On Order Types of Spatiotemporal Dimensions

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Abstract

It has been suggested that mathematical truth can be accounted for or explained in terms of infinitary computations. Certain Zenonian arguments against the possibility of performing such computations raise problems about the structure of time and space. We study the kinds of spatiotemporal structures that would allow for infinitary computations, and identify where further work is needed.

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Chapter 0

Introduction

This study has two main aims. One aim is to explain a few of the ways in which infinite sequences of events are of philosophical interest. The other aim is to investigate the various senses in which it is possible for infinite sequences of events to occur. In particular, we examine which kinds of spatiotemporal structures allow for the occurrence of such sequences.

These two aims are interconnected. The plausibility of philosophical views that make use of infinite sequences of events depends in part on the senses in which it is possible for such sequences to occur. In this way, the philosophical import of these sequences depends in part on the senses in which it is possible for them to occur. Also, questions about which sorts of infinite sequences of events could possibly occur are themselves of interest, tracing back to several arguments of Zeno of Elea. Once these arguments are made more precise, strengthened, and generalized, they raise problems about the order types of possible spatiotemporal dimensions, and about how to conceive of spatiotemporal locations and continuous motion. As we will see, these Zenonian arguments depend on assumptions about the relationship between the order type of the temporal dimension and the order types of the spatial dimensions.

Infinite sequences of events are worthy of philosophical study for several reasons. First, it has been suggested that the conceptual possibility of performing infinite sequences of computations allows for accounts of (our knowledge of) various portions of mathematical truth. Mathematical truth, as well as these kinds of accounts of it, and the issues surrounding these accounts, are topics of independent interest. On such accounts, the truth of a mathematical statement, or our knowledge of its truth, is analyzed or explained in terms of the existence of a formal proof — finite *or infinite* in length — of that statement. Verifying a step in such a proof can be seen as a task, so that verifying an infinite-length proof requires completing an infinite sequence of tasks. These kinds of accounts of (our knowledge of) mathematical truth can be motivated by Zermelo's views about infinitary languages and infinitary logics, which allow infinitely long expressions and infinitely long proofs, respectively. Zermelo's perspective sheds light on many metamathematical results of recursion theory, proof theory, and model theory, which show certain limitations of finitary logical systems.

Mathematical properties of infinite-length proofs also bear on debates in the foundations of mathematics. For example, mathematical theories differ with respect to their provability strength, and various foundational positions disagree about how much strength is acceptable. Certain theorems about infinitary proofs can be seen as indicating that if one accepts a particular infinitary notion of provability, then one can accept the notion of truth in any mathematical theory that has a particular degree of provability strength. Of course, each such theorem is itself provable only in theories with a sufficient degree of provability strength. This presents a number of issues about the philosophical implications of proof-theoretic results.

Another reason that infinite sequences of events are worth studying is that they raise difficult problems involving various concepts of *time*, *space*, *motion*, *continuity*, *divisibility*, *set*, and *class*, among others. For example, there is a family of paradoxes, called dichotomy arguments, concerning motion through infinitely divisible space and time. Analysis of dichotomy arguments leads naturally to considering spatiotemporal structures in which dimensions differ from one another in various ways with respect to their order types. These kinds of possible structural "mismatches" between time and space give rise to problems about the nature and definiteness of spatiotemporal locations. Making sense of locations in these sorts of "mismatched" spaces, and of continuous motion through them, involves subtle mathematical issues.

Uncountable sequences of events are of special interest. Performing uncountably many tasks in principle allows one to come to know certain non-trivial propositions that could not be known merely by performing countably many tasks. But not every way that time could be structured allows for the completion of that many tasks. We will argue that the order type of the temporal dimension constrains what can be known, by limiting the types of task-sequences that can be performed. In our example, only if the structure of time allows for the completion of uncountably many tasks can a particular situation involving $\aleph_0$ physical objects be known to obtain.

Chapter 1 discusses accounts of first-order arithmetical truth in terms of infinitary proof, and covers a few related issues. We address:

- ≻ What reasons are there for accepting infinitary notions of proof in foundational debates?
- ≻ How can truth in first-order arithmetic be accounted for using infinitary proofs?
- ≻ How do considerations about effectiveness bear on various notions of infinitary proof?

Chapter 2 analyzes paradoxes involving the infinite divisibility of time and space, and discusses various order types that those dimensions could instantiate. We address:

- ≻ How can we make sense of motion through infinitely divisible space and time?
- ≻ Is it possible for the dimensions of time and space to differ structurally from one another?

Chapter 3 argues that the structure of time has epistemological consequences, constraining which propositions are knowable. We address:

- ≻ How can the mathematical structure of time limit the scope of what is knowable?
- ≻ What kinds of propositions can we come to know only by doing uncountably many tasks?

It is worth making a few remarks at this point. Given a general interest in infinite sequences of events, we should be permissive about what we count as an "event". The type of event that we will often be concerned with is the performance of some kind of task. Sometimes these tasks will be the performance of a kind of computation. Other times these tasks will be the performance of a certain type of movement. We do not wish to count every kind of motion as involving a task, though we will count the occurrence of any kind of motion as being an event. We require that each event in an event-sequence (i.e., a sequence of events) take its own non-trivial interval (i.e., an interval containing more than one point) of time, with the intervals being pairwise non-overlapping. We also require that an event-sequence be linearly ordered in time, though it need not be well-ordered in time.[1]

A few definitions will be useful.

- A countably infinite sequence of events is a *superevent.*
  If each event corresponds to a unique task, then the sequence is called a *supertask.*
- An uncountable sequence of events is a *hyperevent.*
  If each event corresponds to a unique task, then the sequence is called a *hypertask.*
- A hyperevent that includes a distinct event for each ordinal number is an *ultraevent.*
  If each event corresponds to a unique task, then the sequence is called an *ultratask.*
- For each infinite linear order type τ,
  a τ-sequence (i.e., a sequence of order type τ) of events is a τ-*event.*
  If each event corresponds to a unique task, then the sequence is called a τ-*task.*

The presentation in Chapter 2 is unorthodox in that it avoids topology, algebra, and measure theory, which are often important in discussions of space and time. But many problems involving infinite sequences of events can be formulated without mentioning those areas of mathematics. These problems have to do with the order types of the dimensions of space and time, rather than with their topological, algebraic, or metrical structure. For this reason, we will set aside issues involving topologies, measures, and metrics. We should explain this choice here.

Standardly, a sequence of tasks is required to occur entirely within a *finite-length* of time, so that the task-sequence is completed in a finite-length duration. It would typically be required

[1] Recall that a *linear ordering* is total, transitive, and antisymmetric. A linear order is a *well-order* iff each of its non-empty subcollections has a minimal element under the ordering.

that under an appropriate notion of *measure*, the interval of time which may be taken to complete a sequence of tasks is *finite* in measure. Most of the issues that we will be concerned with do not depend on whether the interval of time that is taken to finish the task-sequence has a finite measure. What is important is that the sequence occurs entirely within an interval of time that

(**i**) contains sufficiently many pairwise non-overlapping subintervals, with each being "long enough" to allow for the completion of one task,

and

(**ii**) is bounded, meaning that there are intervals that lie entirely before the interval, and ones that lie entirely after the interval.

Requirement **i** need not be understood in terms of a measure. Given our purposes, for an interval of time to be "long enough" to allow for the doing of a single task, it suffices that the interval is non-trivial. For example, this rules out completing a task in a single moment (i.e., a point in time), but it does not rule out completing a task in an infinitesimal interval of time.

Requirement **ii** ensures that there are times that are before any task in the sequence begins, and that there are times that are after every task in the sequence ends. In these respects, measure-theoretic requirements — e.g., that the interval be finite in measure — seem irrelevant.

We will also be ignoring issues involving the various possible topologies of spatiotemporal structures.[2] Our study of spatiotemporal dimensions will be mainly order-theoretic, which simplifies the presentation. For example, what it is for a position function in this context to be *continuous* can be defined purely in terms of order, without using topology. We will see that the additional structure imposed by a topology and a metric is inessential to certain easily stated problems about infinite divisibility, spatiotemporal locations, and continuous motion.

[2] Allowing topological spaces to be proper-class-sized and even collection-sized, then using the order and product topologies, all of these spatiotemporal structures are Hausdorff spaces. Also, none are required to be locally Euclidean.

Our discussion will be phrased in terms of points and intervals. But one can also consider pointless analogs of these structures as locales. See Picado & Pultr (2012). In this setting, the pointless analog of the Banach-Tarski theorem fails. For more discussion, see Bauer (2016).

Chapter 1

Portions of mathematical truth

The main purpose of this chapter is to motivate the study of infinite sequences of computations. First we motivate and describe a way of understanding first-order arithmetical truth in terms of proofs that can be countably infinite in length. We consider whether such accounts are extensionally adequate, in terms of capturing every true statement of first-order arithmetic. Then we criticize several arguments purporting to show that such an account fails to handle certain arithmetical sentences that Peano Arithmetic neither proves nor refutes. Finally, we argue that performing a hypertask is compatible with the truths of ZFC.

1.1 True arithmetic

If we count *checking a step in a proof* as a task, then checking each step in a proof that is countably infinite in length may be seen as amounting to a supertask, in which we check a countably infinite sequence of such steps. The conceptual possibility of performing supertasks allows for one kind of account of the notion *truth in T*, where T is in a certain class of theories. A *theory* is a set of sentences (of some language) that is deductively closed (under that theory's corresponding notion of provability).

Theories of particular interest to us will be ones in the language of first-order arithmetic.[1] More generally, we will be interested in characterizing the class of theories whose notions of truth can be adequately accounted for in terms of the possibility of performing supertasks. This approach can be extended to hypertasks, thereby including proofs of uncountable lengths, in

[1] An interesting question is how these classes of theories relate to the various hierarchies measuring the strength of theories, such as interpretability hierarchies (using various notions of interpretability), consistency strength hierarchies (relative to various notions of provability), hierarchies that classify theories by their arithmetical consequences, and others. For more discussion, see Lindström (2017, Chapter 6) and Visser (2006).

order to account for notions of truth in consistent theories that are stronger than those which can be accounted for in terms of proofs that are countable in length.

Given this approach to accounting for truth in a theory, there is a kind of situation that we should be careful to avoid. We are interested in notions of truth in various *consistent* theories, i.e., theories in which no sentence of the theory's language is such that it is both *provable* in that theory and *refutable* in that theory. Note that the consistency of a theory is defined in terms of provability *in that theory.* There are consistent theories which when altered to have a stronger notion of provability — so that additional sentences are provable — result in a theory that is inconsistent. We should be careful to avoid trying to use an inconsistent theory to account for notions of truth in its consistent subtheories.[2]

How might infinite sequences of computations support or explain the notion of *truth in arithmetic*, i.e., arithmetical truth? In 1921, Weyl mentioned the possibility of (dis)proving certain statements in the language of first-order number theory by completing an ω-sequence of computations. After a discussion of Zeno's paradox of Achilles and the tortoise, Weyl (2009) writes:

> ...if the segment of length 1 really consists of infinitely many subsegments of length $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$, as of 'chopped-off' wholes, then it is incompatible with the character of the infinite as the 'incompletable' that Achilles should have been able to traverse them all. If one admits this possibility [of traversing a physical distance that consists of infinitely many subsegments], then there is no reason why a machine should not be capable of completing an infinite sequence of distinct acts of decision within a finite amount of time; say, by supplying the first result after $\frac{1}{2}$ minute, the second after another $\frac{1}{4}$ minute, the third $\frac{1}{8}$ minute later than the second, etc. In this way it would be possible ... to achieve a traversal of all natural numbers and thereby a sure yes-or-no decision regarding any existential question about natural numbers! (p. 42)

For certain simple sentences of arithmetic, each numerical instance of the sentence can be tested sequentially. In order for this idea to support or explain *truth in a theory* T, it must apply to *every* sentence of T.

When T is a theory, an adequate notion of *truth in* T is thought to require that the theorems of T be definite (or determinate) in the following respect: T must partition the collection of all sentences of the language of T into exactly two subcollections — one consisting of those sentences which are provable in T, and the other consisting of those sentences which are refutable in T. However, a consistent collection of axioms of a language $\mathcal{L}$ (or the corresponding theory) may be *negation-incomplete*, meaning that there is an $\mathcal{L}$-sentence σ such that neither σ nor $\neg\sigma$ is provable from those axioms under the corresponding notion of provability.[3] It is thought that

[2] E.g., if we begin with a consistent ω-inconsistent theory, and expand it to allow certain infinitary rules of proof, the resulting theory is inconsistent. We define ω-inconsistency in §1.3.1, footnote 47.

[3] Otherwise the collection of axioms, or the resulting theory, is said to be *negation-complete*.

whenever a theory T is negation-incomplete, the notion of *truth in* T is not adequately clear. Benacerraf & Putnam (1984) express this thought as follows:[4]

> In [first-order] number theory too [(as in set theory)] there are statements that are neither provable nor refutable from the axioms of present-day mathematics [(if they are jointly consistent)]. Intuitionists might agree that this shows (not by itself, of course, but together with other considerations) that we do not have a clear notion of "truth" in [first-order] number theory, and that our notion of a "totality of all integers" is not precise. Most mathematicians would reject this conclusion. Yet most mathematicians feel that the notion of an "arbitrary set" is somewhat unclear. What is the reason for this difference in attitude?
>
> Perhaps the reason is that a verification/refutation procedure is inconceivable for number theory only if we require that the procedure be effective. If we take the stand that "non-constructive procedures" — i.e., procedures that require us to perform infinitely many operations in a finite time — are conceivable, . . . then we can say that there does "in principle" exist a verification/refutation procedure for number theory. . . . Of course, [a procedure like] this requires working forever, or else completing an infinite series of operations in a finite time. . . . What this shows is: The notion of "truth" in number theory is not a dubious one if the notion of a completed actually infinite series (of, say, definitely specifiable physical operations) is itself not dubious.
>
> . . . In short, if you understand such notions as *counting*, *adding*, *multiplying*, and *seeing if two numbers are equal*, [then] we can explain to you the notion of a "true statement of number theory" (though not if you consistently "boggle" at all quantifications over an infinite domain) . . . (pp. 19–20)

Setting aside the issue of whether this kind of account of truth can also apply to various set theories, let us stay with the case of first-order number theory. We will motivate and develop one such account of first-order arithmetical truth in this section and the one following.

True arithmetic (TA) is the set of all sentences of the language $\mathcal{L}_A$ of first-order arithmetic that are true in the standard model $\mathfrak{N}$ of the Dedekind-Peano axioms (including every instance of the first-order induction schema) of *Peano arithmetic* (PA).[5]

One would like to characterize TA with a recursive set of codes of axioms whose closure under syntactic logical entailment is TA.[6] But the set of codes of the sentences in TA is not recursive or even recursively enumerable, so capturing TA requires some degree of additional means.[7] A

[4] Unfortunately, Benacerraf & Putnam use "series" instead of "sequence".

[5] TA is $\mathrm{Th}(\mathfrak{N})$, the theory of $\mathfrak{N}$.

[6] We assume throughout that PA is consistent.

[7] We will ignore deviant codings, e.g., under which all and only sentences in TA are coded by even natural numbers. See Feferman (1960).

We will also ignore deviant definitions of 'axiom' and 'proof'. Feferman (1990) mentions that

> . . . the applicability of Gödel's theorem on the underivability in a consistent [formal] system

recurring theme will be that PA is not strong enough to provide adequate grounds on which to base an account of first-order arithmetical truth.

Definitions

Let L denote standard classical finitary first-order logic, with equality. In order to include infinitary logics in our discussion, we need to define a few terms not defined by Gödel (1931).

A *formal language* consists of a recursively enumerable set of wffs, which are finite sequences of symbols.[8]

A *formal system* S consists of:

(i) a formal language $\mathcal{L}_S$,

(ii) a recursively enumerable set A_S of axioms (which are certain $\mathcal{L}_S$-sentences),

(iii) a recursive relation $\vdash_S$ which determines the proofs in S.

A *semi-formal system* is like a formal system except that nothing need be recursive, or even recursively enumerable.

We will say that a (semi-)formal system is *fully recursive* iff the sets corresponding to its wffs and its axioms are both recursive, and its provability-relation is recursive as well.

For each $n \in \mathbb{N}$, let $\overline{n}$ be the numeral for n.

For each $\mathcal{L}_A$-wff φ, let $\#(\varphi)$ be the code of φ, and let $\ulcorner \varphi \urcorner$ be the numeral for $\#(\varphi)$.[9]

In order to explain how PA falls short of capturing first-order arithmetical truth, we need to mention a few metamathematical results. A corollary of Rosser's (1936) strengthening of Gödel's (1931) first incompleteness theorem is that for any sufficiently strong consistent formal system S that is fully recursive:[10]

> S of the consistency statement Con_S depends essentially on *how* S is presented. That is, ...for suitable S, another presentation S^* of S could be given, with the same set of theorems, for which $S^* \vdash \mathrm{Con}_{S^*}$. In *Takeuti 1955* this was done by changing the set of rules generating the theorems, in *Feferman 1960* by changing the description of the set of axioms, and in *Kreisel 1965* by changing the description of the set of proofs. (p. 282)

[8] For each positive integer n, we write $\varphi(v_1, \ldots, v_n)$ to indicate that φ has exactly n free variables.

[9] From the context, it should be clear which coding this is relative to.

[10] In 1947, Turing (Ince, 1992) remarks that:

> ...if a machine is expected to be infallible, it cannot also be intelligent. There are several mathematical theorems which say almost exactly that. (p. 124)

There is an $\mathcal{L}_S$-wff $\varphi(v_1)$ such that $\forall n \in \mathbb{N},\ \vdash_S \varphi(\overline{n})$, but $\nvdash_S \forall x\, \varphi(x)$.

This property of (semi-)formal systems is called *ω-incompleteness.* In such a case, if

$$\nvdash_S \neg\forall x\, \varphi(x),$$

then S is negation-incomplete. PA's ω-incompleteness is witnessed by PA's Gödel sentences (among others), which are neither PA-provable nor PA-refutable.[11]

We have just stated a *syntactic* form of the first incompleteness theorem for $\mathcal{L}_A$-theories. Some *semantic* forms of the first incompleteness theorem state that for certain $\mathcal{L}_A$-theories, there are $\mathcal{L}_A$-sentences that are true in $\mathfrak{N}$ but are not provable in that theory. Before stating generalizations of these theorems, we need a few more definitions.

First we define the arithmetic hierarchy of $\mathcal{L}_A$-formulas.

A Σ_0-*formula*, Π_0-*formula*, or Δ_0-*formula* is an $\mathcal{L}_A$-wff that contains no unbounded quantifiers.

For each $n \geq 1$, we define Σ_n-formulas and Π_n-formulas inductively.

For each $n \geq 1$, an $\mathcal{L}_A$-wff is a Σ_{n+1}-*formula* iff it has the form

$$\exists \vec{x}\ \varphi(\vec{x}, y),$$

where φ is a Π_n-formula.

For each $n \geq 1$, an $\mathcal{L}_A$-wff is a Π_{n+1}-*formula* iff it has the form

$$\forall \vec{x}\ \varphi(\vec{x}, y),$$

where φ is a Σ_n-formula.

For each $n \geq 1$, an $\mathcal{L}_A$-wff is a Δ_n-*formula* iff it is PA-provably equivalent to both a Σ_n-formula and a Π_n-formula.

For example, here are the special cases of Π_1-formulas, Σ_1-formulas, and Π_2-formulas.

A Π_1-*formula* is an $\mathcal{L}_A$-wff of the form

$$\forall y\ \psi(\vec{x}, y),$$

where ψ contains no unbounded quantifiers.

A Σ_1-*formula* is an $\mathcal{L}_A$-wff of the form

$$\exists y\ \psi(\vec{x}, y),$$

[11] We sometimes speak of a (semi-)formal system as if it were just the set of sentences that are provable from its axioms, relative to that system's notion of a *proof.*

where ψ contains no unbounded quantifiers.

A Π_2-*formula* is an $\mathcal{L}_A$-wff of the form

$$\forall y\ \psi(\vec{x}, y),$$

where ψ is a Σ_1-formula.

Recall that for any function f and $X \subseteq dom(f)$,

$$f[X] := \big\{f(x) \colon x \in X\big\}.$$

Recall that a countable theory T is *recursively enumerable* iff $\#[T]$ is recursively enumerable.

For each $n \in \mathbb{N}$, an $\mathcal{L}_A$-theory T is Σ_n-*sound* iff for every Σ_n-sentence σ,

$$\vdash_T \sigma \quad \Longrightarrow \quad \mathfrak{N} \models \sigma.$$

For each $n \in \mathbb{N}$, an $\mathcal{L}_A$-theory T is Π_n-*complete* iff it is negation-complete with respect to the set of Π_n-sentences, i.e., for every Π_n-sentence σ,

$$\vdash_T \sigma \quad \text{or} \quad \vdash_T \neg\sigma.$$

Otherwise we say that T is Π_n-*incomplete*.

A countable theory T is *definable in* $\mathfrak{N}$ iff $\#[T]$ is definable in $\mathfrak{N}$.

For each $n \in \mathbb{N}$, an $\mathcal{L}_A$-theory T is Σ_n-*definable* iff there is a set T' of $\mathcal{L}_A$-sentences such that both

$$\big\{\sigma \colon T' \vdash_{\mathsf{L}} \sigma\big\} = T,$$

and there is a Σ_n-formula $\varphi(v_1)$ such that $\forall m \in \mathbb{N}$

$$m \in \#\big[T'\big] \quad \Longleftrightarrow \quad \mathfrak{N} \models \varphi(\overline{m}).$$

I.e., there is a set T' of $\mathcal{L}_A$-sentences whose closure under L is T, and the set of codes of the sentences in T' is definable in $\mathfrak{N}$ by a Σ_n-formula.

For each $n \in \mathbb{N}$, the set of $\overline{\Pi}_n$-*sentences* is the closure of the set of Π_n-sentences under conjunction, disjunction, existential quantification and bounded universal quantification.

For each $n \in \mathbb{N}$, an $\mathcal{L}_A$-theory is $\overline{\Pi}_n$-*complete* iff it is negation-complete with respect to the set of $\overline{\Pi}_n$-sentences. Otherwise it is $\overline{\Pi}_n$-*incomplete*.

For each $n \in \mathbb{N}$, an $\mathcal{L}_A$-theory T is n-*consistent* iff for each Σ_n-formula $\varphi(v_1)$,

$$\forall n \in \mathbb{N} \quad \vdash_T \varphi(\overline{n}) \quad \Longrightarrow \quad \nvdash_T \exists x\ \neg\varphi(x).$$

Finally, let Q denote Robinson arithmetic.

1.1.1 Forms of imbalance

A nice way of understanding semantic forms of the first incompleteness theorem is that they result from an imbalance between the *syntactic* (proof) power of L and the *semantic* (expressive) power of $\mathcal{L}_A$. For an example of a distinction along these lines, Carnap (1931) distinguishes between logicism's *definabilty* thesis (that mathematical concepts are explicitly definable in terms of purely logical concepts), and logicism's *provability* thesis (that mathematical theorems are derivable from purely logical theorems by purely logical means).

One semantic form of the first incompleteness theorem about $\mathcal{L}_A$-theories states that for each $n \geq 1$:[12]

For each Σ_n-sound Σ_{n+1}-definable $\mathcal{L}_A$-theory T, there is a $\overline{\Pi}_{n+1}$-sentence that is true in $\mathfrak{N}$ but neither T-provable nor T-refutable.

In the case of $n = 1$, this implies that: if S is a set of $\mathcal{L}_A$-sentences, and

(i) $\#[S]$ is definable in $\mathfrak{N}$ by a Σ_2-formula,

and

(ii) for each Σ_1-sentence σ,

$$S \vdash_{\mathsf{L}} \sigma \quad \Longrightarrow \quad \mathfrak{N} \models \sigma,$$

then there is a $\overline{\Pi}_2$-sentence σ such that

$$\mathfrak{N} \models \sigma,$$
$$S \nvdash_{\mathsf{L}} \sigma,$$
$$S \nvdash_{\mathsf{L}} \neg\sigma.$$

In this case, L-provability is not strong enough to ensure the $\overline{\Pi}_2$-completeness of

$$\{\sigma : S \vdash_{\mathsf{L}} \sigma\}$$

when the set S of $\mathcal{L}_A$-axioms meets conditions i and ii. In this sense, there is an imbalance between the (semantic) expressiveness of $\mathcal{L}_A$ and the (syntactic) L-provability relation.

The purely *syntactic* forms of the first incompleteness theorem can be viewed analogously as resulting from an imbalance between *provability* in L and *representability*, which is a purely

[12] See Theorem 2.5 in Salehi & Seraji (2016).
Generalizing semantic versions of the first incompleteness theorem in the setting of institution theory is an open problem. For a discussion of institution theory, see Diaconescu (2008).

syntactic notion. For example, one syntactic form of the first incompleteness theorem about $\mathcal{L}_A$-theories states that for each $n \geq 1$:[13]

> Each n-consistent Σ_{n+1}-definable $\mathcal{L}_A$-theory that extends Q is $\overline{\Pi}_{n+1}$-incomplete.

In the case of $n = 1$, this implies that: if S is a set of $\mathcal{L}_A$-sentences, and

(1) $\#[S]$ is definable in $\mathfrak{N}$ by a Σ_2-formula,

(2) for each $\sigma \in \mathsf{Q}$,

$$S \vdash_{\mathsf{L}} \sigma,$$

and

(3) for each Σ_1-formula $\varphi(v_1)$,

$$\forall n \in \mathbb{N} \quad S \vdash_{\mathsf{L}} \varphi(\overline{n}) \quad \Longrightarrow \quad S \nvdash_{\mathsf{L}} \exists x\, \neg\varphi(x),$$

then there is a $\overline{\Pi}_2$-sentence σ such that

$$S \nvdash_{\mathsf{L}} \sigma,$$

$$S \nvdash_{\mathsf{L}} \neg\sigma.$$

In this case, provability in L is not strong enough to ensure the $\overline{\Pi}_2$-completeness of

$$\{\sigma \colon S \vdash_{\mathsf{L}} \sigma\}$$

when the set S of $\mathcal{L}_A$-axioms meets conditions 1, 2, and 3. This too is an imbalance between the expressiveness of $\mathcal{L}_A$ and the L-provability relation.

Representation systems

Here is a nice illustration of an imbalance in several theorems that apply to countable theories even in languages other than $\mathcal{L}_A$. Smullyan (1961, Chapter 3) defines a *representation system* as an ordered sextuple

$$\langle E, S, T, R, \mathscr{P}, \Phi \rangle,$$

where:

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[13] See Theorem 4.3 in Salehi & Seraji (2016).

E is a countably infinite set of *expressions*,

$S \subseteq E$ is the set of *sentences*,

$T \subseteq S$ is the set of *theorems* (i.e., those sentences that that system proves),

$R \subseteq S$ is the set of *refutable sentences* (i.e., those sentences that that system refutes),

$\mathscr{P} \subseteq E$ is the set of *unary predicates*, and

$$\Phi\colon E \times \mathbb{N} \longrightarrow E$$

is a total "representation" function such that

$$\forall H \in \mathscr{P} \quad \forall n \in \mathbb{N} \quad \Phi\big(\langle H, n\rangle\big) \in S.$$

For any $A \subseteq \mathbb{N}$, a unary predicate H *represents* A iff

$$A = \Big\{ n \in \mathbb{N}\colon \Phi\big(\langle H, n\rangle\big) \in T \Big\}.$$

A representation system is *consistent* iff

$$T \cap R = \emptyset.$$

A representation system is *complete* iff

$$T \cup R = S.$$

Smullyan (1961) proves three main theorems that are relevant here:

(1) The complement (with respect to $\mathbb{N}$) of the set of codes of the provable sentences of a representation system is not representable in that system.

(2) If the set of codes of the refutable sentences of a representation system is representable in that system, then that system is either inconsistent or incomplete.

(3) If a representation system can represent a subset of $\mathbb{N}$ that is both

(i) a superset of the set of codes of that system's refutable sentences, and

(ii) disjoint from the set of codes of that system's provable sentences,

then that system is incomplete.

The point is that these three theorems are syntactic, and each can be seen as describing a kind of imbalance between what is *provable* in a representation system and what is *representable* in one.

This notion of an imbalance between provability and representability also applies to cases in which our language is infinitary, and even uncountable. Questions about this kind of (im)balance make sense even when both the provability strength of a system and its representability strength are characterized by infinitary notions. In a manner of speaking, provability-strength and representability-strength come in degrees, even when they are infinitary.

We can generalize Smullyan's notion of a *representation system* to include languages of any infinite cardinality. Instead of taking a predicate to represent a subset of $\mathbb{N}$, we can take a predicate to represent a subset of an infinite cardinal κ, with κ equal to — or otherwise depending on — the cardinality of the set of expressions. Instead of coding syntactic entities like formulas, proofs, etc. using natural numbers, we can code them using ordinal numbers (including non-recursive ordinals).[14] One can then prove generalizations of the Tarski, Gödel, and Rosser theorems, respectively, about representability in these transfinite representation systems.[15]

Löwenheim-Skolem theorems

The Löwenheim-Skolem theorems can also be understood in terms of a syntactic-semantic imbalance in set-sized first-order languages that are finititary. First we should define what it is for a system to be "finitary".

A language is *finitary* iff it has only countably many formulas, and each formula is finite in length.

A logic is *finitary* iff each of its proofs are finite in length, and finite in width (i.e., each step in each proof requires only finitely many premises).

A system is *finitary* iff both its language and its logic are finitary.

To explain how the Löwenheim-Skolem theorems indicate an imbalance between the *syntactic* power of such systems and their *semantic* power, we will state a version of these theorems.

---

[14] Recall that an ordinal α is *recursive* iff there is a recursive well-ordering, with order type α, of a subset of $\mathbb{N}$.

[15] For an extension of the incompleteness theorems to the context of transfinite computation, see Patarin (2012).

Löwenheim-Skolem Theorems

Let $\mathcal{L}$ be any set-sized first-order language that is finitary.

Let κ be a cardinal such that

$$|\mathcal{L}| \leq \kappa.$$

Let $\mathfrak{A}$ be an $\mathcal{L}$-structure such that

$$\aleph_0 \leq |\mathfrak{A}| \leq \kappa.$$

Downward: $\mathfrak{A}$ has elementary substructures of every cardinality λ for

$$\aleph_0 \leq \lambda \leq \kappa.$$

Upward: $\mathfrak{A}$ has elementary extensions of every cardinality greater than κ, and of proper-class-size.

In the important special case of finitary first-order languages that are countably infinite:

For each infinite model $\mathfrak{M}$ for a finitary first-order countably infinite language $\mathcal{L}$, there are $\mathcal{L}$-models of every infinite cardinality that are elementarily equivalent to $\mathfrak{M}$, and $\mathcal{L}$-models of proper-class-size that are elementarily equivalent to $\mathfrak{M}$.

The Löwenheim-Skolem theorems show that no set of sentences of any finitary first-order system can distinguish between various *infinite sizes* of mathematical structures. If a set of sentences of such a language is satisfied by an infinite model (for that language), then that set of sentences is also satisfied by elementarily equivalent models (for that language) of every infinite cardinality, and of proper-class-size.

One way to reduce these various kinds of imbalance is to move to a logic with the same *semantic* power as that of L, but with somewhat more *syntactic* power. In particular, we will consider expanding PA by allowing proofs to have infinite ordinal-lengths, with a rule of inference using an ω-sequence of premisses.[16] The resulting theory does not avoid every kind of imbalance that we have described. For example, it still has elementarily equivalent models of different infinite sizes.[17] But unlike PA, the theory that we will consider is negation-complete.[18]

[16] In 1934, Carnap (2003, §14) allowed proofs of infinite lengths in his Language I, using a rule that is similar to the ω-rule that we will define in §1.2.

[17] Recall that TA has nonstandard models.

[18] Two relevant questions are:

Now we motivate the move away from PA by describing a few of its shortcomings.

1.1.2 Recursiveness outstrips PA

This section discusses limitations of PA having to do with *recursiveness*, in order to motivate considering stronger theories in order to capture TA. In §1.2, we will describe theories with the same axioms as PA but with stronger provability relations.

There is a precise sense in which PA does not fully capture the notion of "being a total recursive function". In order to see this, we will need a few definitions.

> Recall that a k-ary relation R on $\mathbb{N}^k$ is *definable in* $\mathfrak{N}$ iff
>
> there is an $\mathcal{L}_A$-wff $\varphi(v_1, \ldots, v_k)$ such that $\forall n_1, \ldots, n_k \in \mathbb{N}$,
>
> $$\langle n_1, \ldots, n_k \rangle \in R \iff \mathfrak{N} \models \varphi(\overline{n_1}, \ldots, \overline{n_k}),$$
>
> in which case we say that φ *defines* R.

For each k-ary total recursive function

$$f \colon \mathbb{N}^k \longrightarrow \mathbb{N},$$

there is an Σ_1-formula that defines (the graph of) f. Recall also that every total recursive function is representable in PA by a Σ_1-formula. In particular, for each unary total recursive function f,

there is a Σ_1-formula $\varphi(v_1, v_2)$ that defines f such that $\forall x, y \in \mathbb{N}$,

$$f(x) = y \implies \vdash_{\mathsf{PA}} \varphi(\overline{x}, \overline{y}),$$

$$f(x) \neq y \implies \vdash_{\mathsf{PA}} \neg\varphi(\overline{x}, \overline{y}).$$

A total recursive function that is not PA-provably recursive

Kirby & Paris (1982) showed that there are true finitary statements, which are number-theoretic (rather than metamathematical) in character, that are not PA-provable. In particular, there is a unary total recursive function — called the Goodstein function — that PA does not prove is total, even though the function is definable in $\mathfrak{N}$.[19]

Q1: What are the various senses in which a system can be "balanced"?

Q2: How can we make sense of what it is for a system to be "maximally balanced"?

[19] For other such functions, see Kaye (1991, Chapter 14) and Hájek & Pudlák (2017, Chapter IV, §3).

Let the Goodstein function be defined by the Σ_1-formula $G(v_1)$, named after Goodstein's theorem.

Now, the Π_2-sentence that expresses *the fact that the Goodstein function is a total function* is true in the standard model, i.e.,

$$\mathfrak{N} \models \forall x\, \exists! y\; G(x) = y.$$

PA proves each numerical instance of that sentence;

$$\forall n \in \mathbb{N} \quad \vdash_{\mathsf{PA}}\; \exists! y\; G(\overline{n}) = y, \text{ so}$$

$$\forall n \in \mathbb{N} \quad \vdash_{\mathsf{PA}}\; \exists y\; G(\overline{n}) = y, \text{ but}$$

$$\nvdash_{\mathsf{PA}}\; \forall x\; \exists y\; G(x) = y.$$

This is another instance of PA's ω-incompleteness. In this sense, PA does not prove anything that is equivalent to a statement that expresses the fact that the Goodstein function is total.[20] As we will say, the Goodstein function is not PA-*provably* total. For any particular number, PA proves that when one inputs that number to the Goodstein function, there is a unique number that the function outputs. But PA cannot prove that for any number that one inputs to the Goodstein function, the function outputs a number. In this way, the total recursiveness of the Goodstein function is beyond PA.

Total recursive functions that are not PA-provably total yield arithmetical statements that are neither PA-provable nor PA-refutable. Each of those statements expressess that one of those functions is total.

PA cannot fully capture the notion of *being a total recursive function.* It can be shown that the predicate "is the code of a total recursive function" is not representable in PA. If that predicate were representable in PA, then there would be an $\mathcal{L}_A$-formula $\varphi(v_1)$ such that $\forall n \in \mathbb{N}$,

$$n \text{ is the code of a total recursive function} \quad \Longrightarrow \quad \vdash_{\mathsf{PA}}\; \varphi(\underline{n}),$$

$$n \text{ is } \textit{not} \text{ the code of a total recursive function} \quad \Longrightarrow \quad \vdash_{\mathsf{PA}}\; \neg\varphi(\underline{n}).$$

But this could not be the case. Firstly, the set of indices of total recursive functions is not recursive.[21] So the characteristic function of *the set of indices of total recursive functions* is

[20] Since the Goodstein function is total, one might think that

the statement that *the Goodstein function is total*

is equivalent to

the statement that $0 = 0$.

In this sense, there is a PA-provable sentence that is "equivalent" to the statement that the Goodstein function is total. But we have in mind a stronger notion of equivalence.

[21] This is a corollary of Rice's Theorem. See Theorem II.2.9 in Odifreddi (1992, §II.2).

not a total recursive function. But the total recursive functions correspond exactly to the functions that are representable in PA. So the characteristic function of *the set of indices of total recursive functions* is not representable in PA. Thus, the predicate "is the code of a total recursive function" is not representable in PA.

Consistent decidable $\mathcal{L}_A$-theories that are not PA-provably decidable

There are other limitations of PA involving the notion of *recursiveness*. For example, there are unary total recursive functions (besides the Goodstein function) that not PA-provably total, and which are more relevant for our purposes. One such function is the characteristic function of the set of codes of the sentences in a particular consistent recursive $\mathcal{L}_A$-theory. First, a definition.

Recall that a countable theory T is *recursive* iff $\#[T]$ is recursive.

There is a precise sense in which PA does not fully capture the notion of "being a decidable $\mathcal{L}_A$-theory". In particular, there are consistent recursive $\mathcal{L}_A$-theories which are not PA-*provably recursive*. In other words, PA does not prove any statement that expresses the fact that "the set of *codes of the theorems of that theory* is recursive".[22] What we want to show is that there are consistent recursive $\mathcal{L}_A$-theories t such that the characteristic function of *the set of codes of the sentences in* t is not PA-provably total.

First we show that the set of codes of recursive $\mathcal{L}_A$-theories is not recursively enumerable. For each $n \in \mathbb{N}$, let T_n denote

the theory that is enumerated by the Turing machine whose index is n.

Denote the set of all such theories by T, so that

$$T := \{T_n : n \in \mathbb{N}\}$$

Now we define the set of indices of Turing machines that enumerate the codes of theories that are recursive. Define

$$R := \big\{n \in \mathbb{N} \colon T_n \text{ is recursive}\big\}.$$

Lemma: R is not recursively enumerable.

Proof (sketch): Let F be the set of indices of total recursive functions. It can be shown that F is many-one reducible to R, meaning that there is a unary total recursive function f such that $\forall n \in \mathbb{N}$,

$$n \in F \quad \Longleftrightarrow \quad f(n) \in R.$$

[22] This limitation of PA is relevant even though no such $\mathcal{L}_A$-theory has all of its theorems true in $\mathfrak{N}$.

If R were recursively enumerable, then F too would be recursively enumerable. But a simple diagonal argument shows that F is not recursively enumerable. Thus, R is not recursively enumerable either. $\square$

Now we show that there is a recursive $\mathcal{L}_A$-theory that is not PA-provably recursive. Define the subset of R that consists of the indices of Turing machines that enumerate $\mathcal{L}_A$-theories for which the characteristic function of *the set of codes of sentences that are in that theory* is PA-provably total. To this end, define

$$R^{\mathsf{PA}} := \{n \in R\colon \vdash_{\mathsf{PA}} \text{“}T_n \text{ is recursive”}\}.$$

For PA to prove that “T_n is recursive” is for PA to prove, of the characteristic function c_f of the set of codes of the theorems of T_n, that c_f is total. R^{PA} is a recursively enumerable subset of R, since the set of codes of theorems of PA is recursively enumerable. By the lemma above, R is not recursively enumerable. Thus,

$$R \setminus R^{\mathsf{PA}} \neq \emptyset.$$

I.e., there is a recursive $\mathcal{L}_A$-theory that is not PA-provably recursive.

Now we show that such theories are consistent. Let z be the index of a Turing machine that enumerates a recursive theory such that the characteristic function of the set of codes of the theorems of that theory is not PA-provably total. Suppose for reductio that T_z is inconsistent. Then since T_z proves the sentence

$$0 \neq 0,$$

PA proves that T_z is inconsistent. But then T_z is PA-provable recursive, which contradicts that T_z is not PA-provable recursive. Therefore T_z is consistent.

This is what we intended to show. There are consistent recursive $\mathcal{L}_A$-theories that are not PA-provably recursive. Such theories are consistent and decidable, but are not PA-provably decidable.

PA cannot fully capture the notion of *being a recursive $\mathcal{L}_A$-theory*. In order for PA to fully capture that notion, every recursive $\mathcal{L}_A$-theory t would have to be such that the characteristic function of *the set of codes of the sentences in t* is PA-provably total. But not every recursive $\mathcal{L}_A$-theory has that property.

Further, it can be shown that the predicate “is the code of a recursive $\mathcal{L}_A$-theory” is not representable in PA. The set of codes of recursive $\mathcal{L}_A$-theories is not recursive, so the characteristic function of that set is not total recursive. Since every total function that is representable in PA is recursive, the characteristic function of *the set of codes of recursive $\mathcal{L}_A$-theories* is not representable in PA. Thus, the predicate “is the code of a recursive $\mathcal{L}_A$-theory” is not representable in PA.

PA is inadequate for capturing certain facts about total recursive functions, and for capturing certain facts about consistent recursive $\mathcal{L}_A$-theories. There are many other limitations of PA. L too has many well known expressive limitations. The next subsection covers only a few such limitations.

1.1.3 Finitistic prejudice

Now we turn to Zermelo's doubts about the adequacy of finitary logics for capturing mathematical truth. This section briefly sketches those parts of Zermelo's views that relate most directly to infinitary proofs. References to Zermelo's writings are from Ebbinghaus, Kanamori & Fraser (2010). We also follow Ebbinghaus (2015, §§4.8–4.10). Our main concern here is to give a rough idea of Zermelo's thoughts on these issues in order to motivate the kind of account of first-order arithmetical truth that we discuss in §1.2.

Taylor (Ebbinghaus, Kanamori & Fraser, 2010, p. 304) writes that "the real issue dividing Zermelo from his contemporaries is his ultra-liberal conceptions of constructibility, provability, and definability". For Zermelo, "theorems are truth functional consequences of axioms" (Ebbinghaus, Kanamori & Fraser, 2010, p. 305), rather than merely finite-length statements for which there are finite-length proofs.

Zermelo holds that finitary axiomatic systems cannot capture the full richness of mathematics, and that only *infinitary* languages are adequate for that purpose. Certain proofs require applications of "infinitary induction", which is a rule of inference that uses infinitely many premisses. We quote Zermelo (*s1921*):

> Every genuine mathematical proposition is "infinitary" in character, that is, it is concerned with an *infinite* domain and is to be considered a collection of infinitely many "elementary propositions". ... Since infinitary propositions can never be derived from finitary ones, the "axioms" of any mathematical theory, too, must be infinitary ... Traditional "Aristotelian" logic is, according to its nature, finitary, and hence not suited for the foundation of mathematical science. Whence the necessity of an extended "infinitary" or "Platonic" logic that rests on some kind of infinitary "intuition" — as, e.g., in connection with the question of the "axiom of choice" ... Every mathematical proposition must be considered a collection of (infinitely many) elementary propositions, ... and every deduction of a proposition from other propositions, in particular every "proof", is nothing but a "regrouping" of the underlying elementary propositions. (p. 307)

We will not dwell on the details of the passage just quoted. But it is clear that Zermelo conceives of mathematical proofs as infinitary entities. Zermelo (*s1929b*) also stresses that only axiomatic systems that are *negation-complete* are adequate for mathematics:

> ...the ideal of a mathematical discipline would be a system of propositions that already contains all [of the] propositions [that are] derivable from it by purely logical

> means, that is, a "logically complete system". A "complete" system is, e.g., the totality of all logical consequences derivable from a given system of basic assumptions, an "axiom system". But not every "complete system" is necessarily determined by means of a finite number of axioms. One and the same complete system can be given by means of several, and even infinitely many, different axiom systems, such as Euclidean geometry or the arithmetic of the reals and that of the complex numbers. A complete system is therefore the "invariant", so to speak, of all equivalent axiom systems, and the question of the "independence" of the axioms does not concern it. A complete system is related to every axiom system determining it like a "field" is related to its "basis", and the investigation of such "complete systems" may hold promise of greater generality and clarity similar to that inherent in the transition from algebraic equations to algebraic fields. (p. 377)

Zermelo has in mind axiomatic systems that contain infinite-length *propositions* as well as infinite-length proofs. He understands quantifiers as set-sized conjunctions or disjunctions, i.e., with set-many conjuncts or disjuncts. Zermelo (*s1929b*) writes that:

> . . . true mathematics is infinitistic according to its nature and rests on the assumption of infinite domains; it may even be called the "logic of the infinite". (p. 383)

Ebbinghaus (2015) describes Zermelo's conception of mathematical propositions and proofs as follows:

> In essence, a proposition or a proof of it is not a syntactical string of signs, but an ideal (infinitary) object. This is in total contrast to the concepts underlying Gödel's [incompleteness] result[s]. With Gödel the finitary representability of the system in question is indispensable. For only then can one code axioms, propositions, and proofs in an effective way by natural numbers, and the presupposed arithmetical power of the system then allows provability to be treated in the system itself. . . . For Zermelo this procedure renders [the] incompleteness [of finitary axiomatic systems] a trivial fact: As the set of provable propositions in Gödel's [finitistic] sense is countable and as there are uncountably many true infinitary propositions, there is of course a true [infinitary] proposition which is not provable in Gödel's [finitistic] sense. (p. 227)

Given any finitary axiomatic system that meets the hypothesis of various forms of the first incompleteness theorem, Gödel's method allows one to define a *finitary* proposition that is neither provable nor disprovable in that system. In any case, Zermelo sees the imbalance between L's syntactic power and its semantic power as essentially tied to its finitary restrictions — i.e., the requirement that formulas and proofs be finite in length. He views several theorems as demonstrating that finitary axiomatic systems have unacceptable limitations. We will mention four such theorems, but many others can be seen in this light too.

Firstly, there is the downward Löwenheim-Skolem theorem.[23] Applied to countable languages, one form of this theorem states that every set of sentences in a countable *finitary*

[23] For a generalization of the downwards Löwenheim-Skolem theorem, see Găină (2015).

first-order language that has a model has a countable model. From Zermelo's perspective, this feature of finitary first-order languages is objectionable because it means that such languages cannot uniquely characterize (up to isomorphism) an *uncountable* mathematical structure. No set of sentences of such a language is satisfied in all and only the uncountable models of that language. Nor can such languages characterize the notion of being a *countable* mathematical structure. No set of sentences of such a language is satisfied in all and only the *countable* models of that language.

While Skolem saw the *countable* vs. *uncountable* distinction as being relative to a model of first-order set theory, Zermelo rejected such a view. But Zermelo (*1930a*) does hold that the *set* vs. *proper class* distinction is relative to a model of the second-order axioms of set theory.

Hodges (2008, p. 267) mentions that Skolem "didn't believe in the existence of uncountable sets". Zermelo (*s1930d*) writes that:

> For *Skolem*, it is supposed to be possible to represent set theory *in its entirety* already in a *countable* model, and, e.g., the problem of the cardinality of the continuum already loses its real significance for him. (p. 439)

Zermelo (*s1930d*) also mentions that:

> ... *Cantor's* (generalized) conjecture [GCH] (according to which the power set of any set is supposed to always be of the immediately succeeding cardinality) does *not* depend on the choice of the model, but that it is decided (as true or false) once and for all by means of our [second-order] axiom system. (p. 437)

Secondly, there is the upward Löwenheim-Skolem theorem.[24] Applied to countable languages, one version of this theorem can be stated as follows.

> Let κ be any infinite cardinal. Then for each cardinal $\lambda > \kappa$,
> every set of sentences in a countable *finitary first-order language* that has a κ-size model has a λ-size model.

Such languages are unable to uniquely characterize (up to isomorphism) mathematical structures of any particular infinite cardinal size. For each infinite cardinal κ, no set of sentences of such a language is satisfied in all and only the κ-size models of that language. This is another way in which the expressiveness of such finitary languages is limited.

[24] Generalizing the upwards Löwenheim-Skolem theorem in the setting of institution theory is an open problem.

Thirdly, there is the compactness theorem for L.[25] This theorem states that for every set S of sentences in a finitary first-order language, if every finite subset of S has a model, then S has a model.[26] Recall that this theorem implies that every set of finitary first-order sentences that has arbitrarily large finite models has an infinite model. In classical model theory, compactness is seen as a very nice property to have around for various constructions. But whenever the compactness property holds, no set of sentences can uniquely characterize the notion of a *finite* mathematical structure. No set of sentences of such a language is satisfied in all and only the *finite* models of that language. On Zermelo's view, this severe expressive limitation of finitary first-order languages is unacceptable.[27]

Fourthly, there is Gödel's (1931) original formulation of the first incompleteness theorem. In his October 12, 1931 letter to Gödel, Zermelo (*s1931b*, p. 489) writes:

> As with the Richard and Skolem paradoxes, the mistake is due to the (incorrect) assumption that it is possible to represent every mathematically definable concept by means of a "finite combination of signs" (in accordance with a *fixed* [axiomatic] system!). In other words, [it is due to] what I call the "finitistic prejudice". Actually things are quite different, and a reasonable "meta-mathematics" will only be possible once we have overcome this prejudice, a matter which I have made my special duty.

Zermelo took incompleteness to be an undesirable feature of *finitary* axiomatic systems that are non-trivial, in the sense of being both consistent and sufficiently strong. In his October 29, 1931 letter to Gödel, Zermelo (*s1931d*) writes that:

> ...all [that] you prove in your [1931] paper amounts to what I, too, always stress, namely that a "finitistically restricted" proof schema *is not sufficient* for "deciding" the propositions of an uncountable mathematical system. ... I, at the very outset, proceeded from a *more general* [infinitistic proof] schema, ... And in *this* [infinitistic] system now really *any* proposition whatsoever is decidable! (p. 501)

Taylor (Ebbinghaus, Kanamori & Fraser, 2010) explains the sense in which every proposition of such a system is decided by that system:

> In *1931d* Zermelo writes in effect that, in one of his own systems of infinitely long propositions [and infinitely long proofs], each and every (conditional) proposition is decidable in the sense that, "under the validity of [its antecedent], the validity of [its consequent] can be made *obvious*". The discussion there suggests that indeed, a

[25] This theorem generalizes to a compactness theorem for a particular class of infinitary logics. See Barwise & Feferman (2017, §5.6 and §6.2).

[26] See Chang & Keisler (2012, §2.1). In 1930, Gödel proved that the compactness property holds for countable sets of such sentences. The same result is nearly proven in Skolem (1922).

[27] Zermelo should have complained about the compactness of L. I imagine that he did.

> decision procedure for logical consequence is what Zermelo offers, despite its involving potentially infinitely many steps. (pp. 535–536)

Taylor (Ebbinghaus, Kanamori & Fraser, 2010, p. 537) also mentions that Zermelo's *1932b* paper "reiterates his claim that all propositions of a system S are decidable, where the decision procedure involves reasoning about something such as an infinite truth table or semantic tableau".

The "finitary prejudice" that Zermelo mentions is a prejudice against infinitary proofs and infinitary propositions. Zermelo (*1932a*, pp. 543–545) writes:

> Proceeding from the assumption that it should be possible to represent all mathematical concepts and theorems by means of a *fixed finite system of signs*, we inevitably run into the well-known "*Richard's antinomy*" already in the case of the arithmetical continuum. This antinomy, which seemed long buried, has recently celebrated a merry resurrection in the form of *Skolemism*, i.e. the doctrine that *every* mathematical theory, including set theory, can be realized in a *countable model*. . . . But a healthy "metamathematics", a true "logic of the infinite", will only become possible once we have *definitively renounced* the assumption characterized above, which I would like to call the "*finitistic prejudice*". Mathematics, generally speaking, is *not* really concerned with "combinations of signs", as some assume, but with *conceptually ideal relations* among the elements of a conceptually posited *infinite manifold*.

Zermelo (*1932a*) adds that the existence of propositions that are undecidable relative to some particular finitary axiomatic system does not touch on the deeper issue of *absolutely* undecidable propositions.

> This whole argument [for Gödel's first incompleteness theorem], in my opinion, only serves as evidence for the inadequacy of *any* "finitistic" proof theory without, however, providing the means to remove this ill. Such relativistic considerations in no way touch on the real question as to whether there are absolutely undecidable propositions or absolutely unsolvable problems in mathematics. (p. 547)

This subsection mentioned only a few natural mathematical concepts that are not expressible in finitary first-order systems. There are many other such concepts. We now turn to infinitary logics.

1.2 Infinitary logics

This section aims to motivate the study of infinite sequences of tasks by considering several accounts or explanation of first-order arithmetical truth.

The unrestricted ω-rule

Since TA is negation-complete, it is natural to consider adding to PA — in hopes of capturing TA — the *ω-rule for* PA:[28]

$$\boxed{\begin{array}{c}\text{For each } \mathcal{L}_A\text{-wff } \varphi(v_1),\\ \text{if } \forall n\in\mathbb{N} \quad \vdash_{\mathsf{PA}} \varphi(\overline{n}), \text{ then let } \vdash_{\mathsf{PA}} \forall x\, \varphi(x).\end{array}}$$

Let PA_ω be the semi-formal system that is like PA but allows proofs to contain any ordinal-length sequence of applications of the ω-rule.[29] PA_ω-proofs correspond to ordinal-height trees of formulae.[30] PA_ω is negation-complete.

Every $\sigma\in\mathsf{TA}$ is PA_ω-provable, i.e.,

$$\mathsf{TA} \subseteq \left\{\mathcal{L}_A\text{-sentences } \sigma\colon\ \vdash_{\mathsf{PA}_\omega} \sigma\right\}.$$

If PA_ω is consistent, then

$$\left\{\mathcal{L}_A\text{-sentences } \sigma\colon\ \vdash_{\mathsf{PA}_\omega} \sigma\right\} = \mathsf{TA}.$$

This is the sense in which PA_ω captures TA.

The recursive ω-rule

Moving to a semi-formal system that allows proofs of infinite length, one would like to get an "effective grip" on what counts as a proof in that system. Given a total recursive coding, one way to allow each infinite proof that uses the ω-rule to be represented by a *finite* object (i.e., a natural number) is to exclude any application of the ω-rule to an ω-sequence of formulas whenever the codes of their PA_ω-proofs form a non-recursive subset of $\mathbb{N}$. If i is the index of a unary total recursive function f such that

$$\forall n\in\mathbb{N} \quad f(n) \text{ is the code of a } \mathsf{PA}_\omega\text{-proof of } \varphi(\overline{n}),$$

then a PA_ω-proof of $\forall x\, \varphi$ can be identified with the finite number that codes the ordered pair

[28] Tarski (1931, p. 260) in a footnote claims to have "pointed out the importance of the rule of infinite induction in the year 1926". In 1934, Carnap (2003, §14) made use of such a rule in his Language I.

[29] The subscript in 'PA_ω' refers to the unrestricted ω-rule.

[30] Derevyankina (1974) mentions that:

> The principal shortcoming of such a method of investigation of completeness consists in the fact that although the deduction tree . . . is an effective object of study, the predicate "being a deduction tree" is not an arithmetic[al] predicate. (p. 210)

$$\langle i, \#(\forall x\,\varphi)\rangle.$$

In this way, such functions are used to encode ω-sequences of recursively codable PA_ω-proofs. This restriction to the ω-rule for PA yields the *recursive ω-rule for* PA:[31]

> For each $\mathcal{L}_A$-wff $\varphi(v_1)$,
> if there is a unary total recursive function f such that
> $\forall n\in\mathbb{N} \quad f(n)$ is the code of a PA_ω-proof of $\varphi(\overline{n})$,
> then let $\vdash_{\mathsf{PA}} \forall x\ \varphi(x)$.

Let $\mathsf{PA}_{\rho\text{-}\omega}$ be the system that is like PA except that it allows a proof to contain any number of applications of the recursive ω-rule.[32] Not all $\mathsf{PA}_{\rho\text{-}\omega}$-proofs are recursive, since some have ordinal heights that are so large that no total recursive function can code such a proof.

Let PA_{ω^2} be the semi-formal system that is like PA except that it allows a proof to contain up to (but not including) ω^2 applications of the recursive ω-rule.[33] This means that each PA_{ω^2}-proof is a $\mathsf{PA}_{\rho\text{-}\omega}$-proof that uses the recursive ω-rule fewer than ω^2 times in total.

Goldfarb (1975) showed that each sentence of TA is derivable in PA_{ω^2}, i.e.,

$$\mathsf{TA} \subseteq \Big\{\mathcal{L}_A\text{-sentences } \sigma\colon \vdash_{\mathsf{PA}_{\omega^2}} \sigma\Big\}.$$

Thus, PA_{ω^2} is negation-complete, and every PA_ω-provable sentence is PA_{ω^2}-provable. In particular, $\forall\sigma \in \mathsf{TA}$, $\exists n \in \mathbb{N}$ such that there is a PA_{ω^2}-proof of σ that includes no longer than an $(\omega\times n)$-sequence of applications of the recursive ω-rule. Thus, for each $\sigma\in\mathsf{TA}$, proving σ from within PA_{ω^2} requires at most finitely many ω-sequences of applications of the recursive ω-rule.[34] We now consider the ways in which infinitary proofs can be used to account for or explain first-order arithmetical truth.

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[31] Unfortunately this rule is sometimes called the *effective* ω-rule.

[32] The subscript in '$\mathsf{PA}_{\rho\text{-}\omega}$' refers to the recursive ω-rule.

[33] The subscript in 'PA_{ω^2}' refers to the limit on the ordinal-length of the total sequence of applications of the recursive ω-rule that a PA_{ω^2}-proof may contain.

[34] Derevyankina (1974) writes that:

> ... Belyakin [[1967]] has described an algorithm that assigns (in the semi-formal system considered by him [that includes a recursive ω-rule]) to any arithmetic formula φ a primitive recursive tree D_φ of search for the deduction of φ. If φ is a true formula, then D_φ will have a chain condition, i.e., it goes over automatically into the tree of deduction of φ, with the height of D_φ being smaller than ω^2 (for any true formula φ).
>
> With a minor modification it is possible to obtain a similar result also for an ordinary arithmetic system to which we adjoin an ω-rule, i.e., to each formula φ we assign a tree $\mathfrak{D}_\varphi$ of deduction of φ in this system; for true arithmetic formulas the heights of their deduction trees remain smaller than ω^2. (p. 220)

1.2.1 Recursive proofs using the recursive ω-rule

This subsection describes several accounts of first-order arithmetical truth. We need not endorse these accounts in order to motivate the study of infinite sequences. The issues surrounding these accounts are interesting enough in their own right and depend on mathematical facts about the strength of infinitary proofs. We will raise a number of questions about these accounts of truth in TA and our knowledge of TA.

Truth in TA

The system PA_{ω^2} is of particular interest because given any $\varphi \in \mathsf{TA}$, there is a PA_{ω^2}-proof of φ that is recursive (i.e., a PA_{ω^2}-proof which is encoded by a recursive function).[35]

There are several ways in which an account of *truth in* TA can employ the negation-completeness of a system like PA_{ω^2}. For example, one such account, the *recursive truth account* (RTA), can be stated as:

> What it is for a first-order arithmetical sentence to be "true" is *for there to be a recursive* PA_{ω^2}*-proof of that sentence.*

Note that RTA does not require that there be a *single* recursive function that encodes a PA_{ω^2}-proof of each sentence of TA. Instead, it suffices that for each sentence of TA, there is some or other recursive function that encodes a PA_{ω^2}-proof of that sentence.

This kind of account analyzes a notion of *truth* in terms of a particular notion of *proof.* On other views of mathematical truth, the fact that there is a proof of φ is to be explained — or otherwise accounted for — in terms of the fact that φ is true. In contrast, RTA reverses the direction of explanation. We take this to be a problem for RTA.

Mints (1991) writes that:

> Belyakin [1967] obtained bounds in the completeness theorem for the recursive omega rule for classical arithmetic by describing a (primitive recursive) canonical proof search tree T_A for any formula A and proving that if A is true, then T_A is well-founded with ordinal length less than ω^2. (p. 404)

[35] See Theorem 6.1.12 of Girard (1987, p. 356), from which we get that for each $\mathcal{L}_A$-sentence σ:

$$\sigma \in \mathsf{TA} \quad \Longleftrightarrow \quad \text{there is a recursive } \mathsf{PA}_\omega\text{-proof of } \sigma.$$

Stronger still, each $\sigma \in \mathsf{TA}$ has a Kálmar-elementary proof tree of height $\omega \cdot k(\sigma)$ in PA_ω, where $k(\sigma) \in \mathbb{N}$ depends on the complexity of σ. See the method of proof of Theorem 8.2 in Takeuti (2013, pp. 41–45) using reduction trees. Also see López-Escobar (1967).

1.2.2 Circularity

RTA is objectionably circular. The account claims that for each $\varphi \in \mathsf{TA}$, the truth of φ consists in the fact that

$$\textit{there is a recursive } \mathsf{PA}_{\omega^2}\textit{-proof of } \varphi.$$

As we will explain, each instance Φ of RTA's analysans is itself equivalent to a sentence of TA that expresses the fact that Φ. The truth of those $\mathcal{L}_A$-sentences must be accounted for too. RTA purports to account for the truth of *every* sentence of TA. Thus, RTA must not help itself to the truth of a sentence of TA. RTA takes for granted part of what it aims to account for.

We want to show that for each $\varphi \in \mathsf{TA}$, there is a $\varphi' \in \mathsf{TA}$ that expresses the fact that there is a recursive PA_{ω^2}-proof of φ, and do so without running afoul of Tarski's undefinability theorem.

Applied to $\mathcal{L}_A$, Tarski's undefinability theorem states that:

> There is no $\mathcal{L}_A$-formula $\varphi(v_1)$ such that for every $\mathcal{L}_A$-sentence σ,
>
> $$\mathfrak{N} \models \sigma \quad \Longleftrightarrow \quad \mathfrak{N} \models \varphi(\ulcorner \sigma \urcorner).$$

Since the predicate

"is the code of an $\mathcal{L}_A$-sentence that is true in $\mathfrak{N}$"

is coextensive with the predicate

"is the code of an $\mathcal{L}_A$-sentence for which there is a recursive PA_{ω^2}-proof",

by Tarski's undefinability theorem, the latter predicate is not definable in $\mathfrak{N}$. But we will not try to show that there is a single truth predicate for every $\mathcal{L}_A$-sentence, since there is no such predicate. Rather, we will show that for each initial segment of the arithmetical hierarchy, there is a truth predicate for that segment.

Note that for each $n \in \mathbb{N}$, the notion of *truth for Σ_n-sentences* is definable in $\mathfrak{N}$. For each $n \in \mathbb{N}$, denote the theory that is like PA but restricts the induction schema to Σ_n-formulas by $I\Sigma_n$.[36] For each $n \in \mathbb{N}$, there is a Σ_n-formula $Sat_{\Sigma_n}(v_1, v_2)$ such that for each $\mathcal{L}_A$-formula $\varphi(v_1)$,

$$I\Sigma_n \vdash_{\mathsf{L}} \forall x \left(Sat_{\Sigma_n}(\ulcorner \varphi(y) \urcorner, x) \;\leftrightarrow\; \varphi(x) \right).$$[37]

Then for each $n \in \mathbb{N}$, there is an $\mathcal{L}_A$-formula $Tr_{\Sigma_n}(v_1)$ such that for each Σ_n-sentence σ,

$$\mathfrak{N} \models \sigma \quad \Longleftrightarrow \quad \mathfrak{N} \models Tr_{\Sigma_n}(\ulcorner \sigma \urcorner).$$

[36] See Hájek & Pudlák (2017, Chapter I).

[37] See Kaye (1991, §9.3).

Thus, for each $n \in \mathbb{N}$, the set of codes of true Σ_n-sentences is definable in $\mathfrak{N}$.[38]

Now define a hierarchy of $\mathcal{L}_A$-theories, each closed under L. The zeroth theory is PA^0_ρ, which is the same as PA.[39] The next theory PA^1_ρ is like PA^0_ρ except that it allows proofs to use at most one application of the recursive ω-rule. And so on for each finite ordinal. For each successor ordinal α, $\mathsf{PA}^{\alpha+1}_\rho$ is the closure under L of the union of PA^α_ρ with the set of those sentences provable from PA^α_ρ using exactly one application of the recursive ω-rule. For each limit ordinal λ, PA^λ_ρ is the closure under L of the union of every PA^α_ρ for all $\alpha < \lambda$. More precisely, define

$$\mathsf{PA}^0_\rho := \mathsf{PA}.$$

For each ordinal α, define

$\mathsf{PA}^{\alpha+1}_\rho :=$ the closure under L of the union of PA^α_ρ with the set of all $\mathcal{L}_A$-sentences of the form $\forall x\ \psi(x)$ for which both

(1) $\forall n \in \mathbb{N} \quad \psi(\overline{n}) \in \mathsf{PA}^\alpha_\rho$,

and

(2) there is a unary total recursive function f such that $\forall n \in \mathbb{N} \quad f(n)$ is the code of a PA^α_ρ-proof of $\psi(\overline{n})$.

For each limit ordinal λ, define

$$\mathsf{PA}^\lambda_\rho := \text{the closure under } \mathsf{L} \text{ of } \bigcup_{\alpha<\lambda} \mathsf{PA}^\alpha_\rho.$$

Proposition: For each $n \in \mathbb{N}$, there is an ordinal $\alpha < \omega^2$ such that:[40]

(i) each true Σ_n-sentence has a recursive PA^α_ρ-proof,

(ii) each false Σ_n-sentence has a recursive PA^α_ρ-refutation,

and

(iii) $\#\left[\mathsf{PA}^\alpha_\rho\right]$ is definable in $\mathfrak{N}$.

[38] By Post's Theorem, for each $n \in \mathbb{N}$, $\exists m > n$ such that the set of codes of true Σ_n-sentences is definable in $\mathfrak{N}$ by a Σ_m-formula.

[39] The subscript in 'PA^0_ρ' refers to the recursive ω-rule.

Define the function

$$\mu\colon \mathbb{N} \longrightarrow \omega^2$$

so that for each $n\in\mathbb{N}$,

$$\mu(n) := \text{the least ordinal } \alpha \text{ such that}$$

(i) each true Σ_n-sentence has a recursive $\mathsf{PA}^{\alpha}_{\rho}$-proof,

(ii) each false Σ_n-sentence has a recursive $\mathsf{PA}^{\alpha}_{\rho}$-refutation,

and

(iii) $\#\left[\mathsf{PA}^{\alpha}_{\rho}\right]$ is definable in $\mathfrak{N}$.

Fix an $n\in\mathbb{N}$ and let φ be a true Σ_n-sentence that is not Σ_{n-1}. We want to show that there is a true $\mathcal{L}_A$-sentence that expresses the fact that there is a recursive PA_{ω^2}-proof of φ. We claim that it suffices to show that for some $m\in\mathbb{N}$, there is a true $\mathcal{L}_A$-sentence that expresses the fact that

$$\varphi \in \mathsf{PA}^{m}_{\rho}.$$

By the proposition above,

$$\#\left[\mathsf{PA}^{\mu(n)}_{\rho}\right]$$

is definable in $\mathfrak{N}$. Let $M(v_1)$ be an $\mathcal{L}_A$-formula (depending on n) such that $\forall x\in\mathbb{N}$,

$$\mathfrak{N} \models M(\overline{x}) \quad\iff\quad x \in \#\left[\mathsf{PA}^{\mu(n)}_{\rho}\right].$$

Then the sentence

$$\varphi' := M(\ulcorner\varphi\urcorner)$$

expresses the fact that there is a recursive $\mathsf{PA}^{\mu(n)}_{\rho}$-proof of φ. Thus, φ' expresses the fact that there is a recursive PA_{ω^2}-proof of φ.[41]

Notice that from the statement that

[40] Theorem 9 in Murawski (2006) is somewhat related, though it uses the unrestricted ω-rule.

[41] This also shows that an account on which for each $n\in\mathbb{N}$,

what it is for a Σ_n-sentence that is not Σ_{n-1} to be “true” is
for there to be a recursive $\mathsf{PA}^{\mu(n)}_{\rho}$*-proof of that sentence*

is objectionably circular.

$$\varphi' \text{ expresses the fact that there is a recursive } \mathsf{PA}_\rho^{\mu(n)}\text{-proof of } \varphi,$$

we inferred that

$$\varphi' \text{ expresses the fact that there is a recursive } \mathsf{PA}_{\omega^2}\text{-proof of } \varphi.$$

It does not in general follow from

(i) φ' expresses the fact that P

and

(ii) $P \implies Q$

that

(iii) φ' expresses the fact that Q.

But our inference above is unobjectionable because every $\mathsf{PA}_\rho^{\mu(n)}$-proof counts by definition as a PA_{ω^2}-proof.

Here is another way to see the way in which RTA is circular. Let $\sigma \in \mathsf{TA}$. Let σ' be an $\mathcal{L}_A$-sentence that expresses the fact that there is a recursive PA_{ω^2}-proof of σ. RTA attempts to account for the truth of σ in terms of the truth of σ'. But $\sigma' \in \mathsf{TA}$, so RTA must also account for the truth of σ'. It seems that this must continue *ad infinitum*.

There are *proper* subsets S of TA such that for each $\sigma \in \mathsf{TA}$,

$$\exists \sigma' \in S \text{ that expresses the fact that there is a recursive } \mathsf{PA}_{\omega^2}\text{-proof of } \sigma.$$

For example, the sentence

$$0 = 0$$

is in TA, but that sentence does not express a fact about the existence of any PA_{ω^2}-proof, recursive or not.

But this does not show that there is such a subset of TA that is deductively closed under L. This motivates two questions.

Q3: Is there a proper subtheory T of TA such that for each $\sigma \in \mathsf{TA}$,
T proves a sentence that expresses the fact that there is a recursive $\mathsf{PA}_{\rho\text{-}\omega}$-proof of σ?

Q4: Is there a proper subtheory T of TA such that for each $\sigma \in \mathsf{TA}$,
T proves a sentence that expresses the fact that there is a recursive PA_{ω^2}-proof of σ?

At best, RTA might be seen as reducing the problem of accounting for truth in TA to the problem of accounting for truth in such an $S \subset \mathsf{TA}$. But RTA is not an adequate account of truth in any such S. And an account of the notion of truth in such an S that is in the spirit of RTA would be circular. I.e., an account that claims that

for a first-order arithmetical sentence to be "true in S" is
for there to be a recursive PA_{ω^2}*-proof of that sentence*

is circular. Even though its analysans does not presuppose a notion of *truth in* TA, it presupposes a notion of *truth in* S.

We have not formalized the notion of a *fact.*[42] Nor have we formalized what it is for a sentence to *express* a fact.[43] Even so, we can raise a problem for RTA as follows. RTA must account for the truth of every $\sigma \in \mathsf{TA}$. At stage 1, form $S_1 \subset \mathsf{TA}$ as follows. For each $\sigma \in \mathsf{TA}$, include in S_1 the sentence that expresses the fact that there is a recursive PA_{ω^2}-proof of σ. RTA accounts for truth in TA by taking for granted the truth of each sentence in S_1. But how is RTA to account for the truth of each sentence of S_1? At stage 2, form $S_2 \subset \mathsf{TA}$ as follows. For each $\sigma \in S_1$, include in S_2 the sentence that expresses the fact that there is a recursive PA_{ω^2}-proof of σ. Now RTA accounts for the truth of the sentences in S_1 by taking for granted the truth of each sentence in S_2. And so on *ad infinitum.*

More precisely, define

$$S_0 := \mathsf{TA}.$$

For each $n \in \mathbb{N}$, define

$$S_{n+1} := \Big\{\sigma \in \mathsf{TA}\colon\ \exists \psi \in S_n \text{ such that } \sigma \text{ expresses the fact that there is a recursive } \mathsf{PA}_{\omega^2}\text{-proof of } \psi\Big\}.$$

Given our assumptions about the "expresses" relation that holds between certain sentences and certain facts, each S_i is countably infinite. This is because S_0 is countably infinite, and each S_{i+1} contains a unique sentence for each sentence in S_i. This motivates asking:

Q5: Is it the case that $\forall n, m \in \mathbb{N}$,

$$S_m \subset S_n \quad \Longleftrightarrow \quad n < m\ ?$$

When a sentence σ is in S_i but not in S_{i+1}, we can think of σ as being excluded at that stage of the construction from those sentences the truth of which RTA must at that stage account for. Whether a sentence can reappear after it is excluded is determined by the answer to Q3. But

[42] For the purposes of thinking about accounting for *truth in* TA, we might choose a particular language — e.g., that of third-order arithmetic — and choose a model of that language, and then take each "fact" to correspond to a sentence (of that language) that is true in that model (of that language).

[43] Each sentence should "express" *at most* one fact. We might also want each fact to be "expressed" by at most one sentence. This can be done as follows. Partition the class of sentences into equivalence classes under the relation of logical equivalence. From each equivalence class of sentences, choose a canonical representative. Have *some but not all* of those canonical representatives "express" facts.

that answer does not determine whether *every* sentence in TA is such that at some stage, it is excluded. This motivates asking:

Q6: Which sentences are in

$$\bigcap_{i\in\mathbb{N}} S_i \ ?$$

There is another problem with RTA. Each recursive PA_{ω^2}-proof is coded by a unary total recursive function, otherwise that proof would not be recursive. But given a unary total recursive function f, *not all* consistent $\mathcal{L}_A$-theories prove a statement that expresses the fact that f is total. Thus, RTA must either:

(**i**) presuppose a notion of truth in a consistent $\mathcal{L}_A$-theory T such that for each $\sigma \in \mathsf{TA}$, there is a total recursive function f that both codes a recursive PA_{ω^2}-proof of σ and is T-provably total, or

(**ii**) for each $\sigma \in \mathsf{TA}$, presuppose a notion of truth in a consistent $\mathcal{L}_A$-theory T such that there is a total recursive function f that both codes a recursive PA_{ω^2}-proof of σ and is T-provably total.

This is a problem for RTA because RTA must explain what it is for an arbitrary $\mathcal{L}_A$-sentence to be true without presupposing a notion of *truth in* an $\mathcal{L}_A$-theory. An adequate account of *truth in* TA must apply to every $\mathcal{L}_A$-sentence, including those sentences in such theories as those mentioned in **i**, and those mentioned in **ii**.

We should also mention a possible objection to the motivational force of RTA. One might worry that RTA fails to motivate the study of supertasks, on the following grounds. Given any unary total recursive function f and any $n \in \mathbb{N}$ as input, only finitely many steps are required in order to compute the value $f(n)$. So one might think that no supertasks are required.

However, for many $\sigma \in \mathsf{TA}$ (including those σ that are sufficiently complex), every PA_{ω^2}-proof of σ that is encoded by a recursive function involves infinitely many applications of the recursive ω-rule. Even though the function encoding an infinite proof of σ is recursive, verifying the infinite proof encoded by that function requires performing infinitely many tasks, since each one of infinitely many steps must be checked.

Infinite-length proofs are central to such an account of first-order arithmetical truth. There are systems stronger than TA that prove sentences for which there is no "finitely specifiable" proof (from that system's axioms). Analogous accounts of truth in some of these theories can be used to motivate the study of uncountable-length proofs and hypertasks.

1.2.3 Knowledge of TA

Now we consider an account of our *knowledge* of first-order arithmetical truth. One such account can be stated as:

first-order arithmetical truth is epistemically secured by our knowledge of the fact that for each $\varphi \in \mathsf{TA}$, there is a recursive PA_{ω^2}-proof of φ.

If our knowledge of the existence of *a recursive* PA_{ω^2}*-proof for each sentence of* TA is supposed to justify our knowledge of TA, one must ask what secures our knowledge of the former. How much more secure is our knowledge of the existence of *a recursive* PA_{ω^2}*-proof for each sentence of* TA as compared to our knowledge of TA? In particular:

How strong of a (natural) metatheory is needed to prove that "each $\varphi \in \mathsf{TA}$ has a recursive PA_{ω^2}-proof"?

PA does not prove any statement that expresses the fact that "each $\varphi \in \mathsf{TA}$ has a recursive PA_{ω}-proof". Since the function defining the predicate "is the code of a sentence that has a recursive PA_{ω}-proof" is not recursive, that function is not representable in PA. It is not the case even that for each $\varphi \in \mathsf{TA}$, PA proves a statement that expresses the fact that φ has a recursive PA_{ω^2}-proof.

One way of comparing our knowledge of the fact that "each $\varphi \in \mathsf{TA}$ has a recursive PA_{ω^2}-proof" with our knowledge of PA is to see how much transfinite induction must be added to PA so that for each $\varphi \in \mathsf{TA}$, the resulting system proves a statement that expresses the fact that there is a recursive PA_{ω^2}-proof of φ.[44] This motivates asking:

Q7: What is the least ordinal α such that for each $\varphi \in \mathsf{TA}$,
PA with induction up to α proves a statement that expresses the fact that
there is a recursive PA_{ω^2}-proof of φ?

It is natural to ask how the answer to Q7 changes when we relax the requirement that the proof of φ have ordinal-height less than ω^2. This motivates:

Q8: What is the least ordinal α such that for each $\varphi \in \mathsf{TA}$,
PA with induction up to α proves a statement that expresses the fact that
there is a recursive $\mathsf{PA}_{\rho\text{-}\omega}$-proof of φ?

Another way of analyzing our knowledge of the fact that "each $\varphi \in \mathsf{TA}$ has a recursive PA_{ω^2}-proof" is given by reverse mathematics.[45] We can ask:

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[44] A canonical encoding of ordinals — in this case those less than ϵ_0 — can be obtained using Cantor normal form.

[45] See Simpson (2010).

Q9: What is the reverse-mathematical strength of a sentence that expresses the fact that each $\varphi \in \mathsf{TA}$ has a recursive PA_{ω^2}-proof?

Again we can relax the requirement that the proof of φ have ordinal-height less than ω^2. This motivates asking:

Q10: What is the reverse-mathematical strength of a sentence that expresses the fact that each $\varphi \in \mathsf{TA}$ has a recursive $\mathsf{PA}_{\rho\text{-}\omega}$-proof?

Each of those facts might have more than one "natural" translation. But answers to questions like these shed light on our knowledge of TA and its relation to our knowledge of the negation-completeness of systems like PA_{ω^2} and $\mathsf{PA}_{\rho\text{-}\omega}$.

1.3 Effective prejudice

This section criticizes several arguments from Cotogno (2015) that can be seen as aiming against accounts of first-order arithmetical truth that involve infinitary proofs. We can take Cotogno to be arguing that no account of *truth in* TA in terms of infinite-length computations can handle certain arithmetical sentences that are neither PA-provable nor PA-refutable. Cotogno's arguments rely on considerations about effectiveness and the ω-rule.

In the passages from Cotogno that we quote, we will add charitable clarifications, subscripts and underlines as needed, enclosed in square brackets. Note that Cotogno sometimes uses Tarski's (1933) term "infinite induction" for the ω-rule, and other times he uses Rosser's (1937) term "Carnap's rule" for the ω-rule.

Cotogno's main point is that infinite sequences of computations cannot be used to decide arbitrary arithmetical sentences. In his words:

> . . . the undecidability of PA is not suppressed by infinite induction, in the general case; even if physicists could design some actual infinity machine [that was capable of completing ω-sequences of computations], one would still remain unable to reach a decision on arbitrary [first-order] arithmetic sentences. (p. 281)

Cotogno explains that "the ω-rule does not cope with *all* [of the] objects entailing the undecidability of PA" (p. 275). He writes:

> . . . we shall criticize directly the assumption that [the] decidability of PA would be secured by concluding a denumerable succession of operations in a finite span of time. We shall observe that Gödel's and Rosser's [PA-]unprovable sentences remain undecidable even if the system allows infinite induction, as long as it is consistent. A similar conclusion obtains by approaching undecidability through the infinite succession of [PA-]unprovable sentences engendered by [one version of] the Yablo paradox. (p. 277)

Cotogno seems to be claiming that if PA_ω is consistent, then certain sentences which are PA_ω-provable but not PA-provable are not decidable, even if we allow applications of the ω-rule. In his words:

> ... the restoration of [negation-]completeness in PA_ω ... does not amount by itself to a restoration of decidability: it tells us what could be ideally grasped *if* we had infinite operating capabilities, but [it] has no proper algorithmic content. (p. 277)

On Cotogno's view, this is because:

> ... there is no way of ensuring [that] the ω-rule has sufficient generality to cope with everything that can be expressed within universal systems; on the contrary, some arithmetized constructions turn out to be refractory to ω instruments. (p. 278)

Background

In order to comment on Cotogno's claims, we need to cover a bit of background.

Recall that the *Church-Turing Thesis* (CT) is the claim that

$\forall n \in \mathbb{N}$, every effectively computable total function from $\mathbb{N}^{n+1}$ to $\mathbb{N}$ is total recursive.[46]

Assuming CT, the set of codes of the theorems of PA_ω is not effectively decidable, if PA_ω is consistent. Since PA_ω's provability relation is not recursive, if PA_ω is consistent then the set of the codes of the theorems of PA_ω is not recursive either. By CT, if PA_ω is consistent, then there is no effective procedure such that for each $\mathcal{L}_A$-sentence σ, it determines which one of

$$\vdash_{\mathsf{PA}_\omega} \sigma,$$
$$\nvdash_{\mathsf{PA}_\omega} \sigma$$

holds. Under these assumptions, the set of codes of the theorems of PA_ω is not effectively decidable.

What is at issue is whether every $\mathcal{L}_A$-sentence can be decidable by infinite sequences of computations. Cotogno seems to be arguing that there are arithmetical sentences which could not be decided even by those means. The next four subsections consider Cotogno's arguments in detail. In §1.3.5 we discuss the import of Cotogno's main claims and arguments.

1.3.1 A Gödel sentence for PA

Cotogno's first argument is for the conclusion that the ω-rule would not help in defining an infinite decision procedure for PA-provability. The argument uses one of PA's Gödel sentences G_{PA}. For each $n \in \mathbb{N}$, Cotogno writes $\underline{n}$ for the numeral for n. We quote Cotogno (2015):

[46] Several claims have been called a "Church-Turing Thesis", but this one is most relevant for us.

> ... [Kleene defines G_{PA} as] $\forall y \neg A(\underline{k}, y)$, where $A(x,y) = Prov_{PA}(sub(x, \ulcorner \dot{x} \urcorner, x), y)$ and $\underline{k}$ is the numeral of $k = \ulcorner \forall y \neg A(x,y) \urcorner$ (Kleene 1974[, p. 207]). A is the self-referencing form of the [standard] *proof predicate* $Prov_{PA}$, so $G_{[PA]}$ is asserting its own [PA-]unprovability via $\underline{k}$ ... The assumption under scrutiny is that the ω-rule would cover all [of the] true [first-order] arithmetic sentences, including the ones that are *proven* [to be PA-]undecidable. Gödel ... could not help viewing the *semantic* part of the First [Incompleteness] Theorem — i.e., the conclusion that $G_{[PA]}$ is a true sentence of PA — as a *mental* inference from the denumerable succession of premisses
>
> $$\vdash_{[PA]} \neg A(\underline{k}, 0), \quad \vdash_{[PA]} \neg A(\underline{k}, 1), \quad \ldots \quad , \quad \vdash_{[PA]} \neg A(\underline{k}, \underline{n}), \quad \ldots \tag{2}$$
>
> to the universal sentence [$\mathfrak{N} \models$] $\forall y \neg A(\underline{k}, y)$, for all $n \geq 0$ [sic] (Gödel 1988, [p. 44,] footnote 48[a]). Within PA_ω, one could think of generating all [of the] premisses [in] (2), and then applying infinite induction to conclude $\vdash_{[PA_\omega]} G_{[PA]}$; this does make sense, but the [PA_ω-]'proof' [of G_{PA}] thus obtained remains a [merely] Platonic entity, which cannot be turned into computational instructions. If it could, [then] it would be represented by a Gödel number g, so we would have $\vdash_{[PA_\omega]} A(\underline{k}, \underline{g})$. Logic then yields $\vdash_{[PA_\omega]} \exists b\, A(\underline{k}, b)$, and then $\vdash_{[PA_\omega]} \neg \forall b \neg A(\underline{k}, b)$, that is, $\vdash_{[PA_\omega]} \neg G_{[PA]}$; [but] this is contradicting $\vdash_{[PA_\omega]} G_{[PA]}$ in PA_ω just as it does in PA ... [The ω-rule schema], in other words, is no *syntactic* rule. One may accept its meaning, but its content is not captured by any finite string of symbols — the '...' used to write down the ω-rule [schema] must be understood as an actually infinite sequence of dots, so that no Gödel number g could ever represent them. As long as Gödel numbers are a stylized expression of actual programs, there is no way [that] we could use infinite induction to define a decision method [for $\vdash_{PA}$], even if we had an infinitely powerful machine at our disposal. (p. 278)

PA_ω proves G_{PA}

Before analyzing that passage, let us rehearse a proof that G_{PA} is PA_ω-provable. Relative to a given coding, for each n that codes an $\mathcal{L}_A$-wff, let F_n be the formula whose code is n. By Kleene's (1976, p. 206) lemma 21, there is a Gödel numbering for $\mathcal{L}_A$-wffs and PA-proofs under which there is an $\mathcal{L}_A$-predicate $A(v_1, v_2)$ such that

$$y \text{ is the code of a } \mathsf{PA}\text{-proof of } F_x(\underline{x}) \implies \vdash_{\mathsf{PA}} A\big(\underline{x}, \underline{y}\big),$$

$$y \text{ is } \textit{not} \text{ the code of a } \mathsf{PA}\text{-proof of } F_x(\underline{x}) \implies \vdash_{\mathsf{PA}} \neg A\big(\underline{x}, \underline{y}\big).$$

With Cotogno, define

$$k := \#\big(\forall y \neg A(x,y)\big).$$

Thus,

$$F_k(x) = \forall y \neg A(x,y).$$

Define

$$G_{\mathsf{PA}} := F_k(\underline{k}) = \forall y\, \neg A(\underline{k}, y).$$

Assuming that PA is ω-consistent,[47] by Kleene's theorem 28 (pp. 207–208),

$$\nvdash_{\mathsf{PA}} G_{\mathsf{PA}}.$$

Then, $\forall n \in \mathbb{N}$, n is not the code of a PA-proof of G_{PA}. By ω applications of lemma 21,

$$\forall n \in \mathbb{N} \quad \vdash_{\mathsf{PA}} \neg A(\underline{k}, \underline{n}).$$

Thus,

$$\forall n \in \mathbb{N} \quad \vdash_{\mathsf{PA}_\omega} \neg A(\underline{k}, \underline{n}).$$

By the ω-rule for PA_ω,

$$\vdash_{\mathsf{PA}_\omega} \forall y\, \neg A(\underline{k}, y).$$

Thus,

$$\vdash_{\mathsf{PA}_\omega} G_{\mathsf{PA}}.$$

The set of codes of PA_ω-proofs

We can creatively extrapolate from the quoted passage of Cotogno (p. 278) the following argument. Assume that PA_ω is consistent. Suppose for reductio that the relation "being the code of a PA_ω-proof" is recursive. Since every recursive relation is numeralwise expressible in PA,[48] and every relation that is numeralwise expressible in PA is numeralwise expressible in PA_ω too, there is a coding of PA_ω-proofs (and $\mathcal{L}_A$-wffs) under which $\forall x, y \in \mathbb{N}$:

$$y \text{ is the code of a } \mathsf{PA}_\omega\text{-proof of } F_x(\underline{x}) \implies \vdash_{\mathsf{PA}_\omega} A\big(\underline{x}, \underline{y}\big),$$

$$y \text{ is } \textit{not} \text{ the code of a } \mathsf{PA}_\omega\text{-proof of } F_x(\underline{x}) \implies \vdash_{\mathsf{PA}_\omega} \neg A\big(\underline{x}, \underline{y}\big).$$

Let g be the code of a PA_ω-proof of G_{PA}. Thus,

---

[47] An $\mathcal{L}_A$-theory T is *ω-consistent* iff for every $\mathcal{L}_A$-wff $\varphi(v_1)$,

$$\forall n \in \mathbb{N} \quad \vdash_T \varphi(\underline{n}) \quad \implies \quad \nvdash_T \exists x\, \neg\varphi(x).$$

T is *ω-inconsistent* iff it is not ω-consistent.

[48] See Corollary 3.25 in Mendelson (2015, p. 191).

$$\vdash_{\mathsf{PA}_\omega} A(\underline{k}, \underline{g}),$$

$$\vdash_{\mathsf{PA}_\omega} \exists b\, A(\underline{k}, b),$$

$$\vdash_{\mathsf{PA}_\omega} \neg\forall b\, \neg A(\underline{k}, b),$$

$$\vdash_{\mathsf{PA}_\omega} \neg G_{\mathsf{PA}},$$

$$\vdash_{\mathsf{PA}_\omega} \neg G_{\mathsf{PA}} \;\wedge\; G_{\mathsf{PA}}.$$

Thus PA_ω is inconsistent. But we assumed the consistency of PA_ω. Contradiction. By reductio, the relation "being the code of a PA_ω-proof" is not recursive. So if PA_ω is consistent, then the relation "being the code of a PA_ω-proof" is not recursive. Here ends the extrapolated argument.

Since there are uncountably many PA_ω-proofs, the conclusion of the extrapolated argument should not be surprising. By Goldfarb's (1975) result, G_{PA} is provable in PA_{ω^2} too, so we can give an argument using PA_{ω^2} that is analogous to the one that we extrapolated.

The set of codes of PA_{ω^2}-proofs

Here is a creative extrapolation of a slightly more interesting argument from the same quoted passage. Assume that PA_{ω^2} is consistent. Suppose for reductio that "being the code of a PA_{ω^2}-proof" is a recursive relation. Since every recursive relation is numeralwise expressible in PA, there is a coding of PA_{ω^2}-proofs (and $\mathcal{L}_A$-wffs) under which $\forall x, y \in \mathbb{N}$:

$$y \text{ is the code of a } \mathsf{PA}_{\omega^2}\text{-proof of } F_x(\underline{x}) \implies \vdash_{\mathsf{PA}} A(\underline{x}, \underline{y}),$$

$$y \text{ is } \textit{not} \text{ the code of a } \mathsf{PA}_{\omega^2}\text{-proof of } F_x(\underline{x}) \implies \vdash_{\mathsf{PA}} \neg A(\underline{x}, \underline{y}).$$

Following Cotogno, let g be the code of a PA_{ω^2}-proof of G_{PA}. Thus,

$$\vdash_{\mathsf{PA}} A(\underline{k}, \underline{g}).$$

Since (as shown previously)

$$\forall n \in \mathbb{N} \quad \vdash_{\mathsf{PA}} \neg A(\underline{k}, \underline{n}),$$

and in particular

$$\vdash_{\mathsf{PA}} \neg A(\underline{k}, \underline{g}).$$

So

$$\vdash_{\mathsf{PA}} \neg A(\underline{k}, \underline{g}) \;\wedge\; A(\underline{k}, \underline{g}).$$

Thus, PA is inconsistent. So PA_{ω^2} is inconsistent. But we assumed the consistency of PA_{ω^2}. Contradiction. By reductio, the relation "being the code of a PA_{ω^2}-proof" is not recursive. So if PA_{ω^2} is consistent, then the relation "being the code of a PA_{ω^2}-proof" is not recursive.

1.3.2 A Rosser sentence for PA

Cotogno's second argument uses a Rosser sentence R_{PA} for PA:

> Rosser's construction [of R_{PA}] hinges on the *refutation predicate* $Ref_{PA}(x, y)$, where y is [the numeral for] the Gödel number of a formal [PA-]proof of the negation of the sentence ... [whose] Gödel number is [indicated by the numeral] x. The construction starts from the open formula $\forall y\,(A(x, y) \to [\exists z]\,(z \leq y \,\wedge\, B(x, z)))$, of Gödel number h, where [the] predicate $B(x, y) = Ref_{PA}(sub(x, \ulcorner \dot{x} \urcorner, x), y)$ is the dual of $A(x, y)$ for refutation. Then, replacing the free variable x with the numeral [for] h produces the self-referencing sentence $R_{[PA]} = \forall y\,(A(\underline{h}, y) \;\to\; \exists z\,(z \leq y \;\wedge\; B(\underline{h}, z)))$ ...
>
> ... [To show that $\nvdash_{PA} R_{PA}$,] one assumes $\vdash_{PA} R_{[PA]}$; therefore, for some [integer] $v > 0 \;\; \vdash_{PA} A(\underline{h}, \underline{v})$. Consistency [of PA] then ensures that exactly v non-refutation theorems are provable [in PA]:
>
> $$\vdash_{PA} \neg B(\underline{h}, 0), \quad \ldots \quad , \quad \vdash_{PA} \neg B(\underline{h}, \underline{v}) \tag{3}$$
>
> These yield $\vdash_{PA} \forall c\,(c \leq \underline{v} \to \neg B(\underline{h}, c))$; by the hypothesis, then, we have: $\vdash_{PA} \exists b\,(A(\underline{h}, b) \wedge \neg\exists c\,(c \leq b \wedge B(\underline{h}, c)))$. Next, logic yields $\vdash_{PA} \exists b\,\neg(\neg A(\underline{h}, b) \vee \exists c\,(c \leq b \wedge B(\underline{h}, c)))$ and [then] $\vdash_{PA} \neg\forall b\,(\neg A(\underline{h}, b) \vee \exists c\,(c \leq b \wedge B(\underline{h}, c)))$. The last proposition actually amounts to $\vdash_{PA} \neg R_{[PA]}$, against the hypothesis; hence, $\nvdash_{PA} R_{[PA]}$. An analogous argument then shows $\nvdash_{PA} \neg R_{[PA]}$. Since succession (3) contains in any case a finite number of sentences, it cannot be generalized; unlike succession (2), it is indifferent to infinite induction.
>
> $R_{[PA]}$ states that [the existence of] a [numeral for the code of a PA-]proof of itself entails [that there is] a shorter [numeral for the code of a PA-]refutation [of itself]. ... In this case as well [as with succession (2)], an infinity of premisses
>
> $$\vdash_{[PA]} A(\underline{h}, \underline{y}) \;\to\; \exists z\,(z \leq \underline{y} \;\wedge\; B(\underline{h}, z)) \tag{4}$$
>
> for all $y \geq 0$ could [be used to] prove [in PA_ω] that $R_{[PA]}$ is a true sentence ... Such [a] 'proof', however, could not be formalized in PA_ω; otherwise, it would have its own Gödel number, and that would lead again to a contradiction. (pp. 278–279)

We take Cotogno to be arguing by reductio that there could not be a recursive coding of PA_ω-proofs. The more interesting argument is about PA_{ω^2} (rather than PA_ω). But before giving that argument, let us show that R_{PA} is PA_{ω^2}-provable.

PA_{ω^2} proves R_{PA}

By Kleene's lemma 21, there is a Gödel numbering for $\mathcal{L}_A$-wffs and PA-proofs under which there is an $\mathcal{L}_A$-predicate $B(v_1, v_2)$ such that

$$y \text{ is the code of a } \mathsf{PA}\text{-proof of } \neg F_x(\underline{x}) \implies \vdash_{\mathsf{PA}} B\big(\underline{x}, \underline{y}\big),$$

$$y \text{ is } \textit{not} \text{ the code of a } \mathsf{PA}\text{-proof of } \neg F_x(\underline{x}) \implies \vdash_{\mathsf{PA}} \neg B\big(\underline{x}, \underline{y}\big).$$

With Cotogno, define

$$h := \#\Big(\forall y \left(A(x,y) \ \to \ \exists z \leq y \ B(x,z)\right)\Big).$$[49]

Thus,

$$F_h(x) = \forall y \left(A(x,y) \ \to \ \exists z \leq y \ B(x,z)\right).$$

Define

$$R_{\mathsf{PA}} := F_h(\underline{h}) = \forall y \left(A(\underline{h},y) \ \to \ \exists z \leq y \ B(\underline{h},z)\right).$$

Assuming that PA is consistent, by Kleene's theorem 29 (p. 208),

$$\nvdash_{\mathsf{PA}} R_{\mathsf{PA}}.$$

So, $\forall y \in \mathbb{N}$, y is *not* the code of a PA-proof of R_{PA}. By ω applications of Kleene's lemma 21,

$$\forall y \in \mathbb{N} \quad \vdash_{\mathsf{PA}} \neg A\big(\underline{h}, \underline{y}\big).$$

So for any $\mathcal{L}_A$-wff $\varphi(v_1)$,

$$\forall y \in \mathbb{N} \quad \vdash_{\mathsf{PA}} A\big(\underline{h}, \underline{y}\big) \ \to \ \varphi\big(\underline{y}\big).$$

Taking

$$\varphi(y) \ = \ \exists z \leq y \ B(\underline{h}, z),$$

we get

$$\forall y \in \mathbb{N} \quad \vdash_{\mathsf{PA}} A\big(\underline{h}, \underline{y}\big) \ \to \ \exists z \leq \underline{y} \ B(\underline{h}, z).$$

Then

$$\forall y \in \mathbb{N} \quad \vdash_{\mathsf{PA}_\omega} A\big(\underline{h}, \underline{y}\big) \to \exists z \leq \underline{y} \ B(\underline{h}, z).$$

By the ω-rule for PA_ω,

$$\vdash_{\mathsf{PA}_\omega} \forall y \left(A(\underline{h}, y) \ \to \ \exists z \leq y \ B(\underline{h}, z)\right).$$

So

$$\vdash_{\mathsf{PA}_\omega} R_{\mathsf{PA}}.$$

By Goldfarb's (1975) result,

$$\vdash_{\mathsf{PA}_{\omega^2}} R_{\mathsf{PA}}.$$

[49] Because of the bounded quantifier, an *abbreviation* of an $\mathcal{L}_A$-wff occurs in that string.

The set of codes of PA_{ω^2}-proofs

We can creatively extrapolate from the quoted passage of Cotogno (pp. 278–279) the following argument. Assume that PA_{ω^2} is consistent. Suppose for reductio that the relation "being the code of a PA_{ω^2}-proof" is recursive. Since every recursive relation is numeralwise expressible in PA, there is a coding of PA_{ω^2}-proofs (and $\mathcal{L}_A$-wffs) under which $\forall x, y \in \mathbb{N}$:

$$y \text{ is the code of a } \mathsf{PA}_{\omega^2}\text{-proof of } F_x(\underline{x}) \implies \vdash_{\mathsf{PA}} A\big(\underline{x}, \underline{y}\big),$$

$$y \text{ is } \textit{not} \text{ the code of a } \mathsf{PA}_{\omega^2}\text{-proof of } F_x(\underline{x}) \implies \vdash_{\mathsf{PA}} \neg A\big(\underline{x}, \underline{y}\big).$$

Let r be the code of a PA_{ω^2}-proof of R_{PA}. Thus,

$$\vdash_{\mathsf{PA}} A(\underline{h}, \underline{r}).$$

Since (as shown previously)

$$\forall y \in \mathbb{N} \quad \vdash_{\mathsf{PA}} \neg A\big(\underline{h}, \underline{y}\big),$$

we have

$$\vdash_{\mathsf{PA}} \neg A(\underline{h}, \underline{r}).$$

So

$$\vdash_{\mathsf{PA}} \neg A(\underline{h}, \underline{r}) \ \wedge \ A(\underline{h}, \underline{r}).$$

Thus, PA is inconsistent. So PA_{ω^2} is inconsistent. Contradiction. By reductio, the relation "being the code of a PA_{ω^2}-proof" is not recursive. So if PA_{ω^2} is consistent, then the relation "being the code of a PA_{ω^2}-proof" is not recursive.

This is the best that we can do to fill in the line of reasoning that Cotogno suggests can be given using R_{PA}.

1.3.3 A consistency statement for PA

Many $\mathcal{L}_A$-sentences express the consistency of PA, and many sentences are PA-provably equivalent to ones that do so. Cotogno suggests that an argument that is analogous to the one using R_{PA} can be given using an $\mathcal{L}_A$-sentence Con_{PA} that expresses the consistency of PA.

> PA_ω ... might well take care of typical [$\mathcal{L}_A$-]sentences [that are] undecidable in PA, such as the statement of Fermat's Last Theorem, but [PA_ω] has nothing more than the original PA [has] for *deciding* odd specimens such as $G_{[PA]}$ or $R_{[PA]}$, if [Peano] arithmetic is consistent.

> Rosser (1937) admitted that PA_ω can prove the consistency of PA: the Carnap rule would draw the consistency sentence Con_{PA} from infinitely many premisses of [the] form $\neg Ref_{PA}(\ulcorner ax_{PA} \urcorner, \underline{n})$, where ax_{PA} are the Peano axioms. Formalizing this argument and representing it by a Gödel number is, however, prevented by Gödel['s] Second [Incompleteness] Theorem; again, all premises are required in actu. (p. 279)

Perhaps Cotogno has in mind an ω-sequence of $\mathcal{L}_A$-sentences which are equivalent to:

$$\begin{gathered}\text{it is not the case that } 0 \text{ is the code of a } \mathsf{PA}\text{-refutation of } \mathsf{PA}\text{'s axioms,}\\ \vdots \\ \text{it is not the case that } n \text{ is the code of a } \mathsf{PA}\text{-refutation of } \mathsf{PA}\text{'s axioms,}\\ \vdots\end{gathered}$$

Part of the claim would need to be that each of these statements is equivalent to a PA-provable sentence, and that the universal generalization

$$\forall n \in \mathbb{N}, \text{ it is not the case that } n \text{ is the code of a } \mathsf{PA}\text{-refutation of } \mathsf{PA}\text{'s axioms}$$

is expressed by a sentence that is PA_ω-provable.

Cotogno's argument seems to aim at showing that a particular PA_ω-proof of Con_{PA} cannot be encoded by a natural number. It is not clear how to make sense of

$$\ulcorner ax_{PA} \urcorner,$$

since the axioms of PA are countably infinite in number, and PA is not finitely axiomatizable. But Cotogno's argument is easily fixed to avoid this problem.

Take

$$Con_{\mathsf{PA}} := \forall x \ \neg A\big(\ulcorner 0 \neq 0 \urcorner, x\big).$$

PA_ω **proves** Con_{PA}

First we show that Con_{PA} is PA_ω-provable. For each natural number, PA proves that that number is not the code of a PA-proof of the sentence

$$0 \neq 0.$$

I.e.,

$$\forall n \in \mathbb{N} \quad \vdash_{\mathsf{PA}} \neg A\big(\ulcorner 0 \neq 0 \urcorner, \underline{n}\big).$$

Since PA_ω has the ω-rule,

$$\vdash_{\mathsf{PA}_\omega} \forall x \ \neg A\big(\ulcorner 0 \neq 0 \urcorner, x\big).$$

Cotogno's argument using Con_{PA}

Cotogno's argument might go as follows. If this PA_ω-proof of Con_{PA} "could be formalized", then it would have a Gödel number. Suppose that k is the code of a PA_ω-proof of Con_{PA}. Then

$$\vdash_{\mathsf{PA}} A(\ulcorner Con_{\mathsf{PA}} \urcorner, \underline{k}).$$

Thus, $\vdash_{\mathsf{PA}} Con_{\mathsf{PA}}$. By the second incompleteness theorem, PA is inconsistent. But by assumption, PA is consistent. Contradiction. So no natural number codes a PA_ω-proof of Con_{PA}.

This is the best that we can do to fill in the line of reasoning using Con_{PA}.

1.3.4 A Yablo sequence for PA

Cotogno's third argument uses an ω-sequence of $\mathcal{L}_A$-sentences:[50]

> $$\varphi_0,\ \varphi_1,\ \dots\ ,\ \varphi_i,\ \dots \qquad (5)$$
>
> where each instance φ_i, for any $i \geq 0$, is a Δ_1 sentence stating that all [of] its successors [in (5)] are [PA-]refutable: $\varphi_i = \forall j\,(j > \underline{i} \to \exists z\, Ref_{PA}(\ulcorner \dot{\varphi_j} \urcorner, z))\ \dots$
>
> **Proposition 2** *If PA is ω-consistent*, then [each of the] sentences in [the] succession [in] Eq. 5 are undecidable [in PA].
>
> ...First, let us assume $\vdash_{PA} \varphi_i$, where φ_i is an arbitrary member of (5), [with] $i \geq 0$: by [PA's] ω-consistency, each existential clause $\exists j\,(j > \underline{i} \to \exists z\, Ref_{PA}(\ulcorner \dot{\varphi_j} \urcorner, z))$ should be satisfied by some integer, so we may assert $\vdash_{PA} \neg\varphi_j$ for all $j > i$. Then, let us consider the subsequent sentence φ_{i+1}: all [of] its successors [in (5)] are [PA-] refutable, and this yields $\vdash_{PA} \varphi_{i+1}$, a contradiction.
>
> Conversely, let us assume $\vdash_{PA} \neg\varphi_i$, [for some] $i \geq 0$. We are thus assuming that $\exists j\, \neg(j > \underline{i} \to \exists z\, Ref_{PA}(\ulcorner \dot{\varphi_j} \urcorner, z))$ is [PA-]proven; logic turns this into $\exists j\,(j > \underline{i}\ \wedge\ \forall z\, \neg Ref_{PA}(\ulcorner \dot{\varphi_j} \urcorner, z))$. In other words, we are holding that no integer z codes a [PA-]refutation of φ_j for some $j > i$; since PA is assumed [to be] ω-consistent, this can count as a proof by enumeration of φ_j. A contradiction then obtains again for φ_{j+1}, by repeating the first part of the argument...
>
> In this case as well, we can see that undecidability is not eliminated by PA_ω: each sentence φ_i is by definition [a] universal [generalization], and [is] therefore indifferent to infinite induction. Even if one could sweep [through] the whole [sequence in] (5) in a single pass, that would make no difference for deciding each φ_i; undecidability is a built-in aspect of the succession [in (5)], and is not affected by the way [in which] one scans its elements — be it one at the time, or all of them in [a] batch. (p. 280)

The second half of Cotogno's argument for proposition 2 fails. There he needs to show that

[50] The sequence is structurally related to a version of Yablo's paradox. See Cook (2014).

$$\forall n \in \mathbb{N} \quad \nvdash_{\mathsf{PA}} \neg\varphi_n.$$

He fixes an $i \in \mathbb{N}$ and assumes for reductio that

$$\vdash_{\mathsf{PA}} \neg\varphi_i.$$

It then follows from his definitions that

$$\vdash_{\mathsf{PA}} \exists j > \underline{i} \ \forall z \, \neg Ref_{\mathsf{PA}}\Big(\ulcorner \dot{\varphi_j} \urcorner, z\Big).$$

Then we have that for some fixed integer $j > i$,

$$\vdash_{\mathsf{PA}} \forall z \, \neg Ref_{\mathsf{PA}}\big(\ulcorner \varphi_j \urcorner, z\big).$$

Here Cotogno claims that since PA is ω-consistent, "this can count as a proof by enumeration of φ_j", i.e.,

$$\vdash_{\mathsf{PA}} \varphi_j.$$

PA's ω-consistency cannot help at this point in the reasoning. The sentence

$$\forall z \, \neg Ref_{\mathsf{PA}}\big(\ulcorner \varphi_j \urcorner, z\big)$$

is PA-provably equivalent to an $\mathcal{L}_A$-sentence expressing the consistency of PA. If PA cannot refute φ_j, then it cannot prove $\neg\varphi_j$, so it cannot prove $0 \neq 0$. If PA could prove $0 \neq 0$, then it could prove any $\mathcal{L}_A$-formula, including $\neg\varphi_j$. So

$$\vdash_{\mathsf{PA}} \forall z \, \neg Ref_{\mathsf{PA}}\big(\ulcorner \varphi_j \urcorner, z\big) \quad \Longrightarrow \quad \vdash_{\mathsf{PA}} Con_{\mathsf{PA}}.$$

Then by the second incompleteness theorem, PA is inconsistent and therefore ω-inconsistent. By reductio, PA does not prove φ_j. This would be one way to finish the proof of proposition 2.

Part of what is causing trouble is the definition of sequence 5. Let us explain a way that the intended sequence can be defined correctly.

A sequence of Yablo sentences for PA

Following Kurahashi (2014), provided that a PA-provability predicate

$$Pr_{\mathsf{PA}}(v_1) \ := \ \exists y \, Prf_{\mathsf{PA}}(v_1, y),$$

$Prf_{\mathsf{PA}}(v_1, v_2)$ is a *standard proof predicate for* PA iff for any $\mathcal{L}_A$-wffs $\varphi(v_1)$ and $\psi(v_1)$:

(i)	Prf_{PA} represents in PA a primitive recursive subset of $\omega \times \omega$,
(ii)	$\vdash_{\mathsf{PA}} \varphi(x) \quad \Longleftrightarrow \quad Pr_{\mathsf{PA}}(\ulcorner \varphi(\dot{x}) \urcorner) \in \mathsf{TA}$,
(iii)	$\vdash_{\mathsf{PA}} \varphi(x) \quad \Longrightarrow \quad \vdash_{\mathsf{PA}} Pr_{\mathsf{PA}}(\ulcorner \varphi(\dot{x}) \urcorner)$,
(iv)	$\vdash_{\mathsf{PA}} Pr_{\mathsf{PA}}(\ulcorner \varphi(\dot{x}) \to \psi(\dot{x}) \urcorner) \to \Big(Pr_{\mathsf{PA}}(\ulcorner \varphi(\dot{x}) \urcorner) \to Pr_{\mathsf{PA}}(\ulcorner \psi(\dot{x}) \urcorner) \Big)$,
(v)	$\varphi(x)$ is a Σ_1-formula $\quad \Longrightarrow \quad \vdash_{\mathsf{PA}} \varphi(x) \to Pr_{\mathsf{PA}}(\ulcorner \varphi(\dot{x}) \urcorner)$.

An $\mathcal{L}_A$-wff $Y(v_1)$ is a *Yablo formula of* PA iff

$$\text{for some standard proof predicate } Pr_{\mathsf{PA}}(v_1) \text{ for } \mathsf{PA},$$
$$\vdash_{\mathsf{PA}} \forall n \Big(Y(n) \;\leftrightarrow\; \forall m > n \; \neg Pr_{\mathsf{PA}}(\ulcorner Y(\dot{m}) \urcorner) \Big).$$

We can correctly define the sequence that Cotogno intended by (5) by using an instance of the fixed point lemma. Kurahashi's lemma 2 (p. 1002) states:

> For each $\mathcal{L}_A$-wff $\varphi(v_1, v_2, v_3)$,
> we can effectively find an $\mathcal{L}_A$-wff $\psi(v_1)$ such that
> $$\vdash_{\mathsf{PA}} \forall x \Big(\psi(x) \;\leftrightarrow\; \forall y \; \varphi\big(x, y, \ulcorner \psi(\dot{y}) \urcorner\big) \Big).$$

Taking

$$\varphi(x, j, u) :=$$
$$j > x \;\to\; \exists z \, Ref_{\mathsf{PA}}(u, z),$$

there is an $\mathcal{L}_A$-wff $Y(v_1)$ such that

$$\vdash_{\mathsf{PA}} \forall x \left(Y(x) \;\leftrightarrow\; \forall j > x \; \exists z \, Ref_{\mathsf{PA}}\Big(\ulcorner Y(\dot{j}) \urcorner, z\Big) \right).$$

For each $i \in \mathbb{N}$ define

$$\varphi_i := Y(\underline{i}).$$

For each $i \in \mathbb{N}$, φ_i is PA-*provably equivalent* to

$$\forall j > \underline{i} \quad \exists z \, Ref_{\mathsf{PA}}\Big(\ulcorner Y(\dot{j}) \urcorner, z\Big).$$

It is not the case that each φ_i is *numerically identical* to the corresponding instance of that formula, as Cotogno suggests. Thus we have the ω-sequence of Yablo sentences for PA

$$Y(0), \quad Y(\underline{1}), \quad Y(\underline{2}), \quad \dots \quad , \quad Y(\underline{n}), \quad \dots \tag{5'}$$

each of which expresses the claim that each of its successors in that sequence is PA-refutable.

PA neither proves nor refutes any of these Yablo sentences

Let us show that each sentence in that sequence is neither PA-provable nor PA-refutable. First we show that if PA is consistent, then PA does not prove any sentence in $5'$. Suppose for reductio that

$$\exists x \in \mathbb{N} \quad \vdash_{\mathsf{PA}} \; Y(\underline{x}).$$

Fix an $i \in \mathbb{N}$ such that

$$\vdash_{\mathsf{PA}} \; Y(\underline{i}).$$

By Kurahashi's lemma 2,

$$\vdash_{\mathsf{PA}} \; Y(\underline{i}) \; \leftrightarrow \; \forall j > \underline{i} \; \exists z \, Ref_{\mathsf{PA}}(\ulcorner Y(\dot{j})\urcorner, z).$$

So

$$\vdash_{\mathsf{PA}} \; \forall j > \underline{i} \; \exists z \, Ref_{\mathsf{PA}}\Big(\ulcorner Y(\dot{j})\urcorner, z\Big).$$

But then

$$\vdash_{\mathsf{PA}} \; \forall j > \underline{i+1} \; \exists z \, Ref_{\mathsf{PA}}\Big(\ulcorner Y(\dot{j})\urcorner, z\Big).$$

By Kurahashi's lemma 2,

$$\vdash_{\mathsf{PA}} \; Y(\underline{i+1}) \; \leftrightarrow \; \forall j > \underline{i+1} \; \exists z \, Ref_{\mathsf{PA}}\Big(\ulcorner Y(\dot{j})\urcorner, z\Big).$$

So

$$\vdash_{\mathsf{PA}} \; Y(\underline{i+1}).$$

Since

$$\vdash_{\mathsf{PA}} \; \forall j > \underline{i} \; \exists z \, Ref_{\mathsf{PA}}\Big(\ulcorner Y(\dot{j})\urcorner, z\Big)$$

(as shown earlier), we have

$$\vdash_{\mathsf{PA}} \ \exists z\, Ref_{\mathsf{PA}}\big(\ulcorner Y(\underline{i+1})\urcorner, z\big).$$

Thus,

$$\vdash_{\mathsf{PA}} \ \neg Y(\underline{i+1}).$$

So

$$\vdash_{\mathsf{PA}} \ \neg Y(\underline{i+1}) \ \wedge \ Y(\underline{i+1}).$$

So PA is inconsistent, which contradicts our assumption that it was consistent. Therefore,

$$\forall x \in \mathbb{N} \quad \nvdash_{\mathsf{PA}} \ Y(\underline{x}).$$

Next we show that PA does not disprove any sentence in $5'$. Suppose for reductio that

$$\exists x \in \mathbb{N} \quad \vdash_{\mathsf{PA}} \ \neg Y(\underline{x}).$$

Let $i \in \mathbb{N}$ be such that

$$\vdash_{\mathsf{PA}} \ \neg Y(\underline{i}).$$

Since $Y(\underline{i})$ is a Yablo formula of PA, there is a standard proof predicate $Pr_{\mathsf{PA}}(v_1)$ for PA such that

$$\vdash_{\mathsf{PA}} \ \neg Y(\underline{i}) \ \leftrightarrow \ \neg \forall m > \underline{i} \quad \neg Pr_{\mathsf{PA}}\big(\ulcorner Y(\dot{m})\urcorner\big).$$

So

$$\vdash_{\mathsf{PA}} \ \neg \forall m > \underline{i} \quad \neg Pr_{\mathsf{PA}}\big(\ulcorner Y(\dot{m})\urcorner\big).$$

Then

$$\vdash_{\mathsf{PA}} \ \exists m > \underline{i} \quad Pr_{\mathsf{PA}}\big(\ulcorner Y(\dot{m})\urcorner\big).$$

Since

$$\exists m > \underline{i} \quad Pr_{\mathsf{PA}}\big(\ulcorner Y(\dot{m})\urcorner\big)$$

is equivalent to a Σ_1-forumla, and PA is Σ_1-sound,

$$\exists m \in \mathbb{N} \quad \vdash_{\mathsf{PA}} \ Pr_{\mathsf{PA}}\big(\ulcorner Y(\underline{m})\urcorner\big).$$

Since Pr_{PA} is a standard proof predicate for PA,

$$\exists m \in \mathbb{N} \quad \vdash_{\mathsf{PA}} \ Y(\underline{m}).$$

This contradicts the conclusion of the previous reductio argument. Therefore,

$$\forall n \in \mathbb{N} \quad \nvdash_{\mathsf{PA}} \ \neg Y(\underline{n}).$$

Thus, no sentence in sequence $5'$ is PA-provable or PA-refutable.

PA_ω refutes each of these Yablo sentences

Cotogno (2015, p. 279) claims that each of these Yablo sentences is "indifferent to infinite induction", i.e., the ω-rule. Since PA_ω proves every $\mathcal{L}_A$-sentence that is true in $\mathfrak{N}$, it should not be surprising that PA_ω refutes each of these Yablo sentences. We will show that the negation of each φ_i is PA_ω-provable, i.e.,

$$\forall n \in \mathbb{N} \quad \vdash_{\mathsf{PA}_\omega} \neg Y(\underline{n}).$$

Work backwards. Fix an integer $i \in \mathbb{N}$. We need to show that

$$\vdash_{\mathsf{PA}_\omega} \neg\forall j \left(j > \underline{i} \;\rightarrow\; \exists z\, Ref_{\mathsf{PA}}\left(\ulcorner Y(\dot{j})\urcorner, z\right)\right).$$

This is equivalent to

$$\vdash_{\mathsf{PA}_\omega} \exists j \left(j > \underline{i} \;\wedge\; \neg\exists z\, Ref_{\mathsf{PA}}\left(\ulcorner Y(\dot{j})\urcorner, z\right)\right).$$

It suffices to show that for some integer $j > i$,

$$\vdash_{\mathsf{PA}_\omega} \neg\exists z\, Ref_{\mathsf{PA}}\left(\ulcorner Y\left(\underline{j}\right)\urcorner, z\right).$$

This is equivalent to showing that for some integer $j > i$,

$$\vdash_{\mathsf{PA}_\omega} \forall z\, \neg Ref_{\mathsf{PA}}\left(\ulcorner Y\left(\underline{j}\right)\urcorner, z\right).$$

Thus, it suffices to show that for some integer $j > i$,

$$\vdash_{\mathsf{PA}_\omega} \forall z\, \neg Prf_{\mathsf{PA}}\left(\ulcorner \neg Y\left(\underline{j}\right)\urcorner, z\right).$$

Thus, it would suffice to show that for some integer $j > i$, each of the following hold:

$$\left.\begin{array}{r} \vdash_{\mathsf{PA}_\omega} \neg Prf_{\mathsf{PA}}\left(\ulcorner \neg Y\left(\underline{j}\right)\urcorner, 0\right) \\ \vdash_{\mathsf{PA}_\omega} \neg Prf_{\mathsf{PA}}\left(\ulcorner \neg Y\left(\underline{j}\right)\urcorner, \underline{1}\right) \\ \vdots \\ \vdash_{\mathsf{PA}_\omega} \neg Prf_{\mathsf{PA}}\left(\ulcorner \neg Y\left(\underline{j}\right)\urcorner, \underline{n}\right) \\ \vdots \end{array}\right\} (*)$$

Now work forwards. For any $j \in \mathbb{N}$, each $x \in \mathbb{N}$ is such that PA proves that x is not the code of a PA-proof of $\neg Y(\underline{j})$. Thus, for some integer $j > i$,

$$\forall n \in \mathbb{N} \quad \vdash_{\mathsf{PA}} \neg Prf_{\mathsf{PA}}\left(\ulcorner \neg Y\left(\underline{j}\right)\urcorner, \underline{n}\right).$$

So we have that for some integer $j > i$,

$$\forall n \in \mathbb{N} \quad \vdash_{\mathsf{PA}_\omega} \neg Prf_{\mathsf{PA}}\Big(\ulcorner \neg Y\big(\underline{j}\big)\urcorner, \underline{n}\Big).$$

This gives us each of the statements in the sequence * above. Therefore,

$$\forall n \in \mathbb{N} \quad \vdash_{\mathsf{PA}_\omega} \neg Y(\underline{n}).$$

This is what we wanted to show. Notice that it follows that

$$\vdash_{\mathsf{PA}_\omega} \forall x \neg Y(x).$$

1.3.5 The import of effectiveness

This subsection briefly discusses the importance of considerations about effectiveness to accounts of *truth in* TA that involve infinitary notions of proof.

Recall that Cotogno's first three arguments respectively use G_{PA}, R_{PA}, and Con_{PA}. Let Ψ be any $\mathcal{L}_A$-sentence such that

$$\nvdash_{\mathsf{PA}} \Psi \quad \text{but} \quad \vdash_{\mathsf{PA}_\omega} \Psi.$$

Cotogno might give analogous arguments for the conclusion that Ψ cannot be decided by an infinite sequence of compuations.

A crucial step in those arguments is the conditional claim that

if a PA_ω-proof of Ψ could be turned into computational instructions,
then that proof would have a Gödel number.

We can make some sense of Cotogno's reasoning that aims to show that no PA_ω-proof of Ψ has a Gödel number. Though that conclusion can be established by simpler reasoning.

If we allow computational instructions to require performing a supertask, then a PA_ω-proof of Ψ *can* be turned into computational instructions. This is shown by the fact that there is a recursive PA_{ω^2}-proof of Ψ. This means that there is an infinitary proof of Ψ, from the axioms of PA, using fewer than ω^2 applications of the recursive ω-rule, and this proof can be coded by a unary total recursive function. Thus, if we were given an $\mathcal{L}_A$-sentence σ (or its code), and we were given the index of a unary total recursive function f, and we had a countably infinite reserve of memory, then a countable sequence of computations (i.e., a supertask) would allow us to verify whether f in fact codes a recursive PA_{ω^2}-proof of σ.

Of course, this does not require there to be an effective method for deciding, of any given $\mathcal{L}_A$-sentence σ, whether there is a recursive PA_{ω^2}-proof of σ. Assuming CT, this is ruled out by syntactic forms of the first incompleteness theorem. The set of codes of $\mathcal{L}_A$-sentences that are PA_ω-provable is not recursive or even recursively enumerable. Assuming CT, nor is there an effective method such that given any $\mathcal{L}_A$-sentence σ,

$$\text{if } \sigma \in \mathsf{TA}, \text{ then it produces the code of a } \mathsf{PA}_\omega\text{-proof of } \sigma,$$

$$\text{if } \sigma \notin \mathsf{TA}, \text{ then it produces the code of a } \mathsf{PA}_\omega\text{-proof of } \neg\sigma.$$

We do not see how to make sense of Cotogno's arguments in a way that supports his claim that supertasks would not allow one to decide arbitrary $\mathcal{L}_A$-sentences. The fact that the set of codes of PA_{ω^2}-proofs is not recursive does not threaten accounts of *truth in* TA such as RTA, which claims that the "truth" of an $\mathcal{L}_A$-sentence φ consists in the existence of a recursive PA_{ω^2}-proof of φ. Accounts like RTA have serious problems, but this is not one of them.

1.4 Arithmetical truth beyond $\mathcal{L}_A$

PA_{ω^2} suffices for capturing TA. In principle, any sentence of first-order arithmetic can be (dis)proven by a countable sequence of computations.[51] Thus, one way of understanding the notion of *first-order arithmetical truth* is in terms of possible supertasks. Even if this is not a tenable philosophical account of *truth in* TA, it is at least an extensionally adequate characterization of TA.

But TA falls *far* short of arithmetical truth proper. If there are truths of arithmetic that are not expressible in $\mathcal{L}_A$, then TA does not exhaust arithmetical truth, and 'true arithmetic' is a misleading term for TA. There are only countably many $\mathcal{L}_A$-sentences, so TA is only countably infinite. For example, if for each non-recursive $S \subset \mathbb{N}$ there is a distinct true arithmetical statement (in an infinitary or higher-order language) about S, then there are uncountably many truths (expressible in that language) about arithmetic. TA only captures countably many of those truths.

A similar point can be made about PA's first-order induction schema, which gives an induction axiom for each $\mathcal{L}_A$-predicate. The set of all such predicates is only countably infinite, but there are uncountably many subsets of $\mathbb{N}$ to which induction ought to apply. PA's first order induction schema only applies to those (countably many) subsets of $\mathbb{N}$ that are definable by $\mathcal{L}_A$-formulas. This gives another reason to look at theories beyond PA, languages beyond $\mathcal{L}_A$, and logics beyond L.

1.5 Mathematical constructions

This section argues that if ZFC is consistent, then performing hypertasks is compatible with the truths of ZFC. This point is rather modest, because (if ZFC is consistent, then) a wide variety of propositions are compatible with the truths of ZFC.

[51] $\#[\mathsf{TA}]$ is Σ^1_1, as is the set of codes of false $\mathcal{L}_A$-sentences. See Theorem IV.1.14 in Odifreddi (1992).

Certain mathematical constructions can be seen as involving infinite sequences of tasks. For example, the *axiom of countable choice* states that for each countable family F of non-empty sets, there is a "choice" function

$$C\colon F \longrightarrow \bigcup F$$

for which

$$\forall S \in F \quad C(S) \in S.$$

I.e., for any such F, there is a "choice" function that maps each $S \in F$ to an element of S. Mormann (2010) writes that the axiom of countable choice, which he calls "CAC",

> ... may be reformulated as the assertion that the "ideal mathematical agent" Mat has the capacity of carrying out supertasks of the following kind: Given a countable family $F = \{S_1, S_2, \ldots\}$ of nonempty sets, let s_i be the task of choosing an element from S_i. Then the countable axiom of choice asserts that Mat ... is able to perform the totality of the countably many tasks s_i in a finite time. To be specific, one may assume that Mat works in a similar fashion as Achilles in that he needs for the first choice s_1 the time s ($s =$ [1] second), for the second choice s_2 the time $\frac{1}{2}s$, for the third choice s_3 the time $\frac{1}{4}s$, and so on. Thereby, the construction of a choice function ... can be conceived [of] as [a] supertask that is carried out in 2 seconds. Something like this is assumed, at least implicitly, when "real mathematicians" invoke (CAC) in their proofs which are thought to be carried out in a finite amount of time. Using (CAC) they feel entitled to the contention of having available a choice function C for any countable family F of non-empty sets. Analogously, the general axiom of choice (AC) may be conceived [of] as the assertion that the "hypertasks" of constructing choice functions for arbitrary families F of nonempty sets in a finite amount of time are possible. (pp. 3–4)

The *axiom of global choice* (GC) states that every class of non-empty sets has a (class-sized) choice function. Analogously, GC may be seen as allowing ultratasks.

We have seen that one way of understanding TA is in terms of possible supertasks. The cumulative hierarchy can be understood in terms of sequences of tasks too. Specifically, each set can be thought of as the result of a sequence of stages of set-formation, as in the iterative conception of the universe of sets.

Potter (2004) mentions the connection between well-ordered hypertasks and the (possible) structure of time:[52]

> When we discussed the constructivist understanding of the process of set formation, we noted the difficulty that this conception would apparently limit us to finite sets. In

[52] Potter nonstandardly uses the term "supertask" to mean any well-ordered infinite sequence of tasks, whether countable or not.

> order to liberate the constructivist from this limitation, we examined the possibility of appealing to supertasks — processes carried out repeatedly with increasing speed, so that an infinite number may be carried out within a finite period of time. The proposition ... [that every well-ordered subset of the real line is countable] shows, however, that this method has a limit. This is because the tasks performed in a supertask are well-ordered in time. As a result, if we assume that the ordering of time is correctly modelled as a continuum [i.e., like the real number system as treated in classical mathematics], we can conclude that any supertask contains only countably many subtasks. (p. 177)

It is not clear why a constructivist appeal in this context to infinite sequences of tasks must assume that time is structured like $\mathbb{R}$, or some other linear continuum that would be incompatible with completing a hypertask in a finite length of time. There are linear continua other than $\mathbb{R}$ such that possible worlds in which *time is structured like one of those continua* allow for well-ordered hypertasks to be completed in a finite length of time. For example, it has been argued that one such continuum is the surreal number line.[53] A possible world with surreal time allows for ultratasks to be completed in any non-trivial interval of time.

1.5.1 Possibility relative to a consistent system

Given a consistent (semi-)formal system, we can define a notion of modality under which what is "possible" (relative to that system) is whatever is "compatible with" that system. For example, we can define what is possible with respect to L as follows.

Say that a proposition p is L-*possible* iff

p is compatible with the truths of L.

Likewise, we can define what is possible with respect to ZFC.

Say that a proposition p is ZFC-*possible* iff

p is compatible with the truths of ZFC.

ZFC-possibility is stricter than L-possibility. Not every L-possible proposition is ZFC-possible. For example, the sentence

$$0 \in 0$$

expresses a proposition that is L-possible. But that proposition is not ZFC-possible — in ZFC, the $\in$-relation is irreflexive.[54]

[53] See Al-Dhalimy & Geyer (2016).

[54] Similarly, the sentence "$0 < 0$" expresses a proposition that is L-possible but not compatible with the truths of classical mathematics.

What is it for a proposition to be "compatible with" the truths of ZFC? This notion can be characterized syntactically roughly as follows. Suppose that a sentence s expresses a proposition p. Form a new language by expanding the language of ZFC to include the basic notions used in s. Then, p is *compatible with* the truths of ZFC iff

ZFC+s does not prove a contradiction (using L) in the new language.

I.e., ZFC is consistent with s (under L).

If a system S is inconsistent, then no proposition is S-possible. So any claim that a proposition is ZFC-possible should be preceded by (something that is equivalent to) the assumption that ZFC is consistent. We will omit this qualification from here on.

1.6 Hypertasks

This section argues that performing a hypertask is ZFC-possible. In particular, we will consider the hypertask of "listing" the real numbers.

Cantor's theorem implies that for any set S, there is no surjection from S to $\mathcal{P}(S)$.[55] A special case of this is that there is no surjection from $\mathbb{N}$ to $\mathbb{R}$, or from ω to 2^ω.[56] One might think that since $\mathbb{R}$ is uncountable, the reals cannot be listed — i.e., it is not possible to enumerate all of them in a finite length of time. We think otherwise. Although the reals cannot be listed in an ω-length sequence (one by one), an infinite list need not have order type ω, or even a countable order type.

For time to be structured in a way that is compatible with performing a hypertask, a finite length of time must contain uncountably many non-trivial pairwise non-overlapping subintervals.

Premise 1: It is ZFC-possible for a finite length of time to contain an ω_1-sequence of non-trivial pairwise non-overlapping subintervals.[57]

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[55] Lawvere's fixed point theorem generalizes various diagonalization theorems. See Karimi & Salehi (2017).

[56] Recall that 2^ω is the set of all total functions from ω to $\{0, 1\}$.

[57] We are only considering one-dimensional structures for time. A one-dimensional topological space T satisfies the *countable chain condition* iff every collection of pairwise disjoint non-empty open subsets of T is countable. See Jech (2006, Chapter 4).

Premise 1 implies that

it is ZFC-possible for time *not* to be structured like a one-dimensional topological space that satisfies the countable chain condition.

Under the order topology, $\mathbb{R}$ satisfies the countable chain condition. So Premise 1 implies that

By premise 1, let I be a finite-length temporal interval that contains, for each ordinal $\alpha < \omega_1$, a non-trivial subinterval I_α, with these subintervals being pairwise non-overlapping, and for any ordinals α, β such that

$$\alpha < \beta < \omega_1,$$

I_α temporally precedes I_β.

Assuming that ZF is consistent, the continuum hypothesis (CH) is ZFC-possible.[58] ZF+CH proves that there is a bijection between ω_1 and $\mathbb{R}$.

> **Premise 2**: It is ZFC-possible that there is a bijection between ω_1 and $\mathbb{R}$.

Let such a bijection be

$$f : \omega_1 \longrightarrow \mathbb{R}.$$

The third premise concerns subdividing intervals of time. Listing a real number amounts to an ω-task, e.g., listing each digit in the binary decimal expansion of that real. If time were dense, then any non-trivial interval of time would contain an ω-sequence of non-trivial, pairwise non-overlapping subintervals, and an ω-task could be done in any non-trivial interval of time.

> **Premise 3**: It is ZFC-possible to list any real number in any non-trivial interval of time.

The possibilities mentioned in premisses 1–3 are jointly compatible with the truths of ZFC.

> **Premise 4**: The possibilities mentioned in premisses 1–3 are jointly ZFC-compossible.

---

it is ZFC-possible for time *not* to be structured like
$\mathbb{R}$ under the order topology.

Since every topological space that is separable satisfies the countable chain condition, Premise 1 also implies that

it is ZFC-possible for time *not* to be structured like
a one-dimensional topological space that is separable.

Assuming ZFC, there are many one-dimensional Hausdorff spaces H that can be equipped with an appropriate notion of distance such that a finite-length interval of H contains uncountably many non-trivial pairwise non-overlapping subintervals. Something similar can be done using locales.

[58] ZFC+GCH is consistent relative to ZF.

For each ordinal $\alpha < \omega_1$, during I_α, list the real number $f(\alpha)$. Thus, each real can be listed during I. So, it is ZFC-possible for there to be a "listing" of all real numbers — a listing in which each real is preceded by only countably many others in the listing — and for an "enumeration" of all the reals to occur within a finite length of time.

Conclusion: It is ZFC-possible to list the reals.

It is not essential to this kind of argument that we use ω_1, but that is the simplest case. If it is ZFC-possible that there is an ordinal α such that both

(i) there is a bijection between α and $\mathbb{R}$,

and

(ii) a finite-length interval of time contains an α-sequence of non-trivial pairwise non-overlapping subintervals,

and this is ZFC-compossible with it being the case that any real number can be listed during any non-trivial interval of time, then listing the reals in a finite length of time is ZFC-possible.

The argument can be strengthened to apply regardless of the cardinality of $\mathbb{R}$. Let κ be whichever cardinal is such that

$$2^{\aleph_0} = \kappa.$$[59]

[59] ZFC places only very weak restrictions on the value of $2^{\aleph_0}$. Let S be a class linearly ordered by $<$. Recall that $s \subseteq S$ is *cofinal in S* iff

$$\forall x \in S \quad \exists y \in s \quad y > x.$$

When the axiom of choice (AC) holds, the cofinality of a class S, written $cf(S)$, is the least cardinal κ such that there is a κ-sized subset of S that is cofinal in S. By the Zermelo-König inequality, when AC holds, every cardinal κ is such that

$$cf(2^\kappa) > \kappa.$$

Recall that a cardinal is *regular* iff it is a fixed-point of the cofinality function cf. Easton (1970) showed that there are many ordinals α for which it is consistent relative to ZFC that

$$2^{\aleph_0} = \aleph_\alpha.$$

In particular, if M is a countable model of ZFC+GCH, and g is a class function in M that maps each infinite regular cardinal to an infinite cardinal, and

$$\text{for any infinite regular cardinals } \kappa, \lambda\text{:}$$
$$\kappa \leq \lambda \quad \Longrightarrow \quad g(\kappa) \leq g(\lambda),$$

and

One might hold that it is *necessarily* true that

$$2^{\aleph_0} = \kappa,$$

and that therefore it is necessarily true that there is a bijection between κ and $\mathbb{R}$. If it is ZFC-compossible that

(i) a finite-length interval of time contains
a κ-sequence of non-trivial pairwise non-overlapping subintervals,

and

(ii) any real number can be listed during any non-trivial interval of time,

then one should also accept that it is ZFC-possible to list the reals.

The argument shows that it is ZFC-possible to perform a hypertask. It is compatible with that conclusion that in the *actual* world, the structure of time does not allow hypertasks.

---

for each infinite regular cardinal κ:

$$cf\big(g(\kappa)\big) > \kappa,$$

then there is a class forcing extension of M in which each regular cardinal κ is such that

$$2^\kappa = g(\kappa).$$

Chapter 2

Spatiotemporal structures

"The history of Zeno's paradoxes is largely the history of concepts of infinity."
— Cajori (1920, p. 7)

Infinite sequences of events raise many philosophical and mathematical problems. Among these are several paradoxes inspired by Zeno of Elea. The study of such event-sequences can be both *motivated by* and *applied to* a family of arguments known as Zeno's dichotomy. These are typically presented as arguments for the impossibility of motion. Adding a premise to the effect that a certain kind of motion is possible, they become arguments for a contradiction. Without interpretative or historical concerns, this chapter characterizes and evaluates a few novel variations and generalizations of those arguments.

We begin by showing how the structure of dichotomy arguments relates to the property of incompleteness, which motivated the study of infinitary proofs in Chapter 1. In analyzing various dichotomy arguments, we take up two main issues. A key premise of these arguments is that for various notions of possibility, it is not possible for a superevent to occur. Also key is a premise to the effect that motion through infinitely divisible time and space necessarily involves a superevent. First, we consider what support can be given for the premise that it is not possible for a certain kind of superevent to occur. Second, we investigate which kinds of spatiotemporal dimensions would allow for the occurrence of such event-sequences, and in particular, how the order types of those dimensions determine the kind of event-sequence that motion through those dimensions involves.

§2.6 describes a conception of dimensions on which there is no limit placed on how finely they can be subdivided. We raise two problems for standard views of spatiotemporal dimensions. One problem has to do with the arbitrariness of the order type of the sequence of moments in time. The other problem has to do with whether what we call *quasi-regions* correspond to spatiotemporal locations.

2.1 Incompleteness

The incompleteness property of certain logical systems is closely related to the structure of several Zenonian arguments about motion through infinitely divisible space and time. This section explains how several of these arguments can be naturally represented as incomplete logical systems. As we will see, these systems crucially lack infinitary rules of inference.[1] Adding a certain infinitary rule of inference to these systems results in systems that are negation-complete.[2]

Achilles and the tortoise

In this example, assume that Achilles begins a race 100 units of distance ahead of the tortoise, and that throughout the race, the velocity of Achilles is ten times the tortoise's velocity. A Zenonian argument suggests that Achilles never reaches the tortoise, because:

> Once Achilles covers 100 units, the tortoise has covered 110;
>
> Once Achilles covers 110 units, the tortoise has covered 111;
>
> Once Achilles covers 111 units, the tortoise has covered $111\frac{1}{10}$;
>
> and so on *ad infinitum*.

The statement that

once Achilles covers $111\frac{1}{9}$ units, the tortoise has covered $111\frac{1}{9}$

is true in the intended interpretation. But that statement is also *unprovable from within* the Zenonian argument without using a rule of inference that requires infinitely many premisses.

The unprovable statement can be defined in terms of the ω-sequence of statements above. Beth (1966) notes that this anticipates a diagonal definition made relative to an ω-sequence:

> As to Zeno's method of proof, it is interesting that in some cases he anticipates Cantor's diagonal principle. In deriving the Achilles paradox, for instance, he introduces, starting from his opponent's thesis [that space is infinitely divisible], an analytic expression for the [spatial] points which are successively reached by Achilles; then, by virtue of this expression, he constructs [a definition of] a new [spatial] point for which this expression does not hold, and which is nevertheless reached by Achilles. (p. 492)

---

[1] López-Escobar (1977, p. 75) suggests that "some of the paradoxes of antiquity were paradoxes because of (implicit) uses" of rules of inference that require infinitely many premisses.

[2] As mentioned in Chapter 1, something similar to this is the case with PA. Allowing infinite-length proofs and sufficiently many uses of various forms of the ω-rule yields a negation-complete system.

López-Escobar (1977) formalizes the Zenonian argument about Achilles and the tortoise as an incomplete logical system in the following way.

> *Formal language of the theory of A & T*
>
> (a) *Individual constants*: $\ulcorner r \urcorner$ for each rational number r,
>
> (b) *Relational constant*: P.
>
> *Axiomatization of A & T*
>
> (a) *Axiom*: $P(\ulcorner 0 \urcorner, \ulcorner 100 \urcorner)$
>
> (b) *Rule of inference*:
>
> $$P(\ulcorner x \urcorner, \ulcorner y \urcorner) \quad \Longrightarrow \quad P\Big(\ulcorner y \urcorner, \ulcorner y + \tfrac{1}{10}(y - x) \urcorner\Big)$$
>
> *Interpretation*: "$P(\ulcorner r \urcorner, \ulcorner s \urcorner)$" is interpreted (in the classical) model [as saying] that Achilles is at a distance r from the starting point of the race when the tortoise is at a distance s from the starting point of the race. (p. 75)

López-Escobar (1977) remarks that:

> An easy induction on the length of derivations in *A & T* shows that for no [rational] number r is the sentence $P(\ulcorner r \urcorner, \ulcorner r \urcorner)$ provable in *A & T*. On the other hand it is well known (nowadays) that $P\Big(\ulcorner 111\frac{1}{9} \urcorner, \ulcorner 111\frac{1}{9} \urcorner\Big)$ is true in the classical model. The fact that $P\Big(\ulcorner 111\frac{1}{9} \urcorner, \ulcorner 111\frac{1}{9} \urcorner\Big)$ is true in some models but not provable in the formal system is, at least since 1930, not paradoxical and it is doubtful that it was the incompleteness of the formal system *A & T* that bothered the Greeks.
>
> The problem had to do much more with the underivability in *A & T* of the following [infinitary] rule [of inference]:
>
> *From*: $$\frac{P(\ulcorner x_0 \urcorner, \ulcorner y_0 \urcorner), \quad P(\ulcorner x_1 \urcorner, \ulcorner y_1 \urcorner), \ \ldots \ , \ P(\ulcorner x_n \urcorner, \ulcorner y_n \urcorner), \ \ldots}{P(\ulcorner x \urcorner, \ulcorner y \urcorner)}$$
>
> *To conclude*: $P(\ulcorner x \urcorner, \ulcorner y \urcorner)$
>
> *Where*: $$x = \lim_{n\to\infty} x_n \quad \textit{and} \quad y = \lim_{n\to\infty} y_n.$$
>
> For it is precisely such a rule which focuses on both the problem of formally expressing the argument used in the paradox of Achilles and the Tortoise as well as on the problem of how a finitary statement requires infinitely many assumptions for its derivation. (pp. 75–76)

To derive the conclusion that Achilles catches the tortoise, we first derive the ω-sequence of sentences

$$\left.\begin{array}{r} P(\ulcorner 0 \urcorner, \ulcorner 100 \urcorner) \\ P(\ulcorner 100 \urcorner, \ulcorner 110 \urcorner) \\ P(\ulcorner 110 \urcorner, \ulcorner 111 \urcorner) \\ \vdots \end{array}\right\} (*)$$

and then apply an infinitary rule of inference to derive from the ω-sequence of sentences in * that

$$P\Big(\ulcorner 111\tfrac{1}{9} \urcorner, \ulcorner 111\tfrac{1}{9} \urcorner\Big).$$

2.1.1 Logical environments

López-Escobar (1993) formalizes a similar Zenonian argument about "a hypothetical race between Achilles and Hector" (p. 7). We need a few definitions for the next two examples.

> A *logical environment*
>
> $$\Lambda := \big(\mathfrak{L}, \models, \mathfrak{A}, \vdash\big)$$
>
> consists of:
>
> (i) a language $\mathfrak{L}$,
>
> (ii) a set $\mathfrak{A}$ of sentences of $\mathfrak{L}$, called axioms, "which are accepted, without any doubts, as having the property of truth" (p. 6),
>
> (iii) a concept $\models$ of "truth" for the sentences of $\mathfrak{L}$,
>
> (iv) a concept $\vdash$ of "derivation" for the sentences of $\mathfrak{L}$.

The following definitions are given in terms of a logical environment

$$\Lambda = \big(\mathfrak{L}, \models, \mathfrak{A}, \vdash\big).$$

A sentence $\sigma \in \mathfrak{L}$ is *true in* Λ (i.e., $\models_\Lambda \sigma$) iff

$$\sigma \in \models.$$

A sentence $\sigma \in \mathfrak{L}$ is *provable in* Λ (i.e., $\vdash_\Lambda \sigma$) iff

$$\mathfrak{A} \vdash \sigma.$$

Λ is *sound* iff every sentence that is provable in Λ is true in Λ, i.e.,

$$\text{for every sentence } \sigma \in \mathfrak{L},$$

$$\vdash_\Lambda \sigma \quad \Longrightarrow \quad \models_\Lambda \sigma \ .$$

Λ is *complete* iff

Λ is sound and every sentence that is true in Λ is provable in Λ.[3]

Λ is *incomplete* iff

Λ is sound and there is a sentence that is true in Λ but not provable in Λ.[4]

Achilles and Hector

López-Escobar (1993) formalizes the Zenonian argument about the race between Achilles and Hector in an incomplete logical environment Λ.[5] He writes:

> What appears paradoxical about Zeno's argument is that, on the one hand, Achilles does catch up [to] Hector and on the other hand the [Zenonian] argument suggests that Hector will **always** be ahead of Achilles. What the incompleteness of the logical environment Λ shows is that Zeno's argument, although sound, is not powerful enough to capture all [of] the true statements about the race, i.e. true statements in Λ. In fact the "provable" statements of Zeno's argument are but a [proper] subset of the statements specifying the simultaneous positions of Achilles and Hector.
>
> Thus, with 100% hindsight we see that historically Zeno had the "first incompleteness theorem". There is another similarity between ... "Zeno's incompleteness" and Gödel's first incompleteness theorem; namely that adding the "Gödel sentence" [of Λ] as a further axiom does not eliminate the incompleteness [of Λ]. ... Thus, given any finite set $\mathcal{S}$ of true sentences of Λ one can determine another true statement σ [of Λ] such that σ is not derivable from $\mathcal{S}$ [in Λ]. (pp. 9–10)

The Zenonian incompleteness result corresponds to the fact that Λ is incomplete. What López-Escobar refers to as "the Gödel sentence of Λ" in the passage quoted above is a statement that expresses the fact that Achilles catches up to Hector. Let us turn to a formalization of a dichotomy argument as an incomplete logical environment.

2.1.2 Incompleteness in a dichotomy argument

A special case of the race between Achilles and the tortoise is one in which the tortoise is always at rest. Dichotomy arguments can be thought of as concerning this kind of situation.

López-Escobar (1993, p. 5) cites a dichotomy argument from Kirk and Raven (1957, p. 293):

---

[3] Standardly, the definition of a "complete" logical system does not require soundness.

[4] Standardly, the definition of an "incomplete" logical system does not require soundness.

[5] We omit the details of Λ. Instead we will consider the formalization of a dichotomy argument in the next subsection.

> It is impossible [for you] to traverse the stadium; because before you reach the far end [of the stadium], you must first reach the half-way point; [and] before you reach the half-way point you must [first] reach the point half way to it; and so on ad infinitum.

López-Escobar (1993) represents this dichotomy argument as a logical environment

$$\Sigma := \big(\mathfrak{L}_2, \models_2, \mathfrak{A}_2, \vdash_2\big).$$

$\mathfrak{L}_2$ is a first-order language suitable for discussing the non-negative rationals, and contains:

> - A set of individual variables: $x, \ldots$.
> - The function symbols: $+$, $\times$, $/$ for *plus*, *times* and *division* respectively.
> - The binary relation symbols: $<$, $\leq$.
> - The individual constants: **0**, **1**, **2**.
> - The unary predicate N for the natural numbers.
> - The binary relation of equality between terms: $=$.
> - The usual sentential connectives: $\wedge$, $\vee$, $\neg$, and $\supset$. (pp. 10–11)

$\mathfrak{L}_2$ also contains:

- The first-order quantifiers $\forall$, $\exists$.
- The binary relation symbol $\wp$.

For any terms t, x of $\mathfrak{L}_2$, the interpretation of

$$\wp(t, x)$$

is that

Achilles has covered x stades in t seconds.[6]

To properly define Σ, we need to fill in some details that López-Escobar (1993, §6) omits. Define the first-order structure

$$\mathfrak{T}_2 := \big(\mathbb{Q}, \mathbb{N}, +, \times, /, <, \leq, 0, 1, 2, R'\big),$$

where for any non-negative rationals t, x:

$$\langle t, x\rangle \in R' \quad \text{iff} \quad \text{Achilles has covered } x \text{ stades in } t \text{ seconds.}$$

Define $\models_2$ so that for each sentence $\sigma \in \mathfrak{L}_2$,

[6] A stade, or stadion, is a unit of distance.

$$\models_2 \sigma \quad \text{iff} \quad \mathfrak{T}_2 \text{ satisfies } \sigma.$$

A sentence $\sigma \in \mathfrak{L}_2$ is *arithmetical* iff σ contains no occurrences of the symbol '$\wp$'.

Define the first-order structure

$$\mathfrak{Q} := \big(\mathbb{Q}, \mathbb{N}, +, \times, /, <, \leq, 0, 1, 2\big).$$

An arithmetical sentence $\sigma \in \mathfrak{L}_2$ is an *arithmetical truth of* Σ iff σ is atomic and true in $\mathfrak{Q}$.[7]

Assume that Achilles can traverse a stade in 20 seconds. Then the *Zenonian axiom for* Σ is:

$$\wp(\mathbf{20}, \mathbf{1}).$$

Let $\mathfrak{A}_2$ be the set containing every arithmetical truth of Σ and the Zenonian axiom for Σ.

The *Zenonian rule of inference for* Σ is:

$$\begin{aligned} &\textbf{from}: \ \wp(t, x) \\ &\textbf{infer}: \ \exists t' \Big(t' < t \ \wedge \ \wp\big(t', \tfrac{x}{2}\big)\Big). \end{aligned}$$

Σ-*proofs* are natural deduction derivations (of finite length), from the set $\mathfrak{A}_2$ of axioms, that may use the rules of L (with equality) and the Zenonian rule of inference for Σ.

Define $\vdash_\Sigma$ so that for each sentence $\sigma \in \mathfrak{L}_2$,

$$\Gamma \vdash_\Sigma \sigma \quad \text{iff} \quad \begin{array}{l}\text{there is a } \Sigma\text{-proof of } \sigma \\ \text{whose undischarged assumption formulas are in } \Gamma \cup \mathfrak{A}_2.\end{array}$$

Theorem 1′: Σ is sound.

Proof: Use induction on the length of Σ-proofs. □

To show that Σ is incomplete, we will follow the presentation in López-Escobar (1993, §6).

Since Achilles reaches the end of the stadium, the sentence

$$\exists t \ \wp(t, \mathbf{1})$$

is true in Σ.

Since Achilles started to traverse the stadium, the sentence

$$\exists t \ \wp(t, \mathbf{0})$$

is true in Σ.

It is easy to show the following:

---

[7] Note that what López-Escobar calls the "arithmetical truths of Σ" are quantifier-free.

$$\vdash_\Sigma \exists t \; \wp(t, \tfrac{\mathbf{1}}{\mathbf{1}}),$$
$$\vdash_\Sigma \exists t \; \wp(t, \tfrac{\mathbf{1}}{\mathbf{2}}),$$
$$\vdash_\Sigma \exists t \; \wp(t, \tfrac{\mathbf{1}}{\mathbf{4}}),$$
$$\vdash_\Sigma \exists t \; \wp(t, \tfrac{\mathbf{1}}{\mathbf{8}}),$$
$$\vdots$$

Theorem 3: For each rational $k \geq 0$,

$$\exists t \; \wp(t, \mathbf{k}) \text{ is provable in } \Sigma \quad \text{iff} \quad \exists n \in \mathbb{N} \quad \frac{1}{2^n} = k.$$

Proof: For the $\Rightarrow$ direction, use the completeness theorem for L. □

There is no $n \in \mathbb{N}$ such that

$$\frac{1}{2^n} = 0.$$

By Theorem 3,

$$\nvdash_\Sigma \; \exists t \; \wp(t, \mathbf{0}).$$

Thus, Σ is incomplete.

2.2 Some general forms of dichotomy arguments

We can formulate a simple version of Zeno's dichotomy as an argument for a contradiction, on the basis of three claims:

P_1: Motion is possible.

P_2: Necessarily, motion involves a superevent.

P_3: No superevent is possible.

Before we refine this argument, a number of remarks will be helpful. Each premise involves the same variety of modality. But there are different readings of these claims depending on which kind of modality we understand them as mentioning. There are various kinds of *logical* possibility, various kinds of *mathematical* possibility, and there are *conceptual*, *metaphysical*, *nomic*, and *epistemic* kinds of possibility, among others. Much can be said about dichotomy arguments even without specifying which kind of modality we take them to involve, because some remarks apply regardless of which of these modalities we have in mind.

In any case, we can reason to a contradiction trivially. Abbreviating "motion occurs" with M, and "a superevent occurs" with S, the premisses are structured as follows.

P$_1$: $\Diamond M$

P$_2$: $\Box\big(M \rightarrow S\big)$

P$_3$: $\neg\Diamond S$

By P$_1$, there is a possible world w in which motion occurs. By P$_2$, every possible world in which motion occurs is one in which a superevent occurs. So in w, a superevent occurs. But by P$_3$, there is no possible world in which a superevent occurs. Contradiction.[8]

Since P$_1$, P$_2$, and P$_3$ are together inconsistent, we ought to reject at least one of the argument's premisses or inferences. One should diagnose the mistake involved and explain why it was initially tempting. Ideally, one's treatment of the dichotomy should be consistent with one's treatments of other paradoxical arguments about space and time.

A premise like P$_3$ — but mentioning tasks rather than events — appears in a number of rational reconstructions of specific instances of Zeno's dichotomy.[9] Arguments against the possibility of performing a supertask were advanced by Black (1951) and Thomson (1954), but were shown to be invalid by Benacerraf (1962).[10] Benacerraf noticed that some supertasks are such that it is underdetermined by L what would result from their respective completion.

P$_2$ states that every possible world in which motion occurs is one in which a superevent occurs. In particular, P$_2$ implies that in the actual world, motion involves the occurrence of a superevent. If there is a possible world in which motion occurs without involving a superevent, then P$_2$ is false. For example, perhaps a possible world in which neither time nor space are infinitely divisible is one in which motion need not involve a superevent.

We can give other dichotomy arguments that avoid these shortcomings.

Strengthening the simple dichotomy argument

There are several easy ways to strengthen the simple version of the dichotomy.

One way is to weaken P$_2$, and then adjust P$_1$ similarly. This gives:

[8] The modal principles needed to derive a contradiction from these premisses are relatively weak.

[9] The arguments are treated systematically by Grünbaum (1968, 1973). More recent work is surveyed by Earman & Norton (1996) and Koetsier & Allis (1997).

[10] This exchange is covered by Chihara (1965), Grünbaum (1969), and Salmon (2001). We survey several related arguments in §2.4.

P_4: Motion *through infinitely divisible space and time* is possible.

P_5: Necessarily, motion *through infinitely divisible space and time* involves a superevent.

P_6: No superevent is possible.

The phrase "infinitely divisible space and time" is intended to indicate the infinitely divisibility of *both* space and time. If motion through space and time that is not infinitely divisible is possible *and* does not involve a superevent, then P_5 is weaker than P_2.

Alternatively, we could weaken P_3 and leave P_1 and P_2 unchanged. This gives:

P_7: Motion is possible.

P_8: Necessarily, motion involves a superevent.

P_9: No superevent *of the kind that motion necessarily involves* is possible.

If there are superevents that are not of the kind that motion necessarily involves, then P_9 is weaker than P_3.

Incorporating all of these changes gives:

P_{10}: Motion through infinitely divisible space and time is possible.

P_{11}: Necessarily, motion through infinitely divisible space and time involves a superevent.

P_{12}: No superevent of the kind that
motion *through infinitely divisible space and time* necessarily involves
is possible.

If there is a kind of superevent that

(**i**) is necessarily involved in motion through infinitely divisible space and time,

and

(**ii**) is not necessarily involved in every kind of motion,

then P_{12} is weaker than P_9. For example, if motion through infinitely divisible space and time necessarily involves an ω-sequence of motions, then there is a kind of superevent that satisfies condition **i**. If it is also the case that motion through discrete (i.e., only *finitely* divisible) space and time need not involve a superevent, then there is a kind of superevent that satisfies condition

ii — namely, a superevent consisting of an ω-sequence of movements. In that case, P_{12} is weaker than P_9.

There is another way to weaken the premisses of this version of the dichotomy argument. The notions of modality that are used in the premisses do not need to be the same — it suffices that they be appropriately related to one another. In the analog of P_{12}, we will move the negation inside the scope of the modal operator, which then switches from a $\Diamond$ to a $\Box$. This gives:

There are notions of possibility $\Diamond_1$, $\Diamond_2$, $\Diamond_3$,
and an infinite linear order type τ
for which these five statements hold:

P_{13}: $\Diamond_1\big($Motion in infinitely divisible space and time occurs.$\big)$

P_{14}: If there are any $\Diamond_1$-worlds in which
motion in infinitely divisible space and time occurs,
then at least one of them is also a $\Diamond_2$-world.[11]

P_{15}: $\Box_2\big($Motion in infinitely divisible space and time involves a τ-event.$\big)$

P_{16}: If there are any $\Diamond_2$-worlds in which a τ-event occurs,
then at least one of them is also a $\Diamond_3$-world.

P_{17}: $\Box_3\big($No τ-event occurs.$\big)$

To reach a contradiction, reason as follows. P_{13} and P_{14} jointly imply that

$$\Diamond_2\big(\text{Motion in infinitely divisible space and time occurs.}\big).$$

P_{15} and the previous line jointly imply that

$$\Diamond_2\big(\text{A } \tau\text{-event occurs.}\big).$$

P_{16} and the previous line jointly imply that

$$\Diamond_3\big(\text{A } \tau\text{-event occurs.}\big).$$

The previous line contradicts P_{17}.

In order to further refine these premisses, we will be more specific about infinite divisibility.

[11] A $\Diamond_i$-world is a world that is possible with respect to the notion of possibility $\Diamond_i$.

2.2.1 Infinite divisibility

For simplicity, we will consider only those possible worlds with one spatial dimension and one temporal dimension. As abbreviations, call the temporal dimension $\mathcal{T}$ and the spatial dimension $\mathcal{S}_1$.[12]

$\mathcal{S}_1$ (and likewise $\mathcal{T}$) might be taken to consist of points, intervals, regions, parts, locations, places, positions, stretches, chunks, pieces, lengths, distances, etc. To be neutral among such views, we will use the term *constituent*. $\mathcal{S}_1$ (and likewise $\mathcal{T}$) might be taken to have two or more kinds of constituents, with some of those kinds being more ontologically fundamental.

What will we mean by saying that $\mathcal{S}_1$ (or likewise $\mathcal{T}$) is "infinitely divisible"? We can state the infinite divisibility of $\mathcal{S}_1$ in a way that applies regardless of the kind of constituents that $\mathcal{S}_1$ is taken to have. For example, these definitions do not depend on whether constituents are points or regions. This can be done in terms of a 2-place reflexive relation *overlaps*, which is defined in terms of constituents of $\mathcal{S}_1$.

Let $\prec$ indicate the irreflexive transitive relation "is to the left of", and likewise let $\succ$ indicate "is to the right of".

For any constituents α, β, we say that α is *adjacent* to β iff there is no constituent that is between α and β.[13]

When $\mathcal{S}_1$ contains at least two constituents that are pairwise non-overlapping and pairwise non-adjacent, the infinite divisibility of $\mathcal{S}_1$ can be understood as a kind of density requirement:

> For any non-overlapping non-adjacent constituents α, β of $\mathcal{S}_1$, there is a constituent γ of $\mathcal{S}_1$ such that:
>
> **(i)** γ does not overlap α or overlap β,
>
> **(ii)** γ is not adjacent to α or adjacent to β,
>
> **(iii)** γ lies (entirely) between α and β.[14]

This kind of statement of divisibility can be given a sort of Aristotelian reading on which the divisibility of $\mathcal{S}_1$ is potentially infinite but not actually infinite.[15] For example, on one such reading, for any non-overlapping non-adjacent constituents α, β, it is *possible* that there is a constituent which meets conditions **i**, **ii**, and **iii**.

Now we define a notion of infinite divisibility that we will call "ω-divisibility", that cannot

---

[12] These are not names for spatiotemporal dimensions of the actual world.

[13] The relation "is adjacent to" on the collection of constituents of $\mathcal{S}_1$ is reflexive. Likewise for $\mathcal{T}$.

[14] Either $\alpha \prec \gamma \prec \beta$, or $\alpha \succ \gamma \succ \beta$.

[15] See Hellman & Shapiro (2017, Chapter 3).

be given a straightforward reading in the Aristotelian spirit. Our notion also generalizes in an obvious way to stronger kinds of divisibility.

Define $\mathcal{S}_1$ is ω-*divisible* as:

> There are at least two non-overlapping non-adjacent constituents of $\mathcal{S}_1$, and
> for any non-overlapping non-adjacent constituents α, β of $\mathcal{S}_1$,
> there is a sequence $\langle x_i \rangle_{i \in \omega}$ of pairwise non-overlapping, pairwise non-adjacent constituents of $\mathcal{S}_1$ ordered as:
>
> $$\alpha,\ x_0,\ x_1,\ x_2,\ \ldots\ ,\ x_n,\ \ldots\ ,\ \beta.$$

For example, suppose that $\mathcal{S}_1$ consists only of points. It is natural to assume that for any points x, y in $\mathcal{S}_1$, if $x \neq y$, then x and y do not overlap.[16] Let 0 and 1 be distinct points in $\mathcal{S}_1$ such that $0 \prec 1$. In this case, if $\mathcal{S}_1$ is ω-divisible then there is a sequence

$$\langle x_i \rangle_{i \in \omega}$$

of points in $\mathcal{S}_1$ such that

$$0 \prec x_0 \prec x_1 \prec x_2 \prec\ \ldots\ \prec x_n \prec\ \ldots\ 1.$$

In this example, in order for an object to move from the point 0 to the point 1, it must pass through an ω-sequence of points: one point, then another point, then another, and so on.

Switching our choices of α and β, and letting $\succ$ indicate the "is to the right of" relation (which is the inverse of $\prec$), there is a sequence

$$\langle y_i \rangle_{i \in \omega}$$

of points in $\mathcal{S}_1$ such that

$$1 \succ y_0 \succ y_1 \succ y_2 \succ\ \ldots\ \succ y_n \succ\ \ldots\ 0,$$

which is equivalent to

$$0\ \ldots\ \prec y_n \prec\ \ldots\ \prec y_2 \prec y_1 \prec y_0 \prec 1.$$

Thus, in order to move from 0 to 1, the object also must pass through the points of the sequence

$$\langle y_i \rangle_{i \in \omega}$$

in reverse order, with no first member.[17]

[16] This property does not hold for some other kinds of constituents, like e.g. intervals.

[17] In that order, the points form an ω^*-sequence. See §2.5.2 and §2.5.3.

2.2.2 An ω-divisibility argument

Abusing notation, let $\mathcal{T}\times\mathcal{S}_1$ be the plane formed by $\mathcal{T}$ and $\mathcal{S}_1$.

We will say that $\mathcal{T}\times\mathcal{S}_1$ is ω-*divisible* iff both $\mathcal{T}$ and $\mathcal{S}_1$ are ω-divisible.

There is another way to weaken several premisses of the dichotomy argument. If *some but not all* types of superevents are impossible, and *some but not all* kinds of motion would involve such (impossible) types of superevents, then *some but not all* kinds of motion are impossible. With this in mind, we can formulate a more general dichotomy argument in terms of ω-divisibility.

There are notions of possibility $\Diamond_1$, $\Diamond_2$, $\Diamond_3$,
a type μ (of motion), and an infinite linear order type τ
for which these five statements hold:[18]

P_{18}: $\Diamond_1\big($Type μ motion in an ω-divisible $\mathcal{T}\times\mathcal{S}_1$ occurs.$\big)$

P_{19}: If there are any $\Diamond_1$-worlds in which
type μ motion in an ω-divisible $\mathcal{T}\times\mathcal{S}_1$ occurs,
then at least one of them is also a $\Diamond_2$-world.

P_{20}: $\Box_2\big($Type μ motion in an ω-divisible $\mathcal{T}\times\mathcal{S}_1$ involves a τ-event.$\big)$

P_{21}: If there are any $\Diamond_2$-worlds in which a τ-event occurs,
then at least one of them is also a $\Diamond_3$-world.

P_{22}: $\Box_3\big($No τ-event occurs.$\big)$

Questions related to premisses like P_{22} — but mentioning tasks rather than events — are addressed by the authors mentioned at the beginning of §2.2. For example: Which notions of possibility allow for the performance of a supertask?

Questions surrounding premisses like P_{20} have been neglected, perhaps because dichotomy arguments were not sufficiently precisified, strengthened, and generalized. For example, one such question is:

Conceiving of $\mathcal{T}$ and $\mathcal{S}_1$ as non-empty dense linear orders without endpoints, and identifying motions with position functions:

What is the relationship between the order types of $\mathcal{T}$ and $\mathcal{S}_1$
and the order type of the event-sequence that motion through $\mathcal{T}\times\mathcal{S}_1$ involves?

[18] This version of the dichotomy argument is generalized in §2.5.1.

When $\mathcal{T}$ and $\mathcal{S}_1$ are not order-isomorphic, the answer is not obvious. To begin to address this question, we need to be more precise about the order-structure of $\mathcal{T}\times\mathcal{S}_1$.

2.3 Continua for time and space

This section discusses a class of ordered structures that we will use to represent the dimensions of time and space. We will also explain several ways in which it is possible — at least conceptually — for the order types of these dimensions to differ from one another.

Particular dichotomy arguments can vary along three parameters:

- the variety of modality that is mentioned,
- the kind of motion that is mentioned,
- the order type of the event-sequence that is mentioned.

With respect to each variety of modality, one can ask a number of questions:

Q11: Which mathematical structures is it possible for an ω-divisible $\mathcal{T}\times\mathcal{S}_1$ to have?

Q12: In each possible ω-divisible $\mathcal{T}\times\mathcal{S}_1$, which kinds of motion are possible?

Q13: Which order types do possible event-sequences have?

Q14: For which values of μ and τ is P_{20} true?

A special case of Q14 is:

> **Q15**: Which order types do event-sequences that are necessarily involved in every possible motion through every possible ω-divisible $\mathcal{T}\times\mathcal{S}_1$ have?

I.e., with respect to each variety of modality: which linear order types τ are such that *necessarily*, every possible motion through any possible ω-divisible $\mathcal{T}\times\mathcal{S}_1$ involves a τ-event?

A related question is:

> **Q16**: How does the structure of the spatiotemporal dimensions in a possible world determine (or constrain) the kinds of motion which may occur in that world?

A spatiotemporal structure is mathematically compatible with the occurrence of *some but not all* kinds of motion. For example, suppose that $\mathcal{T}$ is (structured like) an ordered field. Then an α-event may occur in $\mathcal{T}\times\mathcal{S}_1$ only if a bounded interval of $\mathcal{T}$ contains an α-sequence of non-trivial pairwise non-overlapping subintervals.[19] Since no bounded subinterval of $\mathbb{R}$ is the union

[19] When $\mathcal{T}$ is an ordered field, this requirement is equivalent to the condition that there be an order-embedding of α into $\mathcal{T}$.

of uncountably many non-trivial pairwise non-overlapping subintervals, it is not possible for a hyperevent to occur in a world in which time is structured like $\mathbb{R}$ (under its usual ordering). This rules out certain kinds of motion: e.g., if $\mathcal{T}$ is structured like $\mathbb{R}$, it is not mathematically possible — even allowing arbitrarily large but finite velocities and accelerations — for an object to make uncountably many pairwise disjoint stops, with each stop lasting for some non-trivial interval of time.

With respect to the metaphysical question

What is the nature of motion?,

Russell (2013, p. 65) in his 1901 paper *Mathematics and the Metaphysicians* writes that:

> . . . a body in motion is just as truly where it is as a body at rest. Motion consists merely in the fact that bodies are sometimes in one place and sometimes in another, and that they are at intermediate places at intermediate times.

In 1903, Russell (2015, p. 480) writes that motion "consists *merely* in the occupation of different places at different times, subject to continuity". A natural way of spelling out part of the continuity requirement on motion, in the simple case of a point particle moving through two dimensions, is the following.

Let t, t' be moments with $t < t'$. Let x, y be distinct spatial points.

If a point particle p moves from x at t to y at t', then two conditions hold:

(i) At each time in (t, t'), p occupies exactly one spatial point between (and including) x and y.

(ii) Each spatial point strictly between x and y is occupied by p at some or other time in $[t, t']$.

Treating $\mathcal{T} \times \mathcal{S}_1$ as a mathematical structure, it is common to treat every kind of motion through $\mathcal{T} \times \mathcal{S}_1$ as a position function, i.e., a mapping from the constituents of an interval of $\mathcal{T}$ into the constituents of an interval of $\mathcal{S}_1$.[20] On this picture, the claim that

it is mathematically possible for motion of kind μ to occur in $\mathcal{T} \times \mathcal{S}_1$

is expressed by a formal mathematical statement.

[20] On Russell's conception of motion, a position function must be a *bijective* mapping between an interval of $\mathcal{T}$ and an interval of $\mathcal{S}_1$. We return to this issue in §2.3.2.

There are various kinds of mathematical possibility, depending on what is counted as "mathematics".[21] From certain sufficiently strong background theories, some mathematical statements are provably independent of sufficiently strong consistent theories (e.g., ZFC, if it is consistent). Thus, relative to our (choice of) logical and mathematical assumptions at the levels of object-theory, meta-theory, etc., it may be provably independent of our axioms whether a particular spatiotemporal structure is mathematically compatible with the occurrence of a certain kind of motion. If a formal statement which expresses that mathematical compatibility is independent of our axioms, it may even be that our meta-theory is unable to prove the independence of that statement from our axioms. The independence phenomena in set theory seem to carry over to the truths about which propositions are mathematically possible with respect to any sufficiently strong consistent formal system.

2.3.1 Line-structures

To discuss some of the ways that it is conceptually possible for time and space to be structured, we will use a few definitions.

A *line-structure* is a non-empty collection of points under a dense linear ordering without endpoints.

An interval of a line-structure is *non-trivial* iff it contains more than one point.

We will say that non-trivial bounded intervals come in four *types*; where $a < b$, these are:

$$\begin{aligned} \textit{open}: &\ (a, b) \\ \textit{closed}: &\ [a, b] \\ \textit{open-closed}: &\ (a, b] \\ \textit{closed-open}: &\ [a, b) \end{aligned}$$

A line-structure is *locally uniform* iff

all of its non-trivial bounded open intervals are order-isomorphic.[22]

A line-structure is *locally symmetric* iff

each of its non-trivial bounded open intervals (a, b)
is order-isomorphic to the inverse order of (a, b).[23]

[21] For example, one can ask what is mathematically possible with respect to a particular consistent theory or (semi-)formal system, similar to what we did in §1.5.1.

[22] The global analog of local uniformity is stronger. A line-structure is *globally uniform* iff it is order-isomorphic to all of its non-trivial bounded open subintervals.

[23] A line-structure is *globally symmetric* iff it is order-isomorphic to its inverse order.

For any collection C, we write $[a,b)_C$ for

$$\{x \in C : a \leq x < b\},$$

and similarly for $(a,b]_C$, $(a,b)_C$, and $[a,b]_C$.

For any collection C, we write $C_{<x}$ for

$$\{y \in C : y < x\},$$

and similarly for $C_{\leq x}$, $C_{>x}$, and $C_{\geq x}$.

We will treat $\mathcal{T}$ and $\mathcal{S}_1$ each as locally uniform, locally symmetric line-structures. We are treating the dimensions of time and space respectively as consisting of points, and being ω-divisible, containing at least $\aleph_0$ points in each non-trivial interval. Local uniformity ensures that every non-trivial bounded interval of time (respectively: space) is structurally homogenous with respect to its order type. Local symmetry ensures that every non-trivial bounded interval of time (respectively: space) is structurally homogenous with respect to direction.

2.3.2 Mismatches between time and space

It is conceptually possible for time to have a different mathematical structure than that of the spatial dimension(s). In the context of dichotomy arguments, this possibility matters for assessing premisses like P_{20}, which state that a certain kind of motion through a certain kind of $\mathcal{T} \times \mathcal{S}_1$ necessarily involves an event-sequence of a certain order type.

We will consider two ways that $\mathcal{T}$ and $\mathcal{S}_1$ might be incompatible with respect to their order types. Then we will explain how they can be combined.

If we identify motions with position functions, conceptual difficulties arise even with simple uniform motion when a non-trivial bounded closed (respectively: open) interval of $\mathcal{T}$ and a non-trivial bounded closed (respectively: open) interval of $\mathcal{S}_1$ are not order-isomorphic to one another.

For example, consider an object moving uniformly from left to right through the interval $[0,1]_{\mathcal{S}_1}$ from an initial moment t_i until a final moment $t_f > t_i$, with position function

$$s \colon \left[t_i, t_f\right] \longrightarrow [0,1].$$

If the spatial interval $[0,1]_{\mathcal{S}_1}$ is order-embeddable into $\left[t_i, t_f\right]$, but not vice versa, then there is some moment

$$t \in \left(t_i, t_f\right)$$

at which s is undefined — i.e., t is not in the domain of s. The object is not assigned a location at time t by s. If we insist that the object has a location at t, we might say that because the motion is smooth, the object's location at t must be to the right of every point in

$$s\big[[t_i, t)\big]$$

and to the left of every point in

$$s\Big[(t, t_f]\Big].$$

But no points in $\mathcal{S}_1$ answer to that description. Perhaps the object nevertheless has a location at that time, in which case there are locations *of* $\mathcal{S}_1$ that do not correspond to any point *in* $\mathcal{S}_1$. As we will see in §2.6, we can make sense of this way of speaking.[24]

Type I mismatch

For a simple example of this first kind of situation, treat $\mathcal{T}$ like $(\mathbb{R}, <)$ and $\mathcal{S}_1$ like $(\mathbb{Q}, <)$. Taking $t_i = 0$ and $t_f = 1$, define a partial position function

$$s\colon [0,1]_{\mathbb{R}} \longrightarrow [0,1]_{\mathbb{Q}}.$$

Suppose that the motion is uniform (at a velocity of 1 time unit per space unit), and that s is defined only for rational numbers in $[0,1]_{\mathbb{R}}$.

For each $t \in dom(s)$, define the partial function

$$s(t) := \begin{cases} t & \text{if } t \in [0,1]_{\mathbb{Q}} \\ \uparrow & \text{otherwise.}^{25} \end{cases}$$

Since

$$s\left(\frac{\sqrt{2}}{2}\right)\uparrow,$$

where is the object at

$$t = \frac{\sqrt{2}}{2}\ ?$$

If the object has a location at that time, then that location does not correspond to any point in $\mathcal{S}_1$. We will suggest that that location lies strictly between every rational

$$x < \frac{\sqrt{2}}{2}$$

[24] In this two-dimensional space, we suggest identifying this spatial location with

$$\Big\langle \big\langle \mathcal{S}_{1_{<t}}, \mathcal{S}_{1_{>t}} \big\rangle \Big\rangle.$$

[25] Recall that the up-arrow indicates when a partial function is not defined for a particular element of its domain. The down-arrow indicates when a function *is* defined for a particular element of its domain.

and every rational

$$x > \frac{\sqrt{2}}{2}.$$

Type II mismatch

Now we consider a simple example of the second kind of situation. If $\big[t_i, t_f\big]$ is order-embeddable into the spatial interval $[0,1]_{\mathcal{S}_1}$ but not vice versa, then there is some spatial point

$$x \in (0,1)_{\mathcal{S}_1}$$

such that at no time between t_i and t_f does the object occupy x, despite occupying points to the left of (and arbitrarily close to) x and points to the right of (and arbitrarily close to) x.[26] In this case the object seems to skip from places to the left of x to places to the right of x, even though there is a sense in which its motion is smooth — it passes through every spatial point between $s(t_i)$ and $s\big(t_f\big)$. Despite there being no point in $\big[t_i, t_f\big]$ at which the object occupies x, perhaps there is some time *of* $\mathcal{T}$ at which it occupies x. Or so we will suggest.

For a simple example of this second kind of situation, treat $\mathcal{T}$ like $(\mathbb{Q},<)$, and $\mathcal{S}_1$ like $(\mathbb{R},<)$. Taking $t_i = 0$ and $t_f = 1$, define a position function

$$s\colon [0,1]_{\mathbb{Q}} \longrightarrow [0,1]_{\mathbb{R}}.$$

Suppose that the motion is uniform (at a velocity of 1 spatial unit per temporal unit), and that

$$\forall t \in dom(s) \quad s(t) := t.$$

Since only rational numbers are in the image of s, when does the object occupy the spatial point

$$x = \frac{\sqrt{2}}{2}\ ?$$

Combining mismatches

It is conceptually possible for these two kinds of "mismatch" to occur together.

> We will say that line-structures l_1 and l_2 are *locally incomparable* iff
>
> for any non-trivial bounded open interval of l_1,
> and any non-trivial bounded open interval of l_2,
> neither of those intervals order-embeds into the other.[27]

[26] For any spatial point $y < x$, there is a spatial point $z > y$ which the object eventually occupies.

In other words, if $a < b$ and $c < d$, then no interval of the form (a, b) in either dimension order-embeds into any interval of the form (c, d) of the other dimension.[28]

When $(C, <)$ is a linearly ordered class, we say that a line-structure *locally embeds C* iff

$(C, <)$ order-embeds into each of that line-structure's non-trivial bounded open intervals.

Real analysis could provide an empirically adequate method of approximation for mechanics in some possible worlds in which spatiotemporal dimensions are not isomorphic to $\mathbb{R}$ but locally embed $\mathbb{R}$ and are locally Euclidean. For this reason, line-structures that locally embed $\mathbb{R}$ are especially interesting models of spatiotemporal dimensions.[29]

We will say that a line-structure is *nice* iff

it is locally uniform, locally symmetric, and locally embeds $\mathbb{R}$.[30]

In order for there not to be any "local mismatch" between $\mathcal{T}$ and $\mathcal{S}_1$, it must be that for any non-trivial bounded open interval of $\mathcal{T}$, and any non-trivial bounded open interval of $\mathcal{S}_1$, each of those intervals order-embeds into the other.[31]

The possibility of a local mismatch between $\mathcal{T}$ and $\mathcal{S}_1$ can be used to argue against premisses like P_{11}, which states that

Necessarily, motion through infinitely divisible space and time involves a superevent.

For example, as we will see in §2.5.5, the fact that

$\mathcal{T} \times \mathcal{S}_1$ is ω-divisible

does not in general imply that

motion through $\mathcal{T} \times \mathcal{S}_1$ involves an ω-event.

We will return to problems about motion through mismatched dimensions after considering what support can be given for various instances of P_{22}.

[27] Two line-structures are *globally incomparable* iff neither of them order-embeds into the other.

[28] This property will be important for the divisibility argument in §2.5.

[29] Both kinds of mismatch can occur even when $\mathcal{T}$ and $\mathcal{S}_1$ each locally embed $\mathbb{R}$.

[30] **Q17**: Is it consistent relative to ZFC that there are, up to order-isomorphism, *infinitely many* nice line-structures that are pairwise locally incomparable? *Uncountably many*? *Proper-class-many*?

[31] This condition does not preclude a "global mismatch" between $\mathcal{T}$ and $\mathcal{S}_1$.

2.4 The possibility of ω-tasks

This section surveys five arguments for the conclusion that it is not possible to perform certain kinds of ω-tasks. Those five arguments respectively use the following five considerations:

- No task in an ω-task brings us "closer" to completing the sequence.
- If an ω-task were completed, it would be completed "suddenly".
- No task in an ω-task is such that finishing that task completes the sequence.
- No point in a monotonic ω-sequence is such that reaching it gets you to an endpoint.
- If a supertask were completed, infinitely many failures would amount to a success.

These points will be explained in this section.

Borrowing a phrase from Thomson (1967, p. 187), the dichotomy arguments suggest that "there should be a difficulty about understanding how" an object can pass through a bounded monotone ω-sequence of points in $\mathcal{S}_1$. As we will see, the difficulty is conceptual and has to do with the order type ω. Thomson (1967) writes:

> What causes trouble is doubtless that (in the kind of sequence [that] we are talking of) there is no last term, so that one does not see what finishing consists in in such a case. But, in the context of the Race Course anyway, this trouble is illusory; finishing consists in occupying the limit point of the class [of spatial points in the sequence], and we may suppose that *this is* the last point occupied. (p. 188)

The Zenonian worry that Thomson refers to is about how it is possible to traverse a (non-empty) sequence of points in which there is no last point. Thomson dismisses the worry on the basis that the last point to be occupied is the limit point of the point-sequence, the limit point being the $(\omega + 1)$-th point. But an $(\omega + 1)$-task is completed only if an ω-task is completed. In order for the object to occupy the limit point of the sequence — or to occupy *any* point that follows every point in the sequence — an ω-task must first be completed. We should not dismiss the worry on these grounds as Thomson does.

The Zenonian claims that it is not clear enough what completing an ω-sequence of traversals consists in. We can reply that completing an ω-task consists in finishing every task in the sequence, and that an ω-task is completed once there comes a time by which every task in the sequence has been finished. The Zenonian can and should concede this, but has good reason to remain unsatisfied. There is still a conceptual difficulty with understanding how every task in an ω-sequence of them can be — or would be — completed.

Let us explain part of what is puzzling about the completion of an ω-task. Every task in an ω-task is preceded by only finitely many tasks in the sequence, but followed by infinitely many others in the sequence. Whichever task in the ω-task has just been finished, an ω-sequence of other tasks remains to be completed.

In one sense of the word "closer", for each positive integer n, finishing the first n tasks brings one no closer to completing the sequence of tasks. This is because immediately after finishing any particular task in the sequence, there are infinitely many tasks left to finish, just as there were before that particular task was begun. Since each task in the sequence is preceded by only finitely many tasks, no task in the sequence brings us closer (in that sense of "closer") to completing the sequence. At any time during the doing of an ω-task, infinitely many tasks remain to be finished before the whole ω-task can be completed.

Think of the closed-open interval to be crossed as consisting of an ω-sequence

$$\langle i_n \rangle_{n\in\mathbb{N}}$$

of pairwise disjoint closed-open subintervals, with for each $n\in\mathbb{N}$,

$$i_n := \big[x_n, x_{n+1}\big),$$

with i_n immediately to the left of i_{n+1}. Once the first subinterval i_0 has been crossed, every subinterval in the subsequence

$$\langle i_{n+1} \rangle_{n\in\mathbb{N}}$$

must be crossed in order for the entire interval to be crossed. At every time throughout the crossing of the subintervals one at a time, infinitely many subintervals remain to be crossed in order for the entire interval to have been crossed.[32]

Because of this feature, we might say that an ω-task, if completed, is completed *suddenly*. At no point *during* the doing of an ω-task have infinitely many of its tasks been finished, but suddenly, once the ω-task has been completed, infinitely many tasks have been finished. Part of the conceptual difficulty is understanding how this could happen. The coherence of formal models or mathematical descriptions of supertasks does not by itself alleviate the difficulty. What is needed is some kind of an explanation.

Groarke (1982, pp. 69–70) describes an argument for the conclusion that traversing a monotonic ω-sequence of spatial points is impossible:[33]

> . . . the crossing of the [infinite series of] distances one by one is alleged to be impossible because the series contains no last element. In attempting to traverse the

[32] Given certain assumptions which we will discuss in §2.5.5, crossing i_0 already involves a superevent. One way in which reverse forms of the dichotomy argument — which mention an ω^*-sequence of points or subintervals — are simpler than standard forms of the dichotomy is that they lack this feature.

[33] Unfortunately, Groarke uses "series" instead of "sequence".

> distances, one therefore crosses a particular distance and continually proceeds to another distance which [too] must be traversed. One never reaches a last distance [in the series], its completion, and the consequent completion of the series. It allegedly follows that one cannot complete [the traversal of] the [infinite] series [of distances].

Since the sequence of subintervals does not contain a final member (and is non-empty), each subinterval in the sequence is followed by a unique successor in the sequence. There is no subinterval in the ω-sequence such that crossing it completes the traversal of the entire interval. Groarke explains that:

> ...the problem with our attempt to complete [the traversal of] Zeno's sequence of distances is precisely that we cross the distances in the sequence a finite number at a time — i.e. one by one. The question at issue is how we can, proceeding in this way, ever come to a point where we have crossed an infinite number of distances, and the claim in question is the claim that we cannot do so because we never reach a distance which counts as the last element in an infinite sequence. The claim that we can cross an infinite sequence of distances despite its lack of a last element, and the bald assertion that we can thereby reach the limit point of Zeno's sequence, both beg the very question at issue. One needs an argument to establish that these claims are true ... (p. 70)[34]

To describe an asymmetry in the order type ω, a few definitions will be helpful.

A *proper initial segment* of an ω-sequence $\langle i_n \rangle_{n \in \mathbb{N}}$ is a subsequence

$$\langle i_0, \ldots, i_j \rangle,$$

for some $j \in \mathbb{N}$.

A *final segment* of an ω-sequence $\langle i_n \rangle_{n \in \mathbb{N}}$ is a subsequence

$$\left\langle i_k, i_{k+1}, i_{k+2}, \ldots \right\rangle,$$

for some $k \in \mathbb{N}$.

Every proper initial segment of an ω-sequence has a final member, but every final segment of an ω-sequence lacks a final member. This asymmetry is a feature of the order type ω. It is difficult to understand how the sequential completion of proper initial segments of an ω-sequence of tasks — with each proper initial segment having a final task — could amount to the completion of the entire ω-sequence of tasks, which has no final task. Groarke writes that:

> ...no act within the sequence brings it to completion. It is as though one tried to accomplish a task by repeatedly performing actions which failed to complete the task in question. (p. 73)

[34] This point is analogous to one made by Chihara (1965, p. 81) about ω^*-tasks. See our §2.5.2.

Each task in the ω-sequence is a failure in that finishing that task fails to complete the ω-task. It seems that if the traversal of an ω-sequence of points *were* to occur, then infinitely many individual failures would have amounted to a success. And for the reasons mentioned earlier, it is not clear even how an ω-sequence of *failures* could be completed.

The completion of an ω-task is conceptually puzzling. But this does not constitute good grounds for denying that the completion of an ω-task is possible — whether conceptually, or under various other notions of possibility. Treating $\mathcal{T}$ and $\mathcal{S}_1$ as order-isomorphic line-structures and identifying motions with position functions has a number of puzzling consequences, one of which is that superevents occur if motion occurs. In these ways of modeling space, time, and motion, continuous motion always involves (at least) a superevent.

2.5 Variations of dichotomy arguments

This section surveys a number of arguments that are related to dichotomy arguments.

Granting that it is possible in a finite length of time to traverse an ω-sequence of distinct spatial points, there are other conceptual difficulties with the occurrence of infinite sequences of events. For example, at no stage of an ω-sequence of movements does one move from a place to the intended endpoint. How can one arrive at a point p merely by moving to points that fall short of p?

Thomson (1967) calls the spatial points in the ω-sequence

$$\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \ldots$$

Z-points:

> The [dichotomy] paradox says that to get [from 0] to 1 the man must pass over all [of] the Z-points. This is indeed necessary, but it is also sufficient. There is no question of his having to pass over all [of] the Z-points and then do something else. It is not even true that he must pass over all [of] the Z-points and then finish. For it is plainly impossible for him to have passed over all [of] the Z-points without having passed over 1. To pass over all [of] the Z-points and to get to 1 are the same thing. (p. 188)

Thomson is implicitly assuming that in $\mathcal{S}_1$, the sequence of Z-points converges, and that it converges to 1. In a structure like, e.g.,

$$(\mathbb{R}, 0, 1, <, +, \cdot),$$

the sequence of Z-points converges to an element of the field. But there are other number systems in which the sequence of Z-points does not converge to any point.[35] Thus, whether

[35] For example, there are number systems — containing infinitesimals — in which no bounded monotone ω-sequence of points converges to a point.

passing through all of the Z-points suffices for reaching the intended endpoint of the motion depends on the mathematical structure of $\mathcal{S}_1$. In particular, this depends on whether bounded monotone ω-sequences of points converge to a point in $\mathcal{S}_1$.

Convergence in physical space

Groarke (1982, p. 71) argues that there is a difficulty in understanding how traversing the sequence of Z-points could be sufficient for reaching its limit point:

> ...given that the crossing of a sequence of intervals does not take one to a point outside [of] the sequence, the crossing of the sequence [that] Zeno proscribes can never take one to the endpoint of a motion (for this point lies outside [of] all [of] the intervals [that are] within the sequence). Any attempt to complete the motion by crossing the intervals one by one therefore seems bound to failure (though it is also necessary for the completion of the motion). It is as though one tried to move something to a point b by repeatedly moving it to points that all fall short of b.

If the sequence of Z-points converges in $\mathcal{S}_1$, and if one completes the traversal of the sequence of Z-points, then one is able to reach a point merely by traversing subintervals *none of which contain the limit point of the sequence*. But it seems as though success (in reaching the intended endpoint) could not be achieved by infinitely many failures (to reach the intended endpoint). Groarke (1982, p. 72) puts this challenge in the form of a question:

> Why is it that the crossing of a sequence of spatial intervals can, by itself, take one to a point outside all [of] the intervals in the sequence? In completing the sequence one crosses the individual intervals and does nothing else besides. How then can one reach a point that one doesn't reach in crossing [any] one of the intervals in question?

How can one succeed in getting to a place merely by going to places *other* than that place? It is natural to think that in order to reach a final destination in physical space, eventually one must move *from* some place other than the final destination *to* the final destination. But this never happens if $\mathcal{S}_1$ is ω-divisible. If the sequence of Z-points converges in $\mathcal{S}_1$, then it suffices for reaching 1 (from 0) that one moves to spatial points other than 1. No point is adjacent to any other point, so in particular, the limit point is not adjacent to any of the Z-points. In this sense, each point in $\mathcal{S}_1$ is "isolated" from every other point in $\mathcal{S}_1$.

This conceptual problem about convergence is not adequately answered by the stipulation that in $\mathcal{S}_1$, the sequence of Z-points converges to a point. Assuming that $\mathcal{S}_1$ is (at least locally) structured like a Dedekind complete ordered field, it is a consequence of the least-upper-bound property that traversing the sequence of Z-points brings one to occupy the limit point.[36] But the

[36] For a discussion of Dedekind completeness, see Propp (2013).

conceptual difficulty has to do with how a dimension of *physical* space could ever be structured in such a way that a bounded monotone ω-sequence of spatial points *would* converge to a point outside of the sequence. The concern is not about consistency, or about the mathematical coherence of structures in which such sequences converge. Rather, the concern is about how we can understand such structures as models of space and time in which motion occurs.[37]

We can restate the question as follows. Let x and y be distinct non-adjacent places in an ω-divisible $\mathcal{S}_1$. Leave it open whether $\mathcal{S}_1$ and $\mathcal{T}$ consist of points. In order for an object to move in a continuous — in the informal sense of that word — path p from x to y, where p contains x and y only at its start and end (respectively), the object must move to places which are not y, until it finally reaches y. How can the object reach y merely by moving to places *other than* y? Let p' be the result of removing y from p. How can traversing p' suffice for reaching y? How can an object that traverses p' *thereby* traverse p?

Either

(**i**) there is a "last" place in p such that it comes right before (and is adjacent to) y,

or

(**ii**) there is no such place.

If **i** holds, then $\mathcal{S}_1$ is not dense with respect to places, which is a necessary condition for it being ω-divisible with respect to places. When space is treated as being *dense* with respect to places (i.e., between any two distinct places in space, there is another one), there is no place (except perhaps y) that is adjacent to y, and **ii** holds. But if **ii** holds, then we must explain how, in a physical space, moving only to places *other than* y could suffice for reaching y.

2.5.1 Ultraevents

Here is another kind of worry that is similar to Zeno's dichotomy argument as it is sometimes reconstructed. In order to traverse a path p, an object must first traverse a proper initial sub-path p_0 of p, and then traverse a proper initial sub-path p_1 of the remaining portion of p, and so on for sub-paths indexed by each natural number. For any proper initial part (or segment) of p that has been traversed so far, there remains a further part (or segment) of p that must be traversed before the whole of p is traversed.

This can also be formulated in terms of *portions* of a path. For example, in order to traverse a non-empty path, the first half of that path must be traversed, and then the first half of the remaining portion must be traversed, and so on.

This idea is easily extended to ordered structures that are more finely subdivided than $(\mathbb{Q}, <)$. For example, the real-closed ordered (proper class) Field of surreal numbers is called **No**.[38] A

[37] This problem does not arise for discrete (i.e., non-dense) dimensions.

[38] See Alling (1987, Chapter 4) and Gonshor (1986, Chapter 2).

subclass of **No** is the proper class On of all ordinals. If $\mathcal{S}_1$ were (even locally) structured like the surreal number line, then in order for an object to move from the spatial point 0 to the spatial point 1 within a finite length of time, it would have to pass through every point in the On-length sequence

$$\left\langle \frac{\alpha}{\alpha+1} \right\rangle_{\alpha \in On}.$$

If we count passing through each of these spatial points as a distinct event, then proper-class-many events would need be completed in order to move through the unit interval. If this occurs in a bounded interval of time, then this counts as an ultraevent.[39]

2.5.2 Non-wellfounded sequences

Recall that a linear order $(C, <)$ is *wellfounded* iff every non-empty subcollection of C has a minimal element under $<$.

One type of non-empty linear order that is non-wellfounded is one that lacks a least element under its ordering, i.e., it has no initial member in the order. It is not clear how sequences of events with that kind of order type ever begin. Chihara (1965) expresses puzzlement about this:

> In some respects, Zeno's dichotomy paradox is even more puzzling [than the paradox of Achilles and the tortoise]. If Achilles is to traverse the [spatial] interval $(0, 1)$, he must first traverse the interval $\left(0, \frac{1}{2}\right)$. But before he could finish traversing that interval, he would have to traverse $\left(0, \frac{1}{4}\right)$, and so on ad infinitum. It thus appears that Achilles must do something analogous to counting all [of] the natural numbers in *reverse* order, counting 2 before 1, 3 before 2, 4 before 3, and so forth. But there is an obvious difficulty in the supposition that someone (even someone superhuman) might have completed such a task. In this case, the problem is not "How did he finish?" but rather "How did he start?" And the answer we are looking for is not: "Very fast indeed!"
>
> The type of problem uncovered by Zeno is not restricted to motion in space: one can easily generate similar difficulties in a variety of forms. For example, one might very well argue that it is impossible for Achilles to stand on his head for a full minute . . . or gain a pound of weight in a continuous fashion. Naturally, what is desired is some general method of dealing with such difficulties. (p. 81)

For example, under assumptions which we will discuss in §2.5.5, in possible worlds in which $\mathcal{S}_1$ is structured like $\mathbb{R}$, in order for Achilles to traverse a non-trivial spatial interval, something that is analogous to counting the elements of $\mathbb{N}$ *in reverse order* must occur. This would be a non-wellfounded superevent. Under similar assumptions, in possible worlds in which $\mathcal{S}_1$ is

[39] For more discussion, see Al-Dhalimy & Geyer (2016).

structured like **No**, something that is analogous to counting the ordinals in reverse order would occur. This would be a non-wellfounded ultraevent.

2.5.3 The lapse of time

One of the simplest dichotomy arguments does not involve motion at all, and only concerns the passage of infinitely divisible time.[40] We describe a version of this argument below.

For each order type τ, let τ^* be the reverse order type of τ. For example, ω^* is the order type of $(\mathbb{N}, >)$.

Assume that $\mathcal{T}$ is ω-divisible. Let I be a non-trivial interval of time. Then there is an ω^*-sequence

$$\langle I_n \rangle_{n \in \mathbb{N}}$$

of pairwise non-overlapping subintervals of I, where $\forall j, k \in \mathbb{N}$:

$$I_j \textit{ temporally precedes } I_k \quad \Longleftrightarrow \quad j > k.$$

Letting $\prec$ indicate "temporally precedes", we have:

$$\overbrace{\ldots \prec I_n \prec \ldots \prec I_2 \prec I_1 \prec I_0}^{I}$$

In order for the interval I to elapse, I_0 must elapse. But before I_0 begins to elapse, I_1 must elapse. And before I_1 begins to elapse, I_2 must elapse. And so on. Each one of these subintervals of I is temporally preceded by (an ω^*-sequence of) other subintervals of I. No subinterval in this sequence temporally precedes every other subinterval in the sequence — there is no earliest subinterval in the sequence. The argument concludes that it is impossible for any nontrivial interval of ω-divisible time to elapse, because no such interval can even begin to elapse. In order for such an interval to begin to elapse, it must already have begun to elapse.

Here we used ω^*, but the argument generalizes to arbitrary infinite linear order types.[41]

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[40] In 1929, Whitehead (1979) writes:

> ...the introduction of motion brings in irrelevant details. The true difficulty is to understand how the arrow [of Zeno] survives the lapse of time. (pp. 68–69)

[41] This can be done with the notion of C-divisibility defined in §2.5.5.

2.5.4 Recursive prejudice

One might argue against the possibility of performing any well-ordered hypertask on the basis of considerations about recursiveness. For each variety of modality, there is an instance of the following argument:

$\mathrm{P_i}$: It is possible to perform an α-task *only if* α is recursive.

$\mathrm{P_{ii}}$: Every recursive ordinal is countable.

$\therefore$ It is possible to perform an α-task only if α is countable.

Such arguments are question begging against those who admit the possibility of hypertasks. On what basis can one argue for the various versions of $\mathrm{P_i}$? Granting $\mathrm{P_{ii}}$, which is true by definition, $\mathrm{P_i}$ is stronger than the conclusion. The conception of *recursiveness* from classical computability theory is by definition limited to the realm of the countable, but the notion of a *recursive function* is generalized in higher recursion theory to include total functions with uncountable domains. Notions of possibility that trivially exclude uncountable well-orderings certainly do not allow for the performance of an α-task for uncountable α. But versions of the conclusion that involve such a notion of possibility are trivial. More interesting instances of the argument involve notions of possibility that do not rule out hypertasks from the start.

Recalling Zermelo's term "finitistic prejudice", for a prejudice against what is non-finitistic, we might call what is expressed by various instances of this argument "recursive prejudice", which exceeds what might be called "countable prejudice".[42]

2.5.5 Divisibility arguments

This subsection defines *divisibility arguments*, which use a generalization of ω-divisibility.

When $(C, <)$ is an infinite linearly ordered class, we say that $\mathcal{T} \times \mathcal{S}_1$ is C-*divisible* iff both $\mathcal{T}$ and $\mathcal{S}_1$ locally embed $(C, <)$.

C-divisibility covers order types of arbitrary infinite linearly ordered classes, so we can think of C as any infinite linear order type. C-divisibility ensures that every non-trivial bounded open interval of $\mathcal{T}$ or $\mathcal{S}_1$ can in a manner of speaking be subdivided by the order type of C.

Using the notion of C-divisibility, the dichotomy arguments generalize as follows.

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[42] We address what might be called "set prejudice" in §2.6.

There are notions of possibility $\Diamond_1$, $\Diamond_2$, $\Diamond_3$,
a type μ (of motion), and infinite linear order types C, τ
for which these five statements hold:

P$_{23}$: $\Diamond_1\big($Type μ motion in a C-divisible $\mathcal{T}\times\mathcal{S}_1$ occurs.$\big)$

P$_{24}$: If there are any $\Diamond_1$-worlds in which
type μ motion in a C-divisible $\mathcal{T}\times\mathcal{S}_1$ occurs,
then at least one of them is also a $\Diamond_2$-world.

P$_{25}$: $\Box_2\big($Type μ motion in a C-divisible $\mathcal{T}\times\mathcal{S}_1$ involves a τ-event.$\big)$

P$_{26}$: If there are any $\Diamond_2$-worlds in which a τ-event occurs,
then at least one of them is also a $\Diamond_3$-world.

P$_{27}$: $\Box_3\big($No τ-event occurs.$\big)$

This argument has a number of important kinds of special cases. For example, there are instances that satisfy one or more of the following conditions:

- $\Diamond_1 = \Diamond_2 = \Diamond_3$.
- Type μ includes every instance of motion.
- $C = \tau$.
- $C = \omega$.
- $C = \omega^*$.
- C is non-wellfounded.
- C is a proper class.
- $\mathcal{T}$ and $\mathcal{S}_1$ are order-isomorphic to $\mathbb{R}$.
- $\mathcal{T}$ and $\mathcal{S}_1$ are order-isomorphic to **No**.
- $\mathcal{T}$ and $\mathcal{S}_1$ are order-isomorphic.
- $\mathcal{T}$ and $\mathcal{S}_1$ are locally incomparable.

To resolve all of the instances of this general divisibility argument, one must show that

there are no values of $\Diamond_1$, $\Diamond_2$, $\Diamond_3$, μ, C, τ for which P$_{23}$–P$_{27}$ hold.

Equivalently, one has to show that

for any values of $\Diamond_1$, $\Diamond_2$, $\Diamond_3$, μ, C, τ, at least one of P_{23}–P_{27} fails.

Divisibility arguments merit further thought. For example, one can ask:

- With respect to various notions of modality, for which values of C and τ is P_{25} true?
- What is the relationship between C-divisibility and the order type of the event-sequence that motion through a C-divisible $\mathcal{T}\times\mathcal{S}_1$ involves?
- For various notions of modality, what conditions must the order types of $\mathcal{T}$ and $\mathcal{S}_1$ meet so that motion through $\mathcal{T}\times\mathcal{S}_1$ requires an event-sequence of a given order type?

Now we explain why certain instances of P_{25} are false.

Divisibility and mismatches

The general divisibility argument does not assume that there is a $\Diamond_2$-world in which $\mathcal{T}$ and $\mathcal{S}_1$ are order-isomorphic. How then might one argue for instances of P_{25}?

Recall how such premisses would be employed in a simple divisibility argument. For each combination of a modality and an infinite linear order type C, there is an instance of the following argument schema:

P_Δ: Motion through a C-divisible $\mathcal{T}\times\mathcal{S}_1$ is possible.

P_Ω: Necessarily, motion through a C-divisible $\mathcal{T}\times\mathcal{S}_1$ involves a C-event.

P_C: No C-event is possible.

$\therefore$ $\bot$

Consider how one would argue for P_Ω. We might begin with the following argument:

p_1: $\mathcal{T}\times\mathcal{S}_1$ (exists and) is C-divisible.

p_2: Motion through $\mathcal{T}\times\mathcal{S}_1$ occurs.

$\therefore$ A C-event occurs. (p_3)

Let w be a possible world. Assuming that both p_1 and p_2 hold in w, one would argue that p_3 holds in w too. Conditionalization would yield

$$(p_1 \wedge p_2) \;\rightarrow\; p_3.$$

Then the necessitation rule would give

$$\Box\big((p_1 \wedge p_2) \;\rightarrow\; p_3\big),$$

which is equivalent to P_Ω.

Recall that we have been assuming that $\mathcal{T}$ and $\mathcal{S}_1$ are locally uniform locally symmetric line-structures. As we will see, that assumption does not support the inference from p_1 and p_2 to p_3. Let us examine that inference more closely.

Suppose that p_1 and p_2 hold in a possible world w, and that a particular motion in w is given by a position function

$$s\colon [a,b] \longrightarrow [c,d]$$

with $a < b$ and $c < d$, as usual. Define

$$s(a) := c,$$
$$s(b) := d.$$

To argue for p_3, consider the open interval $(c,d)_{\mathcal{S}_1}$. Since $\mathcal{T}\times\mathcal{S}_1$ is C-divisible, $(C,<)$ order-embeds into $(c,d)_{\mathcal{S}_1}$. Let

$$f\colon C \longrightarrow (c,d)_{\mathcal{S}_1}$$

be an order-embedding. Then for each $x \in C$,

reaching the spatial point $f(x)$

counts as an event, and is one that must occur in order for the moving object to reach d. As we will see, this does not guarantee that the motion described by s involves an event-sequence of order type C. The reason is that when $\mathcal{T}$ and $\mathcal{S}_1$ are locally incomparable, it might not be the case that for each $x \in C$, there is a moment at which the object is at $f(x)$.

On the Russellian conception of motion, at least two conditions must be met in order for s to correspond to a "smooth non-stop motion". For simplicity, suppose that the object is a point particle. The first condition is that there must be a change in position.

There must be distinct moments at which the object occupies distinct positions, i.e.,

$$\exists t,t' \in [a,b] \quad \big(t < t' \quad\wedge\quad s(t)\!\downarrow \quad\wedge\quad s(t')\!\downarrow \quad\wedge\quad s(t) < s(t')\big).$$

The second condition is that the motion be "smooth". In the context of locally uniform locally symmetric line-structures, we require the following.

If an object occupies two distinct positions at two distinct times,
then there is a time that is strictly between those two times,
at which the object occupies a position strictly between those two positions, i.e.,

$$\forall t,t' \in [a,b] \quad \Big(\big(t < t' \quad \wedge \quad s(t)\downarrow \quad \wedge \quad s(t')\downarrow \quad \wedge \quad s(t) \neq s(t') \big) \ \rightarrow$$
$$\exists x \in (t,t') \quad \big(s(x)\downarrow \quad \wedge \quad s(t) < s(x) < s(t') \big) \Big).$$

Consider whether moving from c to d involves a C-event. If we assume that

$\mathcal{T}$ and $\mathcal{S}_1$ (exist and) are structured like order-isomorphic line-structures

in conjunction with p_1 and p_2, then $(a,b)_\mathcal{T}$ is order-isomorphic to $(c,d)_{\mathcal{S}_1}$, and s is an order-isomorphism and thus bijective. At each moment in the open interval of time, the point particle in motion occupies some point in the spatial dimension. But if neither of $(a,b)_\mathcal{T}$ and $(c,d)_{\mathcal{S}_1}$ order-embeds into the other, and the object's motion meets the two conditions mentioned above, then there is no way for the moving object to have a spatial position at every moment in $(a,b)_\mathcal{T}$, and it cannot be that for every point in $(c,d)_{\mathcal{S}_1}$, there is some moment at which the moving object occupies that spatial point.

The point is that $\mathcal{T}$ and $\mathcal{S}_1$ being C-divisible line-structures does not ensure that motion (from one spatial point to another) involves an event-sequence of order type C, if, for each spatial point in the C-ordered class of spatial points chosen, there must be a moment (i.e., a point) in time at which the moving object occupies that spatial point.

To further strengthen the general divisibility argument, we will use two definitions.

We will say that line-structures l_1 and l_2 are *locally comparable* iff

for any non-trivial bounded open interval of l_1,
and any non-trivial bounded open interval of l_2,
each of those intervals order-embeds into the other.

We say that $\mathcal{T} \times \mathcal{S}_1$ is *locally comparable* iff $\mathcal{T}$ and $\mathcal{S}_1$ are locally comparable.

Now we add the assumption of local comparability to P_{23}, P_{24}, and P_{25}. This gives:

There are notions of possibility $\Diamond_1$, $\Diamond_2$, $\Diamond_3$,
a type μ (of motion), and infinite linear order types C, τ
for which these five statements hold:

P_{28}: $\Diamond_1\big($Type μ motion in a locally comparable C-divisible $\mathcal{T}\times\mathcal{S}_1$ occurs.$\big)$

P_{29}: If there are any $\Diamond_1$-worlds in which
type μ motion in a locally comparable C-divisible $\mathcal{T}\times\mathcal{S}_1$ occurs,
then at least one of them is also a $\Diamond_2$-world.

P_{30}: $\Box_2\big($Type μ motion in a locally comparable C-divisible $\mathcal{T}\times\mathcal{S}_1$ involves a τ-event.$\big)$

P_{31}: If there are any $\Diamond_2$-worlds in which a τ-event occurs,
then at least one of them is also a $\Diamond_3$-world.

P_{32}: $\Box_3\big($No τ-event occurs.$\big)$

Notice that P_{30} is weaker than P_{25}.

A general lapse of time argument

Finally, the lapse of time argument from §2.5.3 generalizes as follows:

There are notions of possibility $\Diamond_1$, $\Diamond_2$, $\Diamond_3$,
and infinite linear order types C, τ
for which these five statements hold:

P_{33}: $\Diamond_1\big(\mathcal{T}$ is C-divisible.$\big)$

P_{34}: If there are any $\Diamond_1$-worlds in which $\mathcal{T}$ is C-divisible,
then at least one of them is also a $\Diamond_2$-world.

P_{35}: $\Box_2\big($If $\mathcal{T}$ is C-divisible, then a τ-event occurs.$\big)$

P_{36}: If there are any $\Diamond_2$-worlds in which a τ-event occurs,
then at least one of them is also a $\Diamond_3$-world.

P_{37}: $\Box_3\big($No τ-event occurs.$\big)$

One can ask, for example: For which values of $\Box_2$, C, and τ does P_{35} hold?

2.6 Maximality and indefinite extensibility

In order to explain what it is for a line-structure to be "more finely subdivided" than another, we will use the informal notion of a *collection*, which is thought to be "indefinitely extensible".[43] Some collections are not sets, so we will abuse the symbols of the language of set theory. We might instead have used the symbols

$$\doteq,\ \dot{\forall},\ \dot{\exists},\ \dot{\in},\ \dot{\{},\ \dot{\}},\ \dot{\cup},\ \dot{\cap},\ \dot{<},$$

etc. when dealing with collections that are not sets, or with collections in general.

We need a few definitions. Assume that our first-order language has $=$ and $<$.

Let $(C, <_C)$ be a line-structure. When L and R are subsets of C, and

$$\forall l \in L \quad \forall r \in R \quad l <_C r,$$

then we write $L <_C R$.

A *cut of* $(C, <_C)$ is an ordered pair $\langle L, R \rangle$ such that

$$L \cup R = C,$$

$$L \cap R = \emptyset,$$

$$L \neq \emptyset \neq R,$$

$$L <_C R.$$

A cut partitions the elements of a line-structure into two collections L and R such that $L < R$.

Let a cut $\langle L, R \rangle$ of $(C, <_C)$ be *left-closed* iff

$$\exists l \in L \quad \forall x \in L \quad x \leq_C l;$$

otherwise it is *left-open*.

Similarly, let a cut $\langle L, R \rangle$ of $(C, <_C)$ be *right-closed* iff

$$\exists r \in R \quad \forall x \in R \quad x \geq_C r;$$

otherwise it is *right-open*.

A *void of* $(C, <_C)$ is a cut $\langle L, R \rangle$ of $(C, <_C)$ that is both left-open and right-open — L has no greatest member and R has no least member.

The next definition specifies when cuts of a line-structure are equivalent.

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[43] Dummett (1996) writes:

> An indefinitely extensible concept is one such that, if we can form a definite conception of a totality all of whose members fall under that concept, [then] we can, by reference to that totality, characterize a larger totality all of whose members fall under it. (p. 441)

For any cuts $\langle L_x, R_x \rangle$ and $\langle L_y, R_y \rangle$ of $(C, <_C)$, define

$$\langle L_x, R_x \rangle \equiv \langle L_y, R_y \rangle$$

as:

$$L_x = L_y \quad \vee \quad \exists z \in C \; \big(L_x \cap R_y = \{z\}\big) \quad \vee \quad \exists z \in C \; \big(R_x \cap L_y = \{z\}\big).$$

A *break* $\langle L, R \rangle$ *of* $(C, <_C)$ is a cut of $(C, <_C)$ that is either left-closed and right-open, or left-open and right-closed — either L has no greatest member and R has a least, or L has a greatest member and R has no least.

Given any line-structure $(C, <_C)$ that has voids, we can expand it to another line-structure $(C', <_{C'})$ by "filling in" each of its voids. For each void $\langle L, R \rangle$ of $(C, <_C)$, we can insert a point p so that

$$\forall l \in L \quad \forall r \in R \quad l <_{C'} p <_{C'} r.$$

C' is the union of C with the set of every such added point, and $<_{C'}$ is the appropriate extension of $<_C$.

A line-structure l has a *void-completion* $v(l)$, which is the result of filling in every void in l with a point and appropriately extending the linear ordering.

A line-structure is *void-complete* iff it has no voids.

If instead we fill each void with a copy of $(C, <_C)$ (or a copy of any line-structure), the result will also be a line-structure.

Given any line-structure $(C, <_C)$, we can expand it to another line-structure $(C', <_{C'})$ as follows. For each break $\langle L, R \rangle$ of $(C, <_C)$, we insert a copy of a line-structure in between L and R. But the resulting line-structure has breaks of its own. Every line-structure has breaks.

Whether or not a line-structure has any voids, it can be extended to a "more finely subdivided" line-structure. Take any line-structure and replace each of its points, or fill some (or all) of its cuts with a line-structure, and the result will also be a line-structure. When the resulting line-structure is not order-isomorphic to the original, then we say that it is *more finely subdivided* than the original line-structure.

Compare how finely points are packed into $\mathbb{R}$ with how finely they are packed into **No**. $\mathbb{R}$ is Dedekind complete and has no voids. **No** has voids; each void in **No** is not right next to any surreal number. One sense in which **No** is more "dense" than $\mathbb{R}$ is that **No** fills in all of the breaks that $\mathbb{R}$ has, as well as filling in breaks which $\mathbb{R}$ does *not* have. But **No**, just like its Dedekind completion, has breaks of its own.[44] No non-empty dense linear ordering without endpoints, whether set-sized or not, is "maximally dense" in the sense of having no breaks.

[44] Fornasiero (2004, §§1.9–1.10) discusses the Dedekind completion of **No**.

In 1907, Hausdorff generalized the order type of the rationals.[45] For each ordinal α, an η_α-set is a linear order type such that for any subsets L, R of an η_α-set whose cardinalities are less than $\aleph_\alpha$, if every member of L is less than every member of R, then there is an element that is greater than every member of L and less than every member of R.[46] For each ordinal α, Hausdorff defines a linear order type η_α such that

$$\forall L, R \subseteq \eta_\alpha \quad \Big(\big(|L| + |R| \leq \aleph_\alpha \quad \wedge \quad L < R \big) \quad \rightarrow \quad \exists x \in \eta_\alpha \quad \forall l \in L \quad \forall r \in R \quad l < x < r \Big).$$

For example, η_0 is the order type of $(\mathbb{Q}, <)$. There are η-sets of every infinite cardinality.[47] Ordered lexicographically,

$$\bigcup_{\alpha \in On} \eta_\alpha$$

is order-isomorphic to **No**, and might be called the η-class η_{On}. Each η-set is an η-class, and each η-class is a line-structure. Each η-class is maximal in the sense that any linear order of equal or lesser size order-embeds into it.

2.6.1 The non-infinite surreals

No contains infinite numbers. If we treat $\mathcal{T}$ or $\mathcal{S}_1$ as structured like **No**, which has more structure than just its order, then they will have points that are infinitely distant from one another. One could instead treat $\mathcal{T}$ or $\mathcal{S}_1$ as having only points that are finitely distant from one another, while still being structured like **No** "locally". The point is that one can make sense of $\mathcal{T}$ or $\mathcal{S}_1$ being as finely divisible as **No** even if spatiotemporal points are only finitely distant from one another.

Define the proper class of *non-infinite surreals* (i.e., those surreals that are either infinitesimal or finite) to be

$$No_{\not\infty} := \big\{ x \in No \colon \exists n \in \mathbb{N} \ \ |x| < n \big\}.$$

Let $\mathbf{No}_{\not\infty}$ be the structure

$$\big(No_{\not\infty}, 0, 1, <, +, \cdot \big),$$

which is the reduct of **No** that omits the infinite surreals.

Recall that $gcd(x, y) = z$ iff

$$z|x \quad \wedge \quad z|y \quad \wedge \quad \forall d \, \big((d|x \wedge d|y) \ \rightarrow \ d|z \big).^{48}$$

[45] See Plotkin (2005, p. 150).

[46] For a discussion of the related notion of saturation, see Chang & Keisler (2012, Chapter 5).

[47] For more discussion, see Alling (1962).

[48] As usual, $z|x$ iff $\exists s \ \ z \cdot s = x$.

$\mathbf{No}_{\not\infty}$ is a proper class GCD domain — a nonzero commutative Ring in which

$$\forall x \neq 0 \quad \forall y \neq 0 \quad x \cdot y \neq 0,$$

$$\forall x \neq 0 \quad \forall y \neq 0 \quad \exists z \; gcd(x, y) = z.$$

$\mathbf{No}_{\not\infty}$ is not even a Field — no infinitesimal in $\mathbf{No}_{\not\infty}$ has a multiplicative inverse in $\mathbf{No}_{\not\infty}$. But locally, $\mathbf{No}_{\not\infty}$ is structurally similar to **No**, in that every non-trivial bounded open interval of $\mathbf{No}_{\not\infty}$ is order-isomorphic to **No**.

2.6.2 A hierarchy conception

Is it *metaphysically* possible for a spatiotemporal dimension to be structured like $\mathbf{No}_{\not\infty}$? Even if the notion of metaphysical possibility is clear enough, appealing to the mathematical possibility of such a situation is not sufficient to establish that it is a metaphysical possibility. Huemer (2016) makes a similar point about logical possibility:

> No one should believe that a proposition is metaphysically possible merely because it is logically possible. Similarly, here are two other types of bad reasons for ascribing metaphysical possibility:
>
> First, the argument from mathematical systems: in this type of argument, one cites the existence of a coherent mathematical system as proof (or at least evidence) that it is [metaphysically] possible for something in reality to satisfy the system. This is fallacious. Mathematical systems in modern times are constrained by nothing other than mere consistency and mathematicians' sense of what is *interesting* — they are not constrained by metaphysical possibility. Thus, for example, the fact that one can develop a coherent mathematical system in which one talks about 'infinite numbers' such as $\aleph_0$, does not show that any such numbers exist or could exist. Nor, similarly, does the existence of a consistent mathematical system for talking about 'infinitesimal numbers' mean that there are or could be any infinitesimal quantities.
>
> Second, the argument by analogy: A is possible, and there is an analogy to be drawn between A and B; therefore, B is possible. This is usually fallacious, particularly when the analogy is based upon mere similarity of mathematical structure. (p. 105)

It is conceivable that $\mathcal{T}$ and $\mathcal{S}_1$ are more finely subdivided than $\mathbf{No}_{\not\infty}$.[49] The notion of

[49] In the 1890s, Peirce (1976) writes:

> *Definition* 96. A whole is said to be *indefinitely divisible*, when any *homogeneous parts* of which it may be conceived to consist, consist each of them, of smaller homogeneous parts; and the word *divisible* is not understood to imply that the parts can be separated from one another.
>
> *Illustration* 40. The range of values from 0 to 1 is indefinitely divisible. For, no matter how small the partial ranges may be taken of which it is conceived as consisting, each of these consists of other intervals still smaller. (p. 167)

collection of points which subdivide a linear continuum is indefinitely extensible. We defined "line-structure" in terms of "collection". Despite the limitations of standard set theories with respect to proper classes, the indefinite extensibility of *collection* entails the indefinite extensibility of *line-structure*.

Potter (2004) describes a conception of an indefinitely divisible continuum:[50]

> ...if we ...treat infinitesimals as real [objects], we ...have no reason not to posit a new level of objects which are infinitesimal relative to the non-standard field. ...
>
> We have evidently embarked now on a process which we can iterate in much the same manner as we iterated the construction of the set-theoretic hierarchy. If we do this, we arrive at a conception of the continuum which is very different from, and far richer than, the Weierstrassian one. ...[T]he proposal now under consideration is that we should conceive of the continuum as indefinitely *divisible* in much the same way as the [set-theoretic] hierarchy is indefinitely extensible ...(pp. 146–147)

If we take seriously the thought that a spatiotemporal dimension can *always* be further subdivided, one possible view is that a dimension corresponds to an absolutely infinite hierarchy of ever more finely subdivided linearly ordered collections of points, perhaps with some additional structure at each level of the hierarchy. Such *hierarchy conceptions* of spatiotemporal dimensions are in some respects analogous to Zermelo's view of the universe(s) of set theory.[51]

We will mention two conceptual difficulties with the standard non-hierarchical conception of spatiotemporal dimensions. On the standard conception, a dimension is a "fixed" collection of points — in the sense of being static or completed — with some additional structure (topological, metrical, algebraic). But there is a limit to how finely the dimension is subdivided by points. Consequently, there is no location between a point x in the dimension, and all of the points in that dimension that are to the right of x. No tuple of cuts corresponds to a location, and an object moving from a point x to a point to the right of x does not "pass through" any (locations that correspond to) tuples of cuts.[52] This is the first difficulty.

Generalizing the notion of a point, define a *quasi-region* to be an ordered tuple, of the appropriate arity for the given space, where each entry can be either a coordinate *or* a cut. Now we can ask: Which collections of quasi-regions correspond to spatiotemporal locations? On the standard conception, only those quasi-regions that are points correspond to a spatiotemporal location.

It is natural to think that a moving object passes through not only (locations that correspond to) points but (locations that correspond to) quasi-regions that are not points, as well. If a void

[50] For a discussion of nonstandard extensions of $\mathbb{R}$, see Hall & Todorov (2015).

[51] A hierarchy conception may be relativized to a model of a set theory.

[52] Here, we are not thinking of points as being tuples of cuts.

or a break $\langle L, R \rangle$ in $\mathcal{S}_1$ corresponds to a determinate spatial location, then moving from a point in L to a point in R requires moving through that location. Voids and breaks of $\mathcal{S}_1$ can be seen as analogous to non-trivial intervals of $\mathcal{S}_1$, in that they can potentially be subdivided, unlike a point(-singleton).

Suppose that one conceives of spatiotemporal dimensions in the standard way mentioned above, and thinks that there is a limit to how finely they are subdivided by points. Then it must be explained why spatiotemporal dimensions have the particular order type that they do. Why aren't they *more* finely subdivided than they are? And why aren't they *less* finely subdivided than that? The standard conception has to defend itself against charges of arbitrariness. This is the second difficulty.

Chapter 3

Doing the uncountable

In Chapter 1 we saw how accounts of (our knowledge of) truth in first-order arithmetic can make use of supertasks. This chapter explains how hypertasks could in principle yield knowledge of physical situations. We argue that whether certain propositions are knowable depends on whether the temporal dimension allows for hypertasks.[1]

Motivation

Since the 1960s, mathematicians have developed generalizations of classical computability theory that study computations of transfinite length.[2] Philosophers have paid little attention to epistemic questions about what results from performing infinite sequences of tasks, and in particular, uncountable task-sequences. For example, how might we know what would result from the performance of a supertask, or even a hypertask?

In almost every supertask or hypertask discussed, exactly what occurs at each stage is known beforehand. A well known example of such a supertask is one first stated in 1953 by Littlewood (1986, p. 26).[3] Ross (2012, pp. 46–48) describes a supertask in which what occurs at each stage is *not* known beforehand.[4] If what results from performing such a sequence of tasks depends only

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[1] Since the propositions in question do not concern the structure of time, the claim is not trivial.

[2] See Sacks (2017) and Hinman (2017).

[3] This supertask was later discussed by Allis & Koetsier (1991).

[4] Ross describes a variant of Littlewood's supertask. In Ross's case, at each stage, what occurs depends on a random selection among $\aleph_0$ possibilities. Thus, there are $2^{\aleph_0}$ possible ω-tasks, and uncountably many distinct possible end results.

Probability measures that are *real*-valued assign outcomes that are infinitely unlikely (in the sense of occurring in only countably many out of uncountably many possibilities in total) a value of zero. However, there are reasons to sometimes prefer probability measures that can take (nonzero) infinitesimal values. See Benci, Horsten & Wenmackers (2016).

on what occurs at each stage — holding fixed some possible physical laws — then information about what occurs at each stage of the sequence is crucial for determining what would result from performing the entire sequence of tasks. A natural question is:

> Without complete knowledge of what occurs at each stage of an infinite sequence of tasks, to what extent can we determine what would result from performing that task-sequence?

This question motivates thinking about the kinds of idealized situations that we will analyze.

Main claim

This chapter shows that the structure of time can have epistemological consequences. We describe a way in which the mathematical structure of time constrains what is possible to know. In particular, the temporal continuum's cardinal characteristics and order-theoretic properties impose limitations on which kinds of sequences of tasks can be completed in a bounded interval of time. This in turn may limit which propositions are knowable.

We describe a situation, based on Littlewood's supertask, in which *knowing that a particular state obtains* requires that an uncountable sequence of tasks be completed. Only those possible worlds in which

a bounded interval of time contains uncountably many
non-trivial pairwise non-overlapping subintervals

allow hypertasks.[5] For example, no possible world in which time is structured like $\mathbb{R}$ is one in which a hypertask is performed. We exhibit a simple proposition such that *whether we can know that that proposition holds* depends on whether the structure of time is compatible with the completion of a hypertask.

There are propositions p that satisfy the condition

"p is knowable in a possible world w only if a hypertask may be performed in w"

in uninteresting ways. For example, assume that for every possible world w,

every proposition that is knowable in w is true in w.

Then the proposition that *a hypertask may be performed in w* satisfies the condition. We will describe a proposition that satisfies the condition in a more interesting way.

[5] This fails if we allow that a task may be completed instantaneously (i.e., in a single moment). Recall that a *trivial* interval either contains only one point, or is empty.

3.1 A button and a rule

Assume that $\mathcal{T}$ is a non-empty dense linear order without endpoints. Treat $\mathcal{T}$ as the ordered collection — it may be a proper class — of every moment in time. We assume that $\mathcal{T}$ is locally uniform, meaning that all of its bounded intervals of the form (a, b), where $a < b$, are order-isomorphic to one another. We also assume that $\mathcal{T}$ is locally symmetric, meaning that each of its bounded intervals of the form (a, b), where $a < b$, is order-isomorphic to that interval's inverse order.

Assume that there is an urn which initially contains some (un)known number of marbles, and that there is a button which can be pressed instantaneously and obeys the following rule.[6]

> **For any moment t, if the button is (being) pressed at t, then:**
> *if there is at least one marble in the urn at t, then exactly one marble is removed instantaneously from the urn at t; otherwise, no marble is removed from the urn at t.*

If a press of the button removes a marble from the urn, we do not know which marble was removed by that press. Finally, no marble ever changes with respect to whether it is in the urn unless that is a result of some press(es) of the button. For any moment t, if the button is not (being) pressed at t, then no marble is removed at t.

Define a *moment-set* to be a bounded subcollection of $\mathcal{T}$ that represents the moments t such that we intend *to press the button at t*. The variable 'M' will always stand for a moment-set. If M is our moment-set, then every moment $t \notin M$ is such that we intend *not to press the button at t*. As we will see, it may not be possible to press the button at every moment that we *intend* to do so.

Define a *pressed-set* to be a bounded subcollection of $\mathcal{T}$ that represents all and only the moments at which the button is *actually* pressed.[7] The variable 'P' will always stand for a pressed-set. We will see that some moment-sets do not determine a unique pressed-set — some moment-sets are compatible with more than one pressed-set. Also, some pressed-sets do not uniquely determine which marbles they will remove on a given occasion.

3.2 Finitely many objects in $\mathbb{R}$-like time

We first consider cases in which the urn initially contains finitely many marbles. Several remarks there will also apply to cases in which the urn initially contains infinitely many marbles.

[6] This idealization may be relaxed by allowing that the button can be pressed in any non-trivial interval of time.

[7] We are not requiring that pressed-sets or moment-sets be definable in any particular language.

Depending on the structure of $\mathcal{T}$ and on how many marbles are initially contained in the urn, a moment-set M may result in a pressed-set that is a proper subset of M. For example, suppose that $\mathcal{T}$ is structured like $\mathbb{R}$. What happens if we try to hold the button down beginning at $t = 0$ and ending at $t = 1$? In this case, our moment-set is $[0, 1]_{\mathbb{R}}$.

Suppose that the urn initially contains exactly two marbles, and that we do not know this beforehand. The first press occurs at $t = 0$, when one marble is removed. Then only one marble remains in the urn. The remaining *intended* presses are to occur at every moment in $(0, 1]_{\mathbb{R}}$. But that interval has no earliest moment. If the remaining marble is removed, it is removed at a unique moment. If the button were being held down at every moment in $(0, 1]_{\mathbb{R}}$, then for any moment in that interval, there is an earlier moment in that interval by which the final marble should already have been removed.

Because of the rule that governs the button, if the button is (being) pressed at some moment t, and no marble is removed at t, then the urn must not contain any marbles at t. So if the urn did contain at least one marble at t, and no marble was removed at t, then it must be that the button was not (being) pressed at t. Thus, in cases in which $\mathcal{T}$ is structured like $\mathbb{R}$ and exactly two marbles are initially in the urn, if we *try* to hold the button down during every moment in $[0, 1]_{\mathbb{R}}$, we would not succeed in holding the button down during that interval, i.e., pressing the button at every moment in the interval.[8]

One way of describing this situation is to say that the button would *pop up* immediately after $t = 0$ when a marble is removed. If we do not press the button again after it pops up, the button will be in the unpressed position at every moment in $(0, 1]_{\mathbb{R}}$, so the remaining marble is not removed at any moment in that interval. In this case, the moment-set $[0, 1]_{\mathbb{R}}$ results in the pressed-set $\{0\}$. We could, of course, press the button again at any moment in $(0, 1]_{\mathbb{R}}$. But doing so would require that we choose some moment in that interval at which to press the button again. The moment-set $[0, 1]_{\mathbb{R}}$ by itself does not determine a choice of such a moment.

Similar situations can occur even if we do not try to hold the button down. For example, suppose again that $\mathcal{T}$ is structured like $\mathbb{R}$, and that for some unknown positive integer n, there are exactly n marbles initially in the urn. In this case, because of the rule that governs the button, it cannot be pressed at all and only every moment in a non-empty non-wellfounded class of moments.

For example, consider the moment-set

$$M := \left\{ \frac{1}{2^n} : n \in \mathbb{N} \right\}.$$

Every moment in M is preceded by infinitely many other moments in M. But only finitely many marbles are initially contained in the urn. Under these conditions, M results in the pressed-set

[8] The same reasoning applies when the number of marbles that the urn initially contains is countable and greater than 2.

$\emptyset$. For any moment t, a marble can be removed at t only if the result of *adding one to the cardinality of the class of all presses that have occurred before* t is less than or equal to the cardinality of the class of all marbles that were initially in the urn. The class of presses made *before the urn contains zero marbles* cannot be greater in cardinality than the class of marbles initially in the urn. Thus, none of the intended presses could be carried out.

Suppose that we know that the urn initially contains some positive finite number n of marbles, but that we do not know the value of n. We wish to remove every marble in such a way that we will know beforehand that we will succeed in doing so.

Q18 (informally): Which moment-sets would ensure, regardless of which finite number of marbles are initially in the urn, that every marble would be removed?

Conjecture: The answer to Q18 is that it is necessary and sufficient that a moment-set have an initial segment of order type ω.

With $\mathcal{T}$ structured like $\mathbb{R}$, and an unknown positive finite number of marbles initially in the urn, in order to know beforehand that every marble would be removed as a result of a moment-set, we must know that we would succeed in pressing the button infinitely many times. Since there are only finitely many marbles, infinitely many of those pressings will not remove any marble. But there would be no press such that we know that that press will fail to remove a marble. We will see that depending on the structure of $\mathcal{T}$, an analogous phenomenon can occur at the countable/uncountable level when we have $\aleph_0$ marbles.

3.3 A formalization

First we formalize what can happen given a pressed-set, ignoring which moment-sets could have resulted in that pressed-set. We will use the "is earlier than" relation $<$ on $\mathcal{T}$, and that relation's inverse, the "is later than" relation $>$ on $\mathcal{T}$.

For any $S \subseteq \mathcal{T}$ and $x \in \mathcal{T}$, we write $S_{<x}$ for

$$\{y \in S\colon y < x\},$$

and similarly for $S_{\leq x}$, $S_{>x}$, and $S_{\geq x}$.

Outcomes

Let κ be a cardinal that represents the collection of marbles initially in the urn. Let P be a pressed-set. Given a triple $\langle \mathcal{T}, \kappa, P \rangle$, we define a total function

$$R\colon \mathcal{T} \longrightarrow \mathcal{P}(\kappa)$$

that assigns to each $t \in \mathcal{T}$ some subset of κ that represents the collection of marbles that have been removed *at or before* t. For some such triples, there will be more than one of these mappings.

> We write $\alpha @ \beta$ (read "α is removed at β") for
>
> $$\alpha \in R(\beta) \quad \wedge \quad \forall q \in \mathcal{T}_{<\beta} \;\; \alpha \notin R(q).$$
>
> We write $\alpha \not@ \beta$ (read "α is *not* removed at β") for $\neg(\alpha @ \beta)$.

R is defined as follows.

> For each $t \in \mathcal{T}$, we require that $t \mapsto R(t)$ in such a way that:
>
> no marble is removed before any presses have occurred, i.e.,
>
> (1) $\forall t \in \mathcal{T} \; \big((\forall x \in P \;\; t < x) \quad \rightarrow \quad R(t) = \emptyset \big)$
>
> once a marble is removed, it remains removed thereafter, i.e.,
>
> (2) $\forall x, y \in \mathcal{T} \; \big(x < y \quad \rightarrow \quad R(x) \subseteq R(y) \big)$
>
> no marble is removed at any moment unless the button is pressed at that moment, i.e.,
>
> (3) $\forall t \in \mathcal{T} \; \big((\exists x \in \kappa \;\; x @ t) \quad \rightarrow \quad t \in P \big)$
>
> if there is a marble not removed by any prior press, then this press removes one, i.e.,
>
> (4) $\forall t \in P \; \Big(\big(\exists x \in \kappa \;\; \forall s \in P_{<t} \;\; x \not@ s \big) \quad \rightarrow \quad \exists x \in \kappa \;\; x @ t \Big)$
>
> and only one marble, if any, is removed at a time, i.e.,
>
> (5) $\forall t \in P \; \big((\exists x \in \kappa \;\; x @ t) \quad \rightarrow \quad \exists ! x \in \kappa \;\; x @ t \big)$.

We say that any such function R is an *outcome* for $\langle \mathcal{T}, \kappa, P \rangle$. Each outcome encodes which marbles, if any, were removed from the urn at each moment as a result of that outcome. The graph of an outcome can be linearly ordered in the obvious way using the $<$ relation.

We can then prove that whenever a marble is removed, the number of marbles removed so far is equal to the number of presses made so far. I.e., for every $\langle \mathcal{T}, \kappa, P \rangle$, every outcome R for that triple is such that

$$\forall t \in P \; \Big((\exists x \in \kappa \;\; x @ t) \quad \rightarrow \quad \big| R(t) \big| = \big| P_{\leq t} \big| \Big).$$

We can also prove that if no marble is removed when the button is pressed, then no marbles remain in the urn. I.e., for every $\langle \mathcal{T}, \kappa, P\rangle$, every outcome R for that triple is such that

$$\forall t \in P\ \big((\neg \exists x \in \kappa\ \ x@t) \quad \rightarrow \quad R(t) = \kappa\big).$$

Q19 (informally): Which moment-sets are such that before we begin any presses, we can know that *if we try to press the button at exactly those times, then whatever sequence of pressings that we actually succeed in making will remove every marble?*

We formalize Q19 below.

Compatibility

We need to formalize the relationship between moment-sets and pressed-sets. A moment-set can be "compatible with" multiple pressed-sets. In any given situation, only one pressed-set actually results. But on another occasion with the same initial conditions, a different pressed-set may result.

For example, consider the triple $\langle \mathbb{R}, \aleph_0, M\rangle$ with the moment-set

$$M := \left\{\frac{n}{n+1} : n \in \mathbb{N}\right\} \cup [1,2]_{\mathbb{R}}.$$

If every marble is such that it is removed before $t = 1$, then M results in the pressed-set $P = M$. In this case, the button does not "pop up". Alternatively, if, e.g., every even-numbered marble — but no odd-numbered marble — is such that it is removed before $t = 1$, then M results in the pressed-set $P = M_{\leq 1}$. This is the case in which the button "pops up". More specifically, at every time $t > 1$, the button is in the unpressed position.

Informally, saying that each of the pressed-sets $P = M$ and $P = M_{\leq 1}$ is "compatible with" that triple means that, given $\mathcal{T}$, κ, and the intention to press the button at all and only the moments in M, and given the rule that governs the button, it is possible to press the button at all and only the moments in that pressed-set.

Extending outcomes

Next we define what it is for an outcome to be an "extension" of another outcome. We can extend an outcome (in the sense of extending a function) by enlarging the domain to include a moment that is later than every other moment that was originally in the domain.

Let R_1 be an outcome for $\langle \mathcal{T}, \kappa, P \rangle$. Let $t \in \mathcal{T}$ be such that

$$\forall x \in P \quad x < t.$$

Let R_2 be an outcome for

$$\big\langle \mathcal{T},\ \kappa,\ P \cup \{t\} \big\rangle.$$

Then we say that R_1 *extends to* R_2 (or that R_2 *extends* R_1) iff

$$R_2 \restriction P = R_1.$$

Now we define what it is for a pressed-set to be "compatible with" a triple.

Let $\mathcal{T}$ be a non-empty dense linear ordering without endpoints, let κ be a cardinal, let M be a moment-set, and let $P \subseteq M$. Then we say that a pressed-set P is *compatible with* $\langle \mathcal{T}, \kappa, M \rangle$ iff $\forall t \in M$:

$$t \in P \iff \text{there is an outcome for } \big\langle \mathcal{T}, \kappa, P_{<t} \big\rangle \text{ that extends to an outcome for } \big\langle \mathcal{T},\ \kappa,\ P_{<t} \cup \{t\} \big\rangle.$$

Winning outcomes, winning pressed-sets, and winning moment-sets

Finally we define what it is for:

(i) an *outcome* to remove every marble,

(ii) a triple containing a *pressed*-set to "ensure" that every marble is removed,

(iii) a triple containing a *moment*-set to "ensure" that every marble will be removed.

We will use the term 'wins' in each of these three types of cases.

When R is an outcome for $\langle \mathcal{T}, \kappa, P \rangle$, we say that R *wins* iff

$$\exists t \in \mathcal{T} \quad R(t) = \kappa.$$

I.e., at some time, every marble has been removed.

We say that a triple $\langle \mathcal{T}, \kappa, P \rangle$ *wins* iff

there is an outcome for that triple,
and every outcome for that triple wins.

We say that a triple $\langle \mathcal{T}, \kappa, M \rangle$ *wins* iff

there is a pressed-set that is compatible with that triple,
and for every pressed-set P that is compatible with that triple,
$\langle \mathcal{T}, \kappa, P \rangle$ wins.

Q18 (formally): Which order types of moment-sets M are such that
$\forall n \in \mathbb{N}$, the triple $\langle \mathbb{R}, n, M \rangle$ wins?

Proposition: For any moment-set $M \subset \mathbb{R}$:

$$M \text{ has an initial segment of order type } \omega \quad \Longleftrightarrow \quad \text{for each } n \in \mathbb{Z}^+,\ \langle \mathbb{R}, n, M \rangle \text{ wins.}$$

Proof: The $\Rightarrow$ direction is trivial. For $\Leftarrow$, use induction on n to show that $\forall n \in \mathbb{Z}^+$ and for any moment-set $M \subset \mathbb{R}$,

$$\langle \mathbb{R}, n, M \rangle \text{ wins} \quad \Longrightarrow \quad M \text{ has an initial segment of order type } n.$$

Then in order for it to be the case that $\forall n \in \mathbb{Z}^+$, $\langle \mathbb{R}, n, M \rangle$ wins, M must have initial segments of every finite order type. $\square$

We end this section with an open question.

Q19 (formally): Given $\mathcal{T}$ and κ, which order types of moment-sets M are such that
$\langle \mathcal{T}, \kappa, M \rangle$ wins?

3.4 $\aleph_0$ objects in $\mathbb{R}$-like time

Suppose that the urn contains $\aleph_0$ marbles. We can think of each marble as corresponding to a unique natural number, and each natural number as corresponding to a unique marble. Then each possible emptying of the urn corresponds to a unique (up to order-isomorphism) linear ordering on $\mathbb{N}$, i.e., a countably infinite order type. Each such ordering has the order type of the collection of moments at which marbles were removed, and encodes which marble was removed at each position in that ordering.

An interesting special case of Q19 is:

Q20: Which order types of moment-sets M are such that $\langle \mathbb{R}, \aleph_0, M \rangle$ wins?

Proposition: For no moment-set $M \subset \mathbb{R}$ does $\langle \mathbb{R}, \aleph_0, M \rangle$ win.

Proof: In order for $\langle \mathbb{R}, \aleph_0, M \rangle$ to win, M must have initial segments of every countable wellfounded order type. So M would need to have an initial segment with order type ω_1. But no subset of $\mathbb{R}$ has that order type. □

If the urn initially contains $\aleph_0$ marbles, then in order to know that we have removed every marble, we must perform uncountably many attempted removals. If the order type of $\mathcal{T}$ does not allow for hypertasks, then we cannot know that we have removed every marble. Thus, whether we can know that

every marble has been removed

depends on whether the structure of time is compatible with the completion of a hypertask.

3.5 Other temporal structures

We end this chapter with two open questions about the relationship between the order type of $\mathcal{T}$ and the order types of winning moment-sets.

First, we can ask: What structures of time are compatible with there being a moment-set that would ensure that when the urn initially contains $\aleph_0$ marbles, every marble would be removed? This motivates:

Q21: Which order types of $\mathcal{T}$ are such that there is a moment-set M for which both

(i) there is a pressed-set that is compatible with $\langle \mathcal{T}, \aleph_0, M \rangle$,

and

(ii) every pressed-set P that is compatible with $\langle \mathcal{T}, \aleph_0, M \rangle$ is such that $\langle \mathcal{T}, \aleph_0, P \rangle$ wins?

Second, we can ask: Given the structure of time, how many marbles could be initially in the urn, with there still being a moment-set that would ensure that every marble would be removed? This motivates:

Q22: Given $\mathcal{T}$, for which infinite cardinals κ is there a moment-set M such that both

(i) there is a pressed-set that is compatible with $\langle \mathcal{T}, \kappa, M \rangle$,

and

(ii) every pressed-set P that is compatible with $\langle \mathcal{T}, \kappa, M \rangle$ is such that $\langle \mathcal{T}, \kappa, P \rangle$ wins?

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