# Invariant Euler-Lagrange Equations for Variational Problems Defined over Framed Curves in Two and Three Dimensions

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# Dedication

*To my wife Therese.*

## Abstract

This thesis focuses on deriving and understanding the invariant Euler-Lagrange equations for variational problems defined over framed curves in two and three dimensions. We make use of the moving frame machinery developed by Fels and Olver ([FO99]) along with the structure of the invariant variational complex as derived by Kogan and Olver([KO01]). In [KO03] Kogan and Olver combined these tools in order to develop a procedure for deriving the Euler-Lagrange equations for variational problems that admit symmetries. It will be this procedure that we invoke to achieve our goals. In the two dimensional case, we derive the equations in two sets of coordinates. The difference between our choice of coordinate systems will involve how we represent a frame. In three dimensions, the choice of a coordinate system can drastically change the difficultly of various calculations. In order to fully analyze the three-dimensional case, we will make use of the insights gained in the two-dimensional case. We conclude the thesis by considering restricted framed curves and how restrictions can alter the invariant Euler-Lagrange equations. Finally, it should be noted that the computations needed to write down the invariant Euler-Lagrange equations of interest will be lengthy and difficult to fully write out. These calculations were carried out using code written in the Python programming language. The code used for the work in this thesis can be found on `https://github.com/broom010/Lie_Symmetry`.

# Contents

# Chapter 1

# Introduction

In the late nineteenth century, Sophus Lie laid the foundation for using symmetry methods to study differential equations and variational problems. The introduction of continuous groups and infinitesimal methods to study such problems eventually lead to the creation of Lie theory. While working on variational problems, Lie made the important observation that if $G$ is a symmetry group of a variational problem, then the associated functional may be rewritten in terms of the differential invariants of the symmetry group. Further, the Euler-Lagrange equations corresponding to the original functional will inherit $G$ as a symmetry group. When the Euler-Lagrange equations are rewritten in terms of the differential invariants of $G$, we will refer to these equations as the *invariant* Euler-Lagrange equations.

Kogan and Olver were the first to introduce a general procedure for producing invariant Euler-Lagrange equations for an invariant variational problem using moving frames and the structure of the variational bicomplex (see [KO03]). Chapters two, three, four, and five will serve as background material. First, we will introduce the machinery needed to understand and carry out the necessary computations by discussing jet spaces, moving frames, and the invariant variational bicomplex. Next, we will give a formula for the invariant Euler-Lagrange equations for an invariant variational problem defined over a curve in higher dimensions. After discussing the background material, we then move on to framed curves.

Informally, a framed curve is a curve in $\mathbb{R}^n$ with an orthogonal frame attached at each point along the curve. In chapter six, we will introduce framed curves and present results about variational problems defined over them. After setting the ground work, we go on to construct differential invariants, invariant differential operators, and invariant differential forms for framed curves in two dimensions. Next, we use the machinery of the invariant variational bicomplex to derive the invariant Euler-Lagrange equations for variational problems defined over two-dimensional framed curves. We then rework our analysis in a second set of coordinates in order to highlight how the procedure changes and to provide a second way of working with framed curves. The coordinate system that we work with in the latter half of chapter six will serve as a foundation for extending our work to three dimensions. Finally, we will conclude chapter six with example problems that show the computational benefit of using the invariant Euler-Lagrange equations rather than their classical counterpart.

Many of the results from the latter half of chapter six will have direct counterparts in the analysis that we carry out for three-dimensional framed curves in chapter seven. Yet, there will be subtle differences that must be addressed in order to calculate certain quantities. The computations needed for deriving the invariant Euler-Lagrange equations for three-dimensional framed curves are cumbersome, and as such, they are primarily carried out using code written in the Python programming language. All of the code and work required to produce the invariant Euler-Lagrange equations for both two and three-dimensional framed curves can be found in the IPython notebooks on the following webpage `https://github.com/broom010/Lie_Symmetry`.

Formulas from chapter seven provide a template for studying invariant variational problems for general framed curves in three dimensions, and we adapt the various formulas to restricted framed curves in chapter eight. An explicit derivation of the invariant Euler-Lagrange equations for restricted framed curves will be given for Frenet-Serret framed curves in two and three dimensions. Interestingly, the invariant Euler-Lagrange equations in this case match the results given for plane and space curves ([KO03]). The final example will provide the invariant Euler-Lagrange equations for framed curves with the restriction that the first frame vector is the unit tangent to the curve.

Although the analysis from chapters six, seven, and eight are interesting in their own right, our true motivation for determining the invariant Euler-Lagrange equations is their possible utility in applications. A possible application of our work lies in studying the global geometry of DNA; in general it is quite difficult to determine the global geometry of a long sequence of base pairs. In [Gra16], Grandchamp studied this geometry using multi-scale modelling in which the double helix structure of DNA is treated as two framed curves with the mechanical properties of a hyper-elastic rod. The likely equilibrium states of the double rod structure are determined by minimizing a functional defined over the framed curves. The specific functional used in Grandchamp's work is $SE(3)$-invariant, which means the equations that we obtain in chapters seven and eight can be applied to his work. It is our hope that the work presented here may provide computational benefits to a plethora of applications involving invariant variational problems defined over framed curves.

# Chapter 2

# Jets and Jet Bundles

Jet spaces play an important role in the geometric study of differential equations, and were first introduced by Ehresmann in [Ehr51].

## 2.1 Introduction

**Definition 2.1.1.** Let $X$, $U$ be smooth manifolds and $f_1$, $f_2$ be smooth maps from $X$ to $U$. We say that $f_1$ and $f_2$ are ***n*-equivalent** (for $0 \leq n < \infty$) at $p \in X$ if

1. $f_1(p) = f_2(p)$, and
2. for every smooth map $g : U \to \mathbb{R}$ and smooth curve $\gamma : [a, b] \to X$, with $0 \in [a, b]$ and $\gamma(0) = x$, we have:

$$\left.\frac{d^r}{dt^r}\right|_{t=0} (g \circ f_1 \circ \gamma)(t) = \left.\frac{d^r}{dt^r}\right|_{t=0} (g \circ f_2 \circ \gamma)(t) \qquad \text{for all } 1 \leq r \leq n.$$

It is easily seen that $n$-equivalence defines an equivalence relation on the set of smooth maps from $X$ to $U$. We will also refer to this equivalence relation as $n$-th order contact. With this in mind, we now define the $n$-th jet of a smooth function $f : X \to U$.

**Definition 2.1.2.** The equivalence class of smooth maps from $X$ to $U$ that are $n$-equivalent to $f$ at $p$ is the called the ***n*-jet of *f***, and is denoted by $j^n_p f$. We refer to $n$ as the order of the jet $j^n_p f$.

The space of all $n$-jets at a point $p \in X$ is one of the first building blocks for the geometry of differential equations. This space is a manifold in its own right, and its dimension can be directly computed using the dimensions of $X$ and $U$.

**Definition 2.1.3.** If $X$ and $U$ are smooth manifolds of dimension $p$ and $q$ respectively, then the ***n*-th order jet space at *p*** given by

$$J^n_p(X, U) := C^\infty(X, U)/\sim$$

is a smooth manifold of dimension

$$qp^{(n)} \equiv q\binom{p+n}{n},$$

where $\sim$ denotes equivalence under $n$-th order contact.

Next we construct the jet bundle of $X \times U$.

**Definition 2.1.4.** Let $X$ and $U$ be smooth manifolds. We define the **jet bundle of $\boldsymbol{X \times U}$** as the fiber bundle given by

$$J^n(X, U) \xrightarrow{\pi_n} X$$

where

$$J^n(X, U) = \bigsqcup_{p \in X} J^n_p(X, U)$$

Local coordinates on $J^n(X, U)$ will take the form $(\boldsymbol{x}, \boldsymbol{u}^{(n)})$, where $\boldsymbol{x} = (x^1, \ldots, x^p)$, $\boldsymbol{u} = (u^1, \ldots, u^q)$, and $\boldsymbol{u}^{(n)}$ denotes the tuple of all possible derivatives of the $u^\alpha$'s with respect to the $x^i$'s. Thus, two maps agree up to $n$-th contact at a point $p$ if in local coordinates, all of their partial derivatives of order less than or equal to $n$ are the same at $p$. Given a function $f : X \to U$, one can extend or *prolong* $f$ to a section of $J^n(X, U)$, denoted by $\mathtt{pr}^{(n)} f : X \to J^n(X, U)$. We will call such a function the $n$-th prolongation of $f$, and we define it in the following way.

**Definition 2.1.5.** Let $X, U$ be smooth manifolds and $\boldsymbol{u} = f(\boldsymbol{x})$ be a smooth function, where $f : X \to U$. The ***n*-th prolongation of *f*** is the section of $J^n(X, U)$ given by

$\boldsymbol{u}^{(n)} : X \to J^n(X, U)$, where

$$u_J^\alpha = \frac{\partial}{\partial x^J} f^\alpha(x).$$

There is a natural projection from higher order jet spaces to lower order ones. This projection is denoted by $\pi_n^{k+n} : J^{k+n}(X, U) \to J^n(X, U)$, where $\pi(x, u^{(n+k)}) = (x, u^{(n)})$ is defined by truncation.

As of now, we have made sure to consider $n < \infty$. In order to deal with infinite jet spaces, we define $J_p^\infty(X, U)$ and $J^\infty(X, U)$ as inverse limits of $J_p^n(X, U)$ and $J^n(X, U)$ respectively under the projection maps $\pi_k^{k+1}$ taking $k+1$ jets to $k$ jets, i.e:

$$J^\infty(X, U) := \varprojlim_n J^n(X, U).$$

**Definition 2.1.6.** A smooth function, $F : J^n(X, U) \to \mathbb{R}$, defined on an open subset of $J^n(X, U)$. is called a **differential function**. We will call $F$ a differential function of order $n$ if it depends on $x$, $u$, and partial derivatives of $u$ with respect to $x$ up to order $n$.

Using the natural projection $\pi_n^{n+k}$ from $J^{n+k}(X, U)$ to $J^n(X, U)$ allows us to view any $n$-th order differential function as an $(n+k)$-th order differential function. However, usual convention is to view the order of a differential function as the order of the maximal derivative coordinate upon which $F$ depends.

**Remark.** As we progress, we will generally use local coordinate expressions for particular jet spaces as needed. Due to the uses of jet spaces in applications, we will refer to $X$ as a space of independent variables and $U$ as a space of dependent variables. When working locally, we will denote local coordinates on $X$ by $(x^1, \ldots, x^p)$ and local coordinates on $U$ by $(u^1, \ldots, u^q)$. In this setting, local coordinates on $J^n(X \times U)$ are given by the coordinates on $X$, $U$, and all partial derivatives of $u$ with respect to $x$ having order less than or equal to $n$.

## 2.2 Prolongation of Group Actions and Vector Fields

We build on the ideas of the previous section by next defining the prolongation of group actions and infinitesimal generators. In order to be clear about the local nature of our results, we review the definition of a local group of transformations.

**Definition 2.2.1.** A **local group of transformations** acting on a smooth manifold $M$ is a triple $(G, U, \Phi)$, where $G$ is a local Lie group [Lee13, p. 532], $U \subset G \times M$ is an open set containing $\{e\} \times M$, and $\Phi : U \to M$ is a smooth map satisfying the following properties:

i. If $(g, x)$, $(g, \Phi(h, x))$, and $(g \cdot h, x)$ are in $U$, then

$$\Phi(g, \Phi(h, x)) = \Phi(g \cdot h, x).$$

ii. The map $\Phi(e, \cdot) : M \to M$ is the identity map on $M$.

iii. If $(g, x) \in U$, then $(g^{-1}, \Phi(g, x)) \in U$ and

$$\Phi(g^{-1}, \Phi(g, x)) = x.$$

In the previous definition $U$ is called the domain of definition of the group action. Throughout this paper, we will denote $\Phi(g, x)$ by $g \cdot x$ when this expression is defined, and we will often refer to $(G, U, \Phi)$ simply as $G$. If $U = G \times M$, $(G, M, \Phi)$ is called a global group of transformations. As noted in the introduction, we will use the term Lie group rather than local group of transformations. This convention will allow us to trim certain statements, and focus on the results and major themes.

**Definition 2.2.2.** Let $G$ be a (local) Lie group acting on $X \times U$. The **prolonged action of $\boldsymbol{G}$** to $J^n(X, U)$ is defined by:

$$g^{(n)}(\boldsymbol{x}, \boldsymbol{u}^{(n)}) = \mathtt{pr}^{(n)}(g \cdot f)(\boldsymbol{x}).$$

Where $f$ is chosen to be a smooth function such that $\mathtt{pr}^{(n)} f(\boldsymbol{x}) = \boldsymbol{u}^{(n)}$, for instance $f(\boldsymbol{x})$ can be taken to be the $n$-th Taylor polynomial whose derivatives match with $\boldsymbol{u}^{(n)}$. We will refer to the prolonged action of $G$ on $J^n(X, U)$ as a $G^{(n)}$-action.

If a Lie group $G$ acts on a manifold $M$, we say that a function $f : M \to \mathbb{R}$ is invariant under the action of $G$ if $f(g \cdot x) = f(x)$ for all $x \in M$. Typically, we will refer to $f$ as an **invariant** of $G$. An **$\boldsymbol{n}$-th order differential invariant** is a differential function $f : J^n(X, U) \to \mathbb{R}$ that is invariant under the prolonged action of $G$ on $J^n(X, U)$ for all $g \in G$ and all $(x, u^{(n)})$ for which $g^{(n)} \cdot (x, u^{(n)})$ is defined. Next, we define the prolongation of infinitesimal generators.

**Definition 2.2.3.** Let $V$ be an open subset of $X \times U$, $\boldsymbol{v}$ be a smooth vector field on $V$, and $exp(t\boldsymbol{v})$ be the corresponding one-parameter group of diffeomorphisms generated by $\boldsymbol{v}$. The ***n*-th order prolongation of $\boldsymbol{v}$** to $J^n(X \times U)$ is the infinitesimal generator of the prolonged one-parameter group $\mathtt{pr}^{(n)}[\exp(t\boldsymbol{v})]$, i.e.

$$\mathtt{pr}^{(n)}\boldsymbol{v}\big|_{(x,u^{(n)})} = \frac{d}{dt}\bigg|_{t=0} \mathtt{pr}^{(n)}[\exp(t\boldsymbol{v})](x, u^{(n)}).$$

The following corollary gives a linear criterion for determining whether a differential function is invariant.

**Proposition 2.2.4.** *Let $G$ be a connected group of transformations acting on $X \times U$. A function $I : J^n(X, U) \to \mathbb{R}$ is an $n$-th order differential invariant of the action if and only if*

$$\boldsymbol{pr}^{(n)}(\boldsymbol{v})[I] = 0 \qquad \textit{for every infinitesimal generator } \boldsymbol{v}.$$

Due to its use later, we now develop a method for finding the explicit formula for the prolongation of a vector field.

**Definition 2.2.5.** Let $F(\boldsymbol{x}, \boldsymbol{u}^{(n)})$ be a smooth function from $J^n(X, U)$ to $\mathbb{R}$, defined on an open subset $V \subset J^n(X, U)$. Let $V'$ be the subset of $J^{n+1}(X, U)$ such that $V' = \pi^{-1}(V)$, where $\pi : J^{n+1}(X, U) \to J^n(X, U)$ is the natural projection between these jet spaces. Then the ***i*-th total derivative of $\boldsymbol{F}$** is the unique smooth function defined on $V'$ such that

$$D_i F(x, \mathtt{pr}^{(n+1)} f(x)) = \frac{\partial}{\partial x^i}[F(x, \mathtt{pr}^{(n)} f(x))]$$

for any smooth function $f(\boldsymbol{x}) : X \to U$.

**Proposition 2.2.6.** *Let $F(\boldsymbol{x}, \boldsymbol{u}^{(n)})$ be a smooth function from $J^n(X, U)$ to $\mathbb{R}$, defined on an open subset $V \subset J^n(X, U)$. Then the $i$-th total derivative of $F$ is explicitly given by:*

$$D_i F = \frac{\partial F}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{J} u^{\alpha}_{J,i} \frac{\partial P}{\partial u^{\alpha}_{J}}, \qquad |J| \leq n,$$

*where,*

$$u^\alpha_{J,i} = \frac{\partial u^\alpha_J}{\partial x^i}, \qquad \textit{for each multi-index } J = (j_1, \ldots, j_k).$$

Now we give a formula for computing prolongations of vector fields on $X \times U$.

**Theorem 2.2.7.** *[Olv93, Theorem 2.3.6] Let*

$$\boldsymbol{v} = \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi_\alpha(x,u)\frac{\partial}{\partial u^\alpha}$$

*be a vector field on an open subset $V \subset X \times U$. The $n$-th prolongation of $\boldsymbol{v}$ is given by*

$$\boldsymbol{pr}^{(n)}\boldsymbol{v} = \boldsymbol{v} + \sum_{\alpha=1}^{q}\sum_{J} \phi^J_\alpha(x,u^{(n)})\frac{\partial}{\partial u^\alpha_J}, \qquad |J| \leq n \tag{2.1}$$

*where*

$$\phi^J_\alpha = D_J\left(\phi_\alpha - \sum_{i=1}^{p}\xi^i u^\alpha_i\right) + \sum_{i=1}^{p}\xi^i u^\alpha_{J,i}. \tag{2.2}$$

The following example illustrates this process.

**Example 1.** Let $v = -u\dfrac{\partial}{\partial x} + x\dfrac{\partial}{\partial u}$ be a vector field on $X \times U$, where $X \cong \mathbb{R}$, $U \cong \mathbb{R}^2$, and we take $x$ to be the independent variable and $y, u$ to be the dependent variables. Then

$$\mathtt{pr}^{(3)}v = -u\frac{\partial}{\partial x} + x\frac{\partial}{\partial u} + \phi^x\frac{\partial}{\partial u_x} + \phi^{xx}\frac{\partial}{\partial u_{xx}} + \phi^{xxx}\frac{\partial}{\partial u_{xxx}},$$

where

$$\begin{aligned}
\phi^x &= D_x(x + uu_x) - uu_{xx} = 1 + u_x^2,\\
\phi^{xx} &= D_{xx}(x + uu_x) - uu_{xxx} = 3u_xu_{xx},\\
\phi^{xxx} &= D_{xxx}(x + uu_x) - uu_{xxxx} = 3u_{xx}^2 + 4u_xu_{xxx}.
\end{aligned}$$

Therefore

$$\mathtt{pr}^{(3)}v = -u\frac{\partial}{\partial x} + u\frac{\partial}{\partial x} + (1 + u_x^2)\frac{\partial}{\partial u_x} + (3u_xu_{xx})\frac{\partial}{\partial u_{xx}} + (3u_{xx}^2 + 4u_xu_{xxx})\frac{\partial}{\partial u_{xxx}}.$$

This example corresponds to the prolongation for the infinitesimal generator of an action of $SO(2)$ on $\mathbb{R} \times \mathbb{R}$. Expressions like the above will show up when we consider the prolonged infinitesimal generators of $SE(3)$ acting on framed curves.

Next let us briefly consider a result that will give us a theoretical foothold on how to generate a set of functionally independent differential invariants. The following result we take from [Olv95].

**Theorem 2.2.8.** *Let $G$ be a Lie group acting transitively without pseudo-stabilization* (3.1.5) *on a space $X \times U \cong \mathbb{R} \times \mathbb{R}^q$. Then there are $q+1$ fundamental independent differential invariants $I, J_1, \ldots, J_q$, meaning that, locally every differential invariant of $G$ can be written as a function of these invariants and certain derivatives of them. These derivatives are given by $\mathscr{D}^k J_\nu$ where $\mathscr{D} = (D_x I)^{-1} D_x$.*

Before we move on, it is necessary to say a word about the operators described at the end of the previous theorem. These operators are called invariant differential operators, and such operators can be viewed as dual vector fields to horizontal contact invariant coframes. We will discuss these operators in more details when we discuss the variational bicomplex in chatper 5. In light of this, for this we take the viewpoint that these operators are differential operators that take $n$-th order differential invariants to $(n+1)$-st order differential invariants.

# Chapter 3

# Moving Frames

## 3.1 Actions, Orbits, and Isotropy Groups

**Definition 3.1.1.** Let $G$ be a local group of transformations on a smooth manifold $M$. An **orbit of $G$** is a nonempty set $\mathcal{O}$ satisfying the following conditions:

i. If $x \in \mathcal{O}$ and $g \cdot x$ is defined, then $g \cdot x \in \mathcal{O}$.

ii. $\mathcal{O}$ is minimal in the sense that if $\mathcal{O}' \subset \mathcal{O}$ satisfies condition i., then $\mathcal{O}' = \varnothing$ or $\mathcal{O}' = \mathcal{O}$.

We denote the orbit of $G$ containing $x$ by $\mathcal{O}_x$. If $G$ is a local group of transformations, $\mathcal{O}_x$ is realized as the following set

$$\mathcal{O}_x = \{g_1 \cdot \cdots \cdot g_n \cdot x \mid n \geq 1, g_i \in G, \text{ and } g_1 \cdot \cdots \cdot g_n \cdot x \text{ is defined}\}.$$

If $G$ is a Lie group, then orbits are defined in familiar way as

$$\mathcal{O} = \{g \cdot x \mid g \in G\}.$$

**Definition 3.1.2.** Let $G$ be a Lie group acting on a smooth manifold $M$. Given $x \in M$, the **isotropy subgroup of $x$** is defined by

$$G_x = \{g \mid g \cdot x = x\}.$$

**Definition 3.1.3.** Let $G$ be a Lie group acting on a smooth manifold $M$. Given a subset $S \subset M$, the **isotropy subgroup of $S$** is defined by

$$G_S = \{g \mid g \cdot S = S\}.$$

The **global isotropy subgroup of $S$** is defined by

$$G_S^* = \{g \mid g \cdot x = x \text{ for all } x \in S\}.$$

Having defined orbits and isotropy groups, we now distinguish between different types of actions.

**Definition 3.1.4.** Let $G$ be a local group on transformations acting on $M$, then:

i. The action of G is called **transitive** if there is only one orbit, i.e. $\mathcal{O}_x = M$.

ii. The action of $G$ is called **semi-regular** if all orbits $\mathcal{O}$ have the same dimension as submanifolds of $M$.

iii. The action of $G$ is called **regular** if the action is semi-regular, and for each $x \in M$, there is an arbitrarily small neighborhood whose intersection with each orbit is a pathwise connected subset.

iv. The action of $G$ is called **free** if $G_x = \{e\}$ for all $x \in M$.

v. The action of $G$ is called **locally free** if $G_x$ is a discrete subgroup for all $x \in M$.

Throughout our discussion, we will only consider regular actions and we will denote the dimension of the orbits of the action of $G^{(n)}$ by $s_n$. Then the sequence $\{s_n\}_{n=1}^{\infty}$ is bounded above by $r$, and must reach a maximal value. This value is called the **stable orbit dimension**, and the order of the jet space at which this happens is called the **order of stabilization**.

**Definition 3.1.5.** Let $G$ be a Lie group acting semi-regularly on a manifold $M$. We say that the action of $G$ **pseudo-stabilizes** at order $k$ if the dimension of the orbits of $G^{(k)}$ satisfy $s_k = s_{k+1} < s_{k+2}$.

Theorem 5.37 in [Olv95] generalizes a theorem of Ovsinnikov [Ovs82], and it states that if $s_k = s_{k+1}$ and $s_n = s_{n+1}$ for some $n > k$, then $s_n = s_m$ for all $m \geq n$. This shows that pseudo-stabilization can occur at most once. Throughout our discussion, we will only consider Lie group actions that are transitive, locally free, and regular with the assumption that the action does not pseudo-stabilize. Details for the intransitive case and the inclusion of pseudo-stabilization can be found in [FO99, Olv95].

## 3.2 Moving Frames

The theory of moving frames is a powerful tool for studying the equivalence problems, finding differential invariants, and developing invariant algorithms. In this section, we study a modern approach to moving frames as developed in [FO99].

**Definition 3.2.1.** Given a Lie group, $G$, acting on a manifold, $M$, a **moving frame** on $M$ with respect to $G$ is a right $G$-equivariant map from $M$ to $G$, i.e. $\rho : M \to G$ is a right moving frame if and only if

$$\rho(g \cdot z) = \rho(z) \cdot g^{-1}.$$

The following theorem show that existence of moving frames is closely related to the type of action that $G$ has on $M$.

**Theorem 3.2.2.** *Let $G$ be a Lie group acting on a smooth manifold $M$. A (local) moving frame exists in a neighborhood of a point $z \in M$ if and only if the $G$ action is* locally free *and* regular *near $z$. If the action of $G$ is regular, $\rho$ is uniquely defined by the cross-section.*

In the above theorem, the term local moving frame refers to the fact that the moving frame is a map from $M$ to a neighborhood of the identity in $G$.

The method developed in [FO99] for constructing a moving frame relies on a local coordinate cross-section to a prolonged group action. There are many such cross-sections that one can define, but in practice, a cross-section is usually chosen in a way that reduces computation as much as possible.

**Definition 3.2.3.** Let $G$ be a Lie group acting on an $m$-dimensional smooth manifold $M$ and with $r$-dimensional orbits. A **(local) cross-section** to the group orbits is an

$(m-r)$-dimensional submanifold $\mathcal{K} \subset M$ that intersects each orbit transversally. A (local) cross-section, $\mathcal{K}$, is called **regular** if in addition, $\mathcal{K}$ intersects each orbit in at most one point.

If $G$ is a Lie group that acts regularly and locally freely on $M$, then we can construct a moving frame point-wise using the following process:

1. Given $z \in M$, define a regular local cross-section, $\mathcal{K}$, in a neighborhood of $z$.
2. Define $k := \pi(z)$ to be the unique element contained in $\mathcal{K} \cap \mathcal{O}_z$.
3. By the local freeness of the action, there exist a unique $g$ in a neighborhood of the identity in $G$ such that $g \cdot z = k$.
4. Define $\rho(z)$ by setting $\rho(z) = g$.

Using the ideas from the construction presented above, we now prove theorem 3.2.2.

*Proof.* ($\Rightarrow$) Suppose $\rho : M \to G$ is equivariant, let $z \in M$, and $g \in G_z := \{h \in G : h \cdot z = z\}$. Using the equivariance of $\rho$, we have the following condition locally:

$$\rho(z) = \rho(gz) = \rho(z) \cdot g^{-1}$$

This shows that $g = e$ and therefore $G_z = \{e\}$ for all $z \in M$ i.e. $G$ acts locally freely on $M$. To prove regularity, we consider an arbitrary $z \in M$ and a sequence $z_n = g_n \cdot z \in \mathcal{O}_z$ converging to $z$. Using continuity and equivariance, we have

$$\rho(z) = \lim_{n\to\infty} \rho(z_n) = \lim_{n\to\infty} \rho(g_n \cdot z) = \rho(z) \cdot \lim_{n\to\infty} g_n^{-1}.$$

Therefore,

$$\lim_{n\to\infty} g_n^{-1} = e.$$

Thus the action of $G$ is regular.

($\Leftarrow$) Given a Lie group $G$ acting freely and regularly on $M$, there exists a foliation of $M$ and a one-dimensional submanifold $\mathcal{K}$ intersecting each leaf in exactly one point. Given a $z \in M$, we consider the map $\pi : M \to \mathcal{K}$ to be the canonical projection along the leaf containing $z$. Using the local freeness of $G$, we define $\rho(z) = g$, where $g$ is the element of $G$ such that $g \cdot z = \pi(z)$. To show that $\rho$ is right invariant, we begin by letting $z \in M$ and $h \in G$ and label $z' = hz$. Next take $g'$ to be the element of $G$ such that $g'z' = \pi(z)$. Since $gh^{-1}z' = gh^{-1}(hz) = gz = \pi(z)$, it follows that $g' = gh^{-1}$. Using this, we obtain:

$$\rho(h \cdot z) = \rho(z') = g' = g \cdot h^{-1} = \rho(z)h^{-1}$$

Thus $\rho$ is right equivariant. $\square$

Given a Lie group $G$ acting smoothly on a manifold $M$, the orbits of the prolonged action on $J^n(M)$ is bounded by the dimension of $G$. In many cases, the prolonged action of $G$ becomes free and regular on an open dense set $\mathcal{V}^n \subset J^n(M)$ for $n >> 0$. The set $\mathcal{V}^n$ for which this happens is called the set of *regular jets*. Throughout our work, we will not explicitly reference $\mathcal{V}^n$, but one should be aware that our analysis and work will only apply to regular jets. For more technical details about this point, see [Olv00].

Next we introduce the notion of invariantization. According to Theorem 3.2.2, a function $I : J^n \to \mathbb{R}$ is a differential invariant if and only if it is constant on the orbits of the prolonged group action. Since locally, each orbit meets a cross-section at a unique point, the value of an invariant can be determined by its value on the cross-section. This leads to the process known as invariantization.

**Definition 3.2.4.** The **invariantization** of a function $F : M \to \mathbb{R}$ with respect to a cross-section $K$ is the unique invariant function, $I = \iota(F)$, that coincides with $F$ on the cross-section, i.e. $I\mid_K = F\mid_K$.

**Proposition 3.2.5.** *If $\rho$ is a moving frame with respect to a cross section, then the invariantization of a differential function with respect to this moving frame is given by the formula*

$$\iota(F(\boldsymbol{x}, \boldsymbol{u}^{(n)})) = F(\rho(\boldsymbol{x}, \boldsymbol{u}^{(n)}) \cdot (\boldsymbol{x}, \boldsymbol{u}^{(n)})). \tag{3.1}$$

In particular, every $n$-th order differential invariant can be locally expressed as a function of the fundamental $n$-th order differential invariants.

**Corollary 3.2.6.** *Let $G$ be a Lie group with a transitive, locally free, and regular prolonged action on $X \times U$ that does not pseudo-stabilize, where* $\dim(X) = \dim(U) = 1$. *Let* *the prolonged G-orbits of $J^k$ have constant dimension $r$ for $k \geq r-1$, and let $K \subset J^k$ be a cross-section to the orbits given by the following equations:*

$$x = c_1, \qquad u = c_2, \qquad \ldots, \qquad u_{(r-1)x} = c_{r-1}.$$

*Then the following is a complete list of functionally independent differential invariants on $J^k$, $(k > r-1)$:*

*i.* $H_0(x, u^{(k)}) = c_1$,

*ii.* $I_i(x, u^{(k)}) = \iota(u_{ix}) = c_i$ *(for* $(1 < i \leq r-1)$*),*

*iii.* $I_j(x, u^{(k)}) = \iota(u_{jx})$ *(for* $r-1 < j \leq k$*),*

*where* $u_{ix} = \dfrac{d^i u}{dx^i}$ *and* $u_{jx} = \dfrac{d^j u}{dx^j}$.

The invariants given in *i.* and *ii.* are called phantom invariants because they are constant. The full list of invariants described in the corollary above are referred to as normalized invariants. A moving frame can be used to easily compile a list of normalized invariants by applying (3.1) to various jet coordinates. However, it is important to note that invariantization does not commute with differentiation.

The recurrence relations developed in [FO99] provide the key to efficiently relating the normalized invariants to a second list of differential invariants, called the curvature invariants, which arise from invariant arc length differentiation. The recurrence relations are vital to understanding the algebra of differential invariants and invariant differential operators. With them, one can find a minimal set of generating invariants and determine the syzygies between differential invariants. While discussing these topics would be interesting, a full consideration is outside the intent of this paper.

The invariant arc length differentiation that was alluded to above is also constructed from geometric objects that arise in the variational bicomplex of $J^n(X, U)$. For our purposes, we will view them as operators $\mathcal{D}_i = JD_i$, where $J$ is a certain relative differential invariant ([Olv95, p. 149]). The recurrence relations give the correction factors that relate $\mathcal{D}_i \circ \iota$ and $\iota \circ D_i$. Specifically, if $F(\boldsymbol{x}, \boldsymbol{u}^{(n)})$ is a differential function and $\iota(F)$, its moving frame invariantization, then the recurrence relations are

$$\mathcal{D}_i[\iota(F)] = \iota[D_i(F)] + \sum_{\ell=1}^{r} R_i^\ell \, \iota[\mathtt{pr}\, \boldsymbol{v}_\ell(F)], \tag{3.2}$$

where $\{v_\ell\}_{\ell=1}^r$ is a basis for the infinitesimal generators of the corresponding group action, and $R_i^\ell$ are certain differential invariants. We refer to the $R_i^\ell$'s as the Maurer-Cartan invariants.

The Maurer-Cartan invariants can be found through directly computing the coefficients of the pullbacks of the Maurer-Cartan forms from $G$ to $J^n(X, U)$ via the moving frame. However, this is process can be greatly simplified by observing that the Maurer-Cartan invariants can be directly solved for by applying the recurrence relations to the phantom invariants. In fact, one can solve for the Maurer-Cartan invariants without ever finding an explicit moving frame.

The following example illustrates the above mentioned process.

**Example 2.** Let $G$ be the three parameter Lie group acting on $\mathbb{R}^2$ via

$$(\lambda, a, b) \cdot (x, u) = (\lambda x + a, \lambda^{-1} u + b) := (y, w),$$

where $y$, and $v$ denote the transformed variables. The following is a basis for the infinitesimal generators of this action:

$$\boldsymbol{v}_1 = \frac{\partial}{\partial x}, \qquad \boldsymbol{v}_2 = \frac{\partial}{\partial u}, \qquad \boldsymbol{v}_3 = x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}.$$

Prolonging these vector fields, we have

$$\begin{aligned}
\mathtt{pr}\, \boldsymbol{v}_1 &= \frac{\partial}{\partial x}, \\
\mathtt{pr}\, \boldsymbol{v}_2 &= \frac{\partial}{\partial u}, \\
\mathtt{pr}\, \boldsymbol{v}_3 &= x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u} - 2u_x\frac{\partial}{\partial u_x} - 3u_{xx}\frac{\partial}{\partial u_{xx}} - \cdots .
\end{aligned}$$

For convenience, we will set our cross-section to be $\mathcal{K} = \{x = 0, u = 0, u_x = 1\}$ with $u_x > 0$. It should be noted that using $u_x = 0$ defines a singular subset, hence the reason for choosing $u_x = 1$. The implicit differentiation operator $d/dy$ is given by

$$\frac{d}{dy} = \lambda^{-1}\frac{d}{dx}.$$

Thus

$$D_y^n = \lambda^{-n} D_x^n, \qquad \text{and} \qquad w_{ny} = \frac{u_{nx}}{\lambda^{n+1}}.$$

The normalization equation $y = 0$, $v = 0$, and $v_y = 1$ give us

$$\lambda x + a = 0, \qquad \lambda^{-1}u + b = 0, \qquad \lambda^{-2}u_x = 1.$$

Therefore our (right) moving frame is given by:

$$(\lambda, a, b) = \rho(x, u^{(n)}) := (u_x^{1/2}, -xu_x^{1/2}, -uu_x^{-1/2}).$$

Using this moving frame, we define the first non-trivial invariant by

$$\kappa = \iota(u_{xx}) = w_{yy} = \frac{u_{xx}}{u_x^{3/2}}.$$

The invariant arc length derivative is given by $\mathcal{D} = \iota(D_x) = u_x^{-1/2}D_x$. The recurrence relations for this example take on the following form:

$$\mathcal{D}_x[\iota(F)] = \iota[D_x(F)] + R^1\iota(\texttt{pr}\,\boldsymbol{v}_1(F)) + R^2\iota(\texttt{pr}\,v_2(F)) + R^3\iota(\texttt{pr}\,v_3(F)).$$

To solve for the Maurer-Cartan invariants $R^1$, $R^2$, and $R^3$, we apply this formula to the phantom invariants and use the fact that $\mathcal{D}(\iota(x)) = \mathcal{D}(\iota(u)) = \mathcal{D}(\iota(u_x)) = 0$. This is summarized by the equation:

$$\begin{aligned}
0 &= \iota(1) + R^1\iota(\texttt{pr}\,\boldsymbol{v}_1(x)) + R^2\iota(\texttt{pr}\,v_2(x)) + R^3\iota(\texttt{pr}\,v_3(x)),\\
0 &= \iota(u_x) + R^1\iota(\texttt{pr}\,\boldsymbol{v}_1(u)) + R^2\iota(\texttt{pr}\,v_2(u)) + R^3\iota(\texttt{pr}\,v_3(u)),\\
0 &= \iota(u_{xx}) + R^1\iota(\texttt{pr}\,\boldsymbol{v}_1(u_x)) + R^2\iota(\texttt{pr}\,v_2(u_x)) + R^3\iota(\texttt{pr}\,v_3(u_x)).
\end{aligned}$$

Using the fact that $\iota(x) = \iota(u) = 0$, $\iota(1) = \iota(u_x) = 1$, and $\iota(u_{xx}) = \kappa$, we get

$$\begin{aligned} 0 &= \iota(1) + R^1, \\ 0 &= \iota(u_x) + R^2, \\ 0 &= \iota(u_{xx}) - 2\kappa R^3. \end{aligned}$$

Solving for $R^i$, we have:

$$R^1 = -1, \qquad R^2 = -1, \qquad R^3 = \frac{1}{2}\kappa.$$

Therefore

$$\mathcal{D}_x[\iota(F)] = \iota[D_x(F)] - \iota(\mathtt{pr}\,\boldsymbol{v}_1(F)) - \iota(\mathtt{pr}\,v_2(F)) + \frac{\kappa}{2}\iota(\mathtt{pr}\,v_3(F)),$$

where $F$ is an arbitrary differential function.

With the recurrence relations in hand, knowledge of the invariant arc length derivative is not necessary in order to apply it. Focusing on the normalized invariants, we can take $I_2$ and $I_3$ to be fundamental invariants and $\mathscr{D}_x = (D_x I_2)^{-1} D_x$ to be the fundamental invariant differential operator. If we work with the curvature invariants, we can take $\kappa$ to be a fundamental differential invariant and $\mathcal{D}_s = u_x^{-1/2} D_x$ to be the invariant differential operator. The recurrence relations then give the relations between these sets of invariants.

# Chapter 4

# The Invariant Variational Bicomplex

## 4.1 Variational Bicomplex

A necessary ingredient for our study of variational symmetries is the variational bicomplex (for details see [Vin84a, Vin84b, And92]). This tool was developed to handle variational problems on manifolds in a geometric setting. Eventually, we will use the structure of the variational bicomplex along with the theory of moving frames to introduce an invariant complex that will serve to give us a similar geometric grounding for invariant variational problems. For the full details of the invariant variational bicomplex see [KO01, KO03].

Before we begin, it is important to note that we will focus our discussion of the invariant variational bicomplex to the case in which there is only a single independent variable. This will simplify many of the formulas found in [KO03]. To begin our journey let us define the horizontal and contact subbundles of $TJ^\infty(X,U)$ and $T^*J^\infty(X,U)$.

### 4.1.1 Contact and Horizontal Forms

Differential forms on $J^\infty(M)$ naturally split into two flavors, the contact forms and horizontal forms. We begin with a definition of contact forms and give a generating set for all contact forms.

**Definition 4.1.1.** A contact one-form on $J^\infty(M)$ is a one-form, $\theta$, such that

$$(f^{(n)})^*(\theta) = \theta|_{j^\infty f(p)} = 0$$

for all smooth functions $f : X \to U$ and all $p \in X$.

The subbundle of $T^*J^\infty(M)$ spanned by contact one-forms is called the vertical or contact subbundle, and it is denoted by $\Omega^1_V(J^\infty)$. The horizontal subbundle of $T^*J^\infty(M)$ is the span of the coordinate one-forms $dx^i$, $1 \le i \le p$. We denote this subbundle by $\Omega^1_H(J^\infty)$.

Working out expressions in coordinates, we find that any contact one-form can be written as a linear combination of the following basic contact forms:

$$\theta^\alpha_J = du^\alpha_J - \sum_{i=1}^{p} u^\alpha_{J,i} dx^i, \qquad \alpha = 1, \dots, q, \qquad 0 \le |J|, \tag{4.1}$$

where $u^\alpha_{J,i} = \partial u^\alpha_J / \partial x^i$ as before.

The expression in (4.1) can be viewed, in some sense, as subtracting away terms involving horizontal one-forms from $du^\alpha_J$, i.e. removing any $dx^i$'s. With this in mind, we see that $T^*J^\infty(M)$ naturally splits as

$$T^*J^\infty(M) = \Omega^1_H(J^\infty) \oplus \Omega^1_V(J^\infty).$$

Thus any $\omega \in T^*J^\infty(M)$ has a unique decomposition

$$\omega = \pi_V(\omega) + \pi_H(\omega),$$

where $\pi_V : T^*J^\infty(M) \to \Omega^1_V(J^\infty)$, and $\pi_H : T^*J^\infty(M) \to \Omega^1_H(J^\infty)$. Note that this is not true for finite order jet bundles due to the fact that decomposing $\omega \in T^*J^n(M)$ results in $\pi_H(\omega) \in T^*J^{n+1}(M)$.

This splitting of differential forms into horizontal and contact forms induces a splitting of the exterior differentiation operator into $d = d_H + d_V$, where $d_H(\omega) = \pi_H(d(\omega))$ and $d_V(\omega) = \pi_V(d\omega)$. The fact that $d$ is a closed operator requires the following relations between $d_H$ and $d_V$:

$$d_H \circ d_H = 0, \qquad d_V \circ d_V = 0, \qquad d_H \circ d_V = -d_V \circ d_H.$$

This results in two chain complexes with an anti-commutivity relation. Given a function $F : J^\infty(M) \to \mathbb{R}$, we can calculate $d_H(F)$ and $d_V(F)$ via

$$d_H F = \sum_{i=1}^{p} D_i F dx \tag{4.2}$$

and

$$d_V F = \sum_{\alpha=1}^{q} \sum_{J} \frac{\partial F}{\partial u_J^\alpha} \theta_J^\alpha, \qquad 1 \leq |J|, \tag{4.3}$$

where $D_i$ is as previously defined.

**Definition 4.1.2.** A total differential operator on $J^\infty(M)$, is a vector field that lies in in the annihilator of $\Omega_V^1(J^\infty)$.

**Remark.** The differential operators $D_i$ for $1 \leq i \leq p$ form a basis for the vector space of total differential operators. That is, every total differential operator is of the form:

$$\mathcal{D} = \sum_{i=1}^{p} \Phi^i(x, u^{(n)}) D_i$$

for smooth functions $\Phi^1, \ldots, \Phi^p$.

### 4.1.2 Frames, Coframes, and Total Differential Operators

We define the *horizontal tangent space* of $J^\infty(M)$ at $p$ to be the space spanned by the operators $D_i$ for $1 \leq i \leq p$, and we define the *vertical tangent space* of $J^\infty(M)$ at $p$ to be the span of

$$\left.\frac{\partial}{\partial u_J^\alpha}\right|_p \qquad \text{for} \qquad 1 \leq i \leq p.$$

The horizontal and vertical tangent bundles of $J^\infty(M)$ are defined in the obvious way, and are denoted by $T_H J^\infty(M)$ and $T_V J^\infty(M)$, giving the decomposition

$$TJ^\infty(M) = T_H J^\infty(M) \oplus T_V J^\infty(M).$$

Horizontal and vertical vector fields are taken to be continuous sections of $T_H J^\infty(M)$ and $T_V J^\infty(M)$, and we denote the sets of all such smooth sections by: $\mathfrak{X}_H$ and $\mathfrak{X}_V$ respectively. A horizontal frame for $J^\infty(M)$ is a collection of $p$ horizontal vector fields that are linearly independent at each $p \in M$.

**Definition 4.1.3.** A horizontal coframe for $J^\infty(M)$ is a collection of $p$ horizontal one-forms, $\omega^1, \ldots, \omega^p$, that are linearly independent, i.e. $\omega^1 \wedge \cdots \wedge \omega^p \neq 0$ on every open subset of $J^\infty(M)$.

Given a frame or coframe, we construct the dual coframe or frame by imposing the requirement that

$$\langle \omega^i ; v_j \rangle = \delta_i^j .$$

Concretely, if we are given a horizontal coframe

$$\omega^i = \sum_{j=1}^{p} P_j^i(d, u^{(n)}) dx^i, \qquad 1 \leq i \leq p,$$

then the corresponding dual frame is:

$$\mathcal{D}_j = \sum_{i=1}^{p} Q_j^i(x, u^{(n)}) D_i, \qquad 1 \leq j \leq p, \tag{4.4}$$

where

$$Q_j^i = (P_i^j)^{-1}.$$

The operators above satisfy the formula

$$d_H\, F = \sum_{j=1}^{p} (\mathcal{D}_j F) \omega^j. \tag{4.5}$$

Adapting this formula to a contact one-forms, $\theta$, we have

$$d_H\, \theta = \sum_{j=1}^{p} \omega^j \wedge \mathcal{D}_j \theta. \tag{4.6}$$

The splitting of $T^* J^\infty(M)$ and the decomposition of exterior differentiation induces a bi-grading on the space of differential forms on $J^\infty(M)$. The resulting structure is known as the variational bicomplex, and it plays a role in the theory of calculus of variations. Our next goal is to combine the ideas developed in this section with the theory of moving frames.

## 4.2 The Invariant Variational Complex

We now turn our attention to using the theory of moving frames to build an invariant version of the variational bicomplex. Following the regularization procedure in [KO03], we let $\pi : \mathcal{B} = G \times M \to M$ be the trivial right principal $G$ bundle over $M$, and define the lifted action of $G$ as

$$g \cdot (h, z) = (h \cdot g^{-1}, g \cdot z).$$

Extending the regularization construction to $\pi : \mathcal{B}^{(\infty)} = G \times J^{\infty}(M) \to J^{\infty}(M)$, the lifted action is $w : \mathcal{B}^{\infty} \to J^{\infty}(M)$, given by:

$$w(g, z^{(\infty)}) = g \cdot (h, z^{(\infty)}) = (h \cdot g^{-1}, g^{(\infty)} \cdot z^{(\infty)}).$$

The projection map along with this lifted action produces a double fibration of $B^{\infty}$ over $J^{\infty}$. Using a moving frame $\rho : J^{\infty}(M) \to G$, we can define a $G$-equivariant section of the projection map by setting $\sigma(z^{(\infty)}) = (\rho(z^{(\infty)}), z^{(\infty)})$. The section $\sigma : J^{\infty} \to \mathcal{B}^{\infty}$ will serve a similar role to $\rho : J^{\infty}(M) \to G$ in defining an invariantization map.

### 4.2.1 Invariantization of Differential Functions

**Definition 4.2.1.** A smooth locally defined function $F : V \subset \mathcal{B}^{\infty} \to \mathbb{R}$ is called a lifted differential invariant if it is invariant with respect to the lifted $G$-action.

Using a moving frame section $\sigma : J^{\infty}(M) \to \mathcal{B}^{\infty}$ we can define the invariantization map $\iota : C^{\infty}(J^{\infty}(M)) \to C^{\infty}(J^{\infty}(M))$ as follows:

$$\iota(F) = F \circ w \circ \sigma, \tag{4.7}$$

where $w$ is the lifted $G$ action defined above. One should note that if $I : J^{\infty} \to \mathbb{R}$ is an invariant, then $I(g \cdot z^{(\infty)}) = I(z^{(\infty)})$. Applying the invariantization $\iota$ to $I$ gives

$$\iota(I(z^{(\infty)})) = I(\rho(z^{(\infty)}) \cdot z^{(\infty)}) = I(z^{(\infty)}).$$

Thus invariantization is a projection map from $C^\infty(J^\infty)$ to the set of differential invariants.

### 4.2.2 Invariantization of Differential Forms

The space of differential one-forms on $\mathcal{B}^\infty$ can be decomposed into lifted horizontal forms, lifted contact forms, and the Maurer-Cartan forms. This decomposition can be written as:

$$\Omega^1(\mathcal{B}^\infty) = \Omega^1_H(\mathcal{B}^\infty) \oplus \Omega^1_V(\mathcal{B}^\infty) \oplus \Omega^1_G(\mathcal{B}^\infty).$$

This decomposition leads to a quasi-tricomplex on $\Omega(\mathcal{B}^\infty) = \bigoplus_k \Omega^k(\mathcal{B}^\infty)$. The details of this can be found in [KO03]. Our discussion here will pertain to the decomposition of invariant differential forms on $\mathcal{B}^\infty$.

**Definition 4.2.2.** Let $G$ be a Lie group acting on a manifold $M$. A differential form $\omega$ on a manifold $M$ is said to be invariant with respect to the $G$-action if $g^*(\omega \mid_{g\cdot x}) = \omega \mid_x$ for all $g \in G$.

Invariant differential forms on $\mathcal{B}^\infty$ are those that are invariant with respect to the lifted $G$-action.

**Definition 4.2.3.** Let $\pi_J : \Omega \to \Omega_J$ be the projection from

$$\Omega(\mathcal{B}^\infty) = \bigoplus_k \Omega^k(\mathcal{B}^\infty)$$

to

$$\Omega_J(\mathcal{B}^\infty) = \bigoplus_k \Omega^k_H(\mathcal{B}^\infty) \oplus \Omega^k_V(\mathcal{B}^\infty).$$

**Definition 4.2.4.** The invariantization of a differential form $\eta \in \Omega_J(\mathcal{B}^\infty)$ is defined to be the invariant differential form

$$\iota(\eta) = \sigma^*(\pi_J(w^*(\eta))). \tag{4.8}$$

Theorem 4.5 from [KO03] shows that the invariantization of the vertical and horizontal subbundles of $\Omega_J(\mathcal{B}^\infty)$ form an invariant coframe on the regular subset $\mathcal{V}^\infty \subset J^\infty$. We denote the space of all invariant jet forms on $\mathcal{B}^\infty$ by $\widetilde{\Omega}_J(\mathcal{B}^\infty)$. Using the bigrading on $\Omega_J(\mathcal{B}^\infty)$, we denote the space of jet forms that are the wedge product of $r$ horizontal forms and $s$ contact forms by $\Omega_J^{r,s}(\mathcal{B}^\infty)$, and we let $\pi_{r,s} : \Omega \to \Omega^{r,s}$ be the standard projection. In a similar way, we use the bigrading on $\widetilde{\Omega}_J(\mathcal{B}^\infty)$ to define $\widetilde{\Omega}_J^{r,s}(\mathcal{B}^\infty)$ and $\widetilde{\pi}_{r,s} : \widetilde{\Omega} \to \widetilde{\Omega}^{r,s}$.

With this notation, we note that the invariantization map $\iota : \Omega_J^{r,s}(\mathcal{B}^\infty) \to \widetilde{\Omega}_J^{r,s}(\mathcal{B}^\infty)$ is an exterior algebra morphism. In local coordinates, the invariantization of $dx^i$ decomposes as:

$$\iota(dx^i) := \varpi^i = \sigma^*(d_H X^i) + \sigma^*(d_V X^i),$$

where uppercase symbols indicate transformed variables. Restricting to the case of a single independent variable, $x$, yields

$$\iota(dx) = \varpi = ds + \eta, \qquad \text{where} \qquad ds = \sigma^*(d_H X), \qquad \text{and} \qquad \eta = \sigma^*(d_V X). \tag{4.9}$$

The invariantization of a basic contact forms is given by

$$\vartheta_n^\alpha = \sigma^*(d_V U_n^\alpha - U_{n+1}^\alpha d_V X). \tag{4.10}$$

### 4.2.3 Recurrence Formulas for Invariant Differentiation

When working with invariants and invariant differential forms, one quickly finds that both regular and exterior differentiation do not commute with the operation of invariantization. We previously discussed how to account for this when invariantly differentiating invariant functions. In this section, we discuss how to apply exterior differentiation to invariant differential forms. An important observation is that given any invariant differential form $\widetilde{\omega} \in \widetilde{\Omega}^{r,s}$,

$$d\widetilde{\omega} \in \widetilde{\Omega}^{r+1,s} \oplus \widetilde{\Omega}^{r,s+1} \oplus \widetilde{\Omega}^{r-1,s+2}.$$

This allows one to write a decomposition of the differential as

$$d = d_{\mathcal{H}} + d_{\mathcal{V}} + d_{\mathcal{W}},$$

where

$$\begin{aligned} d_{\mathcal{H}} &: \widetilde{\Omega}^{r,s} \to \widetilde{\Omega}^{r+1,s}, \\ d_{\mathcal{V}} &: \widetilde{\Omega}^{r,s} \to \widetilde{\Omega}^{r,s+1}, \\ d_{\mathcal{W}} &: \widetilde{\Omega}^{r,s} \to \widetilde{\Omega}^{r-1,s+2}. \end{aligned}$$

The formulas for how to apply these operators relies on a decomposition of the invariant Maurer-Cartan forms. Let $\mu^1, \ldots, \mu^r$ denote the Maurer-Cartan forms dual to the infinitesimal generators. In the case of a single independent variable, $\sigma^*\mu^\ell$ decomposes as

$$\sigma^*\mu^\ell = \nu^\ell = \gamma^\ell + \varepsilon^\ell, \quad \text{where} \quad \gamma^\ell = C^\ell \varpi, \quad \varepsilon^\ell = \sum_{\alpha,n} E_\alpha^{\ell,n}\, \vartheta_n^\alpha,$$

where $C^\ell$ and $E_\alpha^{\ell,n}$ are differential invariants which depend on $\mu^\ell$. In order to make this decomposition useful, we will let $\lambda^\ell, \eta^\ell$, and $\beta^\ell$ be differential forms such that,

$$\iota(\lambda^\ell) = \nu, \qquad \lambda^\ell = \eta^\ell + \beta^\ell, \qquad \eta^\ell = A^\ell dx, \quad \beta^\ell = \sum_{\alpha,n} = B_\alpha^{k,n}\, \theta_n^\alpha,$$

and

$$\gamma^\ell = \iota(\alpha^\beta), \qquad \varepsilon = \iota(\beta^\ell), \qquad C^\ell = \iota(A^\ell), \qquad E_\alpha^{\ell,n} = \iota(B_\alpha^{\ell,n}).$$

The one-forms $\lambda^\ell, \eta^\ell$, and $\beta^\ell$ are not uniquely defined, but in practice, one uses phantom invariants to derive one-forms that fit the relations above. Using such differential forms, we have the following

$$d_{\mathcal{H}}\, \iota(\omega) = d_{\mathcal{H}}\, \widetilde{\omega} = \iota\left( d_H\, \omega + \sum_{\ell=1}^{r} \eta^\ell \wedge \pi_{r,s}[\boldsymbol{v}_\ell(\omega)] \right), \tag{4.11}$$

$$d_{\mathcal{V}}\, \iota(\omega) = d_{\mathcal{V}}\, \widetilde{\omega} = \iota\left( d_V\, \omega + \sum_{\ell=1}^{r} \Big( \beta^\ell \wedge \pi_{r,s}[\boldsymbol{v}_\ell(\omega)] + \eta^\ell \wedge \pi_{r-1,s+1}[\boldsymbol{v}_\ell(\omega)] \Big) \right), \tag{4.12}$$

$$d_{\mathcal{W}}\,\iota(\omega) = d_{\mathcal{W}}\,\widetilde{\omega} = \iota\left(\sum_{\ell=1}^{r} \beta^{\ell} \wedge \pi_{r-1,s+1}[\boldsymbol{v}_{\ell}(\omega)]\right), \tag{4.13}$$

where $\boldsymbol{v}_{\ell}(\omega)$ denotes the Lie derivative of $\omega$ with respect to $\boldsymbol{v}_{\ell}$.

For an invariant function, $I = \iota(F)$, we have the following formulas

$$\begin{aligned} d_{\mathcal{H}}\, I &= d_{\mathcal{H}}\,\iota(F) = \iota\left(d_H\, F + \sum_{\ell=1}^{r} \boldsymbol{v}_{\ell}(F)\,\eta^{\ell}\right), \\ d_{\mathcal{V}}\, I &= d_{\mathcal{V}}\,\iota(F) = \iota\left(d_V\, F + \sum_{\ell=1}^{r} \boldsymbol{v}_{\ell}(F)\,\beta^{\ell}\right), \\ d_{\mathcal{W}}\, I &= 0. \end{aligned}$$

If our infinitesimal generators are given by

$$\boldsymbol{v}_{\ell} = \xi_{\ell}\,\frac{\partial}{\partial x} + \sum_{\alpha,n} \varphi^{\alpha}_{n,\ell}\,\frac{\partial}{\partial u^{\alpha}_{n}},$$

then applying these formulas to the fundamental differential invariants yields:

$$d_{\mathcal{H}}\, H = d_{\mathcal{H}}\,\iota(x) = \varpi + \sum_{\ell=1}^{r} \iota(\xi_{\ell})\,\gamma^{\ell}, \tag{4.14}$$

$$d_{\mathcal{H}}\, I^{\alpha}_{n} = d_{\mathcal{H}}\,\iota(u^{\alpha}_{n}) = I^{\alpha}_{n+1}\varpi + \sum_{\ell=1}^{r} \iota(\varphi^{\alpha}_{n,\ell})\,\gamma^{\ell}, \tag{4.15}$$

$$d_{\mathcal{V}}\, H = d_{\mathcal{V}}\,\iota(x) = \sum_{\ell=1}^{r} \iota(\xi_{\ell})\,\varepsilon^{\ell}, \tag{4.16}$$

$$d_{\mathcal{V}}\, I^{\alpha}_{n} = d_{\mathcal{V}}\,\iota(u^{\alpha}_{n}) = \vartheta^{\alpha}_{n} + \sum_{\ell=1}^{r} \iota(\varphi^{\alpha}_{n,\ell})\,\varepsilon^{\ell}. \tag{4.17}$$

Using the above formulas along with the general recurrence formulas, we write the recurrence formulas for the derivatives of the invariant horizontal form. These are given by

$$d_{\mathcal{H}}\,\varpi = \sum_{\ell=1}^{r} \iota(D_k \xi^i_\ell)\,\gamma^\ell \wedge \varpi^\ell, \tag{4.18}$$

$$d_{\mathcal{V}}\,\varpi = \sum_{\ell=1}^{r} \left[ \iota\left(D_x \xi_\ell\right) \varepsilon^\ell \wedge \varpi + \sum_{\alpha} \iota \left( \frac{\partial \xi_\ell}{\partial u^\alpha} \right) \gamma^\ell \wedge \vartheta^\alpha \right], \tag{4.19}$$

$$d_{\mathcal{W}}\,\varpi = \sum_{\ell=1}^{r} \sum_{\alpha} \iota \left( \frac{\partial \xi_\ell}{\partial u^\alpha} \right) \varepsilon \wedge \vartheta^\alpha. \tag{4.20}$$

Before we can express the recurrence formulas for invariant differentiation of contact forms, we will introduce the invariant differential operator $\mathcal{D}_s$. Adapting the various formulas from [KO03, p. 19] to the case of a single independent variable, we have

$$\mathbb{D}_x = D_x + \sum_{\ell=1}^{r} A^\ell \boldsymbol{v}_\ell,$$

and we define $\mathcal{D}_s$ as the invariant differential operator defined by

$$\mathcal{D}_s\,\iota(F) = \iota(\mathbb{D}_x\,F).$$

Applying $\widetilde{\pi}_{1,1}$ to formula (4.6) yields

$$d_{\mathcal{H}}\,\vartheta = \varpi \wedge \mathcal{D}_s\,\vartheta \qquad \text{for all } \vartheta \in \widetilde{\Omega}^{0,1}, \tag{4.21}$$

where $\mathcal{D}_s$ acts by Lie differentiation. Combining the above equation with (4.11) we have

$$\mathcal{D}_x\,\vartheta^\alpha_n = \vartheta^\alpha_{n+1} + \sum_{\ell=1}^{r} C^\ell \iota(\boldsymbol{v}_\ell(\theta^\alpha_n)). \tag{4.22}$$

The following formula will be used to calculate the Lie differentiation of a contact form with respect to an infinitesimal generator:

$$\boldsymbol{v}_\ell(\theta^\alpha_n) = d\varphi^\alpha_{n,\ell} - \varphi^\alpha_{n+1,\ell}\,dx - u^\alpha_{n+1}\,d\xi_\ell. \tag{4.23}$$

We conclude this chapter with a note about the *quasi-tricomplex* that arises from the decomposition of the differential, i.e. $d = d_{\mathcal{H}} + d_{\mathcal{V}} + d_{\mathcal{W}}$. The term “quasi” refers to the fact that $d_{\mathcal{V}}$ is not necessarily closed. Indeed we only have $d_{\mathcal{V}}^2 + d_{\mathcal{H}}\, d_{\mathcal{W}} + d_{\mathcal{W}}\, d_{\mathcal{H}} = 0$. For projectable group actions $d_{\mathcal{W}} = 0$, and our tri-complex becomes a legitimate invariant variational bicomplex. Yet, the edge complex of the quasi-tricomplex is always a true complex. The author finds it quite remarkable how Kogan and Olver combined the notions of the variational complex with invariantization to produce the various formulas above. In the next chapter we explore how these tools can be used to derive the invariant Euler-Lagrange equations.

# Chapter 5

# The Invariant Euler-Lagrange Equations

## 5.1 Introduction

Variational calculus is a cornerstone of many physical theories including classical mechanics, quantum field theory, and string theory. In these theories, one defines an action functional, $S[\boldsymbol{u}]$, in terms of a Lagrangian density function, $L : J^n(X,U) \to \mathbb{R}$, where often $X \subset \mathbb{R}$. This functional is usually of the form

$$S[\boldsymbol{u}] = \int_{x_1}^{x_2} L(x, \boldsymbol{u}^{(n)}(x))\, dx.$$

In classical mechanics, one takes $L$ to be the difference between the kinetic and potential energy of a system. Most physical applications only involve the first or second order jet space, i.e $L : J^1(X,U) \to \mathbb{R}$ or $L : J^2(X,U) \to \mathbb{R}$. Hamilton's principle of least action states that the equations of motion for classical systems can be derived by minimizing the action functional $S[L]$. Two reasons why this formalism has continued to play a prominent role in physics is due to the flexibility in the choosing coordinate systems and the ability to apply Noether's celebrated first theorem which links symmetries to conservation laws. In the next section, we will give a definition of a symmetry group of a variational problem of a single independent variable. For the rest of our discussion, we exclusively work with $X \subset \mathbb{R}$.

A line of reasoning is that one can experimentally test for symmetries of a physical system by verifying conservation laws. This gives a sensible process for determining symmetries of a system. However, it is often difficult to postulate an action functional, i.e. a Lagrangian density function, from first principles. Lie was the first to discover that every $G$-invariant variational problem can be rewritten in terms of the differential invariants of the $G$-action. This fact is often used by theoretical physicist to propose specific $G$-invariant action functionals for various field theories. This is but one of a myriad of reasons for studying invariant variational problems.

The possible extrema of a variational problem are found by solving the Euler-Lagrange equations. These equations can be derived using the geometric structure of the variational bicomplex. In short, the Euler-Lagrange equations are given by

$$E_\alpha(L) = 0,$$

where $E_\alpha$ are the classical Eulerian operators:

$$E_\alpha(L) = \sum_J (-D)_J \frac{\partial L}{\partial u_J^\alpha}. \tag{5.1}$$

It is show in [Olv93, Theorem 4.14] that a variational symmetry group is also a symmetry group of the associated Euler-Lagrange equations. Therefore, under certain conditions, one can rewrite the Euler-Lagrange equations in terms of the fundamental differential invariants of the group action, meaning that the equations be of the form

$$F(I_1, \ldots, I_k) = 0,$$

where $I_1, \ldots, I_k$ form a complete set of functionally independent differential invariants. The process of finding such an $F$ was not known until Olver and Kogan showed in [KO03] that of the invariant Euler-Lagrange equations can be given by

$$W\left[\mathcal{A}^*\mathcal{E}(\widetilde{L}) - \mathcal{B}^*\mathcal{H}(\widetilde{L})\right] = 0, \tag{5.2}$$

where $\widetilde{L}$ is the invariantization of the Lagrangian, $\mathcal{E}(\widetilde{\mathcal{L}})$ is the invariantized Eulerian, $\mathcal{H}(\widetilde{L})$ is a certain invariantized Hamiltonian, $\mathcal{A}^*$, $\mathcal{B}^*$ are specific invariant differential operators, and $W$ is a matrix of relative invariants. Formulas for these operators will be given in section (5.3).

## 5.2 Symmetries of a Variational Problem

Generalizing these ideas, we define a variational problem to comprise of finding the extrema of a functional

$$\mathcal{L}[u] = \int_D L(x, u^{(n)})\, dx,$$

where $D \subset X$ is an open, connected subset with smooth boundary, $f : D \to U$ is a smooth function over $D$ such that $u = f(x)$, and $L : J^n(X, U) \to \mathbb{R}$ is a smooth differential function. Additionally one may impose conditions on which class of function that $\mathcal{L}$ should be extremized over. The set of such functions is referred to as the class of admissible functions. The only restrictions that we shall consider will involve boundary conditions.

**Definition 5.2.1.** A **variational symmetry group** of the functional

$$\mathcal{L}[u] = \int_{D_0} L(x, u^{(n)})\, dx$$

is a local group of transformations, $G$, acting on $D_0 \times U$ with the property that for every $g \in G$ and every subdomain $D \subset D_0$ we have

$$g \cdot \mathcal{L}[u] := \int_{\widetilde{D}} L(\tilde{x}, \mathtt{pr}^{(n)} \tilde{f}(\tilde{x}))\, d\tilde{x} = \int_D L(x, \mathtt{pr}^{(n)} f(x))\, dx,$$

when the expressions and integrals are properly defined. Note that we have used $\widetilde{D}, \tilde{x}$, and $\tilde{f}$ to denote the corresponding transformations by $g$ in $G$. For the precise conditions see [Olv93, Def. 4.10]

A theorem due originally to Lie (see [Olv95, Theorem 7.27]) tells us that any $G$-invariant variational problem can be rewritten in terms of an invariant Lagrangian and a contact-invariant volume form. Thus, for case of a single independent variable, one finds that $\mathcal{L}[u]$ can be rewritten in the form

$$\mathcal{L}[u] = \int \widetilde{L}(I_1, \ldots, I_k)\, \varpi,$$

where $I_1, \ldots, I_k$ form a complete set of functionally independent differential invariants. The term $\varpi$ in the above equation is the invariantization of $dx$, and is referred to as the invariant horizontal one-form. The expression $\widetilde{L}\,\varpi$ is referred to as a *Lagrangian form.* Next, we will write out the explicit formulas for the invariant Euler-Lagrange equations for variational problems in a single independent variable.

## 5.3 Invariant Euler-Lagrange Equations of a Variational Problem

Using integration by parts and the various decompositions obtained from invariant variational bicomplex, one can derive the invariant Euler-Lagrange equations (5.2). In general, these equations cannot be derived by simply applying an Eulerian operator to $\widetilde{L}$. In the case of a single variable, there will exists a set of generating invariants $\kappa^1, \ldots, \kappa^m$ such that all higher order invariants can be generated via invariant differentiation by $\mathcal{D}_s$.

We denote the basic higher order invariants by $\kappa_j^\alpha = (\mathcal{D}_s)^j\,\kappa^\alpha$. Note that the $\alpha$ index indicates that the generating invariants will often be tied to the dependent variable $u^\alpha$, but in most cases $\kappa_j^\alpha \neq \iota(u_j^\alpha)$. For convenience, we will use $\kappa^{(n)}$ to denote this set of generating differential invariants.

We have seen that any $G$-invariant variational problems, $\int L\,dx$, can be rewritten in terms of a Lagrangian form, $\widetilde{L}\,\varpi$, i.e.

$$\int L(x, u^{(n)})\,dx = \int \widetilde{L}(\kappa^{(n)})\,\varpi.$$

In order to proceed in our journey to writing out the invariant Euler-Lagrange equations, we must define several invariant operators. We start with the invariant Euler operators, $\mathcal{E}_\alpha$. These operators are similar to the classical Euler operators defined in eq5.1, and they are given by

$$\mathcal{E}_\alpha(\widetilde{L}) = \sum_{j=0}^{\infty}(-\mathcal{D}_s)^j\,\frac{\partial \widetilde{L}}{\partial \kappa_j^\alpha}. \tag{5.3}$$

The next ingredient we need is the invariant Hamiltonian operator, $\mathcal{H}$, given by:

$$\mathcal{H}(\widetilde{L}) = \sum_{\alpha=1}^{m} \sum_{i>j} \kappa_{i-j}^{\alpha} \, (\mathcal{D}_s)^j \, \frac{\partial \widetilde{L}}{\partial \kappa_i^{\alpha}} - \widetilde{L}. \tag{5.4}$$

Next we define the invariant Eulerian and Hamiltonian operators $\mathcal{A}$ and $\mathcal{B}$. These are given by the following equations:

$$d_{\mathcal{V}}\, \kappa^{\alpha} = \sum_{j=1}^{q} \mathcal{A}_{ij}(\vartheta^j), \qquad d_{\mathcal{V}}\, \varpi = \sum_{j=1}^{q} \mathcal{B}_j(\vartheta^j) \wedge \varpi, \qquad i = 1, ..., m. \tag{5.5}$$

We will use $\mathcal{A}^*$ and $\mathcal{B}^*$ to denote the formal adjoints of $\mathcal{A}$ and $\mathcal{B}$ respectively. This operation amounts to taking the transpose of the corresponding matrix, replacing all occurrences of $\mathcal{D}_s$ with $-\mathcal{D}_s$, and using the Leibniz rule to collect various terms. For details see [Olv93, p. 328].

Lastly we define a matrix, $W$, of relative invariants that is used to transform between certain two-forms and their invariant counterparts. $W$ is the matrix such the following equation holds

$$\pi_{(1,1)}(\vartheta^i \wedge \varpi) = \sum_{j=1}^{q} W_{i,j}\, \theta^j \wedge \, dx. \tag{5.6}$$

In many cases, the matrix $W$ is of full rank and can be disregarded. When this happens the invariant Euler-Lagrange equations are simply given by:

$$\mathcal{A}^* \mathcal{E}(\widetilde{L}) - \mathcal{B}^* \mathcal{H}(\widetilde{L}) = 0,$$

where $\mathcal{E}(\widetilde{L}) = (\mathcal{E}_1(\widetilde{L}), \ldots, \mathcal{E}_m(\widetilde{L}))^T$. It turns out that $W$ plays a non-trivial role in determining the invariant Euler-Lagrange equations for variational problems over framed curves.

For full details as well as a discussion of the multivariate Lagrangians we refer the reader to [KO03, p. 25-49].

# Chapter 6

# Two-Dimensional Framed Curves

## 6.1 Introduction

We begin this chapter by defining a framed curve. Roughly speaking, a framed curve is a curve through Euclidean space with the additional information of a positively oriented orthogonal basis for the ambient space at each point on the curve.

The underlying manifold structure for this space will be denoted by $\mathcal{F}^n$. In coordinates, we have

$$\mathcal{F}^n = \{(\boldsymbol{x}, F) : \boldsymbol{x} \in \mathbb{R}^n, F \in SO(n)\}.$$

The canonical projection from $\mathcal{F}^n$ to $\mathbb{R}^n$ is denoted by $\pi : \mathcal{F}^n \to \mathbb{R}^n$. With this background, we formally define a framed curve as follows.

**Definition 6.1.1.** A **framed curve** is a simple smooth curve $C : [a, b] \to \mathcal{F}^n$ that is transverse to the fibers of $\pi^{-1}(\{\boldsymbol{z}\})$, for all $\boldsymbol{z} \in \mathbb{R}^n$.

The transversality condition above is included so that (locally) a framed curve projects to a curve in Euclidean space. We will specifically work with framed curves that are parameterized by $x^1$, meaning they will be thought of as tuples given by

$$(\boldsymbol{x}(x^1), F(x^1)).$$

We define the action of $SE(n)$ on the space of framed curves in $\mathbb{R}^n$ to be given pointwise by

$$(\boldsymbol{a}, T) * (\boldsymbol{x}, F) = (T\boldsymbol{x} + \boldsymbol{a}, TF).$$

It is worth noting that we can identify $\mathcal{F}^n \cong SE(n)$. Thus the action given above can be viewed as a free and transitive right action of $SE(n)$ on itself. In the two dimensional case, we will take coordinates for our framed curve to be $(x, u(x), \phi(x))$. Here $\phi$ coordinatizes SO(2) in the standard way, i.e. $\phi$ corresponds to the matrix

$$\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}.$$

We will take the coordinates for SE(2) acting on a framed curve to be $(a, b, \psi)$. Using Cartan's convention, transformed variables will be indicated their uppercase variations. In this convention, we have the following:

$$X = x\cos\psi - u\sin\psi + a, \qquad U = x\sin\psi + u\cos\psi + b, \qquad \Phi = \phi + \psi.$$

The prolonged group transformations are found by successively applying the following implicit differentiation operator:

$$D_X = \frac{1}{\cos\psi - u_x\sin\psi} D_x.$$

This gives the following formulas for the first few transformed jet coordinates:

$$U_X = \frac{\sin\psi + u_x\cos\psi}{\cos\psi - u_x\sin\psi},$$

$$U_{XX} = \frac{u_{xx}}{(\cos\psi - u_x\sin\psi)^3},$$

$$\Phi_X = \frac{\phi_x}{\cos\psi - u_x\sin\psi},$$

and

$$\Phi_{XX} = \frac{\phi_{xx}(\cos\psi - u_x\sin\psi) + \phi_x u_{xx}\sin\psi}{(\cos\psi - u_x\sin\psi)^3}.$$

Applying the method of moving frames with the cross-section $X = 0, U = 0$, and $\Phi = 0$, gives

$$\rho(a, b, \psi) = (-x\cos\phi - u\sin\phi, x\sin\phi - u\cos\phi, -\phi).$$

Thus, the first few normalized differential invariants are given by:

$$I_1 = \iota(U_X) = \frac{u_x\cos\phi - \sin\phi}{\cos\phi + u_x\sin\phi},$$

$$I_2 = \iota(U_{XX}) = \frac{u_{xx}}{(\cos\phi + u_x\sin\phi)^3},$$

$$J_1 = \iota(\Phi_X) = \frac{\phi_x}{\cos\phi + u_x\sin\phi},$$

and

$$J_2 = \iota(\Phi_{XX}) = \frac{\phi_{xx}(\cos\phi + u_x\sin\phi) - \phi_x u_{xx}\sin\phi}{(\cos\phi + u_x\sin\phi)^3}.$$

In the next section we prolong the infinitesimal generators and derive the recurrence relations for differential invariants.

## 6.2 The Recurrence Relations

A basis for the set of infinitesimal generators of this group action is given by:

1. $\boldsymbol{v}_1 = \partial_x$,
2. $\boldsymbol{v}_2 = \partial_u$,
3. $\boldsymbol{v}_3 = -u\partial_x + x\partial_u + \partial_\phi$.

The prolongation of these infinitesimal generators is given by

$$\begin{aligned}\mathtt{pr}(\boldsymbol{v}_1) =& \boldsymbol{v}_1,\\ \mathtt{pr}(\boldsymbol{v}_2) =& \boldsymbol{v}_2,\end{aligned}$$

and

$$\begin{aligned}\texttt{pr}(\boldsymbol{v}_3) =& \boldsymbol{v}_3 + (1+u_x^2)\frac{\partial}{\partial u_x} + 3u_x u_{xx}\frac{\partial}{\partial u_{xx}} \\ &+ \sum_{n=4}^{\infty}\left(\sum_{\substack{i=1,\\ 2i\neq n}}^{\lfloor n/2\rfloor}\binom{n}{i}u_i u_{n-i} + \frac{\delta_{2i}^n}{2}\binom{n}{i}u_i^2\right)\frac{\partial}{\partial u_n} \\ &+ \sum_{n=1}^{\infty}\left(\sum_{i=0}^{n-1}\binom{n}{i}\phi_{i+1}u_{n-i}\right)\frac{\partial}{\partial \phi_n},\end{aligned}$$

where $u_i$ and $\phi_i$ denote $D_x^n u$ and $D_x\phi$ respectively and $\delta_{2i}^n$ is the Kronecker delta.

The recurrence relations are derived using the fact that $\mathcal{D}_s(\iota(I)) = 0$ for all phantom invariants. This leads to the following system of equations:

$$\begin{aligned}0 &= \mathcal{D}_s(\iota(x)) = \iota(1) + R^1 + R^3\cdot\iota(-u),\\ 0 &= \mathcal{D}_s(\iota(u)) = \iota(u_x) + R^2 + R^3\cdot\iota(x),\\ 0 &= \mathcal{D}_s(\iota(\phi)) = \iota(\phi_x) + R^3\iota(1)).\end{aligned}$$

Solving these equations, we have

$$R^1 = -1, \qquad R^2 = -\iota(u_x) =: -\kappa, \qquad R^3 = -\iota(\phi_x) =: -\mu.$$

According to formula (3.2), the invariant total derivative of an invariant is given as follows:

$$\mathcal{D}_s[\iota(F)] = \iota\left(D_x(F)\right) - \iota\left(\texttt{pr}\,\boldsymbol{v}_1(F)\right) - \kappa\cdot\iota\left(\texttt{pr}\,\boldsymbol{v}_2(F)\right) - \mu\cdot\iota\left(\texttt{pr}\,\boldsymbol{v}_3(F)\right), \qquad (6.1)$$

where $F$ is an arbitrary differential function. Applying the above relation to $\{u_x, u_{xx}, \dots\}$ and $\{\phi_x, \phi_{xx}, \dots\}$, while using the notation $I_n = \iota(u_n x)$, and $J_n^i = \iota(\phi_n)$, we obtain the following curvature invariants:

$$
\begin{aligned}
&\kappa = I_1 = \iota(u_x), \\
&\kappa_s = \iota(D(I_1)) = I_2 - J_1(1 + I_1^2), \\
&\kappa_{ss} = \iota(D(I_2)) = I_3 - J_1(3I_1I_2), \\
&\ldots, \\
&\mu = J_1 = \iota(\phi_x), \\
&\mu_s = \iota(D(J_1)) = J_2 - I_1J_1^2, \\
&\mu_{ss} = \iota(D(J_2)) = J_3 - I_2J_1^2 - 2I_1J_1J_2, \\
&\ldots, \\
&I_1 = \kappa, \\
&I_2 = \kappa_s + \mu(1 + \kappa^2), \\
&I_3 = \kappa_{ss} + 3\kappa\mu(1 + \kappa^2), \\
&\ldots, \\
&J_1 = \mu, \\
&J_2 = \mu_s + \kappa\mu^2, \\
&J_3 = \mu_{ss} + \mu_s\kappa_s + \mu\mu_s + \mu\mu_s\kappa^2 + 2\kappa\mu\mu_s + 2\kappa^2\mu^3, \\
&\text{etc.}
\end{aligned}
$$

The relations above allow one to easily convert curvature invariants, which may arise in calculations, to normal invariants which can be readily expressed in coordinates. In the next section, we will derive the invariant Euler-Lagrange equations.

## 6.3 The Invariant Euler-Lagrange Equations

We begin by computing the Lie derivatives of the basic contact forms with respect to the infinitesimal generators of the group. We will use the following notation for these contact forms:

$$\theta_n^u = du_{nx} - D_{(n+1)x}[u]\,dx \qquad \text{and} \qquad \theta_n^\phi = d\phi_{nx} - D_{(n+1)x}[\phi]\,dx.$$

Note that $\boldsymbol{v}_1(\theta^u_n) = \boldsymbol{v}_1(\theta^\phi_n) = \boldsymbol{v}_2(\theta^u_n) = \boldsymbol{v}_2(\theta^\phi_n) = 0$. Next we calculate the Lie derivatives of $\theta^u_0$ and $\theta^\phi_0$ with respect to $\boldsymbol{v}_3$. Here we get:

$$\begin{aligned}
\boldsymbol{v}_3(\theta^u) &= d(x) - [(1+u_x^2)dx - u_x\,du] \\
&= dx - dx - u_x^2\,dx + u_x\,du \\
&= u_x\theta^u. \\
\boldsymbol{v}_3(\theta^\phi) &= d(1) - [\phi_x u_x\,dx - \phi_x\,du] \\
&= -\phi_x u_x\,dx + \phi_x\,du \\
&= \phi_x\theta^u.
\end{aligned}$$

Using the recurrence formula (4.22) we have:

$$\begin{aligned}
\mathcal{D}_s(\vartheta^\alpha_n) = \iota(\mathbb{D}_x(\theta^\alpha_n)) &= \iota(D_x\theta^\alpha_n - \boldsymbol{v}_1(\theta^\alpha_n) - u_x\boldsymbol{v}_2(\theta^\alpha_n) - \phi_x\boldsymbol{v}_3(\theta^\alpha_n)) \\
&= \vartheta^\alpha_{n+1} - \mu\cdot\iota(\boldsymbol{v}_3\,\theta^\alpha_n).
\end{aligned}$$

Applying the above equation to $\vartheta^u$ and $\vartheta^\phi$ gives the following set of equations:

$$\mathcal{D}_s\vartheta^u = \vartheta^u_1 - \mu\cdot\iota(\boldsymbol{v}_3\theta^u) = \vartheta^u_1 - \kappa\mu\vartheta^u, \tag{6.2}$$

$$\mathcal{D}_s\vartheta^\phi = \vartheta^\phi_1 - \mu\cdot\iota(\boldsymbol{v}_3\theta^\phi) = \vartheta^\phi_1 - \mu^2\vartheta^u. \tag{6.3}$$

Next we calculate the vertical differentiation of the fundamental curvature invariants using the recurrence formula (4.17) along with equations (6.2) and (6.3). Before we do this, we must determine the one-forms $\varepsilon^1$, $\varepsilon^2$, and $\varepsilon^3$ that arise in (4.17). Using the phantom invariants to solve for the unknown invariant contact forms, $\varepsilon^i$ yields

$$\varepsilon^1 = 0, \qquad \varepsilon^2 = -\vartheta^u_0, \qquad \text{and} \qquad \varepsilon^3 = -\vartheta^\phi_0.$$

Hence, the invariant vertical derivatives of the fundamental curvature invariants are:

$$\begin{aligned} d_\nu \kappa &= \iota(\theta_1^u - \boldsymbol{v}_3(u_x)\theta^\phi) && (6.4)\\ &= \iota(\theta_1^u - (1+u_x^2)\theta^\phi) && (6.5)\\ &= \vartheta_1^u - (1+\kappa^2)\vartheta^\phi && (6.6)\\ &= (\mathcal{D}_s + \mu\kappa)\vartheta^u - (1+\kappa^2)\theta^\phi, && (6.7) \end{aligned}$$

and

$$\begin{aligned} d_\nu \mu &= \iota(\theta_1^\phi - \boldsymbol{v}_3(\phi_x)\theta^\phi) && (6.8)\\ &= \iota(\theta_1^\phi - \phi_x u_x \theta^\phi) && (6.9)\\ &= \vartheta_1^\phi - \kappa\mu\vartheta^u && (6.10)\\ &= \mu^2\vartheta^u + (\mathcal{D}_s - \kappa\mu)\vartheta^\phi. && (6.11) \end{aligned}$$

Using formula (5.5) yields

$$\mathcal{A} = \begin{bmatrix} \mathcal{D}_s + \kappa\mu & -(1+\kappa^2) \\ \mu^2 & \mathcal{D}_s - \kappa\mu \end{bmatrix}.$$

Hence

$$\mathcal{A}^* = \begin{bmatrix} -\mathcal{D}_s + \kappa\mu & \mu^2 \\ -(1+\kappa^2) & -\mathcal{D}_s - \kappa\mu \end{bmatrix}. \tag{6.12}$$

According to formula (4.19) , the invariant vertical differential of $\varpi$ is

$$d_\nu \varpi = \iota[(-\phi_x)dx \wedge (-\theta^u)) + \iota(-\theta^\phi \wedge (-u_x)dx] = -\mu\vartheta^u \wedge \varpi + \kappa\vartheta^\phi \wedge \varpi.$$

Using formula (5.5) again gives

$$\mathcal{B} = \begin{bmatrix} -\mu & \kappa \end{bmatrix}, \qquad \text{and} \qquad \mathcal{B}^* = \begin{bmatrix} -\mu \\ \kappa \end{bmatrix}. \tag{6.13}$$

For the ensuing discussion, we will define the operator

$$\mathcal{G}(\widetilde{L}) = \mathcal{A}^* \begin{bmatrix} \mathcal{E}_\kappa(\widetilde{L}) \\ \mathcal{E}_\kappa(\widetilde{L}) \end{bmatrix} - \mathcal{B}^* \mathcal{H}(\widetilde{L}). \tag{6.14}$$

Putting all of our work together, we see that the invariant Euler-Lagrange equations are given by:

$$W \cdot \mathcal{G}(\widetilde{L}) = W \left( \begin{bmatrix} -\mathcal{D}_s + \kappa\mu & \mu^2 \\ -(1+\kappa^2) & -\mathcal{D}_s - \kappa\mu \end{bmatrix} \begin{bmatrix} \mathcal{E}_\kappa(\widetilde{L}) \\ \mathcal{E}_\mu(\widetilde{L}) \end{bmatrix} - \begin{bmatrix} -\mu \\ \kappa \end{bmatrix} \mathcal{H}(\widetilde{L}) \right) = 0, \tag{6.15}$$

where $W$ is the matrix defined by equation (5.6). The invariant forms that appear in (5.6) are:

$$\vartheta_0^u = \frac{\theta_0^u}{\cos\phi + u_x \sin\phi},$$

$$\vartheta_0^\phi = -\frac{\phi_x \sin\phi}{\cos\phi + u_x \sin\phi}\,\theta_0^u + \theta_0^\phi,$$

$$\varpi = (\cos\phi + u_x \sin\phi)\,dx.$$

Therefore

$$W = \begin{bmatrix} 1 & -\phi_x \sin\phi \\ 0 & \cos\phi + u_x \sin\phi \end{bmatrix}. \tag{6.16}$$

### 6.3.1 Rank and Solvability

In many cases, the matrix $W$ as described above is full rank, however, this matrix need not be full rank. In fact for our system, $W$ is singular if and only if $ds = 0$. The geometric interpretation of this condition is that our frame must satisfy $\phi = \pm\frac{\pi}{2}$. When $W$ is singular, its null space is parameterized by $\{(0,c)^T : c \in \mathbb{R}\}$. Thus there are two possible ways for the invariant Euler-Lagrange equations to be satisfied. Firstly, $u(x)$ and $\phi(x)$ will solve the invariant Euler Lagrange equations if $\mathcal{G}(\widetilde{L}) = 0$, or secondly, they if they satisfy $\mathcal{G}(\widetilde{L}) \neq 0$ and $\mathcal{G}(\widetilde{L})$ lies in the null space of $W$. As we will see in the examples, it is important to be mindful of the possibility that $W$ can be singular.

It should be noted that variational problems for framed curves provide the first example of a case where $W$ can be singular. However, we consider such examples degenerate. All degenerate examples that we have encountered lead to invariant Euler-Lagrange equations that do not have smooth solutions. In section 6.5, we will explore several example problems, one of which will lead to $W$ being rank one. Before we explore these examples, let us extend our analysis to account for the use of matrix coordinates for the frame portion of a framed curve.

## 6.4 Invariant Euler-Lagrange Equations in Matrix Coordinates

The analysis for three-dimensional framed curves will be easier to carry out in matrix coordinates. In order to motivate some of the notation and computations, we first work out the easier case of two dimensions. With in mind, the action of $SE(2)$ on $\mathcal{F}^2$ is given by:

$$(a, b, T) * (x, u(x), F(x)) = (\boldsymbol{T}^1\, \boldsymbol{z} + a, \boldsymbol{T}^2\, \boldsymbol{z} + b, \boldsymbol{T}^2\, \boldsymbol{z} + c, TF)$$

where $\boldsymbol{T}^i$ denotes the $i$-th row of $T$, $\boldsymbol{z}(x) = (x, u(x))^\top$, and $T$ and $F$ are the $2 \times 2$ matrices given by

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}, \qquad \text{and} \qquad F(x) = \begin{bmatrix} F_{11}(x) & F_{12}(x) \\ F_{21}(x) & F_{22}(x) \end{bmatrix}.$$

This fits with our previous analysis by setting

$$F(x) = \begin{bmatrix} F_{11}(x) & F_{12}(x) \\ F_{21}(x) & F_{22}(x) \end{bmatrix} = \begin{bmatrix} \cos\phi(x) & -\sin\phi(x) \\ \sin\phi(x) & \cos\phi(x) \end{bmatrix}.$$

Taking a coordinate cross-section $\mathcal{K} = \{(0, I)\}$, we have the following right equivariant moving frame:

$$T = F^{-1} = F^\top, \qquad (a, b) = -F^\top \boldsymbol{z}.$$

We will use the notation that uppercase letters will denote transformed variables, and lowercase letters will denote domain variables (except in the case of matrices, but this should be evident from context). To this end, we have

$$X = T_{11}x + T_{12}u + a,$$

$$U = T_{21}x + T_{22}u + b,$$

$$\widetilde{F} = TF.$$

To simplify certain expressions, we introduce the notation:

$$\boldsymbol{z}_{nx} = \frac{d^n}{dx^n}\,\boldsymbol{z}.$$

With this in mind, the prolonged group transformations are found by successively applying the following implicit differentiation operator:

$$D_X = \left(\frac{1}{\boldsymbol{T}^1\,\boldsymbol{z}_x}\right) D_x.$$

Applying this to $U$ we have:

$$U_X = \frac{D_x U}{\boldsymbol{T}^1\,\boldsymbol{z}_x} = \frac{\boldsymbol{T}^2\,\boldsymbol{z}_x}{\boldsymbol{T}^1\,\boldsymbol{z}_x}$$

and

$$U_{XX} = \frac{D_x U_X}{\boldsymbol{T}^1\,\boldsymbol{z}_x} = \frac{(\boldsymbol{T}^1\,\boldsymbol{z}_x)(\boldsymbol{T}^2\,\boldsymbol{z}_{xx}) - (\boldsymbol{T}^2\,\boldsymbol{z}_x)(\boldsymbol{T}^1\,\boldsymbol{z}_{xx})}{(\boldsymbol{T}^1\,\boldsymbol{z}_x)^3}.$$

For $\widetilde{F}$, we have:

$$\widetilde{F}_X = \frac{RF_x}{\boldsymbol{T}^1\,\boldsymbol{z}_x}$$

and

$$\widetilde{F}_{XX} = \frac{(\boldsymbol{T}^1\,\boldsymbol{z}_x)RF_{xx} - (\boldsymbol{T}^1\,\boldsymbol{z}_{xx})RF_x}{(\boldsymbol{T}^1\,\boldsymbol{z}_x)^3} = \frac{R}{(\boldsymbol{T}^1\,\boldsymbol{z}_x)^3}\left((\boldsymbol{T}^1\,t)F_{xx} - (\boldsymbol{T}^1\,\boldsymbol{z}_{xx})F_x\right).$$

Using the moving frame to substitute for the parameters of $SE(2)$, we have the following differential invariants:

$$I_1 = \frac{(\boldsymbol{F}_2)^\top \boldsymbol{z}_x}{(\boldsymbol{F}_1)^\top \boldsymbol{z}_x}, \qquad I_2 = \frac{((\boldsymbol{F}_1)^\top \boldsymbol{z}_x)((\boldsymbol{F}_2)^\top \boldsymbol{z}_{xx}) - ((\boldsymbol{F}_2)^\top \boldsymbol{z}_x)((\boldsymbol{F}_1)^\top \boldsymbol{z}_{xx})}{((\boldsymbol{F}_1)^\top \boldsymbol{z}_x)^3},$$

$$J_1 = \frac{F^\top F_x}{(\boldsymbol{F}_1)^\top \boldsymbol{z}_x}, \qquad J_2 = \frac{F^\top}{((\boldsymbol{F}_1)^\top \boldsymbol{z}_x)^3}\Big(((\boldsymbol{F}_1)^\top t)F_{xx} - ((\boldsymbol{F}_1)^\top \boldsymbol{z}_{xx})F_x\Big),$$

$$\vdots \qquad\qquad\qquad \vdots$$

where $(\boldsymbol{F}_i)^\top$ is row vector given by transposing the $i$-th column of $F$.

### Maurer-Cartan Invariants

Next we find the Maurer-Cartan invariants using the recurrence relations. The infinitesimal generators for the action are given by:

1. $\boldsymbol{v}_1 = \partial_x$,
2. $\boldsymbol{v}_2 = \partial_u$,
3. $\boldsymbol{v}_3 = u\partial_x - x\partial_u - F_{21}\partial_{11} - F_{22}\partial_{12} + F_{11}\partial_{21} + F_{12}\partial_{22}$.

where $\partial_{ij}$ denotes the operator $\frac{\partial}{\partial F_{ij}}$. The prolongations of these generators are given by the following:

1. $\mathtt{pr}(\boldsymbol{v}_1) = \boldsymbol{v}_1$,
2. $\mathtt{pr}(\boldsymbol{v}_2) = \boldsymbol{v}_2$,
3. $\mathtt{pr}(\boldsymbol{v}_3) = \boldsymbol{v}_3 + (1+u_x^2)\partial_{u_x} + (F_{11,x}u_x - F_{21,x})\partial_{11,x} + (F_{12,x}u_x - F_{22,x})\partial_{12,x}$
$\quad + (F_{21,x}u_x + F_{11,x})\partial_{21,x} + (F_{22,x}u_x + F_{12,x})\partial_{22,x} + \cdots$.

Recall that if $Q$ is a differential function and $\iota(Q)$ is its moving frame invariantization, then:

$$D_x[\iota(Q)] = \iota[D_x(Q)] + \sum_{i=1}^{3} R^i \iota\left[\mathtt{pr}\boldsymbol{v}_i(W)\right].$$

To solve for the Maurer-Cartan invariants, $R^i_\sigma$, we apply the above equation to the phantom invariants. Our 2D representation of SO(2) gives an overdetermined system of equations, however, the structure of $\mathfrak{so}(2)$ ensures that have a consistent set of equations that can be solved for the Maurer-Cartan invariants. Here is a summary of the resulting equations:

$$\begin{aligned} 0 &= 1 + R^1, \\ 0 &= \iota(u_x) + R^2, \end{aligned}$$

and

$$0 = \iota(F_x) + R^3 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Since $F(x)$ is a curve in SO(2), we see that $F_x$ lies in its tangent space, i.e. $\mathfrak{so}(2)$. This coupled with the fact that $\mathtt{pr}\boldsymbol{v}_3(F) \in \mathfrak{so}(2)$ shows that these equations are indeed consistent. Solving these equations for $R^1$ and $R^2$, we get $R^1 = -1$ and $R^2 = -\iota(u_x) =: -I_1$. The final Maurer-Cartan invariant can be taken to be any linear combination

$$c_1 \cdot \iota(F_{12,x}) - c_2 \cdot \iota(F_{21,x})$$

which satisfies the equation $c_1 + c_2 = 1$. Simple choices include $(c_1, c_2) = (1, 0)$, $(c_1, c_2) = (0, 1)$, or $(c_1, c_2) = (\frac{1}{2}, \frac{1}{2})$. We will take $R^3 = -F_{21,x}$. This choice is made in order to keep our analysis similar to that of previous section.

We will denote the normalized invariants by

$$I_n = \iota\left(\frac{d^n u}{dx^n}\right), \qquad \text{and} \qquad J_{ij,n} = \iota\left(\frac{d^n F_{ij}}{dx^n}\right).$$

The fundamental curvature invariants will be denoted as:

$$\kappa^u = I_1, \qquad \text{and} \qquad \kappa^{ij} = \iota(J_{ij,1}).$$

We will attach a subscript $s$ to a curvature invariant to denote when an invariant arclength derivative has been taken, e.g. $\kappa^u_s = \mathcal{D}_s \kappa^u$. At this point, we should note that the symmetry of $F_x$ will ensure that $\kappa^{12} = -\kappa^{21}$ and $\kappa^{11} = \kappa^{22} = 0$. This means that a complete set of functionally independent invariants is given by $\{I_n, J_{21,n} : n \in \mathbb{N}\}$.

### 6.4.1 Contact Forms in Matrix Coordinates

In the literature, there are various definitions of the/a Maurer-Cartan form. We distinguish between the various definitions in the following way:

**Definition 6.4.1.** A right (left) Maurer-Cartan form on a Lie group $G$ is a right-invariant (left-invariant) differential one-form on $G$.

**Definition 6.4.2.** The right (left) canonical Maurer-Cartan form on a Lie group, $G$, is the unique right-invariant (left-invariant) $\mathfrak{g}$-valued one-form on $G$ such that $\omega_G\big|_e : T_eG \to \mathfrak{g}$ is the identity. We may identify $\omega_G$ as a smooth section of $\bigwedge^1 T^*G \otimes \mathfrak{g}$.

For matrix Lie groups, the right canonical Maurer-Cartan form is given by

$$\omega_G = dg\, g^{-1}. \tag{6.17}$$

A basis for the set of all Maurer-Cartan forms is given by a maximal set of linearly independent entries of $\omega_G$.

For the frame portion of a framed curve $\omega_G$ is given by:

$$\omega_G = dg\, g^{-1} = \begin{bmatrix} -F_{21}\, dF_{12} + F_{22}\, dF_{11} & F_{11}\, dF_{12} - F_{12}\, dF_{11} \\ -F_{21}\, dF_{22} + F_{22}\, dF_{21} & F_{11}\, dF_{22} - F_{12}\, dF_{21} \end{bmatrix}. \tag{6.18}$$

If we pull back the coordinates of $\omega_G$ via the standard embedding $\tau : \mathrm{SO}(2) \to M_{2\times 2}(\mathbb{R})$ we get

$$\tau^*(\omega_G) = \begin{bmatrix} 0 & -d\phi \\ d\phi & 0 \end{bmatrix}.$$

Using $\omega_G$, we find that the invariant contact form for the frame portion of our curve is given by:

$$\theta_0 = dg\, g^{-1} - g_x g^{-1}\, dx = \begin{bmatrix} 0 & \theta_0^1 \\ \theta_0^2 & 0 \end{bmatrix}, \tag{6.19}$$

where

$$\theta_0^1 = -F_{12}dF_{11} + F_{11}dF_{12} + (F_{11x}F_{12} - F_{12x}F_{11})\, dx, \tag{6.20}$$

$$\theta_0^2 = F_{22}dF_{21} - F_{21}dF_{22} - (F_{21x}F_{22} - F_{22x}F_{21})\, dx. \tag{6.21}$$

Pulling back the coordinates of $\theta_0$ via the embedding $\tau : \mathrm{SO}(2) \to M_{2\times 2}(\mathbb{R})$ we get:

$$\tau^*(\theta_0) = \begin{bmatrix} 0 & -\theta_0^\phi \\ \theta_0^\phi & 0 \end{bmatrix}$$

which matches our previous results.

### 6.4.2 Lie Derivatives of Contact Forms

Taking the Lie derivative of the coordinates $\theta_0$ with respect to $\boldsymbol{v}_3$ gives

$$\boldsymbol{v}_3(\theta_0) := \begin{bmatrix} 0 & \boldsymbol{v}_3(\theta_0^1) \\ \boldsymbol{v}_3(\theta_0^2) & 0 \end{bmatrix} = \begin{bmatrix} 0 & \tilde{\theta}_0^1 \\ \tilde{\theta}_0^2 & 0 \end{bmatrix}, \tag{6.22}$$

where

$$\tilde{\theta}_0^1 = (F_{11}F_{12,x} - F_{12}F_{11,x})\, \theta_0^u + F_{22}\, \theta_0^{11} - F_{21}\, \theta_0^{12} + F_{12}\, \theta_0^{21} - F_{11}\, \theta_0^{22}, \tag{6.23}$$

$$\tilde{\theta}_0^2 = (F_{22}F_{21,x} - F_{21}F_{22,x})\, \theta_0^u + F_{22}\, \theta_0^{11} - F_{21}\, \theta_0^{12} + F_{12}\, \theta_0^{21} - F_{11}\, \theta_0^{22}, \tag{6.24}$$

$$\theta_0^{ij} := dF_{ij} - F_{ij,x}\, dx. \tag{6.25}$$

Calculating the pullback of these one-forms by $\tau$ yields:

$$\tau^*(\tilde{\theta}_0^1) = -\phi_x \theta_0^u \tag{6.26}$$

and

$$\tau^*(\tilde{\theta}_0^2) = \phi_x \theta_0^u \tag{6.27}$$

as expected.

### 6.4.3 Relations and Structure for Contact Forms

Since Maurer-Cartan form, $\omega_G$, is a smooth section of $\bigwedge^1 T^*SO(2) \otimes \mathfrak{so}(2)$, it inherits the various relations of $\mathfrak{so}(2)$, i.e. $\omega_G = -\omega_G^T$. Using this fact and the various relations imposed on $F$ and $F_x$ yields the following formulas:

$$\theta_0^{ij} = -\theta_0^{ji}, \tag{6.28}$$

$$\theta_0^1 = -\theta_0^2, \tag{6.29}$$

$$\tilde{\theta}_0^1 = -\tilde{\theta}_0^2, \tag{6.30}$$

$$\tilde{\theta}_0^1 = (F_{11}F_{12,x} - F_{12}F_{11,x})\,\theta_0^u, \tag{6.31}$$

$$\tilde{\theta}_0^2 = (F_{22}F_{21,x} - F_{21}F_{22,x})\,\theta_0^u. \tag{6.32}$$

The above relations show that $\theta_0^2$ forms a coframe for $SO(2)$ and that

$$\boldsymbol{v}_3(\theta_0^2) = (F_{22}F_{21,x} - F_{21}F_{22,x})\,\theta_0^u. \tag{6.33}$$

Applying the invariantization operator, we have

$$\iota(\boldsymbol{v}_3(\theta_0^2)) = \iota((F_{22}F_{21,x} - F_{21}F_{22,x})\,\theta_0^u) = \iota(F_{21,x})\,\vartheta^u = \kappa^{21}\,\vartheta^u. \tag{6.34}$$

Before moving on, we make one final remark about the invariantization of $\theta_0^1$ and $\theta_0^2$. Note that we may rewrite these contact forms as follows

$$\theta_0^1 = F_{11}\theta_0^{12} - F_{12}\theta_0^{11}, \tag{6.35}$$

$$\theta_0^2 = F_{22}\theta^{21} - F_{21}\theta_0^{22}. \tag{6.36}$$

By the rewrite rule, we conclude that

$$\iota(\theta_0^1) = \vartheta_0^{12}, \tag{6.37}$$

$$\iota(\theta_0^2) = \vartheta_0^{21}. \tag{6.38}$$

This observations is key to the interpretation and application of (4.17) in matrix coordinates.

### 6.4.4 Invariant Eulerian and Hamiltonian

We are have all of the tools necessary to derive the invariant Eulerian and Hamiltonian operators. Using the relations that we found above along with (6.2), we have

$$\mathcal{D}_s\vartheta^u = \vartheta_1^u - \kappa^{21}\cdot\iota(\boldsymbol{v}_3\theta^u) = \vartheta_1^u - \kappa^u\kappa^{21}\vartheta^u, \tag{6.39}$$

$$\mathcal{D}_s\vartheta^2 = \vartheta_1^2 - \kappa^{21}\cdot\iota(\boldsymbol{v}_3\theta^2) = \vartheta_1^2 - (\kappa^{21})^2\vartheta^u. \tag{6.40}$$

Next we calculate the vertical differentiation of the fundamental curvature invariants using the recurrence formula (4.17) along with equations (6.39) and (6.40). Before we do this, we must determine the one-forms $\varepsilon^1$, $\varepsilon^2$, and $\varepsilon^3$ that arise in (4.17). Using the phantom invariants to solve for the unknown invariant contact forms, $\varepsilon^i$ yields

$$\varepsilon^1 = 0, \qquad \varepsilon^2 = -\vartheta_0^u, \qquad \text{and} \qquad \varepsilon^3 = -\vartheta_0^{21} = -\vartheta_0^2. \tag{6.41}$$

Hence, the invariant vertical derivatives of the fundamental curvature invariants are:

$$\begin{aligned} d_\nu\kappa^u &= \iota(\theta_1^u - \boldsymbol{v}_3(u_x)\theta^2) && (6.42)\\ &= \iota(\theta_1^u - (1+u_x^2)\theta^2) && (6.43)\\ &= \vartheta_1^u - (1+(\kappa^u)^2)\vartheta^2 && (6.44)\\ &= (\mathcal{D}_s + \kappa^{21}\kappa)\vartheta^u - (1+(\kappa^u)^2)\vartheta^2 && (6.45)\end{aligned}$$

and

$$\begin{aligned} d_\nu\kappa^{21} &= \iota(\theta_1^2 - \boldsymbol{v}_3(F_{21,x})\theta^2) && (6.46)\\ &= \iota(\theta_1^2 - (u_xF_{21,x} + F_{11,x})\theta^2) && (6.47)\\ &= \vartheta_1^2 - \kappa^u\kappa^{21}\vartheta^u && (6.48)\\ &= (\kappa^{21})^2\vartheta^u + (\mathcal{D}_s - \kappa^u\kappa^{21})\vartheta^2. && (6.49)\end{aligned}$$

Using formula (5.5) yields

$$\mathcal{A} = \begin{bmatrix} \mathcal{D}_s + \kappa^u\kappa^{21} & -(1+\kappa^2) \\ (\kappa^{21})^2 & \mathcal{D}_s - \kappa^u\kappa^{21} \end{bmatrix}. \tag{6.50}$$

Hence

$$\mathcal{A}^* = \begin{bmatrix} -\mathcal{D}_s + \kappa^u \kappa^{21} & (\kappa^{21})^2 \\ -(1+\kappa^2) & -\mathcal{D}_s - \kappa^u \kappa^{21} \end{bmatrix}. \tag{6.51}$$

According to formula (4.19) , the invariant vertical differential of $\varpi$, we have

$$d_\nu \varpi = -\kappa^{21} \vartheta^u \wedge \varpi + \kappa^u \vartheta^\phi \wedge \varpi. \tag{6.52}$$

Therefore using formula (5.5) again gives

$$\mathcal{B} = \begin{bmatrix} -\kappa^{21} & \kappa^u \end{bmatrix}, \qquad \text{and} \qquad \mathcal{B}^* = \begin{bmatrix} -\kappa^{21} \\ \kappa^u \end{bmatrix}. \tag{6.53}$$

Using formula (5.6) we find that $W$ is given by:

$$W = (F_{11} + F_{21} u_x) \begin{bmatrix} W_{11} & W_{12} \\ 0 & 1 \end{bmatrix}, \tag{6.54}$$

where

$$W_{11} = \frac{F_{11} F_{22} - F_{12} F_{21}}{F_{11} + F_{21} u_x}, \tag{6.55}$$

$$W_{12} = -F_{21} \frac{F_{11,x} F_{12} - F_{21,x} F_{22}}{F_{11} + F_{21} u_x}. \tag{6.56}$$

This shows us that as long as $\boldsymbol{F}_1^\top \boldsymbol{z}_x$ does not vanish, then $W$ has full rank. Since $\boldsymbol{z}_x$ is proportional to the unit tangent to the curve, this condition means that $\boldsymbol{F}_1$ must not lie in the normal plane to the curve. When this condition holds, the invariant Euler-Lagrange equations are given by

$$\mathcal{A}^* \mathcal{E} - \mathcal{B}^* \mathcal{H} = 0. \tag{6.57}$$

Thus, the work in matrix coordinates follows a very similar path to our previous analysis. The reason for rederiving these formulas in matrix coordinates is two fold. First it is due in part to the fact that it is relatively easy to see the correspondence between the two approaches in two dimensions. Secondly, it Gives us an alternative way to discuss restricted frames. In many applications, a framed curve will require one of the frame vectors to be the unit tangent to the curve. In the chapter eight, we discus some subtleties that arise in working with restricted framed curves. When we move to three dimensions, we will work exclusively in matrix coordinates.

## 6.5 Examples

Now that we have the explicit formulas for calculating the invariant Euler-Lagrange equations, we will explore a few examples. These examples are meant to demonstrate the computational benefit of working with the invariant Euler-Lagrange equations as oppose to their classically derived counterparts.

**Example 3.** For our first example, we will consider the problem of minimizing

$$\int ds = \int (\cos\phi + u_x \sin\phi) dx.$$

In this case, the invariant Lagrangian is given by $\widetilde{L} = 1$, therefore,

$$\mathcal{E}_\kappa(\widetilde{L}) = \mathcal{E}_\mu(\widetilde{L}) = 0, \qquad \text{and} \qquad \mathcal{H}(\widetilde{L}) = -1.$$

Plugging this into our equation for $\mathcal{G}(\widetilde{L})$ gives:

$$\mathcal{G}(\widetilde{L}) = \begin{bmatrix} -\mu \\ \kappa \end{bmatrix}.$$

This leads to the following set of equations:

$$\mu = 0 \qquad \text{and} \qquad \kappa = 0 \qquad (\textit{when } W \textit{ has full rank}).$$

In the original coordinates, these equations are:

$$\frac{-\sin\phi + u_x\cos\phi}{\cos\phi + u_x \sin\phi} = 0 \qquad \text{and} \qquad \frac{\phi_x}{\cos\phi + u_x\sin\phi} = 0.$$

Computing the Euler-Lagrange equations from first principles, we get:

$$-\sin\phi + u_x\cos\phi = 0 \qquad \text{and} \qquad -\phi_x\cos\phi = 0.$$

The solutions to both sets of equations are provided when $\phi$ is constant and $u$ is a linear function of $x$ with the corresponding slope depending on $\phi$.

**Example 4.** For our second example, we will consider the problem of minimizing $\int \kappa\, ds$. In this case, we have

$$\int \kappa\, ds = \widetilde{L}\, ds = \int \frac{u_x \cos\phi - \sin\phi}{\cos\phi + u_x \sin\phi}(\cos\phi + u_x \sin\phi)dx$$

$$= \int (u_x \cos\phi - \sin\phi)dx = \int L\, dx$$

Calculating the invariant Eulerians and Hamiltonian, we get:

$$\mathcal{E}_\kappa(\widetilde{L}) = 1, \qquad \mathcal{E}_\mu(\widetilde{L}) = 0, \qquad \text{and} \qquad \mathcal{H}(\widetilde{L}) = -\kappa.$$

Plugging this into our equation for $\mathcal{G}(\widetilde{L})$ gives:

$$\mathcal{G}(\widetilde{L}) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

Since $\mathcal{G}(\widetilde{L}) \neq 0$, we must conclude that $W$ is singular and $\mathcal{G}(\widetilde{L})$ is in the null space of $W$. This means that the invariant Euler-Lagrange equations for this Lagrangian are given by:

$$u_x = -\cot\phi, \qquad \sin\phi = 0.$$

Computing the Euler-Lagrange equations for the functional

$$\int (u_x \cos\phi - \sin\phi)dx$$

leads to precisely the same equations. However it is clear that the Euler-Lagrange equations do not have a smooth solution. As discussed earlier, this example is considered degenerate.

**Example 5.** For our third example, we will consider the problem of minimizing $\int \mu\, ds$.

In this case, we have

$$\int \mu\, ds = \widetilde{L}\, ds = \int \frac{\phi_x}{\cos\phi + u_x \sin\phi}(\cos\phi + u_x \sin\phi)dx$$

$$= \int \phi_x\, dx = L\, dx$$

Calculating the invariant Eulerians and Hamiltonian, we get:

$$\mathcal{E}_\kappa(\widetilde{L}) = 0, \qquad \mathcal{E}_\mu(\widetilde{L}) = 1, \qquad \text{and} \qquad \mathcal{H}(\widetilde{L}) = -\mu.$$

Plugging this into our equation for $\mathcal{G}(\widetilde{L})$ gives:

$$\mathcal{G}(\widetilde{L}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since $\mathcal{G}(\widetilde{L}) = 0$ regardless of $u(x)$ and $\phi(x)$, we conclude that $\widetilde{L}$ is a null Lagrangian. This means that any choice of $u$ and $\phi$ that satisfy the boundary conditions will minimize the functional $\int \widetilde{L}\, ds$. Applying the fundamental theorem of calculus shows that $\int \phi_x\, dx$ only depends on boundary conditions, thus we can immediately conclude that $L$ is a null Lagrangian. In this case, the invariant bicomplex machinery is excessive. As such, this example shows that it is still worth while to consider the original problem before indiscriminately applying a method with the hope of simplifying computations.

**Example 6.** For our final example, we will consider the problem of minimizing $\int \kappa^2\, ds$. In this case, we have

$$\int \kappa^2\, ds = \int \frac{(u_x \cos\phi - \sin\phi)^2}{u_x \sin\phi + \cos\phi}\, dx.$$

Calculating the invariant Eulerians and Hamiltonian, we get:

$$\mathcal{E}_\kappa(\widetilde{L}) = 2\kappa, \qquad \mathcal{E}_\mu(\widetilde{L}) = 0, \qquad \text{and} \qquad \mathcal{H}(\widetilde{L}) = -\kappa^2.$$

Plugging this into our equation for $\mathcal{G}(\widetilde{L})$ gives:

$$\mathcal{G}(\widetilde{L}) = \begin{bmatrix} -2\kappa_s \\ -\kappa^2 \end{bmatrix}.$$

Assuming that $\mathcal{G}(\widetilde{L})$ does not lie in the null space of $W$ gives the equations

$$\kappa_s = I_2 - J_1(1 + I_1^2) = 0, \qquad \text{and} \qquad \kappa^2 = I_1^2 = 0.$$

This reduces to

$$I_2 - \mu = 0, \qquad \text{and} \qquad \kappa = 0.$$

When simplified and expressed in coordinates, these equation are given by:

$$u_x = \tan\phi, \qquad \text{and} \qquad u_{xx} - \phi_x(\cos\phi + u_x \sin\phi) = 0.$$

The above equations require that $\phi_x = u_{xx} = 0$ and $u_x = \tan\phi$. The classical Euler-Lagrange equations are rather unruly, but we include them here to show the benefit of using the invariant approach to solving this variational problem.

$$\begin{aligned} 0 = \frac{\partial L}{\partial u} - \frac{d}{dx}\frac{\partial L}{\partial u_x} &= \frac{(u_x\cos\phi - \sin\phi)\left(2\phi_x u_x \cos 2\phi - 2\phi_x \sin 2\phi + u_{xx}\sin 2\phi\right)}{2\left(u_x\sin\phi + \cos\phi\right)^2} \\ &+ \frac{(u_x \sin 2\phi + \cos 2\phi + 3)\left(-\phi_x u_x \sin\phi - \phi_x\cos\phi + u_{xx}\cos\phi\right)}{2\left(u_x\sin\phi + \cos\phi\right)^2} \\ &+ \frac{(u_x\cos\phi - \sin\phi)\left(u_x\sin 2\phi + \cos 2\phi + 3\right)}{2\left(u_x\sin\phi + \cos\phi\right)^3} \\ &\times \left(-2\phi_x u_x\cos\phi + 2\phi_x\sin\phi - 2u_{xx}\sin\phi\right) \end{aligned}$$

and

$$\begin{aligned} 0 = \frac{\partial L}{\partial \phi} - \frac{d}{dx}\frac{\partial L}{\partial \phi_x} = -&\frac{\left(2\left(u_x\sin\phi + \cos\phi\right)^2 + \left(u_x\cos\phi - \sin\phi\right)^2\right)}{\left(u_x\sin\phi + \cos\phi\right)^2} \\ &\times \left(u_x\cos\phi - \sin\phi\right). \end{aligned}$$

Although these equations are complicated, we can see by inspection that our previous solution does indeed satisfy the above equations.

**Remark.** At first glance, this final example may look similar to the problem of minimizing the Euler-Bernoulli energy functional for elastica i.e. $\frac{1}{2}\int \kappa^2\, ds$. However it should be noted that in the Euler-Bernoulli functional, $\kappa = \iota(u_{xx})$. This is not the case for framed curves $(\kappa = \iota(u_x))$. The study of elastica for framed curves instead involves considering the functional

$$\frac{1}{2}\int \kappa_s^2\, ds.$$

While this would be an interesting example to study, time does not permit a detailed discussion of this problem.

In this chapter, we developed the tools to study invariant variational problems for two-dimensional framed curves that admit $SE(2)$ symmetry. Example 4 showed us the computational benefit of using the invariant equations along with the various relations among normal and curvature invariants. With these results behind us, we now move on to the much harder case of three-dimensions.

# Chapter 7

# Three-Dimensional Framed Curves

## 7.1 Introduction

Following the methods set out the the previous chapter, we now derive the invariant Euler-Lagrange equations for general three-dimensional framed curves. As before, we take the action of $SE(3)$ on $\mathcal{F}^3$ to be given by:

$$(a, b, c, T) * (x, y(x), u(x), F(x)) = (\boldsymbol{T}^1 \cdot \boldsymbol{z} + a, \boldsymbol{T}^2 \cdot \boldsymbol{z} + b, \boldsymbol{T}^3 \cdot \boldsymbol{z} + c, TF)$$

where $\boldsymbol{z}(x) = (x, y(x), u(x))^\top$, $T$ and $F$ are the $3 \times 3$ matrices given by

$$T = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}, \qquad \text{and} \qquad F(x) = \begin{bmatrix} F_{11}(x) & F_{12}(x) & F_{13}(x) \\ F_{21}(x) & F_{22}(x) & F_{23}(x) \\ F_{31}(x) & F_{32}(x) & F_{33}(x) \end{bmatrix},$$

and $\boldsymbol{T}^i$ denotes the $i$-th row of $T$. Taking the coordinate cross-section $\mathcal{K} = \{(0, I)\}$, we produce the following right equivariant moving frame:

$$T = F^{-1} = F^\top \qquad \text{and} \qquad (a, b, c) = -F^\top \boldsymbol{z}.$$

Again, we will use the notation that uppercase letters will denote transformed variables, and lowercase letters will denote domain variables (except in the case of matrices, but this should be evident from context). To this end, we have

$$X = T_{11}\,x + T_{12}\,y + T_{13}\,u + a,$$

$$Y = T_{21}\,x + T_{22}\,y + T_{23}\,u + b,$$

$$U = T_{31}\,x + T_{32}\,y + T_{33}\,u + c,$$

$$\widetilde{F} = TF.$$

Borrowing notation from the two-dimensional case, we set

$$\boldsymbol{z}_{nx} = \frac{d^n}{dx^n}\,\boldsymbol{z}.$$

Then the prolonged group transformations are found by successively applying the following implicit differential operator:

$$D_X = \left(\frac{1}{\boldsymbol{T}^1\,\boldsymbol{z}_x}\right) D_x.$$

Applying this to $Y$ and $U$ we have:

$$Y_X = \frac{D_x Y}{\boldsymbol{T}^1\,\boldsymbol{z}_x} = \frac{\boldsymbol{T}^2\,\boldsymbol{z}_x}{\boldsymbol{T}^1\,\boldsymbol{z}_x},$$

$$Y_{XX} = \frac{D_x Y_X}{\boldsymbol{T}^1\,\boldsymbol{z}_x} = \frac{(\boldsymbol{T}^1\,\boldsymbol{z}_x)(\boldsymbol{T}^2\boldsymbol{z}_{xx}) - (\boldsymbol{T}^2\,\boldsymbol{z}_x)(\boldsymbol{T}^1\,\boldsymbol{z}_{xx})}{(\boldsymbol{T}^1\,\boldsymbol{z}_x)^3},$$

$$U_X = \frac{D_x U}{\boldsymbol{T}^1\,\boldsymbol{z}_x} = \frac{\boldsymbol{T}^3\,\boldsymbol{z}_x}{\boldsymbol{T}^1\,\boldsymbol{z}_x},$$

$$U_{XX} = \frac{D_x U_X}{\boldsymbol{T}^1\,\boldsymbol{z}_x} = \frac{(\boldsymbol{T}^1\,\boldsymbol{z}_x)(\boldsymbol{T}^3\,\boldsymbol{z}_{xx}) - (\boldsymbol{T}^3\,\boldsymbol{z}_x)(\boldsymbol{T}^1\,\boldsymbol{z}_{xx})}{(\boldsymbol{T}^1\,\boldsymbol{z}_x)^3}.$$

For $\widetilde{F}$, we have:

$$\widetilde{F}_X = \frac{TF_x}{\boldsymbol{T}^1\,\boldsymbol{z}_x}$$

and

$$\widetilde{F}_{XX} = \frac{(\boldsymbol{T}^1\,\boldsymbol{z}_x)\,TF_{xx} - (\boldsymbol{T}^1\,\boldsymbol{z}_{xx})\,TF_x}{(\boldsymbol{T}^1\,\boldsymbol{z}_x)^3} = \frac{T}{(\boldsymbol{T}^1\,\boldsymbol{z}_x)^3}\left((\boldsymbol{T}^1\boldsymbol{z}_x)F_{xx} - (\boldsymbol{T}^1\,\boldsymbol{z}_{xx}F_x\right).$$

Using the moving frame to substitute for the parameters of $SE(2)$, we have the following differential invariants:

$$\begin{aligned}
I_1^y &= \frac{\boldsymbol{F}_2^\top\,\boldsymbol{z}_x}{\boldsymbol{F}_1^\top\,\boldsymbol{z}_x}, & I_2^y &= \frac{(\boldsymbol{F}_1^\top\,\boldsymbol{z}_x)(\boldsymbol{F}_2^\top\,\boldsymbol{z}_{xx}) - (\boldsymbol{F}_2^\top\,\boldsymbol{z}_x)(\boldsymbol{F}_1^\top\,\boldsymbol{z}_{xx})}{(\boldsymbol{F}_1^\top\,\boldsymbol{z}_x)^3},\\
I_1^u &= \frac{\boldsymbol{F}_3^\top\,\boldsymbol{z}_x}{\boldsymbol{F}_1^\top\,\boldsymbol{z}_x}, & I_2^u &= \frac{(\boldsymbol{F}_1^\top\,\boldsymbol{z}_x)(\boldsymbol{F}_3^\top\,\boldsymbol{z}_{xx}) - (\boldsymbol{F}_3^\top\,\boldsymbol{z}_x)(\boldsymbol{F}_1^\top\,\boldsymbol{z}_{xx})}{(\boldsymbol{F}_1^\top\,\boldsymbol{z}_x)^3},\\
J_1 &= \frac{F^\top F_x}{\boldsymbol{F}_1^\top\,\boldsymbol{z}_x}, & J_2 &= \frac{F^\top}{(\boldsymbol{F}_1^\top\,\boldsymbol{z}_x)^3}\Big((\boldsymbol{F}_1^\top\boldsymbol{z}_x)F_{xx} - (\boldsymbol{F}_1^\top\,\boldsymbol{z}_{xx}F_x\Big)),\\
&\ \vdots & &\ \vdots
\end{aligned}$$

where $\boldsymbol{F}_j$ is the $j$-th column of $F$.

## 7.2 The Recurrence Relations

Next we find the Maurer-Cartan invariants using the recurrence relations. The infinitesimal generators for the action are given by:

1. $\boldsymbol{v}_1 = \partial_x$,
2. $\boldsymbol{v}_2 = \partial_y$,
3. $\boldsymbol{v}_3 = \partial_u$,
4. $\boldsymbol{v}_4 = -y\partial_x + x\partial_y - F_{21}\partial_{11} - F_{22}\partial_{12} - F_{23}\partial_{13} + F_{11}\partial_{21} + F_{12}\partial_{22} + F_{13}\partial_{23}$,
5. $\boldsymbol{v}_5 = -u\partial_x + x\partial_u - F_{31}\partial_{11} - F_{32}\partial_{12} - F_{33}\partial_{13} + F_{11}\partial_{31} + F_{12}\partial_{32} + F_{13}\partial_{33}$,

6. $\boldsymbol{v}_6 = -u\partial_y + y\partial_u - F_{31}\partial_{21} - F_{32}\partial_{22} - F_{33}\partial_{23} + F_{21}\partial_{31} + F_{22}\partial_{32} + F_{23}\partial_{33}$.

where again $\partial_{ij}$ denotes the operator $\dfrac{\partial}{\partial F_{ij}}$.

The prolongations of these generators are given by the following:

1. $\mathtt{pr}^{(\infty)}(\boldsymbol{v}_1) = \boldsymbol{v}_1$,
2. $\mathtt{pr}^{(\infty)}(\boldsymbol{v}_2) = \boldsymbol{v}_2$,
3. $\mathtt{pr}^{(\infty)}(\boldsymbol{v}_3) = \boldsymbol{v}_3$,
4. $$\begin{aligned}\mathtt{pr}^{(1)}(\boldsymbol{v}_4) = \boldsymbol{v}_4 &+ (1+y_x^2)\partial_{y_x} + y_x u_x \partial_{u_x} + (F_{11,x}y_x - F_{21,x})\partial_{11,x} \\ &+ (F_{12,x}y_x - F_{22,x})\partial_{12,x} + (F_{13,x}y_x - F_{23,x})\partial_{13,x} \\ &+ (F_{21,x}y_x + F_{11,x})\partial_{21,x} + (F_{22,x}y_x + F_{12,x})\partial_{22,x} \\ &+ (F_{23,x}y_x + F_{13,x})\partial_{23,x} + y_x F_{31,x}\,\partial_{31,x} \\ &+ y_x F_{32,x}\,\partial_{32,x} + y_x F_{33,x}\,\partial_{33,x},\end{aligned}$$
5. $$\begin{aligned}\mathtt{pr}^{(1)}(\boldsymbol{v}_5) = \boldsymbol{v}_5 &+ y_x u_x \partial_{y_x} + (1+u_x^2)\partial_{u_x} + (F_{11,x}u_x - F_{31,x})\partial_{11,x} \\ &+ (F_{12,x}u_x - F_{32,x})\partial_{12,x} + (F_{13,x}u_x - F_{33,x})\partial_{13,x} \\ &+ u_x F_{21,x}\,\partial_{21,x} + u_x F_{22,x}\,\partial_{22,x} + u_x F_{23,x}\,\partial_{23,x} \\ &+ (F_{31,x}u_x + F_{11,x})\partial_{31,x} + (F_{32,x}u_x + F_{12,x})\partial_{32,x} \\ &+ (F_{33,x}u_x + F_{13,x})\partial_{33,x},\end{aligned}$$
6. $$\begin{aligned}\mathtt{pr}^{(1)}(\boldsymbol{v}_6) = \boldsymbol{v}_6 &- u_x\partial_{y_x} + y_x\partial_{u_x} - F_{31,x}\;\partial_{21,x} - F_{32,x}\;\partial_{22,x} - F_{33,x}\;\partial_{23,x} \\ &+ F_{21,x}\;\partial_{31,x} + F_{22,x}\;\partial_{32,x} + F_{23,x}\;\partial_{33,x}.\end{aligned}$$

Similar to before, we solve for the Maurer-Cartan invariants, $R^i$, by applying the recurrence relations (3.2) to the phantom invariants. Here is a summary of the resulting equations:

$$\begin{aligned}0 &= 1 + R^1, \\ 0 &= \iota(y_x) + R^2 \\ 0 &= \iota(u_x) + R^3,\end{aligned}$$

and

$$0 = \iota(F_x) + R^4 \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + R^5 \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + R^6 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Using relations from $\mathfrak{so}(3)$, we can take:

$$\begin{aligned} R^1 &= -1, \\ R^2 &= -\iota(y_x) =: -I_1^y, \\ R^3 &= -\iota(u_x) =: -I_1^u, \\ R^4 &= -\iota(F_{21,x}) =: -J_1^{21}, \\ R^5 &= -\iota(F_{31,x}) =: -J_1^{31}, \\ R^6 &= -\iota(F_{32,x}) =: -J_1^{32}. \end{aligned}$$

Following our previous convention, we will denote the normalized invariants by

$$I_n^y = \iota\left(\frac{d^n y}{dx^n}\right), \qquad I_n^u = \iota\left(\frac{d^n u}{dx^n}\right), \qquad \text{and} \qquad J_n^{ij} = \iota\left(\frac{d^n F_{ij}}{dx^n}\right).$$

The fundamental curvature invariants will be denoted as:

$$\kappa^y = I_1^y, \qquad \kappa^u = I_1^u, \qquad \text{and} \qquad \kappa^{ij} = \iota(J_1^{ij}).$$

We will attach a subscript $s$ to a curvature invariant to denote when an invariant arclength derivative has been taken, e.g. $\kappa_s^u = \mathcal{D}_s \kappa^u$. Since $F_x$ is antisymmetric, $\kappa^{ij} = -\kappa^{ij}$, hence $\kappa^{ii} = 0$. This means that a complete set of functionally independent invariants is given by $\{I_n^y, I_n^y, J_{21,n}, J_{31,n}, J_{32,n} : n \in \mathbb{N}\}$. According to formula (3.2), the invariant total derivative of an invariant is given as follows:

$$\begin{aligned} \mathcal{D}_s[\iota(F)] =& \iota\left(D_x(F)\right) - \iota\left(\mathtt{pr}\,\boldsymbol{v}_1(F)\right) - \kappa^y\,\iota\left(\mathtt{pr}\,\boldsymbol{v}_2(F)\right) - \kappa^u\,\iota\left(\mathtt{pr}\,\boldsymbol{v}_3(F)\right) \\ & - \kappa^{21}\,\iota\left(\mathtt{pr}\,\boldsymbol{v}_4(F)\right) - \kappa^{31}\,\iota\left(\mathtt{pr}\,\boldsymbol{v}_5(F)\right) - \kappa^{21}\,\iota\left(\mathtt{pr}\,\boldsymbol{v}_6(F)\right), \end{aligned} \tag{7.1}$$

where $F$ is an arbitrary differential function.

## 7.3 The Invariant Euler-Lagrange Equations

### 7.3.1 Contact Forms in Matrix Coordinates

For the frame portion of a framed curve the canonical Maurer-Cartan form, $\omega_G$, is given by:

$$\omega_G = dg\, g^{-1} = \begin{bmatrix} 0 & -\omega_1 & -\omega_2 \\ \omega_1 & 0 & -\omega_3 \\ \omega_2 & \omega_3 & 0 \end{bmatrix}$$

where

$$\omega_1 = F_{11}dF_{21} + F_{12}dF_{22} + F_{13}dF_{23}, \tag{7.2}$$

$$\omega_2 = F_{11}dF_{31} + F_{12}dF_{32} + F_{13}dF_{33}, \tag{7.3}$$

$$\omega_3 = F_{21}dF_{31} + F_{22}dF_{32} + F_{23}dF_{33}. \tag{7.4}$$

Using $\omega_G$, we find that the invariant contact forms for the frame portion of our curve are the linearly independent components of

$$\theta = dg\, g^{-1} - g_x g^{-1}\, dx = \begin{bmatrix} 0 & -\theta^1 & -\theta^2 \\ \theta^2 & 0 & -\theta^3 \\ \theta^2 & \theta^3 & 0 \end{bmatrix}$$

where

$$\theta^1 = F_{11}dF_{21} + F_{12}dF_{22} + F_{13}dF_{23} - (F_{11}F_{21x} + F_{12}F_{22x} + F_{13}F_{23x})\, dx, \tag{7.5}$$

$$\theta^2 = F_{11}dF_{31} + F_{12}dF_{32} + F_{13}dF_{33} - (F_{11}F_{31x} + F_{12}F_{32x} + F_{13}F_{33x})\, dx, \tag{7.6}$$

$$\theta^3 = F_{21}dF_{31} + F_{22}dF_{32} + F_{23}dF_{33} - (F_{21}F_{31x} + F_{22}F_{32x} + F_{23}F_{33x})\, dx. \tag{7.7}$$

### 7.3.2 Lie Derivatives of Contact Forms

In this section we calculate the Lie derivatives of various contact forms. Since $\boldsymbol{v_1}$, $\boldsymbol{v_2}$, and $\boldsymbol{v_3}$ correspond to simple translations, it is easy to see that they annihilate all contact forms. Applying $\boldsymbol{v_4}$, $\boldsymbol{v_5}$, and $\boldsymbol{v_6}$ to the first order contact forms $\theta_0^y$ and $\theta_0^u$ yields:

$$\begin{aligned} \boldsymbol{v_4}(\theta_0^y) &= y_x\theta_0^y, & \iota(\boldsymbol{v_4}(\theta_0^y)) &= \kappa^y\vartheta_0^y, \\ \boldsymbol{v_5}(\theta_0^y) &= y_x\theta_0^u, & \iota(\boldsymbol{v_5}(\theta_0^y)) &= \kappa^y\vartheta_0^u, \\ \boldsymbol{v_6}(\theta_0^y) &= -\theta_0^u, & \iota(\boldsymbol{v_6}(\theta_0^y)) &= -\vartheta_0^u, \end{aligned}$$

$$\begin{aligned} \boldsymbol{v_4}(\theta_0^u) &= -u_x\theta_0^y, & \iota(\boldsymbol{v_4}(\theta_0^u)) &= -\kappa^u\vartheta_0^y, \\ \boldsymbol{v_5}(\theta_0^u) &= u_x\theta_0^u, & \iota(\boldsymbol{v_5}(\theta_0^u)) &= \kappa^u\vartheta_0^u, \\ \boldsymbol{v_6}(\theta_0^u) &= \theta_0^y, & \iota(\boldsymbol{v_6}(\theta_0^u)) &= \vartheta_0^y. \end{aligned}$$

Applying $\boldsymbol{v_4}$, $\boldsymbol{v_5}$, and $\boldsymbol{v_6}$ to $\theta_0^1$, $\theta_0^2$, and $\theta_0^3$ gives

$$\begin{aligned} \boldsymbol{v_4}(\theta_0^1) &= (F_{11}F_{21,x} + F_{12}F_{22,x} + F_{13}F_{23,x})\theta_0^y, \\ \boldsymbol{v_5}(\theta_0^1) &= (F_{11}F_{21,x} + F_{12}F_{22,x} + F_{13}F_{23,x})\theta_0^u + \theta_0^3, \\ \boldsymbol{v_6}(\theta_0^1) &= 0, \end{aligned}$$

$$\begin{aligned} \boldsymbol{v_4}(\theta_0^2) &= (F_{11}F_{31,x} + F_{12}F_{32,x} + F_{13}F_{33,x})\theta_0^y - \theta_0^3, \\ \boldsymbol{v_5}(\theta_0^2) &= (F_{11}F_{31,x} + F_{12}F_{32,x} + F_{13}F_{33,x})\theta_0^u, \\ \boldsymbol{v_6}(\theta_0^2) &= 0, \end{aligned}$$

$$\begin{aligned} \boldsymbol{v_4}(\theta_0^3) &= (F_{21}F_{31,x} + F_{22}F_{32,x} + F_{23}F_{33,x})\theta_0^y, \\ \boldsymbol{v_5}(\theta_0^3) &= (F_{21}F_{31,x} + F_{22}F_{32,x} + F_{23}F_{33,x})\theta_0^u + \theta_0^1, \\ \boldsymbol{v_6}(\theta_0^3) &= 0. \end{aligned}$$

Applying the invariantization map to our results leads to the following equations:

$$\begin{aligned} &\iota(\boldsymbol{v_4}(\theta_0^1)) = \kappa^{21}\vartheta_0^y, && \iota(\boldsymbol{v_4}(\theta_0^2)) = \kappa^{31}\vartheta_0^y - \vartheta_0^3, && \iota(\boldsymbol{v_4}(\theta_0^3)) = \kappa^{32}\vartheta_0^y, \\ &\iota(\boldsymbol{v_5}(\theta_0^1)) = \kappa^{21}\vartheta_0^u + \vartheta_0^3, && \iota(\boldsymbol{v_5}(\theta_0^2)) = \kappa^{31}\vartheta_0^u, && \iota(\boldsymbol{v_5}(\theta_0^3)) = \kappa^{32}\vartheta_0^u + \vartheta_0^1, \\ &\iota(\boldsymbol{v_6}(\theta_0^1)) = 0, && \iota(\boldsymbol{v_6}(\theta_0^2)) = 0, && \iota(\boldsymbol{v_6}(\theta_0^3)) = 0. \end{aligned}$$

### 7.3.3 Rewriting Higher Order Contact One-Forms

In this section, we develop the machinery to the invariant vertical differential of expression and to rewrite higher order invariant contact one-forms in terms of $\theta_0^u$ and $\theta_0^{ij}$. To do this, we use the following recurrence formula:

$$\mathcal{D}_s(\vartheta_n^\alpha) = \iota(D_x\theta_n^\alpha) - \kappa^{21}\,\iota(\boldsymbol{v_4}(\theta_n^\alpha)) - \kappa^{31}\,\iota(\boldsymbol{v_5}(\theta_n^\alpha)) - \kappa^{32}\,\iota(\boldsymbol{v_6}(\theta_n^\alpha)).$$

Using this, we have the following expressions for $\vartheta_1^u$, $\vartheta_1^u$, and $\vartheta_1^{ij}$:

$$\begin{aligned}
\vartheta_1^y &= (\mathcal{D}_s + \kappa^y\kappa^{21})\vartheta_0^y + (\kappa^y\kappa^{31} - \kappa^{32})\vartheta_0^u,\\
\vartheta_1^u &= (\kappa^{32} - \kappa^{21}\kappa^u)\vartheta^y + (\mathcal{D}_s + \kappa^u\kappa^{21})\vartheta_0^u,\\
\vartheta_1^1 &= (\kappa^{21})^2\vartheta_0^y + \kappa^{21}\kappa^{31}\vartheta_0^u + \mathcal{D}_s\vartheta_0^1 + \kappa^{31}\vartheta_0^3,\\
\vartheta_1^2 &= \kappa^{21}\kappa^{31}\vartheta_0^y + (\kappa^{31})^2\vartheta_0^u + \mathcal{D}_s\vartheta_0^2 - \kappa^{31}\vartheta_0^3,\\
\vartheta_1^3 &= \kappa^{21}\kappa^{32}\vartheta_0^y + \kappa^{31}\kappa^{32}\vartheta_0^u + \kappa^{31}\vartheta_0^1 + \mathcal{D}_s\vartheta_0^3.
\end{aligned}$$

These expressions will allow us to replace higher order contact one-forms that may arise in the invariant Euler-Lagrange equations.

### 7.3.4 The Invariant Eulerian and Hamiltonian Operators

Using the expressions and tools developed in the previous section, we can now apply the recurrence formula to the set of fundamental curvature invariants. The first step is to determine the one-forms $\varepsilon^i$, for $1 \le i \le 6$ in the following recurrence formula:

$$\begin{aligned}
d_{\mathcal{V}}\, I = d_{\mathcal{V}}\,\iota(W) = \iota(d_V W) &+ \iota(\mathtt{pr}\boldsymbol{v_1}(W))\,\varepsilon^1 + \iota(\mathtt{pr}\boldsymbol{v_2}(W))\,\varepsilon^2 + \iota(\mathtt{pr}\boldsymbol{v_3}(W))\,\varepsilon^3\\
&+ \iota(\mathtt{pr}\boldsymbol{v_4}(W))\,\varepsilon^4 + \iota(\mathtt{pr}\boldsymbol{v_5}(W))\,\varepsilon^5 + \iota(\mathtt{pr}\boldsymbol{v_6}(W))\,\varepsilon^6.
\end{aligned}$$

Again, we use the phantom invariants to solve for the unknown invariant contact forms, $\varepsilon^i$. Doing so produces

$$\varepsilon^1 = 0, \quad \varepsilon^2 = -\vartheta_0^y, \quad \varepsilon^3 = -\vartheta_0^u \quad \varepsilon^4 = -\vartheta_0^{21}, \quad \varepsilon^5 = -\vartheta_0^{31}, \quad \text{and} \quad \varepsilon^6 = -\vartheta_0^{32}. \tag{7.8}$$

Similar to the two-dimensional case, we observe that $\vartheta_0^{21} = \vartheta_0^1$, $\vartheta_0^{31} = \vartheta_0^2$, and $\vartheta_0^{32} = \vartheta_0^3$. This leads to the following equations:

$$\begin{aligned} d_{\mathcal{V}}(\kappa^y) =& (\mathcal{D}_s + \kappa^y\kappa^{21})\vartheta_0^y + (\kappa^y\kappa^{31} - \kappa^{32})\vartheta_0^u \\ &- (1 + (\kappa^y)^2)\vartheta_0^1 - \kappa^y\kappa^u\vartheta_0^2 + \kappa^u\vartheta_0^3, \end{aligned} \tag{7.9}$$

$$\begin{aligned} d_{\mathcal{V}}(\kappa^u) =& (\kappa^{32} - \kappa^u\kappa^{21})\vartheta_0^y + (\mathcal{D}_s + \kappa^u\kappa^{31})\vartheta_0^u \\ &- \kappa^y\kappa^u\vartheta_0^1 - (1 + (\kappa^u)^2)\vartheta_0^2 - \kappa^y\vartheta_0^2, \end{aligned} \tag{7.10}$$

$$\begin{aligned} d_{\mathcal{V}}(\kappa^{21}) =& (\kappa^{21})^2\vartheta_0^y + \kappa^{21}\kappa^{31}\vartheta_0^u + (\mathcal{D}_s - \kappa^{21}\kappa^y)\vartheta_0^1 \\ &- (\kappa^{32} + \kappa^{21}\kappa^u)\vartheta_0^2 + \kappa^{31}\vartheta_0^3, \end{aligned} \tag{7.11}$$

$$\begin{aligned} d_{\mathcal{V}}(\kappa^{31}) =& \kappa^{21}\kappa^{31}\vartheta_0^y + (\kappa^{31})^2\vartheta_0^u + (\kappa^{32} - \kappa^{31}\kappa^y)\vartheta_0^1 \\ &+ (\mathcal{D}_s - \kappa^{31}\kappa^u)\vartheta_0^2 - \kappa^{21}\vartheta_0^3, \end{aligned} \tag{7.12}$$

$$\begin{aligned} d_{\mathcal{V}}(\kappa^{32}) =& \kappa^{21}\kappa^{32}\vartheta_0^y + \kappa^{31}\kappa^{32}\,\vartheta_0^u + (\kappa^{31} - \kappa^{32}\kappa^y)\vartheta_0^1 \\ &+ \left(\kappa^{21} - \kappa^{32}\kappa^u\right)\vartheta_0^2 + \mathcal{D}_s\vartheta_0^3. \end{aligned} \tag{7.13}$$

Using these expressions, it follows that the invariant Eulerian operator is given by:

$$\mathcal{A} = \begin{bmatrix} \mathcal{D}_s + \kappa^{21}\kappa^y & \kappa^{31}\kappa^y - \kappa^{32} & -1 - \left(\kappa^y\right)^2 & -\kappa^u\kappa^y & \kappa^u \\ -\kappa^{21}\kappa^u + \kappa^{32} & \mathcal{D}_s + \kappa^{21}\kappa^u & -\kappa^u\kappa^y & -1 - \left(\kappa^u\right)^2 & -\kappa^y \\ \left(\kappa^{21}\right)^2 & \kappa^{21}\kappa^{31} & \mathcal{D}_s - \kappa^{21}\kappa^y & -\kappa^{21}\kappa^u & 2\kappa^{31} \\ \kappa^{21}\kappa^{31} & \left(\kappa^{31}\right)^2 & -\kappa^{31}\kappa^y & \mathcal{D}_s - \kappa^{31}\kappa^u & -\kappa^{21} - \kappa^{31} \\ \kappa^{21}\kappa^{32} & \kappa^{31}\kappa^{32} & \kappa^{31} - \kappa^{32}\kappa^y & \kappa^{21} - \kappa^{32}\kappa^u & \mathcal{D}_s \end{bmatrix}$$

and

$$\mathcal{A}^* = \begin{bmatrix} -\mathcal{D}_s + \kappa^{21}\kappa^y & -\kappa^{21}\kappa^u + \kappa^{32} & \left(\kappa^{21}\right)^2 & \kappa^{21}\kappa^{31} & \kappa^{21}\kappa^{32} \\ \kappa^{31}\kappa^y - \kappa^{32} & -\mathcal{D}_s + \kappa^{21}\kappa^u & \kappa^{21}\kappa^{31} & \left(\kappa^{31}\right)^2 & \kappa^{31}\kappa^{32} \\ -1 - \left(\kappa^y\right)^2 & -\kappa^u\kappa^y & -\mathcal{D}_s - \kappa^{21}\kappa^y & -\kappa^{31}\kappa^y & \kappa^{31} - \kappa^{32}\kappa^y \\ -\kappa^u\kappa^y & -1 - \left(\kappa^u\right)^2 & -\kappa^{21}\kappa^u & -\mathcal{D}_s - \kappa^{31}\kappa^u & \kappa^{21} - \kappa^{32}\kappa^u \\ \kappa^u & -\kappa^y & 2\kappa^{31} & -\kappa^{21} - \kappa^{31} & -\mathcal{D}_s \end{bmatrix}.$$

The expression for $\mathcal{A}^*$ was carefully determined using the symbolic package *Sympy* in the python programming language. For details of this calculation, please visit the Lie_Symmetry project page on the Github account `https://github.com/broom010/Lie_Symmetry`. The calculation for producing $\mathcal{A}$ can be found in the Ipython notebook labeled "Chapter_7".

Next we work out the invariant Hamiltonian operator. The calculation is as follows:

$$\begin{aligned} d_{\mathcal{V}}\varpi =& \iota(D_x\xi^4)\,\varepsilon^4 \wedge \varpi + \iota(D_x\xi^4)\,\varepsilon^4 \wedge \varpi \\ &+ \iota\left(\frac{\partial \xi^4}{\partial y}\right)\gamma^4 \wedge \vartheta_0^y + + \iota\left(\frac{\partial \xi^5}{\partial u}\right)\gamma^5 \wedge \vartheta_0^u \\ =& -\kappa^{21}\,\vartheta_0^y \wedge \varpi - \kappa^{31}\,\vartheta_0^u \wedge \varpi + \kappa^y\,\vartheta_0^1 \wedge \varpi + \kappa^u\,\vartheta_0^2 \wedge \varpi. \end{aligned} \tag{7.14}$$

Therefore

$$\mathcal{B} = \begin{bmatrix} -\kappa^{21} & -\kappa^{31} & \kappa^y & \kappa^u & 0 \end{bmatrix}$$

and

$$\mathcal{B}^* = \mathcal{B}^\top.$$

Using formula (5.6) we find that $W$ is given by:

$$W = (F_{11} + F_{21}y_x + F_{31}u_x)\begin{bmatrix} W_{11} & W_{12} & W_{13} & W_{14} & W_{15} \\ W_{21} & W_{22} & W_{23} & W_{24} & W_{25} \\ 0 & 0 & F_{33} & -F_{32} & F_{31} \\ 0 & 0 & F_{23} & -F_{22} & F_{21} \\ 0 & 0 & F_{13} & -F_{12} & F_{11} \end{bmatrix} \tag{7.15}$$

where

$$W_{11} = \frac{F_{11}F_{22} - F_{12}F_{21} - F_{21}F_{32}u_x + F_{22}F_{31}u_x}{F_{11} + F_{21}y_x + F_{31}u_x}, \tag{7.16}$$

$$W_{12} = \frac{F_{11}F_{23} - F_{13}F_{21} - F_{21}F_{33}u_x + F_{23}F_{31}u_x}{F_{11} + F_{21}y_x + F_{31}u_x}, \tag{7.17}$$

$$W_{13} = -\frac{F_{21}\left(F_{11x}F_{12} + F_{21x}F_{22} + F_{31x}F_{32}\right)}{F_{11} + F_{21}y_x + F_{31}u_x}, \tag{7.18}$$

$$W_{14} = -\frac{F_{21}\left(F_{11x}F_{13} + F_{21x}F_{23} + F_{31x}F_{33}\right)}{F_{11} + F_{21}y_x + F_{31}u_x}, \tag{7.19}$$

$$W_{15} = -\frac{F_{21}\left(F_{12x}F_{13} + F_{22x}F_{23} + F_{32x}F_{33}\right)}{F_{11} + F_{21}y_x + F_{31}u_x}, \tag{7.20}$$

$$W_{21} = \frac{F_{11}F_{32} - F_{12}F_{31} + F_{21}F_{32}y_x - F_{22}F_{31}y_x}{F_{11} + F_{21}y_x + F_{31}u_x}, \tag{7.21}$$

$$W_{22} = \frac{F_{11}F_{33} - F_{13}F_{31} + F_{21}F_{33}y_x - F_{23}F_{31}y_x}{F_{11} + F_{21}y_x + F_{31}u_x}, \tag{7.22}$$

$$W_{23} = -\frac{F_{31}\left(F_{11x}F_{12} + F_{21x}F_{22} + F_{31x}F_{32}\right)}{F_{11} + F_{21}y_x + F_{31}u_x}, \tag{7.23}$$

$$W_{24} = -\frac{F_{31}\left(F_{11x}F_{13} + F_{21x}F_{23} + F_{31x}F_{33}\right)}{F_{11} + F_{21}y_x + F_{31}u_x}, \tag{7.24}$$

$$W_{25} = -\frac{F_{31}\left(F_{12x}F_{13} + F_{22x}F_{23} + F_{32x}F_{33}\right)}{F_{11} + F_{21}y_x + F_{31}u_x}. \tag{7.25}$$

At first this may seem quite daunting, but the structure of $W$ makes it easy to determine when it is singular. The determinant of the lower right three-by-three block of $W$ is simply

$$-\left(F_{11} + F_{21}y_x + F_{31}u_x\right)^3. \tag{7.26}$$

More remarkably, the determinant of the upper left two-by-two block of $W$ simplifies to $F_{11} + F_{21}y_x + F_{31}u_x$. Thus $W$ is singular precisely when

$$F_{11} + F_{21}y_x + F_{31}u_x = 0. \tag{7.27}$$

The details of this simplification can be found in the "Chapter_7" Ipython notebook found on the webpage `https://github.com/broom010/Lie_Symmetry`. This shows us that as long as $\boldsymbol{z}_x$ does not vanish, the invariant Euler-Lagrange equations are given by

$$\mathcal{A}^*\mathcal{E} - \mathcal{B}^*\mathcal{H} = 0. \tag{7.28}$$

This is in line with our findings for the two-dimensional case, and it greatly simplifies the invariant Euler-Lagrange equations for non-singular curves. Next let us explore a few simple examples.

## 7.4 Examples

Due to the computational difficulties that arise in dealing with variational problems for three-dimensional framed curves, we will focus on two simple examples that show how the machinery that we developed can aid in solving such problems.

**Example 7.** For our first example, we will consider the problem of minimizing

$$\int ds = \int (F_{11} + F_{21}y_x + F_{31}u_x)dx.$$

In this case, the invariant Lagrangian is given by $\widetilde{L} = 1$, therefore,

$$\mathcal{E}(\widetilde{L}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix}^\top, \qquad \text{and} \qquad \mathcal{H}(\widetilde{L}) = -1.$$

Plugging this into our equation for $\mathcal{G}(\widetilde{L})$ gives:

$$\mathcal{G}(\widetilde{L}) = \begin{bmatrix} -\kappa^{21} & -\kappa^{31} & \kappa^y & \kappa^u & 0 \end{bmatrix}^\top.$$

This leads to the following result:

$$\kappa^y = \kappa^u = \kappa^{21} = \kappa^{31} = 0.$$

**Example 8.** For our second example, we will consider the problem of minimizing $\int \kappa^y \, ds$. In this case, we have

$$\int \kappa^y \, ds = \widetilde{L} \, ds = \int \frac{F_{12} + F_{22}y_x + F_{32}u_x}{F_{11} + F_{21}y_x + F_{31}u_x}(F_{11} + F_{21}y_x + F_{31}u_x)dx$$

$$= \int (F_{12} + F_{22}y_x + F_{32}u_x)dx$$

Calculating the invariant Eulerians and Hamiltonian, we get:

$$\mathcal{E}(\widetilde{L}) = \begin{bmatrix}1 & 0 & 0 & 0 & 0\end{bmatrix}^\top, \qquad \text{and} \qquad \mathcal{H}(\widetilde{L}) = -\kappa^u.$$

Plugging this into our equation for $\mathcal{G}(\widetilde{L})$ gives:

$$\mathcal{G}(\widetilde{L}) = \begin{bmatrix}0 & -\kappa^{32} & -1 & 0 & \kappa^u\end{bmatrix}^\top.$$

Since $\mathcal{G}(\widetilde{L}) = 0$ gives rise to a contradiction, we must conclude that $W$ is singular and $\mathcal{G}(\widetilde{L})$ is in the null space of $W$. This is another example for which $W$ is singular. As before, this example is considered degenerate and there is no smooth solution to the invariant Euler-Lagrange equations.

# Chapter 8

# Conclusion

In order to complete our derivation of the invariant Euler-Lagrange equations in the three-dimensional case, we required a suitable set of coordinates for the frame portion of our framed curve. When approaching this choice, one might select explicit coordinates in the form of Euler angles or Cardan angles—one might even make use of quaternions. In practice, we have found that it is quite difficult to use these coordinate systems to derive the various quantities that arise in the invariant bicomplex. The complicated nature of the expressions that arise in specifying the action of $SO(3)$ leads to unwieldy formulas for invariants, invariant differential forms, and infinitesimal generators.

Rather than choose explicit coordinates, we decided to work with general matrix coordinates subject to the relations of $SO(3)$. The benefits of this approach were first realized after re-examining the two-dimensional case, where the correspondence between the geometric objects from the variational bicomplex in the various coordinate systems became apparent. In the three-dimensional case, the final key needed for using matrix coordinates came from the relations among the entries of the canonical Maurer-Cartan form. Due to these relations, we are able to simplify the various Lie derivatives of the fundamental contact forms. Making use of the replacement theorem and the relations of $SO(3)$ allowed us to express the invariant Euler-Lagrange equations in a reasonable manner.

In chapter eight, we found that the equations that arise from the variational bicomplex can be simplified in the case of restricted framed curves. Simplifying the expressions for the invariant vertical derivatives of the fundamental curvature invariants led to relations among the invariant contact forms. A key observation that allowed us to modify our more general results to the case of restricted framed curves was the fact that the non-vanishing curvature invariants still generated the algebra of differential invariants. Our final example in chapter eight is by far the more interesting result for two reasons: first, it appears to be a genuinely new result; second, it is natural to require that one of the frame vectors be the unit tangent to the curve.

The explicit variational problems, presented in our examples, demonstrate the computational benefits of using the invariant Euler-Lagrange equations. However, these examples are relatively simple and do not lend themselves to important applications. With that said, applications that might benefit from our work include the analysis of global geometry in DNA ([Gra16]) as well as the study of the mechanics of Möbius bands ([FF16]). In both cases, one examines $SE(3)$-invariant variational problems over framed curves; therefore, our formulas from chapters seven and eight apply to these applications.

Beyond our work on Euler-Lagrange equations, there are many other interesting problems yet to be studied in the context of $SE(3)$ acting on framed curves. For example, the operators $\mathcal{A}$ and $\mathcal{B}$ that we derived in chapter seven can be used to study the evolution of differential invariants under invariant submanifold flows. It is also possible that our description of the invariants of $SE(3)$ acting on framed curves might someday aid in the creation of invariant numerical methods (see [Olv09, p. 31-42]). In a recent development, the invariant Euler-Lagrange equations for rotation-minimizing framed curves has been detailed by Mansfield and Rojo-Echeburua in a preprint on the arXiv ([MRE19]). These types of frames have potential applications in studying the geometry of proteins and polymers, of which the results are both impressive and intriguing. It would be interesting to explore connections between their work and our own; however, time dictates that this must be an endeavor for another day.

# References

[And92] Ian M. Anderson. Introduction to the variational bicomplex. In *Mathematical aspects of classical field theory (Seattle, WA, 1991)*, volume 132 of *Contemp. Math.*, pages 51–73. Amer. Math. Soc., Providence, RI, 1992.

[Ehr51] Charles Ehresmann. Les prolongements d'une variété différentiable. I. Calcul des jets, prolongement principal. *C. R. Acad. Sci. Paris*, 233:598–600, 1951.

[FF16] Roger Fosdick and Eliot Fried, editors. *The mechanics of ribbons and Möbius bands*. Springer, Dordrecht, 2016. Previously published in J. Elasticity **119** (2015), no. 1–2; see special issue foreword [ MR3326179].

[FO99] Mark Fels and Peter J. Olver. Moving coframes. II. Regularization and theoretical foundations. *Acta Appl. Math.*, 55(2):127–208, 1999.

[Gra16] Alexandre Emmanuel Grandchamp. On the statistical physics of chains and rods, with application to multi-scale sequence-dependent dna modelling. Technical report, EPFL, 2016.

[KO01] Irina A. Kogan and Peter J. Olver. The invariant variational bicomplex. In *The geometrical study of differential equations (Washington, DC, 2000)*, volume 285 of *Contemp. Math.*, pages 131–144. Amer. Math. Soc., Providence, RI, 2001.

[KO03] Irina A. Kogan and Peter J. Olver. Invariant Euler-Lagrange equations and the invariant variational bicomplex. *Acta Appl. Math.*, 76(2):137–193, 2003.

[Lee13] John M. Lee. *Introduction to smooth manifolds*, volume 218 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2013.

[MRE19] E.L. Mansfield and A. Rojo-Echeburua. On the use of the rotation minimising frame for variational systems with euclidean symmetry. *arXiv preprint arXiv:1903.03139*, 2019.

[Olv93] Peter J. Olver. *Applications of Lie groups to differential equations*, volume 107 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1993.

[Olv95] Peter J. Olver. *Equivalence, invariants, and symmetry*. Cambridge University Press, Cambridge, 1995.

[Olv00] Peter J. Olver. Moving frames and singularities of prolonged group actions. *Selecta Math. (N.S.)*, 6(1):41–77, 2000.

[Olv09] Peter J Olver. Lectures on moving frames. *Retrieved from the University of Minnesota Digital Conservancy, http://hdl.handle.net/11299/46960.*, 2009.

[Ovs82] L. V. Ovsiannikov. *Group analysis of differential equations*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1982. Translated from the Russian by Y. Chapovsky, Translation edited by William F. Ames.

[Vin84a] A. M. Vinogradov. The $\mathcal{C}$-spectral sequence, Lagrangian formalism, and conservation laws. I. The linear theory. *J. Math. Anal. Appl.*, 100(1):1–40, 1984.

[Vin84b] A. M. Vinogradov. The $\mathcal{C}$-spectral sequence, Lagrangian formalism, and conservation laws. II. The nonlinear theory. *J. Math. Anal. Appl.*, 100(1):41–129, 1984.