on H^ℓ.

Next, we add the equation $\Delta v_* - v_* + v_*^2 = 0$ to the right-hand side of (2.2.1) and precondition with the operator $(\mathcal{M}^\varepsilon)^{-1}$. Set $\alpha^\varepsilon = L^\varepsilon_{\mathrm{cc}} + \frac{a^h_{101}}{\alpha} L^\varepsilon_{\mathrm{ch}}$, $\beta^\varepsilon = -L^\varepsilon_{\mathrm{cc}} + \frac{a^h_{200}}{-\beta} L^\varepsilon_{\mathrm{ch}}$, and we find

$$\begin{aligned}
0 =& (\mathcal{M}^\varepsilon)^{-1}\left[(1-\mathcal{M}^\varepsilon)\Delta v_* - \mathcal{M}^\varepsilon \Delta w + \alpha^\varepsilon (v_* + w) - v_* + \beta^\varepsilon (v_* + w)^2 + v_*^2 + \mathcal{R}\right] \\
=& [(\mathcal{M}^\varepsilon)^{-1} - 1]\mathcal{M}^\varepsilon \Delta v_* + (\mathcal{M}^\varepsilon)^{-1}\left[(\alpha^\varepsilon - 1)v_* + (\beta^\varepsilon + 1)v_*^2 + \mathcal{R}\right] + \\
& - \Delta w + (\mathcal{M}^\varepsilon)^{-1}\left[\alpha^\varepsilon w + \beta^\varepsilon(2v_* w + w^2)\right], \\
=:& F_1(w;\varepsilon) + F_2(w;\varepsilon) =: F(w;\varepsilon).
\end{aligned} \tag{2.2.4}$$

By Lemma 2.3, we have that F maps $H^\ell_\Gamma(\mathbb{R}^n)$ to $H^{\ell-2}_\Gamma(\mathbb{R}^n)$. Our goal is to set up a Newton iteration scheme to solve $F(w,\varepsilon) = 0$ for w in terms of ε as a fixed point problem.

Following the strategy of Lemma 2.1, we shall show:

(i) $\|F(0,\varepsilon)\|_{H^{\ell-2}} \to 0$ as $\varepsilon \to 0$;

(ii) $F(w,\varepsilon)$ is continuously differentiable in w and $D_w F(0,\varepsilon) : H^\ell_\Gamma(\mathbb{R}^n) \to H^{\ell-2}_\Gamma(\mathbb{R}^n)$ is uniformly invertible in ε.

For (i), we note that

$$F(0,\varepsilon) = F_2(0;\varepsilon) = [(\mathcal{M}^\varepsilon)^{-1} - 1]\mathcal{M}^\varepsilon d\Delta v_* + (\mathcal{M}^\varepsilon)^{-1}[(\alpha^\varepsilon - 1)v_* + (\beta^\varepsilon + 1)v_*^2 + \mathcal{R}(v_*, \psi; \varepsilon)].$$

From Lemma 2.4, $\Delta v_* \in H^\ell(\mathbb{R}^n)$ for all ℓ. Since $\mathcal{M}^\varepsilon$ takes $H^\ell(\mathbb{R}^n)$ into itself and is uniformly bounded in ε, we conclude from Lemma 2.3 that $\|[(\mathcal{M}^\varepsilon)^{-1} - 1]\mathcal{M}^\varepsilon \Delta v_*\|_{H^\ell} \to 0$ as $\varepsilon \to 0$.

Moreover, by (2.1.10), it holds that $\|\alpha^\varepsilon v - v\|_{H^\ell} \to 0$ and $\|\beta^\varepsilon v + v\|_{H^\ell} \to 0$ as $\varepsilon \to 0$, for any $v \in H^\ell(\mathbb{R}^n)$. Moreover, the remainder $\mathcal{R}(v_*, \psi; \varepsilon)$ satisfies $\|\mathcal{R}\|_{H^\ell} = \mathrm{O}(\varepsilon^2)$ as shown above. Hence, we conclude that

$$\|F(0;\varepsilon)\|_{H^{\ell-2}} = \|F_2(0;\varepsilon)\|_{H^{\ell-2}} \to 0,$$

as $\varepsilon \to 0$, which proves (i).

For (ii), we first verify that F is continuously differentiable in w from $H^\ell(\mathbb{R}^n)$ to $H^{\ell-2}(\mathbb{R}^n)$. Indeed, take $h, w_0 \in H^\ell(\mathbb{R}^n)$ with w_0 fixed. We observe that $D_wF(w_0;\varepsilon)h : H^\ell(\mathbb{R}^n) \to H^{\ell-2}(\mathbb{R}^n)$ is given by

$$D_wF(w_0;\varepsilon)h = -\Delta h + (\mathcal{M}^\varepsilon)^{-1}\left[(a^\varepsilon h + 2v_*\beta^\varepsilon h + 2w_0 h) + D_w\mathcal{R}h\right],$$

which depends continuously on w_0 since the superposition operator induced by $\mathcal{N}$ is of class $\mathscr{C}^1$.

Now, at $w_0 = 0$, we see that, $D_wF(0;\varepsilon)h \to -\Delta h + h - 2v_*h = \mathcal{L}h$ in $H^{\ell-2}(\mathbb{R}^n)$ as $\varepsilon \to 0$ for $h \in H^\ell$ because $\|D_w\mathcal{R}h\|_{H^\ell} = \mathrm{O}(\varepsilon^2)$ as noted above. By Lemma 2.4, the operator $\mathcal{L} : H^\ell_\Gamma(\mathbb{R}^n) \to H^{\ell-2}_\Gamma(\mathbb{R}^n)$ is bounded invertible. We notice that $D_wF(0;\varepsilon)$ respects the symmetry and is a small perturbation of $\mathcal{L}$, therefore invertible with uniform bounds on the inverse for ε small enough. This shows (ii).

We now set up the Newton iteration scheme, by iterating the map $\tilde{\mathcal{S}}$, defined as

$$\tilde{\mathcal{S}}(w;\varepsilon) = w - D_wF(0;\varepsilon)^{-1}[F(w;\varepsilon)].$$

Note that $\tilde{\mathcal{S}}$ respects the symmetry as well, $\tilde{\mathcal{S}} : H^\ell_\Gamma(\mathbb{R}^n) \to H^{\ell-2}_\Gamma(\mathbb{R}^n)$. Therefore, we can proceed as in Lemma 2.1 to obtain $w = w(\varepsilon)$ which solves $F(w(\varepsilon);\varepsilon) = 0$ for ε small enough and satisfies $\|w(\varepsilon)\|_{H^\ell} \to 0$ as $\varepsilon \to 0$. ■

Finally, we prove Theorem 1.

Proof of Theorem 1. We revert to tildes for the rescaled variables. From Proposition 2.5, we know that (2.2.1) has a solution of the form $\tilde{v}_\mathrm{c}(\cdot) = v_*(\cdot) + w(\cdot;\varepsilon)$. Together with $\tilde{v}_\mathrm{h} = \psi(\tilde{v}_\mathrm{c}, \varepsilon)$, reverting the rescaling, we obtain $v_\mathrm{c}(\cdot) = -\frac{\alpha}{\beta}\mu\tilde{v}_\mathrm{c}(\sqrt{\alpha\mu}\cdot)$ and $v_\mathrm{h}(\cdot) = \alpha\mu\tilde{v}_\mathrm{h}(\sqrt{\alpha\mu}\cdot)$ as solutions to (2.1.2) and (2.1.3).

Now, recall that $V = (v_\mathrm{c}, v_\mathrm{h})^T$, and that the original variable U is obtained as $U = QV$, where Q is defined in Lemma 1.1. We conclude that $U(\cdot) = v_\mathrm{c}(\cdot)\boldsymbol{e} + v_\perp(\cdot)$, where $v_\perp$ takes values in the orthogonal complement of $\boldsymbol{e}$. The behavior of $v_\mathrm{c}, v_\perp$ as $\mu \to 0$ is a direct consequence of Lemma 2.1 and Proposition 2.5. Finally, we restore the original variable $x = T_0 y$, thus getting the desired form of the bifurcating solution.

■

2.3 Applications

We outline how our main result applies rather immediately to a variety of specific model equations.

Neural fields. In the simplest setup, neural field equations involve nonlocal coupling through a sigmoidal response function S, with dynamics

$$u_t = -u + \mathcal{K} * S(u;\mu),$$

where $\mathcal{K}$ is a not necessarily positive convolution kernel sampling input from firing neighboring neurons, and S is the firing rate depending on the state u of neurons, typically a sigmoidal, strictly monotone function [3]. The sign of $\mathcal{K}$ may change depending on an excitatory or inhibitory coupling. Vector-valued generalizations of the equations have been proposed to model functionally different populations of neurons.

Looking for stationary spike-like solutions of this equation, we set $u_t = 0$ and substitute $U = S(u;\mu)$, with inverse $u = \Psi(U;\mu)$, obtaining

$$-U + \mathcal{K} * U + (U - \Psi(U;\mu)) = 0,$$

which is of the form (1.2.1). Assuming $\int \mathcal{K} = 1$, saddle-node bifurcations occur when the nonlinearity $U - \Psi(U;\mu)$ has a double zero, which is equivalent to a double zero in $-u + S(u;\mu)$. Hypotheses (L) and (TC) directly translate into assumptions on $\mathcal{K}$ and S. A generic saddle-node bifurcation can be easily transformed into a transcritical bifurcation as outlined in the discussion following Hypothesis (TC).

The spikes constructed in this fashion would be expected to be unstable, of Morse index 1, their stable manifold separating spatially uniformly quiescent and spatially uniformly excited populations of neurons.

Material science. Phase separation in multi-component alloys has been modeled by free energy functionals for concentrations of species $W(u)$, $u \in \mathbb{R}^k$, together with a local or nonlocal interaction term. For nonlocal interactions, together with an H^{-1}-gradient flow, one obtains nonlocal Cahn-Morral systems

$$u_t = -\Delta(-u + J * u - W'(u)),$$

see [17]. Equilibria satisfy

$$-u + J * u - W'(u) = \mu,$$

with chemical potential $\mu \in \mathbb{R}^k$. Saddle-node bifurcations in $W'(u) + \mu = 0$ now lead to bifurcation of spikes as constructed here. We also note that anisotropic versions, respecting discrete crystallographic symmetries Γ, have also been proposed, at least in a context of local coupling [18].

Dispersive solitary waves. In a slightly different direction, nonlocal coupling can encode dispersion, such as in models for shallow water waves generalizing KdV

$$u_t = (Mu - u^2)_x,$$

where Mu is a nonlocal pseudo-differential operator generalizing ∂_{xx} in the KdV equation. Traveling waves satisfy

$$Mu + cu - u^2 = 0,$$

which in the case of $M = I_k + \mathcal{K}*$ reduces to the problem studied here [6]. Other examples include systems of nonlinear Schrödinger equations with nonlocal dispersion,

$$\mathrm{i}v_t = -v + \mathcal{K} * v + N(v) \in \mathbb{R}^k,$$

with possible nonlocal nonlinear dispersion $N(v)$. Here, the simplest case $v = \mathrm{e}^{\mathrm{i}\mu t}u,\ u \in \mathbb{R},\ N(u) = u|u|^2$, leads to

$$-u + \mathcal{K} * u - \mu u + u^3 = 0,$$

which is amenable to the analysis presented here, slightly changing scalings to account for the cubic nonlinearity.

2.4 Discussion

We presented a direct and simple approach to the bifurcations of localized spikes in non-locally coupled systems. While somewhat simpler and more general than approaches based on spatial dynamics, it does not offer insight into uniqueness questions, and, arguably, relies on an a priori understanding of the resulting phenomena. Our assumptions appear to be sharp in terms of localization of the convolution kernel. Interesting questions arise when studying kernels with less smoothness; see for instance [19] for a numerical exploration of kernel regularity on phenomena.

Large μ. Of course, the bifurcation theoretic approach here is limited to small values of the bifurcation parameter μ. Quite different phenomena are to be expected for large values of μ as the simple scalar example

$$-u + \mathcal{K} * u + \mu u - u^2 = 0, \tag{2.4.1}$$

shows. For $\mu = 1/\varepsilon$, we can scale $\varepsilon u = v$ and obtain

$$v - v^2 + \varepsilon(-v + \mathcal{K} * v) = 0.$$

At $\varepsilon = 0$, we have solutions $v(x) = 1$ for $x \in \Omega$, $v(x) = 0$ otherwise, for any measurable Ω. The linearization at such solutions in $L^\infty(\mathbb{R}^n)$ is invertible as a multiplication operator with values ± 1, and the solutions therefore can be continued in ε. This plethora of solution does of course not exist in the case of diffusive, local coupling; see [20]. The transition can in some cases be understood as a depinning transition of interfaces as analyzed in [19].

Tail expansions. At leading order, the solutions we find here are exponentially localized, with exponential rate of order $\sqrt{\mu}$. As is clear from the simple example (2.4.1), the actual solution will typically not be exponentially localized. To see this, it suffices to observe that $-u + \mu u - u^2$ is exponentially localized when u is, but $\mathcal{K} * u$ is not, for instance when both u and $\mathcal{K}$ are positive, and $\mathcal{K}$ does not decay exponentially.

Somewhat more precisely, our analysis shows that the leading order correction to the solution of

$$-u + \mathcal{K} * u - \mu u + u^2 = 0,$$

can be found by substituting $u(x) = \mu u_*(\sqrt{\mu}x) + \mu w(\sqrt{\mu}x)$, $u_*'' - u_* + u_*^2 = 0$, finding at leading order

$$w'' - w + 2u_* w = u_*'' - \frac{1}{\mu}(-u_* + \mathcal{K}_{\sqrt{\mu}} * u_*).$$

The right-hand side is algebraically localized with decay behavior dominated by $\mathcal{K}(x)\int u_*$. Solving for w and comparing decay rates shows the same behavior for the corrector w. In principle, one can in this way obtain higher-order algebraic expansions for the decay of u, assuming an expansion for $\mathcal{K}$ in terms of x^{-k}.

Periodic spike patterns. The analysis of the bifurcation towards spatially periodic patterns is much simpler, due to the fact that the convolution acts as a compact perturbation of the identity. Therefore, classic Lyapunov-Schmidt bifurcation analysis will yield bifurcation of spatially periodic patterns, after restricting to appropriate symmetry planforms, such as hexagonal or square lattices in $\mathbb{R}^2$. More interesting and relevant for our technical questions here is the case of large spatial period, say $L = L_0/\sqrt{\mu}$. Imposing such boundary conditions allows for an analysis completely analogous to our analysis here, with linear Fourier multipliers restricted to periodic boundary conditions, thus allowing for the same bounds and convergence estimates. The solutions in the rescaled system would be a periodic solution to

$$\Delta u - u + u^2 = 0,$$

which one can obtain, using a variety of methods, for instance bifurcation and global continuation [21]. Analyzing the maximum A of the spike as a function of L, one then finds a stronger dependence of A on L for weakly decaying kernels by inspecting the residual as described above. In particular, one expects pulse interaction in the "weak", well-separated regime to be algebraic rather than exponential when kernels have algebraic tails.

Traveling waves. In some cases, relevant coherent structures may not be stationary in time, such that we need to make assumptions on temporal dynamics. Considering for instance the neural field model

$$u_t = -u + \mathcal{K} * S(u),$$

we find the traveling-wave equation

$$cu_x - u + \mathcal{K} * S(u) = 0.$$

Applying $(1 - c\partial_x)^{-1}$ gives

$$-u + \tilde{K}_{\mathrm{c}} * S(u) = 0, \qquad \tilde{K}_{\mathrm{c}} = (1 - c\partial_x)^{-1} * \mathcal{K}.$$

Assuming that $\mathcal{K}$ is of class $W^{2,1}$, say, $\tilde{\mathcal{K}}_{\mathrm{c}}$ is differentiable in c as a function in $W^{1,1}$, with expansion $\tilde{\mathcal{K}}_{\mathrm{c}} = \mathcal{K} + c\mathcal{K}'$. In the long-wavelength scaling, $\mathcal{K}'$ converges to ∂_x, such that we obtain at leading order, after scaling the model equation

$$u'' + \tilde{c}u' + u - u^2 = 0.$$

This strategy has been carried out in the case of exponentially localized kernels in [11, §4.2], and we expect that the methods here allow for an adaptation to kernels with second moments, only.

Stability. Focusing on the existence problem of stationary solution, we did not discuss dynamics in most of the exposition. From that perspective, a first relevant question would be the stability of bifurcating solutions. It is worth noting that an important tool for this analysis, the Evans function [22, 23, 8] is not available for the nonlocal equations considered here. On the other hands, an analysis as in [24, 25], exploiting Evans function analysis for the leading order expansion combined with a perturbation argument as presented here should yield stability information for the type of solutions constructed here for nonlocal equations. More ambitiously, it would be interesting to construct weak interaction manifolds [26, 8, 27], for spikes with algebraic tail decay constructed here.

Chapter 3

Passage through fold

In this chapter we prove Theorem 3 and Corollary 3.2. We first prepare with the setup of the problem in the following sections.

Outline The reminder of this chapter is organized as follows. In Section 3.1 we explain how we divide up the passage time into 3 different regions A, B, and C to prepare for the proof of Theorem 3. Then in Sections 3.2, 3.3, and 3.4, we construct the solutions. Finally, in Section 3.5, we show how to piece together all the parts to arrive at our main results.

Notation Throughout this chapter we will use the following notations.

- We use $A \lesssim B$ to indicate that there is a constant C such that $A \leq C \cdot B$, independent of the properties of A and B.
- We use the brackets $\langle x \rangle$ to represent the expression $\sqrt{x^2+1}$.
- We use $\mathcal{C}^k(a,b)$ to denote the space of functions with k continuous derivatives on the interval (a,b) for $k \geq 0$.
- We denote the Euclidean norm in Euclidean space as $|\cdot|$, the norm in a general Banach space X as $\|\cdot\|_X$, and we use $\|A\|_{X\to Y}$ to denote the operator norm for the linear operator $A : X \to Y$ which maps from the function space X into Y.

3.1 Division of regions and the rescaling of time

We now introduce the ansatz. We divide the time from $t = 0$ to $t = T$ into three "Regions" where we take a different ansatz on each region.

We first start with the exit time $t = T$ when the trajectory hits the section Δ_{out}. Also, we use T to mark the rightmost boundary of the region A, where our proposed solution takes the form

$$u_A(t; u_0) = u_*(t; u_0) + w_r(t; u_0).$$

Here, the function $u_* = u_*(t; u_0)$ is defined as:

$$u_*(t; u_0) := \varepsilon^{1/3} u_R(\varepsilon^{1/3}(t - \varepsilon^{-1}\delta_-); u_0), \tag{3.1.1}$$

where $u_R = u_R(s; u_0)$ is the family of solutions to the Riccati equation which were shown to exist in Proposition 3.1. This family solve the initial value problem

$$\begin{aligned} \frac{d}{dt} u_*(t; u_0) &= \mu(t) + u_*^2(t; u_0), \\ \mu(t) &= \varepsilon t - \delta_-, \\ u_*(\varepsilon^{-1}\delta_-; u_0) &= \varepsilon^{\frac{1}{3}} u_0, \end{aligned} \tag{3.1.2}$$

with ε and u_0 as parameters. The function w_r is a correction term whose properties will be given in later sections.

In region A, the ansatz $u_*(t)$ is merely a rescaled version of the Riccati solution $u_R(s; u_0)$, with the asymptotic expansion as the first part of Proposition 3.1. When $u_*(t)$ is controlled by the other part of the asymptotic expansion, we will need to adaptively change our ansatz or function space and later to "glue" solutions. An intuitive, but rather arbitrary place to switch regions is at $t = \varepsilon^{-1}\delta_-$. This marks the start of region B, where our choice of solution is as follows:

$$u_B(t) = \bar{u}_*(t) + w_\ell(t).$$

Where $\bar{u}_*$ is the function

$$\bar{u}_*(t) = u_*(t;\bar{u}_0) = \varepsilon^{1/3}u_R(\varepsilon^{1/3}(t-\varepsilon^{-1}\delta_-);\bar{u}_0) = \varepsilon^{1/3}\bar{u}_R(\varepsilon^{1/3}(t-\varepsilon^{-1}\delta_-)), \tag{3.1.3}$$

and $\bar{u}_*$ solves the equation

$$\begin{aligned} \frac{d}{dt}\bar{u}_*(t) &= \mu(t) + \bar{u}_*^2(t), \\ \mu(t) &= \varepsilon t - \delta_-, \\ \bar{u}_*(\varepsilon^{-1}\delta_-) &= \varepsilon^{1/3}u_0. \end{aligned} \tag{3.1.4}$$

Therefore, $\bar{u}_*$ is also a rescaled version of the special solution to the Riccati equation. Similar to the situation in region A, w_ℓ is a correction term whose properties will be described later.

The next piece of the ansatz will be used to connect the piece u_B, which roughly follows the special Riccati solution $\bar{u}_R$, to the attracting branch S_a of the critical manifold S_0, a natural "gluing point" is where the error between $\bar{u}_*(t)$ and $u_s(t)$ is small. Recall from (1.3.18) and (1.3.11), we have

$$u_s(t) = -\sqrt{\delta_- - \varepsilon t} + \mathrm{O}(\delta_- - \varepsilon t),$$

and

$$\bar{u}_*(t) = -\sqrt{\delta_- - \varepsilon t} + \mathrm{O}(\varepsilon(\delta_- - \varepsilon t)^{-1}),$$

as $\varepsilon \to 0$. Hence we define t^* so that

$$\delta_- - \varepsilon t^* = \varepsilon(\delta_- - \varepsilon t^*)^{-1}, \text{ or } t^* = \varepsilon^{-1}(\delta_- + \varepsilon^{1/2}). \tag{3.1.5}$$

We then choose t^* as a natural transition point from region B to the last region, region C, where it covers the rest of the passage time until at $t = 0$. The corresponding solution will take the form

$$u_C(t) = u_s(t) + w_s(t),$$

where $w_s(t)$ is yet another correction term whose properties will be discussed later.

In summary, region A, B and C are divided as follows:

$$\begin{aligned}
&\text{Region A:} \quad (\varepsilon^{-1}\delta_-, T), \quad \text{solution in region A:} \quad u_A(t) = u_*(t) + w_r(t),\\
&\text{Region B:} \quad (t^*, \varepsilon^{-1}\delta_-), \quad \text{solution in region B:} \quad u_B(t) = \bar{u}_*(t) + w_\ell(t),\\
&\text{Region C:} \quad (0, t^*), \quad \text{solution in region C:} \quad u_C(t) = u_*(t) + w_s(t).
\end{aligned} \tag{3.1.6}$$

Now we can briefly describe the strategy of our proof. We plug our ansatz into (1.3.7) to derive the equation for the corrections w_r, w_ℓ and w_s, choose appropriate function space with norms and set up the equations for the corrections as fixed point equations on these function spaces. A main technical part of our proof consists of appropriately rescaling the time $t \in (0, T)$ so that we gain hyperbolicity in the sense that the linearized operator at the ansatz becomes Fredholm in the new time scale. This is the key observation in our approach, comparable to the blow-up approach which also gains hyperbolicity via a carefully chosen change of variables. Having solved for the correction terms w_r, w_ℓ and w_s, we can then collect information about their asymptotic expansion to confirm the corresponding solution has the properties we need.

Next, we describe how we transform time t into time σ to gain hyperbolicity. This is demonstrated in the following steps.

(i) Step 1. Define ψ as

$$\psi = \varepsilon^{1/3}(t - \varepsilon^{-1}\delta_-),$$

(ii) Step 2. Fix $M > 0$ large, define σ as

$$\psi = \psi(\sigma; u_0) = \begin{cases} -(-\frac{3}{2}\sigma)^{2/3}, & \text{for } \sigma \leq -M, \\ \Omega_\infty(u_0) - e^{-\sigma}, & \text{for } \sigma \geq M, \end{cases} \tag{3.1.7}$$

Here Ω_∞ is the blow-up time for u_R found in Proposition 3.1.

Also, we denote $\varphi(\sigma) := \frac{d}{d\sigma}\psi(\sigma)$, which satisfies

$$\varphi(\sigma) = \begin{cases} (-\frac{3}{2}\sigma)^{-1/3}, & \text{for } \sigma \leq -M, \\ e^{-\sigma}, & \text{for } \sigma \geq M. \end{cases} \tag{3.1.8}$$

(iii) Step 3. For $\sigma \in (-M, M)$, we define $\psi(\sigma)$ as the straight line connecting the two points $(M, \Omega_\infty - e^{-M})$ and $(-M, -(\frac{3}{2}M)^{2/3})$. As a result, if we define $\sigma_m = \sigma_m(u_0)$ as the value of σ such that $\psi(\sigma_m; u_0) = 0$, then we have

$$\frac{|\sigma_m - M|}{M} = \left| \frac{(\frac{3M}{2})^{2/3} - (\Omega_\infty - e^{-M})}{(\frac{3M}{2})^{2/3} - (\Omega_\infty + e^{-M})} - 1 \right| \leq CM^{-2/3},$$

for some constant C independent of u_0.

Therefore we can write

$$\sigma_m = M + M_r, \qquad |M_r| \leq CM^{1/3}. \tag{3.1.9}$$

After defining the time transformation, the original exit time $t = T$ will be transformed into $\sigma = \sigma_T$ under this change of variable, and similarly the boundary between region A and region B is transformed from $t = \varepsilon^{-1}\delta_-$ to $\sigma = \sigma_m$ and the boundary between region B and region C is transformed from $t = t^*$ to $\sigma = \sigma^*$, and $t = 0$ into $\sigma = \sigma_0$. We will see later that the $\sigma_0, \sigma^*, \sigma_m, \sigma_T$ satisfy:

$$|\sigma_0| = \mathrm{O}(\delta_-^{3/2}\varepsilon^{-1}), \quad |\sigma^*| = \mathrm{O}(\varepsilon^{-1/4}), \quad \sigma_m = \mathrm{O}(1), \quad \sigma_T = \mathrm{O}(-\log(\varepsilon^{1/3}\delta_+^{-1})).$$

In summary, the regions in time σ are as follows:

$$\begin{aligned} &\text{Region A:} \quad (\sigma_m, \sigma_T), \\ &\text{Region B:} \quad (\sigma^*, \sigma_m), \\ &\text{Region C:} \quad (\sigma_0, \sigma^*), \end{aligned} \tag{3.1.10}$$

and Figure 2 summarizes the relationship between the two time scales t and σ, as well as the corresponding regions divided.

3.2 Region A

Region A corresponds to the time-t interval $\{t : t > \varepsilon^{-1}\delta_-\}$. In this section we will give the precise form of the solution which solves (1.3.2) on this region.

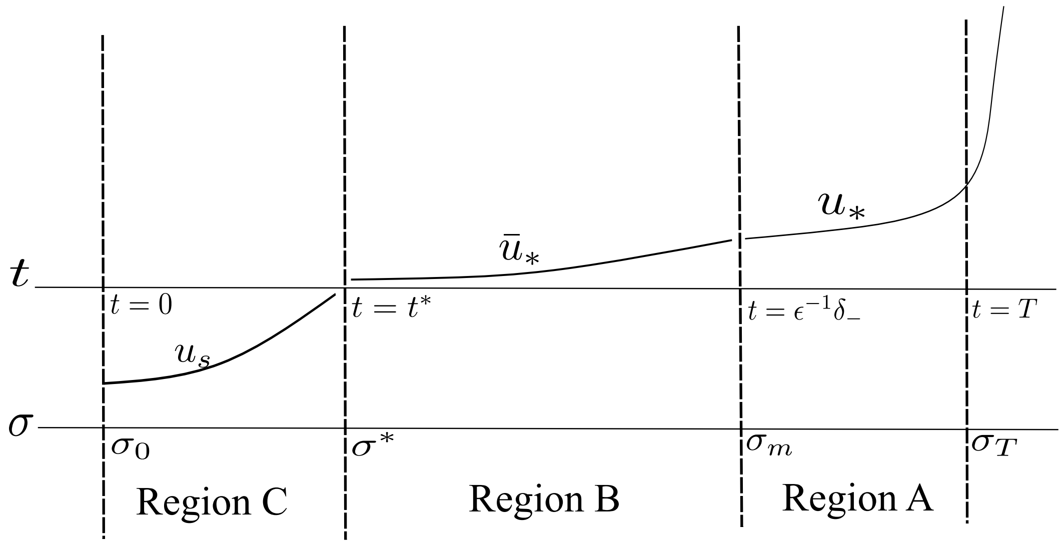

Figure 2: Time scales used to divide regions A, B, and C

3.2.1 Solution in region A

Theorem 4 *Fix $\alpha > 0, \delta_-, \delta_+ > 0, \eta > 0$ small enough. Then there exists $\varepsilon_A > 0$ and a constant $C = C(\alpha, \delta_-, \delta_+, \eta)$, such that for all $0 < \varepsilon < \varepsilon_A$, and all $|u_0 - \bar{u}_0| < \eta$, there exist a time $T = T(\varepsilon; u_0)$, such that a solution of the form*

$$u_A(t; u_0) = u_*(t; u_0) + w_r(t; u_0), \tag{3.2.1}$$

to (1.3.19) *exists on the time interval $t \in (\varepsilon^{-1}\delta_-, T)$, where*

$$u_*(t; u_0) = \varepsilon^{1/3} u_R(\varepsilon^{1/3}(t - \varepsilon^{-1}\delta_-); u_0), \tag{3.2.2}$$

and $u_R(\cdot; u_0)$ is the family of solutions to Riccati equation that were shown to exist in Proposition 3.1.

Moreover, we have

(i) $T = T(\varepsilon; u_0) = \varepsilon^{-1}\delta_- + \varepsilon^{-1/3}\Omega_\infty(u_0) - \delta_+^{-1} + T_r$ with $T_r = \mathrm{O}(\varepsilon^{2/3}\delta_+^{-3})$ where the constant $\Omega_\infty(u_0)$ is defined in Proposition 3.1;

(ii) $w_r(T; u_0) = 0$ and $u_*(T, u_0) = \delta_+$;

(iii) if we define

$$T_\infty = T_\infty(\varepsilon; u_0) := \varepsilon^{-1}\delta_- + \varepsilon^{-1/3}\Omega_\infty(u_0), \tag{3.2.3}$$

then w_r is continuous with $|w_r(t; u_0)| \le C|T_\infty - t|^{\alpha-2}$ for $t \in (\varepsilon^{-1}\delta_-, T)$;

(iv) the function $\Psi : (\bar{u}_0-\eta, \bar{u}_0+\eta) \to \mathbb{R}$ defined by $\Psi(u_0) = w_r(\varepsilon^{-1}\delta_-; u_0)$ is Lipschitz continuous, with Lipschitz constant

$$|Lip_{u_0}\Psi| \le C\varepsilon^{(2-\alpha)/3}.$$

We will prove this theorem in the following sections.

3.2.2 The exit time $T(u_0)$

Take $\eta > 0$ so that the function $u_R = u_R(\cdot; u_0)$ exists for $|u_0 - \bar{u}_0| < \eta$ as demonstrated in Proposition 3.1. The exit time T is then defined by the condition

$$\delta_+ = u_*(T; u_0) = \varepsilon^{1/3}u_R(\varepsilon^{1/3}(T - \varepsilon^{-1}\delta_-); u_0).$$

Recall the expansion for u_R as formula given in (1.3.13). If we define $\psi_T = \varepsilon^{1/3}(T - \varepsilon^{-1}\delta_-)$, then ψ_T satisfies

$$\frac{1}{\Omega_\infty - \psi_T} + (\Omega_\infty - \psi_T)r(\Omega_\infty - \psi_T) = \varepsilon^{-1/3}\delta_+,$$

from which we obtain the leading order expansion $\Omega_\infty - \psi_T = \mathrm{O}(\varepsilon^{1/3}\delta_+^{-1})$. A fixed point argument gives

$$\Omega_\infty - \psi_T = \varepsilon^{1/3}\delta_+^{-1} + \mathrm{O}(\varepsilon\delta_+^{-3}).$$

Hence, the expansion for $T = T(\varepsilon; u_0)$ becomes

$$T = T(\varepsilon; u_0) = \varepsilon^{-1}\delta_- + \varepsilon^{-1/3}\Omega_\infty(u_0) - \delta_+^{-1} + T_r(\varepsilon; u_0), \tag{3.2.4}$$

with $|T_r| \le C\varepsilon^{2/3}\delta_+^{-3}$, for some constant C independent of u_0, as $\varepsilon \to 0$.

3.2.3 Equation for w_r and rescaling

We now substitute $u = u_* + w_r$ into (1.3.2), and obtain the equation for w_r,

$$w_r' - 2u_* w_r = w_r^2 + f(u_* + w_r, \mu; \varepsilon) =: R_r(w_r; \varepsilon, u_0). \tag{3.2.5}$$

Moreover, we enforce the boundary condition $u(T; u_0) = \delta_+$, which gives the boundary condition for w_r at $t = T$:

$$w_r(T; u_0) = 0. \tag{3.2.6}$$

Therefore, we need to solve (3.2.5) on the interval $t \in (\varepsilon^{-1}\delta_-, T)$, with boundary condition (3.2.6).

Next, we rescale (3.2.5) into σ-time by using the rescaling map introduced in Section 3.1 and obtain

$$\left(\frac{d}{d\sigma} - a(\sigma; u_0)\right) W_r = \varepsilon^{-1/3} \varphi \mathcal{R}_r(W_r; \varepsilon, u_0). \tag{3.2.7}$$

We have the following properties.

- The term $a(\sigma; u_0)$ satisfies

  $$a(\sigma; u_0) := 2\varphi(\sigma) u_R(\psi(\sigma; u_0); u_0) = 2 + \mathrm{O}(e^{-2\sigma}) \text{ as } \sigma \to \infty, \tag{3.2.8}$$

  where ψ, φ were defined in (3.1.7) and (3.1.8), respectively. We remark that this convergence as $\sigma \to \infty$ is uniform in u_0 due to the definition of our time-rescaling.

- The function $W_r(\sigma)$ is the rescaled version of $w_r(t)$ in the σ-variable,

  $$W_r(\sigma) := w_r(\varepsilon^{-1/3}\psi(\sigma) + \varepsilon^{-1}\delta_-) = w_r(t).$$

  Similarly, U_* is the rescaled version of u_*, with

  $$U_*(\sigma; \varepsilon, u_0) := u_*(\varepsilon^{-1/3}\psi(\sigma) + \varepsilon^{-1}\delta_-; \varepsilon, u_0) = \varepsilon^{1/3} u_R(\psi(\sigma; u_0); u_0) = u_*(t; \varepsilon, u_0).$$

- The function $\mathcal{R}_r$ is a rescaled version of R_r such that

  $$\mathcal{R}_r(W_r; \varepsilon, u_0) := W_r^2 + f(U_* + W_r, \mu; \varepsilon).$$

In order to obtain the boundary condition corresponding to (3.2.6), we need to know the corresponding time-σ for the time-t interval $t \in (\varepsilon^{-1}\delta_-, T)$.

At $t = \varepsilon^{-1}\delta_-$, the corresponding σ time is at $\sigma = \sigma_m$, from its definition in Section 3.1, with

$$\sigma_m = M + \mathrm{O}(M^{1/3}) = \mathrm{O}(1),$$

independent of ε.

At $t = T$, we have $\varepsilon^{1/3}(T - \varepsilon^{-1}\delta_-) = \Omega_\infty - \varepsilon^{1/3}\delta_+^{-1} + \varepsilon^{1/3}T_r = \psi(\sigma_T) = \Omega_\infty - e^{-\sigma_T}$ from (3.2.4), hence, for ε small enough, we find that the σ-time corresponding to $t = T$ is

$$\sigma_T = \sigma_T(u_0;\varepsilon) = -\log(\varepsilon^{1/3}(\delta_+^{-1} - T_r)) = -\log(\varepsilon^{1/3}\delta_+^{-1}) - \log(1 - \delta_+ T_r(u_0;\varepsilon)), \quad (3.2.9)$$

where $T_r(u_0;\varepsilon)$ is defined in (3.2.4).

In conclusion, the problem we want to solve is

$$\begin{aligned} \frac{d}{d\sigma}W_r - a(\sigma;u_0)W_r &= \varepsilon^{-1/3}\varphi\mathcal{R}_r(W_r), \text{ for } \sigma \in (\sigma_m, \sigma_T), \\ W_r(\sigma_T) &= 0. \end{aligned} \quad (3.2.10)$$

3.2.4 Linear equation and norms

Our goal now is to solve (3.2.10) on an appropriate function space. to do so, we first slightly enlarge the time interval (σ_m, σ_T) where the boundary value problem is posed.

From the definition of σ_T in (3.2.9), we see that

$$|\sigma_T - (-\log(\varepsilon^{1/3}\delta_+^{-1}))| \leq |\log(1 - \delta_+ T_r)| \leq C|\delta_+ T_r| \leq C\varepsilon^{2/3}\delta_+^{-2}, \quad (3.2.11)$$

for some constant C independent of u_0 provided that $|u_0 - \bar{u}_0| < \eta$.

We now define $\sigma_{\inf}$ and $\sigma_{\sup}$ as follows:

$$\sigma_{\inf} = \inf_{|u_0 - \bar{u}_0| < \eta} \sigma_m(u_0), \qquad \sigma_{\sup} = \sup_{|u_0 - \bar{u}_0| < \eta} \sigma_T(u_0).$$

From the definition of σ_m in Section 3.1 and (3.2.11), we have

$$\sigma_{\inf} = M + \mathrm{O}(M^{1/3}), \qquad \sigma_{\sup} = -\log(\varepsilon^{1/3}\delta_+^{-1}) + \mathrm{O}(\varepsilon^{2/3}\delta_+^{-2}).$$

Fix $\alpha > 0$, and introduce the following weighted function space:

$$\mathcal{C}_r = \mathcal{C}_r(\sigma_{\inf}, \sigma_{\sup}) = \left\{ w(\sigma) \in \mathcal{C}(\sigma_{\inf}, \sigma_{\sup}) : \sup_{\sigma_{\inf} \le \sigma \le \sigma_{\sup}} \left| \varepsilon^{(\alpha-2)/3} e^{(\alpha-2)\sigma} w(\sigma) \right| < \infty \right\}.$$

Next we study the linear operator $A_r = A_r(u_0, \varepsilon)$ with

$$A_r w = \left(\frac{d}{d\sigma} w - a(\sigma; u_0) w, w(\sigma_T) \right),$$

which is defined on a dense subset $\mathcal{D}(A_r) \subset \mathcal{C}_r$.

Proposition 2.1 *Let $\eta > 0$ be as in Proposition 3.1 so that the family of solutions $u_R(\cdot; u_0)$ exists for $u_0 \in I_r$ where $I_r = \{u_0 : |u_0 - \bar{u}_0| < \eta\}$. There then exist $\varepsilon_1 > 0$ such that the operator $A_r(u_0, \varepsilon) : \mathcal{D} \subset \mathcal{C}_r \to \mathcal{C}_r \times \mathbb{R}$ is invertible, with its inverse*

$$A_r^{-1}(u_0, \varepsilon) : \mathcal{C}_r \times \mathbb{R} \to \mathcal{C}_r$$

bounded uniformly for all $u_0 \in I_r$ and $\varepsilon \in (0, \varepsilon_1)$.

Moreover, we have the following estimate on the Lipschitz constant for the map $u_0 \mapsto A_r^{-1}(u_0, \varepsilon)$.

$$\|A_r^{-1}(\tilde{u}_0, \varepsilon) - A_r^{-1}(u_0, \varepsilon)\|_{\mathcal{C}_r \times \mathbb{R} \to \mathcal{C}_r} \le C|\tilde{u}_0 - u_0| \tag{3.2.12}$$

for some constant C independent of ε.

Proof. Consider the conjugate operator of A_r, given by

$$\begin{aligned} \tilde{A}_r v &= \left(e^{(\alpha-2)\sigma} \left(\frac{d}{d\sigma} - a(\sigma; u_0) \right) e^{(2-\alpha)\sigma} v, v(\sigma_T) \right) \\ &= \left(\frac{d}{d\sigma} v - (\alpha + a(\sigma; u_0) - 2) v, v(\sigma_T) \right), \end{aligned}$$

which acts on $v(\sigma) \in \mathcal{C}^1(\sigma_{\inf}, \sigma_{\sup}) \subset \mathcal{C}(\sigma_{\inf}, \sigma_{\sup})$, with $v(\sigma) = \varepsilon^{(\alpha-2)/3} e^{(\alpha-2)\sigma} w(\sigma)$.

Consider the system

$$\frac{d}{d\sigma}w - a(\sigma;u_0)w = f,\ \text{for}\ \sigma \in (\sigma_m, \sigma_T),$$
$$w(\sigma_T) = w_T.$$

Which can be written as

$$A_r w = (f, w_T)\ \text{with}\ f \in \mathcal{C}_r, w_T \in \mathbb{R}.$$

The equivalent conjugate system satisfied by v is

$$\frac{d}{d\sigma}v - (\alpha + a(\sigma;u_0) - 2)v = \tilde{f},\ \text{for}\ \sigma \in (\sigma_m, \sigma_T),$$
$$w(\sigma_T) = v_T,$$

equivalently written as

$$\tilde{A}_r v = (\tilde{f}, v_T), \tag{3.2.13}$$

with $\tilde{f}(\sigma) = \varepsilon^{(\alpha-2)/3}e^{(\alpha-2)\sigma}f(\sigma) \in \mathcal{C}(\sigma_{\text{inf}}, \sigma_{\text{sup}})$, and $v_T = \varepsilon^{(\alpha-2)/3}e^{(\alpha-2)\sigma_T}w_T$.

Then, from (3.2.8) we see that the first component of $\tilde{A}_r$ converges uniformly in u_0 and ε as $\sigma \to \infty$ to the asymptotic operator

$$v \mapsto \left(\frac{d}{d\sigma} - \alpha\right)v.$$

Since $\alpha > 0$, we may apply Lemma A.2.1 to (3.2.13) from the Appendix with $L = \sigma_T$ to conclude that there exists a constant C independent of ε, u_0 with

$$\|w\|_{\mathcal{C}_r} = |v|_\infty \leq C(|\varepsilon^{(\alpha-2)/3}e^{(\alpha-2)\sigma}f|_\infty + |v_T|) \leq C(\|f\|_{\mathcal{C}_r} + |w_T|). \tag{3.2.14}$$

Notice that $|v_T| \leq w_T$ by the asymptotic expansion of $\sigma_T = \mathrm{O}(-\log(\varepsilon^{1/3}))$. We therefore conclude that A_r is uniformly invertible in the parameters u_0, ε.

To show the Lipschitz estimates, we observe that if we take $f \in \mathcal{C}_r$ and $u_0, \tilde{u}_0 \in (\bar{u}_0 - \eta, \bar{u}_0 + \eta)$, then we have

$$(A_r^{-1} - \tilde{A}_r^{-1})f = \tilde{A}_r^{-1}(\tilde{A}_r - A_r)A_r^{-1}f,$$

where we abbreviated $A_r^{-1} = A_r^{-1}(u_0, \varepsilon)$ and $\tilde{A}_r^{-1} = A_r^{-1}(\tilde{u}_0, \varepsilon)$ and similarly for A_r. Since we just showed that A_r^{-1} is uniformly bounded in u_0 and ε, it suffices to estimate the Lipschitz constant for the map

$$u_0 \mapsto A_r(u_0, \varepsilon).$$

However, by the definition of A_r, we have

$$\|A_r - \tilde{A}_r\|_{\mathcal{C}_r \to \mathcal{C}_r \times \mathbb{R}} \leq \sup_{\sigma \in [\sigma_{\inf}, \sigma_{\sup}]} |a(\sigma; u_0) - a(\sigma; \tilde{u}_0)|.$$

where $a(\sigma; u_0) = 2\varphi(\sigma) u_R(\psi(\sigma; u_0); u_0)$ was defined in (3.2.8). Now we use the asymptotic expansion (1.3.14) of u_R

$$u_R(s; u_0) = \frac{1}{\Omega_\infty - s} + (\Omega_\infty - s) r(\Omega_\infty - s; u_0),$$

with the remainder function r smooth in both of its arguments, to observe that

$$\sup_{\sigma \in [\sigma_{\inf}, \sigma_{\sup}]} |a(\sigma; u_0) - a(\sigma; \tilde{u}_0)| \lesssim \sup_{\sigma \in [\sigma_{\inf}, \sigma_{\sup}]} |r(e^{-\sigma}; u_0) - r(e^{-\sigma}; \tilde{u}_0)| \leq C|u_0 - \tilde{u}_0|$$

for some constant C independent of ε. This shows the estimate (3.2.12) and concludes the proof. ∎

3.2.5 Nonlinear estimates

In this section we estimate the nonlinear term

$$\mathcal{R}_r(W_r; \varepsilon, u_0)(\sigma) = W_r^2(\sigma) + f(U_*(\sigma; \varepsilon, u_0) + W_r(\sigma), \mu; \varepsilon),$$

in the $\mathcal{C}_r$-norm to prove the following.

Proposition 2.2 *If $W_r \in \mathcal{C}_r$, then $\varepsilon^{-1/3}\varphi\mathcal{R}_r(W_r) \in \mathcal{C}_r$, and*

$$\|\varepsilon^{-1/3}\varphi\mathcal{R}_r(W_r)\|_{\mathcal{C}_r} = \mathrm{O}(\delta^\alpha). \tag{3.2.15}$$

where $\alpha > 0$ is a parameter in the definition of the function space $\mathcal{C}_r$. Moreover, we *have the following Lipschitz constant estimate for the map $W_r \mapsto \varepsilon^{-1/3}\varphi\mathcal{R}_r(W_r)$.*

$$\|\varepsilon^{-1/3}\varphi(\mathcal{R}_r(\widetilde{W}_r) - \mathcal{R}_r(W_r))\|_{\mathcal{C}_r} \leq C\delta^{1-\alpha}\|\widetilde{W}_r - W_r\|_{\mathcal{C}_r}, \tag{3.2.16}$$

for some constant C independent of ε, δ.

Proof. Proposition 1.3.13 shows that

$$U_*(\sigma; u_0) = \varepsilon^{\frac{1}{3}}(e^{\sigma} + e^{-\sigma}r(e^{-\sigma}; u_0)) \text{ as } \sigma \to \infty, \tag{3.2.17}$$

with the remainder function $r(\cdot; u_0)$ smooth in both arguments. Therefore we have

$$|U_*(\sigma)| \lesssim \varepsilon^{\frac{1}{3}}e^{\sigma} \leq \varepsilon^{1/3}e^{\sigma_{\sup}} = \mathrm{O}(\delta_+),$$

for all $\sigma \in [\sigma_{\inf}, \sigma_{\sup}]$.

From the definition of the time rescaling in Section 3.1 we have

$$\mu = \varepsilon t - \delta_- = \varepsilon^{2/3}\psi(\sigma) \lesssim \varepsilon^{2/3},$$

for all $\sigma \in [\sigma_{\inf}, \sigma_{\sup}]$, since $|\psi(\sigma)| \leq \Omega_\infty(u_0)$.

Since $W_r \in \mathcal{C}_r$, we have

$$|W_r(\sigma)| \lesssim \varepsilon^{\frac{2-\alpha}{3}}e^{(2-\alpha)\sigma} \ll \varepsilon^{1/3}e^{\sigma}, \text{ for } \sigma \in [\sigma_{\inf}, \sigma_{\sup}].$$

which implies $|W_r(\sigma)| \lesssim |U_*(\sigma)|$ for $\sigma \in [\sigma_{\inf}, \bar{\sigma}_{\sup}]$, using the asymptotic expansion (3.2.17).

Now we use the expansion (1.3.8)

$$f(u, \mu; \varepsilon) = \mathrm{O}(\varepsilon(1 + u + \mu + u^2), u\mu, \mu^2, u^3)$$

and the fact that U_*, W_r, μ are all small in sup norm, to observe that

$$f(U_* + W_r, \mu; \varepsilon) = \mathrm{O}(\varepsilon, (U_* + W_r)\mu, \mu^2, (U_* + W_r)^3) = \mathrm{O}(\varepsilon, U_*\mu, \mu^2, U_*^3). \tag{3.2.18}$$

Using these facts, we have

$$\|\varepsilon^{-\frac{1}{3}}\varphi W_r^2\|_{\mathcal{C}_r} = \sup_{\sigma\in(\sigma_{\inf},\sigma_{\sup})} |\varepsilon^{-\frac{1}{3}}\varphi W_r| \lesssim \varepsilon^{\frac{1-\alpha}{3}} e^{(1-\alpha)\sigma} \lesssim \varepsilon^{\frac{1-\alpha}{3}} e^{(1-\alpha)\sigma_{\sup}} = \mathrm{O}(\delta_+^{1-\alpha}), \tag{3.2.19}$$

and

$$\|\varepsilon^{-1/3}\varphi f(U_* + W_r, \mu; \varepsilon)\|_{\mathcal{C}_r} \leq \|\varepsilon^{\frac{2}{3}}\varphi\|_{\mathcal{C}_r} + \|\varepsilon^{-\frac{1}{3}}\varphi U_*\mu\|_{\mathcal{C}_r} + \|\varepsilon^{-\frac{1}{3}}\varphi\mu^2\|_{\mathcal{C}_r} + \|\varepsilon^{-\frac{1}{3}}\varphi U_*^3\|_{\mathcal{C}_r}.$$

However we have

$$\|\varepsilon^{\frac{2}{3}}\varphi\|_{\mathcal{C}_r} = \sup_{\sigma\in(\sigma_{\inf},\sigma_{\sup})} |\varepsilon^{\frac{2}{3}}\varphi\varepsilon^{(\alpha-2)/3}e^{(\alpha-2)\sigma}| \lesssim \varepsilon^{\alpha/3}e^{(\alpha-3)\sigma_m} = \mathrm{O}(\varepsilon^{\alpha/3}),$$

$$\|\varepsilon^{-\frac{1}{3}}\varphi U_*\mu\|_{\mathcal{C}_r} = \sup_{\sigma\in(\sigma_{\inf},\sigma_{\sup})} |\varepsilon^{-\frac{1}{3}}\varphi U_*\mu\varepsilon^{(\alpha-2)/3}e^{(\alpha-2)\sigma}| \lesssim \varepsilon^{\alpha/3}e^{(\alpha-2)\sigma_m} = \mathrm{O}(\varepsilon^{\alpha/3}),$$

$$\|\varepsilon^{-\frac{1}{3}}\varphi\mu^2\|_{\mathcal{C}_r} \lesssim \|\varepsilon^{-\frac{1}{3}}\varphi\varepsilon\|_{\mathcal{C}_r} \text{ since } \mu^2 = \mathrm{O}(\varepsilon^{4/3}),$$

and

$$\|\varepsilon^{-\frac{1}{3}}\varphi U_*^3\|_{\mathcal{C}_r} \lesssim \varepsilon^{\frac{\alpha}{3}}e^{\alpha\sigma} \lesssim \varepsilon^{\frac{\alpha}{3}}e^{\alpha\sigma_{\sup}} = \mathrm{O}(\delta_+^{\alpha}).$$

Therefore, we conclude that

$$\|\varepsilon^{-1/3}\varphi f(U_* + W_r, \mu; \varepsilon)\|_{\mathcal{C}_r} = \mathrm{O}(\delta_+^{\alpha}) \tag{3.2.20}$$

Combining estimates (3.2.19) and (3.2.20) we conclude that

$$\|\varepsilon^{-1/3}\varphi\mathcal{R}_r(W_r)\|_{\mathcal{C}_r} = \max\{\mathrm{O}(\delta_+^{\alpha}), \mathrm{O}(\delta_+^{1-\alpha})\},$$

thus for $\alpha \leq 1/2$, it follows that $\|\varepsilon^{-1/3}\varphi\mathcal{R}_r(W_r)\|_{\mathcal{C}_r} = \mathrm{O}(\delta_+^{\alpha})$. This shows the estimate (3.2.15).

We now show the Lipschitz constant estimate (3.2.16) for the map

$$W_r \mapsto \varepsilon^{-1/3}\varphi\mathcal{R}_r(W_r).$$

First recall (3.2.18), which implies that

$$\begin{aligned}\|\varepsilon^{-1/3}\varphi(f(U_*+W_r,\mu;\varepsilon)-f(U_*+\widetilde{W}_r,\mu;\varepsilon))\|_{\mathcal{C}_r} &\le C\sup_{\sigma\in(\sigma_{\inf},\sigma_{\sup})}|\varepsilon^{-1/3}\varphi\mu|\|W_r-\widetilde{W}_r\|_{\mathcal{C}_r}\\ &\le C\varepsilon^{1/3}\|W_r-\widetilde{W}_r\|_{\mathcal{C}_r}\end{aligned}$$

for some constant C independent of ε and δ.

On the other hand, we have

$$\|\varepsilon^{-1/3}\varphi(\widetilde{W}_r^2-W_r^2)\|_{\mathcal{C}_r}\le C\sup_{\sigma\in(\sigma_{\inf},\sigma_{\sup})}|\varepsilon^{-1/3}(W_r+\tilde{W}_r)|\|W_r-\widetilde{W}_r\|_{\mathcal{C}_r}\le C\delta^{1-\alpha}\|W_r-\widetilde{W}_r\|_{\mathcal{C}_r}$$

for some constant C independent of ε and δ.

Since $\mathcal{R}_r(W_r)=W_r^2+f(U_*+W_r;\mu,\varepsilon)$, we therefore conclude that

$$\|\varepsilon^{-1/3}\varphi(\mathcal{R}_r(\widetilde{W}_r)-\mathcal{R}_r(W_r))\|_{\mathcal{C}_r}\le C\delta^{1-\alpha}\|\widetilde{W}_r-W_r\|_{\mathcal{C}_r},$$

for some constant C independent of ε and δ, which is the estimate (3.2.16). This concludes the proof. ∎

3.2.6 Fixed point argument and the proof of Theorem 4

In this section we prove Theorem 4 by setting up an appropriate fixed point argument.

Proof of Theorem 4. Assertions (i) and (ii) in the theorem have been demonstrated in Sections 3.2.2 and 3.2.3. Assertion (iii) is a direct consequence of the fact that $W_r\in\mathcal{C}_r$. To prove a solution W_r exists, observe that (3.2.7)

$$\left(\frac{d}{d\sigma}-a(\sigma;u_0)\right)W_r=\varepsilon^{-1/3}\varphi\mathcal{R}_r(W_r;\varepsilon,u_0),$$

with the boundary condition $W_r(\sigma_T) = 0$ can be rewritten as

$$\begin{aligned}(0,0) &= \left(\frac{d}{d\sigma}W_r - aW_r - \varepsilon^{-1/3}\varphi\mathcal{R}_r(W_r), W_r(\sigma_T)\right)\\ &= \left(\frac{d}{d\sigma}W_r - aW_r, W_r(\sigma_T)\right) - \left(\varepsilon^{-1/3}\varphi\mathcal{R}_r(W_r), 0\right)\\ &= A_r W_r - \left(\varepsilon^{-1/3}\varphi\mathcal{R}_r(W_r), 0\right).\end{aligned}$$

We now use Proposition 2.1 to precondition the above equation with the operator A_r^{-1} to obtain the equivalent equation

$$W_r = \mathcal{S}_r(W_r; \varepsilon, u_0) := A_r^{-1}(\varepsilon^{-1/3}\varphi\mathcal{R}_r(W_r), 0). \tag{3.2.21}$$

Note that by Proposition 2.1 and Proposition 2.2, $\mathcal{S}_r$ is well defined and maps $\mathcal{C}_r$ into $\mathcal{C}_r$ with ε and u_0 as parameters. Moreover, we have the following.

- At $W_r = 0$ it holds that

$$\begin{aligned}\|\mathcal{S}_r(0; \varepsilon, u_0)\|_{\mathcal{C}_r} &= \|A_r^{-1}(\varepsilon^{-1/3}\varphi\mathcal{R}_r(0), 0)\|_{\mathcal{C}_r}\\ &\leq \|A_r^{-1}\|\|(\varepsilon^{-1/3}\varphi\mathcal{R}_r(0), 0)\|_{\mathcal{C}_r}\\ &\lesssim \|\varepsilon^{-1/3}\varphi U_*^3\|_{\mathcal{C}_r} \leq C_1\delta_+^\alpha,\end{aligned}$$

  for some constant C_1 independent of ε and u_0.

- The map $\mathcal{S}_r : \mathcal{C}_r \to \mathcal{C}_r$ is well defined and smooth in W_r.

- The derivative of $f(U_* + W_r, \mu; \varepsilon)$ with respect to W_r satisfies

$$D_{W_r} f(U_* + W_r, \mu; \varepsilon) = \mathrm{O}(\mu, U_*^2).$$

  The linearization of $\mathcal{S}_r$ at $W_r = 0$, $D_{W_r}\mathcal{S}_r(0; \varepsilon, u_0)$ satisfies

$$\|D_{W_r}\mathcal{S}_r(0; \varepsilon, u_0)\|_{\mathcal{C}_r \to \mathcal{C}_r} \lesssim \sup_\sigma |\varepsilon^{-1/3}\varphi(\mu + U_*^2)| = \mathrm{O}(\delta_+).$$

Moreover, for $\|W_r\|_{\mathcal{C}_r} \lesssim \delta_+^\alpha$, we have

$$\begin{aligned}\|D_{W_r}\mathcal{S}_r(W_r;\varepsilon,u_0)\|_{\mathcal{C}_r\to\mathcal{C}_r} &\leq \|D_{W_r}\mathcal{S}_r(0;\varepsilon,u_0)\|_{\mathcal{C}_r\to\mathcal{C}_r} + \sup_\sigma |\varepsilon^{-1/3}\varphi W_r| \\ &\lesssim \mathrm{O}(\delta_+) + \|W_r\| \sup_\sigma |\varepsilon^{-1/3}\varphi\varepsilon^{\frac{2-\alpha}{3}} e^{(2-\alpha)\sigma}| \\ &\lesssim (\delta_+ + \delta_+^\alpha \delta_+^{1-\alpha}) \leq C_2\delta_+,\end{aligned}$$

for some constant C_2 independent of ε and u_0.

Then, consider $W_r \in \mathcal{C}_r$ with $\|W_r\|_{\mathcal{C}_r} \leq 2C_1\delta_+^\alpha$, we have

$$\begin{aligned}\|\mathcal{S}_r(W_r)\|_{\mathcal{C}_r} &\leq \|\mathcal{S}_r(W_r) - \mathcal{S}_r(0)\|_{\mathcal{C}_r} + \|\mathcal{S}_r(0)\|_{\mathcal{C}_r} \\ &\leq C_2\delta_+\|W_r\|_{\mathcal{C}_r} + C_1\delta_+^\alpha \\ &\leq 2C_1C_2\delta_+^{1+\alpha} + C_1\delta_+^\alpha \leq 2C_1\delta_+^\alpha,\end{aligned} \tag{3.2.22}$$

for $\delta_+ < (2C_2)^{-1}$. Therefore, the map $\mathcal{S}_r$ maps the ball $\mathcal{B}_{2C_1\delta_+^\alpha}(0)$ in $\mathcal{C}_r$ into itself. We may apply Banach's fix point theorem to conclude the existence of a fixed point for the map $\mathcal{S}_r$, uniformly in ε, u_0. The fixed point W_r consequently solves the boundary value problem (3.2.10) as desired.

Finally, to prove item (iv) we need to estimate the Lipschitz constant for the map

$$\Psi : u_0 \mapsto w_r(\varepsilon^{-1}\delta_-; u_0) = W_r(\sigma_m; u_0),$$

which maps from a small interval I containing $\bar{u}_0$ to $\mathbb{R}$. We can write Ψ as the composition of two maps $\Psi = \Psi_1 \circ \Psi_2$ where $\Psi_2 : I \to \mathcal{C}_r$ is the map

$$u_0 \mapsto W_r(\sigma; u_0),$$

and $\Psi_1 : \mathcal{C}_r \to \mathbb{R}$ is the evaluation map

$$W_r(\sigma; u_0) \mapsto W_r(\sigma_m, u_0).$$

To estimate $\mathrm{Lip}_{u_0}\Psi$, we need to estimate $\mathrm{Lip}_{u_0}\Psi_2$ and $\mathrm{Lip}_{W_r}\Psi_1$.

For $\mathrm{Lip}_{u_0}\Psi_2$, it suffices to estimate the following two quantities

$$C_1 = \mathrm{Lip}_{W_r}\mathcal{S}_r, \text{ and } C_2 = \mathrm{Lip}_{u_0}\mathcal{S}_r.$$

Because W_r is the fixed point of the map $\mathcal{S}_r$, this implies

$$\mathrm{Lip}_{u_0}\Psi_2 \le C_2/(1 - C_1).$$

From the Lipschitz estimate (3.2.16), we see that

$$C_1 \lesssim \mathrm{Lip}_{W_r}|\varepsilon^{-1/3}\varphi R_r(W_r)| = \mathrm{O}(\delta_+^{1-\alpha}).$$

To estimate C_2. We notice that

$$\mathrm{Lip}_{u_0}\mathcal{S}_r \le \mathrm{Lip}_{u_0}A_r^{-1} \cdot \sup_{\sigma} |\varepsilon^{-1/3}\mathcal{R}(W_r)| + \sup_{u_0} \|A_r^{-1}\| \cdot \mathrm{Lip}_{u_0}|\varepsilon^{-1/3}\varphi\mathcal{R}(W_r)|$$

First from the estimate (3.2.12) in Proposition 2.1, we have

$$\mathrm{Lip}_{u_0}A_r^{-1} \cdot \sup |\varepsilon^{-1/3}\mathcal{R}(W_r)| \lesssim \sup_{\sigma} |\varepsilon^{-1/3}\mathcal{R}(W_r)| \lesssim \sup_{\sigma} |\varepsilon^{-1/3}U_*^3| = \mathrm{O}(\delta_+^2).$$

Then we observe that

$$\|A_r^{-1}\| \cdot \mathrm{Lip}_{u_0}|\varepsilon^{-1/3}\varphi\mathcal{R}(W_r)| \lesssim \ \mathrm{Lip}_{u_0}|\varepsilon^{-1/3}\varphi U_*^3(\sigma; u_0)|.$$

However if $u_0, \tilde{u}_0$ are two different numbers near $\bar{u}_0$, then

$$\|\varepsilon^{-1/3}\varphi[U_*^3(\sigma; u_0) - U_*^3(\sigma; \tilde{u}_0)]\|_{\mathcal{C}_r} \le \|\varepsilon^{-1/3}\varphi U_*^2\|_{\mathcal{C}_r} \sup |U_*(\sigma; u_0) - U_*(\sigma; \tilde{u}_0)|.$$

Moreover, by Proposition 1.3.13, $U_*(\sigma; u_0) = \varepsilon^{1/3}(e^{\sigma} + e^{-\sigma}r(e^{-\sigma}; u_0))$ for σ large, hence

$$\partial_{u_0}U_*(\sigma; u_0) \lesssim \varepsilon^{1/3}.$$

On the other hand

$$\|\varepsilon^{-1/3}\varphi U_*^2\|_{\mathcal{C}_r} = \mathrm{O}(\varepsilon^{(\alpha-1)/3}),$$

so that we can conclude that

$$\|\varepsilon^{-1/3}\varphi(U_*^3(\sigma;u_0)-U_*^3(\sigma;\tilde{u}_0))\|_{\mathcal{C}_r}\lesssim \varepsilon^{\alpha/3}|u_0-\tilde{u}_0|.$$

Therefore, we have that

$$C_2=\mathrm{Lip}_{u_0}\mathcal{S}_r=\mathrm{O}(\delta_+^2),$$

and hence

$$\mathrm{Lip}_{u_0}\Psi_2\le C_2/(1-C_1)=\mathrm{O}(\delta_+^2).$$

On the other hand, the evaluation map Ψ_1 is a linear map which satisfies

$$\begin{aligned}|\Psi_1(W)-\Psi_1(\widetilde{W})|&=|W(\sigma_m)-\widetilde{W}(\sigma_m)|\\&\le\varepsilon^{(2-\alpha)/3}e^{(2-\alpha)\sigma_m}\|W-\widetilde{W}\|_{\mathcal{C}_r}\\&\lesssim\varepsilon^{(2-\alpha)/3}\|W-\widetilde{W}\|_{\mathcal{C}_r},\end{aligned}$$

for $W,\widetilde{W}$ in $\mathcal{C}_r$. Therefore,

$$\mathrm{Lip}_{W_r}\Psi_1=\mathrm{O}(\varepsilon^{(2-\alpha)/3}).$$

Combining these two estimates we conclude that

$$\mathrm{Lip}_{u_0}\Psi\le\left(\mathrm{Lip}_{u_0}\Psi_2\right)\left(\mathrm{Lip}_{W_r}\Psi_1\right)=\mathrm{O}(\delta_+^2\varepsilon^{(2-\alpha)/3})=\mathrm{O}(\varepsilon^{(2-\alpha)/3}),$$

which completes the proof of Theorem 4. ■

3.3 Region B

Region B corresponds to the time-t interval $\{t:t^*<t<\varepsilon^{-1}\delta_-\}$. In this section, we give the precise form of the solution in this region and prove the needed properties.

3.3.1 Solution in region B

Theorem 5 *Fix $\delta_-,\delta_+>0,\alpha>0$ small enough. Then there exists $\varepsilon_B>0$ and a constant $C=C(\delta,_-,\delta_+,\alpha)$ such that for all $0<\varepsilon<\varepsilon_B$, and all w^* with $|w^*|\le$*

$C\varepsilon^{1/2-\alpha/3}$, *a solution* u_B *of the form*

$$u_B(t) = \bar{u}_*(t) + w_\ell(t), \tag{3.3.1}$$

to (1.3.19) *with the initial condition*

$$u(t^*) = \bar{u}_*(t^*) + w^*, \qquad \mu(t^*) = \varepsilon t^* - \delta_- \tag{3.3.2}$$

exists on the time interval $t \in (t^*, \varepsilon^{-1}\delta_-)$. *Here*

$$\bar{u}_*(t) = \varepsilon^{1/3}\bar{u}_R(\varepsilon^{1/3}(t - \varepsilon^{-1}\delta_-)),$$

and $\bar{u}_R(\cdot)$ *is the special solution of the Riccati equation with asymptotics* (1.3.11)*. Moreover, the correction function* w_ℓ *is continuous with*

$$|w_\ell(t)| \leq C\varepsilon^{(2-\alpha)/3}|\varepsilon^{1/3}(t - \varepsilon^{-1}\delta_-) + 1|, \tag{3.3.3}$$

for $t \in (t^*, \varepsilon^{-1}\delta_-)$.

We will prove this theorem in the rest of this section.

3.3.2 Equation of W_ℓ and rescaling

As before, we substitute $u = \bar{u}_* + w_\ell$ into (1.3.7) to obtain the equation for w_ℓ.

$$w_\ell' - 2\bar{u}_* w_\ell = w_\ell^2 + f(\bar{u}_* + w_\ell, \mu; \varepsilon) =: R_\ell(w_\ell, \mu; \varepsilon). \tag{3.3.4}$$

We want to solve this equation on $t \in (t^*, \varepsilon^{-1}\delta_-)$. Following previous steps, we next rescale the equation and obtain

$$\frac{d}{d\sigma}W_\ell - b(\sigma)W_\ell = \varepsilon^{-1/3}\varphi\mathcal{R}_\ell(W_\ell, \mu, \varepsilon) \tag{3.3.5}$$

We have the following observations.

- The equation is posed on $\sigma \in (\sigma^*, \sigma_m(\bar{u}_0))$ where $\sigma^* < 0$ and $|\sigma^*| = \mathrm{O}(\varepsilon^{-1/4})$. While $\sigma_m(\bar{u}_0) =: \bar{\sigma}_m$ is defined in Section 3.1.

- The term $b(\sigma)$ satisfies

$$b(\sigma) := 2u_R(\psi(\sigma;\bar{u}_0))\varphi(\sigma) = -2 + \mathrm{O}(|\sigma|^{-1}) \text{ as } \sigma \to -\infty.$$

  We remark that the convergence is uniform in ε.

- The function $W_\ell(\sigma)$ is the rescaled version of $w_\ell(t)$ in the σ-variable, with

$$W_\ell(\sigma) := w_\ell(\varepsilon^{-1/3}\psi(\sigma) + \varepsilon^{-1}\delta_-) = w_\ell(t).$$

  Similarly, $\bar{U}_*$ is the rescaled version of $\bar{u}_*$, with

$$\bar{U}_*(\sigma) := \bar{u}_*(\varepsilon^{-1/3}\psi(\sigma) + \varepsilon^{-1}\delta_-) = \varepsilon^{1/3}\bar{u}_R(\psi(\sigma)) = \bar{u}_*(t).$$

- The function $\mathcal{R}_\ell$ is a rescaled version of R_ℓ such that

$$\mathcal{R}_\ell(W_r;\varepsilon,u_0) := W_\ell^2 + f(\bar{U}_* + W_\ell,\mu;\varepsilon).$$

We also need to specify the boundary value at the left end point $\sigma = \sigma^*$. For $w^* \in \mathbb{R}$ whose order of magnitude will be determined later, we pose the following problem for W_ℓ

$$\begin{aligned} &\frac{d}{d\sigma}W_\ell - b(\sigma)W_\ell = \varepsilon^{-1/3}\varphi\mathcal{R}_\ell(W_\ell), \text{ for } \sigma \in (\sigma^*,\bar{\sigma}_m),\\ &W_\ell(\sigma^*) = w^*. \end{aligned} \tag{3.3.6}$$

3.3.3 Linear equation and norms

Fix $\alpha > 0$, the proof of Theorem 5 consists of solving (3.3.5) via a fixed point argument on the following function space

$$\mathcal{C}_\ell = \mathcal{C}_\ell(\sigma^*,\bar{\sigma}_m) = \left\{ w(\sigma) \in \mathcal{C}(\sigma^*,\bar{\sigma}_m) : \sup_{\sigma^*\le\sigma\le\bar{\sigma}_m} |\varepsilon^{(\alpha-2)/3}\langle\sigma\rangle^{-2/3}w(\sigma)| < \infty \right\}.$$

We also study the linear operator A_ℓ with

$$A_\ell w = \left(\frac{d}{d\sigma}w - b(\sigma)w, w(\sigma^*)\right),$$

which is defined on a dense subset $\mathcal{D}(A_\ell) \subset \mathcal{C}_\ell$.

Proposition 3.1 *Fix $\varepsilon_2 > 0$, there is a constant C such that the operator $A_\ell : \mathcal{D}(A_\ell) \subset \mathcal{C}_\ell \to \mathcal{C}_\ell \times \mathbb{R}$ is invertible. Moreover, the inverse $A_r^{-1}(\varepsilon) : \mathcal{C}_\ell \times I_\ell \to \mathcal{C}_\ell$ is bounded uniformly for ε with $0 < \varepsilon < \varepsilon_2$, where I_ℓ is the interval*

$$I_\ell := \{w^* : |w^*| \leq C\varepsilon^{1/2-\alpha/3}\}.$$

Proof. Similar to the proof of Proposition 2.1, we set $v(\sigma) = \varepsilon^{(\alpha-2)/3}\langle\sigma\rangle^{-2/3}w(\sigma)$ and consider the conjugate linear operator

$$\begin{aligned}\tilde{A}_\ell v &= \left(\langle\sigma\rangle^{-2/3}\left(\frac{d}{d\sigma} - b(\sigma)\right)\langle\sigma\rangle^{2/3}v, v(\sigma^*)\right)\\ &= \left(\frac{d}{d\sigma}v - \tilde{b}(\sigma)v, v(\sigma^*)\right).\end{aligned}$$

The operator is defined on a dense subset of $\mathcal{C}(\sigma^*, \bar{\sigma}_m)$, with $\tilde{b}$ satisfying

$$\tilde{b}(\sigma) = b(\sigma) - \frac{2}{3}\langle\sigma\rangle^{-1} = -2 + \mathrm{O}(|\sigma|^{-1}) \to -2,$$

as $\sigma \to -\infty$.

The equivalent conjugate equation of

$$A_\ell w = (f, w^*) \text{ with } f \in \mathcal{C}_\ell, \text{ and } w^* \in \mathbb{R},$$

is

$$\tilde{A}_\ell v = (\varepsilon^{(\alpha-2)/3}\langle\sigma\rangle^{-2/3}f, v^*). \tag{3.3.7}$$

With $\varepsilon^{(\alpha-2)/3}\langle\sigma\rangle^{-2/3}f \in C, v^* = \varepsilon^{(\alpha-2)/3}\langle\sigma^*\rangle^{-2/3}w^*$.

Therefore, we may apply Lemma 2.1 to (3.3.7) to conclude that there exist a constant

C independent of ε such that

$$\|w\|_{\mathcal{C}_\ell} = |v|_\infty \leq C(|\varepsilon^{(\alpha-2)/3}\langle\sigma\rangle^{-2/3} f|_\infty + |v^*|) = C(\|f\|_{\mathcal{C}_\ell} + \varepsilon^{\alpha/3-1/2}|w^*|). \qquad (3.3.8)$$

Therefore, there is $\varepsilon_* > 0$ small enough so that for all $\varepsilon \in (0, \varepsilon_2)$ the operator A_ℓ is uniformly invertible in $\varepsilon \in (0, \varepsilon_2)$, provided that $w_0 \in I_\ell$. ■

3.3.4 Nonlinear estimates

Next, we will estimate the nonlinear term

$$\mathcal{R}_\ell(W_\ell) = W_\ell^2 + f(\bar{U}_* + W_\ell, \mu; \varepsilon),$$

in the $\mathcal{C}_\ell$ norm.

Proposition 3.2 *If $W_\ell \in \mathcal{C}_\ell$, then $\varepsilon^{-1/3}\varphi\mathcal{R}_\ell(W_\ell(\sigma)) \in \mathcal{C}_\ell$ and*

$$\|\varepsilon^{-1/3}\varphi\mathcal{R}_\ell(W_\ell)\|_{\mathcal{C}_\ell} = \mathrm{O}(\varepsilon^{\alpha/3}). \qquad (3.3.9)$$

Proof. From the asymptotic expansion (1.3.11), we have that

$$\bar{U}_*(\sigma) = \varepsilon^{1/3}\bar{u}_R(\psi(\sigma)) \lesssim |\varepsilon\sigma|^{1/3} \leq \varepsilon^{1/4}, \qquad (3.3.10)$$

and

$$\varphi(\sigma) \lesssim \langle\sigma\rangle^{-1/3}, \qquad (3.3.11)$$

for $\sigma^* \leq \sigma \leq \bar{\sigma}_m$.

Also, for σ in this range, we have $\mu = \varepsilon t - \delta_- = \varepsilon^{2/3}\psi(\sigma)$ by definition of ψ in (3.1.7). Since $|\psi(\sigma)| \leq |\sigma|^{2/3}$, we have that

$$|\mu| \lesssim |\varepsilon\sigma|^{2/3} \leq \varepsilon^{1/2}.$$

If $W_\ell \in \mathcal{C}_\ell$, then it is true that

$$|W_\ell(\sigma)| \lesssim \varepsilon^{(2-\alpha)/3}\langle\sigma\rangle^{2/3} \ll |\varepsilon\sigma|^{1/3},$$

which implies that $|W_\ell(\sigma)| \lesssim |\bar{U}_*(\sigma)|$ for $\sigma \in [\sigma^*, \bar{\sigma}_m]$ by the asymptotics (3.3.10).

Now, again, we use the expansion (1.3.8) for f to observe that

$$f(\bar{U}_* + W_\ell, \mu; \varepsilon) = \mathrm{O}(\varepsilon, (\bar{U}_* + W_\ell)\mu, \mu^2, (\bar{U}_* + W_\ell)^3) = \mathrm{O}(\varepsilon, \bar{U}_*\mu, \mu^2, \bar{U}_*^3). \qquad (3.3.12)$$

Finally recall $|\sigma^*| = \mathrm{O}(\varepsilon^{-1/4})$ and $|\bar{\sigma}_m| = \mathrm{O}(1)$. From these facts we have

$$\|\varepsilon^{-1/3}\varphi W_\ell^2\|_{\mathcal{C}_\ell} \lesssim \sup_{\sigma\in(\sigma^*,\bar{\sigma}_m)} \varepsilon^{(1-\alpha)/3}\langle\sigma\rangle^{1/3} \leq \varepsilon^{(1-\alpha)/3}\langle\sigma^*\rangle^{1/3} = \mathrm{O}(\varepsilon^{(3-4\alpha)/12}). \qquad (3.3.13)$$

From (3.3.12) it follows that

$$\|\varepsilon^{-1/3}\varphi f(\bar{U}_*+W_\ell, \mu; \varepsilon)\|_{\mathcal{C}_\ell} \leq \|\varepsilon^{2/3}\varphi\|_{\mathcal{C}_\ell} + \|\varepsilon^{-1/3}\varphi\bar{U}_*\mu\|_{\mathcal{C}_\ell} + \|\varepsilon^{-1/3}\varphi\mu^2\|_{\mathcal{C}_\ell} + \|\varepsilon^{-1/3}\varphi\bar{U}_*^3\|_{\mathcal{C}_\ell}.$$

However, we have

$$\|\varepsilon^{2/3}\varphi\|_{\mathcal{C}_\ell} \lesssim \sup_{\sigma\in(\sigma^*,\bar{\sigma}_m)} \varepsilon^{-1/3}\varphi(\sigma)\varepsilon(\varepsilon^{(\alpha-2)/3}\langle\sigma\rangle^{-2/3}) \lesssim \varepsilon^{\alpha/3}\langle\bar{\sigma}_m\rangle^{-1} = \mathrm{O}(\varepsilon^{\alpha/3}),$$

$$\|\varepsilon^{-1/3}\varphi\bar{U}_*\mu\|_{\mathcal{C}_\ell} \lesssim \sup_{\sigma\in(\sigma^*,\bar{\sigma}_m)} \varepsilon^{-1/3}\varphi|\varepsilon\sigma|^{2/3}|\varepsilon\sigma|^{1/3}\varepsilon^{(\alpha-2)/3}\langle\bar{\sigma}_m\rangle^{-2/3} = \mathrm{O}(\varepsilon^{\alpha/3}),$$

$$\|\varepsilon^{-1/3}\varphi\mu^2\|_{\mathcal{C}_\ell} \lesssim \|\varepsilon^{2/3}\varphi\|_{\mathcal{C}_\ell} = \mathrm{O}(\varepsilon^{\alpha/3}), \text{ since } \mu^2 \lesssim \varepsilon,$$

and

$$\|\varepsilon^{-1/3}\varphi\bar{U}_*^3\|_{\mathcal{C}_\ell} \lesssim \sup_{\sigma\in(\sigma^*,\bar{\sigma}_m)} \varepsilon^{(\alpha-2)/3}\langle\sigma\rangle^{-2/3}|\varepsilon\sigma|^{2/3} = \mathrm{O}(\varepsilon^{\alpha/3}).$$

Therefore we conclude that

$$\|\varepsilon^{-1/3}\varphi f(\bar{U}_* + W_\ell, \mu; \varepsilon)\|_{\mathcal{C}_\ell} = \mathrm{O}(\varepsilon^{\alpha/3}). \qquad (3.3.14)$$

Combining estimates (3.3.13) and (3.3.14) we conclude that $\|\varepsilon^{-1/3}\varphi\mathcal{R}_\ell(W_\ell)\|_{\mathcal{C}_\ell} = \mathrm{O}(\varepsilon^{\alpha/3})$ if $W_\ell \in \mathcal{C}_\ell$, which finishes the proof. ■

3.3.5 Fixed point argument and the proof of Theorem 5

We are now ready to prove Theorem 5.

Proof of Theorem 5. The proof consists of rewriting (3.2.10) as a fixed point equation. Using the linear operator A_ℓ, we rewrite (3.3.6)

$$\frac{d}{d\sigma}W_\ell - b(\sigma)W_\ell = \varepsilon^{-1/3}\varphi\mathcal{R}_\ell(W_\ell),$$
$$W_\ell(\sigma^*) = w^*.$$

as

$$\begin{aligned}(0,0) &= \left(\frac{d}{d\sigma}W_\ell - bW_\ell - \varepsilon^{-1/3}\varphi\mathcal{R}_\ell(W_\ell), W_\ell(\sigma^*) - w^*\right)\\ &= \left(\frac{d}{d\sigma}W_\ell - bW_\ell, W_\ell(\sigma^*)\right) - \left(\varepsilon^{-1/3}\varphi\mathcal{R}_\ell(W_\ell), w^*\right)\\ &= A_\ell W_\ell - \left(\varepsilon^{-1/3}\varphi\mathcal{R}_\ell(W_\ell), w^*\right).\end{aligned}$$

By Proposition 3.1 we precondition the above equation with the operator A_ℓ^{-1} to obtain the equivalent equation

$$W_\ell = \mathcal{S}_\ell(W_\ell; w^*, \varepsilon) := A_\ell^{-1}(\varepsilon^{-1/3}\varphi\mathcal{R}_\ell(W_\ell), w^*). \tag{3.3.15}$$

From Propositions 3.1 and 3.2, we conclude the following.

- At $W_\ell = 0$ it holds that

$$\begin{aligned}\|\mathcal{S}_\ell(0; w^*, \varepsilon)\|_{\mathcal{C}_\ell} &= \|A_\ell^{-1}\left(\varepsilon^{-1/3}\varphi\mathcal{R}_\ell(W_\ell), w^*\right)\|_{\mathcal{C}_\ell}\\ &\lesssim \|\varepsilon^{-1/3}\varphi\mathcal{R}_\ell(0)\|_{\mathcal{C}_\ell}\\ &\lesssim \|\varepsilon^{-1/3}\varphi\bar{U}_*^3\|_{\mathcal{C}_\ell} \le C_1\varepsilon^{\alpha/3},\end{aligned}$$

  for some constant C_1 independent of ε, w^*.

- The map $\mathcal{S}_\ell : \mathcal{C}_\ell \to \mathcal{C}_\ell$ is well defined and smooth in W_ℓ.

- The derivative of $f(\bar{U}_* + W_\ell, \mu; \varepsilon)$ with respect to W_ℓ satisfies

$$D_{W_\ell} f(\bar{U}_* + W_\ell, \mu; \varepsilon) = \mathrm{O}(\mu, \bar{U}_*^2).$$

  The linearization of $\mathcal{S}_\ell$ at $W_\ell = 0$, $D_{W_\ell}\mathcal{S}_\ell(0; w^*, \varepsilon)$ satisfies

$$\|D_{W_\ell}\mathcal{S}_\ell(0; w^*, \varepsilon)\|_{\mathcal{C}_\ell \to \mathcal{C}_\ell} \lesssim \sup_\sigma |\varepsilon^{-1/3}\varphi(\mu + \bar{U}_*^2)| = \mathrm{O}(\varepsilon^{1/4}).$$

  Moreover, for W_r with $\|W_\ell\|_{\mathcal{C}_r} = \mathrm{O}(\varepsilon^{\alpha/3})$ and $|w^*| = \mathrm{O}(\varepsilon^{1/2-\alpha/3})$, we have

$$\|D_{W_\ell}\mathcal{S}_\ell(W_\ell; w^*, \varepsilon)\|_{\mathcal{C}_\ell \to \mathcal{C}_\ell} \le \|D_{W_\ell}\mathcal{S}_\ell(W_\ell; w^*, \varepsilon)\|_{\mathcal{C}_\ell \to \mathcal{C}_\ell} + \sup_\sigma |\varepsilon^{-1/3}\varphi W_\ell| \le C_2\varepsilon^{1/4},$$

  for some constant C_2 independent of w^*, ε.

Hence, fix $w^* = \mathrm{O}(\varepsilon^{1/2-\alpha/3})$ and by a similar estimate to (3.2.22), we have that $\mathcal{S}_\ell$ maps the ball $\mathcal{B}_{2C_1\varepsilon^{\alpha/3}}(0)$ in $\mathcal{C}_r$ into itself for $\varepsilon \le (2C_2)^{-4}$. We may apply Banach's fixed point theorem to the map $\mathcal{S}_\ell$ to obtain a solution of (3.3.6) with $W_\ell \in \mathcal{C}_\ell$. Scaling back from σ to t, we obtain claim (3.3.3), which concludes the proof. ■

3.4 Region C

This region corresponds to the time-t interval $\{t : 0 < t < t^*\}$. We will give the precise form of the solution in this region and prove the needed properties.

3.4.1 Solution in region C

Theorem 6 *Fix $\delta_- > 0, \alpha > 0$ small enough. Then there exists $\varepsilon_C > 0$ and a constant $C = C(\delta_-, \alpha)$ such that for all $0 < \varepsilon < \varepsilon_C$, and all w_0 with $|w_0| \le C\varepsilon^{1-\alpha/3}$, there exists a solution u_C of the form*

$$u_C(t) = u_s(t) + w_s(t), \tag{3.4.1}$$

to (1.3.19) *with the initial condition*

$$u(0) = u_s(0) + w_0, \qquad \mu(0) = -\delta_-, \tag{3.4.2}$$

on the time interval $t \in (0, t^*)$. *Here,* u_s *is the function defined in* (1.3.17) *and the correction function* w_s *is continuous with*

$$|w_s(t)| \le C\varepsilon^{1-\alpha/3}(\varepsilon t - \delta_-)^{-1}, \tag{3.4.3}$$

for $t \in (t^*, \varepsilon^{-1}\delta_-)$.

This theorem will be proved in the remainder of this section.

3.4.2 Equation of W_s and rescaling

Once again, we substitute u_C into (1.3.7) and obtain the equation satisfied by w_s

$$\frac{d}{dt}w_s - 2u_s w_s = w_s^2 + f(u_s + w_s, \mu; \varepsilon) - f(u_s, \mu; 0) - \frac{d}{dt}u_s =: R_s(w_s), \tag{3.4.4}$$

for $t \in (0, t^*)$.

Next we rescale the above equation to obtain

$$\frac{d}{d\sigma}W_s - c(\sigma)W_s = \varepsilon^{-1/3}\varphi\mathcal{R}_s(W_s). \tag{3.4.5}$$

We note the following observations.

- The equation is posed on $\sigma \in (\sigma_0, \sigma^*)$. Where $\sigma^* < 0$ with $|\sigma^*| = \mathrm{O}(\varepsilon^{-1/4})$, and $\sigma_0 = -\frac{2}{3}\delta_-^{3/2}\varepsilon^{-1}$.
- The term $c(\sigma)$ satisfies

$$c(\sigma) = 2\varepsilon^{-\frac{1}{3}}U_s(\sigma)\varphi(\sigma) = -2 + \mathrm{O}(\varepsilon^{1/3}|\sigma|^{1/3}),$$

  as $\sigma \to -\infty$.

- The function $W_s(\sigma)$ is the rescaled version of $w_s(t)$ in the σ-variable, with

$$w_s(t) = w_s(\varepsilon^{-1/3}\psi(\sigma) + \varepsilon^{-1}\delta_-) = W_s(\sigma).$$

Similarly we obtain the rescaled version of $u_s(t)$

$$U_s(\sigma) := u_s(\varepsilon^{-1/3}\psi(\sigma) + \varepsilon^{-1}\delta_-) = -\left(\frac{3}{2}\varepsilon\sigma\right)^{1/3} + \mathrm{O}(|\varepsilon\sigma|^{2/3}). \tag{3.4.6}$$

where the asymptotic expansion follows from (1.3.18) and the time rescaling.

- The function $\mathcal{R}_s$ is likewise a rescaled version of R_s such that

$$\mathcal{R}_s(W_s;\varepsilon) = W_s^2 + f(U_s + W_s, \mu; \varepsilon) - f(U_s, \mu; 0) - \varepsilon^{1/3}\varphi^{-1}\frac{d}{d\sigma}U_s(\sigma).$$

Once again we need to specify the boundary value at the left end point $\sigma = \sigma_0$. We therefore consider the following problem,

$$\begin{aligned} \frac{d}{d\sigma}W_s - c(\sigma)W_s &= \varepsilon^{-1/3}\varphi\mathcal{R}_s(W_s), \text{ for } \sigma \in (\sigma_0, \sigma^*), \\ W_s(\sigma_0) &= w_0, \end{aligned} \tag{3.4.7}$$

for given, sufficiently small w_0 to be determined later.

3.4.3 Linear equation and norms

The proof of Theorem 6 will be complete if we are able to solve (3.4.7) using a fixed point argument similar to that used in region A and B. Fix $\alpha > 0$, we first introduce the following function space

$$\mathcal{C}_s = \mathcal{C}_s(\sigma_0, \sigma^*) = \left\{ w(\sigma) : \sup_{\sigma_0 \le \sigma \le \sigma^*} |\varepsilon^{\frac{\alpha}{3}-1}\langle\varepsilon\sigma\rangle^{\frac{2}{3}} w(\sigma)| < \infty \right\}.$$

Similarly we study the linear operator A_s with

$$A_s w = \left(\frac{d}{d\sigma}w - cw, w(\sigma_0)\right),$$

defined on the dense subset $\mathcal{D}(A_s) \subset \mathcal{C}_s$.

Proposition 4.1 *There exist $\varepsilon_3 > 0$, and a constant C so that the operator $A_s : \mathcal{D}(A_s) \subset \mathcal{C}_s \to \mathcal{C}_s \times \mathbb{R}$ is invertible. The inverse $A_s^{-1}(\varepsilon) : \mathcal{C}_s \times I_s \to \mathcal{C}_s$, with $I_s = \{w_0 : |w_0| \le C\delta_-^{-1}\varepsilon^{1-\alpha/3}\}$, is bounded uniformly for ε with $0 < \varepsilon < \varepsilon_3$.*

Proof. Unlike previous cases, Lemma 2.1 cannot be directly used for the operator A_s since from the asymptotic expansion of c we see that $c(\sigma)$ does not converge to -2 as $\sigma \to -\infty$. In fact, $c(\sigma)$ diverges to $-\infty$ as $\sigma \to -\infty$. However, for $\sigma \in (\sigma_0, \sigma^*)$, we have

$$|c(\sigma) - (-2)| \lesssim |\varepsilon\sigma|^{1/3} \lesssim \delta_-^{1/2},$$

hence for δ_- small, A_s is a small perturbation of the operator

$$L_s : w \mapsto \left(\frac{d}{d\sigma}w + 2w, w(\sigma_0)\right),$$

uniformly in ε.

To conclude the invertibility of L_s on the weighted space $\mathcal{C}_s$, let $v(\sigma) = \varepsilon^{\alpha/3-1}\langle\varepsilon\sigma\rangle^{2/3}w(\sigma)$ and consider the conjugate linear operator

$$\begin{aligned}\tilde{L}_s : v \mapsto &\left(\langle\varepsilon\sigma\rangle^{-2/3}\left(\frac{d}{d\sigma} + 2\right)\langle\varepsilon\sigma\rangle^{2/3}v(\sigma), v(\sigma_0)\right)\\ &= \left(\left(\frac{d}{d\sigma} + 2 + \frac{2}{3}\varepsilon\langle\varepsilon\sigma\rangle^{-1}\right)v, v(\sigma^*)\right),\end{aligned}$$

defined on $\mathcal{C}^1(\sigma_0, \sigma^*) \subset \mathcal{C}(\sigma_0, \sigma^*)$.

Hence, for $f \in \mathcal{C}_s$, the equivalent conjugate linear equation of

$$L_s w = (f, w_0),$$

is

$$\tilde{L}_s v = (\tilde{f}, v_0),$$

with $v_0 = \varepsilon^{\alpha/3-1}\langle\varepsilon\sigma_0\rangle^{2/3}w_0$ and $\tilde{f} = \varepsilon^{\alpha/3-1}\langle\varepsilon\sigma\rangle^{2/3}f$, which is a differential equation of the form

$$\left(\frac{d}{d\sigma} + 2 + \mathrm{O}(\varepsilon)\right)v = f, \qquad v(\sigma_0) = v_0.$$

The linear part of this equation is yet another small perturbation of the linear operator $\frac{d}{d\sigma} + 2$ on the uniform space $\mathcal{C}(\sigma_0, \sigma^*)$. Integrating this equation yields

$$|v|_\infty \le C(|v_0| + |f|_\infty),$$

for some constant C independent of ε. Equivalently, in terms of w we have

$$\|w\|_{\mathcal{C}_s} \le C(|\delta_- \varepsilon^{\alpha/3-1} w_0| + \|f\|_{\mathcal{C}_s}). \tag{3.4.8}$$

Therefore, there is $\varepsilon^* > 0$ small enough so that for all $\varepsilon \in (0, \varepsilon^*)$ the operator L_s is invertible with uniform bounds for all $w_0 \in I_s$. ■

3.4.4 Nonlinear estimates

Next we estimate the nonlinear term $\varepsilon^{-1/3}\varphi\mathcal{R}_s(W_s(\sigma))$ in the $\mathcal{C}_s$ norm. Similar to Propositions 2.2 and 3.2, we have

Proposition 4.2 *If $W_s \in \mathcal{C}_s$, then $\varepsilon^{-\frac{1}{3}}\varphi\mathcal{R}_s(W_s(\sigma)) \in \mathcal{C}_s$ and*

$$\|\varepsilon^{-\frac{1}{3}}\varphi\mathcal{R}_s(W_s)\|_{\mathcal{C}_s} = \mathrm{O}(\delta_-^{1/2}). \tag{3.4.9}$$

Proof. First recall that

$$\varepsilon^{-1/3}\varphi\mathcal{R}_s(W_s;\varepsilon) = \varepsilon^{-1/3}\varphi\left[W_s^2 + f(U_s + W_s, \mu; \varepsilon) - f(U_s, \mu; 0)\right] - \frac{d}{d\sigma}U_s(\sigma),$$

then from (3.1.8) we have

$$|\varphi(\sigma)| \lesssim |\sigma|^{-1/3},$$

and from (3.1.7), we see that $\mu = \varepsilon t - \delta_- = \varepsilon^{2/3}\psi(\sigma)$ satisfies

$$|\mu| \lesssim |\varepsilon\sigma|^{2/3}.$$

From (3.4.6), we have that

$$U_s(\sigma) = -\left(\frac{3}{2}\varepsilon\sigma\right)^{1/3} + \mathrm{O}(|\varepsilon\sigma|^{2/3}),$$
$$\frac{d}{d\sigma}U_s(\sigma) = -\frac{1}{2}\varepsilon(\varepsilon\sigma)^{-2/3} + \mathrm{O}(\varepsilon|\varepsilon\sigma|^{-1/3}),$$

and for $W_s \in \mathcal{C}_s$, it holds that

$$|W_s(\sigma)| \lesssim \varepsilon^{1-\alpha/3}\langle\varepsilon\sigma\rangle^{-2/3}.$$

Hence we have the following estimates:

$$\left\| \frac{d}{d\sigma} U_s(\sigma) \right\|_{\mathcal{C}_s} \lesssim \sup_{\sigma \in (\sigma_0, \sigma^*)} \varepsilon^{\alpha/3-1} \langle \sigma \rangle^{2/3} \varepsilon |\varepsilon\sigma|^{-2/3} = \mathrm{O}(\varepsilon^{\alpha/3}), \tag{3.4.10}$$

$$\|\varepsilon^{-1/3} \varphi W_s^2(\sigma)\|_{\mathcal{C}_s} \lesssim \sup_{\sigma \in (\sigma_0, \sigma^*)} \varepsilon^{-1/3} |\sigma|^{-1/3} \varepsilon^{1-\alpha/3} \langle \varepsilon\sigma \rangle^{-2/3} = \mathrm{O}(\varepsilon^{\frac{1}{4}-\alpha/3}). \tag{3.4.11}$$

Next we estimate the term $\varepsilon^{-1/3}\varphi[f(U_s + W_s, \mu; \varepsilon) - f(U_s, \mu; 0)]$ as follows. First, we rewrite the term $f(U_s + W_s, \mu; \varepsilon) - f(U_s, \mu; 0)$ as $f_1 + f_2$, where

$$f_1 = f(U_s + W_s, \mu; \varepsilon) - f(U_s + W_s, \mu; 0),$$

and

$$f_2 = f(U_s + W_s, \mu; 0) - f(U_s, \mu; 0).$$

Recall the expansion $f(u, \mu; \varepsilon) = \mathrm{O}(\varepsilon(1+u+\mu+u^2), u\mu, \mu^2, u^3)$, which we use to obtain

$$f_1 \leq C\varepsilon,$$

and

$$f_2 \leq C(W_s\mu + U_s W_s^2 + U_s^2 W_s + W_s^3),$$

for some constant C independent of ε.

For f_1 we have

$$\|\varepsilon^{-1/3}\varphi f_1\|_{\mathcal{C}_s} \lesssim \sup_{\sigma \in (\sigma_0, \sigma^*)} \varepsilon^{2/3} \varphi \varepsilon^{\alpha/3-1} |\varepsilon\sigma|^{2/3} \lesssim \varepsilon^{\alpha/3} |\varepsilon\sigma_0|^{1/3} = \mathrm{O}(\varepsilon^{\alpha/3}). \tag{3.4.12}$$

For f_2 we have

$$\|\varepsilon^{-1/3}\varphi W_s\mu\|_{\mathcal{C}_s} \lesssim \sup_{\sigma \in (\sigma_0, \sigma^*)} \varepsilon^{-1/3} |\sigma|^{-1/3} |\mu| \lesssim |\varepsilon\sigma_0|^{1/3} = \mathrm{O}(\delta_-^{1/2}),$$

$$\|\varepsilon^{-1/3}\varphi U_s^2 W_s(\sigma)\|_{\mathcal{C}_s} \lesssim \sup_{\sigma \in (\sigma_0, \sigma^*)} \varepsilon^{-1/3} |\sigma|^{-1/3} |\varepsilon\sigma|^{2/3} \lesssim |\varepsilon\sigma_0|^{1/3} = \mathrm{O}(\delta_-^{1/2}),$$

$$\|\varepsilon^{-1/3}\varphi U_s W_s^2(\sigma)\|_{\mathcal{C}_s} \lesssim \sup_{\sigma\in(\sigma_0,\sigma^*)} \varepsilon^{-1/3}|\sigma|^{-1/3}|\varepsilon\sigma|^{1/3}\varepsilon^{1-\alpha/3}|\varepsilon\sigma|^{-2/3} = \mathrm{O}(\varepsilon^{1/2-\alpha/3}),$$

and

$$\|\varepsilon^{-1/3}\varphi W_s^3(\sigma)\|_{\mathcal{C}_s} \lesssim \sup_{\sigma\in(\sigma_0,\sigma^*)} \varepsilon^{-1/3}|\sigma|^{-1/3}(\varepsilon^{1-\alpha/3}|\varepsilon\sigma|^{-2/3})^2 = \mathrm{O}(\varepsilon^{3/4-2\alpha/3}).$$

Hence

$$\|\varepsilon^{-1/3}\varphi f_2\|_{\mathcal{C}_s} = \mathrm{O}(\delta_-^{1/2}), \tag{3.4.13}$$

combining estimates (3.4.10), (3.4.11), (3.4.12), and (3.4.13) we conclude that

$$\|\varepsilon^{-1/3}\varphi\mathcal{R}_s(W_s)\|_{\mathcal{C}_s} = \mathrm{O}(\delta_-^{1/2}),$$

which finishes the proof. ■

3.4.5 Fixed point argument and the proof of Theorem 6

In this section we prove Theorem 6.

Proof of Theorem 6. Using the linear operator A_s, we rewrite (3.4.7)

$$\frac{d}{d\sigma}W_s - c(\sigma)W_s = \varepsilon^{-1/3}\varphi\mathcal{R}_s(W_s), \text{ for } \sigma \in (\sigma_0, \sigma^*),$$
$$W_s(\sigma_0) = w_0.$$

as

$$\begin{aligned}(0,0) &= \left(\frac{d}{d\sigma}W_s - cW_s - \varepsilon^{-1/3}\varphi\mathcal{R}_s(W_s), W_s(\sigma_0) - w_0\right)\\ &= \left(\frac{d}{d\sigma}W_s - cW_s, W_s(\sigma^*)\right) - \left(\varepsilon^{-1/3}\varphi\mathcal{R}_s(W_s), w_0*\right)\\ &= A_sW_s - \left(\varepsilon^{-1/3}\varphi\mathcal{R}_s(W_s), w_0\right).\end{aligned}$$

Using Proposition 4.1 we precondition the above equation with the operator A_s^{-1} to obtain the equivalent equation

$$W_s = \mathcal{S}_s(W_s; w_0, \varepsilon) := A_s^{-1}(\varepsilon^{-1/3}\varphi\mathcal{R}_s(W_s), w_0). \tag{3.4.14}$$

From Propositions 4.1 and 4.2, we conclude the following.

- At $W_s = 0$ it holds that

$$\begin{aligned}\|\mathcal{S}_s(0; w_0, \varepsilon)\|_{\mathcal{C}_s} &= \|A_s^{-1}\left(\varepsilon^{-1/3}\varphi\mathcal{R}_s(0), w_0)\right)\|_{\mathcal{C}_s} \\ &\lesssim \|\varepsilon^{-1/3}\varphi\mathcal{R}_s(0)\|_{\mathcal{C}_s} \\ &\lesssim \|\varepsilon^{2/3}\varphi\|_{\mathcal{C}_s} \leq C_1\varepsilon^{\alpha/3}.\end{aligned}$$

  for some constant C_1 independent of ε, w_0.

- The map $\mathcal{S}_s : \mathcal{C}_s \to \mathcal{C}_s$ is well defined and smooth in W_s.

- The derivative of $f(U_s + W_s, \mu; \varepsilon)$ with respect to W_s satisfies

$$D_{W_s} f(U_s + W_s, \mu; \varepsilon) = \mathrm{O}(\mu, U_s^2).$$

  The linearization of $\mathcal{S}_s$ at $W_s = 0$, $D_{W_s}\mathcal{S}_s(0; w_0, \varepsilon)$ satisfies

$$\|D_{W_s}\mathcal{S}_s(0; w_0, \varepsilon)\|_{\mathcal{C}_s \to \mathcal{C}_s} \lesssim \sup_{\sigma} |\varepsilon^{-1/3}\varphi(\mu + U_s^2)| = \mathrm{O}(\delta_-^{1/2}).$$

  Moreover, for $\|W_s\| = \mathrm{O}(\varepsilon^{\alpha/3})$ and $|w_0| = \mathrm{O}(\varepsilon^{1-\alpha/3})$, we have

$$\|D_{W_s}\mathcal{S}_s(W_s; w_0, \varepsilon)\|_{\mathcal{C}_s \to \mathcal{C}_s} \lesssim \|D_{W_s}\mathcal{S}_s(0; w_0, \varepsilon)\|_{\mathcal{C}_s \to \mathcal{C}_s} + \sup_{\sigma} |\varepsilon^{-1/3}\varphi W_s|) \leq C_2\delta_-^{1/2},$$

  for some constant C_2 independent of ε.

Hence, assuming $w_0 = \mathrm{O}(\varepsilon^{1-\alpha/3})$ and using an estimate similar to (3.2.22), we have that $\mathcal{S}_s$ maps the ball $\mathcal{B}_{2C_1\varepsilon^{\alpha/3}}(0)$ in $\mathcal{C}_s$ into itself for $\delta_- \leq (2C_2)^{-2}$. We may apply Banach's fixed point theorem to the map $\mathcal{S}_s$ to obtain a solution of (3.4.7) with $W_s \in \mathcal{C}_s$. Scaling back from σ to t, we obtain claim (3.4.3), which concludes the proof.

■

3.5 Gluing

So far we have showed that solutions of the form u_A, u_B, and u_C exist on regions A, B, and C, respectively. To prove Theorem 3, we will need to glue solutions u_A, u_B and u_C

at $t = t^*$ and $t = t_*$.

In region C, Theorem 6 shows that (1.3.2) has a solution of the form

$$u_C(t) = u_s(t) + w_s(t; w_0),$$

provided that we pick the initial condition $w_0 = \mathrm{O}(\delta_-^{-1}\varepsilon^{1-\alpha/3})$.

At t^*, the right end point of region C, the values $u_s(t^*)$ and $w_s(t^*; w_0)$ satisfy

$$\begin{aligned} u_s(t^*) &= -\sqrt{\delta_- - \varepsilon t^*} + \mathrm{O}(|\delta_- - \varepsilon t^*|) = -\varepsilon^{1/4} + \mathrm{O}(\varepsilon^{1/2}), \\ w_s(t^*) &\lesssim \varepsilon^{1-\alpha/3}(\varepsilon t^* - \delta_-)^{-1} = \mathrm{O}(\varepsilon^{1/2-\alpha/3}), \end{aligned}$$

therefore

$$u_C(t^*) = u_s(t^*) + w_s(t^*; w_0) = -\varepsilon^{1/4} + \mathrm{O}(\varepsilon^{1/2-\alpha/3}). \tag{3.5.1}$$

Next, we go to region B. At t^*, we notice the ansatz $\bar{u}_*(t^*)$ has the following expansion in ε,

$$\bar{u}_*(t^*) = \varepsilon^{1/3}\bar{u}_R(\varepsilon^{1/3}(t^* - \varepsilon^{-1}\delta_-)) = -\varepsilon^{1/4} + \mathrm{O}(\varepsilon^{1/2}). \tag{3.5.2}$$

Hence, if we set $w^* = u_C(t^*) - \bar{u}_*(t^*)$, then by (3.5.1) and (3.5.2) we have $w^* = \mathrm{O}(\varepsilon^{1/2-\alpha/3})$. We may then apply Theorem 5 with the initial condition $u(t^*) = \bar{u}_*(t^*) + w^*$ to get a solution of (1.3.2) of the form

$$u_B(t) = \bar{u}_*(t) + w_\ell(t; w^*),$$

in region B. With this choice, we find the desired continuity condition for gluing,

$$u_C(t^*) = u_B(t^*). \tag{3.5.3}$$

Again from Theorem 5, we conclude that at the point $t = \varepsilon^{-1}\delta_-$, the values of $\bar{u}_*(\varepsilon^{-1}\delta_-)$ and $w_\ell(\varepsilon^{-1}\delta_-; w^*)$ satisfy

$$\begin{aligned} \bar{u}_*(\varepsilon^{-1}\delta_-) &= \varepsilon^{1/3}\bar{u}_R(0) = \varepsilon^{1/3}\bar{u}_0, \\ w_\ell(\varepsilon^{-1}\delta_-) &= \mathrm{O}(\varepsilon^{(2-\alpha)/3}). \end{aligned}$$

Therefore we have

$$u_B(\varepsilon^{-1}\delta_-) = \bar{u}_*(\varepsilon^{-1}\delta_-) + w_\ell(\varepsilon^{-1}\delta_-) = \varepsilon^{1/3}\bar{u}_0 + \mathrm{O}(\varepsilon^{(2-\alpha)/3}). \tag{3.5.4}$$

Finally in region A, Theorem 4 shows that a solution to (1.3.2) of the form $u_A(t; u_0) = u_*(t; u_0) + w_r(t; u_0)$ exists for $t \in (\varepsilon^{-1}\delta_-, T)$ with $u_A(T; u_0) = \delta_-$, provided that u_0 is close to $\bar{u}_0$. Moreover, at $t = \varepsilon^{-1}\delta_-$, the values $u_*(\varepsilon^{-1}\delta_-; u_0)$ and $w_r(\varepsilon^{-1}\delta_-; u_0)$ satisfy

$$\begin{aligned} u_*(\varepsilon^{-1}\delta_-; u_0) &= \varepsilon^{1/3}u_R(0; u_0) = \varepsilon^{1/3}u_0, \\ w_r(\varepsilon^{-1}\delta_-; u_0) &= \mathrm{O}(\varepsilon^{(2-\alpha)/3}). \end{aligned}$$

Therefore we have

$$u_A(\varepsilon^{-1}\delta_-; u_0) = u_*(\varepsilon^{-1}\delta_-; u_0) + w_r(\varepsilon^{-1}\delta_-; u_0) = \varepsilon^{1/3}u_0 + \mathrm{O}(\varepsilon^{(2-\alpha)/3}). \tag{3.5.5}$$

Since we require the solution to be continuous at $\varepsilon^{-1}\delta_-$, we must require the following matching condition

$$u_A(\varepsilon^{-1}\delta_-; u_0) = u_B(\varepsilon^{-1}\delta_-). \tag{3.5.6}$$

Using (3.5.4) and (3.5.5), this amounts to solving the equation

$$0 = \varepsilon^{1/3}(u_0 - \bar{u}_0) + w_r(\varepsilon^{-1}\delta_-; u_0) - w_\ell(\varepsilon^{-1}\delta_-), \tag{3.5.7}$$

in the variable u_0. Let $\phi(\varepsilon; u_0) := w_r(\varepsilon^{-1}\delta_-; u_0) - w_\ell(\varepsilon^{-1}\delta_-)$. We conclude from Theorem 4 and Theorem 5 that

$$\phi(\varepsilon; u_0) = \mathrm{O}(\varepsilon^{(2-\alpha)/3}),$$

uniformly in u_0 and

$$\mathrm{Lip}_{u_0}\phi(\varepsilon; u_0) = \mathrm{O}(\varepsilon^{(2-\alpha)/3}).$$

Hence we divide the right hand side of (3.5.7) by $\varepsilon^{1/3}$, and apply the implicit function theorem around the point $(u_0 = \bar{u}_0, \varepsilon = 0)$ to conclude that there exists a branch $u_0 = u_0(\varepsilon)$ with $u_0(\varepsilon) = \bar{u}_0 + \mathrm{O}(\varepsilon^{(1-\alpha)/3})$ which solves the matching condition (3.5.6).

In conclusion, we have proved the following theorem, which implies Theorem 3.

Theorem 7 *Let $u_-(\mu)$ be the attracting branch of the critical manfold defined in* (1.3.17) *and $\bar{u}_0$ be the value defined in* (1.3.12)*. Fix $\alpha > 0$ and $\delta_-, \delta_+ > 0$ small enough, there exist $\varepsilon_0 > 0$, and a constant $C = C(\delta_-, \delta_+, \alpha)$, such that for all $0 < \varepsilon < \varepsilon_0$ and u_i with $|u_i - u_-(-\delta_-)| \leq C\varepsilon^{1-\alpha/3}$, there exist a branch of initial conditions $u_0 = u_0(\varepsilon) = \bar{u}_0 + \mathrm{O}(\varepsilon^{(1-\alpha)/3})$, such that a solution of* (1.3.19) *with the initial condition* (1.3.20) *of the following form*

$$u(t) = \begin{cases} u_A(t; u_0(\varepsilon)) = u_*(t; u_0(\varepsilon)) + w_r(t; u_0(\varepsilon)), \\ u_B(t) = \bar{u}_*(t) + w_\ell(t), \\ u_C(t) = u_s(t) + w_s(t), \end{cases} \tag{3.5.8}$$

and

$$\mu(t) = \varepsilon t - \delta_-, \tag{3.5.9}$$

exists on $t \in (0, T)$. Moreover, all of the correction functions w_r, w_ℓ and w_s satisfy the properties stated in Theorems 4, 5 and 6, respectively.

Proof of Theorem 3 and 3.2. Theorem 3 follows from Theorem 7 almost directly. We only need to show the expansions (1.3.22) and (1.3.23).

Recall that $\Omega_\infty(u_0)$ depends smoothly on u_0 by Proposition 3.1. Hence, there is a constant C such that

$$|\Omega_\infty(u_0) - \Omega_\infty(\bar{u}_0)| \leq C|u_0 - \bar{u}_0|,$$

for u_0 close to $\bar{u}_0$. Now we substitute the branch $u_0(\varepsilon)$ with $u_0(\varepsilon) = \bar{u}_0 + \mathrm{O}(\varepsilon^{(1-\alpha)/3})$ and use the expansion (3.2.4). We find

$$\begin{aligned} T &= \varepsilon^{-1}\delta_- + \varepsilon^{-1/3}\Omega_\infty(u_0(\varepsilon)) + \delta_+^{-1} + \mathrm{O}(\varepsilon^{2/3}) \\ &= \varepsilon^{-1}\delta_- + \varepsilon^{-1/3}(\Omega_\infty(\bar{u}_0) + \mathrm{O}(\varepsilon^{(1-\alpha)/3})) + \delta_+^{-1} + \mathrm{O}(\varepsilon^{2/3}) \\ &= \varepsilon^{-1}\delta_- + \varepsilon^{-1/3}\Omega_0 + \mathcal{H}(\varepsilon), \end{aligned}$$

with $\mathcal{H}(\varepsilon) = \mathrm{O}(\varepsilon^{-\alpha/3})$.

Next we examine the dependence of $\mathcal{H}$ on the initial condition u_i as in (1.3.23).

First, consider the following maps:

$$
\begin{aligned}
u_i &\mapsto u_C(t; u_i), \\
u_C(t; u_i) &\mapsto u_C(t^*; u_i), \\
u_C(t^*; u_i) := u^* &\mapsto u_B(t; u^*), \\
u_B(t; u^*) &\mapsto u_B(\varepsilon^{-1}\delta_-; u^*).
\end{aligned}
$$

All maps are smooth either because they are evaluation maps or by the smooth dependence of solutions on the initial conditions. Hence, if we set $\mathcal{H}_1$ to be the map $u_i \mapsto u_B(\varepsilon^{-1}\delta_-; u^*)$, it can be seen from (3.2.14),(3.3.8), (3.4.8) and the definition of the norms that

$$
\begin{aligned}
\|u_C - \tilde{u}_C\|_{\mathcal{C}_s} &\leq C\varepsilon^{\alpha/3-1}|u_i - \tilde{u}_i|, \\
|u_C(t^*) - \tilde{u}_C(t^*)| &\leq C\varepsilon^{1/2-\alpha/3}\|u_C - \tilde{u}_C\|_{\mathcal{C}_s}, \\
\|u_B - \tilde{u}_B\|_{\mathcal{C}_\ell} &\leq \varepsilon^{\alpha/3-1/2}|u_C(t^*) - \tilde{u}_C(t^*)|, \\
|u_B(\varepsilon^{-1}\delta_-) - \tilde{u}_B(\varepsilon^{-1}\delta_-)| &\leq C\varepsilon^{(2-\alpha)/3}\|u_B - \tilde{u}_B\|_{\mathcal{C}_\ell},
\end{aligned}
$$

which implies

$$
\text{Lip } \mathcal{H}_1 \leq C\varepsilon^{-1/3}, \tag{3.5.10}
$$

for some constant C independent of ε.

Next, recall that the branch $u_0 = u_0(\varepsilon)$ was solved by the condition (3.5.7), from which it follows that

$$
u_0 + \varepsilon^{-1/3} w_r(\varepsilon^{-1}\delta_-; u_0) = \bar{u}_0 - \varepsilon^{-1/3} w_\ell(\varepsilon^{-1}\delta_-).
$$

Therefore, if we set $\mathcal{H}_2$ to be the map $w_\ell(\varepsilon^{-1}\delta_-) \mapsto u_0$, then

$$
\text{Lip } \mathcal{H}_2 \leq C\varepsilon^{-1/3}, \tag{3.5.11}
$$

for some constant C independent of ε.

Moreover, we have that the map $\mathcal{H}_3$

$$
u_0 \mapsto \Omega_\infty(u_0) = \Omega_\infty(u_0) \tag{3.5.12}
$$

satisfies

$$\text{Lip } \mathcal{H}_3 \leq C, \tag{3.5.13}$$

for some constant C independent of ε from Proposition 3.1.

Finally, we have $\mu(T) = \varepsilon T - \delta_- = \varepsilon^{2/3}\Omega_\infty(u_0) + \mathrm{O}(1)$. Therefore we conclude that the map $\mathcal{H}$

$$u_i \mapsto \mu(T) \tag{3.5.14}$$

satisfies

$$\text{Lip } \mathcal{H} \leq \varepsilon^{2/3} \, (\text{Lip } \mathcal{H}_3) \, (\text{Lip } \mathcal{H}_2) \, (\text{Lip } \mathcal{H}_1) \leq C, \tag{3.5.15}$$

for some constant C independent of ε from the estimates (3.5.10), (3.5.11), and (3.5.13).

Now, we can prove Corollary 3.2. Recall $(\mu, u, \varepsilon) \in \mathcal{U}$ where $\mathcal{U}$ is a small neighborhood of the origin $(0, 0, 0)$. Take a compact interval K such that $(\mu, u, \varepsilon) \subset \mathcal{U}$ for all $u \in K$. Consider a trajectory (μ, u) with initial condition $\mu(0) = -2\delta_-$ and $u(0) \in K$. By standard Fenichel theory, at $\varepsilon = 0$, there exist stable foliations with base the lower branch $u_-(\mu)$ of the critical manifold $S_0 = \{(\mu, u) : \mu + u^2 + f(\mu, u, 0) = 0\}$ for $\mu \in [-2\delta_-, -\delta_-]$. Further, for $0 < \varepsilon \ll 1$, the foliation persists and the part of the critical manifold $u_-(\mu)$ perturbs to a nearby slow manifold $u_-^\varepsilon(\mu)$. Moreover, there exist an asymptotic rate $\gamma = \gamma(\delta_-) > 0$ and a constant $C = C(\delta_-)$, independent of ε such that the solution $u(t)$ starts at $\mu(0) = -2\delta_-$ and $u(0) \in K$ converges exponentially to the slow manifold:

$$|u(t) - u_-^\varepsilon(\mu(t))| \leq Ce^{-\gamma t}. \tag{3.5.16}$$

Let T_* be the time of flight for a solution which starts at $\mu = -2\delta_-$ at $t = 0$ to reach $\mu = -\delta_-$. Since $\mu' = \varepsilon$ we see directly that $T_* = \delta_-/\varepsilon$. From the estimate (3.5.16) we see that $|u(T_*) - u_-^\varepsilon(-\delta_-)| \leq Ce^{-\gamma \cdot \delta_-/\varepsilon}$. But again by Fenichel theory, $u_-^\varepsilon(\mu)$ is $\mathrm{O}(\varepsilon)$ distance away from $u_-(\mu)$. This implies

$$|u(T_*) - u_-(-\delta_-)| = \mathrm{O}(\varepsilon),$$

which satisfies the assumption of Theorem 7 if we restart the equation with $u = u(T_*)$ and $\mu = -\delta_-$ at $t = 0$ and follow this trajectory, by Theorem 7 we conclude that the trajectory hit the section Δ_{out} at the point $(h(\varepsilon), \delta_+)$ with $h(\varepsilon)$ has the expansion

(1.3.22).

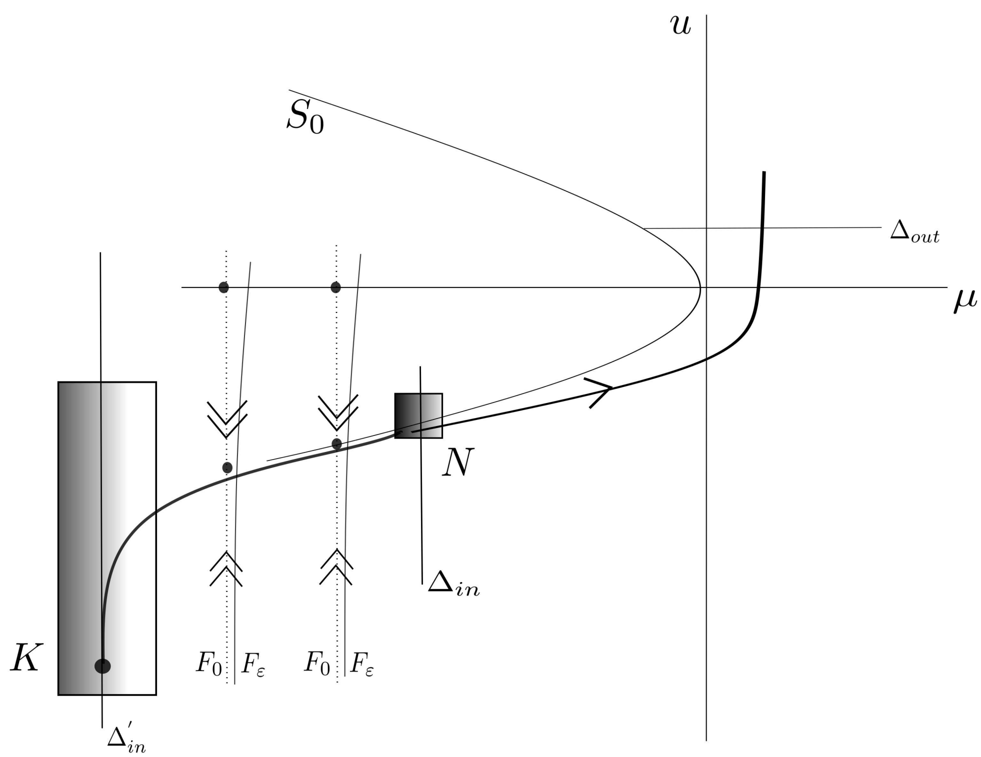

Figure 3: Schematic depiction of region K in Corollary 3.2.

The above argument can be seen more directly in Figure 3, where K is a rectangular region near the section $\Delta'_{in} = \{(-2\delta_-, u)\}$. What Fenichel theory shows is that this region K contracts into a small neighborhood N around near the point $(-\delta_-, u_-(-\delta_-))$, with Lipschitz constant of order $e^{-c/\varepsilon}$ for some constant $c > 0$ independent of ε. Inside the region N, we can apply the conclusion of Theorem 7 to continue the trajectory until it hits the section Δ_{out} at the desired $\mathrm{O}(\varepsilon^{2/3})$ location. ■

Remark: The last part where we used Fenichel theory to track how a solution arrives exponentially close to the slow manifold $u_-^\varepsilon(\mu)$ could also be established as in [31], using functional analytic methods. By using function spaces with suitably centered exponential weights and exploting the location of "touchdown" and "takeoff" between the slow manifold and the solution in stable and unstable manifolds as free parameters, Faye and Scheel were able to track the solution in a way similar to the Exchange Lemma as in [32], [33], and [34].

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Appendix A

Passage through fold

We first show how to extend the asymptotic expansion (1.3.11) to a more general family of solutions.

A.1 A family of solutions to the Riccati equation

Proof of Proposition 3.1. We investigate the dependence on u_0 of Ω_∞. We first add the equation $\frac{d}{ds}s = 1$ to (1.3.10) to obtain an autonomous $2-$dimensional system in the (s, u) plane.

$$\begin{aligned} \frac{d}{ds}s &= 1, \\ \frac{d}{ds}u &= s + u^2. \end{aligned} \tag{A.1.1}$$

Consider a small neighborhood I containing $\bar{u}_0$ on the vertical u-axis. Then $u_R(s; u_0)$ is the trajectory that starts at $u_0 \in I$. The map $P_1 : I \to \mathbb{R}$ defined by $P_1(p) = u(2; p)$ is smooth in p, as the blow up time for $\bar{u}_R(s; \bar{u}_0)$ is $\Omega_0 > 2$. Moreover, the image $P_1(I)$ is a finite interval on the vertical line $s = 2$ containing $\bar{u}_R(2; u_0)$, bounded away from 0, since the trajectory $u_R(s; \bar{u}_0)$ crosses the horizontal axis near $s = 1$ and the vector field goes upwards in the first quadrant of the (s, u)-plane. Figure 4 summarizes how the map P_1 is defined.

Next, denote $\tilde{u}_0 := P_1(u_0)$ for brevity (technically, the interval $P_1(I)$ is a small section of the line $s = 2$, with a slight abuse of notation, we identify $\tilde{u}_0$ with the second

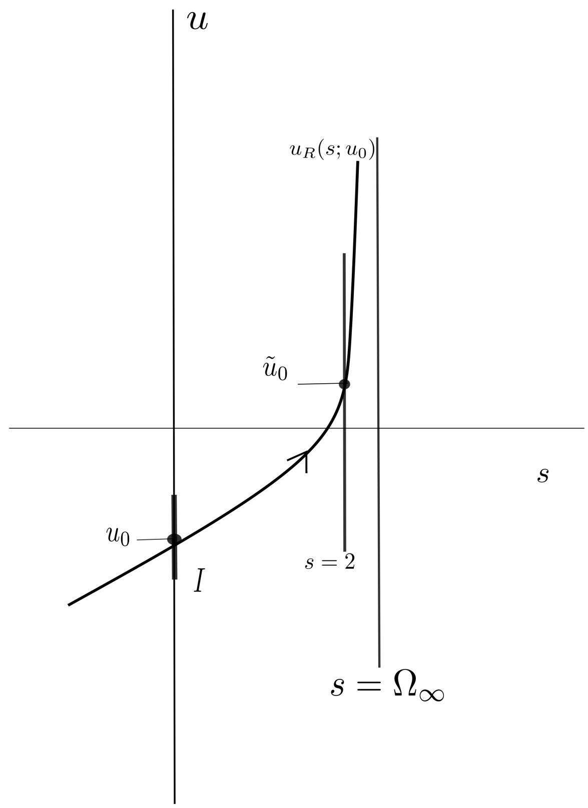

Figure 4: A trajectory $u_R(s;u_0)$ of the vector field (A.1.1) which starts at u_0. The map P_1 takes u_0 to the value $\tilde{u}_0 = u_R(2;u_0)$.

coordinate of the point $P_1(u_0)$). Again, we make a change of variable in (1.3.10) by setting $z(s) = 1/u(s)$. Then z satisfies

$$\begin{aligned} \frac{d}{ds}s &= 1 \\ \frac{d}{ds}z &= -z^2 s - 1. \end{aligned} \tag{A.1.2}$$

Let $J = \{1/\tilde{u}_0 \mid \tilde{u}_0 \in P_1(I)\}$ and let $z(s;1/\tilde{u}_0)$ be the trajectory which starts at $1/\tilde{u}_0$. We claim that $z(s;1/\tilde{u}_0)$ reaches 0 at a finite time $\Omega_\infty = \Omega_\infty(1/\tilde{u}_0)$. To see this, first notice there is no equilibrium for (A.1.2). Then, on the boundary $s = 2$, the vector field takes the form $(1, -2z^2 - 1)$, which makes any trajectory starting at a point on J move down towards the right. Moreover, the vector field $(1, -sz^2 - 1)$ is always pointing down in the first quadrant of the (s, z) plane, so trajectories cannot go upwards. Lastly, the vector field crosses the horizontal axis non-tangentially, it identically equals $(1, -1)$ throughout the line $z = 0$. Hence, any trajectory which starts at a point on J will cross $z = 0$ in finite time at a unique point $\Omega_\infty = \Omega_\infty(1/\tilde{u}_0)$. The dependence of Ω_∞ on $1/\tilde{u}_0$ is smooth by the smooth dependence on initial conditions.

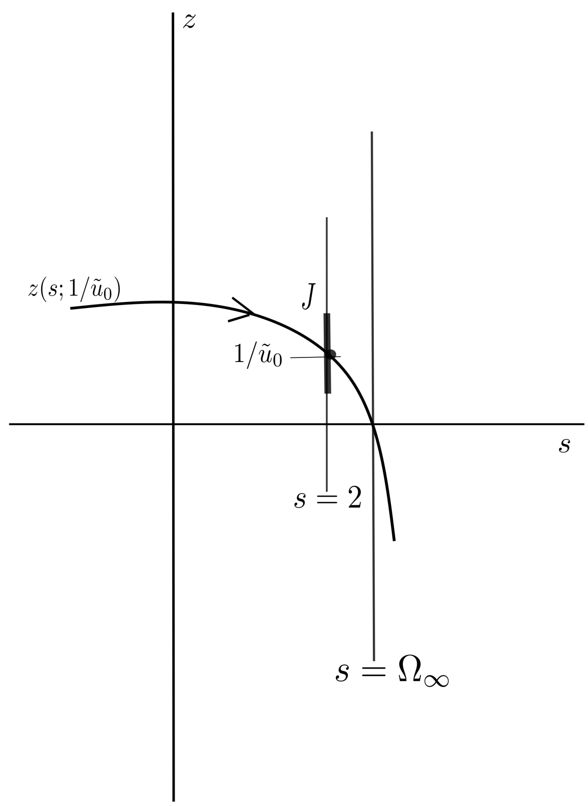

Figure 5: A tracjectory $z(s; 1/\tilde{u}_0)$ of the vector field (A.1.2) that pass through $1/\tilde{u}_0$ will cross the horizontal axis at Ω_∞. The map P_2 maps $1/\tilde{u}_0$ to the value Ω_∞.

We now define another map $P_2 : J \to \mathbb{R}$ by $P_2(1/\tilde{u}_0) = \Omega_\infty(1/\tilde{u}_0)$, Figure 5 also summarizes the construction of the map P_2. We then obtain a smooth map $P : I \to \mathbb{R}$ by the composition

$$P = P_2 \circ f \circ P_1,$$

where $f(z) = 1/z$ is the inversion map. Since each of the maps in the composition is smooth, $P : u_0 \mapsto \Omega_\infty = \Omega_\infty(u_0)$ is smooth as well.

To obtain the asymptotic expansion, we set $\xi = \Omega_\infty - s$. Then $\tilde{z}(\xi) = z(\Omega_\infty - \xi)$ solves

$$\frac{d}{d\xi}\tilde{z} = \tilde{z}^2(\Omega_\infty - \xi) + 1,$$

and $\tilde{z}(0) = 0$.

Hence we can assume the expansion for $\tilde{z}$ near $\xi = 0$ is of the form

$$\tilde{z} = \xi + z_2\xi^2 + z_3\xi^3 + \mathrm{O}(\xi^4),$$

for some constant z_2, z_3. Differentiating this expansion, and using the equation for $\tilde{z}$ and comparing coefficients, we find that $z_2 = 0, z_3 = \Omega_\infty/3$. Changing back from $\tilde{z}(\xi)$ to $z = z(s)$ with $s = \Omega_\infty - \xi$ and recalling $z(s) = 1/u(s)$, we find that $u_R(s; u_0)$ has the expansion (1.3.13) with remainder r satisfying (1.3.14). ∎

A.2 Uniform invertibility of boundary value problems

Here, we show the main perturbation lemma used to prove the invertibility of the linearized operators at the ansatz.

Lemma 2.1 *Consider the following boundary value problems*

$$\dot{u}(\sigma) = a(\sigma)u + f(\sigma), \qquad u(L) = u_L, \tag{A.2.1a}$$

$$\dot{u}(\sigma) = b(\sigma)u + g(\sigma), \qquad u(-M) = u_M, \tag{A.2.1b}$$

where (A.2.1a) *is posed on* $\sigma \in (\sigma_0, L)$ *with* $L > \sigma_0$ *and* (A.2.1b) *is posed on* $\sigma \in (-M, \sigma_0)$ *with* $-M < \sigma_0$.

Assume $a(\sigma), b(\sigma)$ *are continuous functions such that*

$$a(\sigma) \to a_+ > 0, \qquad \sigma \to \infty, \tag{A.2.2a}$$

$$b(\sigma) \to b_- < 0, \qquad \sigma \to -\infty, \tag{A.2.2b}$$

Then (A.2.1) *possesses unique solutions* u_a, u_b *which satisfy*

$$|u_a|_\infty \le C_a(u_L + |f|_\infty), \tag{A.2.3a}$$

$$|u_b|_\infty \le C_b(u_m + |g|_\infty). \tag{A.2.3b}$$

for some constants C_a, C_b *independent of* L *and* M*.*

Proof. We only prove the estimate (A.2.1a) since the other case is similar. Also, without loss of generality, we assume that $\sigma_0 = 0$.

To begin with, consider the asymptotic equation

$$\dot{u} = a_+ u + f(\sigma), \qquad u(L) = u_L. \tag{A.2.4}$$

posed on $\sigma \in [0, L]$. Then the estimate (A.2.3a) holds for (A.2.4) since in this case we have

$$\begin{aligned} u(\sigma) &= e^{a_+(\sigma - L)} u_L + \int_L^t e^{a_+(\sigma - s)} f(s) ds \\ &\le 2|u_L| + \left| \int_L^t e^{a_+(\sigma - s)} ds \right| |f|_\infty \\ &\le 2|u_L| + \frac{1}{a_+} \left| e^{t-L} - 1 \right| |f|\infty \\ &\le 2(|u_L| + |f|_\infty). \end{aligned}$$

Next, give $\eta > 0$ small enough and independent of L, there exist $\sigma_* \le L$ such that $|a(\sigma) - a_+| < \eta$ for all $\sigma > \sigma*$. It is important to note here that one can choose σ_* independent of L as long as L is large enough. A Neumann series argument shows that in this case the operator

$$u \mapsto \left(\frac{d}{dt} u - a(t) u, u(L) \right)$$

is a $\eta-$perturbation of the asymptotic operator

$$u \mapsto \left(\frac{d}{dt} u - a_+ u, u(L) \right)$$

as a densely defined operator on $\mathcal{C}(\sigma_*, L)$. So we conclude that

$$\sup_{\sigma\in(\sigma_*,L)} |u(\sigma)| \leq C(|u|_L + |f|_\infty) \tag{A.2.5}$$

for some constant C independent of L.

Finally, for $\sigma \in (0, \sigma_*)$, the solution is given by the following formula

$$u(\sigma) = \exp\left(\int_{\sigma_*}^{\sigma} a(\tau)d\tau\right) u(\sigma_*) + \int_{\sigma_*}^{\sigma} \exp\left(-\int_{\sigma}^{s} a(\tau)d\tau\right) f(s)ds.$$

Since $\sigma_* < \infty$ and does not depend on L, there exist a constant C_1 independent of L so that

$$\max\left\{\left|\exp\left(\int_{\sigma_*}^{\sigma} a(\tau)d\tau\right)\right|, \left|\int_{\sigma_*}^{\sigma} \exp\left(-\int_{\sigma}^{s} a(\tau)d\tau\right)\right|\right\} \leq C_1.$$

Moreover, the value $u(\sigma_*)$ satisfies

$$u(\sigma_*) \leq \sup_{\sigma\in(\sigma_*,L)} |u(\sigma)| \leq C_2(u_L + |f|_\infty),$$

for some constant C_2 independent of L by (A.2.5). Therefore on $[0, \sigma_*]$ the solution satisfies

$$\sup_{\sigma\in[0,\sigma_*]} |u(\sigma)| \leq C_1C_2(u_L + |f|_\infty) + C_1|f|_\infty \leq C(u_L + |f|_\infty),$$

where the constant C does not depend on L. In summary, we conclude that

$$\sup_{\sigma\in[0,L]} = |u|_\infty \leq C(u_L + |f|_\infty),$$

which is (A.2.3a). ∎