STABILITY ROBUSTNESS OF STATE SPACE SYSTEMS: INTER-RELATIONS BETWEEN THE CONTINUOUS AND DISCRETE TIME CASES

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Abstract: *Motivated by a robust exponential stability problem of a finite dimensional state space system, we chart in matrix terms the gap between the requirements for stability of LTI and NLTV systems. It turns out that these two cases are different for continuous and discrete time systems. This stems from the fact that in general convexisty is not preserved under the Cayley transformation from the left half plane to the unit disc, however in the case of quadratic stability it does. Four levels of stability of convex set of matrices are examined, each is illustrated by an example of a dynamical system.*

I. Introduction

The aim of this work is twofold,

(i) to investigate, in matrix terms, the gap between the requirements for stability robustness of two types of systems: Linear Time-Invariant and Non-Linear Time-Varying;

(ii) to explore some inter-relations n stability analysis of continuous and discrete time systems.

As a point of departure consider a finite dimensional, non-linear, time-varying system of the following form:

$$\dot{x}(t) = [F(t, x(t))]x(t) \quad , \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector and $t \geq 0$ represents time. Here F is a matrix valued function:

$$F : \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbf{X} \subset \mathbb{R}^{n \times n} \quad . \tag{2}$$

F in (1) may be *arbitrary,* provided that: *(i)* its values are in the prescribed convex set $\mathbf{X}$, and *(ii)* that from any initial condition, (1) admits a unique solution $x(t)$, for all $t \geq 0$. We wish to study the following stability robustness problem:

Without specifically knowing F , what conditions should be imposed upon $\mathbf{X}$ in order to guarantee the uniform exponential stability of the system (1-2) ?

For example, the case where $\mathbf{X}$ consists of a norm bounded ball around a "nominal" matrix was discussed in [17] and references herein. Now, if the system is known to be linear time invariant, namely F is a constant matrix, then robust stability amounts to the requirement that the real part of the spectrum of each matrix in $\mathbf{X}$ will be negative. The problem of ascertaining the stability of the convex set $\mathbf{X}$ by its extreme points was addressed in [2]. Clearly, stability of $\mathbf{X}$ is not sufficient in a more general case, see the examples in Section V. If no further restrictions are imposed upon F (like bounding $\|\dot{F}\|$ e.g. [15], [17]), then quadratic stability of $\mathbf{X}$ is a reasonable choice for a sufficient condition for robust Stability. In Section II we discuss the significance of resorting to this stringent type of stability. The problem of robust quadratic stability has received a considerable amount of attention in literature e.g. [1], [6], [8], [10], [12], [13], [16], [18], [19], [24].

Formally, if we denote by $\mathcal{P}$ the set of symmetric positive definite matrices, then the system (1-2) is said to be robustly quadratically stable if there exists a single $P \in \mathcal{P}$ which simultaneously satisfies all the algebraic Lyapunov inclusions

$$-(PA + A^T P + \varepsilon I) \in \mathcal{P} \quad , \quad \text{for all} \quad A \in \mathbf{X} \quad , \tag{3}$$

for some $\varepsilon > 0$. We shall refer to this P as a common Lyapunov solution for the set $\mathbf{X}$. For arbitrary fixed matrix $P \in \mathcal{P}$ we can define the following set

$$\mathcal{A}_P := \{ \ A \in \mathbb{R}^{n \times n} \quad | \quad -(PA + A^T P) \in \mathcal{P} \ \} \ . \tag{4}$$

In most of the references cited above robust quadratic stability is guaranteed by directly verifying that there exists a $P \in \mathcal{P}$ such that $(\mathbf{X} + \varepsilon I) \subseteq \mathcal{A}_P$ for some $\varepsilon > 0$. An easy-to-check, non conservative, sufficient condition for that, has not yet been found. In [3] a somewhat different approach was adopted namely to explore in matrix terms the gap between the two conditions: (3) and stability of $\mathbf{X}$. Here we further elaborate on this point showing that stability robustness consideration of *discrete* time systems emerges in quite a natural way. This enables us to consider distinct levels of stability, where condition (3) is the first one and stability of the convex set $\mathbf{X}$ is the fourth.

The organization of the papers is as follows. In Section III we introduce the four levels of stability and in Section V we explore the differences between then. In Section IV we investigate some of the analogies and differences in the study of stability robustness of continuous and discrete time systems.

II. Motivation for Quadratic Stability

In this section we examine, beyond mathematical convenience, our resort to the stringent condition of quadratic stability. We first remark that for simplicity of exposition throughout this work the discussion is global. However, since in practice quadratic stability is indeed restrictive, in many cases local refinements may be necessary. This can be achieved by substituting (2) by,

$$F \ : \ [0, T) \times \mathcal{S} \ \longrightarrow \ \mathbf{X} \subset \mathbb{R}^{n \times n} \ ,$$

where $T > 0$ and $\mathcal{S} \subseteq \mathbb{R}^n$ is a neighborhood of the origin.

Next, by citing known facts we show that if one wants to maintain: (i) exponential rate of convergence, and (ii) stability robustness, then each of these two requirements at least locally implies quadratic stability.

Almost trivially quadratic stability implies exponential stability. In this context, we wish to emphasize the known fact that *roughly* the converse is true as well. To this end we need the following preliminaries. We shall denote by $B_r \subset \mathbb{R}^n$, $r > 0$ the ball of radius r in $\mathbb{R}^n$, i.e.

$B_r := \{x \in \mathbb{R}^n \mid \|x\| \leq r\}$. The solution of (1) with initial condition $x(t_o) = x_o$ evaluated at the time $t \geq t_o$ will be denoted by $y(t, x_o, t_o)$. Recall that the equilibrium point of (1-2), assumed to be the origin, is said to be exponentially stable if it is possible to find an estimate of the form

$$\|y(t, x_o, t_o)\| \leq \alpha \|x_o\| e^{-\beta(t-t_o)} \quad , \quad \alpha \geq 1 \ , \ \ \beta > 0 \ ,$$

for all $x_o \in B_r$ for some $r > 0$ and for all $t \geq t_o$. Note that α and β may depend on r . If $r \longrightarrow \infty$ the equilibrium is said to be globally exponentially stable or equivalently the system is exponentially stable. The following fact is based upon a combination Theorems 26.5 and 56.1 in [11].

Theorem 1 : [21, Theorem 1.5.1]. *Assume that the vector* $[F(t, x(t))]x(t)$ *in (1) has continuous and bounded first partial derivatives in* x *and is piecewise continuous in* t *for all* $x \in B_r$, $t \geq t_o$. *Then, the following statements are equivalent:*

(i) The origin is an exponentially stable equilibrium point of (1).

(ii) There exists a function $v(t, x(t))$ *and some positive constants* $r', \alpha_1, \alpha_2, \alpha_3, \alpha_4$ *such that, for all* $x \in B_{r'}$, $t \geq t_o$

$$\alpha_1 \|x(t)\|^2 \leq v(t, x(t)) \leq \alpha_2 \|x(t)\|^2$$

$$\dot{v}(t, x(t)) \mid_{Eq.(1)} \leq -\alpha_3 \|x(t)\|^2$$

$$\left| \frac{\partial v(t, x(t))}{\partial x} \right| \leq \alpha_4 \|x(t)\| \ .$$

In particular, the choice of $v = x^T P x$, $P \in \mathcal{P}$ as a Lyapunov function where ε from (3) substitute for α_3 satisfies the conditions of the Theorem.

Quadratic stability not only provides us with an exponential rate of convergence of the solutions $|x(t)|$, but is also intimately related to robust stability, as the following explanation indicates. Consider a time invariant system $\dot{x} = [F(x)]x$ which is linearizable around the origin, and assume that it is just asymptotically, but not quadratically stable. From Lyapunov linearization method it follows that the matrix $F(0)$ necessarily has at least one eigenvalue with zero real part. Hence, one can find arbitrarily small perturbations of $F(0)$ that will render the system unstable. Namely, lack of quadratic stability implies high sensitivity. This observation was discussed in [5].

For simplicity of exposition of the matrix part we relax from now on the condition for quadratic stability and set $\varepsilon = 0$ in (3), or equivalently verify whenever $\mathbf{X} \subseteq \mathcal{A}_P$ for some $P \in \mathcal{P}$.

III. Levels of Stability - Introduction

In this section we introduce the levels of stability of convex set of matrices. To this end we need to resort to some algebraic notions. We start with convex invertible cones of matrices.

Definitions : [3]

- *A set* $\mathbf{X} \subset \mathbb{R}^{n\times n}$ *is said to be invertible if along with any nonsingular matrix* A *in it, it contains also its inverse* A^{-1}.
- *A Convex Invertible Cone of matrices is a set in* $R^{n\times n}$, *closed under addition, matrix inversion and positive scaling.*
- *For* $\mathbf{X} \subset \mathbb{R}^{n\times n}$ *we denote by* **cic** $(\mathbf{X})$ *the convex invertible cone generated by* $\mathbf{X}$, *namely, the smallest* **cic** *containing* $\mathbf{X}$.

Note the difference between the notions of invertible and nonsingular. As an illustration, the set of matrices of the form aI where $1 \leq a$ is a nonsingular cone, but it is not invertible. The sets of stable upper triangular and symmetric positive definite matrices are both non-singular **cic***s* ; while the sets of upper triangular and symmetric positive semidefinite matrices are both singular **cic**s . Convex invertible cones are discussed in [3], [4]. In Proposition 2 below we cite some of the relevant properties of **cic**s of matrices.

In order to gain further insight, we shall examine now the structure of a **cic** generated by several stable scalars. First, it is easy to see that **cic** $(e^{i\theta})$ contains in particular the convex hull of $e^{-i\theta}$ and $e^{i\theta}$ where θ is arbitrary between 0 and 2π . This implies that for a set of m stable scalars, namely $s_k = |s_k|e^{i\theta_k}$, where $\frac{\pi}{2} < \theta_k < \frac{3\pi}{2}$, we have,

$$\mathbf{cic}\ (s_1,\ s_2,\ \dots,\ s_m) = \mathbf{cic}\ (e^{i\theta_0}) = \{\ |s|e^{i\theta}\ \ |\ \ \theta \leq \theta_o := \max_{1\leq k\leq m} |\pi - \theta_k\ |\ \}\ , \tag{5}$$

namely a wedge symmetric with respect to $\mathbb{R}_-$ with angle opening of $2|\theta_o - \pi|$, see figure 1.

FIGURE 1

Intuitively, a **cic** is a natural algebraic set for complex scalars with negative real part. We now introduce the notion of a Convex Unsigned Product **cup** which is the algebraic analog of **cic** in the unit disc.

Definition : *Let* $\mathbf{X} \subset \mathbb{R}^{n\times n}$ *be a set, denote by* $\mathbf{cup}(\mathbf{X})$ *the minimal convex set containing* $\mathbf{X}$ *which is closed under matrix product and sign change.*

Remarks:

(i) To avoid confusion we shall designate by $\hat{s}$ or $\hat{A}$ elements in the unit disc or matrices with spectrum there, i.e. Schur stable.

(ii) Sometimes, for scalars we shall resort to a *complex* version of a **cup** namely the set which is closed under matrix product, convex combination and multiplication by arbitrary complex scalar of unity modulus. For example, if $\hat{x}_1, \hat{x}_2, \hat{x}_3 \in \hat{\mathbf{X}} \subset \mathbb{C}$, then $((\theta e^{i\psi}\hat{x}_1 + (1-\theta)e^{i\nu}\hat{x}_2)x_3) \in \mathbf{cup}(\hat{\mathbf{X}})$ for all $0 \le \theta \le 1$ and all $0 \le \psi, \nu < 2\pi$.

Trivial examples for **cup**s are the unit disc over the scalars, and for matrices the set $\hat{\mathcal{A}}_I$ of strict contractions, namely

$$\hat{\mathcal{A}}_I = \{ \ A \in \mathbb{R}^{n\times n} \ \ | \ \ \|A\| < 1 \ \} \ . \tag{6}$$

To gain further insight into the structure of the complex **cup** note that for an arbitrary set of m scalars $\hat{s}_k \in \mathbb{C}$, where $|\hat{s}_k| \le a < 1$,

$$\mathbf{cup}(\hat{s}_1, \ \hat{s}_2, \ \dots, \hat{s}_m) = \{ s \in \mathbb{C} \ \ | \ \ |s| \le a := \max_{1 \le k \le m} |s_k| \ \} \ , \tag{7}$$

namely a disc of a radius a centered at the origin.

in the sequel we shall make extensive use of the Cayley transformation from the open left half of the complex plane onto the interior of the unit disc, e.g [9, Chapter I.5]. This bilinear map $\mathcal{M}$ and its inverse $\mathcal{IM}$ are defined by,

$$\mathcal{M}(A) := (I + A)(I - A)^{-1} \ , \qquad\qquad A = \mathcal{IM}(\mathcal{M}(A)) := (\mathcal{M}(A) - I)(\mathcal{M}(A) + I)^{-1} \ .$$

$\mathcal{M}$ is a bijection between the sets of stable and Schur stable (i.e. spectral radius less than 1) matrices. For example, if we denote by $\mathcal{T}$ (respectively $\hat{\mathcal{T}}$) the set of stable (Schur stable) upper triangular matrices, then

$$\mathcal{M}(\mathcal{T}) = \hat{\mathcal{T}} \ . \tag{8}$$

Here, we cite some of the relevant properties of this map. Note that (i) for arbitrary nonsingular M we have

$$\mathcal{M}(MAM^{-1}) = M\mathcal{M}(A)M^{-1} \ , \tag{9}$$

and (ii) if A is nonsingular then,

$$\mathcal{M}(A^{-1}) = -\mathcal{M}(A) \ .$$

This observation indicates in a way, why in introducing the notion of **cic** we needed to resort to matrix inversion while in **cup** the analog operation is sign change.

Now, consider the image under the Cayley transformation of a **cic** generated by several stable scalars, (5), i.e. $\mathcal{M}(\ \mathbf{cic}\ (e^{i\theta_o}))$ where $\frac{\pi}{2} < \theta_o < \frac{3\pi}{2}$. It consists of the intersection of two discs of radius $\frac{1}{sin(\theta_o)}$ centered at the points $\{0\ \pm i\frac{cos(\theta_o)}{sin(\theta_o)}\}$. Namely, one disc is the minus $(-)$ of the other, and they always intersect at the points $\{\pm 1 + 0i\}$, see Figure 2.

Figure 2

For the direction from the unit disc to the left half plane, we have due to (7) that the set $\mathcal{IM}(\mathbf{cup}(\hat{s}_1,\ \hat{s}_2,\ \ldots\ ,\ \hat{s}_m))$ is the disc of a radius $\frac{2a}{1-a^2}$ centered at the point $(\frac{a^2+1}{a^2-1} + 0i)$, see Figure 3.

Figure 3

From the definition of a complex **cup** and the previous discussion it follows that for arbitrary scalar $s \in \mathbb{C}$ such that $Re(s) < 0$ the set $\mathbf{cup}(\mathcal{M}(\ \mathbf{cic}\ (s)))$ turns to be *all* the interior of the unit disc. Consequently, $\mathcal{IM}(\mathbf{cup}(\mathcal{M}(\ \mathbf{cic}\ (s))))$ is *all* the open left half plane. This suggests that $\mathbf{cic}\ (\mathcal{IM}(\mathbf{cup}(\mathcal{M}(\ \mathbf{cic}\ (\mathbf{X})))))$ is in general much larger then $\mathbf{cic}\ (\mathbf{X})$ itself. This observation motivates the introduction of level **(b)** of stability below, in addition to the three studied in [3].

In this stage we can present four distinct levels of stability for a convex set of matrices $\mathbf{X} \subset \mathbb{R}^{n\times n}$,

(a): The set $\mathbf{X}$ has a common Lyapunov solution;

(b): The set $\mathbf{cup}(\mathcal{M}(\ \mathbf{cic}\ (\mathbf{X})))$ is Schur stable.

(c): The set $\mathbf{cic}\ (\mathbf{X})$ is stable;

(d): The convex set $\mathbf{X}$ is stable.

In Section V we shall show that although **(a)** $\Longrightarrow$ **(b)** $\Longrightarrow$ **(c)** $\Longrightarrow$ **(d)**, neither of the converse implications is true in general. This will be demonstrated by examples of unstable dynamical systems.

We conclude this section, by stating a dynamical systems motivation for introducing the Cayley transformation. Let

$$x(k+1) = [\hat{F}(k, x(k))]x(k) , \tag{10}$$

be a discrete time system where $x \in \mathbb{R}^n$ is the state vector and $k \in \mathbb{N}$. As in the continuous case $\hat{F}$ is a matrix valued function:

$$\hat{F} \ : \ \mathbb{N} \times \mathbb{R}^n \ \longrightarrow \ \hat{\mathbf{X}} \subset \mathbb{R}^{n \times n} \quad . \tag{11}$$

$\hat{F}$ in (10) may be *arbitrary,* provided that its values are in the prescribed convex set $\hat{\mathbf{X}}$. As in the Introduction one can ask the question what are the conditions to be imposed upon $\hat{\mathbf{X}}$ in order to guarantee the stability of the system (10-11) without specifically knowing $\hat{F}$? Here, we want to address a related question namely, assuming that $\hat{\mathbf{X}}$ is the convex hull of $\mathcal{M}(\mathbf{X})$, under what conditions can we infer from the stability of the continuous time system (1-2) to the discrete time one (10-11)?

Recall that in general convexity is not preserved under the Cayley transformation. Hence, $\mathcal{M}(\mathbf{X})$ itself needs not be convex. This observation will turn to be crucial in the sequel.

IV. Discrete and Continuous Cases - Inter-Relations

Based on the definitions of the previous section we are going to examine some inter-relations between stability and Schur stability of sets. First, we cite the following list of properties of **cic**s . For the sake of completeness, the proof is given in the Appendix.

Proposition 2 : [3]

(i) *For all* $P \in \mathcal{P}$, $\mathcal{A}_P = M^{-1}\mathcal{A}_I M$, *where* M *is an arbitrary matrix such that* $M^T M \ = P$.

(ii) *The set* $\mathcal{A}_P$ *is a maximal open stable* **cic** .

(iii) *The set* $\mathcal{T}$ *of stable upper triangular matrices is a maximal stable* **cic** .

(iv) *For all* $P \in \mathcal{P}$ *the set* $\mathcal{T}$ *neither contains nor is contained in* $\mathcal{A}_P$.

In order to study the discrete version of the above properties, we need to introduce the set $\hat{\mathcal{A}}_P$ which is the analog of $\mathcal{A}_P$ for Schur stability, where Stein equation substitutes for Lyapunov equation. For $P \in \mathcal{P} \subset \mathbb{R}^{n\times n}$ define:

$$\hat{\mathcal{A}}_P := \{ A \in \mathbb{R}^{n\times n} \ \ | \ \ (P - A^T P A) \in \mathcal{P} \ \} \ .$$

In particular this definition conforms with the set of strict contractions $\hat{\mathcal{A}}_I$, (6).

Proposition 3 :

(i) *For all* $P \in \mathcal{P}$, $\hat{\mathcal{A}}_P = M^{-1}\hat{\mathcal{A}}_I M$, *where* M *is an arbitrary matrix such that* $M^T M = P$.

(ii) *For arbitrary* $P \in \mathcal{P}$ *the set* $\hat{\mathcal{A}}_P$ *is a maximal open Schur stable* **cup**.

(iii) *The set* $\hat{\mathcal{T}}$ *is a maximal Schur stable* **cup**.

(iv) *For all* $P \in \mathcal{P}$ *the set* $\hat{\mathcal{T}}$ *neither contains nor is contained in* $\hat{\mathcal{A}}_P$.

From (8) and Propositions 2*(iii)*, 3*(iii)* it follows that in the case of upper triangular matrices the Cayley transformation is a bijection between a maximal stable **cic** and a maximal Schur stable **cup**.

The proof of Proposition 3, given in the Appendix, relies heavily on Proposition 2 and Corollary 5 below, which in turn is based on the following important observation.

Lemma 4 : [23]. *Let* $A \in \mathbb{R}^{n\times n}$ *be a matrix, then for arbitrary* $P \in \mathcal{P}$,

$$-(PA + A^T P) \in \mathcal{P} \quad \Longleftrightarrow \quad (P - (\mathcal{M}(A))^T P \mathcal{M}(A)) \in \mathcal{P} \ .$$

It is interesting to note that P on both sides is identical. This relation was further studied in [22]. Using our notation (4), (12), leads to the following formulation.

Corollary 5 : *For arbitrary* $P \in \mathcal{P}$ *the following relation holds between the set of solutions of the Lyapunov and Stein inclusions,*

$$\mathcal{M}(\mathcal{A}_P) \ = \ \hat{\mathcal{A}}_P \ .$$

The special case $P = I$ was thoroughly studied in [7]. We shall now dwell on the significance of this result to dynamical systems.

We start by remarking that in the case of common Lyapunov/ Stein inclusion, in a way similar to triangular matrices, the Cayley transformation is a bijection between a maximal stable **cic** and a maximal Schur stable **cup**. However, as was already remarked, convexity is not preserved in general under the Cayley transformation. The following example illustrates this point.

Example 6 : (i) Consider the two following matrices,

$$A = \begin{pmatrix} 0.2 & -0.8 \\ 0.8 & -0.21 \end{pmatrix} , \qquad B = \begin{pmatrix} -0.21 & -0.8 \\ 0.8 & 0.2 \end{pmatrix} .$$

Direct calculation reveals that $conv(A,\ B)$ is stable. Applying the Cayley transformation results in,

$$\mathcal{M}(A) \approx \begin{pmatrix} 0.505 & -0.995 \\ 0.995 & -0.005 \end{pmatrix} , \qquad \mathcal{M}(B) \approx \begin{pmatrix} -0.005 & -0.995 \\ 0.995 & 0.505 \end{pmatrix} .$$

Now, the spectral radius of $\frac{1}{2}(\mathcal{M}(A) + \mathcal{M}(B))$ is approximately 1.026, so the convex combination of $\mathcal{M}(A)$ and $\mathcal{M}(B)$ is not Schur stable.

(ii) Consider the two following matrices,

$$\hat{A} = \begin{pmatrix} \frac{1}{3} & 0 \\ \frac{8}{5} & -\frac{1}{5} \end{pmatrix} , \qquad \hat{B} = \begin{pmatrix} \frac{1}{3} & \frac{8}{5} \\ 0 & -\frac{1}{5} \end{pmatrix} .$$

Direct calculation reveals that $conv(\hat{A},\ \hat{B})$ is Schur stable. Applying the inverse Cayley transformation results in,

$$\mathcal{IM}(\hat{A}) = \begin{pmatrix} -0.5 & 0 \\ 3 & -1.5 \end{pmatrix} , \qquad \mathcal{IM}(\hat{B}) = \begin{pmatrix} -0.5 & 3 \\ 0 & -1.5 \end{pmatrix} .$$

Now, $\lambda_{1,2}(\frac{1}{2}(\mathcal{IM}(\hat{A}) + \mathcal{IM}(\hat{B}))) \approx +0.58,\ -2.58$ hence $conv(\mathcal{IM}(\hat{A}),\ \mathcal{IM}(\hat{B}))$ is not stable. □

From Example 6 it is clear that if the system (1-2) is known to be linear time-invariant, namely F is a constant matrix, then its stability does not imply the stability of the system (10-11) where $\hat{\mathbf{X}}$ is taken to be the convex hull of $\mathcal{M}(\mathbf{X})$. Similarly, linear time-invariant (i.e. Schur) stability of (10-11) does not imply the stability of (1-2) where $\mathbf{X}$ is the convex hull of $\mathcal{IM}(\hat{\mathbf{X}})$.

Next, recall that the system (10-11) is said to be robustly quadratically stable if there exists a $P \in \mathcal{P}$ such that $(1+\varepsilon')\hat{\mathbf{X}} \subseteq \hat{\mathcal{A}}_P$ for some $\varepsilon' > 0$. Hence, Corollary 5 implies that unlike mere LTI stability, robust quadratic stability is equivalent for the continuous and discrete time systems.

The difference in behavior between LTI and quadratic stability under the Cayley transformation discussed above, can be reformulated in matrix terms. A set $\mathbf{X}$ in level **(a)** of stability is said to statisfy a *global* Lyapunov condition. By Corollary 5, this property is mapped under the

Cayley transformation to a *global* Stein condition. However, it was shown in [2, Section 4] that a set satisfying a *local* Lyapunov condition, which is intimately related to level **(d)** of stability, is not mapped under the same transformation to a set with *local* Stein property.

V. Levels of Stability

In this section we prove that the four levels of stability of convex sets of matrices, introdurced in Section III, are indeed and still one is implied by the other.

First we show that indeed **(a)** $\Longrightarrow$ **(b)** $\Longrightarrow$ **(c)** $\Longrightarrow$ **(d)**. Let us Assume that condition **(a)** holds, namely $\mathbf{X} \subseteq \mathcal{A}_P$ for some $P \in \mathcal{P}$. Now, from Proposition 2*(ii)* we have that also $\textbf{cic}\ (\mathbf{X}) \subseteq \mathcal{A}_P$. Applying the Cayley transformation on both sides together with Lemma 4 yields $\mathcal{M}(\ \textbf{cic}\ (\mathbf{X})) \subseteq \hat{\mathcal{A}}_P$. From Proposition 3*(ii)* it follows now that also $\textbf{cup}(\mathcal{M}(\ \textbf{cic}\ (\mathbf{X}))) \subseteq \hat{\mathcal{A}}_P$ which in particular implies that $\textbf{cup}(\mathcal{M}(\ \textbf{cic}\ (\mathbf{X})))$ is Schur stable, i.e. condition **(b)** is satisfied. The implications **(b)** $\Longrightarrow$ **(c)** $\Longrightarrow$ **(d)** are trivial.

We shall now demonstrate by examples of dynamical systems that neither of the converse implications is true in general. We start with examples of two unstable systems one is nonlinear time-invariant and the second is linear time-varying. In both cases condition **(d)** does not imply **(c)**.

Example 7 : [17] Consider the following matrices:

$$A_1 = \begin{pmatrix} -4 & 15.9 \\ 0.9 & -4 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} -0.1 & 12 \\ -3 & -0.1 \end{pmatrix}.$$

It is easy to verify that the convex hull of A_1 and A_2 is stable. However, the time-invariant system (1) defined by,

$$F(x) = \begin{cases} A_1 & 1 \geq \frac{x_2}{x_1} \geq \frac{31}{119} \\ A_2 & \text{else} \end{cases}, \tag{13}$$

where $x := (x_1, x_2)^T$, turns out to be unstable. Indeed, since this system is piecewise linear it can be solved explicitly. Starting from $x_1(0) = c,\ x_2(0) = c$, the system switches at time 0.345 and then again at time 0.764. At these points the resulting trajectory passes through

$$(2.3c, 0.6c)^T = e^{0.345 A_1}(c, c)^T \quad , \qquad (-1.11c, -1.11c)^T = e^{0.419 A_2}(2.3c, 0.6c)^T \quad .$$

Namely, at $t = 0.764$, $\|x(t)\| \approx 1.11\|x(0)\|$. In fact, the solution of this system diverges from any nonzero initial point, and in the average, $\|x(t)\| \approx \|x(0)\|e^{0.142t}$ for all $t \geq 0$, see Figure 4.

FIGURE 4

The idea behind this construction is the following. Let $\mathbf{X}$ in this case be the convex hull of A_1 and A_2 . Among all the systems with values in $\mathbf{X}$, the matrix F was chosen so that the quantity $\frac{d}{dt}\|x\|^2$ is maximal at each moment, namely the "least stable" one. Now since $\frac{d}{dt}\|x\|^2 = 2x^T F x$, it is natural to choose $F = A_1$ whenever $x^T(A_1 - A_2)x \geq 0$ and $F = A_2$ otherwise. This leads to (13).

Indeed, although the convex hull of A_1, A_2 is stable, namely condition **(a)** is satisfied, the **cic** generated by these matrices is not stable. For example the eigenvalues of the matrix $(A_1 + 9A_2^{-1})$ are approximately $+0.6, -8.6$.

Example 8 : [20]. Consider the following *linear time varying* system:

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} -1 - 9cos^2 6t + 6sin12t & 12cos^2 6t + \frac{9}{2}sin12t \\ -12sin^2 6t + \frac{9}{2}sin12t & -1 - 9sin^2 6t - 6sin12t \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} .$$

Although for all t the eigenvalues of the system matrix $F(t)$ are $-1, -10$ still, from every nonzero initial condition the solution diverges., i.e. $|x_1(t)| \approx c_1 e^{+2t}$, $|x_2(t)| \approx c_2 e^{+2t}$, where c_1, c_2 depend upon the initial conditions.

The convex invertible cone generated by $F(t)$ where we "freeze" the time is not stable. For example taking

$$F(t=0) = \begin{pmatrix} -10 & 12 \\ 0 & -1 \end{pmatrix} , \qquad F(t = \frac{\pi}{12}) = \begin{pmatrix} -1 & 0 \\ -12 & -10 \end{pmatrix} ,$$

yields $\lambda_{1,2}[F(t = \frac{\pi}{12}) + 10(F(t=0))^{-1}] \approx +6.65, -18.65$.

□

In [3] it was shown that for a pair of 2×2 matrices existence of a stable **cic** , i.e. condition **(c)**, implies already a common Lyapunov solution, i.e. condition **(a)**. In the following example we present a triple of 2×2 matrices that satisfy condition **(c)**, but not condition **(b)**.

Example 9 : Consider the matrices

$$A = \begin{pmatrix} -14 & 13 \\ 0 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} -1 & 0 \\ 13 & -14 \end{pmatrix}, \qquad C = \begin{pmatrix} -1 & 0 \\ 0 & -14 \end{pmatrix}.$$

It was shown in [3] that the **cic** generated by this triple is stable. Now clearly also $\mathcal{M}($ **cic** $(A,\ B,\ C))$ is Schur stable, but as we are to show yet **cup**$(\mathcal{M}($ **cic** $(A,\ B,\ C)))$ is not Schur stable.

Let us define the following linear time-varying periodic system:

$$x(k+1) = [\hat{F}(k)]x(k) \ ,$$

$$\hat{F}(4k) = \mathcal{M}(\frac{1}{\sqrt{14}}A) \quad , \qquad \hat{F}(4k+1) = \mathcal{M}(C^{-1})$$

$$\hat{F}(4k+2) = \mathcal{M}(\frac{1}{\sqrt{14}}B) \quad , \qquad \hat{F}(4k+3) = \mathcal{M}(\frac{1}{14}C) \ ,$$

where $k = 0,\ 1,\ 2,\ \ldots$ The computation is simplified if we use (9), noting that $B = UAU$ and $\frac{1}{14}C = UC^{-1}U$, where $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. If we take $\mathbf{X}$ to be the convex hull of the triple $\{A,\ B,\ C\}$ then clearly $\hat{F} \in$**cup**$(\mathcal{M}($ **cic** $(\mathbf{X})))$. It turns out that for $k = 1,\ 2,\ \ldots$

$$\begin{pmatrix} x_1(4k) \\ x_2(4k) \end{pmatrix} = \gamma^k \begin{pmatrix} x_1(0) \\ \frac{x_1(0)}{2} \end{pmatrix} \ ,$$

where $\gamma = (\frac{2}{15}(15 - 2\sqrt{14})^2 \approx 1.0045$. Hence, the solution diverges for all initial conditions $x_1(0) \neq 0$. □

Finally we show that condition **(b)** does not imply condition **(a)**. To this end we shall examine the choice $\mathbf{X} = \mathcal{T}$, the set of stable upper triangular matrices. From (8) and Propositions 2*(iii)*, 3*(iii)* we have that $\hat{\mathcal{T}} \subseteq$**cup**$(\mathcal{M}($ **cic** $(\mathcal{T})))$ and in particular Schur stable. However Proposition 2*(iv)* states that the set of stable upper triangular matrices does not satisfy condition **(a)**.

We conclude this section by further investigating the difference between conditions **(a)** and **(b)** for the set $\mathcal{T}$. It turns out that on one hand this gap is narrow in matrix terms (Proposition 10), but on the other hand from the dynamical systems point of view it is indeed meaningful (Example 11).

The following proposition show that although $\mathcal{T}$ does not have a common Lyapunov solution (Proposition 2*(iv)*), still one can construct large subsets of $\mathcal{T}$ for which condition **(a)** is satisfied.

Proposition 10 : [3] *A set of stable upper triangular matrices has a common Lyapunov solution provided that the quantity*

$$\tilde{t} := (\max_{1 \le i < j \le n} |\ t_{ij}\ |\ ) \ / \ (- \max_{1 \le i \le n} Re(t_{ii})\ ) \ ,$$

is uniformly bounded.

Strictly speaking, the converse of Proposition 10 is not true, see [3]. However, the following example indicates that roughly this is almost the case.

Example 11 : [14, Exercise 9.1-3]. Consider the following *linear time varying* system:

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} -1 & e^{2t} \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} .$$

Although $F(t) \in \mathcal{T}$ for all t with eigenvalues $-1, -1$ still, for the initial conditions $x(0) = (0,\ 1)^T$ we have $x(t) = (\frac{e^{+t} - e^{-t}}{2},\ e^{-t})^T$, $t \geq 0$ which is diverging. Indeed, the quantity $\tilde{t}$ in Proposition 10 is not bounded. □

VI. Concluding Remarks

We summarize here the major points discussed in this paper. First, whenever exponential stability is desired and only the set $\mathbf{X}$, the range of F , is known quadratic stability is a natural requirement. With this motivation, four distinct levels of stability for a convex set $\mathbf{X} \in \mathbb{R}^{n\times n}$ were studied charting in matrix terms the gap between the stability requirements for LTI and NLTV systems.

The possibility of using the Cayley transformation for deducing robust stability of a discrete time system from a continuous time one was examined. It was shown that while for mere LTI stability this is in general impossible, in the more restrictive case of quadratic stability, the conditions for continuous and discrete time systems are equivalent. Quadratic stability is applicable to non-linear time-varying systems.

Finally we remark that due to the duality between the continuous and discrete time cases, one could equivalently examine the following four levels of stability for a convex set of matrices $\hat{\mathbf{X}} \subset \mathbb{R}^{n\times n}$,

(a'): The set $\hat{\mathbf{X}}$ has a common Stein solution;

(b'): The set **cic** $(\mathcal{IM}(\mathbf{cup}(\hat{\mathbf{X}})))$ is stable.

(c'): The set **cup** $(\hat{\mathbf{X}})$ is Schur stable;

(d'): The convex set $\hat{\mathbf{X}}$ is Schur stable.

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Appendix : Proofs

Proof of Proposition 2 :

(i) First, the assumption implies that $M = UP^{1/2}$ where $U \in \mathbb{R}^{n\times n}$ is an arbitrary orthogonal matrix. Let $A_o \in \mathcal{A}_I$ i.e. $A_o = -Q + S$, where $Q \in \mathcal{P}$, $S + S^T = 0$. Denoting $A = M^{-1}A_oM$ it is easy to verify that,

$$PA + A^TP \;=\; -2P^{1/2}U^TQUP^{1/2} \quad .$$

Now, since $(P^{1/2}U^TQUP^{1/2}) \in \mathcal{P}$ we have that indeed $A \in \mathcal{A}_P$. This proves *(i)*. (A similar statement can be formulated for nonsingular $M \in \mathbb{C}^{n\times n}$ where $M^*M = P$.)

(ii) We shall show that $\mathcal{A}_I$ is a maximal open regular **cic** . Using *(i)* the claim follows for arbitrary $\mathcal{A}_P$. It is straightforward to show that $\mathcal{A}_I$ is a regular **cic** and an open set. Moreover, we shall show that if a matrix $B \notin \overline{\mathcal{A}}_I$, the closure of $\mathcal{A}_I$, then **cic** $(\mathcal{A}_I, B)$ is not regular. To avoid triviality assume that the matrix B is stable. Let us write B as follows:

$$B = \frac{A+A^T}{2} + \frac{A-A^T}{2} = U^TDU + \frac{A-A^T}{2} \qquad ,$$

where U is orthogonal and D is diagonal. Now, since B is stable, but not in $\overline{\mathcal{A}}_I$ it implies that D is invertible but not definite. Let us choose $C \in \mathcal{A}_I$ as

$$C = U^T|D|U + \frac{A^T-A}{2} \qquad .$$

Clearly, the matrix $B + C = U^T(D + |D|)U$ is singular, so *(ii)* is established.

(iii) This proof works over the complex field as well. For an arbitrary stable matrix $A = [a_{i,j}] \notin \mathcal{T}$ we shall construct a matrix $T = [t_{i,j}] \in \mathcal{T}$ such that $A + T$ is singular. First, we may always

take $T = T' + T"$ where $T" \in \mathcal{T}$, $T' = [t'_{i,j}]$ is strictly upper triangular such that $t'_{i,j} = -a_{i,j}$ for all $1 \leq i < j \leq n$, and zero otherwise. Consequently, we have that the matrix $A + T'$ is lower triangular, say

$$A + T' = \begin{pmatrix} a_{1,1} & O \\ \phi & \tilde{A} \end{pmatrix} ,$$

where $\tilde{A} \in \mathbb{C}^{(n-1)\times(n-1)}$ is lower triangular and $\phi \in \mathbb{C}^{n-1}$. If $\tilde{A} = 0$ we are done, so assume this is not the case. Set

$$T" = \begin{pmatrix} -1 & \eta^* \\ 0 & \tilde{T} \end{pmatrix} ,$$

where $\eta \in \mathbb{C}^{n-1}$ and $\tilde{T} \in \mathcal{T}_{n-1}$ are to be determined.

If $\phi \neq 0$ choose $\tilde{T} = -2\|\tilde{A}\|I$ and η so that $\eta^*(\tilde{A} - 2\|\tilde{A}\|I)^{-1}\phi = a_{1,1} - 1$. In this case the matrix $A + T$ is singular and the vector $(1\ ,\ \phi^*(-\tilde{A} + 2\|\tilde{A}\|I)^{-1})^*$ is in its null space.

If $\phi = 0$ then $A + T = (a_{1,1} - 1) \oplus (\tilde{A} + \tilde{T})$ and the problem is reduced to finding $\tilde{T}$ such that the matrix $\tilde{A} + \tilde{T}$ is singular. Since by assumption $\tilde{A}$ is lower triangular, but not diagonal, so the proof can easily be completed by induction and *(iii)* is established.

(iv) First, we show that for an arbitrary fixed $P \in \mathcal{P}$ there is always a matrix $T \in \mathcal{T}$ such that $T \notin \mathcal{A}_P$. To simplify the argument we shall consider only 2×2 matrices. Up to a positive scaling, all matrices in $\mathcal{P}$ can be represented in the form $P = \begin{pmatrix} 1+\alpha & \beta \\ \beta & 1-\alpha \end{pmatrix}$, $\alpha^2 + \beta^2 < 1$, and let $T \in \mathcal{T}_2$, $T = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$. Now if $|\ b\ | >> max\ (\ |a|,\ |c|\ )$, it can be seen that $\frac{1}{b}(PA + A^TP)$ is a small perturbation of the matrix $C = \begin{pmatrix} 0 & 1+\alpha \\ 1+\alpha & 2\beta \end{pmatrix}$. Since $det(C) < 0$ it implies that C is *indefinite* matrix, so it can not be a small perturbation of a definite matrix.

In order to show that $\mathcal{A}_P \not\subset \mathcal{T}$ consider the following construction. For arbitrary $P \in \mathcal{P}$ let T‘ be an arbitrary non-zero strictly *lower* triangular matrix so that $\|T‘\| < \frac{\min_k \lambda_k(P)}{\max_k \lambda_k(P)}$. Then, the matrix $-I + T$‘ is in $\mathcal{A}_P$ but not in $\mathcal{T}$, so the proof is complete. □

Proof of Proposition 3 :

(i) This follows from (9) and Proposition 2*(i)*, still we shall prove it directly. The assumption implies that $M = UP^{\frac{1}{2}}$ where $U \in \mathbb{R}^{n\times n}$ is an arbitrary orthogonal matrix. Let $\hat{A}_o \in \hat{\mathcal{A}}_I$,

namely $(I - \hat{A}_o^T \hat{A}_o) \in \mathcal{P}$. Denoting $\hat{A} = M^{-1} \hat{A}_o M$, it is easy to see that,

$$P - \hat{A}^T P \hat{A} = M^T (I - \hat{A}_o^T \hat{A}_o) M \in \mathcal{P} \ ,$$

hence $A \in \hat{\mathcal{A}}_P$.

(ii) Due to *(i)* all there is to consider is the set $\hat{\mathcal{A}}_I$. The properties of **cup** follow imediately from the fact that $\hat{\mathcal{A}}_I$ is the set of strict contractions. For maximality, assume B is not in the closure of $\hat{\mathcal{A}}_I$, namely $B = U diag\{ \sigma_1, \sigma_2, \ldots , \sigma_n \} V^*$ where U, V are unitary matrices and there exists some $\varepsilon > 0$ s.t. $\sigma_1 \geq 1 + \varepsilon$. Let $A = \frac{2}{2+\varepsilon} V U^*$. Clearly, $A \in \hat{\mathcal{A}}_I$, denoting by $\rho(\bullet)$ the spectral radius, we have $\rho(AB) \geq \frac{2(1+\varepsilon)}{2+\varepsilon} > 1$ and hence $\mathbf{cup}(\hat{\mathcal{A}}_I \cup B)$ is not Schur stable.

(iii) Immediate if we follow the proof of 2*(iii)* where instead of A we use $\mathcal{M}(A)$ which is a Schur stable matrix but not upper triangular, and $\hat{\mathcal{T}}$ substitutes for $\mathcal{T}$, (8).

(iv) Follows trivially from (8), Proposition 2*(iv)* and Corollary 5. Still we shall show it independently. Fix a matrix $P \in \mathcal{P}$. Due to *(i)* if $\hat{A} \in \hat{\mathcal{A}}_P$ it implies that $\|\hat{A}\| \leq \sqrt{\frac{\max_k \lambda_k(P)}{\min_k \lambda_k(P)}}$. Now the matrix $\begin{pmatrix} 0 & 2\sqrt{\frac{\max_k \lambda_k(P)}{\min_k \lambda_k(P)}} \\ 0 & 0 \end{pmatrix}$ is in $\hat{\mathcal{T}}$ but not in $\hat{\mathcal{A}}_P$. □

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