THE ANALYSIS OF COATING FLOWS NEAR THE CONTACT LINE

AVNER FRIEDMAN* AND JUAN J.L. VELÁZQUEZ**

0. Introduction. The term "coating flows" refers to any viscous flow which is used to coat surfaces. Such flows are used for coating photographic films, magnetic tapes, optical devices and, of course, for painting surfaces. Various techniques are used for depositing the fluid on surfaces; the fluid may be Newtonian or Non-Newtonian, often being a 2-phase fluid. We refer to [9] [11] [13] [15] for a general introduction to the subject.

The boundary of the flow region consists of three parts: (i) the part Γ_0 in contact with the container from which the fluid emerges; (ii) the part Γ_1 in contact with air, and (iii) the part Γ_2 in contact with the surface which is being coated. In many applications Γ_2 is a part of a plane, the upper surface of a horizontal substrate moving with a given fixed velocity $\vec{U}$. The surface Γ_1 is a free boundary; it is one of the unknowns of the problem. The other unknowns are the velocity $\vec{v}$ and pressure p in the fluid.

Numerical methods based on finite elements have been developed by Kistler and Scriven [10]. One of the difficulties is to determine the location of the so called *contact line* $\gamma_0 = \overline{\Gamma}_2 \cap \overline{\Gamma}_1$ where the free boundary meets the substrate, and to analyze the shape of Γ_1 near γ_0.

There are a few existence results for steady coating flow problems with Newtonian fluids. These are usually restricted to "quiescent" situations whereby the flow velocity is "very small," an assumption which does not usually cover industrial applications; see [14] [17] and the references given there. Other free boundary problems for viscous flow are treated in [2] [3] [8], but here again there is a "smallness" assumption. We also refer to [4] for additional articles on free boundary problems for viscous flow, mostly by numerical methods.

In this paper we shall consider a 2-dimensional steady Stokesian flow. The free boundary is then a curve $y = f(x)$, $-\infty < x < 0$ with $f(0) = 0$. The top surface of the moving substrate is given $y = 0$, and the contact line γ_0 is identified with the origin (0,0); see Figure 1.

*University of Minnesota, Institute for Mathematics and its Applications, Minneapolis, Minnesota 55455. This work partially supported by the National Science Foundation Grant DMS–86–12880.

**Departamento de Matematica Aplicada, Universidad Complutense, Facultad de Matematicas, 28040 Madrid, SPAIN. Partially supported by CICYT Research Grant PB90–0235 and Fulbright Fellowship.

FIGURE 1

We wish to determine the existence of solutions and the behavior of the free boundary near the contact point γ_0.

The free boundary conditions are

$$T\vec{n} = \sigma\kappa(f)\vec{n}\ ,\ \vec{v}\cdot\vec{n} = 0 \qquad (\sigma > 0) \tag{0.1}$$

where $\vec{n}$ is the outward normal, $\kappa(f)$ the curvature, and T is the stress tensor; σ is the reciprocal of the capillary number. On the substrate we impose the no-slip condition:

$$\vec{v} = \vec{U} \quad \text{on} \quad y = 0\ . \tag{0.2}$$

The no-slip condition causes some difficulties near the contact point γ_0. It was pointed out by several authors (see [6] [9] and the references given there) that if the contact angle at γ_0 is not equal to 0 or π, then the no-slip condition predicts unbounded force about γ_0; such a force is physically impossible. (In this sense the solutions established in [8] and [17] are not physically acceptable.) This motivated several different approaches based on (i) molecular theory near γ_0, (ii) rolling type motion near γ_0, (III) $\vec{v} = \vec{U}$ near γ_0, and other adhoc assumptions (see [6] [9] [12] and the references therein).

However the situation where the free boundary is tangent to the substrate at γ_0 has not been analyzed for coating flow. The purpose of this paper is to show that this situation is consistent with the no-slip condition. More specifically, we establish the existence of smooth solutions satisfying (0.1) and (0.2), with $f'(0) = 0$.

We shall study first the "linearized problem" about $\vec{v} = \vec{U}$ near $(0,0)$. This is an eigenvalue problem. It has two sequences of eigensolutions,

$$(0.3) \qquad \begin{cases} (f_n, \psi_n) \quad \text{with} \quad f_n = (-x)^{\alpha_n} \ , & \alpha_n = n + \dfrac{1}{2} \ , \\ (\widetilde{f}_n, \widetilde{\psi}_n) \quad \text{with} \quad \widetilde{f}_n(x) = (-x)^{\widetilde{\alpha}_n} \ , \ , & \widetilde{\alpha}_n = n - \rho \end{cases}$$

where ρ is determined by

$$(0.4) \qquad \rho \in \left(-\frac{1}{2}\ ,\ \frac{1}{2}\right) \ , \quad \rho = \frac{1}{2\pi i} \log \frac{1 - 2i\lambda}{1 + 2i\lambda} \ , \ \lambda = -\frac{U}{\sigma} \ .$$

For coating flow $U > 0$ so that $0 < \rho < \frac{1}{2}$ (The case $U < 0$, or $-\frac{1}{2} < \rho < 0$, represents a cavity problem with $\{(x,0); x < 0\}$ as the nose or obstacle.)

By superposition we can form linearized solutions (f_0, ψ_0) with

$$(0.5) \qquad f_0 = \sum_n (A_n f_n + B_n \widetilde{f}_n) \ .$$

Suppose, for such a pair,

$$(0.6) \qquad f_0(x) = \begin{cases} A(-x)^{\alpha} + \text{higher order terms near } x = 0 \ , \ \alpha > \dfrac{3}{2} \\ B(-x)^{\beta} + \text{lower order terms near } x = -\infty, \ 0 < \beta < 1 \end{cases}$$

and $f_0(x) > 0$ if $-\infty < x < 0$. Then we prove, and this is the main result of the paper, that there exists a solution to the coating problem (satisfying (0.1) and (0.2)) with free boundary

$$(0.7) \qquad y = \varepsilon f(x) = \varepsilon f_0(x) + \varepsilon^2 f_1\ (x, \varepsilon) \ ,$$

for any $\varepsilon > 0$ small enough, where

$$(0.8) \qquad \begin{aligned} |f_1(x,\varepsilon)| &\le C|x|^{\alpha+\theta} \quad \text{near} \quad x = 0 \ , \qquad \theta > 0 \ , \\ |f_1(x,\varepsilon)| &\le C|x|^{\beta-\delta} \quad \text{near} \quad x = -\infty \ , \quad \delta > 0 \ . \end{aligned}$$

The class of functions f_0 satisfying (0.6) is not at all restrictive: Given any function $g_0(x)$, continuous and positive for $-\infty < x < 0$, such that

$$
\begin{aligned}
g_0(x) &\approx A_0(-x)^\alpha \quad \text{near} \quad x = 0 \ , \\
&\approx B_0(-x)^\beta \quad \text{near} \quad x = -\infty \ ,
\end{aligned}
\tag{0.9}
$$

we can approximate it by functions f_0 having the form (0.5) and satisfying (0.6). (The functions $f_0(x)(-x)^{-\alpha}$ approximate $g_0(x)(-x)^{-\alpha}$ near $x = 0$ and the functions $g_0(x)(-x)^\beta$ approximate $g_0(x)(-x)^\beta$ near $-\infty$.)

The conclusion of the paper is then that the no-slip condition is consistent with the free boundary conditions provided the contact angle is π (as indicated in Figure 1).

In §1 we describe the coating problem in detail and state the main results of the paper. In §2 we study the linearized eigenvalue problem. In §3 we transform the coating problem into an equivalent problem in the fixed domain $\{y > 0\}$; we also outline the proof of existence. The details of the proof are given in §§4–9. In §9 it is also shown that any g_0 as in (0.9) can be approximated by functions f_0 to which the existence theorem can be applied.

In this paper we consider the flow problem in $\{y > 0\}$ and we do not impose boundary conditions at $y = \infty$; this explains why we can obtain a large family of solutions. In a future work we shall extend the methods of this paper to study flows in domains such as $0 < y < h$, where boundary conditions are imposed at $y = h$; for more details see Remark 9.1.

§1. Statement of the main result. We are given velocity $\vec{U} = (U, 0)$ of the substrate surface $\{y = 0\}$ where U is any real number; for the coating problem, $U > 0$. Denote by Ca the capillary number and set

$$\sigma = \frac{1}{Ca} \qquad (\sigma > 0) \ .$$

We wish to determine a free boundary

$$y = f(x) > 0 \quad (-\infty < x < 0) \quad \text{with} \quad f(0) = 0 \ , \ f'(0) = 0$$

and a velocity function $\vec{v}$ and pressure p in the flow region

$$\Omega = \{(x, y) \ ; \ x \in \mathbb{R} \quad \text{and} \quad y > f(x) \quad \text{if} \quad -\infty < x \le 0 \ , \quad y > 0 \quad \text{if} \quad x > 0\} \ ,$$

such that

$$\Delta \vec{v} = \nabla p \quad \text{in} \quad \Omega \ , \tag{1.1}$$

$$\nabla \cdot \vec{v} = 0 \quad \text{in} \quad \Omega \ , \tag{1.2}$$

and

(1.3) $$\vec{v} = \vec{U} \quad \text{on} \quad \Gamma_2 = \{(x,0),\ x > 0\}\ ,$$

(1.4) $$\vec{v} \cdot \vec{n} = 0 \quad \text{on} \quad \Gamma_1 = \{(x, f(x)),\ x < 0\}\ ,$$

(1.5) $$T\vec{n} = \sigma\kappa\vec{n} \quad \text{on} \quad \Gamma_1$$

where T is the stress tensor

$$T_{ij} = -p\delta_{ij} + \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \qquad (x_1 = x, \quad x_2 = y)\ ,$$

$$\vec{n} = \frac{(f_x\ ,\ -1)}{(1+f_x^2)^{1/2}}$$

is the outward normal to Γ_1, and κ is the curvature

$$\kappa = \kappa(f) = -\frac{f_{xx}}{(1+f_x^2)^{3/2}}\ .$$

We refer to problem (1.1)–(1.5) as problem (C). One may introduce the stream function ψ:

$$\vec{v} = (-\psi_y, \psi_x)\ ,$$

and reformulate problem (C) also in terms of (f, ψ).

Introduce the sequences

(1.6) $$\begin{aligned} \alpha_n &= \frac{1}{2} + n \qquad (n = 1, 2, \dots)\ , \\ \widetilde{\alpha}_n &= -\rho + n \quad (n = 1, 2, \dots) \end{aligned}$$

where $\rho \in \left(-\frac{1}{2}\ ,\ \frac{1}{2}\right)$ is uniquely determined by

$$\rho = \frac{1}{2\pi i}\ \log\frac{1-2i\lambda}{1+2i\lambda}\ , \qquad \lambda = -\frac{U}{\sigma}\ .$$

Notice that if $U > 0 \quad (U < 0)$ then $\lambda < 0 \quad (\lambda > 0)$ and $0 < \rho < \frac{1}{2}$ $\left(-\frac{1}{2} < \rho < 0\right)$.

In §2 we prove that if we linearize problem (C) about $\vec{v} = \vec{U}$ near (0,0), then we get a linear homogeneous problem for ψ in $\{y \geq 0\}$. This problem and the associated "free boundary" f have eigensolutions (f_n, ψ_n) and $(\widetilde{f}_n, \widetilde{\psi}_n)$ where

$$f_n(x) = (-x)^{\alpha_n}\ , \quad \widetilde{f}_n(x) = (-x)^{\widetilde{\alpha}_n} \qquad (-\infty < x < 0)\ .$$

We need to introduce some notation:

$$B_\mu(\overline{x},\overline{y}) = \{(x,y);\quad (x-\overline{x})^2 + (y-\overline{y})^2 < \mu^2\}\ ,$$

$$B_\mu(\overline{x}) = B_\mu(\overline{x},0),\quad B_\mu = B_\mu(0,0)\ ,\ B_\mu^+ = B_\mu \cap \{y>0\}\ ,$$

$$\Sigma_{\theta_0}^+ = \{(x,y); x>0,\ 0\le y\le \theta_0 x\}\ ,$$

$$\Sigma_{\theta_0}^- = \{(x,y); x<0,\ 0\le y\le \theta_0 |x|\}\ ,$$

$$\Sigma_{\theta_0} = \Sigma_{\theta_0}^+ \cup \Sigma_{\theta_0}^- \quad \text{where} \quad 0<\theta_0<\frac{\pi}{2}\ .$$

Set $X=(x,y)$, $r=|X|$. In the sequel Ω will denote any one of the sets

$$\Sigma_{\theta_0}^-, \Sigma_\theta, \Sigma_{\theta_0}\cup B_R^+ \quad \text{and} \quad \{y>0\}\backslash B_R\ ,\quad \text{where} \quad 0\le\theta_0<1\ ,\ R>0\ ;$$

here Σ_0 (Σ_0^-) denotes the x-axis (the negative x-axis).

For $X\in\Omega$ and $\gamma\in(0,1)$ we define

$$|D_\Omega^\gamma \varphi(X)| = \sup \frac{|\varphi(\overline{X}) - \varphi(\widetilde{X})|}{|\overline{X}-\widetilde{X}|^\gamma}$$

where the "sup" is taken over $\overline{X},\widetilde{X}$ in Ω such that $|\overline{X}-X| < \dfrac{r}{2}$, $|\widetilde{X}-X| < \dfrac{r}{2}$.

Let α and β be any real numbers. Let $g(r)$ be a continuous positive function for $0<r<\infty$, such that

$$g(r) = r^\alpha \quad \text{if} \quad r<1\ ,\ g(r) = r^\beta \quad \text{if} \quad r>2\ . \tag{1.7}$$

For any integer $m\ge 0$ we define the norm

$$\|\varphi\|_{\Omega,m+\gamma}^g = \sup_\Omega \left\{ \sum_{|k|\le m} \frac{r^{|k|}}{g(r)} |D^k\varphi(X)| + \sum_{|k|=m} \frac{r^{m+\gamma}}{g(r)} |D^k\varphi(X)| \right\}\ .$$

Since the particular choice of g which satisfies (1.7) will not be of any consequence to us, we shall use the less precise but more explicit notation

$$\|\varphi\|_{\Omega,m+\gamma}^{\alpha,\beta} = \|\varphi\|_{\Omega,m+\gamma}^g\ ,\quad g \quad \text{as in (1.7)}\ . \tag{1.8}$$

By superposing eigensolutions of the linearized problem we obtain a linear class of solutions (f_0,ψ_0); we denote this class by $\mathcal{A}$. If (f_0,ψ_0) belongs to $\mathcal{A}$ and

$$\|f_0\|_{\Sigma_0^-,4+\gamma}^{\alpha,\beta} < \infty\ ,\ \|\psi_0\|_{\Sigma_{\theta_0,4+\gamma}^-}^{\alpha,\beta} < \infty$$

for some $\alpha>0,\beta>0,\theta_0>0$, then we say that (f_0,ψ_0) belongs to $\mathcal{A}_{\theta_0,4+\gamma}^{\alpha,\beta}$.

We can now state the main result of this paper.

THEOREM 1.1. *Let α, β be positive real number such that $\alpha > \frac{3}{2}$, $0 < \beta < 1$, and $\alpha = \alpha_n$ or $\alpha = \tilde{\alpha}_n$ for some n. Let θ and δ be any positive numbers such that $\delta < \min(\beta, 1-\beta)$ and $\theta < \alpha - 1$. Finally, let $(f_0, \psi_0) \in \mathcal{A}^{\alpha,\beta}_{\theta_0,4+\gamma}$ for some $0 < \theta_0 < 1$ and γ positive and small. Then, for any sufficiently small $\varepsilon > 0$ there exists a solution $(f_\varepsilon, \psi_\varepsilon)$ of problem (C) with*

$$f_\varepsilon = \varepsilon f_0(x) + \varepsilon^2 f_1(x, \varepsilon) \ , \tag{1.9}$$

$$\psi_\varepsilon = -Uy + \varepsilon \psi_0(x,y) + \varepsilon^2 \psi_1(x,y,\varepsilon) \tag{1.10}$$

where f_1 satisfies:

$$\|f_1\|^{\alpha+\theta,\beta-\delta}_{\Sigma_0^-,4+\gamma} \le C < \infty \tag{1.11}$$

and

$$\widetilde{\psi}_1(x,y,\varepsilon) = \begin{cases} \psi_1(x, y + \varepsilon f_0(x) + \varepsilon^2 f_1(x,\varepsilon), \varepsilon)) & (x < 0) \\ \psi_1(x,y) & (x \ge 0) \end{cases}$$

(which is defined in all of $\{y > 0\}$) satisfies

$$\|\widetilde{\psi}_1\|^{\alpha+\theta,\beta-\delta}_{\Sigma_{\theta_0'} \cup B_R^+,4+\gamma} \le C < \infty \quad \forall \quad \theta_0' < 1 \ , \ R > 0 \ ; \tag{1.12}$$

the constant C is independent of ε

Notice that θ can be chosen such that $\theta > \frac{1}{2}$.

Remark 1.1. The solution ψ is not bounded in general; $\widetilde{\psi}$ may grow exponentially if $x^2 + y^2 \to \infty$ and $|y/x| > 1$. Furthermore, we do not make any assertions about uniqueness. The main point of Theorem 1.1 is to exhibit a large class of smooth solutions of problem (C) in a neighborhood of the contact point. In a future work we shall extend the methods of the present paper to flows lying in regions such as $0 < y < h$, and impose boundary conditions at $y = h$; see Remark 9.1 for more details.

§**2. The eigenvalue problem.** In this section we consider the linearization of problem (C) about $\vec{v} = \vec{U}$ in a neighborhood of $(0,0)$. Set

$$\vec{v} = \vec{U} + \vec{G} \ , \ \vec{G} = (G_x, G_y) \ ,$$

$$\ell_1 = \{(x,0), \ x < 0\} \ , \ l_2 = \{(x,0), \ x > 0\} \ .$$

Then

$$\left.\begin{aligned} &\Delta\vec{G} = \nabla p && (2.1)\\ &\nabla\cdot\vec{G} = 0 && (2.2)\end{aligned}\right\}\quad \text{in the flow region ,}$$

$$\left.\begin{aligned} &\vec{G}\cdot\vec{n} = -\vec{U}\cdot\vec{n} && (2.3)\\ &\vec{\tau}T(\vec{G},p)\vec{n} = 0 && (2.4)\\ &\vec{n}T(\vec{G},p)\vec{n} = \sigma\kappa(f) && (2.5)\end{aligned}\right\}\quad \text{on the free boundary ,}$$

and

$$\vec{G} = 0 \quad \text{on} \quad \ell_2 \ . \tag{2.6}$$

Here $\vec{\tau}$ is obtained by rotating $\vec{n}$ clockwise by $90°$, and

$$\begin{aligned} T(\vec{G},p) &= -\begin{pmatrix} p & 0\\ 0 & p\end{pmatrix} + \begin{pmatrix} \dfrac{\partial G_x}{\partial x} & \dfrac{1}{2}\left(\dfrac{\partial G_x}{\partial y} + \dfrac{\partial G_y}{\partial x}\right)\\ \dfrac{1}{2}\left(\dfrac{\partial G_x}{\partial y} + \dfrac{\partial G_y}{\partial x}\right) & \dfrac{\partial G_y}{\partial y}\end{pmatrix}\\ &\equiv -\begin{pmatrix} p & 0\\ 0 & p\end{pmatrix} + \Pi(\vec{G})\end{aligned}$$

is the stress tensor.

On the free boundary Γ_1

$$\vec{n} = \frac{(f',-1)}{(1+f'^2)^{1/2}} \ ,$$

and since $f'(x) \sim 0$ near $x = 0$, Γ_1 is approximately ℓ_1,

$$\vec{U}\cdot\vec{n} = \frac{Uf'(x)}{(1+f'(x)^2)^{1/2}} \approx Uf'(x) \ ,$$

and

$$\vec{n} \approx -\vec{j} = (0,-1) \ , \ \vec{\tau} \approx \vec{i} = (1,0) \ .$$

Hence, for the linearized problem, (2.3) and (2.4) become

$$G_y = Uf' \qquad \text{on} \quad \ell_2 \ , \tag{2.3$'$}$$

$$\frac{\partial G_x}{\partial y} + \frac{\partial G_y}{\partial x} = 0 \quad \text{on} \quad \ell_2 \ . \tag{2.4$'$}$$

Equation (2.5) can be written as

$$-p + \vec{n}\,\Pi(\vec{G})\vec{n} = \sigma\kappa(f) \ ,$$

and since p is determined only up to an additive constant, this equation is equivalent to

$$-\frac{dp}{ds} + \frac{d}{ds}[\vec{n}\,\Pi(\vec{G})\vec{n}] = \sigma\frac{d}{ds}\kappa(f)$$

where s is the length parameter with $ds/dx > 0$. From (2.1),

$$-\frac{dp}{ds} = (\Delta\vec{G})\cdot\vec{\tau} \approx (\Delta\vec{G})\cdot\vec{i} = \Delta G_x \ .$$

Also $d\vec{n}/ds = -\kappa\vec{\tau}$. Therefore by (2.4) and $\dfrac{\partial}{\partial s} \approx \dfrac{\partial}{\partial x}$,

$$\frac{d\vec{n}}{ds}\Pi(\vec{G})\vec{n} = \vec{n}\,\Pi(\vec{G})\frac{d\vec{n}}{ds} = -\kappa\vec{\tau}\,\Pi(\vec{G})\vec{n} = 0 \ ,$$

$$\vec{n}\left(\frac{d}{ds}\ \Pi(\vec{G})\right)\vec{n} \approx \vec{j}\,\frac{\partial}{\partial x}\ \Pi(\vec{G})\vec{j} = \frac{\partial}{\partial x}\ \frac{\partial G_y}{\partial y} \ .$$

We conclude that

$$-\Delta G_x + \frac{\partial}{\partial x}\ \frac{\partial G_y}{\partial y} = \sigma\frac{d}{dx}\ \kappa(f) \approx -\sigma\frac{d}{dx}f''(x) \ .$$

provided

$$f'(x) = 0(|x|^\nu) \quad \text{for some} \quad \nu > 0 \ .$$

Thus (2.5) becomes

$$(2.5') \qquad -\Delta G_x(x,0) + \frac{\partial^2 G_y}{\partial x \partial y}\ (x,0) = -\sigma f^{(3)}(x) \qquad (x < 0) \ .$$

From (2.2) we know that there exists a stream function ψ:

$$\vec{G} = (-\psi_y, \psi_x)$$

and $\Delta^2\psi = 0$ in the flow region (by taking the curl of (2.1)). Conditions (2.3′)–(2.5′), written for ψ, read:

$$(2.3'') \qquad \psi_x(x,0) = Uf'(x) \quad \text{on} \quad \ell_1 \ ,$$

$$(2.4'')\qquad \psi_{yy}(x,0)-\psi_{xx}(x,0)=0 \quad \text{on} \quad \ell_1 \ ,$$

$$(2.5'')\qquad 2\psi_{xxy}(x,0)+\psi_{yyy}(x,0)=-\sigma f^{(3)}(x) \quad \text{on} \quad \ell_1 \ .$$

Substituting f' from (2.3″) into (2.5″), we get

$$(2.5''')\qquad 2\psi_{xxy}(x,0)+\psi_{yyy}(x,0)=-\frac{\sigma}{U}\ \psi_{xxx}(x,0) \quad \text{on} \quad \ell_1 \ .$$

Set

$$\lambda=-\frac{U}{\sigma}\ .$$

Then the final version of the linearized problem for ψ is the following:

$$(2.7)\qquad \Delta^2\psi=0 \quad \text{in} \quad \{y>0\}\ ,$$

$$(2.8)\qquad \psi_x(x,0)=\psi_y(x,0)=0 \quad \text{if} \quad x>0\ ,$$

and

$$(2.9)\qquad \psi_{yy}-\psi_{xx}=0 \quad \text{at} \quad (x,0),\ x<0\ ,$$

$$(2.10)\qquad \lambda(2\psi_{xxy}+\psi_{yyy})-\psi_{xxx}=0 \quad \text{at} \quad (x,0),\ x<0\ .$$

The corresponding "linearized" free boundary is given by

$$(2.11)\qquad f'(x)=\frac{1}{U}\ \psi_x(x,0) \quad (x<0) \quad \text{with} \quad f(0)=0\ .$$

We may view (2.7)–(2.10) as an eigenvalue problem. It has a sequence of solutions of the form $\psi=r^\beta B(\theta)$ when β varies over the sequence (1.6) ((r,θ) are polar coordinate about the origin). It will be more convenient, however, to use complex notation and write the eigenfunctions ψ in the form

$$(2.12)\qquad \psi(z,\bar z)=Az^\beta+Bz^{\beta-1}\bar z+\widehat{A}\bar z^\beta+\widehat{B}\bar z^{\beta-1}z \quad (z=x+iy)$$

where β and the complex coefficients $A,B,\widehat{A},\widehat{B}$ are to be determined.

We note that any biharmonic function in a disc $|z|<\delta$ has the form

$$r^2h_1+h_2 \qquad (r=|z|)$$

where each h_j is harmonic function and thus has an expansion

$$\Sigma(A^j_m z^m + \overline{A^j_m}\overline{z}^m) \qquad (j = 1, 2)$$

This suggests that ψ should be a superposition of functions of the form (2.12) where β is an integer ≥ 0. Since however the stream function is defined only in $\{y \geq 0\}$ and may have a singularity at $z = 0$, we allow more general terms as in (2.12), where β is still to be determined.

We need to rewrite (2.7)–(2.10) in complex notation using

$$\partial = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \ , \ \overline{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \ ,$$

or

$$\frac{\partial}{\partial x} = \partial + \overline{\partial} \ , \ \frac{\partial}{\partial y} = i(\partial - \overline{\partial}) \ .$$

By direct calculation we find that (2.7)–(2.10) become

$$\partial^2\overline{\partial}^2\psi = 0 \quad \text{if} \quad y > 0 \ , \tag{2.13}$$

$$\partial\psi = 0 \ , \ \overline{\partial}\psi = 0 \quad \text{if} \quad x > 0 \ , \ y = 0 \ , \tag{2.14}$$

$$(\partial^2 + \overline{\partial}^2)\psi = 0 \quad \text{if} \quad x < 0, \ y = 0 \ , \tag{2.15}$$

$$[i\lambda(\partial^3 + 5\partial^2\overline{\partial} - 5\partial\overline{\partial}^2 - \overline{\partial}^3) - (\partial^3 + 3\partial^2\overline{\partial} + 3\partial\overline{\partial}^2 + \overline{\partial}^3)]\psi = 0 \quad \text{if} \quad x < 0, \ y = 0 \ . \tag{2.16}$$

Substituting (2.12) into (2.14) we obtain after some simple calculation,

$$\beta A + (\beta - 1)B + \widehat{B} = 0 \ , \tag{2.17}$$

$$B + \beta\widehat{A} + (\beta - 1)\widehat{B} = 0 \ . \tag{2.18}$$

Next, from (2.15) we obtain after noting that $z = re^{i\pi}$ if $x < 0$,

$$\begin{aligned} &\beta(\beta-1)e^{i\pi(\beta-2)}A + (\beta-1)(\beta-2)e^{i\pi(\beta-4)}B \\ &\quad + \beta(\beta-1)e^{-i\pi(\beta-2)}\widehat{A} + (\beta-1)(\beta-2)e^{-i\pi(\beta-4)}\widehat{B} = 0 \ . \end{aligned} \tag{2.19}$$

Finally (2.16) gives

(2.20)

$$\beta(\beta-1)(\beta-2)e^{i\pi(\beta-3)}(i\lambda-1)A$$
$$+\left[(\beta-1)(\beta-2)(\beta-3)e^{i\pi(\beta-5)}(i\lambda-1)+(\beta-1)(\beta-2)e^{i\pi(\beta-3)}(5i\lambda-3)\right]B$$
$$-\beta(\beta-1)(\beta-2)e^{-\pi(\beta-3)}(i\lambda+1)\widehat{A}$$
$$-\left[(\beta-1)(\beta-2)e^{i\pi(\beta-3)}(5i\lambda+3)+(\beta-1)(\beta-2)(\beta-3)e^{-i\pi(\beta-3)}(i\lambda+1)\right]\widehat{B}=0.$$

In case $\beta=1$ or $\beta=2$ we can compute $A,\widehat{A},B,\widehat{B}$ directly and discover that $\psi\equiv 0$. Hence we may assume that $\beta\neq 1,\beta\neq 2$. Factoring out $\beta-1$ from (2.19) and $(\beta-1)(\beta-2)$ from (2.20), we arrive at

$$\beta e^{i\pi\beta}A+(\beta-2)e^{i\pi\beta}B+\beta e^{-i\pi\beta}\widehat{A}+(\beta-2)e^{-i\pi\beta}\widehat{B}=0\ , \tag{2.21}$$

$$\beta e^{i\pi\beta}z_1+e^{i\pi\beta}z_2B+\beta e^{-\pi\beta}\overline{z}_1\widehat{A}+e^{-i\pi\beta}\overline{z}_2\widehat{B}=0 \tag{2.22}$$

where

$$z_1=i\lambda-1\ ,\ z_2=(\beta-3)(i\lambda-1)+(5i\lambda-3)\ . \tag{2.23}$$

We now concentrate on the systems (2.17), (2.18), (2.21), (2.22) where the coefficients $A,B,\widehat{A},\widehat{B}$ are arbitrary unknown complex numbers. The determinant of the coefficients is

$$\begin{vmatrix} 1 & \beta-1 & 0 & 1 \\ 0 & 1 & 1 & \beta-1 \\ e^{i\pi\beta} & (\beta-2)e^{i\pi\beta} & e^{-i\pi\beta} & (\beta-2)e^{-i\pi\beta} \\ e^{i\pi\beta}z_1 & e^{i\pi\beta}z_2 & e^{-i\pi\beta}\overline{z}_1 & e^{-i\pi\beta}\overline{z}_2 \end{vmatrix}$$

Add to the third row $e^{-i\pi\beta}$ times the first row and $e^{i\pi\beta}$ times the second row. Then add to the fourth row $-e^{i\pi\beta}z_1$ times the first row. The result is

$$\begin{vmatrix} 1 & \beta-1 & 0 & 1 \\ 0 & 1 & 1 & \beta-1 \\ 0 & 0 & 2\cos\pi\beta & 2(\beta-2)\cos\pi\beta \\ 0 & e^{i\pi\beta}(z_2-(\beta-1)z_1) & e^{-i\pi\beta}\overline{z}_1 & e^{-i\pi\beta}\overline{z}_2-e^{i\pi\beta}z_1 \end{vmatrix}$$

If we add to the fourth row $e^{-i\pi\beta}(z_2-(\beta-1)z_1)$ times the second row, we find that the determinant vanishes if and only if either

$$\cos\pi\beta=0 \tag{2.24}$$

or

$$e^{-i\pi\beta}\overline{z}_2 - e^{i\pi\beta} z_1 - (\beta-1)e^{i\pi\beta}(z_2 - (\beta-1)z_1) \; , \tag{2.25}$$
$$-(\beta-2)[e^{-i\pi\beta}\overline{z}_1 - e^{i\pi\beta}(z_2 - (\beta-1)z_1)] = 0 \; .$$

Substituting z_1, z_2 from (2.23) we find that (2.25) is equivalent to

$$e^{2i\pi\beta} = \frac{1+2i\lambda}{1-2i\lambda} \; , \tag{2.26}$$

or

$$\beta = -\rho + n \qquad (n = 1, 2, \dots) \tag{2.27}$$

where ρ is uniquely determined by

$$e^{2i\pi\rho} = \frac{1-2i\lambda}{1+2i\lambda} \; , \; -\frac{1}{2} < \rho < \frac{1}{2} \tag{2.28}$$

or,

$$\rho = \frac{1}{2\pi i} \log \frac{1-2i\lambda}{1+2i\lambda}, \; -\frac{1}{2} < \rho < \frac{1}{2} \; . \tag{2.29}$$

Note that $n \leq 0$ gives infinite velocity for $\vec{v}$ at $(0, 0)$, and is therefore excluded from (2.27).

The solutions to (2.24) are given by

$$\beta = \frac{1}{2} + n \qquad (n = 0, 1, 2, \dots) \; . \tag{2.30}$$

Observe that the two sequences (2.27) and (2.30) are mutually disjoint; (2.30) is a limiting case of (2.27) when $|\lambda| \to \infty$.

Consider first the case (2.30). We proceed to solve the system (2.17), (2.18), (2.21), (2.22) for this case. From (2.17), (2.18) we get

$$A = -\frac{\beta - 1}{\beta} \; B - \frac{\widehat{B}}{\beta} \; , \tag{2.31}$$

$$\widehat{A} = -\frac{\beta - 1}{\beta} \; \widehat{B} - \frac{B}{\beta} \; , \tag{2.32}$$

Since $e^{2\pi i\beta} = -1$, equation (2.21) depends linearly on (2.17), (2.18) and may therefore be dropped. Substituting (2.31), (2.32) into (2.22) we obtain a linear relation between B and $\widehat{B}$, from which we deduce, after some calculation, that

$$\widehat{B} = \frac{1 - \lambda i}{1 + \lambda i} \; B \; . \tag{2.33}$$

Therefore (2.31) and (2.32) become

$$A = -\frac{1}{\beta}\left(\beta - \frac{2\lambda i}{1+\lambda i}\right) B \ , \tag{2.34}$$

$$\widehat{A} = -\frac{1}{\beta}\left[1 + (\beta - 1)\frac{1-\lambda i}{1+\lambda i}\right] B \ . \tag{2.35}$$

We finally need to choose B so as to obtain a real valued solution ψ, i.e., a solution with

$$\overline{A} = \widehat{A} \quad \text{and} \quad \overline{B} = \widehat{B} \ . \tag{2.36}$$

Setting

$$B = |B| e^{i\theta} \ , \tag{2.37}$$

the second relation in (2.36) becomes

$$e^{2i\theta} = \frac{1-\lambda i}{1+\lambda i} \ . \tag{2.38}$$

An easy calculation shows that with this choice of θ, also the first relation in (2.36) is satisfied.

Consider next the case (2.26). Equations (2.31), (2.32) are still valid. Next we use (2.26) in (2.23) and obtain, after some simple calculation which make use of (2.31), (2.32),

$$\widehat{B} = -B \ . \tag{2.39}$$

Consequently, equations (2.31), (2.32) become

$$A = -\frac{\beta - 2}{\beta} B \ , \quad \widehat{A} = \frac{\beta - 2}{2} B \ . \tag{2.40}$$

We now choose B as in (2.37) with θ such that

$$e^{2i\theta} = -1 \ . \tag{2.41}$$

It follows that

$$\overline{B} = \widehat{B} \quad \text{and} \quad \overline{A} = A \ ,$$

so that ψ is real-valued.

We summarize:

THEOREM 2.1. *There exist two sequences of real-valued solutions of (2.13)–(2.16) having the form*

$$\psi_\beta(z,\overline{z}) = A_\beta z^\beta + B_\beta z^{\beta-1}\overline{z} + \overline{A}_\beta \overline{z}^\beta + \overline{B}\overline{z}^{\beta-1} z \tag{2.42}$$

where either

$$\beta A_\beta = \left(\frac{2i\lambda}{1+i\lambda} - \beta\right) B_\beta \ , \quad \beta = \frac{1}{2} + n \qquad (n = 0,1,2,\dots) \ , \tag{2.43}$$

or

$$\beta A_\beta = -(\beta - 2) B_\beta \ , \quad \beta = -\rho + n \qquad (n = 1,2,\dots) \ ; \tag{2.44}$$

here $\overline{A}$ and $\overline{B}$ are the complex conjugates of A and B respectively, B has the form (2.37) and θ is defined by (2.38) if $\beta = \frac{1}{2} + n$ and by (2.41) if $\beta = -\rho + n$.

We shall now superpose these solutions. We begin with the case $\beta = \frac{1}{2} + n$. From (2.43),

$$\frac{\partial}{\partial z}(A_\beta z^\beta) = \frac{2i\lambda}{1+i\lambda} \ B_\beta z^{\beta-1} - \beta B_\beta z^{\beta-1} \ .$$

Setting

$$\Psi_1(z) = \sum_{n=0}^{\infty} A_n z^{\frac{1}{2}+n} \ , \ \Phi_1(z) = \sum_{n=0}^{\infty} B_n z^{\frac{1}{2}+n} \tag{2.45}$$

we then have

$$\frac{\partial}{\partial z} \ \Psi_1(z) = \frac{2i\lambda}{1+i\lambda} \ \frac{\Phi_1(z)}{z} - \frac{\partial}{\partial z} \ \Phi_1(z) \tag{2.46}$$

and, by (2.37), (2.38),

$$\Phi_1(z) = \omega_\lambda z^{1/2} \sum_{n=0}^{\infty} \Gamma_n z^n \ , \ \Gamma_n \in \mathbb{R} \ , \ \omega_\lambda^2 = \frac{1-\lambda i}{1+\lambda i} \ . \tag{2.47}$$

In case $\beta = -\rho + n$ we have, by (2.44),

$$\frac{\partial}{\partial z}(A_\beta z^\beta) = 2\frac{B_\beta z^\beta}{z} - \beta B_\beta z^{\beta-1} \ .$$

Setting

$$\Psi_2(z) = \sum_{n=1}^{\infty} A_n z^{-\rho+n} \ , \ \Phi_2(z) = \sum_{n=1}^{\infty} B_n z^{-\rho+n} \ , \tag{2.48}$$

we have

$$\frac{\partial}{\partial z} \ \Psi_2(z) = 2\frac{\Phi_2(z)}{z} - \frac{\partial}{\partial z} \ \Phi_2(z) \tag{2.49}$$

and, by (2.37), (2.41),

$$\Phi_2(z) = i z^{-\rho} \sum_{n=1}^{\infty} \Gamma_n z^n \ , \ \Gamma_n \in \mathbb{R} \ . \tag{2.50}$$

We summarize:

THEOREM 2.2. *The system (2.13)–(2.16) has solutions of the form*

$$\psi(z,\overline{z}) = 2Re\sum_{j=1}^{2}\left[\Psi_j(z)+\Phi_j(z)\frac{\overline{z}}{z}\right] \tag{2.51}$$

where Φ_1,Φ_2 *are given by (2.47), (2.50) and* $\Psi_1(z),\Psi_2(z)$ *are given by (2.46), (2.49).*

This solution will be sufficiently general for our purposes. Let us compute the linearized free boundary $y=f(x)$ corresponding to ψ. From (2.3″) we have

$$\begin{aligned} f(x) &= \frac{2}{U}\ \mathrm{Re}(\Phi_1+\Phi_2+\Psi_1+\Psi_2) \qquad (z=\overline{z}=x<0)\\ &= \frac{2}{U}\ \mathrm{Re}\sum_{n=1}^{\infty}\left[\omega_\lambda\frac{2i\lambda}{1+i\lambda}\ \Gamma_n\frac{z^{n+\frac12}}{n+\frac12}+2i\widetilde{\Gamma}_n\frac{z^{n-\rho}}{n-\rho}\right]\ ; \end{aligned}$$

We have taken $\Gamma_0=0$, since $f'(x)$ must vanish at $x=0$. Setting $r=-x$ and noting that

$$\mathrm{Re}\left\{\omega_\lambda\frac{2i\lambda}{1+i\lambda}\right\}\neq 0\ ,\ \mathrm{Re}\left\{iz^{-\rho}\right\}\neq 0 \quad \text{if} \quad z=-x\ ,$$

we deduce that

$$f(x)=\sum_{n=1}^{\infty}\left[\Gamma_n r^{n+\frac12}+\widetilde{\Gamma}_n r^{n+\rho}\right] \qquad (\Gamma_n\in\mathbb{R},\widetilde{\Gamma}_n\in R)\ ; \tag{2.52}$$

the coefficients $\Gamma_n,\widetilde{\Gamma}_n$ are not the same as before, but they are nevertheless arbitrary.

§**3. The perturbation problem.** Consider problem (C). We shall try to find a solution with

$$\begin{aligned} &\text{velocity } \vec{v}=\vec{U}+\varepsilon\vec{G}, \quad \text{pressure } \varepsilon p \text{ and free boundary}\\ &\Gamma_1=\{y=\varepsilon f(x), \quad -\infty<x<0\} \end{aligned} \tag{3.1}$$

where $\varepsilon>0$ is sufficiently small and $\vec{G},p,f$ depend on ε. The outward normal is

$$\vec{n}=\frac{(\varepsilon f'(x),-1)}{[1+\varepsilon^2 f'(x)^2]^{1/2}}$$

and the unit tangent vector is taken to be

$$\vec{\tau}=\frac{(1,\varepsilon f'(x))}{[1+\varepsilon^2 f'(x)^2]^{1/2}}\ .$$

Set

$$ds = [1 + \varepsilon^2 f'(x)^2]^{1/2} dx \ ,$$

$$\Omega_\varepsilon = \{(x,y); x < 0, y > \varepsilon f(x)\} \cup \{(x,y); x \geq 0, y > 0\} \ .$$

Then (C) becomes:

$$\Delta \vec{G} = \nabla p \ , \ \nabla \cdot \vec{G} = 0 \quad \text{in the flow domain} \quad \Omega_\varepsilon \tag{3.2}$$

and

$$\vec{G} = 0 \quad \text{on} \quad \Gamma_2 \ , \tag{3.3}$$

$$U(1,0) \cdot (\varepsilon f'(x), -1) + \varepsilon \vec{G}(x) \cdot (\varepsilon f'(x), 1) = 0 \quad \text{on} \quad \Gamma_1 \ , \tag{3.4}$$

$$\left(-pI + \Pi(\vec{G})\right) \vec{n} = -\sigma \frac{f''(x)}{[1 + \varepsilon^2 f'(x)^2]^{3/2}} \ \vec{n} \quad \text{on} \quad \Gamma_1 \ . \tag{3.5}$$

We wish to simplify (3.5). Taking the scalar product with $\vec{n}$ and then differentiating along Γ_1, we get

$$-\frac{dp}{ds} + \frac{d}{ds}\left[\vec{n}\,\Pi(\vec{G})\vec{n}\right] = -\sigma \frac{d}{ds} \frac{f''(x)}{[1 + \varepsilon^2 f'(x)^2]^{3/2}} \ . \tag{3.6}$$

Also, from (3.2),

$$\frac{dp}{ds} = (\Delta \vec{G}) \cdot \vec{\tau} \ .$$

Since $d\vec{n}/ds = \vec{\tau}/R \quad (R = \text{radius of curvature})$ and

$$\vec{\tau}\,\Pi(\vec{G})\vec{n} = \vec{\tau}(-pI + \Pi(\vec{G}))\vec{n} = \vec{\tau} \cdot \frac{(-\sigma f'')}{[1 + \varepsilon^2 f'^2]^{3/2}} \ \vec{n} = 0 \ ,$$

we also have

$$\frac{d\vec{n}}{ds} \ \Pi(\vec{G})\vec{n} = \vec{n}\,\Pi(\vec{G}) \frac{d\vec{n}}{ds} = 0 \ .$$

Hence (3.6) becomes

(3.7)

$$-(\Delta \vec{G}) \cdot \vec{\tau} + \frac{1}{[1 + \varepsilon^2 f'(x)^2]^{1/2}} \vec{n} \left(\frac{d}{dx}\Pi(\vec{G})\right) \vec{n} = -\frac{\sigma}{[1 + \varepsilon^2 f'(x)^2]^{1/2}} \frac{d}{dx} \frac{f''(x)}{[1 + \varepsilon f'(x)^2]^{3/2}} \ \text{on} \ \Gamma_1 \ .$$

From (3.5) we also have

$$\vec{\tau}\,\Pi(\vec{G})\vec{n} = 0 \quad \text{on} \quad \Gamma_1 \ . \tag{3.8}$$

We shall reformulate this problem in terms of the stream function ψ, where

$$\vec{G} = (-\psi_y, \psi_x) \ ;$$

the existence of ψ is equivalent to the condition $\nabla \cdot \vec{G} = 0$.

Equation $\nabla \vec{G} = \nabla p$ is equivalent to

$$\Delta^2 \psi = 0 \quad \text{in} \quad \Omega_\varepsilon \ . \tag{3.9}$$

Conditions (3.3), (3.4) become

$$\psi_x(x,0) = \psi_y(x,0) = 0 \quad \text{if} \quad x > 0 \ , \tag{3.10}$$

$$U f'(x) - \psi_x = \varepsilon f'(x)\psi_y \quad \text{if} \quad y = \varepsilon f(x) \ , \ x < 0 \ . \tag{3.11}$$

Since

$$\Pi(\vec{G}) = \begin{pmatrix} -\psi_{xy} & \frac{1}{2}(\psi_{xx} - \psi_{yy}) \\ \frac{1}{2}(\psi_{xx} - \psi_{yy}) & \psi_{xy} \end{pmatrix}$$

we can easily check that (3.8) reduces to

$$\psi_{yy} - \psi_{xx} = 4\varepsilon\psi_{xy} f'(x) - \varepsilon^2 f'(x)^2(\psi_{xx} - \psi_{yy}) \equiv \varepsilon Q_1(\varepsilon, f, \psi) \quad \text{on} \quad \Gamma_1. \tag{3.12}$$

Next, from (3.7) we get by direct calculation

$$[\psi_{yxx} + \psi_{yyy} - \varepsilon f'(x)(\psi_{xxx} + \psi_{xyy})]$$
$$+ \frac{1}{[1+\varepsilon^2 f'(x)^2]} \ [\psi_{xxy} - \varepsilon f'(x)(\psi_{xxx} - \psi_{yyx}) - \varepsilon^2 f'(x)^2 \psi_{yxx}]$$
$$= -\sigma \frac{d}{dx} \ \frac{f''(x)}{(1+\varepsilon^2 f'(x)^2]^{3/2}} \ ,$$

or

$$\frac{1}{[1+\varepsilon^2 f'(x)^2]}[2\psi_{xxy} + \psi_{yyy} - 2\varepsilon f'(x)\psi_{xxx} + \varepsilon^2 f'(x)^2 \psi_{yyy}$$
$$- \varepsilon^3 f'(x)^3(\psi_{xxx} + \psi_{xyy})] = -\sigma \frac{d}{dx} \frac{f''(x)}{[1+\varepsilon^2 f'(x)^2]^{3/2}} \ .$$

Expanding the right-hand side we can rewrite the last equation in a more convenient way:

$$2\psi_{xxy} + \psi_{yyy} + \sigma f^{(3)}(x) = 2\varepsilon f'(x)\psi_{xxx} - \varepsilon^2 f'(x)^2\psi_{yyy}$$

$$+ \varepsilon^3 f'(x)^3(\psi_{xxx} + \psi_{xyy}) - \sigma f^{(3)}(x)\frac{1-(1+\varepsilon^2 f'(x)^2)^{1/2}}{(1+\varepsilon^2 f'(x)^2)^{1/2}}$$

$$+ 3\sigma\varepsilon^2\frac{f'(x)f''(x)^2}{[1+\varepsilon^2 f'(x)^2]^{3/2}} ,$$

where all the terms on the right-hand side are $O(\varepsilon)$. It will be useful to eliminate $f^{(3)}$ from the left-hand side of the last equation. We do this by differentiating (3.11) twice, then solving for $f^{(3)}$ and substituting into the left-hand side. This results in the equation

(3.13)

$$\lambda(2\psi_{xxy} + \psi_{yyy}) - \psi_{xxx} = 2\varepsilon\lambda f'(x)\psi_{xxx} + \varepsilon f^{(3)}(x)\psi_y$$

$$+ 2\varepsilon f''(x)\psi_{yx} + \varepsilon\lambda f'(x)\psi_{yxx} - \varepsilon^2\lambda f'(x)^2\psi_{yyy}$$

$$+ \varepsilon^3\lambda f'(x)^3(\psi_{xxx} + \psi_{xyy})$$

$$+ U f^{(3)}(x)\frac{1-(1+\varepsilon^2 f'(x)^2)^{1/2}}{(1+\varepsilon^2 f'(x)^2)^{1/2}} - 3U\varepsilon^2\frac{f'(x)f''(x)^2}{(1+\varepsilon^2 f'(x)^2)^{3/2}} \equiv \varepsilon Q_2(\varepsilon, f, \psi) \text{ on } \Gamma_1 ,$$

where $\lambda = -U/\sigma$. We shall refer to problem (C) in its formulation (3.9)–(3.13) also as problem (C_ε).

It is not convenient to work directly with the system (3.9)–(3.13) since the flow region Ω_ε is unknown. We shall therefore perform a change of variables which transforms Ω_ε onto $\{y > 0\}$.

Let $\eta(t)$ be a function satisfying:

$$\eta \in C^5[0,\infty],\ \eta'(t) \le 0, \quad \text{and}$$

(3.14)

$$\eta(t) = 1 \quad \text{if} \quad 0 \le t \le \frac{1}{2}\,\theta_0 \ , \quad \eta(t) = 0 \quad \text{if} \quad t \ge \frac{3}{4}\,\theta_0 \ .$$

We shall be working with positive functions $f(x)$ which approximate $A(-x)^\alpha$ for x near 0 and $B(-x)^\beta$ for x near $-\infty$, where A, B are positive constants and $\alpha > 1$ and $0 < \beta < 1$. Therefore, for any given $\theta_0 > 0$, the free boundary $\{(x, \varepsilon f(x)), -\infty < x < 0\}$ is contained in $\Sigma^-_{\theta_0/4}$ if ε is small enough.

We introduce the function

(3.15)

$$\overline{\psi}(x,y) = \psi\left(x, y + \varepsilon f(x)\eta\left(\frac{y}{|x|}\right)\right) \quad \text{if} \quad x < 0$$

$$= \psi(x,y) \quad \text{if} \quad x \ge 0$$

and note that $\eta = 1$ on the free boundary, so that the free boundary for ψ is mapped onto the negative real axis for $\overline{\psi}$.

We also note that

$$\overline{\psi}(x,y) = \psi(x,y) \quad \text{if} \quad (x,y) \notin \Sigma_{\theta_0}^- .$$

We now wish to rewrite the system (3.9)–(3.13) in terms of $\overline{\psi}$. Take for example equation (3.11). Using the relations

$$\text{(3.16)} \qquad \begin{aligned} \partial_x \overline{\psi} &= \partial_x \psi + \varepsilon \partial_y \psi \cdot \left[f' \eta + \frac{yf}{|x|^2} \, \eta' \right] , \\ \partial_y \overline{\psi} &= \partial_y \psi + \varepsilon \partial_y \psi \cdot \frac{f}{(-x)} \, \eta' \end{aligned}$$

we can rewrite (3.11) in the form

$$\text{(3.17)} \qquad U f'(x) - \overline{\psi}_x = \varepsilon \left\{ \partial_y \psi \cdot \left[f' \eta + \frac{yf}{|x|^2} \, \eta' \right] + f' \partial_y \psi \right\} .$$

We can use (3.16) to rewrite the expression in braces in (3.17) in terms of $\overline{\psi}_x$ and $\overline{\psi}_y$: it becomes a linear combination (with bounded coefficients) of products of f' or f/x by $\overline{\psi}_x$ or $\overline{\psi}_y$. The same calculation can be carried out for the other equations, using the relations

$$\partial^k \overline{\psi} = [\partial - (\partial \eta) \partial_y]^k \psi .$$

System (3.9)–(3.13) then takes the form

$$\text{(3.18)} \qquad \Delta^2 \overline{\psi} = \varepsilon H_4[\overline{\psi}, f] \quad \text{in} \quad \{y > 0\} ,$$

$$\text{(3.19)} \qquad \overline{\psi}_x(x,0) = \overline{\psi}_y(x,0) = 0 \quad \text{if} \quad x > 0 ,$$

$$\text{(3.20)} \qquad U f'(x) - \overline{\psi}_x = \varepsilon H_1[\overline{\psi}, f] \quad \text{at} \quad (x,0), \; x < 0 ,$$

$$\text{(3.21)} \qquad \overline{\psi}_{yy} - \overline{\psi}_{xx} = \varepsilon H_2[\overline{\psi}, f] \quad \text{at} \quad (x,0), \; x < 0 ,$$

$$\text{(3.22)} \qquad \lambda(2\overline{\psi}_{xxy} + \overline{\psi}_{yyy}) - \overline{\psi}_{xxx} = \varepsilon H_3[\overline{\psi}, f] \quad \text{at} \quad (x,0), \; x < 0$$

where $H_j[\overline{\psi}, f]$ is a linear combination (with bounded coefficients) of products of derivatives of f of order $\leq j$ by derivatives of $\overline{\psi}$ of order $\leq j$, and $H_4[\overline{\psi}, f] = 0$ outside $\Sigma_{\theta_0}^-$. The $H_j[\psi, f]$ are linear in the derivatives of $\overline{\psi}$.

The functions f and ψ will actually depend on ε and, to conform to the notation of Theorem 1.1, we set

$$\begin{aligned} \psi &= \psi_0(x,y) + \varepsilon\psi_1(x,y,\varepsilon) \ , \\ f &= f_0(x) + \varepsilon f_1(x,\varepsilon) \ . \end{aligned} \tag{3.23}$$

We can then write

$$\overline{\psi}(x,y) = \overline{\psi}_0(x,y,\varepsilon) + \varepsilon\overline{\psi}_1(x,y,\varepsilon) \tag{3.24}$$

where

$$\overline{\psi}_0(x,y,\varepsilon) = \psi_0\left(x, y + \varepsilon[f_0 + \varepsilon f_1(x,\varepsilon)]\eta\left(\frac{y}{|x|}\right)\right) \ , \tag{3.25}$$

$$\psi_1(x,y,\varepsilon) = \psi_1\left(x, y + \varepsilon[f_0(x) + \varepsilon f_1(x,\varepsilon)]\eta\left(\frac{y}{|x|}\right), \varepsilon\right) \ . \tag{3.26}$$

Note that $\overline{\psi}_1$ coincides in $\Sigma^-_{\theta_0/2}$ with the function $\widetilde{\psi}_1$ (defined in Theorem 1.1).

Since (f_0, ψ_0) is a solution to (2.7)–(2.11), we can rewrite (3.10)–(3.22) as a system of equations for $(f_1, \overline{\psi}_1)$ as follows:

$$\Delta^2\overline{\psi} = \varepsilon H_4^1[\overline{\psi}_0, f_0 + \varepsilon f_1] + \varepsilon^2 H_4^2[\overline{\psi}_1, f_0 + \varepsilon f_1] \quad \text{in} \quad \{y > 0\} \ , \tag{3.27}$$

$$\overline{\psi}_{1,x}(x_1 0) = \overline{\psi}_{1,y}(x,0) = 0 \quad \text{if} \quad x > 0 \ , \tag{3.28}$$

$$\overline{\psi}_{1,yy} - \overline{\psi}_{1,xx} = H_2^1[\overline{\psi}_0, f_0 + \varepsilon f_1] + \varepsilon H_2^2[\overline{\psi}_1, f_0 + \varepsilon f_1] \quad \text{at} \quad (x,0), x < 0 \ , \tag{3.29}$$

$$\lambda(2\overline{\psi}_{1,xxy} + \overline{\psi}_{1,yyy}) - \overline{\psi}_{1,xxx} = H_3^1[\overline{\psi}_0, f_0 + \varepsilon f_1] + \varepsilon H_3^2[\overline{\psi}_1, f_0 + \varepsilon f_1] \quad \text{at} \quad (x,0), x < 0 \tag{3.30}$$

and

$$Uf_1' - \overline{\psi}_{1,x} = H_1^1[\overline{\psi}_0, f_0 + \varepsilon f_1] + \varepsilon H_1^2[\overline{\psi}_1, f_0 + \varepsilon f_1] \quad \text{at} \quad (x,0), x < 0 \ . \tag{3.31}$$

Actually H_4^2, H_2^2 and H_3^2 depend also on $\overline{\psi}_0$ and ε.

LEMMA 3.1. *If*

$$\|f_0\|^{\alpha,\beta}_{\Sigma_0^-,4+\gamma} \le C_0\ , \quad \|\psi_0\|^{\alpha,\beta}_{\Sigma_{\theta_0}^-,4+\gamma} \le C_0 \qquad (C_0 > 1) \tag{3.32}$$

and

$$\|f_1\|^{\alpha,\beta}_{\Sigma_0^-,4+\gamma} \le C^*\ ,\ \|\overline{\psi}_1\|^{\alpha,\beta}_{\Sigma_{\theta_0}^-,4+\gamma} \le C^* \qquad (C^* > C_0)\ , \tag{3.33}$$

then the functions

$$H^1_m = H^1_m[\overline{\psi}_0, f_0 + \varepsilon f_1]\ ,\ H^2_m = H^2_m[\overline{\psi}_1, f_0 + \varepsilon f_1]$$

satisfy:

$$\|H^1_4\|^{2\alpha-5,2\beta-5}_{\Sigma_{\theta_0}^-,\gamma} \le \widetilde{C}(C_0 + \varepsilon C^*),\ \|H^2_4\|^{2\alpha-5,2\beta-5}_{\Sigma_{\theta_0}^-,\gamma} \le \widetilde{C}(C^*)\ , \tag{3.34}$$

$$\|H^1_i\|^{2\alpha-1-i,2\beta-1-i}_{\Sigma_0^-,4-i+\gamma} \le \widetilde{C}\,(C_0 + \varepsilon C^*),\ \|H^2_i\|^{2\alpha-1-i,2\beta-1-i}_{\Sigma_0^-,4-i+\gamma} \le \widetilde{C}(C^*) \quad (0 \le i \le 3) \tag{3.35}$$

where $\widetilde{C}(t)$ is a positive monotone increasing function of t.

Proof. consider the relation (3.31). It is obtained from (3.17) by using (3.23). H^1_1 contains the terms

$$\partial_y\overline{\psi}_0\left(f_0'\eta + \frac{yf_0}{x^2}\,\eta'\right) + f_0'\partial_y\overline{\psi}_0\ . \tag{3.36}$$

Using (3.32), (3.33), the estimate (3.35) for $i = 1,\ j = 1$ readily follows. The proof for $i = 1,\ j = 2$ is similar; H^2_1 contains products of f_0' or f_1' by first derivatives of $\overline{\psi}_0$ or $\overline{\psi}_1$.

The proof of the other estimates in (3.35) and (3.34) can also be established by similar straightforward calculation.

We now outline the proof of Theorem 1.1.

Outline. Denote by $X(C^*)$ the set of pairs $f_1(x), \overline{\psi}_1(x,y)$ satisfying (3.33). For any such pair, consider the problem

$$\Delta^2\Psi = H^1_4[\overline{\psi}_0, f_0 + \varepsilon f_1] + \varepsilon H^2_4[\overline{\psi}_1, f_0 + \varepsilon f_1] \quad \text{in} \quad \{y > 0\}\ , \tag{3.37}$$

$$\Psi_x(x,0) = \Psi_y(x,0) = 0 \quad \text{if} \quad x > 0\ , \tag{3.38}$$

$$(3.39) \qquad \Psi_{yy} - \Psi_{xx} = H_2^1[\overline{\psi}_0, f_0 + \varepsilon f_1] + \varepsilon H_2^2[\overline{\psi}_1, f_0 + \varepsilon f_1] \quad \text{at} \quad (x,0),\ x < 0\ ,$$

$$(3.40) \quad \lambda(2\Psi_{xxy} + \Psi_{yyy}) - \Psi_{xxx} = H_3^1[\overline{\psi}_0, f_0 + \varepsilon f_1] + \varepsilon H_3^2[\overline{\psi}_1, f_0 + \varepsilon f_1] \quad \text{at} \quad (x,0),\ \ x < 0$$

and

$$(3.41) \qquad UF' - \Psi_x = H_1^1[\overline{\psi}_0, f_0 + \varepsilon f_1] + \varepsilon H_1^2[\overline{\psi}_1, f_0 + \varepsilon f_1] \quad \text{at} \quad (x,0),\ x < 0\ .$$

We shall construct (in §§6–8) a special solution $\Psi(x,y)$ of (3.37)–(3.40) and then define $F(x)$ by (3.41), and a mapping T by

$$T(f_1, \overline{\psi}_1) = (F, \Psi)$$

We shall prove that if C^* is chosen large enough then T maps $X(C^*)$ into itself and is a contraction; hence it has a fixed point which is a solution to problem (C_ε).

To construct Ψ we first establish in §§4,5 a useful representation for biharmonic functions in $\{y > 0\}$ satisfying mixed boundary conditions

$$(3.42) \qquad \begin{cases} \psi_{yy} - \psi_{xx} = g_1,\ \lambda(2\psi_{xxy} + \psi_{yyy}) - \psi_{xxx} = g_2 \quad \text{at} \quad (x,0), x < 0, \\ \psi_x(x,0) = \psi_y(x,0) = 0 \quad \text{if} \quad x > 0\ . \end{cases}$$

Consider

$$(3.43) \qquad \Delta^2 \psi = H \quad \text{in} \quad \{y > 0\}$$

together with (3.42), where H and g_1, g_2 satisfy estimates similar to those derived in Lemma 3.1 for the right-hand sides of (3.27), (3.29) and (3.30). In §6 we construct a special solution W of (3.43) such that

$$(3.44) \qquad \|W\|^{\alpha+\theta,\beta-\delta}_{\Sigma_{\theta_0'} \cup B_R^+, 4+\gamma} \leq \widetilde{C}(C^*)$$

for any $0 < \theta < \alpha - 1,\ 0 < \delta < \beta - 1$. In §7 we add to it a biharmonic function φ whose boundary data $\varphi_x(x,0)$, $\varphi_y(x,,0)$ $(-\infty < x < \infty)$ coincide with $-W_x(x,0), -W_y(x,0)$ respectively. In §8 we add to $W + \varphi$ another biharmonic function $\widehat{\varphi}$ so as to obtain a function $\Psi = W + \varphi + \widehat{\varphi}$ satisfying (3.43) and (3.42), as well as an estimate such as (3.44). The function $\widehat{\varphi}$ consists of the special solution constructed in §5 plus an appropriate finite linear combination of eigenfunctions of the linearized problem.

In §9 we apply the above construction to the special case of (3.37)–(3.40) in order to show that the mapping T, which is well defined by the above construction, maps $X(C^*)$ into itself and is a contraction.

We finally prove, also in §9, that the class of functions f_0, with (f_0, ψ_0) as in Theorem 1.1, is dense in a natural weighted norm.

§4. **Mixed boundary conditions.** Consider the problem

$$\Delta^2\varphi_1 = 0 \quad \text{in} \quad \{y > 0\} \ , \tag{4.1}$$

$$\varphi_{1,x}(x,0) = \delta(x), \quad \varphi_{1,y}(x,0) = 0 \quad \text{for} \quad x > 0 \tag{4.2}$$

where $\delta(x)$ is the Dirac function. Using the Fourier transform in x one can easily derive the following formulas (which are special cases of results obtained in [1]).

LEMMA 4.1. *There exists a solution φ_1 of (4.1), (4.2), which is given by*

$$D_x\varphi_1(x,y) = \frac{2}{\pi}\,\frac{y^3}{(x^2+y^2)^2} \ , \tag{4.3}$$

$$D_y\varphi_1(x,y) = -\frac{2}{\pi}\,\frac{xy^2}{(x^2+y^2)^2} \ . \tag{4.4}$$

Consider next the problem

$$\Delta^2\varphi_2 = 0 \quad \text{in} \quad y > 0 \ , \tag{4.5}$$

$$\varphi_{2,x}(x,0) = 0, \quad \varphi_{2,y}(x,0) = \delta(x) \quad \text{for} \quad y = 0 \ . \tag{4.6}$$

LEMMA 4.2. *There exists a solution φ_2 of (4.5), (4.6), given by*

$$D_x\varphi_2(x,y) = -\frac{2}{\pi}\,\frac{xy^2}{(x^2+y^2)^2} \ , \tag{4.7}$$

$$D_y\varphi_2(x,y) = \frac{2}{\pi}\,\frac{x^2y}{(x^2+y^2)^2} \ . \tag{4.8}$$

In this and in the next section we are concerned with the mixed boundary value problem:

$$\Delta^2\psi = 0 \quad \text{in} \quad \{y > 0\} \ , \tag{4.9}$$

$$\psi_x(x,0) = \psi_y(x,0) = 0 \quad \text{if} \quad x > 0 \ , \tag{4.10}$$

$$(\psi_{yy} - \psi_{xx})(x,0) = g_1(x) \quad \text{if} \quad x < 0 \ , \tag{4.11}$$

$$(4.12) \qquad \lambda(2\psi_{xxy} + \psi_{yyy})(x,0) - \psi_{xxx}(x,0) = g_2(x) \quad \text{if} \quad x < 0 \ .$$

Observe that the boundary operators change abruptly from $x > 0$ to $x < 0$. Problems of this type have been considered in earlier papers (see [16] and the references therein) where L^p and C_α estimates were derived. However we shall need much sharper estimates. To derive such estimates (in §8) we shall establish an integral representation for a solution of (4.9)–(4.12). Our approach is motivated by [16;§3].

We first write the boundary conditions in two different forms:

$$(4.13) \qquad \psi_x(x,0) = h_1(x), \ \psi_y(x,0) = h_2(x) \quad (x \in \mathbb{R}) \ ,$$

and

$$(4.14) \qquad \begin{cases} (\psi_{yy} - \psi_{xx})(x,0) = g_1(x) \quad (x \in \mathbb{R}) \ , \\ \lambda(2\psi_{xxy} + \psi_{yyy})(x,0) - \psi_{xxx}(x,0) = g_2(x) \quad (x \in \mathbb{R}) \ , \end{cases}$$

and then try to express the h_i in terms of the g_i. This step will be carried out in the present section. In §5 we shall use the Green function constructed in Lemmas 4.1, 4.2 in order to represent the solution of (4.9)–(4.12) in terms of the functions $g_1(x), g_2(x)$ (defined for $x < 0$, only).

The subsequent calculations are somewhat formal. They can be justified however without my difficulty if h_i and g_i have compact support on decay fast enough at infinity. These conditions on h_i, g_i are satisfied in the actual applications in this paper.

The Fourier transform $\widehat{\psi}(\xi, y)$ satisfies

$$[(i\xi)^2 + D_y^2]\widehat{\psi} = 0 \ .$$

We are interested only in solutions that do not grow exponentially as $y \to \infty$, and therefore take

$$(4.15) \qquad \widehat{\psi}(\xi, y) = A(\xi)e^{-|\xi|y} + B(\xi)ye^{-|\xi|y} \ .$$

We can rewrite (4.13), (4.14) in the form

$$(4.16) \qquad i\xi\widehat{\psi}_x(\xi,0) = \widehat{h}_1(\xi), \ \widehat{\psi}_y(\xi,0) = h_2(\xi)$$

and

$$(4.17) \qquad \begin{cases} (\widehat{\psi}_{yy} - (i\xi)^2\widehat{\psi})(\xi,0) = \widehat{g}_1(\xi) \ , \\ \lambda[2(i\xi)^2\widehat{\psi}_y + \widehat{\psi}_{yyy}](\xi,0) + (i\xi)^3\widehat{\psi}(\xi,0) = \widehat{g}_2(\xi) \ . \end{cases}$$

We want to simplify (4.17) by using (4.15). This can be done along the general procedure used in [1].

Set

$$\widetilde{B}_1(\xi,\tau) = \tau^2 - \xi^2 \ ,$$

$$\widetilde{B}_2(\xi,\tau) = \lambda(2\xi^2\tau + \tau^3) - \xi^3 \ .$$

Notice that

$$\widetilde{B}_1(i\xi, D_y) = -\widetilde{B}_1(\xi, -iD_y) \ ,$$

$$\widetilde{B}_2(i\xi, D_y) = -i\widetilde{B}_2(\xi, -iD_y)$$

and therefore (4.17) can be written in the form

$$\begin{aligned} & -\widetilde{B}_1(\xi,-iD_y)\widehat{\psi} = \widehat{g}_1(\xi) \ , \\ & -i\widetilde{B}_2(\xi,-iD_y)\widehat{\psi} = \widehat{g}_2(\xi) \ . \end{aligned} \tag{4.18}$$

Set

$$M_+(\xi,\tau) = (\tau - i|\xi|)^2 = (\tau^2 - \xi^2) - 2i|\xi|\tau \ .$$

Dividing $\widetilde{B}_i(\xi,\tau)$, as a polynomial in τ, by M_+, we get

$$\widetilde{B}_1(\xi,\tau) = M_+(\xi,\tau) + \widetilde{B}'_1(\xi,\tau), \quad \widetilde{B}'_1(\xi,\tau) = 2i|\xi|\tau \ ,$$

$$\widetilde{B}_2(\xi,\tau) = M_+(\xi,\tau)(\lambda\tau + 2\lambda i|\xi|) + \widetilde{B}'_2(\xi,\tau) \ ,$$

$$\widetilde{B}'_2(\xi,\tau) = -\lambda\xi^2\tau + \xi^3(2\lambda i \operatorname{sgn}\xi - 1) \ .$$

Since, by (4.15), $M_+(\xi,-iD_y)\widehat{\psi} = 0$, the boundary conditions in (4.18) simplify to

$$-\widetilde{B}'_1(\xi,-iD_y)\widehat{\psi} = \widehat{g}_1(\xi) \ ,$$

$$-i\widetilde{B}'_2(\xi,-iD_y)\widehat{\psi} = \widehat{g}_2(\xi)$$

and then, by (4.16),

$$\begin{aligned} & \widehat{g}_1(\xi) = 2(i\xi)(i \operatorname{sgn}\xi)\widehat{h}_2(\xi) \ , \\ & \widehat{g}_2(\xi) = -\lambda(i\xi)^2\widehat{h}_2(\xi) + (i\xi)^2[2\lambda i \operatorname{sgn}\xi - 1]\widehat{h}_1(\xi) \ . \end{aligned} \tag{4.19}$$

We now introduce the Hilbert transform

$$H\varphi(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\varphi(y)}{x-y}\, dy$$

where the integral is taken in the sense of principal value $(P.V.)$. Then

$$\widehat{H\varphi}(\xi) = -(i \operatorname{sgn} \xi)\widehat{\varphi}(\xi) \ .$$

Consequently (4.19) may be written in the form

$$-\frac{2}{\pi} \int_{-\infty}^{\infty} \frac{Dh_2(\xi)}{x-\xi}\, d\xi = g_1(x) \ , \tag{4.20}$$

$$D_x^2 h_1(x) + \frac{2\lambda}{\pi} \int_{-\infty}^{\infty} \frac{D_x^2 h_1(\xi)}{x-\xi}\, d\xi = -\lambda D_x^2 h_2(x) - g_2(x) \ . \tag{4.21}$$

We now assume that

$$h_1(x) = h_2(x) = 0 \quad \text{if} \quad x > 0 \ . \tag{4.22}$$

Then

$$-\frac{2}{\pi} \int_{-\infty}^{0} \frac{D_x h_2(\xi)}{x-\xi}\, d\xi = g_1(x) \ , \tag{4.23}$$

$$D_x^2 h_1(x) + \frac{2\lambda}{\pi} \int_{-\infty}^{0} \frac{D_x^2 h_1(\xi)}{x-\xi}\, d\xi = -\lambda D_x^2 h_2(x) - g_2(x) \ . \tag{4.24}$$

To invert the operators on the left-hand sides we shall use the following well known relations (see [7; pp. 159–160 and 187–190]):

$$\begin{gathered} \text{if} \quad \frac{1}{\pi} \int_{0}^{\infty} \frac{\varphi(y)}{x-y}\, dy = f(x), \quad f \in L^2(0,\infty) \quad \text{then} \\ \varphi(x) = \frac{C}{\sqrt{x}} - \frac{1}{2\pi} \int_{0}^{\infty} \frac{x+y}{x-y} \, \frac{f(y)}{\sqrt{yx}}\, dy \end{gathered} \tag{4.25}$$

where any constant C is possible;

(4.26)
$$\text{if} \quad \varphi(x) - \frac{2\lambda}{\pi} \int_0^\infty \frac{\varphi(y)}{x-y}\, dy = f(x) \ , \ f \in L^2(0,\infty) \quad \text{and}$$
$$\lambda^2 \notin (-\infty,-1) \quad \text{then} \quad \varphi(x) = \frac{f(x)}{1+4\lambda^2} + \frac{2\lambda x^{-\rho}}{(1+4\lambda^2)\pi} \int_0^\infty \frac{f(y)y^\rho}{x-y} dy$$

where ρ is the real number defined in (2.28) or (2.29).

Taking in (4.25) $y=-\xi$, $\varphi(x) = -2Dh_2(-x), f(x) = g_1(-x)$, we obtain from (4.23)

$$-2D_x h_2(x) = \frac{C}{\sqrt{-x}} - \frac{1}{2\pi} \int_{-\infty}^0 \frac{x+\xi}{x-\xi}\ \frac{g_1(\xi)}{\sqrt{-\xi}\ \sqrt{-x}}\ d\xi \ .$$

The last integral can be written as

$$\int_{-\infty}^0 \frac{x-\xi}{x-\xi}\ \frac{g_1(\xi)}{\sqrt{-\xi}\ \sqrt{-x}}\ d\xi + \int_{-\infty}^0 \left(\frac{2\xi}{x-\xi} - \frac{2\xi}{-\xi}\right) \frac{g_1(\xi)}{\sqrt{-\xi}\ \sqrt{-x}}\ d\xi + \int_{-\infty}^0 \frac{2\xi}{(-\xi)}\ \frac{g_1(\xi)}{\sqrt{-\xi}\ \sqrt{-x}}\ d\xi \ .$$

The first and last integrals are equal to const $/\sqrt{-x}$ and the middle integral is equal to

$$-\frac{1}{\sqrt{-x}} \int_{-\infty}^0 \frac{2xg_1(\xi)}{\sqrt{-\xi}\ (x-\xi)}\ d\xi \ .$$

Hence, for a suitable choice of C,

(4.27)
$$D_x h_2(x) = \frac{\sqrt{-x}}{2\pi} \int_{-\infty}^0 \frac{g_1(\xi)}{\sqrt{-\xi}\ (x-\xi)}\ d\xi \ .$$

Similarly, by using (4.26) we obtain from (4.24)

(4.28)
$$D_x^2 h_1(x) = -\frac{1}{1+4\lambda^2}\ [\lambda D_x^2 h_2(x) + g_2(x)]$$
$$+ \frac{2\lambda(-x)^{-\rho}}{(1+4\lambda^2)\pi} \int_{-\infty}^0 \frac{(-\xi)^\rho}{x-\xi} [\lambda D_x^2 h_2(\xi) + g_2(\xi)] d\xi \ .$$

Our goal in the next section is to establish integral representation for a solution of (4.9)–(4.12). This representation is given in Theorem 5.1. The proof is based on Theorem 4.3 below, whose proof however is valid only if

$$0 < \rho < \frac{1}{2} \; . \tag{4.29}$$

We shall first assume that (4.29) is satisfied and prove both Theorem 4.3 and part of Theorem 5.1. We shall then use analytic continuation to extend this part to $-\frac{1}{2} < \rho < 0$, and then proceed to complete the proof of Theorem 5.2 for all $-\frac{1}{2} < \rho < \frac{1}{2}$.

THEOREM 4.3. *Assume that (4.29) holds. Suppose ψ is a bounded solution of (4.9)–(4.12) and set $h_1(x) = \psi_x(x,0)$, $h_2(x) = \psi_y(x,0)$ for $x < 0$. If $h_1(0) = h_1'(0) = 0$ and $h_2(0) = 0$ then*

$$h_2(x) = \int_x^0 \frac{\sqrt{-\eta}}{2\pi}\, d\eta \int_{-\infty}^0 \frac{g_1(\xi)}{\sqrt{-\xi}\,(\eta - \xi)}\, d\xi \; , \tag{4.30}$$

$$\begin{aligned} h_1(x) = {} & \frac{1}{1+4\lambda^2} \int_x^0 d\eta \int_{-\infty}^{\eta} [\lambda D_x^2 h_2(q) + g_2(q)] dq \\ & - \frac{2\lambda}{(1+4\lambda^2)\pi} \int_x^0 d\eta \int_{-\infty}^{\eta} (-q)^{-\rho} dq \int_{-\infty}^0 \frac{(-\xi)^{\rho}}{q-\xi} [\lambda D_x^2 h_2(\xi) + g_2(\xi)] d\xi \; . \end{aligned} \tag{4.31}$$

Proof. Formula (4.30) follows by integrating (4.27). To prove (4.31) we need to show (since $h_1'(0) = 0$) that the integral $\int_{-\infty}^0$ of the right-hand side of (4.28) vanishes. Equivalently, if we set

$$\varphi(x) = \frac{f(x)}{1+4\lambda^2} + \frac{2\lambda x^{-\rho}}{(1+4\lambda^2)\pi} \int_0^{\infty} \frac{f(y) y^{\rho}}{x-y}\, dy$$

where $f(x) = (\lambda D_x^2 h_2 + g_2)(x)$, then we need to show that

$$\int_0^{\infty} \varphi(x) dx = 0 \; . \tag{4.32}$$

We have

$$(4.33)\qquad \begin{aligned}(1+4\lambda^2)\int_0^\infty \varphi(x)dx &= \int_0^\infty f(x)dx + \frac{2\lambda}{\pi}\int_0^\infty x^{-\rho}\int_0^\infty \frac{f(y)y^\rho}{x-y}\,dy\\ &= \int_0^\infty f(x)dx + \frac{2\lambda}{\pi}\int_0^\infty f(y)dy\int_0^\infty \frac{x^{-\rho}y^\rho}{x-y}\,dx\end{aligned}$$

provided we may use Fubini's theorem for the *P.V.* integrals; this will be justified later on.

Since $0<\rho<\frac{1}{2}$ we have (setting $x=uy$)

$$(4.34)\qquad \int_0^\infty \frac{x^{-\rho}y^\rho}{x-y}\,dx = \int_0^\infty \frac{u^{-\rho}}{u-1}\,du\ .$$

Using contour deformation and the residue theorem we also get

$$(1-e^{-2\pi\rho i})\int_0^\infty \frac{u^{-\rho}}{u-1}\,du = \pi i(1+e^{-2\pi\rho i})\ ,$$

or

$$\int_0^\infty \frac{u^{-\rho}}{u-1}\,du = -\frac{\pi}{2\lambda}\ .$$

Thus the left-hand side of (4.34) is equal to $-\frac{\pi}{2\lambda}$ and, therefore, the right-hand side of (4.33) vanishes; this completes the proof of (4.32).

It remains to justify the change of order of integration in (4.33). Setting $h(x,y) = f(y)x^{-\rho}y^\rho$ we have

$$h(x,y) = O\left(\frac{1}{x^\rho}\right)O\left(\frac{1}{y^{(3/2)-\rho}}\right) \quad \text{as} \quad x\to\infty,\ y\to\infty\ .$$

We may write

$$(4.35)\qquad \begin{aligned}&\int_0^\infty dx\int_0^\infty \frac{h(x,y)}{x-y}\,dy\\ &= \int_0^\infty dx\left\{\int_{|y-x|\ge\varepsilon}\frac{h(x,y)}{x-y}dy + \lim_{\delta\to 0}\int_{\delta\le|y-x|<\varepsilon}\frac{h(x,y)}{x-y}dy\right\}\\ &= \int_0^\infty dy\int_{|x-y|\ge\varepsilon}\frac{h(x,y)}{x-y}dx + \int_0^\infty dx\left[\lim_{\delta\to 0}\int_{\delta\le|y-x|<\varepsilon}\frac{h(x,y)}{x-y}dy\right].\end{aligned}$$

From the form of h and its smoothness,

$$\int\limits_{\delta\le|y-x|\le\varepsilon} \frac{h(x,y)}{x-y}\,dy = \int\limits_{\delta\le|y-x|\le\varepsilon} \frac{h(x,y)-h(x,x)}{x-y}\,dy = \sigma(\varepsilon,x)$$

where $\sigma(\varepsilon,x) = O(\varepsilon)$ and, in fact,

$$\int |\sigma(\varepsilon,x)|dx = O(\varepsilon)\ , \quad \text{uniformly in} \quad \delta\ .$$

Hence the last term in (4.35) is $O(\varepsilon)$. Similarly the first term on the right-hand side of (4.35) differs from

$$\int_0^\infty dy \int_0^\infty \frac{h(x,y)}{x-y}\,dx$$

by $O(\varepsilon)$. Taking $\varepsilon \to 0$ in (4.35), the change of order of integration in (4.33) is thereby justified.

Remark 4.1. From (4.30), (4.31) we see that

$$h_1(x) \approx |x|^{1-\rho},\ h_2(x) \approx |x|^{1/2} \quad \text{as} \quad x \to -\infty\ .$$

§5. Representation of the solution of (4.9)–(4.12). Set

$$\Phi(q) = \frac{1}{1+4\lambda^2}\,[\lambda D_x^2 h_2(q) + g_2(q)]\ . \tag{5.1}$$

Then by (4.31).

$$h_1(x) = \int_x^0 dq_1 \int_{-\infty}^{q_1} \Phi(q_2)dq_2 - \frac{2\lambda}{\pi}\int_x^0 dq_1 \int_{-\infty}^{q_1} (-q_2)^{-\rho}dq_2 \int_{-\infty}^0 \frac{(-\xi)^\rho}{q_2-\xi}\,\Phi(\xi)d\xi\ . \tag{5.2}$$

Introduce the contour C and a small circle γ about q_2, as shown in Figure 2.

FIGURE 2

We shall temporarily assume that

$$\text{(5.3)} \qquad \begin{array}{l} \Phi \text{ is holomorphic in a domain } D \text{ containing} \\ C \text{ and the negative } x\text{-axis, and } \Phi(q) \to 0 \\ \text{if } q \in D,\ |q| \to \infty\ . \end{array}$$

Then, by Cauchy's theorem,

$$\text{(5.4)} \qquad \begin{aligned} \oint\limits_C \frac{(-q_2)^{-\rho}(-\xi)^\rho}{q_2 - \xi} \Phi(\xi) d\xi &= P.V. \int\limits_{-\infty}^{0} (e^{2\pi\rho i} - 1) \frac{(-q_2)^{-\rho}(-\xi)^\rho}{q_2 - \xi}\ \Phi(\xi) d\xi \\ &+ \oint\limits_\gamma \frac{(-q_2)^{-\rho}(-\xi)^\rho}{q_2 - \xi}\ \Phi(\xi) d\xi \end{aligned}$$

where we have taken the branch of $(-\xi)^\rho \quad (\xi = x + iy)$ for which

$$\arg(-\xi)^\rho = 2\pi\rho \quad \text{on} \quad \{x < 0,\ y = 0+\}\ ,$$

$$\arg(-\xi)^\rho = 0 \qquad \text{on} \quad \{x < 0,\ y = 0-\}\ .$$

When γ shrinks to q_2 the integral over γ converges to

$$\pi i(e^{2\pi\rho i}+1)\Phi(\xi)\ .$$

Substituting this into (5.4) and using (2.28), we get

$$\frac{2\lambda}{\pi}(e^{2\pi\rho i}-1)^{-1}\oint_C \frac{(-q_2)^{-\rho}(-\xi)^{\rho}}{q_2-\xi} = \frac{2\lambda}{\pi}\ P.V.\int_{-\infty}^{0}\frac{(-q_2)^{-\rho}(-\xi)^{\rho}}{q_2-\xi}\Phi(\xi)d\xi - \Phi(q_2).$$

Substituting this into (5.2) we obtain a simpler expression for h_1:

$$h_1(x) = -\frac{2\lambda}{\pi}(e^{2\pi\rho i}-1)^{-1}\int_x^0 dq_1\int_{-\infty}^{q_1} dq_2\oint_C \frac{(-q_2)^{-\rho}(-\xi)^{\rho}}{q_2-\xi}\Phi(\xi)d\xi\ . \tag{5.5}$$

We shall now use the Green functions φ_1,φ_2 introduced in Lemmas 4.1, 4.2 in order to represent the solution ψ of (4.9)–(4.13) in terms of h_1, h_2 as defined by (5.5) (4.30) for $x<0$ (recall (5.1)). We write

$$\psi = \psi_1+\psi_2 \tag{5.6}$$

where $\Delta^2\psi_1=\Delta^2\psi_2=0$ and

$$D_x\psi_1 = h_1\ ,\ D_y\psi_1 = 0 \quad \text{at} \quad y=0\ ,$$

$$D_x\psi_2 = 0\ ,\ D_y\psi_2 = h_2 \quad \text{at} \quad y=0\ .$$

By Lemma 4.1

$$D_x\psi_1(x,y) = \int_{-\infty}^{0} K_1(x-\gamma,y)h_1(\gamma)d\gamma\ , \tag{5.7}$$

$$D_y\psi_1(x,y) = \int_{-\infty}^{0} K_2(x-\gamma,y)h_2(\gamma)d\gamma \tag{5.8}$$

where

$$\begin{cases} K_1(x,y) = \dfrac{2}{\pi}\ \dfrac{y^3}{(x^2+y^2)^2}\ , \\[2ex] K_2(x,y) = -\dfrac{2}{\pi}\ \dfrac{xy^2}{(x^2+y^2)^2}\ . \end{cases} \tag{5.9}$$

In the sequel we let K denote either K_1 or K_2; $K(x-\gamma,y)$ is a meromorphic function in γ with poles at $\gamma = x \pm iy$.

We wish to evaluate

$$
\begin{aligned}
I &\equiv \int_{-\infty}^{\infty} K(x-\gamma,y)h_1(\gamma)d\gamma \\
&= -\frac{2\lambda}{\pi}(e^{2\pi\rho i}-1)^{-1}\int_{-\infty}^{0} d\gamma K(x-\gamma,y)\int_{\gamma}^{0} dq_1 \int_{-\infty}^{0} dq_2 \oint_C \frac{(-q_2)^{-\rho}(-\xi)^{\rho}}{q_2-\xi}\Phi(\xi)d\xi.
\end{aligned}
\tag{5.10}
$$

We shall use the following relations, valid for any L^1 function g,

$$
\begin{aligned}
&\int_{\gamma}^{0} dq_1 \int_{-\infty}^{q_1} g(q_2)dq_2 = \int_{\gamma}^{0} dq_1 \int_{-\infty}^{\gamma} g dq_2 + \int_{\gamma}^{0} dq_1 \int_{\gamma}^{q_1} dq_2 \\
&= \int_{-\infty}^{\gamma} dq_2 \int_{\gamma}^{0} g dq_1 + \int_{\gamma}^{0} dq_2 \int_{q_2}^{0} g dq_1 = -\gamma \int_{-\infty}^{\gamma} g(q_2)dq_2 - \int_{\gamma}^{0} q_2 g(q_2)dq_2 \ .
\end{aligned}
$$

Using this in (5.10) yields

$$
I = \frac{2\lambda}{\pi}(e^{2\pi\rho i}-1)^{-1}\int_{-\infty}^{0} d\gamma K(x-\gamma,y)\left[\gamma\int_{-\infty}^{\gamma} dq_2 + \int_{0}^{\gamma} q_2 dq_2\right]\oint_C \frac{(-q_2)^{-\rho}(-\xi)^{\rho}}{q_2-\xi}\Phi(\xi)d\xi
$$

and, by changing the order of integration,

$$
I = \frac{2\lambda}{\pi}(e^{2\pi\rho i}-1)^{-1}\int_{-\infty}^{0} d\gamma K(x-\gamma,y)\oint_C d\xi \Phi(\xi)(-\xi)^{\rho}H(\gamma,\xi)
\tag{5.11}
$$

where

$$
\begin{aligned}
&H(\gamma,\xi) = \gamma G(\gamma,\xi) + Q(\gamma,\xi) \ , \ G(\gamma,\xi) = \int_{-\infty}^{\gamma} \frac{(-q_2)^{-\rho}}{q_2-\xi}\, dq_2 \ , \\
&Q(\gamma,\xi) = \int_{\gamma}^{0} \frac{q_2(-q_2)^{-\rho}}{q_2-\xi}\, dq_2 \ .
\end{aligned}
\tag{5.12}
$$

We note that for fixed x, y, ξ,

$$|H(\gamma,\xi)| \leq C|\gamma|^{1-\rho} \ , \ |K(x-\gamma,y)| \leq \frac{C}{|\gamma|^3} \tag{5.13}$$

for $|\gamma|$ large. Using these estimates and (5.3), we may change the order of integration in (5.11):

$$I = \frac{2\lambda}{\pi}(e^{2\pi\rho i}-1)^{-1}\oint_C d\xi \Phi(\xi)(-\xi)^\rho \int_{-\infty}^{0} K(x-\gamma,y)H(\gamma,\xi)d\gamma \ . \tag{5.14}$$

We next introduce a contour δ as shown in Figure 3.

FIGURE 3

For ξ fixed the functions $G(\gamma) = G(\gamma,\xi)$ and $Q(\gamma) = Q(\gamma,\xi)$ are analytic in γ in $\mathbb{R}^2 \setminus \{(x,0), \ x<0\}$. If $Im \ \gamma > 0$, the integral defining $G(\gamma)$ can be taken as any contour (from $-\infty$ to γ) above the x-axis and, if $Im\gamma < 0$, as any contour below the x-axis; the same holds for $Q(\gamma)$. Also, since

$$\arg(-\xi_2 + i0)^\rho = 2\pi\rho \ , \ \arg(-\xi_2 - i0)^\rho = 0 \quad \text{for} \quad \xi_2 > 0 \ ,$$

the functions $G(\gamma), Q(\gamma)$ in (5.12) satisfy

$$G(\gamma + i0) = e^{-2\pi\rho i}G(\gamma - i0), \ Q(\gamma + i0) = e^{-2\pi\rho i} \ Q(\gamma - i0)$$

for $-\infty < \gamma < 0$. It follows that

$$\oint_\delta K(x-\gamma,y)H(\gamma)d\gamma = \int_{-\infty-i0}^{-i0} K(x-\gamma,y)H(\gamma-i0)d\gamma - \int_{-\infty+i0}^{+i0} K(x-\gamma,y)H(\gamma+i0)d\gamma$$

$$= (1-e^{-2\pi\rho i})\int_{-\infty}^{0} K(x-\gamma,y)H(\gamma-i0)d\gamma \ .$$

Consequently,

$$I = \frac{2\lambda}{\pi}(e^{2\pi\rho i}-1)^{-1}(1-e^{-2\pi\rho i})^{-1}\oint_\delta d\gamma K(x-\gamma,y)\oint_C d\xi \Phi(\xi)(-\xi)^\rho H(\gamma,\xi). \tag{5.15}$$

Consider the integral

$$J \equiv \oint_C d\xi \Phi(\xi)(-\xi)^\rho H(\gamma,\xi) = \oint_C d\xi \Phi(\xi)(-\xi)^\rho \Big\{ \gamma \int_{-\infty}^{\gamma} \frac{(-q_2)^{-\rho}}{q_2-\xi}\, dq_2$$

$$+ \int_\gamma^0 \frac{q_2(-q_2)^{-\rho}}{q_2-\xi} dq_2 \Big\}, \quad \gamma \in \delta$$

and deform C into a contour C' as shown in Figure 4.

FIGURE 4

Each time we cross a point q_2 (in δ) we get an additional term due to the pole of $\Phi(\xi)(-\xi)^\rho/(q_2-\xi)$ at $\xi = q_2$. After changing also the order of integration, we end up with

$$J = \gamma \int_{-\infty}^{\gamma} dq_2(-q_2)^{-\rho} \oint_{C'} \frac{d\xi \Phi(\xi)(-\xi)^\rho}{q_2-\xi} + \int_{\gamma}^{\infty} dq_2(-q_2)^{-\rho} q_2 \oint_{C'} \frac{d\xi \Phi(\xi)(-\xi)^\rho}{q_2-\xi}$$

$$+2\pi i\gamma \int_{-\infty}^{\gamma} \Phi(q_2)dq_2 + 2\pi i \int_{\gamma}^{0} q_2\Phi(q_2)dq_2 \ .$$

Substituting this into (5.15) and collapsing C' onto the negative real axis, we obtain

$$I = \frac{2\lambda}{\pi}(e^{2\pi\rho i}-1)^{-1}(1-e^{-2\pi\rho i})^{-1}\Bigg\{\oint_{\delta} K(x-\gamma,y)\left[2\pi i\gamma \int_{-\infty}^{\gamma} \Phi(q_2)dq_2 + \int_{\gamma}^{0} \Phi(q_2)q_2 dq_2\right]$$

$$+ (e^{2\pi\rho i}-1)\Bigg[\gamma \int_{-\infty}^{0} dq_2(-q_2)^{-\rho} \int_{-\infty}^{0} d\xi\Phi(\xi)\frac{(-\xi)^\rho}{q_2-\xi}$$

$$+ \int_{\gamma}^{0} dq_2(-q_2)^{-\rho}q_2 \int_{-\infty}^{0} d\xi\Phi(\xi)\frac{(-\xi)^\rho}{q_2-\xi}\Bigg]\Bigg\} \ .$$

Observe that $K(x-\gamma,y)$ as well as the expression in the first brackets are holomorphic in γ in a domain D_0 containing δ and the negative real axis (the denominator in K is $\neq 0$) and the product is bounded by $C/|\gamma|^{1+\mu}$ for some $\mu > 0$, as $|\gamma| \to \infty$, $\gamma \in D_0$. (here we use the assumption (5.3)). Hence the integral over δ is equal to zero. The result is that

$$I = \frac{2\lambda}{2}(1-e^{-2\pi\rho i})^{-1} \int_{-\infty}^{0} d\xi\Phi(\xi)(-\xi)^\rho \oint_{\delta} K(x-\gamma,y)H(\gamma,\xi)d\gamma \tag{5.16}$$

where H is defined in (5.12).

Finally we deform δ into two circles

$$\omega_1 : |\gamma-(x+iy)| = \varepsilon \ ,$$

$$\omega_2 : |\gamma-(x-iy)| = \varepsilon$$

both in the clockwise direction, so that

$$\oint_{\delta} K(x-y,\gamma)H(\gamma,\xi)d\gamma = \sum_{j=1}^{2} \oint_{\omega_j} K(x-y,y)H(\gamma,\xi)d\gamma \ . \tag{5.17}$$

We shall now use (5.16), (5.17) to evaluate I for K_1. Writing

$$K_1(x-\gamma,y)=-\frac{y}{2\pi}\left[\frac{1}{(\gamma-(x+iy))^2}-\frac{2}{(\gamma-(x+iy))(\gamma-(x-iy))}+\frac{1}{(\gamma-(x-iy))^2}\right]$$

and

$$H(\gamma,\xi)=H(x+iy,\xi)+H'(x+iy,\xi)(\gamma-(x+iy))+\cdots$$

where $H'(\gamma,\xi)=\partial H(\gamma,\xi)/\partial\gamma$ we find, by the residue theorem, that

$$\oint_{\omega_1} K_1(x-\gamma,y)H(\gamma,\xi)d\gamma=-y\left[iH'(x+iy,\xi)-\frac{H(x+iy,\xi)}{y}\right]$$

and similarly,

$$\oint_{\omega_2} K_1(x-\gamma,y)H(\gamma,\xi)d\gamma=-y\left[iH'(x-iy,\xi)+\frac{H(x-iy,\xi)}{y}\right] .$$

Substituting this into (5.17) and observing that $\gamma G'(\gamma,\xi)+Q'(\gamma,\xi)=0$ we get

(5.18)
$$\oint_{\delta} K_1(x-\gamma,y)H(\gamma,\xi)d\gamma=-[xG(x-iy,\xi)-xG(x+iy,\xi)+Q(x-iy,\xi)-Q(x+iy,\xi)] .$$

Substituting this into (5.16) and recalling (5.10), (5.7), we find that

(5.19)
$$D_x\psi_1(x,y)=-\frac{2\lambda}{\pi}(1-e^{-2\pi\rho i})^{-1}\int_{-\infty}^{0} d\xi\Phi(\xi)(-\xi)^\rho[xG(x-iy,\xi)-xG(x+iy,\xi)$$
$$+Q(x-iy,\xi)-Q(x+iy,\xi)] ,$$

where

(5.20)
$$G(x-iy,\gamma)-G(x+iy,\gamma)=\int_{x+iy}^{x-iy}\frac{(-q)^{-\rho}}{q-\xi}\,dq ,$$

(5.21)
$$Q(x-iy,\gamma)-Q(x+iy,\gamma)=-\int_{x+iy}^{x-iy}\frac{(-q)^{-\rho}}{q-\xi}\,dq .$$

Similarly, using

$$K_2(x-\gamma,y)=-\frac{1}{2\pi}\left[\frac{1}{(\gamma-(x+iy))^2}-\frac{2}{(\gamma-(x+iy))(\gamma-(x-iy))}+\frac{1}{(\gamma-(x-iy))^2}\right]$$
$$\cdot[(\gamma-(x+iy))+iy] ,$$

we calculate

$$\oint_{\omega_1} K_2(x-\gamma,y)H(\gamma,\xi)d\xi = yH'(x+iy,\xi)$$

and similarly

$$\oint_{\omega_2} K_2(x-\gamma,y)H(\gamma,\xi)d\xi = -yH'(x-iy,\xi)\ .$$

Substituting this into (5.16) and recalling (5.10), (5.8) and the relation $\gamma G'(\gamma,\xi)+Q'(\gamma,\xi) = 0$ we obtain

$$(5.22)\quad D_y\psi_1(x,y) = -\frac{2\lambda}{\pi}(1-e^{-2\pi\rho i})^{-1}\int_{-\infty}^{0} d\xi\Phi(\xi)(-\xi)^\rho y[G(x-iy,\xi)-G(x+iy,\xi)]\ .$$

So far we have established (5.19), (5.22) for $0<\rho<\frac{1}{2}$. To extend these results to $-\frac{1}{2}<\rho<0$, we use analytic continuation. For any $a>0$ the function a^ρ is an entire holomorphic function in ρ. From (4.27) we have

$$|D^2h_2(x)| \le C/|x|^{3/2} \qquad \text{for} \quad |x| \quad \text{large.}$$

We can therefore extend the definition of $D^2h_1(x)$ to $\rho\in D$ where $D=\left\{-\frac{1}{2}<\operatorname{Re}\rho<\frac{1}{2}\right\}$; for such values of ρ the integral in (4.28) is absolutely convergent, uniformly in ρ, thus defining a holomorphic function in ρ. Since, by assumption, $h_1(0)=h_1'(0)=0$, we clearly have

$$h_1(x,\rho)\equiv h_1(x)=\int_0^x dx'\int_0^{x'} D^2h_1(x'')dx''$$

and so $h_1(x,\rho)$ is also holomorphic in ρ, $\rho\in D$.

Clearly

$$|D_x^2h_1(x,\rho)| \le C|x|^{-1-\operatorname{Re}\{\rho\}}$$

for $|x|$ large, so that

$$(5.23)\qquad |h_1(x,\rho)| \le C|x|^{1-\operatorname{Re}\{\rho\}}\ .$$

Denote the right-hand side of (5.7) by $k(x,y,\rho)$, and the right-hand side of (5.19) by $D_x\psi_1(x,y,\rho)$. Since

$$|K_1(x-\gamma,y)| \le C/|\gamma|^4$$

if $|\gamma|$ is large, the function $k(x,y,\rho)$ is holomorphic in ρ, $\rho \in D$. The function $D_x\psi_1(x,y,\rho)$ can obviously also be extended as holomorphic function in $\rho \in D$. Since we have proved that

$$k(x,y,\rho) = D_x\psi_1(x,y,\rho)$$

if $0 < \rho < \frac{1}{2}$, this relation must hold also for $-\frac{1}{2} < \rho < 0$, by analytic continuation. Thus the function $D_x\psi_1$ defined in (5.7) can be represented in the form (5.19) for all $-\frac{1}{2} < \rho < \frac{1}{2}$.

Similarly we can establish for the function $D_y\psi_1$, defined by (5.8), the representation (5.22) for all $-\frac{1}{2} < \rho < \frac{1}{2}$.

So far we have established (5.19) and (5.22) under the restriction (5.3). In general what we know about Φ is that it is given by (5.1) and therefore

$$|\Phi(\xi)| \le \frac{C}{|\xi|^{3/2}} \quad \text{for} \quad |\xi| \quad \text{large.}$$

If we can approximate $\Phi(\xi)$ by a sequence of functions $\Phi_n(\xi)$, each satisfying (5.3), such that the integrals in (5.19), (5.22) and (5.7), (5.8) corresponding to Φ_n converge to the same integrals corresponding to Φ, then the proof of (5.19) and (5.22) will be completed for general Φ.

To carry out this approximation, we first approximate Φ by functions $\widetilde{\Phi}_n$ with compact support (using cut-off functions), and then proceed to approximate the $\widetilde{\Phi}_n$ by functions Φ_n satisfying (5.3). Suppose

$$\widetilde{\Phi}_n(\xi) = 0 \quad \text{if} \quad \xi < -b \; .$$

For any $\varepsilon > 0$ choose a polynomial $p(\xi)$ such that

$$|p - \widetilde{\Phi}_n| \le \varepsilon \quad \text{in} \quad [-b-1, 0] \; . \tag{5.24}$$

Introduce the function

$$f(\xi) = \frac{1}{2} \left\{ 1 + \tanh\left[A\left(\xi + b + \frac{1}{2}\right)\right]\right\} \; .$$

Then $|f| \le 1$ and

$$\text{if } A \to \infty \text{ then } f(\xi) \to \begin{cases} 1 & \text{if} \quad \xi + b > 0 \; , \\ 0 & \text{if} \quad \xi + b + 1 < 0 \; . \end{cases}$$

Hence, for A large,

$$|fp - \widetilde{\Phi}_n| < 2\varepsilon \quad \text{if} \quad -b < \xi < 0 \; .$$

If $-b-1<\xi<-b$ then $\widetilde{\Phi}_n=0$ and so, by (5.24),

$$|p|<\varepsilon\ ,\quad \text{so that also}\quad |fp|<\varepsilon\ .$$

Consequently

$$|fp-\widetilde{\Phi}_n|<\varepsilon\quad\text{if}\quad -b-1<\xi<-b\ .$$

Finally if A is chosen large enough, depending on p, then

$$|fp-\widetilde{\Phi}_n|=|fp|<\varepsilon\quad\text{if}\quad \xi<-b-1\ .$$

The function $\Phi_n=fp$ provides the desired approximation to $\widetilde{\Phi}_n$ and to Φ; it satisfies (5.3) and, in fact, it decreases exponentially fast at infinity.

We now turn to ψ_2. By Lemma 4.2 and the proof of (4.27)

$$D_x\psi_2(x,y)=-\frac{2}{\pi}\int_{-\infty}^{0}h_2(\xi)\frac{y^2(x-\xi)}{((x-\xi)^2+y^2)^2}\,d\xi\ , \tag{5.25}$$

$$D_y\psi_2(x,ky)=\frac{2}{\pi}\int_{-\infty}^{0}h_2(\xi)\frac{y(x-\xi)^2}{((x-\xi)^2+y^2)^2}\,d\xi \tag{5.26}$$

where

$$h_2(\gamma)=\frac{1}{2\pi}\int_{\gamma}^{0}\sqrt{-q}\,dq\int_{-\infty}^{0}\frac{g_1(\xi)d\xi}{\sqrt{-\xi}\,(q-\xi)}\ .$$

Introduce a contour C as above and assume first that $g_1(\xi)$ is holomorphic in a domain D containing C and the negative x-axis, and that it decreases fast to zero as $\xi\in D, |\xi|\to\infty$. Defining a branch of $\sqrt{-\xi}$ off the negative real axis such that $\arg\sqrt{-\xi_2+i0}=\pi$ $(\xi_2>0)$, $\arg\sqrt{-\xi_2-i0}=0$ $(\xi_2<0)$, we easily find by deforming contours that

$$\oint_C\frac{g_1(\xi)d\xi}{\sqrt{-\xi}\,(\gamma-\xi)}=-2P.V.\int_{-\infty}^{0}\frac{g_1(\xi)d\xi}{\sqrt{-\xi}\,(\gamma-\xi)}\ .$$

Hence

$$h_2(\gamma)=-\frac{1}{4\pi}\int_{\gamma}^{0}\sqrt{-q}\,dq\oint_C\frac{g_1(\xi)}{\sqrt{-\xi}}\,\frac{d\xi}{q-\xi}\ . \tag{5.27}$$

Set

$$K_3(x,y)=-\frac{2}{\pi}\,\frac{xy^2}{(x^2+y^2)^2}\ ,\ K_4(x,y)=\frac{2}{\pi}\,\frac{x^2y}{(x^2+y^2)^2}\ . \tag{5.28}$$

Then the right-hand sides of (5.25), (5.26) have the form

$$I = \int_{-\infty}^{0} d\gamma K(x-\gamma, y) \oint_C \frac{g_1(\xi)}{\sqrt{-\xi}}\, Z(\gamma, \xi) d\xi \tag{5.29}$$

where

$$Z(\gamma, \xi) = \frac{1}{4\pi} \int_0^{\gamma} \sqrt{-q}\, \frac{dq}{q-\xi} \tag{5.30}$$

and $K = K_3$ or $K = K_4$. Note that

$$Z(\gamma + 0i, \xi) = -Z(\gamma - i0, \xi) \quad \text{if} \quad \gamma < 0. \tag{5.31}$$

Next we introduce a contour δ as before. Then, by (5.31),

$$\oint_\delta d\gamma K(x-\gamma, y) Z(\gamma, \xi) = 2 \int_{-\infty - i0}^{0-i0} d\gamma K(x-\gamma, y) Z(\gamma - i0, \xi)$$

and thus

$$I = \frac{1}{2} \oint_C d\xi \frac{g_1(\xi)}{\sqrt{-\xi}} \oint_\delta d\gamma K(x-\gamma, y) Z(\gamma, \xi)\ . \tag{5.32}$$

We next deform C into C' as above. The term which arises as we cross the singularity at a point $\xi = q$ is

$$\oint_\delta d\gamma K(x-\gamma, y) \frac{1}{4\pi} \int_0^{\gamma} g_1(q) dq \equiv \oint_\delta d\gamma F(\gamma)$$

and this is equal to zero by Cauchy's theorem (since $F(\gamma)$ is analytic in D and converges to zero fast enough if $|\gamma| \to \infty$, $\gamma \in D$). We thus get

$$\begin{aligned} I &= \frac{1}{2} \oint_{C'} d\xi \frac{g_1(\xi)}{\sqrt{-\xi}} \oint_\delta d\gamma K(x-\gamma, y) Z(\gamma, \xi) \\ &= \int_{-\infty}^{0} d\xi \frac{g_1(\xi)}{\sqrt{-\xi}} \oint_\delta d\gamma K(x-\gamma, y) Z(\gamma, \xi) \end{aligned} \tag{5.33}$$

after collapsing C' onto the negative real axis.

Finally we may deform δ to obtain

$$\oint_{\delta} d\gamma K(x-\gamma,y)Z(\gamma,\xi) = \sum_{j=1}^{2} \oint_{\omega_j} d\gamma K(x-\gamma,y)Z(\gamma,\xi) \ . \tag{5.34}$$

Since the kernel K_3 coincides with the kernel K_2, we can proceed as in the case of K_2 to derive

$$\oint_{\delta} K_3(x-\gamma,y)Z(\gamma,\xi) = y[Z'(x-iy,\xi) - Z'(x+iy,\xi)]$$

where $Z'(\gamma,\xi) = \partial Z(\gamma,\xi)/\partial\gamma$. Hence

$$D_x\psi_2(x,y) = \int_{-\infty}^{0} d\xi \frac{g_1(\xi)}{\sqrt{-\xi}} \ y[Z'(x-iy,\xi) - Z'(x+iy,\xi)] \tag{5.35}$$

where

$$Z(\gamma,\xi) = \frac{1}{4\pi} \int_{0}^{\gamma} \sqrt{-q} \ \ \frac{dq}{q-\xi} \ , \ Z'(\gamma,\xi) = \frac{1}{4\pi} \ \frac{\sqrt{-\gamma}}{\gamma-\xi} \ . \tag{5.36}$$

Similarly, by the residue theorem,

$$\oint_{\omega_1} K_4(x-\gamma,y)Z(\gamma,\xi)d\gamma = -[Z(x+iy,\gamma) + iyZ'(x+iy,\gamma)]$$

and

$$\oint_{\omega_2} K_4(x-\gamma,y)Z(\gamma,\xi)d\gamma = [Z(x-iy,\gamma) - iyZ'(x-iy,\gamma)] \ .$$

It follows that

$$\begin{aligned} D_y\psi_2(x,y) = \int_{-\infty}^{0} d\xi \frac{g_1(\xi)}{\sqrt{-\xi}} \Big\{ & [Z(x-iy,\gamma) - Z(x+iy,\gamma)] \\ & - iy[Z'(x+iy) + Z'(x-iy)] \Big\} \ , \end{aligned} \tag{5.37}$$

and

$$Z(x-iy,\gamma) - Z(x+iy,\gamma) = \frac{1}{4\pi} \int_{x+iy}^{x-iy} \sqrt{-q} \ \frac{dq}{q-\xi} \ . \tag{5.38}$$

So far we have assumed that $g_1(\xi)$ is holomorphic in the domain D (D as in (5.3)) and $g_1(\xi) \to 0$ fast, if $|\xi| \to \infty, \xi \in D$. Since any g_1 with compact support can be approximated by $g_{1,n}$ analytic in D and decaying to zero exponentially fast as $|\xi| \to \infty$ (the proof is the same as for Φ above), we conclude that (5.35) and (5.37) are true for general g_1.

We summarize:

Theorem 5.1. *Given any continuous functions $g_1(x), g_2(x)$ with compact support in $(-\infty < x \leq 0)$, define functions $h_1(x), h_2(x)$ for $-\infty < x \leq 0$ by (4.27), (4.28) and $h_1(0) = h_1'(0) = 0$, $h_2(0) = 0$. Define also a function $\Phi(x)$, for $-\infty < x < 0$, by (5.1). Then a bounded solution ψ of (4.9)–(4.12) is given by $\psi = \psi_1 + \psi_2$ where ψ_1 and ψ_2 are determined by (5.19), (5.22) and (5.35), (5.37).*

This result is valid actually whenever $g_1(x), g_2(x)$ are decreasing sufficiently fast as $x \to -\infty$.

§6. Special solution of $\boldsymbol{\Delta^2\psi = H}$. Let $\alpha > \dfrac{3}{2}$, $0 < \beta < 1$ and let $H(x,y)$ be a function satisfying:

$$H = 0 \quad \text{if} \quad (x,y) \notin \Sigma^-_{\theta_0} \ , \tag{6.1}$$

$$\|H\|^{2\alpha-5, 2\beta-5}_{\Sigma^-_{\theta_0}, \gamma} \leq N \tag{6.2}$$

where $0 < \theta_0 < 1$, $0 < \gamma < 1$ and N is a positive constant. In this section we shall prove:

Theorem 6.1. *For any $0 < \theta < \alpha - 1, 0 < \delta < \min(\beta, 1-\beta)$, there exists a solution W to*

$$\Delta^2 W = H \quad \textit{in} \quad \{y > 0\} \tag{6.3}$$

such that

$$\|W\|^{\alpha+\theta, \beta-\delta}_{\Sigma_{\theta_0'} \cup B^+_R, 4+\gamma} \leq CN \tag{6.4}$$

for any $0 < \theta_0' < 1$ and $R > 0$, where C is a constant independent of N.

Introduce nonnegative C^∞ functions ζ_1, ζ_2 such that $\operatorname{supp} \zeta_1 \subset B_{4/3}, \operatorname{supp} \zeta_2 \subset \mathbb{R}^2 \backslash B_{5/4}$ and $0 \leq \zeta_1 \leq 1$ and $\zeta_1 + \zeta_2 \equiv 1$. We first construct a solution to

$$\Delta^2 \psi = \zeta_1 H \equiv H_1 \quad \text{in} \quad \{y > 0\} \ . \tag{6.5}$$

We introduce the fundamental solution

$$K|\xi|^2 \log|\xi| \qquad (K \quad \text{constant} \ , \quad \xi = (x,y))$$

of Δ^2 in $\mathbb{R}^2$ and consider the special solution of (6.5)

$$\varphi(\xi) = K \int\limits_{B_2} |\xi - q|^2 \log|\xi - q| H_1(q) dq \ . \tag{6.6}$$

In order to estimate this function we cover the support of H_1 by discs $B_j(x_j) = B_{\rho_i}(x_j, 0)$ where ρ_j and $|x_j|$ decrease exponentially to zero; for instance, if θ_0 is small enough (which may assume to be the case) we can take

$$(6.7) \qquad -x_j = |x_j| = \left(\frac{3}{4}\right)^j \ , \ \rho_j = \frac{1}{2}\left(\frac{3}{4}\right)^j \qquad (j = 0, 1, 2, \dots) \ .$$

Write

$$(6.8) \qquad \varphi = \sum_j \int_{B_{\rho_j}(x_j)} |\xi - q|^2 \log|\xi - q| h_j(q) dq \equiv \sum_j I_j \ .$$

By (6.2),

$$(6.9) \qquad \begin{aligned} &|h_j(q)| \le CN|x_j|^{2\alpha - 5} \ , \\ &|D^\gamma h_j(q)| \le CN|x_j|^{2\alpha - 5 - \gamma} \ . \end{aligned}$$

Substituting $q = \lambda|x_j|$ in I_j we get

$$(6.10) \qquad \begin{aligned} I_j = {} & |x_j|^4 \log|x_j| \int_{B_{1/2}(-1)} \left|\frac{\xi}{|x_j|} - \lambda\right|^2 h_j(\lambda|x_j|) d\lambda \\ & + |x_j|^4 \int_{B_{1/2}(-1)} \left|\frac{\xi}{|x_j|} - \lambda\right|^2 \log\left|\frac{\xi}{|x_j|} - \lambda\right| h_j(\lambda|x_j|) d\lambda \equiv I'_j + I''_j \ . \end{aligned}$$

The first integral can be written as a quadratic polynomial in ξ,

$$\log|x_j| \{a_j|\xi|^2 + b_j \cdot \xi + c_j\}$$

where, by (6.9),

$$|a_j| \le CN|x_j|^{2\alpha - 3} \ , \ |b_j| \le CN|x_j|^{2\alpha - 4} \ , \ |c_j| \le CN|x_j|^{2\alpha - 5} \ .$$

Summing over j we get a quadratic polynomial

$$(6.11) \qquad \sum_j I'_j = a|\xi|^2 + b \cdot \xi + c \ , \quad |a| + |b| + |c| \le CN \ .$$

Next we wish to estimate the I''_j. It will be important to observe that

$$(6.12) \qquad \text{if} \quad \lambda \in B_{1/2}(-1) \quad \text{then} \quad |\lambda| \ge \frac{1}{2} \ .$$

We apply Taylor's formula to

$$g(u) = |u-\lambda|^2 \log |u-\lambda| \qquad (\lambda \in B_{1/2}(-1))$$

for u near 0 and then substitute $u = \xi/|x_j|$. We get

$$\left| |\frac{\xi}{|x_j|} - \lambda|^2 \log |\frac{\xi}{|x_j|} - \lambda| - P_M\left(\frac{\xi}{|x_j|}, \lambda\right) \right| \leq C \left(\frac{\xi}{|x_j|}\right)^{\alpha+\theta} \tag{6.13}$$

where P_M is a polynomial in $\xi/|x_j|$ of degree M and M is such that

$$M < 2\alpha - 1 \leq M + 1 \; ; \tag{6.14}$$

here we can take any $\theta \leq \alpha - 1$. We shall actually need to choose θ such that

$$\theta < \alpha - 1 \; , \; \alpha + \theta > 2 \; , \; \alpha + \theta > M \; , \tag{6.15}$$

which is possible since $\alpha > 3/2$. Applying any k-th derivative to the Taylor expansion of $g(u)$ we also get

$$|\partial_\xi^k \left[|\frac{\xi}{|x_j|} - \lambda|^2 \log |\frac{\xi}{|x_j|} - \lambda| - P_M\left(\frac{\xi}{|x_j|}, \lambda\right) \right]| \leq \frac{C}{|x_j|^k} \left(\frac{\xi}{|x_j|}\right)^{\alpha+\theta-k} \tag{6.16}$$

provided $\xi/|x_j|$ is small. If $\xi/|x_j|$ is large (or just stays away from zero and λ) then (6.16) is still valid since $\alpha+\theta > 2, \alpha + \theta > M$. If $\xi/|x_j|$ is near λ then the left-hand side of (6.13) remains bounded and, by (6.12), it is bounded by the right-hand side for a suitably chosen constant C.

The same considerations apply to (6.16) with $k = 1$. If $k = 2$, however, then we must replace (6.16) by

$$\begin{aligned} |\partial_\xi^2 \left[|\frac{\xi}{|x_j|} - \lambda|^2 \log |\frac{\xi}{|x_j|} - \lambda| - P_M\left(\frac{\xi}{|x_j|}, \lambda\right) \right]| &\leq \frac{C}{|x_j|^2} \left(\frac{|\xi|}{|x_j|}\right)^{\alpha+\theta-2} \\ &+ \frac{C}{|x_j|^2} \log |\frac{\xi}{|x_j|} - \lambda| \end{aligned} \tag{6.17}$$

in order to bound the left-hand side also for $\xi/|x_j|$ near λ.

For $k = 3, 4$ we need to add still larger terms on the right-hand side of (6.16) in order to control the left-hand side for $\xi/|x_j|$ near λ:

$$\begin{aligned} |\partial_\xi^k \left[|\frac{\xi}{|x_j|} - \lambda|^2 \log |\frac{\xi}{|x_j|} - \lambda| - P_M\left(\frac{\xi}{|x_j|}, \lambda\right) \right]| &\leq \frac{C}{|x_j|^k} \left(\frac{|\xi|}{|x_j|}\right)^{\alpha+\theta-k} \\ + \frac{C}{|x_j|^k} \; \frac{1}{|\frac{\xi}{x_j} - \lambda|^{k-2}} \chi_{\left\{ |\frac{\xi}{|x_j|} - \lambda| < \frac{1}{4} \right\}} & \qquad (k = 3, 4) \; . \end{aligned} \tag{6.18}$$

This estimate, although good enough for our purposes in case $k = 3$, will not be sufficiently precise for $k = 4$. We note that any fourth order derivative ∂_u^4 of $|u|^2 \log |u|$ is a kernel $\Omega(u)$ of a singular integral. Using this fact we can write

(6.19)
$$\begin{aligned} |\partial_\xi^k \left[\left|\frac{\xi}{|x_j|} - \lambda\right|^2 \log \left|\frac{\xi}{|x_j|} - \lambda\right| - P_M\left(\frac{\xi}{|x_j|}, \lambda\right)\right] | \\ - \frac{1}{|x_j|^4} \Omega\left(\frac{\xi}{|x_j|} - \lambda\right) \chi_{\left\{\left|\frac{\xi}{|x_j|} - \lambda\right| < \frac{1}{4}\right\}} | \le \frac{C}{|x_j|^4} \left(\frac{|\xi|}{|x_j|}\right)^{\alpha+\theta-4} \end{aligned}.$$

We proceed to estimate the I_j''. Using (6.13) and (6.9) we find that

$$I_j'' = |x_j|^4 \int_{B_{1/2}(-1)} P_M\left(\frac{\xi}{|x_j|}, \lambda\right) h_j(\lambda |x_j|) d\lambda + R_j'' \tag{6.20}$$

where

$$|R_j''| \le CN|x_j|^4 \frac{|\xi|^{\alpha+\theta}}{|x_j|^{\alpha+\theta}} |x_j|^{2\alpha-5} = CN|x_j|^{\alpha-1-\theta} |\xi|^{\alpha+\theta}.$$

Since $\alpha - 1 - \theta > 0$, $\Sigma |R_j''| \le CN|\xi|^{\alpha+\theta}$. The first term on the right-hand side of (6.20) is a polynomial in ξ of degree M with coefficients bounded by

$$CN|x_j|^{2\alpha-1-M} \qquad \text{(using (6.9))}.$$

Since $2\alpha - 1 > M$, we conclude that

$$\left|\sum_j I_j'' - P_M(\xi)\right| \le CN|\xi|^{\alpha+\theta} \tag{6.21}$$

where $P_M(\xi)$ is a polynomial in ξ of degree M.

Similarly to (6.21) we can derive the estimate

$$|\partial_\xi \left[\sum_j I_j'' - P_M(\xi)\right]| \le CN|\xi|^{\alpha-1+\theta}. \tag{6.22}$$

To estimate $\partial_\xi^2 I_j''$ and $\partial_\xi^3 I_j''$ we need to use (6.17) and (6.18) (for $k = 3$) respectively. We shall carry out the details just for $\partial_\xi^3 I_j''$.

We have

$$\partial_\xi^3 I_j'' = \partial_\xi^3 \left[|x_j|^4 \int_{B_{1/2}(-1)} P_M\left(\frac{\xi}{|x_j|}, \lambda\right) h_j(\lambda|x_j|) d\lambda\right] + R_j''$$

where

$$|R_j''| \leq CN|x_j|^4 \frac{|\xi|^{\alpha-3+\theta}}{|x_j|^{\alpha-3+\theta}} \; \frac{1}{|x_j|^3} \; |x_j|^{2\alpha-5} + |\widetilde{R}_j|$$

and

$$|\widetilde{R}_j| \leq CN|x_j|^4 \left[\int\limits_{B_{1/2}(-1)} \frac{1}{|\frac{\xi}{|x_j|} - \lambda|} \chi_{\left\{\left|\frac{\xi}{|x_j|}-\lambda\right|<\frac{1}{4}\right\}} d\lambda \right] \frac{1}{|x_j|^3} \; |x_j|^{2\alpha-5} \; .$$

In the last integral the only possible points ξ for which the integrand does not vanish identically are such that $|\xi| \approx |x_j|$. Consequently

$$|\widetilde{R}_j| \leq C|x_j|^{2\alpha-4} \int\limits_{B_{1/2}(-1)} \frac{1}{|\frac{\xi}{x_j} - \lambda|} \chi_{\left\{\left|\frac{\xi}{|x_j|}-\lambda\right|<\frac{1}{4}\right\}} d\lambda \leq CN|x_j|^{2\alpha-4}$$

$$\leq CN|\xi|^{\alpha-3+\theta}|x_j|^{\alpha-1-\theta} \; .$$

Hence

$$|R_j''| \leq CN|\xi|^{\alpha-3+\theta}|x_j|^{\alpha-1-\theta}$$

and, since $\alpha - 1 > \theta$,

$$\left|\partial_\xi^3[\Sigma I_j'' - P_M(\xi)]\right| \leq CN|\xi|^{\alpha-3+\theta} \; . \tag{6.23}$$

Similarly

$$|\partial_\xi^2[\Sigma I_j'' - P_M(\xi)]| \leq CN|\xi|^{\alpha-2+\theta} \; . \tag{6.24}$$

We proceed to evaluate $\partial_\xi^4 I_j''$. By (6.19),

$$\partial_\xi^4 I_j'' = \partial_\xi^4 \left[|x_j|^4 \int\limits_{B_{1/2}(-1)} P_M\left(\frac{\xi}{|x_j|}, \lambda\right) h_j(\lambda|x_j|) d\lambda \right] + R_j'' + R_j^*$$

where

$$|R_j''| \leq CN|\xi|^{\alpha-4+\theta}|x_j|^{\alpha-1-\theta}$$

and

$$R_j^* = \int\limits_{B_{1/2}(-1)} \Omega\left(\frac{\xi}{|x_j|} - \lambda\right) h_j(\lambda|x_j|) \chi_{\left\{\left|\frac{\xi}{|x_j|}-\lambda\right|<\frac{1}{4}\right\}} d\lambda \; .$$

Using (6.9) we can apply the Schauder estimates as in [5] to conclude that

$$|R_j^*| + |x_j|^\gamma |\partial_\xi^\gamma R_j^*| \leq CN|x_j|^{2\alpha-5} \; .$$

Since $|\xi| \approx |x_i|$ if $R_j^* \neq 0$, it follows that

$$|R_j^*| + |\xi|^\gamma |\partial_\xi^\gamma R_j^*| \leq CN|\xi|^{\alpha-4+\theta}|x_j|^{\alpha-1+\theta}. \tag{6.25}$$

Combining the estimates for R_j'' and R_j^*, we get

$$\begin{aligned} &|\partial_\xi^k[\Sigma I_j'' - P_M(\xi)]| + |\xi|^\gamma|\partial_\xi^{k+\gamma}[\Sigma I_j'' - P_M(\xi)]| \\ &\qquad \leq CN|\xi|^{\alpha-4+\theta} \qquad (|k|=4) \ . \end{aligned} \tag{6.26}$$

Combining the estimates (6.26) with (6.21)–(6.24) and recalling (6.8), (6.10), (6.11), we get the following result:

Lemma 6.2. *The function φ defined by (6.8) in a solution to (6.5) in $\{y > 0\}$, satisfying:*

$$\begin{aligned} &|D_\xi^k[\varphi(\xi) - P_M(\xi]| \leq CNr^{\alpha-|k|+\theta} \qquad (|k| \leq 4) \ , \\ &|D_\xi^{k+\gamma}[\varphi(\xi) - P_M(\xi)]| \leq CNr^{\alpha-4-\gamma+\theta} \quad (|k|=4) \end{aligned} \tag{6.27}$$

in $\{y > 0\}$ where $\xi = (x,y)$, $r = |\xi|$, and $P_M(\xi)$ is a polynomial of degree M with coefficients bounded by CN.

We next turn to solving

$$\Delta^2\psi = \zeta_2 H \equiv H_2 \ . \tag{6.28}$$

Proceeding similarly to (6.6) it seems natural to take

$$\varphi_0(\xi) = K \int\limits_{\{x<-1\}} |\xi - q|^2 \log|\xi - q| H_2(q) dq \ . \tag{6.29}$$

We can cover the support of H_2 by discs $B_{\rho_j}(x_j)$ where

$$|x_j| = \left(\frac{4}{3}\right)^j \ , \ \rho = \frac{1}{2}\left(\frac{4}{3}\right)^j \qquad (j = 1, 2, \dots)$$

provided θ_0 is small enough. Then we split the domain of integration in (6.29) into these discs. However, unlike what we did before, we now must add to the j-th term a special biharmonic function φ_j; otherwise the series will not converge. Thus, we shall replace φ_0 by a somewhat different function

$$\widetilde{\varphi} = \Sigma I_j \tag{6.30}$$

where

$$
(6.31)\qquad
\begin{aligned}
I_j = {} & |x_j|^4 \log|x_j| \int\limits_{B_{1/2}(-1)} \Big|\frac{\xi}{|x_j|} - \lambda\Big|^2 \log \frac{|\frac{\xi}{|x_j|} - \lambda|}{|\lambda|} h_j(\lambda|x_j|) d\lambda \\
& + |x_j|^4 \int\limits_{B_{1/2}(-1)} \Big|\frac{\xi}{|x_j|} - \lambda\Big|^2 \log \frac{|\frac{\xi}{|x_j|} - \lambda|}{|\lambda|} h_j(\lambda|x_j|) d\lambda \equiv I'_j + I''_j \ ;
\end{aligned}
$$

the harmonic function φ_j enter through the factor $1/|\lambda|$ in the "log" terms of both I'_j and I''_j. From (6.2) we have

$$
(6.32)\qquad
\begin{aligned}
&|h_j(q)| \le CN|x_j|^{2\beta-5} \ , \\
&|D^\gamma h_j(q)| \le CN|x_j|^{2\beta-5-\gamma} \ .
\end{aligned}
$$

We can estimate the I'_j as before, simply replacing α by β and noting that $2\beta - 3 < 0$. We then get the same result (6.11) as before. It remains to estimate the I''_j.

We expand

$$
(6.33)\qquad \log|u-\lambda| = \log|u| + \log\Big|\frac{u}{|u|} - \frac{\lambda}{|u|}\Big| = \log|u| + \frac{c_0(\lambda)\cdot u}{|u|^2} + O\left(\frac{1}{|u|^2}\right)
$$

for $|u|$ large and use this expansion to establish the relation, for $\xi/|x_j|$ large,

$$
(6.34)\qquad
\begin{aligned}
& \Big|\frac{\xi}{|x_j|} - \lambda\Big|^2 \log \frac{|\frac{\xi}{|x_j|} - \lambda|}{|\lambda|} - \Big\{ \frac{|\xi|^2}{|x_j|^2} \log \frac{|\xi|}{|x_j|} + \frac{c(\lambda)\cdot\xi}{|x_j|} \log\frac{|\xi|}{|x_j|} \\
& \quad + \frac{\widetilde{c}(\lambda)|\xi|^2}{|x_j|^2} + \frac{\widehat{c}(\lambda)\cdot\xi}{|x_j|} + |\lambda|^2 \log \frac{|\frac{\xi}{|x_j|} - \lambda|}{|\lambda|} \chi_{\left\{\left|\frac{\xi}{|x_j|} - \lambda\right| < \frac{1}{4}\right\}} \Big\} \\
& = O\left(\frac{|\xi|^{\beta-\delta}}{|x_j|^{\beta-\delta}}\right) \quad \text{for any} \quad 0 < \delta < \beta
\end{aligned}
$$

where $c, \widetilde{c}, \widehat{c}$ depend on ξ, x_j, λ and are uniformly bounded. (In fact, the right-hand side may be replaced by $O(\log|\xi|/|x_j|)$. If $\xi/|x_j|$ varies in a bounded set away from 0 then (6.34) is still true since the left-hand side remains bounded. Finally, if $\xi/|x_j|$ tends to zero then the left-hand side converges linearly to zero (it is here that we need the factor $1/|\lambda|$ in "log"), and therefore (6.34) remains valid.

We need to estimate the contributions of the various terms in $\{\cdots\}$ and of the O-term to I''_j;

We begin with

$$|x_j|^4 \frac{|\xi|^2}{|x_j|^2} |\log \frac{|\xi|}{|x_j|}| \, |\int\limits_{B_{1/2}(-1)} h_j(\lambda|x_j|) d\lambda| \tag{6.35}$$
$$\leq CN|x_j|^{2\beta-3}|\xi|^2(|\log|\xi|| + \log|x_j|) \ .$$

The next term contributes

$$CN|x_j|^{2\beta-2}|\xi|(|\log|\xi| \, | + \log|x_j|) \ . \tag{6.36}$$

The contributions from the next two terms in $\{\cdots\}$ are bounded as in (6.35), (6.36), and the last term contribution is bounded by

$$CN|x_j|^4|x_j|^{2\beta-5} = CN|x_j|^{\beta-1} = CN|\xi|^{\beta-\delta}|x_j|^{\beta-1+\delta}$$

since $|\xi| \approx |x_j|$ if the integrand is not identically zero.

Finally the O-term in (6.34) contributes to I_j'' a function bounded by

$$CN|x_j|^4 \frac{|\xi|^{\beta-\delta}}{|x_j|^{\beta-\delta}} \; |x_j|^{2\beta-5} = CN|\xi|^{\beta-\delta}|x_j|^{\beta-1+\delta} \ .$$

Summing over j in the above estimates and choosing $\delta < 1-\beta$, we obtain

$$|\Sigma I_j''| \leq N\{|\xi|^2 \log|\xi| + |\xi|^2 + C|\xi| + O(|\xi|^{\beta-\delta})\} \ . \tag{6.37}$$

Next we wish to estimate derivatives $\partial_\xi^k I_j''$. The procedure is similar to that used for ξ near 0. If we apply ∂_ξ^k to the left-hand side of (6.34) we get the error term

$$\frac{1}{|x_j|^k} \; O\left(\frac{|\xi|^{\beta-|k|-\delta}}{|x_j|^{\beta-|k|-\delta}}\right)$$

provided $|\xi|/|x_j|$ is large. The same is true if $|\xi|/|x_j|$ is near 0 (since $\beta - |k| < 0$). Finally, if $\xi/|x_j|$ is near λ we need to modify the estimate when $|k| \geq 2$ in the same way as before. It will be enough to consider just the case $|k| = 4$. The various terms obtained by applying ∂_ξ^k to the sum in $\{\cdots\}$ in (6.34) can be handled essentially as before, with minor changes. Therefore we concentrate only on the new term which involves a singular integral:

$$|x_j|^4 \frac{1}{|x_j|^4} \int\limits_{B_{1/2}(-1)} \Omega\left(\frac{\xi}{|x_j|} - \lambda\right) \chi_{\left\{\left|\frac{\xi}{|x_j|} - \lambda\right| < \frac{1}{4}\right\}} h_j(\lambda|x_j|) d\lambda \ .$$

By the Schauder estimates [5] this term and $|x_j|^\gamma$ times its derivative ∂_ξ^γ are bounded by

$$CN|x_j|^{2\beta-5-\gamma} = CN|\xi|^{\beta-4-\delta}|x_j|^{\beta-1+\delta-\gamma} ,$$

where we used the fact that $|\xi| \approx |x_j|$ if the integral is $\neq 0$. Since $\beta - 1 + \delta < 0$, we can sum over j and, together with the estimates of the other terms in $\partial_\xi^k\{\cdots\}$, deduce that

$$\left|\partial_\xi^k \left[\sum_j I_j''\right] \right| + |\xi|^\gamma \left|\partial_\xi^{k+\gamma}[\sum_j I_j'']\right| \leq CN|\xi|^{\beta-4-\delta}$$

if $|k| = 4$.

We summarize:

The function $\widetilde{\varphi}$ defined by

$$\widetilde{\varphi}(\xi) = K \sum_j \int_{B_{\rho_j}(x_j)} |\xi - q|^2 \log \frac{|\xi - q|}{|\lambda_j|} \; H_2(q)dq \tag{6.38}$$

with $|\lambda_j| \approx \rho_j$ satisfies:

$$\begin{aligned} &\left|\partial_\xi^k \left\{\widetilde{\varphi} - \left[A_1|\xi|^2 \log|\xi| + A_2|\xi|^2 + A_3 \cdot \xi \log|\xi| + A_4 \cdot \xi + A_5 \log|\xi|\right]\right\}\right| \\ &\qquad \leq CNr^{\beta-|k|-\delta} \quad \text{if} \quad |k| \leq 4 , \end{aligned} \tag{6.39}$$

and

$$\begin{aligned} &\left|\partial_\xi^{k+\gamma} \left\{\widetilde{\varphi} - \left[A_1|\xi|^2 \log|\xi| + A_2|\xi|^2 + A_3 \cdot \xi \log|\xi| + A_4 \cdot \xi + A_5 \log|\xi|\right]\right\}\right| \\ &\qquad \leq CNr^{\beta-4-\gamma-\delta} \quad \text{if} \quad |k| = 4 \end{aligned} \tag{6.40}$$

where $|\xi| = r$ and

$$|A_j| \leq CN . \tag{6.41}$$

Actually we have only proved so far a cruder result than (6.39), namely,

$$|\partial_\xi^k \widetilde{\varphi}| \leq C\left[|\xi|^2 |\log|\xi\ | + |\xi|^2 + |\xi|\ |\log|\xi\ | + |\xi| + r^{\beta-|k|-\delta}\right] .$$

However a little bit more careful analysis shows that the bounds $|\xi|\log|\xi|$ and $|\xi|$ actually come from terms $A_3 \cdot \xi \log|\xi|$ and $A_4 \cdot \xi$. The same remark applies to (6.40).

Fix a point ξ_0 in $\{y < 0\}$ and consider the function

$$\begin{aligned} \widehat{\varphi}(\xi) = \widetilde{\varphi}(\xi) - [&B_1|\xi - \xi_0|^2 \log|\xi - \xi_0| + B_2|\xi - \xi_0|^2 \\ &+ B_3 \cdot (\xi - \xi_0) \log|\xi - \xi_0| + B_4 \cdot (\xi - \xi_0) + B_5 \log|\xi - \xi_0|] \end{aligned} \tag{6.42}$$

We can choose $B_1, B_2, \ldots, B_5$ in a unique way so that all the unbounded terms in $\widetilde{\varphi}$ cancel out. Hence:

LEMMA 6.3. *Suppose let* $0 < \delta < \min(\beta, 1-\beta)$. *Then the function* $\widehat{\varphi}$ *defined by (6.42), where* $\widetilde{\varphi}$ *is defined in (6.38), is a solution to (6.28) satisfying:*

$$
\begin{aligned}
&|D^k\widehat{\varphi}(\xi)| \le CNr^{\beta-|k|-\delta} \qquad (|k| \le 4)\ , \\
&|D^{k+\gamma}\widehat{\varphi}(\xi)| \le CNr^{\beta-4-\gamma-\delta} \quad (|k| = 4)\ .
\end{aligned}
\tag{6.43}
$$

We return to Lemma 6.2, and prove:

LEMMA 6.4. $P_M(\xi)$ *is a biharmonic function.*

Proof. We can write P_M as a sum of homogeneous polynomials μ_n of degree n:

$$P_M(\xi) = \sum_{n=n_0}^{M} \mu_n(\xi)\ .$$

In view of (6.27), the biharmonic functions

$$\varphi_R(\xi) \equiv \frac{1}{R^{n_0}}\ \varphi(R\xi)$$

converge to $\mu_{n_0}(\xi)$ as $R \downarrow 0$; hence $\mu_{n_0}(\xi)$ is biharmonic. Proceeding similarly with $\varphi - \mu_{n_0}$ we discover that μ_{n_0+1} is biharmonic, etc.

Any biharmonic function has the form

$$h_1(z) + \overline{z}h_2(z) + c.c.$$

where h_1, h_2 are holomorphic functions and *c.c.* stands for the complex conjugate. It follows that the biharmonic polynomial $P_M(\xi)$ which occurs in Lemma 6.1 is a finite sum of expressions

$$\psi_j = A_j z^k + \overline{A}_j\overline{z}^k + B_j z^{k-1}\overline{z} + \overline{B}_j\overline{z}^{k-1}z \tag{6.44}$$

where $|A_j|, |B_j|$ are bounded by CN and the bar means *c.c.*. For any such ψ_j and for any $R > 0$ one can construct polynomials $Q_j(z), R_j(z)$ such that

$$\widetilde{\psi}_j \equiv \psi_j - [Q_j(z)e^{-z^2} + \overline{z}R_j(z)e^{-z^2} + c.c.] = O(|z|^R)$$

for $|z| < 1$. Of course we also have

$$\widetilde{\psi}_j \to 0 \text{ exponentially fast if } |z| \to \infty, \quad |Imz| < \theta_0'|\operatorname{Re} z|$$

provided $\theta_0' < 1$; the convergence is in the $(4+\gamma)$-norm.

Note that the function φ constructed in Lemma 6.2 satisfies (6.43) if $|\xi| > 1$ whereas the function $\widehat{\varphi}$ constructed in Lemma 6.3 satisfies (6.27) if $|\xi| < 1$. Consequently, the function

$$W(\xi) = \varphi(\xi) + \widehat{\varphi}(\xi) - \sum_j \left[\left[Q_j(z)e^{-z^2} + \overline{z}R_j(z)e^{-z^2}\right] + c.c.\right] \quad (\xi = (x,y)) \tag{6.45}$$

satisfies the assertions of Theorem 6.1.

§7. **Solution to auxiliary boundary value problems.** In this section we wish to find a solution to

$$\Delta^2\varphi = 0 \quad \text{in} \quad \{y>0\}\ , \tag{7.1}$$

$$\varphi_x(x,0) = W_x(x,0)\ , \quad x\in\mathbb{R}\ , \tag{7.2}$$

$$\varphi_y(x,0) = W_y(x,0)\ , \quad x\in\mathbb{R}\ . \tag{7.3}$$

It will be convenient to use integral representation for φ_x, φ_y rather than directly for φ.

Lemmas 4.1, 4.2 provide us with the representations

$$\frac{\partial\varphi}{\partial x} = \int_{-\infty}^{0} \frac{\partial K_1}{\partial x}(x-s,y)D_xW(s,0)ds + \int_{-\infty}^{0} \frac{\partial K_2}{\partial y}(x-s,y)D_yW(s,0)ds\ , \tag{7.4}$$

$$\frac{\partial\varphi}{\partial y} = \int_{-\infty}^{0} \frac{\partial K_1}{\partial x}(x-s,y)D_xW(s,0)ds + \int_{-\infty}^{0} \frac{\partial K_2}{\partial y}(x-s,y)D_yW(s,0)ds \tag{7.5}$$

where

$$\frac{\partial K_1(x,y)}{\partial x} = \frac{2}{\pi}\ \frac{y^3}{(x^2+y^2)^2}\ , \quad \frac{\partial K_2(x,y)}{\partial y} = \frac{2}{\pi}\ \frac{x^2y}{(x^2+y^2)^2}\ ,$$

$$\frac{\partial K_1(x,y)}{\partial y} = \frac{\partial K_2(x,y)}{\partial x} = -\frac{2}{\pi}\ \frac{xy^2}{(x^2+y^2)^2}\ .$$

From the estimates in Theorem 6.1 we deduce that the integrals in (7.4), (7.5) are convergent and define smooth functions which we tentatively denote by φ_1 and φ_2 respectively. One can verify that

$$\frac{\partial\varphi_1}{\partial y} = \frac{\partial\varphi_2}{\partial x}\ .$$

Hence there exists a function φ such that

$$\frac{\partial\varphi}{\partial x} = \varphi_1\ , \quad \frac{\partial\varphi}{\partial y} = \varphi_2\ . \tag{7.6}$$

By direct computation

$$\Delta\frac{\partial\varphi_1}{\partial x} + \Delta\frac{\partial\varphi_2}{\partial y} = 0\ . \tag{7.7}$$

Hence

$$\Delta^2 \varphi = 0 \ . \tag{7.8}$$

We decompose

$$W_x(x,0) = \Sigma h_j(x,0), \quad W_y(x,0) = \Sigma k_j(x,0)$$

where h_j, k_j are supported in intervals with center $x_j = \pm \left(\frac{3}{4}\right)^j$ and length $\left(\frac{3}{4}\right)^j$; $j = 0, \pm 1, \pm 2, \ldots$, and then split the integral in (7.4) into infinite series,

$$\frac{\partial \varphi}{\partial x} = \Sigma \frac{\partial \psi_j}{\partial x} \tag{7.9}$$

where $\partial \psi_j / \partial x$ is an integral corresponding to the integrands h_j, k_j. Thus

$$\frac{\partial \psi_j}{\partial x} = \int_{-\infty}^{\infty} ds \left[\frac{2}{\pi} \frac{y^3}{[(x-s)^2 + y^2]^2} \ h_j(s,0) - \frac{2}{\pi} \frac{(x-s)y^2}{[(x-s)^2+y^2]^2} \ k_j(s,0) \right] \ .$$

Substituting $s = |x_j|\lambda$, we get

$$\begin{aligned} \frac{\partial \psi_j}{\partial x} = \frac{2}{\pi} \int_{\widehat{B}_{1/2}(-1) \cup \widehat{B}_{1/2}(1)} d\lambda \Bigg\{ & \frac{\left(\frac{y}{|x_j|}\right)^3}{\left[\left(\frac{x}{|x_j|} - \lambda\right)^2 + \left(\frac{y}{|x_j|}\right)^2\right]^2} h_j(|x_j|\lambda, 0) \\ & - \frac{\left(\frac{x}{|x_j|} - \lambda\right)\left(\frac{y}{|x_j|}\right)^2}{\left[\left(\frac{x}{|x_j|} - \lambda\right)^2 + \left(\frac{y}{|x_j|}\right)^2\right]^2} k_j(|x_j|\lambda, 0) \Bigg\} \end{aligned} \tag{7.10}$$

where $\widehat{B}_R(\pm 1)$ denotes the interval with center ± 1 and length $2R$.

Consider first the terms with $j > 0$. By (6.46), if $j > 0$ then

$$|h_j| + |k_j| \le CN|x_j|^{\alpha - 1 + \theta} \ , \quad \text{and similar bounds hold for the } 3+\gamma \text{ derivatives.} \tag{7.11}$$

Expanding the function $u^3/[u^3 + (v-\lambda)^2]^2$ about $u = v = 0$ by Taylor's formula, we find that if $\lambda \in \widehat{B}_{1/2}(+1) \cup \widehat{B}_{1/2}(-1)$ (so that λ stays away from 0) then

$$\left| \frac{\left(\frac{y}{|x_j|}\right)^3}{\left[\left(\frac{x}{|x_j|} - \lambda\right)^2 + \left(\frac{y}{|x_j|}\right)^2\right]^2} - P_{M-1}\left(\frac{x}{|x_j|}, \frac{y}{|x_j|}, \lambda\right) \right| \le C \left(\frac{|\xi|}{|x_j|}\right)^{\alpha - 1 + \theta - \mu} \tag{7.12}$$

where $\xi = (x,y), P_{M-1}$ is polynomial of degree $M-1$ in $\xi/|x_j|$, $\quad M$ is as in (6.14) and μ is a sufficiently small positive number such that

$$\alpha - 1 + \theta - \mu > M - 1$$

(which is possible by (6.14)). This choice enables us to conclude that (7.12) is valid also for $(x/|x_j|, y/|x_j|)$ near ∞ and, in fact, whenever

$$\left(\frac{x}{|x_j|} \ , \ \frac{y}{|y_j|} \right) \notin B_{1/4}(-1,0) \cup B_{3/4}(+1,0) \ .$$

If

$$\left(\frac{x}{|x_j|} \ , \ \frac{y}{|x_j|} \right) \in B_{3/4}(-1,0) \cup B_{3/4}(+1,0)$$

then we need add to the right-hand side the term

$$\frac{\left(\frac{y}{|x_j|}\right)^3}{\left[\left(\frac{x}{|x_j|} - \lambda\right)^2 + \left(\frac{y}{|x_j|}\right)^2\right]^2} \chi_{\left\{ \left(\frac{x}{|x_j|} \ , \ \frac{y}{|x_j|}\right) \in \left[B_{3/4}(-1,0) \cup B_{3/4}(+1,0)\right] \right\}} \ . \tag{7.13}$$

Using (7.11) we then easily find that

$$\begin{aligned} \left| \frac{\partial \psi_j}{\partial x} - N|x_j|^{\alpha-1+\theta} \frac{P_{M-1,j}(x,y)}{|x_j|^{M-1}} \right| &\leq \frac{CN|\xi|^{\alpha-1+\theta-\mu}}{|x_j|^{\alpha-1+\theta-\mu}} |x_j|^{\alpha-1+\theta} \\ &+ \widehat{C}_j N |x_j|^{\alpha-1+\theta} \end{aligned} \tag{7.14}$$

where $\widehat{C}_j$ is a constant c times the integral of the function in (7.13). If this integral is nonzero then $|\xi| \approx |x_j|$ and therefore we can replace the last term on the right-hand side of (7.14) by

$$CN|\xi|^{\alpha-1+\theta-\mu}|x_j|^{\mu} \ .$$

In (7.14) $P_{M-1,j}$ is a polynomial degree M with uniformly bounded coefficients. Summing over all $j = 1, 2, \dots,$ we get

$$\Big| \sum_{j>0} \frac{\partial \psi_j}{\partial x} - P_{M-1}(\xi) \Big| \leq CN|\xi|^{\alpha-1+\theta-\mu} \ . \tag{7.15}$$

Similarly one can establish the corresponding $3+\gamma$ estimates

$$\begin{aligned} &\Big| D^k \left[\sum_{j>0} \frac{\partial \psi_j}{\partial x} - P_{M-1}(\xi) \right] \Big| \leq CN|\xi|^{\alpha-1-|k|+\theta-\mu} \qquad (|k| \leq 3) \ , \\ &\Big| D^{k+\gamma} \left[\sum_{j>0} \frac{\partial \psi_j}{\partial x} - P_{M-1}(\xi) \right] \Big| \leq CN|\xi|^{\alpha-4-\gamma+\theta-\mu} \qquad (|k| = 3); \end{aligned} \tag{7.16}$$

the coefficients of $P_{M-1}(\xi)$ are bounded by CN.

We now proceed with the part of the series (7.9) for which $j \le 0$. Here we use the estimate

$$\frac{\left(\frac{y}{|x_j|}\right)^2}{\left[\left(\frac{x}{|x_j|}-\lambda\right)^2+\left(\frac{y}{|x_j|}\right)^2\right]^2} \le C\frac{|\xi|^{\beta-1-\widetilde{\mu}}}{|x_j|^{\beta-1-\widetilde{\mu}}} \quad \text{if} \quad \left(\frac{x}{|x_j|}, \frac{y}{|x_j|}\right) \notin B_{1/2}(-1,0)\cup B_{1/2}(+1,0)$$

where $\widetilde{\mu}$ is any small positive number. Analogously to (7.14) we get, for $j \le 0$,

$$|\frac{\partial \psi_j}{\partial x}| \le \frac{CN|\xi|^{\beta-1-\widetilde{\mu}}}{|x_j|^{\beta-1-\widetilde{\mu}}}\,|x_j|^{\beta-1-\delta}+\widehat{C}N|x_j|^{\beta-1-\delta}\widehat{\chi} \tag{7.17}$$

where $\widehat{\chi}$ is equal to the integral of (7.13), so that $|\xi| \approx |x_j|$ if the last term in (7.17) is nonzero. We conclude that

$$|\sum_{j\le 0}\frac{\partial \psi_j}{\partial x}| \le C|\xi|^{\beta-1-\delta-\widetilde{\mu}}\,, \quad \text{for any } 0<\widetilde{\mu}<\delta\ . \tag{7.18}$$

Similarly,

$$\begin{aligned} &|D^k\left[\sum_{j\le 0}\frac{\partial \psi_j}{\partial x}\right]| \le CN|\xi|^{\beta-1-|k|-\delta-\widetilde{\mu}} \quad (|k|\le 3)\ ,\\ &|D^{k+\gamma}\left[\sum_{j\le 0}\frac{\partial \psi_j}{\partial x}\right]| \le CN|\xi|^{\beta-4-\gamma-\delta-\widetilde{\mu}} \quad (|k|=3)\ . \end{aligned} \tag{7.19}$$

The estimates (7.16) hold near 0 also for $\sum_{j\le 0}\frac{\partial \psi_j}{\partial x}$ (with another polynomial), since this sum corresponds to integrating over $D_xW(x,0)$ for $|x|>1$. Similarly the estimate (7.19) near ∞ holds for $\sum_{j>0}\frac{\partial \psi_j}{\partial x}$. Since further μ and $\widetilde{\mu}$ can be taken arbitrarily small, they can be absorbed into the definitions of θ and δ, respectively. Thus:

$$\begin{cases} \left|D^k\left[\frac{\partial \varphi}{\partial x}-\widetilde{P}\right](\xi)\right| \le CN|\xi|^{\alpha-1-|k|+\theta} \quad (|k|\le 3)\ ,\\ \left|D^{k+\gamma}\left[\frac{\partial \varphi}{\partial x}-\widetilde{P}\right](\xi)\right| \le CN|\xi|^{\alpha-4-\gamma+\theta} \quad (k=3) \quad \text{if} \quad |\xi|<1\ , \end{cases} \tag{7.20}$$

$$
(7.21) \qquad \begin{cases} \left|D^k \dfrac{\partial \varphi(\xi)}{\partial x}\right| \le CN|\xi|^{\beta-1-|k|-\delta} & (|k| \le 3)\ , \\[2ex] \left|D^{k+\gamma} \dfrac{\partial \varphi(\xi)}{\partial x}\right| \le CN|\xi|^{\beta-4-\gamma-\delta} & (|k| = 3) \quad \text{if} \quad |\xi| \ge 1 \end{cases}
$$

where $\widetilde{P}$ is a polynomial.

Similar estimates can be derived for $\partial\varphi/\partial y$, with another polynomial $\widehat{P}$. From (7.20) and the corresponding estimates for $\partial\varphi/\partial y$ we easily deduce that $\partial\widetilde{P}/\partial y = \partial\widehat{P}/\partial x$ and therefore there exists a polynomial P such that

$$
\frac{\partial P}{\partial x} = \widetilde{P}\ , \ \frac{\partial P}{\partial y} = \widehat{P}\ .
$$

LEMMA 7.1. *The function $\varphi(\xi)$ $(\xi = (x,y))$ satisfies*

$$
(7.22) \qquad \begin{cases} |D^k(\varphi - P)(\xi| \le CN|\xi|^{\alpha-|k|+\theta} & (|k| \le 4)\ , \\ |D^{k+\gamma}(\varphi - P)(\xi)| \le CN|\xi|^{\alpha-4-\gamma+\theta} & (k = 4) \quad \textit{if} \quad |\xi| \le 1,\ y > 0\ , \end{cases}
$$

and

$$
(7.23) \qquad \begin{cases} |D^k\varphi(\xi)| \le CN|\xi|^{\beta-|k|-\delta} & (|k| \le 4)\ , \\ |D^{k+\gamma}\varphi(\xi)| \le CN|\xi|^{\beta-4-\gamma-\delta} & (|k| = 4) \quad \textit{if} \quad |\xi| > 1, y > 0 \end{cases}
$$

where P is a biharmonic polynomial of degree $< \alpha + \theta$ with coefficients bounded by CN, and

$$
(7.24) \qquad P_x(x,0) = P_y(x,0) = 0\ ,\ x \in \mathbb{R}\ .
$$

Proof. We have already proved the assertions (7.22), (7.23). It remains to show that P is biharmonic and that (7.24) is satisfied. Write P is a sum of homogeneous polynomials

$$
(7.25) \qquad \sum_{n=n_0}^{n_1} Q_n \quad , \quad \text{where} \quad n_1 < \alpha + \theta\ .
$$

Consider the functions

$$
\varphi_R(x,y) = \frac{1}{R^{n_0}}\ \varphi(Rx, Ry) \qquad (R \downarrow 0)\ .
$$

Since $n_0 < \alpha + \theta$, we deduce from (7.22) that, as $R \to 0$,

$$
\varphi_R(x,y) \to Q_{n_0}(x,y)
$$

and

$$\frac{\partial \varphi_R(x,0)}{\partial x} \to \frac{\partial Q_{n_0}(x,0)}{\partial x} , \quad \frac{\partial \varphi_R(x,0)}{\partial y} \to \frac{\partial Q_{n_0}(x,0)}{\partial y} .$$

Since by (6.4) and (7.2), (7.3)

$$\frac{\partial \varphi_R(x,0)}{\partial x} \to 0 , \quad \frac{\partial \varphi_R(x,0)}{\partial y} \to 0 \quad \text{if} \quad R \to 0 ,$$

it follows that Q_{n_0} is biharmonic polynomial satisfying (7.24). Applying this proof to $\varphi - Q_{n_0}$, $\varphi - (Q_{n_0} + Q_{n_1})$, etc., Lemma 7.1 follows.

We wish to eliminate P from Lemma 7.1 by modifying the function φ. To do this we need to study biharmonic polynomials whose gradient vanishes on $y = 0$.

LEMMA 7.2. *If φ is a real-valued biharmonic polynomial of degree N with $\nabla\varphi(x,0) = 0$ for all $x \in \mathbb{R}$, then φ has the form*

$$\varphi = -\sum_{n=2}^{N} \frac{n-1}{n} B_n z^n - \sum_{n=2}^{N} \frac{\overline{B}_n}{n} z^n + \overline{z} \sum_{n=2}^{N} B_n z^{n-1} + c.c. \tag{7.26}$$

where B_n are complex numbers, and $\overline{B}_n$ is the complex conjugate of B_n.

Proof. From the paragraph following (2.12) we know that φ must be a linear combinations of functions

$$Az^n + Bz^{n-1}\overline{z} + \widetilde{A}\overline{z}^n + \widetilde{B}\overline{z}^{n-1}z .$$

Equating $\partial\varphi$ and $\overline{\partial}\varphi$ to zero an $z = r$ and $z = re^{i\pi} = -r$ for all $r > 0$, we get the equations

$$nA + (n-1)B + \widehat{B} = 0 ,$$

$$B + (n-1)\widetilde{B} = n\widetilde{A} = 0 ,$$

$$ne^{i\pi(n-1)}A + (n-1)e^{i\pi(n-3)} + e^{-i\pi(n-1)}\widetilde{B} = 0 ,$$

$$e^{i\pi(n-1)}B + (n-1)e^{-i\pi(n-3)} + ne^{-i\pi(n-1)}\widetilde{A} = 0 .$$

We easily compute the solution to this systems:

$$A = -\frac{1}{n}[(n-1)B + \widetilde{B}] ,$$

$$\widetilde{A} = -\frac{1}{n}[(n-1)\widetilde{B} + B]$$

where $B, \widetilde{B}$ are arbitrary. Since φ is real-valued, the conditions $\widetilde{B} = \overline{B}$, $\widetilde{A} = \overline{A}$ must hold, and this yields the n-th degree homogeneous polynomials in (7.26).

By Lemma 7.2, the polynomial P in (7.22) is a linear combination of polynomials

$$-\frac{k-1}{k}\,\Gamma_k z^k - \frac{\overline{\Gamma}_k}{k}\,z^k + \overline{z}\Gamma_k z^{k-1} + c.c. \,. \tag{7.27}$$

We wish to construct a biharmonic function which near the origin coincides with (7.27), up to any given order, and, at the same time, decreases exponentially fast to zero in $|Imz| \le \theta_0' |\operatorname{Re} z|$, $\theta_0' < 1$, as $|z| \to \infty$.

We can write

$$z^k e^{-z^2} = \sum_{j=k}^{\infty} \gamma_j z^j \ , \ \gamma_j \in \mathbb{R} \ .$$

Setting $B_n = n\gamma_n$ we have

$$\sum_{n=k}^{\infty} \frac{B_n}{n}\, z^n = \sum_{n=k}^{\infty} \frac{\overline{B}_n}{n}\, z^n = z^k e^{-z^2} \ .$$

Also

$$\sum_{n=k}^{\infty} B_n z^n = z\frac{d}{dz}(z^k e^{-z^2}) \ ,$$

$$\overline{z}\sum_{n=k}^{\infty} B_n z^{n-1} = \overline{z}\frac{d}{dz}(z^k e^{-z^2}) \ .$$

Hence, as easily computed, the real biharmonic function $\varphi = \varphi_k$ defined by (7.26) with $N = \infty$ and $B_n = n\gamma_n$ satisfies:

$$\varphi_k = k\left[-\frac{k-1}{k}\,\Gamma_k z^k - \frac{\overline{\Gamma}_k}{k}\,z^k + \overline{z}\Gamma_k z^{k-1} + c.c.\right] + \widetilde{P}_{k+2} \tag{7.28}$$

with $\Gamma_k = 1$, where $\widetilde{P}_{k+2}$ is a polynomial of degree $k+2$, and $\varphi_k \to 0$ exponentially fast if $|Imz| \le \theta_0' |\operatorname{Re} z|, |z| \to \infty$.

Similarly, taking

$$iz^k e^{-z^2} = \sum_{j=k}^{\infty} \widetilde{\gamma}_j z^j \ , \ \widetilde{B}_n = n\widetilde{\gamma}_n$$

and forming the corresponding $\widetilde{\varphi}_k$ by (7.26) with $N = \infty$ and B_n replaced by $\widetilde{B}_n$, we obtain (7.28) with φ_k replaced by $\widetilde{\varphi}_k$ and $\Gamma_k = i$. By taking a linear combination with real coefficients of φ_k and $\widetilde{\varphi}_k$ we obtain a new function, which we denote again by φ_k, of the form (7.26) with $N = \infty$ such that it satisfies (7.28) with any given complex coefficients Γ_k; the B_n are also complex.

We now write P as a sum of homogeneous polynomials Q_k, as in (7.25). We apply the above construction to obtain a function φ_{n_0} of the form (7.28) (with $N = \infty$) such that, near the origin,

$$P - \varphi_{n_0} \quad \text{contains only terms of order} \quad \geq n_0 + 1 \ ,$$

and $\varphi_{n_0}(z) \to 0$ exponentially fast if $|z| \to \infty$, $|Imz| \leq \theta_0' |\operatorname{Re} z|$.

Repeating the construction, we get similar functions φ_n for $n = n_0 + 1, \dots, n_1$ such that, near the origin,

$$P - \sum_{n=n_0}^{n_1} \varphi_n \quad \text{contains only terms of order} > n_0 + 1$$

whereas each φ_n decays at ∞ in the same way as φ_{n_0}.

Defining

$$\widetilde{\varphi} = \varphi - \sum_{n=n_0}^{n_1} \varphi_n \tag{7.29}$$

we obtain the following result.

THEOREM 7.3. *The function $\widetilde{\varphi}$ is a solution to (7.1)–(7.3) satisfying:*

$$\|\varphi\|^{\alpha+\theta,\beta-\delta}_{\Sigma_{\theta_0'} \cup B_R^+, 4+\gamma} \leq CN \tag{7.30}$$

for any $0 < \theta_0' < 1$, $R > 0$.

§**8. Solution to inhomogeneous mixed boundary value problem.** In this section we wish to find a special solution to

$$\Delta^2 \varphi = 0 \quad \text{in} \quad \{y > 0\} \ , \tag{8.1}$$

$$\varphi_x(x,0) = \varphi_y(x,0) = 0 \ , \quad x > 0 \ , \tag{8.2}$$

$$(\varphi_{yy} - \varphi_{xx})(x,0) = g_1(x) \ , \quad x < 0 \ , \tag{8.3}$$

$$[\lambda(2\varphi_{xxy} + \varphi_{yyy}) - \varphi_{xxx}](x,0) = g_2(x) \ , \quad x < 0 \tag{8.4}$$

where g_1, g_2 satisfy:

$$\|g_1\|_{\Sigma_0^-,2+\gamma}^{\alpha+\theta-2,\beta-\delta-2} \le N \ , \tag{8.5}$$

$$\|g_2\|_{\Sigma_0^-,3+\gamma}^{\alpha+\theta-3,\beta-\delta-3} \le N \ . \tag{8.6}$$

We chose $\widetilde{\theta} < \theta$ with $\theta - \widetilde{\theta}$ small, and γ positive and small ($\gamma < \theta - \widetilde{\theta}$). Then there exists a positive integer M and $\widetilde{\varepsilon} \in (0,1)$ such that

$$\alpha + \widetilde{\theta} - \frac{1}{2} < M + \widetilde{\varepsilon} < \alpha + \theta - \frac{1}{2} - \gamma \ , \ M + \widetilde{\varepsilon} > 1 \ . \tag{8.7}$$

As in §§4,5 we associate to g_1, g_2 functions $h_1(x), h_2(x)$ defined by (4.27), (4.28) for $x \le 0$ with $h_1(0) = h_1'(0) = h_2(0) = 0$ and a function Φ defined by (5.1). According to Theorem 5.1, there is a solution ψ to (8.1)–(8.4) given by

(8.8)

$\psi = \psi_1 + \psi_2$,

ψ_1 is defined by (5.7), (5.8) or (5.19), (5.22),

ψ_2 is defined by (7.25), (5.26) or (5.35), (5.37).

We wish to evaluate the function ψ and use this to construct the special solution φ.

We begin with ψ_1. By (4.27)

$$2\pi \frac{Dh_2(x)}{\sqrt{-x}} = \int_{-\infty}^{0} \frac{g_1(s)ds}{\sqrt{-\xi}\ (x-s)} \equiv \widehat{h}_2(x) \tag{8.9}$$

We split g_1 into a series Σg_{1j} using a sequence of overlapping intervals $\widehat{B}_{\rho_j}(x_j)$ with centers x_j and length $2\rho_j$, where

$$x_j = -\left(\frac{3}{4}\right)^j \ , \ \rho_j = \frac{1}{2}\left(\frac{3}{4}\right)^j \qquad (j = 0, \pm 1, \pm 2, \dots) \ ;$$

g_{1j} is supported in $\widehat{B}_{\rho_j}(x_j)$. Then

$$\widehat{h}_2(x) = \sum_j \int_{\widehat{B}_{\rho_j}(x_j)} \frac{g_{1j}(s)ds}{\sqrt{-s}\ (x-s)} \ . \tag{8.10}$$

We first consider the case $j > 0$. Clearly

$$\int_{\widehat{B}_{\rho_j}(x_j)} \frac{g_{1j}(s)ds}{\sqrt{-s}\ (x-s)} = \frac{1}{|x_j|^{1/2}} \int_{B_{1/2}(-1)} \frac{g_{1j}(\eta|x_j|)}{\sqrt{-s}} \ \frac{d\eta}{\frac{x}{|x_j|} - \eta} \tag{8.11}$$

and

$$(8.12)\qquad \begin{aligned} &|D^k g_{1j}(\eta|x_j|)| \le CN|x_j|^{\alpha-2-k+\theta} \quad (k=0,1,2)\ , \\ &|D^{2+\gamma} g_{1j}(\eta|x_j|)| \le CN|x_j|^{\alpha-4-\gamma+\theta}\ . \end{aligned}$$

We shall use the expansion

$$(8.13)\qquad \frac{1}{x-a} = -\frac{1}{a}\sum_{n=0}^{m}\left(\frac{x}{a}\right)^n + \left(\frac{x}{a}\right)^{m+1}\frac{1}{x-a}\ .$$

For any $\eta \in \widehat{B}_{1/2}(-1)$, if $\frac{x}{|x_j|} \notin \widehat{B}_{3/4}(-1)$ then, by (8.13),

$$\frac{1}{\frac{x}{|x_j|}-\eta} = \widetilde{P}_{M-2}\left(\frac{x}{|x_j|}\ ,\eta\right) + O\left(|\frac{x}{|x_j|}|^{M-2+\widetilde{\varepsilon}}\right)$$

if $x/|x_j|$ is small enough, where $\widetilde{P}_{M-2}$ is a polynomial of degree $M-2$ in $x/|x_j|$. Since $M+\widetilde{\varepsilon}>1$ (see (8.7)), this relation remains true also for $x/|x_j|$ large and, in fact, as long as $\frac{x}{|x_j|} \notin B_{3/4}(-1)$.

It follows that

$$\frac{1}{|x_j|^{1/2}}\int\limits_{\widehat{B}_{1/2}(-1)} \frac{g_{1j}(\eta|x_j|)}{\sqrt{-\eta}}\ \frac{d\eta}{\frac{x}{|x_j|}-\eta}$$

$$= \frac{1}{|x_j|^{1/2}}\int\limits_{\widehat{B}_{1/2}(-1)} \frac{g_{1j}(\eta|x_j|)}{\sqrt{-\eta}}\ P_{M-2}\left(\frac{x}{|x_j|}\ ,\eta\right) d\eta$$

$$+\frac{1}{|x_j|^{1/2}}\int\limits_{\widehat{B}_{1/2}(-1)} \frac{g_{1j}(\eta|x_j|)}{\sqrt{-\eta}}\ O\left(|\frac{x}{x_j}|^{M-2+\widetilde{\varepsilon}}\right) d\eta\ .$$

Using (8.12) and (8.7) we obtain, upon summing over all $j>0$,

$$P_{M-2}(x) + NO(|x|^{m-2+\widetilde{\varepsilon}})$$

where $P_{M-2}(x)$ is a polynomial of degree $M-2$ with coefficients bounded by CN.

Next we consider the integral on the right-hand side of (8.11) in case $\frac{x}{|x_j|} \in B_{3/4}(-1)$. We apply the Schauder estimates to get

$$\frac{1}{|x_j|^{1/2}}\Big|\int\limits_{B_{1/2}(-1)} \frac{g_{1j}(\eta|x_j|)}{\sqrt{-\eta}}\ \frac{d\eta}{\frac{x}{|x_j|}-\eta}\Big| \le \frac{CN}{|x_j|^{1/2}}|x_j|^{\alpha-2-\gamma'+\theta}$$

for any $\gamma' > 0$. Furthermore, if the integral is actually nonzero then necessarily $|x| \approx |x_j|$. Therefore the last bounded may be replaced by

$$\frac{CN}{|x_j|^{1/2}} \; |x_j|^{\alpha-2-\gamma'+\theta} \Big|\frac{x}{x_j}\Big|^{M-2+\widetilde{\varepsilon}} ,$$

and using (8.9) we get, upon summing over $j > 0$, the bound $NO(|x|^{M-2+\widetilde{\varepsilon}})$.

Thus, altogether,

$$\int_{-\frac{3}{2}}^{0} \frac{g_1(\xi)}{\sqrt{-\xi}} \; \frac{d\xi}{x-\xi} = P_{M-2}(x) + NO(|x|^{M-2+\widetilde{\varepsilon}}) \; . \tag{8.14}$$

Obviously,

$$\int_{-\infty}^{-3/2} \frac{g_1(\xi)}{\sqrt{-\xi}} \; \frac{d\xi}{x-\xi}$$

can be expanded near $x = 0$ using (8.13). Combining this remark with (8.14) and recalling (8.9), we conclude that

$$\frac{Dh_2(x)}{\sqrt{-x}} = P_{M-2}(x) + NO(|x|^{M-2+\widetilde{\varepsilon}}) \tag{8.15}$$

with another polynomial P_{M-2}.

Similarly we can estimate the $2+\gamma$ derivative the left-hand side of (8.15), from which we deduce, after using (8.7), that

$$\begin{aligned} &|D^k[h_2(x) - (-x)^{3/2} P_{M-2}(x)]| \le C|x|^{\alpha-1-k+\widetilde{\theta}} \quad (0 \le k \le 3) \; , \\ &|D^{3+\gamma}[h_2(x) - (-x)^{3/2} P_{M-2}(x)]| \le C|x|^{\alpha-4-\gamma+\theta} \quad \text{if} \quad -1 \le x < 0 \; . \end{aligned} \tag{8.16}$$

We now turn to the case where $|x| \ge 1$.

We shall use the estimates

$$\begin{aligned} &|D^k g_{1j}(\eta|x_j|)| \le CN|x_j|^{\beta-2-k-\delta} \quad (k = 0,1,2) \; , \\ &|D^{2+\gamma} g_{1j}(\eta|x_j|)| \le CN|x_j|^{\beta-4-\gamma-\delta} \end{aligned} \tag{8.17}$$

for $j \le 0$. One of the following cases must hold:

(i) $\quad \dfrac{x}{|x_j|} \in \widehat{B}_{3/4}(-1)$,

(ii) $\quad \dfrac{x}{|x_j|} \notin \widehat{B}_{3/4}(-1)$, and $|\dfrac{x}{x_j}| < \dfrac{5}{6}\,\eta \quad \forall\, \eta \in B_{1/2}(-1)$,

(iii) $\quad \dfrac{x}{|x_j|} \notin \widehat{B}_{3/4}(-1)$, and $|\dfrac{x}{x_j}| > \dfrac{6}{5}\,\eta \quad \forall\, \eta \in B_{1/2}(-1)$.

In case (i) we use the Schauder estimates to get the bound

$$CN|x|^{\beta-\delta-2-\frac{1}{2}+\widehat{\varepsilon}}$$

for any $\widehat{\varepsilon} > 0$. In case (ii) we bound the j-th term by

$$CN|x_j|^{\beta-\delta-2} \quad \text{or} \quad CN|\frac{x}{x_j}|^{\tau}|x_j|^{\beta-\delta-2} \quad (\tau < 0)$$

and choose $\tau = \beta - \delta - \dfrac{5}{2} + \widehat{\varepsilon}$, giving us (upon summation) the same bound as before.

In case (iii) we expand $[(x/|x_j|) - \eta]^{-1}$ by (8.13) with $m = 2$. The error term is bounded by

$$C|\frac{x}{x_j}|^{-3} \quad \text{and therefore also by} \quad C|\frac{x}{x_j}|^{\tau}$$

where $\tau = \beta - \delta - 2 - \dfrac{1}{2} + \widehat{\varepsilon}$. We obtain, upon summation on $j > 0$,

$$\frac{C_1 N}{x} + \frac{C_2 N}{x^2} + NO(|x|^{\beta-\delta-2-\frac{1}{2}+\widehat{\varepsilon}}), \quad C_i \text{ constants.}$$

We conclude that

$$|\frac{Dh_2(x)}{\sqrt{-x}} - \frac{C_1 N}{x} - \frac{C_2 N}{x^2}| \le CN|x|^{\beta-\delta-2-\frac{1}{2}+\widehat{\varepsilon}}\,. \tag{8.18}$$

Similarly we can estimate the $2+\gamma$ derivatives of $Dh_2(x)/\sqrt{-x}$, thereby obtaining:

(8.19)

$$\begin{cases} |D^k[h_2(x) - C_1 N(-x)^{1/2} - C_2 N(-x)^{-1/2}]| \le CN|x|^{\beta-\widetilde{\delta}-1-k} & (0 \le k \le 3)\ , \\ |D^{3+\gamma}[h_2(x) - C_1 N(-x)^{1/2} - C_2 N(-x)^{1/2}]| \le CN|x|^{\beta-\widetilde{\delta}-4-\gamma} & \text{if} \quad -\infty < x < -1 \end{cases}$$

for any $\widetilde{\delta} < \delta$; $|C_i| \le C$.

Recalling (8.5), (8.6), (8.16), we deduce from (5.1) that

$$\begin{cases} |D^k[\Phi(x) - x^{-1/2}P_{M-2}]| \le CN|x|^{\alpha+\widetilde{\theta}-3-k} & (k = 0, 1)\ , \\ |D^{1+\gamma}[\Phi(x) - x^{-1/2}P_{M-2}]| \le CN|x|^{\alpha+\widetilde{\theta}-4-\gamma} & \text{if} \quad -1 \le x < 0 \end{cases} \tag{8.20}$$

and

$$
(8.21)\quad \begin{cases} \left|D^k\left[\Phi(x)-\dfrac{C_1N}{(-x)^{3/2}}-\dfrac{C_2N}{(-x)^{5/2}}\right]\right| \le CN|x|^{\beta-\widetilde{\delta}-3-k} \quad (k=0,1)\ , \\ \left|D^{1+\gamma}\left[\Phi(x)-\dfrac{C_1N}{(-x)^{3/2}}-\dfrac{C_2N}{(-x)^{5/2}}\right]\right| \le CN|x|^{\beta-\widetilde{\delta}-4-\gamma} \quad \text{if} \quad -\infty < x \le -1\ . \end{cases}
$$

We shall use (8.20), (8.21) to estimate ψ_1. Using complex notation we write these equations in the form

$$
(8.22)\qquad D_x\psi_1 = K\int_{-\infty}^{0} d\xi\Phi(\xi)(-\xi)^\rho\left[\frac{z+\overline{z}}{2}\int_{\overline{z}}^{z}\frac{(-q)^{-\rho}}{q-\xi}\,dq-\int_{\overline{z}}^{z}\frac{q(-q)^{-\rho}}{q-\xi}\,dq\right]\ ,
$$

$$
(8.23)\qquad D_y\psi_1 = K\int_{-\infty}^{0} d\xi\Phi(\xi)(-\xi)^\rho\left[\frac{z+\overline{z}}{2}\int_{\overline{z}}^{z}\frac{(-q)^{-\rho}}{q-\xi}\,dq\right]
$$

where K is a constant. It will be convenient to take the contour of integration from $\overline{z}$ to z to be the counter-clockwise traced arc of a circle whose points ζ satisfy: $|\zeta| = |z|$.

The kernels in (8.22), (8.23) are much more regular than what may appear at first glance:

LEMMA 8.1. *Let $a < a' < b' < b$ and denote the function $\Phi(\xi)(-\xi)^{-\rho}$ in (8.22), (8.23) by $\Psi(\xi)$. If $\psi \in C^{1+\gamma}(a,b)$ then $D_x\psi_1$ and $D_y\psi_1$ belong to $C^{3+\gamma}(a',b')$.*

Proof. We compute

$$
\partial_z(D_x\psi_1) = \frac{K}{2}\int_{-\infty}^{0} d\xi\Psi(\xi)\left\{\int_{\overline{z}}^{z}\left[\frac{(-q)^{-\rho}}{q-\xi}\,dq+(\overline{z}-z)\frac{(-z)^{-\rho}}{z-\xi}\right]\right\}\ .
$$

From the Hilbert transform type of the kernel it follows (e.g. by [5]) that $\partial_z(D_x\psi_1)$ is in $C^{1+\gamma}$, since Ψ is in $C^{1+\gamma}$. Next

$$
\partial_{\overline{z}}\partial_z(D_x\psi_1) = \frac{K}{2}\int_{-\infty}^{0} d\xi\Psi(\xi)\left[-\frac{(-\overline{z})^{-\rho}}{\overline{z}-\xi}+\frac{(-z)^{-\rho}}{z-\xi}\right]
$$

and the same argument shows that this function is in $C^{1+\gamma}$, i.e., $\Delta(D_x\psi_1) \in C^{1+\gamma}$.

If we can derive $C^{3+\gamma}$ estimates on $D_x\psi_1$ at $y=0$ then, by elliptic regularity, $D_x\psi_1$ will belong to $C^{3+\gamma}$, as asserted. Thus, it remains to show that

$$
\partial_z^2 D_x\psi_1 \text{ and } \partial_{\overline{z}}^2 D_x\psi_1 \text{ belong to } C^{1+\gamma} \text{ on } y=0.
$$

We first compute

$$\partial_z^2(D_x\psi_1) = \frac{K}{2}\int\limits_{-\infty}^{0} d\xi\Psi(\xi)(\overline{z}-z)\left[\frac{\rho(-z)^{-\rho-1}}{z-\xi} - \frac{(-z)^{-\rho}}{(z-\xi)^2}\right] .$$

The part corresponding to $1/(z-\xi)$ is of course in $C^{1+\gamma}$. For the other part we note that as $y \to 0$, $(\overline{z}-z)/(z-\xi) \to C_0\delta(x-\xi)$. Indeed, for any test function F,

$$\int \frac{z-\overline{z}}{(z-\xi)^2}F(\xi)d\xi = 2iy\int \frac{F(\xi)d\xi}{(x-\xi)^2 - y^2 + 2i(x-\xi)y}$$
$$= 2i\int F(x+\lambda y)\frac{d\lambda}{\lambda^2 - 1 - 2i\lambda} \to C_0F(x) \quad \text{as} \quad y \to 0 ,$$

where C_0 is a constant. It follows that

$$\partial_z^2 D_x\psi_1(x,y) \to C_0K\Psi(x) \quad \text{as} \quad y \to 0$$

and this function is in $C^{1+\gamma}$.

Similarly we compute that

$$\partial_{\overline{z}}^2 D_x\psi_1|_{y=0} = C_1K\Psi(x) .$$

We have thus completed the proof of the lemma for $D_x\psi_1$.

The proof for $D_y\psi_1$ is similar. Here

$$\partial_z(D_y\psi_1) = \frac{K}{2i}\int\limits_{-\infty}^{0} d\xi\Psi(\xi)\left[\int\limits_{\overline{z}}^{z}\frac{(-q)^{-\rho}}{q-\xi}dq + (z-\overline{z})\frac{(-z)^{-\rho}}{z-\xi}\right],$$

$$\partial_{\overline{z}}\partial_z(D_y\psi_1) = \frac{K}{2i}\int\limits_{-\infty}^{0} d\xi\Psi(\xi)\left[-\frac{(-\overline{z})^{-\rho}}{\overline{z}-\xi} - \frac{(-z)^{-\rho}}{z-\xi}\right],$$

and

$$\partial_z^2(D_y\psi_1) = \frac{K}{2i}\int\limits_{-\infty}^{0} d\xi\Psi(\xi)\Bigg\{\left[\frac{(-z)^{-\rho}}{z-\xi} + \frac{(-z)^{-\rho}}{z-\xi}\right]$$
$$+ (z-\overline{z})\left[\frac{\rho(-z)^{-\rho-1}}{z-\xi} - \frac{(-z)^{-\rho}}{(z-\xi)^2}\right]\Bigg\} .$$

All the terms are bounded in the $C^{1+\gamma}$ norm except the one associated with $(-z)^{-\rho}/(z-\xi)^2$, which converges to $-C_1(-x)^{-\rho}\delta(x-\xi)$ for some constant $C_1 \neq 0$.

Set

$$M_1(z,\xi) = K\frac{z+\overline{z}}{2}\int_{\overline{z}}^{z}\frac{(-q)^{-\rho}}{q-\xi}\,dq\ . \tag{8.24}$$

We wish to evaluate the integral

$$\int_{-\infty}^{0} d\xi\Phi(\xi)(-\xi)^{-\rho}M_1(z,\xi) \tag{8.25}$$

which comes from (8.22). First we do this for $|z|<1$. Introduce a "cutoff" function

$$\eta(\xi) = [\tanh(-\xi)]^A \qquad (\xi > 0)$$

where A is a large positive integer. Note that

$$\begin{aligned}\eta(\xi) &\sim (-\xi)^A && \text{if} \quad \xi \to 0\ ,\\ \eta(\xi) &\sim 1-2Ae^{\xi} && \text{if} \quad \xi \to -\infty\ .\end{aligned}$$

We write

$$\Phi = \Phi\eta + \widetilde{\Phi}(1-\eta) + \xi^{-1/2}P_{M-2}(1-\eta) \quad (\widetilde{\Phi} = \Phi - \xi^{-1/2}P_{M-2}) \tag{8.26}$$

where P_{M-2} is the polynomial appearing in (8.20).

One can easily check (with $\widetilde{\theta}$ near $\alpha-1$) that

$$|\Phi(\xi)(-\xi)^{\rho}\eta(\xi)| \leq C|\xi|^{A-1}\ .$$

We can now use the splitting arguments (cf. the derivation of (8.16)) to get

$$\int_{-\infty}^{0} d\xi\Phi(\xi)(-\xi)^{\rho}\eta(\xi)M_1(z,\xi) = \sum_{j=0}^{A-1}\nu_j(z) + O(|z|^{A-\rho})$$

for $|z|<1$, where $\nu_j(z) = |z|^{-\rho}\widetilde{\nu}_j(z)$ and $\widetilde{\nu}_j(z)$ is homogeneous polynomial of degree j in (x,y); the factor $|z|^{-\rho}$ arises from the factor $(-q)^{-\rho}$ in $M_1(z,\xi)$.

Similarly one can prove higher order estimates:

$$
(8.27)\qquad \begin{aligned}
&|D^k \left[\int_{-\infty}^{0} d\xi \Phi(\xi)(-\xi)^\rho \eta(\xi) M_1(z,\xi) - |z|^{-\rho} P_{A-1} \right] | \\
&\qquad \le CN|z|^{A-\rho} \qquad (|k| \le 3) \ , \\
&|D^{k+\gamma} \left[\int_{-\infty}^{0} d\xi \Phi(\xi)(-\xi)^\rho \eta(\xi) M_1(z,\xi) - |z|^{-\rho} P_{A-1} \right] | \\
&\qquad \le CN|z|^{A-\rho-\gamma} \qquad (|k| = 3) \quad \text{if} \quad |z| \le 1.
\end{aligned}
$$

where P_{A-1} is a polynomial of degree $A-1$ with coefficients bounded by CN. Here we use the Schauder estimates established in Lemma 8.1.

The term

$$\int_{-\infty}^{0} d\xi \widetilde{\Phi}(1-\eta) M_1(z,\xi)$$

can be estimated in the same way, making use of (8.20). The result is that

$$
(8.28)\qquad \begin{aligned}
&|D^k \left[\int_{-\infty}^{0} d\xi \widetilde{\Phi}(1-\eta) M_1(z,\xi) - |z|^{-\rho} P_J(z) \right] | \\
&\qquad \le CN|z|^{\alpha+\widetilde{\theta}-1-|k|-\varepsilon^*-\rho} \quad (|k| \le 3), \\
&|D^{k+\gamma} \left[\int_{-\infty}^{0} d\xi \widetilde{\Phi}(1-\eta) M_1(z,\xi) - |z|^{-\rho} P_J(z) \right] | \\
&\qquad \le CN|z|^{\alpha+\widetilde{\theta}-4-\gamma-\varepsilon^*-\rho} \quad (|k| = 3) \quad \text{if} \quad |z| < 1
\end{aligned}
$$

where ε^* is any positive number and $P_J(z)$ is a homogeneous polynomial of degree J in (x,y), $J < \alpha + \widetilde{\theta} - 4 - \varepsilon^*$.

It remains to evaluate

$$(8.29)\qquad \int_{-\infty}^{0} (-\xi)^{-1/2} P_{M-2}(\xi)(-\xi)^\rho (1-\eta(\xi)) M_1(z,\xi) d\xi$$

for $|z| < 1$. A typical term in the polynomial P_{M-2} contributes

$$(8.30)\qquad \int_{\overline{z}}^{z} dq(-q)^{-\rho} \left[\int_{-\infty}^{0} (-\xi)^{\frac{1}{2}+\rho+k} (1-\eta(\xi)) \frac{d\xi}{q-\xi} \right] \frac{K(z+\overline{z})}{2} \ .$$

where $k = -1, 0, \ldots M-3$. We use (8.13) to rewrite the expression in brackets in the form

$$\int_{-\infty}^{0} (-\xi)^{\frac{1}{2}+\rho+k}(1-\eta(\xi)) \left[-\frac{1}{\xi} \sum_{n=0}^{B} \left(\frac{q}{\xi}\right)^{n} + \left(\frac{q}{\xi}\right)^{B+1} \frac{1}{\xi - q} \right] d\xi$$

where B is chosen so that

$$B < \frac{1}{2} + \rho + k < B + 1 \ . \tag{8.31}$$

Each term in Σ_n contributes a homogeneous polynomial times $r^{-\rho}$, after substituting in (8.30). It remains to consider the last term:

$$\begin{aligned} & q^{B+1} \int_{-\infty}^{0} (-\xi)^{\frac{1}{2}+\rho+k}(1-\eta(\xi)) \frac{1}{\xi^{B+1}} \frac{d\xi}{\xi - q} \\ &= q^{B+1} \left\{ \int_{-\infty}^{0} (-\xi)^{\frac{1}{2}+\rho+k} \frac{1}{\xi^{B+1}} \frac{d\xi}{\xi - q} - q^{B+1} \int_{-\infty}^{0} (-\xi)^{\frac{1}{2}+\rho+k} \frac{\eta(\xi)}{\xi^{B+1}} \frac{d\xi}{\xi - q} \right\} \\ &= q^{B+1}(I_1 - I_2) \ . \end{aligned}$$

Suppose $Im q > 0$. Substituting in I_1 $\quad q = r\omega, \ \xi = ru$ we get

$$I_1 = r^{\frac{1}{2}+\rho+k+1} \omega^{B+1} \int_{-\infty}^{0} (-u)^{\frac{1}{2}+\rho+k} \frac{du}{u^{B+1}(u - \omega)} \ .$$

We can deform the u-contour into Γ as shown in Figure 5.

FIGURE 5

Then $|u-\omega|$ remains uniformly positive and by (8.31) the last integral is uniformly convergent. To determine $q^{B+1}I_1$ more sharply, we substitute in the last integral $u=\lambda\omega$. We get

$$q^{B+1}I_1 = q^{\frac{1}{2}+\rho+k+1+B+1}\int\limits_{\Gamma_q}(-1)^{\frac{1}{2}+\rho+k}\frac{\lambda^{\frac{1}{2}+\rho+k}}{\lambda^{B+1}(\lambda-1)}\,d\lambda$$

where Γ_q is obtained from Γ by multiplying by $1/\omega$. Note that Γ_q goes from 0 to ∞ above the x-axis and it avoids the point 1 as long as $Imq>0$. Hence we can modify Γ_q so as to obtain a contour independent of q. Consequently

$$I_1 = cq^{\frac{1}{2}+\rho+k+1+B+1}\,, \quad c \quad \text{constant.} \tag{8.32}$$

To evaluate I_2 we again use (8.13):

$$I_2 = \int\limits_{-\infty}^{0}(-\xi)^{\frac{1}{2}+\rho+k}\frac{\eta(\xi)}{\xi^{B+1}}\left[\frac{1}{\xi}\sum_{j=0}^{L}\left(\frac{q}{\xi}\right)^j-\left(\frac{q}{\xi}\right)^{L+1}\frac{1}{\xi-q}\right]d\xi$$

where

$$\frac{1}{2}+\rho+k-(B+1)<L+1 \quad \text{and} \quad A>B+1+L-\left(\frac{1}{2}+\rho+k\right)\,.$$

The terms in $\sum_j$ give homogeneous polynomials of degree j. In the last term we can deform the contour as before (recall that η is holomorphic); this term is then bounded by $O(q^{L+1})$ and is thus an error term if L is taken large enough.

The above analysis applies also to $Imq<0$ (with corresponding deformation of the contour). Substituting the results for $Imq>0$ and $Imq<0$ into (8.30) we obtain, up to an error term of large order,

$$\Sigma|z|^{\frac{1}{2}}\mu_j(x,y) \qquad (z=r\omega)$$

where μ_j are homogeneous polynomials in (x,y) of degree j with coefficients bounded by CN.

Together with (8.27), (8.28) we conclude that the function (8.26) is equal to a sum

$$r^{-\rho}\sum_j Q_j(x,y)+\sum_j r^{\frac{1}{2}}\mu_j(x,y) \tag{8.33}$$

where Q_j are homogeneous polynomial of degree j, plus an error term which can be estimated in the $3+\gamma$ norm (as in (8.28)).

The same analysis applies when M_1 is replaced by

$$K\frac{z+\overline{z}}{2}\int_{\overline{z}}^{z}\frac{q(-q)^{\rho}}{q-\xi}\,dq\ ,$$

and this concludes the evaluation of $D_x\psi_1$.

Similarly we can evaluate $D_y\psi_1$ and $D_x\psi_2, D_y\psi_2$. We obtain

$$\text{(8.34)}\qquad \begin{cases} |D^k[\psi-\Sigma r^{-\rho}Q_j-\Sigma r^{\frac{1}{2}}\mu_j]|\le Cr^{\alpha+\widetilde{\theta}-|k|-\varepsilon^*} & (|k|\le 4)\ , \\ |D^{k+\gamma}[\psi-\Sigma r^{-\rho}Q_j-\Sigma r^{\frac{1}{2}}\mu_j]|\le Cr^{\alpha+\widetilde{\theta}-4-\gamma-\varepsilon^*} & (|k|\le 4) \end{cases}$$

for $|z|<1$, with different polynomials Q_j and μ_j.

For $|z|>1$ we proceed in a similar way. The terms in (8.26) with factor $1-\eta$ can be estimated directly, and it only remains to evaluate the terms corresponding to $\Phi\eta$. In view of (8.21)

$$\left(\Phi(\xi)-\frac{C_1N}{(-\xi)^{3/2}}-\frac{C_2N}{(-\xi)^{5/2}}\right)\eta(\xi)$$

will again contribute just an error term $O(r^{\beta-\widetilde{\delta}+\varepsilon^*})$. Thus it remains to evaluate

$$K\frac{z+\overline{z}}{2}\int_{\overline{z}}^{z}dq(-q)^{-\rho}\left[\int_{-\infty}^{0}d\xi\frac{(-\xi)^{\rho}}{(-\xi)^{\ell/2}}\,\eta(\xi)\frac{1}{q-\xi}\right]\qquad(\ell=3,5)\ .$$

Using

$$\frac{1}{q-\xi}=-\frac{1}{\xi}\sum_{n=0}^{m}\left(\frac{q}{\xi}\right)^{n}+\left(\frac{q}{\xi}\right)^{m+1}\frac{1}{q-\xi}$$

we get, for each n, a term of the form (8.33). The integral corresponding to

$$\left(\frac{q}{\xi}\right)^{m+1}\frac{1}{q-\xi}$$

can be evaluated as before, making contour deformation (see Figure 5).

We find that

$$\text{(8.35)}\qquad \begin{cases} |D^k[\psi-\Sigma r^{-\rho}\widehat{Q}_j-\Sigma r^{\frac{1}{2}}\widehat{\mu}_j]|\le Cr^{\beta-\widetilde{\delta}-|k|-\varepsilon^*} & (|k|\le 4)\ , \\ |D^k[\psi-\Sigma r^{-\rho}\widehat{Q}_j-\Sigma r^{\frac{1}{2}}\widehat{\mu}_j]|\le Cr^{\beta-\widetilde{\delta}-4-\gamma-\varepsilon^*} & (|k|\le 4) \end{cases}$$

if $|z|>1$.

We now proceed as in the proof of Theorem 7.3 to show that

$$r^{-\rho} Q_j \quad \text{and} \quad r^{1/2} \mu_j$$

satisfy (8.1)–(8.4) with $g_1 \equiv g_2 \equiv 0$; hence they are eigenfunctions of the linearized problem. The same can be proved for

$$r^{-\rho} \widehat{Q}_j \quad \text{and} \quad r^{1/2} \widehat{\mu}_j \ .$$

The function

$$\widehat{\psi} \equiv \psi - \Sigma r^{-\rho} \widehat{Q}_j - \Sigma r^{1/2} \widehat{\mu}_j$$

then satisfies (8.34), for $|z| < 1$, with new polynomials $\widetilde{Q}_j = Q_j - \widehat{Q}_j, \widetilde{\mu}_j = \mu_j - \widehat{\mu}_j$. It also satisfies

$$\text{(8.36)} \qquad \begin{cases} |D^k \widehat{\psi}| \leq C N r^{\beta - \widetilde{\delta} - |k| + \varepsilon^*} & (|k| \leq 4) \ , \\ |D^{k+\gamma} \widehat{\psi}| \leq C N r^{\beta - \widetilde{\delta} - 4 - \gamma + \varepsilon^*} & (|k| = 4) \quad \text{if} \quad |z| > 1 \ . \end{cases}$$

Next as in Lemma 6.4 we can construct a function

$$\text{(8.37)} \qquad \varphi^* = \widehat{\psi} - e^{-z^2} \ \Sigma (A_j r^{-\rho} R_j + B_j r^{1/2} S_j) \ ,$$

where $r^{-\rho} R_j \ , r^{1/2} S_j$ are eigenfunctions of the linearized problem, such that φ^* vanishes at the origin to any given order. We have thus proved the following result:

THEOREM 8.2. *The function φ^* defined in (8.37) is a solution to (8.1)–(8.4) satisfying:*

$$\text{(8.38)} \qquad \|\varphi^*\|^{\alpha + \widetilde{\theta}, \beta - \widetilde{\delta}}_{\Sigma_{\theta_0'} \cup B_R^+, 4+\gamma} \leq C N$$

for any $\widetilde{\theta} < \theta$, $\widetilde{\delta} < \delta$, provided γ is sufficiently small.

Remark 8.1. Given H in (6.3) we have constructed a function φ^* in Theorem 8.2. This construction is uniquely defined independently of H if we fix J in (8.28) (say the largest integer such that $J < \alpha + \widetilde{\theta} - 4 - \varepsilon^*$), the largest indices j in (8.33), (8.35), as well as the order of approximation in choosing Q_j, R_j in (6.45), the φ_n in (7.29) and the R_j, S_j in (8.37). From now on we assume that such a choice has been made in the construction of φ^*. This ensures that φ^* will depend continuously on H, in suitable norm.

§**9. Proof of Theorem 1.1.** We introduce a Banach space of all pairs $(f_1(x), \psi_1(x,y))$ of functions with finite norm

$$\|(f_1, \psi_1)\| \equiv \|f_1\|^{\alpha,\beta}_{\Sigma_0^-, 4+\gamma} + \|\psi_1\|^{\alpha,\beta}_{\Sigma_{\theta_0}^-, 4+\gamma} < \infty \ ,$$

and denote by $X(\mu)$ ($\mu > 0$ the ball of radius μ centered at the origin.

By assumption

$$\|(f_0, \psi_0)\| \le \overline{C} < \infty \ .$$

For any $(f_1, \psi_1) \in X(\mu)$ consider the problem: Find $(F, \Psi) \in X$ satisfying (3.37)–(3.40).

By Lemma 3.1 we can estimate the right-hand side of (3.37) (with C^* replaced by μ). Thus, using Theorems 6.1, 7.3 and 8.2, we can construct a solution Ψ to (3.37)–(3.39) satisfying:

$$\|\Psi\|^{\alpha+\theta,\beta-\delta}_{\Sigma_{\theta_0'} \cup B_R^+, 4+\gamma} \le \widetilde{C}(\overline{C} + \varepsilon\mu)$$

for any $0 < \theta_0' < 1$, $R > 0$; $\widetilde{C}(t)$ is a continuous monotone increasing function. From (3.41) we also get

$$\|F\|^{\alpha+\theta,\beta-\delta}_{\Sigma_0^-, 4+\gamma} \le \widetilde{C}(\overline{C} + \varepsilon\mu) \ .$$

Choosing $\mu = 2[\widetilde{C}(\overline{C}+1)+1]$ and ε sufficiently small (say $\varepsilon\mu < 1$) we see that the mapping T defined by

$$T(f_1, \psi_1) = (F, \Psi)$$

maps $X(\mu)$ into itself.

The solution Ψ was constructed in a very specific way, as explained in Remark 8.1. Thus, if we set

$$T(f_1, \psi_1) = (F_1, \Psi_1), \ T(\widetilde{f}_1, \widetilde{\psi}_1) = (\widetilde{F}_1, \widetilde{\Psi}_1)$$

then Ψ_1 and $\widetilde{\Psi}_1$ were obtained by the same recipe, making the same corrections or subtractions of special solutions. From the structure of the H^i_j in (3.37)–(3.40) we easily see that they satisfy a Lipschitz condition in the appropriate norms (as in (3.34), (3.35)), with coefficient $c\varepsilon\mu$. Hence, by the estimates of Theorems 6.1, 7.3, 8.2 (and Remark 8.1) it follows that

$$\|T(f_1, \psi_1) - T(\widetilde{f}_1, \widetilde{\psi}_1)\| \le C\varepsilon\mu \ .$$

Thus, if ε is small enough then T is a contraction in $X(\mu)$ and, consequently, it has a fixed point. This completes the proof of Theorem 1.1.

We finally wish to prove that the assumption $(f_0, \psi_0) \in \mathcal{A}^{\alpha,\beta}_{\theta_0, 4+\gamma}$ is not very restrictive:

THEOREM 9.1. *Let $g(x)$ be any continuous positive function for $-\infty < x < 0$ such that*

$$\frac{h(x)}{(-x)^\alpha} \to A > 0 \quad \text{if} \quad x \to 0 \ , \tag{9.1}$$

$$\frac{h(x)}{(-x)^\beta} \to B > 0 \quad \text{if} \quad x \to -\infty \ . \tag{9.2}$$

Then for any $\mu > 0$ there exists a pair $(f_0, \psi_0) \in \mathcal{A}^{\alpha,\beta}_{\theta_0, 4+\gamma}$ such that

$$(9.3) \qquad |h(x) - f_0(x)| < \mu g(-x) \ , \ -\infty < x < 0 \ .$$

Here $g(r)$ is the function defined in (1.7).

Proof. Set $k(x) = h(-x) \quad (x > 0)$, so that

$$\frac{k(r)}{r^\alpha} \to A \quad \text{if} \quad r \to 0, \ \frac{k(r)}{r^\beta} \to B \quad \text{if} \quad r \to \infty \ .$$

Consider the holomorphic function

$$M(z) = \int_0^1 t^\gamma e^{-zt} dt \qquad (\gamma > -1) \ .$$

It satisfies

$$M(z) = \frac{1}{z^{\gamma+1}} \int_0^z u^\gamma e^{-u} du \approx \frac{c_0}{z^{\gamma+1}} \quad \text{if} \quad |z| \to \infty, |Imz| < |\operatorname{Re} z| \ ,$$

where $c_0 > 0$.

Set $\tau = \beta - \alpha + (\gamma + 1)$. Then

$$r^\tau M(r) \sim c_0 r^{\beta - \alpha} \quad \text{if} \quad r \to \infty \ .$$

Consider the function

$$(9.4) \qquad Q(z) = (\tanh z)^N z^\tau M(z) \ , \qquad N \gg 1 \ .$$

This is a holomorphic function in $\Sigma^+_{\theta_0}$ for some $\theta_0 > 0$, and $Q(0) = 0$.

Next we introduce the function

$$(9.5) \qquad J(r) = \frac{k(r)}{r^\alpha} - \frac{B}{c_0} Q(r)$$

and note that

$$J(r) \to A \qquad \text{if} \quad r \to 0 \ ,$$

$$J(r) = o(r^{\beta - \alpha}) \quad \text{if} \quad r \to \infty \ .$$

In particular, for any small $\eta > 0$,

$$|J(r)| < \eta r^{\beta-\alpha} \quad \text{if} \quad r \geq R_0 \tag{9.6}$$

where R_0 is large enough.

We now approximate $J(r)$ by a polynomial P in $0 \leq r \leq 4R_0$:

$$\begin{aligned} &|J(r) - P(r)| < \eta_0 \quad \text{if} \quad 0 < r < 4R_0 \ , \\ &P(0) = J(0) = A \end{aligned} \tag{9.7}$$

where η_0 is any given small positive number.

Let

$$S(z) = \frac{1 - \tanh L(z - 3R_0)}{1 - \tanh L(-3R_0)} \quad , \qquad L \gg 1 \ . \tag{9.8}$$

Then $S(r) \approx 1$ if $0 < r < 2R_0$ and so

$$|P(r)S(r) - P(r)| < \eta_0 \ .$$

Recalling (9.7) we deduce that

$$|J(r) - P(r)S(r)| < 2\eta_0 \quad \text{if} \quad 0 < r < 2R_0 \ . \tag{9.9}$$

If $2R_0 < r < 4R_0$ then by (9.6) and (9.7)

$$|P(r)| < \eta_0 + \eta r^{\beta-\alpha} \ ,$$

so that

$$|P(r)S(r)| \leq C(\eta_0 + \eta r^{\beta-\alpha})$$

(since $|S(r)| \leq C$) and, again using (9.6),

$$|J(r) - P(r)S(r)| \leq C(\eta_0 + \eta r^{\beta-\alpha}) \quad (2R_0 < r < 4R_0) \tag{9.10}$$

with another constant C.

Finally, if $r \geq 4R_0$ and $L \gg 1$ then

$$|P(r)S(r)| < \eta r^{\beta-\alpha}$$

and together with (9.6),

$$|J(r) - P(r)S(r)| \leq 2\eta r^{\beta-\alpha} \ , \ r \geq 4R_0 \ . \tag{9.11}$$

We may combine (9.9), (9.10) into

$$|J(r) - P(r)s(r)| < \eta_1 \ , \quad 0 < r < 4R_0 \tag{9.12}$$

where η_1 can be chosen any small positive number. From (9.11), (9.12) it follows that

$$\left|k(r) - r\left[\frac{G}{c_0}(\tanh r)^N r^\tau M(r) + P(r)S(r)\right]\right| < \mu g(r)$$

where μ can be taken arbitrarily small.

We may choose γ such that τ is an integer. Then the expression in brackets is the restriction to $z = r$ of a function holomorphic in z. Denote this function by $W(z)$. Then $f_0(x) = (-x)^\alpha W(-x)$ satisfies (9.3). Furthermore, from the form of $W(-z)$ we see that $(-z)^\alpha W(-z)$ has a series expansion of the form (2.52) with $\widetilde{\Gamma}_n = 0$ if $\alpha = n_0 + \dfrac{1}{2}$ and $\Gamma_n = 0$ if $\alpha = n_0 - \rho$, n_0 positive integer. Hence we can correspond to it an eigensolution (f_0, ψ_0) and, as can easily be checked, (f_0, ψ_0) belongs to $\mathcal{A}^{\alpha,\beta}_{\theta_0,4+\gamma}$.

Remark 9.1. The methods of the present paper should be helpful in studying flow problems in other geometries. Suppose for example that we modify problem (C) by considering the flow only in $\{0 < y < h\}$ (instead of $\{0 < y < \infty\}$) and imposing the condition

$$\vec{v} = \vec{U} + \varepsilon\vec{W} \quad \text{at} \quad \{y = h\}.$$

Using the linearization procedure of Section 3, we arrive at a modified version of the free boundary problem (C_ε) whereby

$$\psi_x = w_2, \ \psi_y = -w_1 \quad \text{on} \quad y = h \quad (W = (w_1, w_2)).$$

For $\varepsilon = 0$ we get a linear problem in $\{0 < y < h\}$ which we expect to have a *unique* solution (f_0, ψ_0) with $f_0(x) \approx A(-x)^\alpha$ as $x \uparrow 0$ (α of the form $n + 1/2$ or $n - \rho$). We anticipate that for suitably chosen w_1, w_2 the function $f_0(x)$ will satisfy $0 < f_0(x) < h$ if $-\infty < x < 0$ and $\lim\limits_{x\to-\infty} f(x) \in (0, h)$. The method of the present paper can probably be extended to establish existence of a solution to the modified problem (C_ε) with a free boundary $\varepsilon f_0(x) + \varepsilon^2 f_1(x, \varepsilon)$. We also anticipate that the solution will be unique.

We expect that the situation where there is a second free boundary $y = \widetilde{f}(x)$ ($a < x < \infty$) with $a > 0$, $\widetilde{f}(a) = h$, $\lim\limits_{x\to-\infty} f(x) \in [0, h)$, can also be handled by the above methods. We hope to establish the above conjectures in a future work. We shall also consider other geometries; for example, a flow between cylinders under small gavitational force.

Remark 9.2. Suppose we wish to solve numerically the coating problem for general Σ_0 by (say) a finite difference method. Then one may try to take very fine mesh near $y = 0$. This however is expensive, and, more importantly, it is known to cause instability in determining the location of the contact point. An alternate approach suggested by Theorem 1.1 is taken in the (not very fine mesh) an eigensolution (f_0, ψ_0), where

$$f_0 = A_0(x_0 - x)^{2-\rho} \qquad (x < x_0) \;, \tag{9.13}$$

with unspecified parameters A_0, x_0 $(A_0 > 0)$ and to determine these parameters so that this solution, in some rectangle $|x - x_0| < a$, $0 \le y \le b$, fits with the finite difference solution of the meshes neighboring the rectangle; the fit is to be taken in some average sense. For a coarser mesh it might be advisable to take

$$f_0 = A_0(x_0 - x)^{2-\rho} + A_1(x_0 - x)^{\frac{5}{2}} \qquad (x < x_0) \tag{9.14}$$

with unspecified parameters A_0, A_1, x_0 $(A_0 > 0)$. We do not know whether Theorem 1.1 extends to the case where

$$f_0(x) \approx A(-x)^{3/2} \quad \text{for} \quad x < 0, |x| \quad \text{small.} \tag{9.15}$$

If indeed it does, then this will suggest that one should take

$$f_0 = A_0(x_0 - x)^{3/2} \quad \text{or} \quad f_0 = A_0(x_0 - x)^{3/2} + A_1(x_0 - x)^{2-\rho} \qquad (x < x_0)$$

instead of (9.13) or (9.14). The case (9.15) will probably have to be studied in order to extend our results to the geometry described in Remark 9.1.

Remark 9.3. The results of this paper and their possible extensions as indicated in Remark 9.1 can probaly be extended to Navier–Stokes flows. The only difference occurs in the linearized equation which (instead of $\Delta\vec{G} = \nabla p$) becomes $\nabla\vec{G} + U\nabla\vec{G} = \nabla p$.

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