The Bernstein Inequality for Formal Power Series

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Peyman Ghahremani

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GENNADY LYUBEZNIK, adviser

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On the mountains of truth you can never climb in vain: either you will reach a point higher up today, or you will be training your powers so that you will be able to climb higher tomorrow.
Friedrich Nietzsche

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Abstract

In this thesis, we give a short proof of the Bernstein inequality for the ring of formal power series. Let $R = k[[x_1, \ldots, x_n]]$ be the ring of formal power series over a field of characteristic zero k and let $D(R,k)$ stand for the ring of $k-$linear differential operators in R. If M is a finitely generated $D(R,k)-$module then $d(M) \geq n$ where $d(M)$ is defined to be the Krull dimension of the graded ring $\mathrm{gr}_\Sigma(D(R,k))/\mathrm{Ann}(\mathrm{gr}_\Gamma(M))$ in which $\Sigma = \Sigma_0 \subseteq \Sigma_1 \subseteq \Sigma_2 \subseteq \cdots$ is the standard filtration on $D(R,k)$ and $\Gamma = \Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \cdots$ is any good filtration on M. This is a celebrated inequality called the Bernstein inequality.

The Bernstein inequality was originally proved by I. N. Bernstein [2] for the ring of polynomials by a beautiful short argument. It was extended to formal power series by J, E, Björk [3]. His proof is far from simple.

In chapter 1 we reproduce Bernstein's simple proof for the ring of polynomials and we are going to see that the proof is short and simple. Unfortunately this simple proof does not extend to formal power series since the Bernstein filtration in the ring of formal power series has quite more difficult structure compared to that of ring of polynomials. To be more precise the Bernstein filtration in the ring of polynomials $T = k[x_1, \ldots, x_n]$ is $\Xi = \Xi_0 \subseteq \Xi_1 \subseteq \Xi_2 \subseteq \cdots$ in which Ξ_j is the $k-$vector space generated by monomials of the form $x_1^{\alpha_1} \cdots x_1^{\alpha_1} \partial_1^{\beta_1} \cdots \partial_n^{\beta_n}$ of total degree j where $\partial_i = \partial/\partial x_i : T \to T$ whereas the Bernstein filtration in the ring of formal power series is $\Sigma = \Sigma_0 \subseteq \Sigma_1 \subseteq \Sigma_2 \subseteq \cdots$ in which Σ_j is the left $R-$module generated by monomials of the form $d_1^{\alpha_1} \cdots d_n^{\alpha_n}$ of total degree j in which $d_i = \partial/\partial x_i : R \to R$. Clearly the Bernstein filtration for the ring of polynomials is of simpler structure. In chapter 2 we give Björk's proof of the Bernstein inequality and as we will see, the proof is utterly difficult as it is comprised of many elegant homological algebra tools. In the third chapter a new and short proof for the Bernstein inequality for formal power series is given which is inspired by [6] and is of elementary nature.

Contents

Chapter 1

Bernstein inequality for the ring of polynomials

In this chapter we provide the proof of the Bernstein inequality for the ring of polynomials. Our references are [8], [4], [3] and [5].

1.1 Preliminaries on Graded Rings and Modules

In this section, we will cover the definition and some properties of graded rings and modules that we are going to use later in this thesis. Some results are analogues of classical theorems of commutative algebra. Most of the results in this section are gathered from [4].

Definition 1.1.1. A ring R is called *graded* if there exists a family of subgroups $\{R_n\}_{n\in\mathbb{Z}}$ of R such that $R = \bigoplus_{n\in\mathbb{Z}} R_n$ (as abelian groups) and $R_nR_m \subseteq R_{n+m}$ for all $n, m \in \mathbb{Z}$.
A graded ring R is said to be *nonnegatively* graded if $R_n = 0$ for all $n \leq 0$. A non-zero element $x \in R_n$ is called a *homogeneous* element of R of degree n and we denote by $\deg(x) = n$.

Remark 1.1.2. *If $R = \bigoplus_{n\in\mathbb{Z}} R_n$ is a graded ring then R_0 is a subring of R, $1 \in R_0$ and R_n is an R_0−module for all $n \in \mathbb{Z}$.*

There are many examples of graded rings. In fact, every ring R is trivially graded by setting $R_0 = R$ and $R_n = 0$ for $n \neq 0$. Other rings with more interesting gradings are polynomial rings.

Example 1.1.3. Let S be a ring and $x_1, \ldots, x_l$ be indeterminates over S. Then the polynomial ring $R = S[x_1, \ldots, x_l]$ is graded ring with

$$R_n = \{ \sum_{m \in \mathbb{N}^l} r_m x_1^{m_1} \cdots x_l^{m_l} : r_m \in S \text{ and } m_1 + \cdots + m_l = n\}$$

Clearly $R_0 = S$ and $\deg(x_i) = 1$ for $i = 1, \ldots, l$. This way of grading the polynomial ring is called the *standard grading*.

Definition 1.1.4. Let $R = \bigoplus_{n \in \mathbb{Z}} R_n$ be graded ring and M an $R-$module. We say that M is a *graded* $R-$module if there exists a family of subgroups $\{M_n\}_{n \in \mathbb{Z}}$ of M with $M = \bigoplus_{n \in \mathbb{Z}} M_n$ (as abelian groups) and $R_n M_m \subseteq M_{n+m}$ for all $n, m \in \mathbb{Z}$. If $u \in M \setminus 0$ then there exist a unique integer s and unique non-zero elements $u_{i_j} \in M_{i_j}$ with $u = u_{i_1} + \cdots + u_{i_s}$. The elements u_{i_j} are called *homogeneous* components of u.

Definition 1.1.5. Let $R = \bigoplus_{n \in \mathbb{Z}} R_n$ be a graded ring and let $M = \bigoplus_{n \in \mathbb{Z}} M_n$, $N = \bigoplus_{n \in \mathbb{Z}} N_n$ be graded $R-$modules. An $R-$homomorphism $f : M \to N$ is said to be graded of degree k if $f(N_n) \subseteq M_{n+k}$ for all n. Denote $\operatorname{Hom}_k(M,N)$ to be the Abelian group of homogeneous homomorphisms of degree k. We define ${}^*\operatorname{Hom}_R(M,N) = \bigoplus_{k \in \mathbb{Z}} \operatorname{Hom}_k(M,N)$ which is a graded $R-$submodule of $\operatorname{Hom}_R(M,N)$. The graded modules M and N are said to be isomorphic if there exists a degree 0 isomorphism between them.

Let $M = \bigoplus_{n \in \mathbb{Z}} M_n$ be a graded module over the graded ring $R = \bigoplus_{n \in \mathbb{Z}} R_n$. Given an integer l we can define the new graded modules $M(l)$ over the graded ring R which is equal to M but with a different grading as follows: for every $n \in \mathbb{Z}$ we define $M(l)_n = M_{n+l}$. We call $M(l)$ as M twisted by l. Clearly if $x \in M$ is homogeneous of degree l then Rx is isomorphic with $(R/\operatorname{Ann}(x))(-l)$ as graded modules.

The following result is well-known in commutative algebra.

Proposition 1.1.6 ([1], Prop. 2.2)**.** *Let $R = \bigoplus_{n \in \mathbb{N}} R_n$ be a nonnegatively graded then the following are equivalent.*

1. *R is a Noetherian ring.*

2. *R_0 is Noetherian and R is a finitely generated R_0-algebra.*

Remark 1.1.7. *Let $R = \bigoplus_{n\geq 0} R_n$ be a nonnegatively graded ring and let $M = \bigoplus_{n\in\mathbb{N}} M_n$ be a finitely generated R-module then M_n is finitely generated as R_0-module for all $n \geq 0$.*

Proof. The graded module M can be generated by finitely many homogeneous elements $u_1,\ldots,u_s$ of degrees $k_1,\ldots,k_s$. Let $n \geq 0$ and $u \in M_n$ be given. There exist homogeneous elements $f_1,\ldots,f_s$ of degrees $n-k_1,\ldots,n-k_s$ with $u = f_1u_1+\ldots+f_su_s$. By 1.1.6, R is finitely generated algebra over R_0 hence there exist $x_1,\ldots,x_t \in R_+$ which generate R as an R_0-algebra. Clearly $M_n = R_{n-k_1}u_1+\cdots+R_{n-k_s}u_s$ where R_j stands for R_0 module generated by monomials of total degree j therefore M_n is finitely generated over R_0. □

Definition 1.1.8. Let R be a Noetherian graded ring. A graded ideal $\mathfrak{m}$ is called **maximal* if every graded ideal strictly containing $\mathfrak{m}$ equals R. The ring R is called **local* if it has a unique *maximal ideal. We shall denote a *local ring R with *maximal ideal $\mathfrak{m}$ by $(R,\mathfrak{m})$.

Lemma 1.1.9. *Let $R = R_0 \oplus R_1 \oplus R_2 \oplus \cdots$ be a nonnegatively graded ring. Then:*

1. *Every *maximal ideal of R is of the form $\mathfrak{m} = \mathfrak{m}_0 \oplus R_1 \oplus R_2 \oplus \cdots$ for some maximal ideal $\mathfrak{m}_0$ in R_0.*

2. *If $(R_0,\mathfrak{m}_0)$ is local then R is *local with *maximal ideal $\mathfrak{m} = \mathfrak{m}_0 \oplus R_1 \oplus R_2 \oplus \cdots$.*

Proof. We only need to show (1). Let $\mathfrak{m} = \mathfrak{m}_0 \oplus R_1 \oplus R_2 \oplus \cdots$. Clearly $\mathfrak{m}$ is a graded ideal of R. If $I = I_0 \oplus I_1 \oplus I_2 \oplus \cdots$ is a graded ideal containing $\mathfrak{m}$ then clearly $m_0 \subseteq I_0$. Since I_0 is an ideal of R_0 we have $I_0 = \mathfrak{m}_0$ or $I_0 = R_0$. In the first case we have $I = \mathfrak{m}$ while in the second case we have $I = R$ therefore $\mathfrak{m}$ is a *maximal ideal of R.
Conversely assume that $\mathfrak{n} = \mathfrak{n}_0 \oplus R_1 \oplus R_2 \oplus \cdots$ is *maximal. Let $\mathfrak{n}_0 \subsetneq I_0$ for some ideal I_0 in R_0 then we have $\mathfrak{n} \subsetneq I = I_0 \oplus I_1 \oplus I_2 \oplus \cdots$ since $\mathfrak{n}$ is *maximal we have $I = R$ whence $I_0 = R_0$. This proves that $\mathfrak{n}_0$ is a maximal ideal in R_0. □

Lemma 1.1.10. *Let R be a graded ring which is not a field and assume that the only graded ideals of R are (0) and R. Then $R = k[t, t^{-1}]$ where $k = R_0$ is a field and t is transcendental over k.*

Proof. Clearly every non-zero homogeneous element of R is a unit therefore if $a \in R_n \setminus 0$ then there exists $b \in R_{-n}$ with $ab = 1$. This proves that R_0 is a field.

Now let n be the smallest positive integer with $R_n \neq 0$. Let $t \in R_n \setminus 0$. If t is algebraic over k then there exists a polynomial $p(x) = a_s x^s + \cdots + a_1 x + a_0 \in k[x]$ such that $p(t) = 0$ whence $t(a_s t^{s-1} + \cdots + a_1) = -a_0$. Since $\deg(t) > 0$ we have $\deg(a_s t^{s-1} + \cdots + a_1) > 0$, therefore $\deg(-a_0) > 0$ which is a contradiction. Thus t is transcendental over k.

Now we show that every homogeneous element of R_m is of the form ct^i for some i. The result it true when $0 \leq m < n$. Now assume that $m \geq n$ and let $u \in R_m$. Clearly $t^{-1}u \in R_{m-n}$ and $0 \leq m - n < m$. Thus by the induction hypothesis $t^{-1}u = ct^i$ for some i. Whence $u = ct^{i+1}$ and we are done. Similarly each every homogeneous element of R_n when $n < 0$ is of the form ct^{-i} for some c and $i \geq 0$. This completes the proof. □

Proposition 1.1.11. *(Graded version of prime avoidance) Let R be a ring and I be a graded ideal generated by homogeneous elements of positive degree. Assume that $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ are homogeneous prime ideals, non of which contain I. Then there exists a homogeneous element $x \in I$ with $x \notin \bigcup_{i=1}^{n} \mathfrak{p}_i$.*

Proof. Without loss of generality we can assume that there is no containment relations among the homogeneous prime ideals $\mathfrak{p}_1, \cdots, \mathfrak{p}_n$ whence for each $1 \leq i \leq n$, $\mathfrak{p}_i$ does not contain $\mathfrak{p}_1 \cap \cdots \cap \hat{\mathfrak{p}}_i \cap \cdots \cap \mathfrak{p}_n$. Therefore there exist homogeneous elements $u_i \notin \mathfrak{p}_i$ with $u_i \in \mathfrak{p}_1 \cap \cdots \cap \hat{\mathfrak{p}}_i \cap \cdots \cap \mathfrak{p}_n$. Since $I \nsubseteq \mathfrak{p}_i$, there exist homogeneous elements $w_i \in I$ of positive degree with $w_i \notin \mathfrak{p}_i$. By replacing w_i with sufficiently large power of it we may assume that $\deg(u_i w_i) > 0$. Let $y_i = u_i w_i$. Clearly $y_i \notin \mathfrak{p}_i$ but $y_i \in I \cap \mathfrak{p}_1 \cap \cdots \cap \hat{\mathfrak{p}}_i \cap \cdots \cap \mathfrak{p}_n$. By taking the powers of y_is we may assume that $\deg(y_i) = \deg(y_j)$ for every $i \neq j$. Set $y = y_1 + \cdots + y_n$. Clearly y is homogeneous with $y \in I$ but $y \notin \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n$. □

Definition 1.1.12. Suppose that M is graded module over a graded ring R and let N be a submodule of M. We denote by N^* the $R-$submodule of M generated by all homogeneous elements of N.

Clearly if $\mathfrak{p}$ is a prime ideal of a graded ring R then the graded ideal $\mathfrak{p}^* \subseteq \mathfrak{p}$ is also prime. It is natural to ask what is the relation between $\mathfrak{p}$ and $\mathfrak{p}^*$. The following results will shed a light on the answer.

Lemma 1.1.13. *Let R be a graded ring and $\mathfrak{p}$ a non-homogeneous prime ideal of R. Then there are no primes between $\mathfrak{p}$ and $\mathfrak{p}^*$.*

Proof. By passing to $R/\mathfrak{p}^*$ we may assume that R is a graded domain and $\mathfrak{p}^* = 0$. Let W be the set of all non-zero homogeneous elements of R. Since $\mathfrak{p} \cap W = 0$, $\mathfrak{p}R_W$ is a non-zero prime ideal of R_W. Clearly every non-zero homogeneous element of R_W is a unit whence by 1.1.10 $R_W = k[t^{-1}, t]$ where k is a field t is transcendental over k. Since $\dim(k[t, t^{-1}]) = 1$ there are no primes between (0) and $\mathfrak{p}R_W$. Whence, there are no primes between $\mathfrak{p}^*$ and $\mathfrak{p}$. □

Theorem 1.1.14. *(Matijevic, Roberts) Let R be a graded ring and $\mathfrak{p}$ a non-homogeneous prime ideal of R. Then* $\mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(\mathfrak{p}^*) + 1$.

Proof. The result is clear when $\mathrm{ht}(\mathfrak{p}^*) = \infty$ so we can assume that $\mathrm{ht}(\mathfrak{p}^*) < \infty$. We will complete the proof by induction on $n = \mathrm{ht}(\mathfrak{p}^*)$. If $n = 0$ then we are done by 1.1.13. Now let that $n > 0$ and let $\mathfrak{q}$ be a prime ideal properly contained in $\mathfrak{p}$. We need to show that $\mathrm{ht}(\mathfrak{q}) \leq n$. We have $\mathfrak{q}^* \subseteq \mathfrak{p}^*$. If $\mathfrak{q}^* = \mathfrak{p}^*$ then $\mathfrak{p}^* = \mathfrak{q}^* \subseteq \mathfrak{q} \subsetneq \mathfrak{p}$. Since there is no prime between $\mathfrak{p}^*$ and $\mathfrak{p}$ by 1.1.13 we have $\mathfrak{q} = \mathfrak{p}^*$ whence $\mathrm{ht}(\mathfrak{q}) \leq n$ and we are done. If $\mathfrak{q}^* \subsetneq \mathfrak{p}^*$ then $\mathrm{ht}(\mathfrak{q}) \leq n$ by the induction hypothesis. □

Proposition 1.1.15. *Let R be a Noetherian graded ring and let $\mathfrak{p}$ be a homogeneous prime ideal of height n. Then there exists a chain of distinct homogeneous prime ideals*

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p}.$$

Proof. We will use induction on $n = \mathrm{ht}(\mathfrak{p})$. If $n = 0$ then there is nothing to prove. Let $n > 0$ and the result holds for all homogeneous prime ideals of height less than n. Let $\mathfrak{q}$ be a prime ideal properly contained in $\mathfrak{p}$ of height $n-1$. If $\mathfrak{q}$ is homogeneous then we are done by the induction hypothesis so suppose that $\mathfrak{q}$ is not homogeneous. By 1.1.14 we have $\operatorname{ht}(\mathfrak{q}^*) = \operatorname{ht}(\mathfrak{q}) - 1 = n-2$. By passing to $R/\mathfrak{q}^*$ we can assume that R is a graded domain and $\mathfrak{p}$ is a homogeneous prime ideal of height 2. It suffices to show that $\mathfrak{p}$ contains a homogeneous prime ideal of height 1. Let $x \in \mathfrak{p}$ be a non-zero homogeneous element of $\mathfrak{p}$. By the Krull's principal ideal theorem, $\mathfrak{p}$ is not a minimal prime divisor of (x) so there exists a prime ideal $\mathfrak{p}_1$ with $(x) \subseteq \mathfrak{p}_1 \subsetneq \mathfrak{p}$. Since (x) is homogeneous we can assume that $\mathfrak{p}_1$ is also homogeneous and clearly it is of height 1. □

Theorem 1.1.16. *Let R be a nonnegatively graded ring. Then*

$$\dim(R) = \sup\{\operatorname{ht}(\mathfrak{m}) : \mathfrak{m} \textit{ is a } {}^*\textit{maximal ideal in } R\}.$$

Proof. There exists a maximal ideal $\mathfrak{n}$ of R with $\dim(R) = \operatorname{ht}(\mathfrak{n})$. If $\mathfrak{n}$ is homogeneous then we are done so we assume that $\mathfrak{n}$ is not homogeneous. By 1.1.14 we have $\operatorname{ht}(\mathfrak{n}) = \operatorname{ht}(\mathfrak{n}^*) + 1$. By 1.1.9 $\mathfrak{n}^*$ is not *maximal whence there exists a *maximal ideal $\mathfrak{m}$ of R with $\mathfrak{n}^* \subsetneq \mathfrak{m}$. We have $\operatorname{ht}(\mathfrak{m}) \geq \operatorname{ht}(\mathfrak{n}^*) + 1 = \operatorname{ht}(\mathfrak{n}) = \dim(R)$. □

Theorem 1.1.17. *Let R is a nonnegatively graded Noetherian ring. Then*

$$\dim(R) = \sup\{\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n : \ \mathfrak{p}_0, \cdots, \mathfrak{p}_n \textit{ are homogeneous primes of } R\}.$$

Proof. Immediate from 1.1.16 and 1.1.15 □

The following lemma is the graded version of Nakayama's lemma.

Lemma 1.1.18. *Let $R = R_0 \oplus R_1 \oplus R_2 \oplus \cdots$ be a nonnegatively graded ring with $(R_0, \mathfrak{m}_0)$ local. By 1.1.9 R is *local with $\mathfrak{m} = \mathfrak{m}_0 \oplus R_1 \oplus R_2 \oplus \cdots$ as its *maximal ideal. Let $M = M_0 \oplus M_1 \oplus M_2 \oplus \cdots$ be a finitely generated nonnegatively graded $R-$module with $\mathfrak{m}M = M$ then $M = 0$.*

Proof. By 1.1.7 each of the M_ns are finitely generated R_0-modules and $\mathfrak{m}M = M$ implies that $\mathfrak{m}_0 M_n = M_n$ for all $n \geq 0$. Now NAK implies that $M_n = 0$ for all $n \geq 0$. □

Proposition 1.1.19. *Let M be a graded module over a graded ring R and let N be a submodule of M. Then:*

1. $\sqrt{\mathrm{Ann}_R(M/N^*)} = (\sqrt{\mathrm{Ann}_R(M/N)})^*$.

2. *If N is a $\mathfrak{p}-$primary submodule of M then N^* is a $\mathfrak{p}^*-$ primary submodule of M.*

Proof. (1) Since $N^* \subseteq N$ we have $\mathrm{Ann}_R(M/N^*) \subseteq \mathrm{Ann}_R(M/N)$ therefore $\sqrt{\mathrm{Ann}_R(M/N^*)} \subseteq (\sqrt{\mathrm{Ann}_R(M/N)})^*$. Now let r be a homogeneous element of degree s in $(\sqrt{\mathrm{Ann}_R(M/N)})^*$. There exists an integer m with $r^m M \subseteq N$. Since r^m is of degree ms we have $r^m M_n \subseteq M_{n+ms} \cap N = N^*_{n+ms}$ for all $n \in \mathbb{Z}$. Whence $\sqrt{\mathrm{Ann}_R(M/N^*)} \supseteq (\sqrt{\mathrm{Ann}_R(M/N)})^*$.
(2) Let r be a zero-divisor on M/N^*. We need to show that r is nilpotent on M/N^*. Let s stand for the non-zero homogeneous components of r. We use induction on s to show that r is nilpotent on M/N^*. If $s = 0$ then $r = 0$ and there is nothing to prove. Let $s > 0$ and the hypothesis of the induction holds for all integers less than s. Let $\overline{x}$ be a non-zero element of M/N^* with $rx \in N^*$. Let r_s and $\overline{x_t}$ be the homogeneous components of highest degree in r and x respectively. We have $r_s x_t \in N^*$. This shows that r_s is a zero-divisor on M/N therefore there exists some integer e with $r_s^e M \subseteq N$. Clearly $r_s^e M_n \subseteq M_{n+es} \cap N$ for all $n \in \mathbb{Z}$ whence $r_s \in \sqrt{\mathrm{Ann}_R(M/N^*)}$.
Now choose t to be an integer with $r_s^t x \in N^*$ but $r_s^{t-1} x \notin N^*$. Let $x' = r_s^{t-1} x$ and $r' = r - r_s$. We have $r'x' = rx' - r_s x' \in N^*$. Hence r' is a zero-divisor on N^*. By induction we have
$r' \in \sqrt{\mathrm{Ann}_R(M/N^*)}$. Since we showed $r_s \in \sqrt{\mathrm{Ann}_R(M/N^*)}$ we have $r \in \sqrt{\mathrm{Ann}_R(M/N^*)}$. This shows that N^* is a primary submodule of M and since N is $\mathfrak{p}-$ primary, by (1), N^* is $\mathfrak{p}^*-$primary. □

Corollary 1.1.20. *Let M be a graded module over a graded ring R and let N be a graded submodule of M. If N has a primary decomposition then all of the primary components of N can be chosen to be homogeneous.*

Proof. Let $N = Q_1 \cap \cdots \cap Q_t$ be a primary decomposition of N. We have

$$N = N^* = (Q_1 \cap \cdots \cap Q_t)^* = Q_1^* \cap \cdots \cap Q_t^*$$

By 1.1.19 the above decomposition is a decomposition into homogeneous primary submodules.

□

Corollary 1.1.21. *Let M be finitely generated graded module over a Noetherian graded ring R. Then $\mathrm{Ass}_R(M)$ is consisted of homogeneous prime ideals. Moreover, if $\mathfrak{p} \in \mathrm{Ass}_R(M)$ then $\mathfrak{p} = (0 :_R g)$ for some homogeneous element $g \in M$.*

Proof. That $\mathrm{Ass}_R(M)$ is consisted of homogeneous prime ideals is immediate from 1.1.20. Now let $\mathfrak{p} \in \mathrm{Ass}_R(M)$ we have $\mathfrak{p} = (0 :_R g)$ for some $g \in M$. Let $g = g_1 + \cdots + g_s$ where g_i are the homogeneous components of g of degrees $k_1 < \cdots < k_s$. Clearly $\bigcap_{i=1}^{s}(0 :_R g_i) \subseteq \mathfrak{p}$. Now let $r \in \mathfrak{p}$ be homogeneous. We have $0 = rg = rg_1 + \cdots + rg_s$. Since each of the elements rg_i are homogeneous and have pairwise distinct degrees we have $rg_i = 0$ for $i = 1, \ldots, s$. Therefore $\mathfrak{p} \subseteq \bigcap_{i=1}^{s}(0 :_R g_i)$ and hence $\mathfrak{p} = \bigcap_{i=1}^{s}(0 :_R g_i)$. Since $\mathfrak{p}$ is a prime ideal we have $\mathfrak{p} = (0 :_R g_j)$ for some $1 \leq j \leq s$. □

Proposition 1.1.22. *Let R be a Noetherian graded ring and M a non-zero finitely generated $R-$module. Then there exists a filtration*

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_r = M$$

such that $M_i/M_{i-1} \cong (R/\mathfrak{p}_i)(l_i)$ for some homogeneous prime ideal $\mathfrak{p}_i$ and integer l_i.

Proof. Let Λ stand for the set of all graded submodules of M which are either zero or have a filtration with above property. Let N be a maximal element of Λ. If $M = N$ the we are done. Suppose $M \neq N$. and let $N' = M/N$. Let $\mathfrak{q} \in \mathrm{Ass}_R(N')$. By 1.1.20, $\mathfrak{q}$ is homogeneous therefore N' has a graded submodule L isomorphic to $(R/\mathfrak{q})(l)$ for some integer l as graded modules. Since L is a graded submodule of M/N, it can be written in the form M'/N where M' is the inverse image of L in M with respect to the natural map $M \to M/N$. Clearly $M' \in \Lambda$ a contradiction. □

1.2 Preliminaries on Hilbert Series

We begin with the definition of *Hilbert Series*. Our main references are [8] and [1]. While the introductory given these two books is quite deep, we modify the results if necessary in such a way that it fits with the objective of our thesis.

Definition 1.2.1. Let $R = k[x_1, \ldots, x_l]$ be the ring of polynomials in l variables over a field k equipped with standard grading and $M = \bigoplus_{n \geq 0} M_n$ a graded $R-$module. Suppose that $\mathrm{Vdim}_k(M_n) < \infty$ for all n. Define the *Hilbert function* $H_M : \mathbb{Z}_{\geq 0} \to \mathbb{Z}$ of M by

$$H_M(n) = \mathrm{Vdim}_k(M_n)$$

for all $n \in \mathbb{Z}_{\geq 0}$. We define *Hilbert series* of M to be

$$P(M,t) = \sum_{n=0}^{\infty} \mathrm{Vdim}_k(M_n) t^n$$

as an element of $\mathbb{Z}[[t]]$.

It is natural to ask if the ring $R = k[x_1, \ldots, x_l]$ possesses a Hilbert series.

Example 1.2.2. Let $R = k[x_1, \ldots, x_l] = \bigoplus_{n=0}^{\infty} R_n$ be the ring of polynomials in l variables over a field k equipped with the standard grading. For $m = (m_1, \ldots, m_l) \in \mathbb{Z}_{\geq 0}^l$ let $\mathbf{x}^m = x^{m_1} \cdots x^{m_l}$. We have:

$$R_n = \{ \sum_{m \in \mathbb{Z}_{\geq 0}^l} r_m \mathbf{x}^m : r_m \in k \text{ and } m_1 + \cdots + m_l = n \}$$

Clearly the vector dimension of each of R_ns over the field k is equal to the number of solutions of $m_1 + \cdots + m_l = n$ in $\mathbb{Z}_{\geq 0}^l$ which is equal to $\binom{l+n-1}{l-1}$. Thus

$$P(R,t) = \sum_{n=0}^{\infty} \binom{l+n-1}{l-1} t^n = (1-t)^{-l}$$

The above example shows that if $R = k[x_1, \ldots, x_l]$ is the ring of power series in l variables over a field k then the Hilbert series of it will be a rational function of t of the form $\frac{f(t)}{(1-t)^l}$ where $f(t)$ is a polynomial in $\mathbb{Z}$. We will see that this is not a coincedence on the following theorem.

Theorem 1.2.3. *Let $R = k[x_1, \ldots, x_l]$ be the ring of polynomials over the field k equipped with standard grading and let $M = \bigoplus_{n=0}^{\infty} M_n$ be finitely generated graded $R-$module. Suppose that* $\mathrm{Vdim}_k(M_n) < \infty$ *for all $n \geq 0$ then $P(M,t) = \frac{g(t)}{(1-t)^l}$ where $g(t) \in \mathbb{Z}[t]$.*

Proof. We proceed by on induction on l. If $l = 0$ then $R = k$. Since M is a finitely generated $R-$module, $M_n = 0$ for large n whence $P(M,t)$ is a polynomial and we are done. Let $l > 0$ and we are done for all integers less than l. We have the following exact sequence

$$0 \to (0 :_M x_l)(-1) \to M(-1) \xrightarrow{x_l} M \to M/x_lM \to 0$$

Therefore for each $n \geq 0$ we have

$$0 \to (0 :_M x_l)_{n-1} \to M_{n-1} \xrightarrow{x_l} M_n \to (M/x_lM)_n \to 0$$

Whence we have

$$\text{Vdim}_k(M_n)t^n - \text{Vdim}_k(M_{n-1})t^n = \text{Vdim}_k((M/x_lM)_n)t^n - \text{Vdim}_k((0 :_M x_l)_{n-1})t^n$$

Summing up these equations over $n \geq 0$ we have

$$P(M,t) - tP(M,t) = P(M/x_lM) - tP((0 :_M x_l))$$

Note that M/x_lM and $(0 :_M x_l)$ are modules over $k[[x_1,\ldots,x_{l-1}]$ there for by the hypothesis of the induction $P(M/x_lM) = \frac{g_1(t)}{(1-t)^{l-1}}$ and $P((0 :_M x_l)) = \frac{g_2(t)}{(1-t)^{l-1}}$ for some $g_1(t), g_2(t) \in \mathbb{Z}[t]$. Therefore we have

$$(1-t)P(M,t) = \frac{g_1(t) - tg_2(t)}{(1-t)^{l-1}}$$

diving the last equation by $1-t$ gives the result. □

Corollary 1.2.4. *Let $R = k[x_1,\ldots,x_l]$ be the ring of polynomials over the field k equipped with standard grading and let $M = \bigoplus_{n=0}^{\infty} M_n$ be a non-zero finitely generated graded $R-$module. Suppose that* $\text{Vdim}_k(M_n) < \infty$ *for all $n \geq 0$ then there exists a unique integer $s = s(M) \leq l$ with $P(M,t) = \frac{g(t)}{(1-t)^s}$ where $g(t) \in \mathbb{Z}[t]$ and $g(1) \neq 0$.*

Proof. By 1.2.3 we have $P(M,t) = \frac{g(t)}{(1-t)^l}$ for some $g(t) \in \mathbb{Z}[t]$. We may write $g(t) = (1-t)^m f(t)$ where $f(1) \neq 0$. We have $P(M,t) = \frac{f(t)}{(1-t)^s}$ where $s = l-m$. If $s \geq 0$ then we are done, if $s < 0$ then clearly $P(M,1) = 0$ which implies that $\sum_{n=0}^{\infty} \text{Vdim}_k(M_n) = 0$ whence $M = 0$ which is a contradiction. The uniqueness of $s = s(M)$ is clear. □

Lemma 1.2.5. *Let $R = k[x_1, \dots, x_l]$ be the ring of polynomials in l variables over a field k equipped with standard grading and let*

$$0 \to M_s \to M_{s-1} \to \cdots \to M_0 \to 0$$

be an exact sequence of graded modules with degree 0 maps. If for each $0 \leq i \leq s$ and $n \geq 0$ $\mathrm{Vdim}_k(M_{in}) < \infty$ then $\sum_{i=1}^{s}(-1)^i P_{M_i}(t) = 0$.

Proof. Since the maps in the above exact sequence are of degree 0, for each $n \geq 0$ we have the following exact sequence of $k-$vector spaces

$$0 \to (M_s)_n \to (M_{s-1})_n \to \cdots \to (M_0)_n \to 0$$

therefore we have $\sum_{i=1}^{s}(-1)^i \mathrm{Vdim}_k((M_i)_n) = 0$. Multiplying the last equation by t^n and adding the equations over n gives the result. □

Proposition 1.2.6. *Let $R = k[x_1, \dots, x_l]$ be the ring of polynomials in l variables over a field k equipped with standard grading and let $M = \bigoplus_{n=0}^{\infty} M_n$ be a graded $R-$module having a Hilbert series of the form*

$$P(M,t) = \frac{f(t)}{(1-t)^s}$$

where $f(t) \in \mathbb{Z}[t]$ and $f(1) \neq 0$. Then there exists a unique polynomial $Q(x) \in \mathbb{Q}[x]$ of degree $s-1$ such that $\mathrm{Vdim}_k(M_n) = Q(n)$ for large n.

Proof. Let $f(t) = a_0 + a_1 t + \cdots + a_r t^r$. We have

$$P(M,t) = \frac{f(t)}{(1-t)^s} = f(t) \sum_{n=0}^{\infty} \binom{n+s-1}{s-1} t^n$$

Comparing the coefficients of t^n for $n \geq r$ we have

$$\mathrm{Vdim}_k(M_n) = \sum_{i=0}^{r} a_i \binom{n+s-i-1}{s-1}$$

Let $Q(x) = \sum_{i=0}^{r} a_i \binom{x+s-i-1}{s-1}$. Clearly $Q(x) \in \mathbb{Q}[x]$, $\mathrm{Vdim}_k(M_n) = Q(n)$ for $n \geq r$ and the degree of $Q(x)$ is at most $s-1$. Note that the coefficient of x^{s-1} in $Q(x)$ is $\frac{a_0+a_1+\cdots+a_r}{(s-1)!} = \frac{f(1)}{(s-1)!}$. Since $f(1) \neq 0$ the degree of $Q(x)$ is $s-1$. This completes the proof. □

The above proposition gives rise to the notion of *Hilbert polynomial*.

Definition 1.2.7. Let $R = k[x_1,\dots,x_l]$ be the ring of polynomials in l variables over a field k equipped with standard filtration and $M = \bigoplus_{n=0}^{\infty} M_n$ a graded $R-$module with Hilbert function $H_M(n)$. A polynomial $Q_M(x) \in \mathbb{Q}[x]$ is called the *Hilbert polynomial* of M if $Q_M(n) = \text{Vdim}_k(M_n)$ for all sufficiently large integers n.

Proposition 1.2.8. *Let $R = k[x_1,\dots,x_l]$ be the ring of power series over a field k equipped with standard grading. Let M be a graded finitely generated $R-$module then M has a Hilbert polynomial $Q_M(x)$ and $\deg(Q_M(x)) = s(M) - 1 \leq l - 1$.*

Proof. Immediate from 1.2.4 and 1.2.6. □

Proposition 1.2.9. *[1, Proposition. 11.3] Let $R = k[x_1,\dots,x_l]$ be the ring of polynomials in l variables over a field k equipped with standard grading, let $M = \bigoplus_{n=0}^{\infty} M_n$ be a finitely generated graded $R-$module and let $x \in R$ be a non-zero divisor graded element. Then $s(M/xM) = s(M) - 1$.*

Definition 1.2.10. Let $R = k[x_1,\dots,x_l] = \bigoplus_{n=0}^{\infty} R_n$ be the ring of polynomials in l variables over a field k equipped with standard filtration and $M = \bigoplus_{n=0}^{\infty} M_n$ a graded $R-$module. An element $x \in R_t$ is said to be superficial of order t if $\text{Ann}_{M_n}(x) = 0$ for sufficiently large n.

The above definition motivates the question that if superficial elements exist. The next proposition shows the existence of superficial elements when M is a finitely generated graded module over $R = k[x_1,\dots,x_l]$.

Proposition 1.2.11. *Let $R = k[x_1,\dots,x_l] = \bigoplus_{n=0}^{\infty} R_n$ be the ring of polynomials in l variables over a field k of characteristic zero equipped with standard grading and let $M = \bigoplus_{n=0}^{\infty} M_n$ be a finitely generated graded $R-$module. Then there exists $1 \leq i \leq l$ such that x_i is superficial for M.*

Proof. Let $R_+ = \bigoplus_{n=1}^{\infty} R_n$ and let $0 = Q_1 \cap \cdots \cap Q_t$ be a minimal primary decomposition of 0 in M. Since M is graded, by 1.1.20 all Q_is can be chosen to be graded. Let $P_i = \sqrt{\text{Ann}_R(M/Q_i)}$

be the prime ideal associated to Q_i for $i = 1, \ldots, t$. Since R_+ is a maximal ideal for R, we can reorder the Q_is so that $R_+ \nsubseteq P_i$ for $i = 1, \cdots, t-1$ and $R_+ = P_t$.

Clearly $R_+ \nsubseteq P_1 \cup \cdots \cup P_{t-1}$ thus $R_1 \nsubseteq P_i \cap R_1$ for any $i = 1, \ldots, t-1$, else $R_+ \subseteq P_i$ which is a contradiction. We claim that $R_1 \nsubseteq \bigcup_{j=1}^{t-1}(P_j \cap R_1)$. If $R_1 = \bigcup_{j=1}^{t-1}(P_j \cap R_1)$ then since k is of characteristic zero and hence infinite, we have written R_1 which is a vector space over k as a finite union of finite number of proper subspaces which is a contradiction. This implies that there exists a homogeneous element $x = x_i$ for some $1 \leq i \leq l$ which is not in $P_1 \cup \cdots \cup P_{t-1}$. We show that x is a superficial element for M.

Since $P_t = \sqrt{\operatorname{Ann}_R(M/Q_t)}$ there exists $k \geq 1$ such that $R_+^k = P_t^k \subseteq \operatorname{Ann}_R(M/Q_t)$. Therefore, $R_+^k M \subseteq Q_t$. Since M is finitely generated over R, there exists a number N such that $M_n \subseteq R_+^k M$ for $n \geq N$ therefore $M_n \subseteq Q_t$ for $n \geq N$. Now we claim that $\operatorname{Ann}_{M_n}(x) = 0$ for $n \geq N$. Let $n \geq N$ and $u \in \operatorname{Ann}_{M_n}(x)$ be given. We have $ux = 0 \in Q_1 \cap \cdots \cap Q_{t-1}$ since $x \notin P_i$ for $i = 1, \cdots, t-1$ we have $u \in Q_1 \cap \cdots \cap Q_t = 0$. This completes the proof. $\square$

Proposition 1.2.12. *[1, Corollary. 11.18] Let* $(A, \mathfrak{m})$ *be a Noetherian local ring and let x be a non zero-divisor element of* $\mathfrak{m}$. *Then* $\dim(A/Ax) = \dim(A) - 1$.

Lemma 1.2.13. *Let* $R = k[x_1, \ldots, x_l] = \bigoplus_{n=0}^{\infty} R_n$ *be the ring of polynomials in l variables over a field k equipped with standard grading, let* $M = \bigoplus_{n=0}^{\infty} M_n$ *be a finitely generated graded* $R-$*module with* $\dim(M) > 0$. *Then for any superficial element* $x \in R_+$ *for M we have*

$$\dim(M/xM) = \dim(M) - 1.$$

Proof. Without loss of generality we may assume that $\operatorname{Ann}_R(M) = 0$ thus $\dim(M/xM) = \dim(R/xR)$. Let $N = R_+$. Since N is maximal ideal of R of height l we have $\dim(R) = l = \dim(R_N)$. Using 1.2.12 we have

$$\dim(R/xR) \geq \dim(R_N/xR_N) = \dim(R_N) - 1 = \dim(R) - 1$$

In order to complete the proof we need to show that $\dim(R/xR) < \dim(R)$. Suppose we have $\dim(R/xR) = \dim(R)$. Therefore there exists a prime ideal $\mathfrak{p}$ containing xR with

$\dim(R/\mathfrak{p}) = \dim(R)$. Since $\mathfrak{p}$ is minimal over $(0) = \mathrm{Ann}_R(M)$, $\mathfrak{p} \in \mathrm{Ass}_R(M)$. By 1.1.21 $\mathfrak{p} = (0 :_R g)$ for some homogeneous element $g \in M$. Since $x \in \mathfrak{p}$ we have $g \in (0 :_M x)$ and clearly $R_+^k g \subseteq (0 :_M x)$ for all $k \geq 1$. Since x is superfluous $(0 :_M x)_n = 0$ for large n hence $R_+^k g = 0$ for large k. Hence $R_+ \subseteq \mathfrak{p}$ and therefore $R_+ = \mathfrak{p}$. We have

$$\dim(M) = \dim(R) = \dim(R/\mathfrak{p}) = \dim(R/R_+) = \dim(k) = 0$$

a contradiction. Hence $\dim(R/xR) < \dim(R)$ as desired. □

Using the above results, we will prove the following theorem which is of great importance in later chapters.

Theorem 1.2.14. *Let $R = k[x_1, \ldots, x_l] = \bigoplus_{n=0}^{\infty} R_n$ be the ring of polynomials in l variables over a field k equipped with standard grading and let $M = \bigoplus_{n=0}^{\infty} M_n$ be a non-zero finitely generated graded $R-$module of dimension $d = \dim(M)$. Then there exists $f(t) \in \mathbb{Z}[t]$ with $f(1) \neq 1$ such that*

$$P(M,t) = \frac{f(t)}{(1-t)^d}.$$

Proof. We prove the theorem using induction on d. If $d = 0$ then there exists a number N such that $M_n = 0$ for $n \geq N$. Thus $P(M,t) = \sum_{i=0}^{t} \mathrm{Vdim}_k(M_i) = f(t)$. Clearly $f(1) = \mathrm{Vdim}_k(M) \neq 0$. Now assume that $d > 0$ and the hypothesis of the induction holds for all non-zero finitely generated graded modules over R. By 1.2.11 there exists $1 \leq i \leq l$ such that x_i is a superficial element for M. Let $x = x_i$. We have the following exist sequence.

$$0 \longrightarrow \mathrm{Ann}_{M_n}(x) \longrightarrow M_n \xrightarrow{x} M_{n+1} \longrightarrow \big(M/xM\big)_{n+1} \longrightarrow 0$$

The above exact sequence implies that

$$\mathrm{Vdim}_k(M_{n+1}) - \mathrm{Vdim}_k(M_n) = \mathrm{Vdim}_k(\big(M/xM\big)_{n+1}) - \mathrm{Vdim}_k(\mathrm{Ann}_{M_n}(x)) \tag{1.1}$$

therefore

$$\mathrm{Vdim}_k(M_{n+1})t^{n+1} - \mathrm{Vdim}_k(M_n)t^{n+1} = \mathrm{Vdim}_k(\big(M/xM\big)_{n+1})t^{n+1} - \mathrm{Vdim}_k(\mathrm{Ann}_{M_n}(x))t^{n+1}$$

Summing these equations up we have

$$P(M,t) - tP(M,t) = P(M/xM,t) - tP(\mathrm{Ann}_M(x),t)$$

Note that by 1.2.13, $\dim(M/xM) = d-1$ whence by the induction hypothesis $P(M/xM,t) = \frac{g_1(t)}{(1-t)^{d-1}}$ for some $g_1(t) \in \mathbb{Z}[t]$ with $g_1(1) \neq 0$. Since x is superficial over M, $P(\mathrm{Ann}_M(x),t) = g_2(t)$ is polynomial in $\mathbb{Z}[t]$. Whence,

$$P(M,t) = \frac{g_1(t) + t(1-t)^{d-1} g_2(t)}{(1-t)^d}$$

Let $g(t) = g_1(t) + t(1-t)^{d-1} g_2(t)$. In order to complete the proof we need to show that $g(1) \neq 0$. If $d > 1$ then $g(1) = g_1(1) \neq 0$. Let $d = 1$. We have

$$g(1) = g_1(1) - g_2(1) = \mathrm{Vdim}_k(M/xM) - \mathrm{Vdim}_k(\mathrm{Ann}_M(x))$$

If $g(1) = 0$ we have $\mathrm{Vdim}_k(M/xM) = \mathrm{Vdim}_k(\mathrm{Ann}_M(x))$. Since M/xM and $\mathrm{Ann}_M(x)$ are both of dimension zero $\mathrm{Vdim}_k((M/xM)_n)$ and $\mathrm{Vdim}_k(\mathrm{Ann}_{M_n}(x))$ are both zero for sufficiently large n thus using equation 1.1 if n is sufficiently large we have

$$\begin{aligned} \mathrm{Vdim}_k(M_{n+1}) &= \textstyle\sum_{i=-1}^{n} (\mathrm{Vdim}_k(M_{i+1}) - \dim_k(M_i)) \\ &= \textstyle\sum_{i=-1}^{n} (\dim_k(M/xM))_{i+1} - \sum_{i=0}^{n} \dim_k(\mathrm{Ann}_{M_i}(x)) \\ &= \mathrm{Vdim}_k(M/xM) - \mathrm{Vdim}_k(\mathrm{Ann}_M(x)) = 0 \end{aligned}$$

This shows that $\mathrm{Vdim}_k(M_n) = 0$ for large n which implies $\dim(M) = 0$, a contradiction whence $g(1) \neq 0$ this completes the proof. □

Corollary 1.2.15. *Let $R = k[x_1,\ldots,x_l]$ be the ring of polynomials over a field k and let M be a finitely generated graded $R-$module. Then $s(M) = \dim(M)$. In particular, $\dim(M) \leq l$ and $\deg(Q_M(x)) = \dim(M) - 1$.*

Proof. Immediate from 1.2.4 and 1.2.14. □

1.3 Preliminaries on $\mathscr{D}-$modules

Let $T = k[x_1, \ldots, x_n]$ be the ring of polynomials over n variables over a field of characteristic zero k. We define the *Weyl algebra* in n variables over the field k to be the ring generated by the $k-$linear derivations on R with multiplications on R and denote it by $\mathscr{D}_n(k)$. In the other words $\mathscr{D}_n(k)$ is the $k-$algebra is generated by $\partial_i = \partial/\partial x_i : T \to T$ and multiplication operators $x_1, \ldots, x_n$. Note that $\partial_i x_j - x_j \partial_i = \delta_{ij}$ hence $\mathscr{D}_n(k)$ is consisted of finite sums of elements of the form $cx^\alpha \partial^\beta$ where $c \in k$.

Proposition 1.3.1. *The family $\{x^\alpha \partial^\beta\}$ is a basis for the Weyl algebra $\mathscr{D}_n(k)$ as a vector space over k.*

Proof. We only need to show that the family $\{x^\alpha \partial^\beta\}$ is linearly independent over k. Let $D = \sum k_{\alpha\beta} x^\alpha \partial^\beta$ be a linear combination of finitely many elements of $\{x^\alpha \partial^\beta\}$. Let $D = 0$ and assume that $k_{\alpha\beta} \neq 0$ for some α and β. Let β^* be a multi-index with $k_{\alpha\beta^*} \neq 0$ for some α and $k_{\alpha\beta} = 0$ when $|\beta| = \beta_1 + \cdots + \beta_n > \beta^*$. We have $D(x^{\beta^*}) = \beta^*! \sum k_{\alpha\beta^*} x^\alpha$. Since the field k has characteristic zero we have $\beta^*! \neq 0$ hence $k_{\alpha\beta^*} = 0$ which is a contradiction. Therefore $\{x^\alpha \partial^\beta\}$ is linearly independent over k. □

In light of the above proposition we define Ξ_v to be the $k-$vector subspace of $\mathscr{D}_n(k)$ generated by elements of the form $x_1^{\alpha_1} \cdots x_n^{\alpha_n} \partial_1^{\beta_1} \cdots \partial_n^{\beta_n}$ with $\sum \alpha_i + \sum \beta_j \leq v$. Clearly $\Xi = \Xi_0 \subseteq \Xi_1 \subseteq \Xi_2 \subseteq \cdots$ is an increasing sequence of finite dimensional $k-$subspaces of $\mathscr{D}_n(k)$ with $\bigcup_{v=0}^{\infty} \Xi_v = \mathscr{D}_n(k)$ and $\Xi_v \Xi_k \subseteq \Xi_{k+v}$. The filtration Ξ is called the standard filtration of $\mathscr{D}_n(k)$. We denote the direct sum $\Xi_0 \oplus (\Xi_1/\Xi_0) \oplus (\Xi_2/\Xi_1) \oplus \cdots$ by $\mathrm{gr}_\Xi(\mathscr{D}_n(k))$. Set $\Xi(v) = \Xi_v/\Xi_{v-1}$ for all $v \geq 0$. Let $D \in \Xi_v$, let $E \in \Xi_k$ and let $\overline{D}$ and $\overline{E}$ stand for the image of D and E in $\Xi(v)$ and $\Xi(k)$ respectively. Since $\Xi_v \Xi_k \subseteq \Xi_{k+v}$, $DE \in \Xi_{k+v}$. We define the product $\overline{DE}$ to be the image of DE in $\Xi(k+v)$. We need to check if this product is well-defined. Let D_1 be another element of Ξ_v with $\overline{D} = \overline{D_1}$. There exists some $G \in \Xi_{v-1}$ with $D - D_1 = G$. We have $DE - D_1E = GE \in \Xi_{v+k-1}$ whence $\overline{DE} = \overline{D_1E}$. This shows that $\overline{DE}$ does not depend on the choice of D. Similarly, it can be shown that $\overline{DE}$ does not depend on the choice of E. Moreover it can be shown that if $D \in \Xi_v$ and $E \in \Xi_k$ then $DE - ED \in \Xi_{k+v-1}$ therefore we have shown the following result.

Proposition 1.3.2. *The ring* $\mathrm{gr}_\Xi(\mathscr{D}_n(k))$ *is a commutative graded ring.*

The next proposition shows that $\mathrm{gr}_\Xi(\mathscr{D}_n(k))$ does have a familiar structure.

Proposition 1.3.3. *The ring* $\mathrm{gr}_\Xi(\mathscr{D}_n(k))$ *is the polynomial ring in* $2n$ *variables with coefficients in* k*.*

Proof. We have $\mathrm{gr}_\Xi(\mathscr{D}_n(k)) = k[\overline{x_1}, \ldots, \overline{x_n}, \overline{\partial_1}, \ldots, \overline{\partial_n}]$. We need to show that $\overline{x_1}, \ldots, \overline{x_n}, \overline{\partial_1}, \ldots, \overline{\partial_n}$ are algebraically independent over k. Let $R = \Sigma k_{\alpha\beta}\overline{x}^\alpha\overline{\partial}^\beta = 0$. We need to show that $k_{\alpha\beta} = 0$ for all α and β. Suppose the contrary therefore there exists α and β with $k_{\alpha\beta} \neq 0$. Let v be the largest integer with $|\alpha| + |\beta| = v$ such that there exists $k_{\alpha\beta} \neq 0$ in $\Sigma k_{\alpha\beta}\overline{x}^\alpha\overline{\partial}^\beta$. Let $D = \Sigma k_{\alpha\beta}x^\alpha\partial^\beta$ and assume that $\overline{D}$ is the image of D in $\Xi(v)$ hence $\overline{D} = \Sigma^* k_{\alpha\beta}\overline{x}^\alpha\overline{\partial}^\beta$ where Σ^* extends over $|\alpha| + |\beta| = v$. Clearly $\overline{D}$ is the $v-$graded component of R and since $R = 0$ we have $\overline{D} = 0$ therefore $D \in \Xi_{v-1}$. This shows that $D = \sum r_{\alpha'\beta'}x^{\alpha'}\partial^{\beta'}$ where $|\alpha'| + |\beta'| < v$ which is a contradiction by 1.3.1. □

Definition 1.3.4. Let M be a $\mathscr{D}_n(k)-$module. A filtration on M is an increasing sequence of $k-$vector subspaces $\Gamma = \Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \cdots$ with:

1. Γ_i is finitely generated for all $i \geq 0$.
2. $\Xi_i\Gamma_j \subseteq \Gamma_{i+j}$ for all integers i and j.
3. $\bigcup_{i=0}^\infty \Gamma_i = M$.

Quite similar to the proof we provided for 1.3.2, it can be shown that if M is an $\mathscr{D}_n(k)-$module equipped with a filtration $\Gamma = \Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \cdots$ then $\mathrm{gr}_\Gamma(M) = \Gamma_0 \oplus (\Gamma_1/\Gamma_0) \oplus (\Gamma_2/\Gamma_1) \oplus \cdots$ is an $\mathrm{gr}_\Xi(\mathscr{D}_n(k))-$module.

Definition 1.3.5. Let M be an $\mathscr{D}_n(k)-$module equipped with a filtration $\Gamma = \Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \cdots$ and let $0 \neq m \in M$. There exists a unique number v with $m \in \Gamma_v \setminus \Gamma_{v-1}$. The image $\gamma(m)$ of m

in Γ_v/Γ_{v-1} is called the $\Gamma-$*symbol of m*. Similaly, for the case that $M=\mathscr{D}_n(k)$ and $0\neq a\in A$ the $\Xi-$symbol of a is denoted by $\xi(a)$ and is called the *principle symbol of a*.

Definition 1.3.6. Let M be an $\mathscr{D}_n(k)-$module equipped with a filtration $\Gamma=\Gamma_0\subseteq\Gamma_1\subseteq\Gamma_2\subseteq\cdots$. The filtration Γ is said to be *good* if $\mathrm{gr}_\Gamma(M)$ is finitely generated as an $\mathrm{gr}_\Xi(\mathscr{D}_n(k))-$module.

Proposition 1.3.7. *Let M be an $\mathscr{D}_n(k)-$module equipped with a filtration $\Gamma=\Gamma_0\subseteq\Gamma_1\subseteq\Gamma_2\subseteq\cdots$. Assume that $\mathrm{gr}_\Gamma(M)$ is a finitely generated $\mathrm{gr}_\Xi(\mathscr{D}_n(k))-$module generated by the $\Gamma-$symbols $\gamma(m_1),\ldots,\gamma(m_t)$. Then M is a finitely generated $\mathscr{D}_n(k)-$module.*

Proof. It is enough to show that Γ_v is generated by $m_1,\ldots,m_t$ for all $v\geq 0$ by induction of v. Let $0\neq m\in M$ and $\gamma(m)\in\Gamma_v/\Gamma_{v-1}$ be the $\Gamma-$symbol of m of degree v in the graded $\mathscr{D}_n(k)-$module $\mathrm{gr}_\Gamma(M)$. Let $\gamma(m_1),\ldots,\gamma(m_t)$ be of degrees $k_1,\ldots,k_t$. There exists elements $a_1,\ldots,a_t\in\mathscr{D}_n(k)$ with principle symbols $\xi(a_1),\ldots,\xi(a_t)$ of degrees $v-k_1,\ldots,v-k_t$ with $\xi(a_1)\gamma(m_1)+\cdots+\xi(a_t)\gamma(m_t)=\gamma(m)$. The result is clear if $v=0$. Let $v>0$ and assume that the hypothesis of the induction holds for all integers less than v. We have $m-a_1m_1-\cdots-a_tm_t\in\Gamma_{v-1}$. Since Γ_{v-1} is generated by $m_1,\ldots,m_t$ we have $m-a_1m_1-\cdots-a_tm_t=b_1m_1+\cdots+b_tm_t$ for some $b_1,\ldots,b_t\in\mathscr{D}_n(k)$. Therefore $m=(a_1+b_1)m_1+\cdots+(a_t+b_t)m_t$. □

Proposition 1.3.8. *Let M be a finitely generated $\mathscr{D}_n(k)-$module then M has a good filtration.*

Proof. Let M be generated by $m_1,\ldots,m_t$ as an $\mathscr{D}_n(k)-$module. For each $v\geq 0$ we set $\Gamma_v=\Xi_vm_1+\cdots+\Xi_vm_t$. Clearly each Γ_v is finitely generated, $\Xi_k\Gamma_v\subseteq\Gamma_{k+v}$ for all $k,v\geq 0$, $\bigcup_{i=0}^\infty\Gamma_i=M$ and $\mathrm{gr}_\Gamma(M)$ is generated by $\gamma(m_1),\ldots,\gamma(m_t)$. □

Proposition 1.3.9. *The ring $\mathscr{D}_n(k)$ is left and right Noetherian.*

Proof. Let I be a left ideal of $\mathscr{D}_n(k)$ and let $\Omega_i=\Xi_i\cap I$ for each $i\geq 0$. We have $\bigcup_{i=0}^\infty\Omega_i=I$ and $\Xi_v\Omega_k\subseteq\Omega_{k+v}$ for all $k,v\geq 0$ hence $\Omega=\Omega_0\subseteq\Omega_1\subseteq\Omega_2\subseteq\cdots$ is a filtration on I and clearly $\mathrm{gr}_\Omega(I)$ is an ideal of $\mathrm{gr}_\Xi(\mathscr{D}_n(k))$. Since by 1.3.3 $\mathrm{gr}_\Xi(\mathscr{D}_n(k))$ is the polynomial ring with $2n$ variables over the field k and hence is Noetherian, $\mathrm{gr}_\Omega(I)$ is finitely generated, therefore by 1.3.7, I is a finitely generated $\mathscr{D}_n(k)-$module this shows that $\mathscr{D}_n(k)$ is left Noetherian. Similarly, it can be shown that $\mathscr{D}_n(k)$ is right Noetherian. □

The following proposition is of great importance and will be used later in this thesis to obtain a new proof for the Bernstein inequality.

Proposition 1.3.10. *Let M be an finitely generated $\mathscr{D}_n(k)-$module. Let $\Gamma = \Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \cdots$ and $\Omega = \Omega_0 \subseteq \Omega_1 \subseteq \Omega_2 \subseteq \cdots$ be two good filtrations on M. Then:*

1. *There exists an integer w such that $\Gamma_v \subseteq \Omega_{v+w}$ and $\Omega_v \subseteq \Gamma_{v+w}$ for all values of v.*
2. $\sqrt{\mathrm{Ann}_{\mathrm{gr}_\Xi(\mathscr{D}_n(k))}(\mathrm{gr}_\Gamma(M))} = \sqrt{\mathrm{Ann}_{\mathrm{gr}_\Xi(\mathscr{D}_n(k))}(\mathrm{gr}_\Omega(M))}$.

Proof. 1. Since $\mathrm{gr}_\Gamma(M)$ if finitely generated as an $\mathrm{gr}_\Xi(\mathscr{D}_n(k))-$module, there exist integers $k_1,\ldots,k_s$ and elements $u_i \in \Gamma_{k_i} \setminus \Gamma_{k_i-1}$ with $1 \leq i \leq s$ such that $\gamma(u_1),\ldots,\gamma(u_s)$ generate $\mathrm{gr}_\Gamma(M)$ as an $\mathrm{gr}_\Xi(\mathscr{D}_n(k))-$module. Let $k_0 = \max\{k_1,\ldots,k_s\}$. For $v \geq k_0$ we set $R_v = \Xi_v\Gamma_0 + \cdots + \Xi_{v-k_0}\Gamma_{k_0}$. We claim that $\Gamma_v \subseteq R_v$ for $v \geq k_0$ by induction. Since $1 \in \Xi_0$, $\Gamma_{k_0} = R_{k_0}$. Assume that $v > k_0$ and $\Gamma_{v-1} \subseteq R_{v-1}$. Let $\theta \in \Gamma_v \setminus \Gamma_{v-1}$. There exist $b_{v-k_1} \in \Sigma_{v-k_1},\ldots,b_{v-k_s} \in \Sigma_{v-k_s}$ such that

$$\gamma(\theta) = \sigma(b_{v-k_1})\gamma(u_1) + \cdots + \sigma(b_{v-k_s})\gamma(u_s)$$

thus

$$\theta \in \Sigma_{v-k_1}\Gamma_{k_1} + \cdots + \Sigma_{v-k_s}\Gamma_{k_s} + \Gamma_{v-1} \subseteq R_v + \Gamma_{v-1} \subseteq R_v + R_{v-1} = R_v$$

whence $\Gamma_v \subseteq R_v$. This completes the induction.

Note that by 1.3.3 $\mathrm{gr}_\Xi(\mathscr{D}_n(k))$ is a finitely generated $k-$algebra, since $\mathrm{gr}_\Gamma(M)$ is a finitely generated $\mathrm{gr}_\Xi(\mathscr{D}_n(k))-$module by 1.1.7, each Γ_i is a finitely generated $k-$vector space. Clearly $\{\Omega_j \cap \Gamma_{k_0}\}_{j\geq 0}$ forms an increasing chain of $k-$vector subspaces of the finite dimensional $k-$vector space Γ_{k_0} thus we have $\Gamma_{k_0} \subseteq \Omega_{w'}$ for some number w'. If $0 \leq j \leq k_0$ and $v \geq k_0$ we have

$$\Xi_{v-j}\Gamma_j \subseteq \Xi_{v-j}\Gamma_{k_0} \subseteq \Xi_{v-j}\Omega_{w'} \subseteq \Xi_v\Omega_{w'} \subseteq \Omega_{v+w'}$$

This shows that $\Gamma_v \subseteq R_v \subseteq \Omega_{v+w'}$ for all $v \geq k_0$. If $v < k_0$ then $\Gamma_v \subseteq \Gamma_{k_0} \subseteq \Omega_{w'} \subseteq \Omega_{v+w'}$. Thus $\Gamma_v \subseteq \Omega_{v+w'}$ for all $v \geq 0$.

Swapping Γ and Ω and repeating the above proof gives a number w'' with $\Omega_v \subseteq \Gamma_{v+w''}$ for all $v \geq 0$. Set $w = \max\{w',w''\}$. This proves 1.

2. Let $f \in \sqrt{\mathrm{Ann}_{\mathrm{gr}_{\Xi}(\mathscr{D}_n(k))}(\mathrm{gr}_{\Gamma}(M))}$ be homogeneous of degree s. There exists an integer $m \geq 1$ with $f^m \in \mathrm{Ann}_{\mathrm{gr}_{\Xi}(\mathscr{D}_n(k))}(\mathrm{gr}_{\Gamma}(M))$ and an element $\beta \in \Xi_s \setminus \Xi_{s-1}$ with $f = \xi(\beta)$. Thus $\beta^m \Gamma_i \subseteq \Gamma_{ms+i-1}$ for every $i \geq 0$. By induction on q this implies that

$$\beta^{mq}\Gamma_i \subseteq \Gamma_{i+msq-q} \tag{1.2}$$

for every $q \geq 1$. By the first part there exists an integer w such that $\Gamma_{i-w} \subseteq \Omega_i \subseteq \Gamma_{i+w}$ for all $i \geq 0$. Together with (1.3) for $q = 2w+1$ we have

$$\beta^{m(2w+1)}\Omega_i \subseteq \beta^{m(2w+1)}\Gamma_{i+w} \subseteq \Gamma_{i+ms(2w+1)-w-1} \subseteq \Omega_{i+ms(2w+1)-1}$$

Thus $\beta^{m(2w+1)}\Omega_i \subseteq \Omega_{i+ms(2w+1)-1}$ for all $i \geq 0$. Therefore $f^{m(2w+1)} \in \mathrm{Ann}_{\mathrm{gr}_{\Xi}(\mathscr{D}_n(k))}(\mathrm{gr}_{\Gamma}(M))$ whence $f \in \sqrt{\mathrm{Ann}_{\mathrm{gr}_{\Xi}(\mathscr{D}_n(k))}(\mathrm{gr}_{\Omega}(M))}$. Hence $\sqrt{\mathrm{Ann}_{\mathrm{gr}_{\Xi}(\mathscr{D}_n(k))}(\mathrm{gr}_{\Gamma}(M))} \subseteq \sqrt{\mathrm{Ann}_{\mathrm{gr}_{\Xi}(\mathscr{D}_n(k))}(\mathrm{gr}_{\Omega}(M))}$. The opposite inclusion follows similarly. □

Definition 1.3.11. Let M be a finitely generated $\mathscr{D}_n(k)-$module. We define its dimension to be the Krull dimension of $\mathrm{gr}_{\Gamma}(M)$ as an $\mathrm{gr}_{\Xi}(\mathscr{D}_n(k))-$module and denote it by $d(M)$ in which $\Gamma = \Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \cdots$ is a good filtration for M.

Theorem 1.3.12. *Let M be a finitely generated $\mathscr{D}_n(k)-$module then its dimension is independent of the good filtration chosen.*

Proof. By 1.3.8 M does have a good filtration $\Gamma = \{\Gamma_i\}_{i\geq 0}$. If $\Omega = \{\Omega_i\}_{i\geq 0}$ is any other good filtration for M then by 1.3.10 we have

$$\dim(\mathrm{gr}_{\Xi}(\mathscr{D}_n(k))/\sqrt{\mathrm{Ann}_{\mathrm{gr}_{\Xi}(\mathscr{D}_n(k))}(\mathrm{gr}_{\Gamma}(M))}) = \dim(\mathrm{gr}_{\Xi}(\mathscr{D}_n(k))/\sqrt{\mathrm{Ann}_{\mathrm{gr}_{\Xi}(\mathscr{D}_n(k))}(\mathrm{gr}_{\Omega}(M))})$$

hence the definition of the dimension of M is independent of the good filtration chosen. □

1.4 Bernstein inequality for the ring of polynomials

In this section we provide the proof for the Bernstein inequality for the ring of polynomials. As we are going to see, the proof is very simple.

Lemma 1.4.1. *Let M be a finitely generated $\mathscr{D}_n(k)-$module with filration $\Gamma = \{\Gamma_i\}_{i\geq 0}$. Assume the $\Gamma_0 \neq 0$ then the $k-$linear transformation*

$$\phi^{(i)} : \Xi_i \to \mathrm{Hom}_k(\Gamma_i, \Gamma_{2i})$$

given by $\phi^{(i)}_a(u) = au$ for all $a \in \Xi_i$ and $u \in \Gamma_i$ is injective.

Proof. We proceed by induction on i to prove this lemma. If $i = 0$ then $\Xi_i = k$. Since $\Gamma_0 \neq 0$ the result follows. Now assume that $i > 0$ and $\phi^{(j)}$ is injective for all values of $j < i$. Let $a \in \Xi_i$ with $\phi^{(i)}_a = 0$ and look for a contradiction. Let $cx^\alpha\partial^\beta$ be an element which appears in a in which $c \in k \setminus 0$ and $|\alpha| + |\beta| = i$. Let $|\alpha| > 0$ and let $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Without loss of generality we may assume that $\alpha_1 > 0$. Clearly $\alpha_1 c\delta\partial^\beta$ is a summand in $[a, \partial_1]$ where $\delta = x_1^{\alpha_1-1} \cdots x^{\alpha_n}$ therefore $[a, \partial_1]$ is a non-zero element in Ξ_{i-1}. Since $a\Gamma_i = 0$ and $\partial_1\Gamma_{i-1} \subseteq \Gamma_i$ we have

$$[a, \partial_1]\Gamma_{i-1} = (a\partial_1 - \partial_1 a)\Gamma_{i-1} \subseteq a\partial_1\Gamma_{i-1} \subseteq a\Gamma_i = 0$$

This is a contradiction since $[a, \partial_1] \in \Xi_{i-1} \setminus 0$. Similarly we can reach a contradiction for the case that $|\beta| > 0$. This completes the proof. □

Theorem 1.4.2. *Let M be a finitely generated $\mathscr{D}_n(k)-$module then $d(M) \geq n$.*

Proof. Since M is finitely generated M has a good filtration by 1.3.8. Let $\Gamma = \{\Gamma_i\}_{i\geq 0}$ be a good filtration for M. We have

$$d(M) = \dim_{\mathrm{gr}(\mathscr{D}_n(k))}(\mathrm{gr}_\Gamma(M))$$

where $\mathrm{gr}(\mathscr{D}_n(k)) = k[\overline{x_1}, \ldots, \overline{x_n}, \overline{\partial_1}, \ldots, \overline{\partial_n}]$. By 1.2.15 $\dim_k(\Gamma_i/\Gamma_{i-1})$ is a polynomial in i of degree $d(M) - 1$ for large i hence $\dim_k(\Gamma_i) = \sum_{j=0}^{i} \dim_k(\Gamma_j/\Gamma_{j-1})$ is a polynomial in i of degree $d(M)$ for large i. By 1.4.1, Ξ_i is a $k-$vector subspace of $\mathrm{Hom}_k(\Gamma_i, \Gamma_{2i})$. Hence $\dim_k(\Xi_i) \leq \dim_k(\mathrm{Hom}_k(\Gamma_i, \Gamma_{2i})$. Since $\dim_k(\mathrm{Hom}_k(\Gamma_i, \Gamma_{2i})) = \dim_k(\Gamma_i)\dim_k(\Gamma_{2i})$ and $\dim_k(\Gamma_i) = \sum_{j=0}^{i} \dim_k(\Gamma_j/\Gamma_{j-1})$ together with $\dim_k(\Xi_i) = \binom{i+2n}{2n}$ are polynomials in i of degrees $d(M)$ and $2n$ for high i respectively, we have $2d(M) \geq 2n$ hence $d(M) \geq n$. □

This simple proof does not extend to formal power series primarily because there is no analogue of the Bernstein filtration for formal power series.

Chapter 2

Bernstein inequality for the ring of power series: Björk's proof

In this chapter we recall Björk's proof of the Bernstein inequality over the ring of power series. The main complication compared to the case of polynomials is that for formal power series one needs to develop an extensive and cumbersome homological theory of filtered rings. Our main references will be [3] and [11].

2.1 Preliminaries on filtered rings

Throughout this section we assume that A be ring equipped with a filtration $\Sigma = \Sigma_0 \subseteq \Sigma_1 \subseteq \Sigma_2 \subseteq \cdots$ with $1 \in \Sigma_0$, $\bigcup_{i=0}^{\infty} \Sigma_i = A$ and $\Sigma_k \Sigma_v \subseteq \Sigma_{k+v}$ for all $k, v \geq 0$. We denote the direct sum $\Sigma_0 \oplus (\Sigma_1/\Sigma_0) \oplus (\Sigma_2/\Sigma_1) \oplus \cdots$ by $\mathrm{gr}_\Sigma(A)$. Similar to the method we used in 1.3 it can be shown that $\mathrm{gr}_\Sigma(A)$ is a graded ring. We assume that $\mathrm{gr}_\Sigma(A)$ is a Noetherian ring.

Definition 2.1.1. Let M be an $A-$module. A *filtration* on M is an increasing chain of Σ_0-submodules $\Gamma = \Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \cdots$ of M with:

1. $\Sigma_i \Gamma_j \subseteq \Gamma_{i+j}$ for all integers i and j.

2. $\bigcup_{i=0}^{\infty} \Gamma_i = M$.

Quite similar to 1.3.2, it can be shown that if M is an $A-$module equipped with a filtration $\Gamma = \Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \cdots$, then $\mathrm{gr}_\Gamma(M) = \Gamma_0 \oplus (\Gamma_1/\Gamma_0) \oplus (\Gamma_2/\Gamma_1) \oplus \cdots$ is an $\mathrm{gr}_\Sigma(A)-$module.

Definition 2.1.2. Let M be an $A-$module equipped with a filtration $\Gamma = \Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \cdots$. The filtration Γ is said to be *good* if the associated graded $\mathrm{gr}_\Sigma(A)-$module $\mathrm{gr}_\Gamma(M) = \Gamma_0 \oplus \Gamma_1/\Gamma_0 \oplus \Gamma_2/\Gamma_1 \oplus \cdots$ is finitely generated.

Repeating the proofs of 1.3.7, 1.3.8, 1.3.9 and 1.3.10 respectively we have the following four results:

Proposition 2.1.3. *Let M be an $A-$module equipped with a filtration $\Gamma = \Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \cdots$ with $\bigcup_{i=0}^{\infty} \Gamma_i = M$. Assume that $\mathrm{gr}_\Gamma(M)$ is a finitely generated $\mathrm{gr}_\Sigma(A)-$module generated by the $\Gamma-$symbols $\gamma(m_1), \ldots, \gamma(m_t)$. Then M is a finitely generated $A-$module.*

Proposition 2.1.4. *Let M be a finitely generated $A-$module then M has a good filtration.*

Proposition 2.1.5. *The ring A is left and right Noetherian.*

Proposition 2.1.6. *Let M be an finitely generated $A-$module. Let $\Gamma = \Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \cdots$ and $\Omega = \Omega_0 \subseteq \Omega_1 \subseteq \Omega_2 \subseteq \cdots$ be two good filtrations on M. Then:*

1. *There exists an integer w such that $\Gamma_v \subseteq \Omega_{v+w}$ and $\Omega_v \subseteq \Gamma_{v+w}$ for all values of v.*
2. $\sqrt{\mathrm{Ann}_{\mathrm{gr}_\Sigma(A)}(\mathrm{gr}_\Gamma(M))} = \sqrt{\mathrm{Ann}_{\mathrm{gr}_\Sigma(A)}(\mathrm{gr}_\Omega(M))}$.

Let M be finitely generated $A-$module. By 2.1.4, M has a good filtration $\Gamma = \Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \cdots$ and by 2.1.6 the dimension of $\mathrm{gr}_\Gamma(M)$ as an $\mathrm{gr}_\Sigma(A)-$module is independent of the good filtration chosen. Whence, we have the following definition.

Definition 2.1.7. The dimension of a finitely generated $A-$module M, denoted by $d(M)$ is the dimension of $\mathrm{gr}_\Gamma(M)$ as an $\mathrm{gr}_\Sigma(A)-$module where Γ is any good filtration for M.

Proposition 2.1.8. *If $0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0$ is an exact sequence of finitely generated $A-$modules then $d(M) = \max\{d(M'), d(M'')\}$*

Proof. Let $\Gamma = \Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \cdots$ be a good filtration for M. For each $i \geq 0$ we define $\Gamma'_i = \alpha^{-1}(\Gamma_i)$ and $\Gamma''_i = \beta(\Gamma_i)$. Clearly $\Gamma' = \{\Gamma'_i\}_{i\geq 0}$ and $\Gamma'' = \{\Gamma''_i\}_{i\geq 0}$ give rise to filtrations on M' and M'' respectively and we have the following exact sequence of $\mathrm{gr}_\Sigma(A)-$modules.

$$0 \to \mathrm{gr}_{\Gamma'}(M') \to \mathrm{gr}_\Gamma(M) \to \mathrm{gr}_{\Gamma''}(M'') \to 0$$

Therefore

$$\begin{aligned} d(M) &= \dim_{\mathrm{gr}_\Sigma(A)}(\mathrm{gr}_\Gamma(M)) \\ &= \max\{\dim_{\mathrm{gr}_\Sigma(A)}(\mathrm{gr}_{\Gamma'}(M')), \dim_{\mathrm{gr}_\Sigma(A)}(\mathrm{gr}_{\Gamma''}(M''))\} \\ &= \max\{d(M'), d(M'')\}. \end{aligned}$$ □

Let M be a finitely generated $A-$module and let $\Gamma = \{\Gamma_v\}_{v\geq 0}$ be a good filtration on M. Since M is a finitely generated $A-$module there exists a free resolution of finitely generated $A-$modules

$$\mathscr{F} = \cdots \to F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{\mu} M \to 0$$

Applying the dual functor $^* = \mathrm{Hom}(-,A)$ to the deleted complex of $\mathscr{F}$ we have the following complex

$$0 \to F_0^* \xrightarrow{d_1^*} F_1^* \xrightarrow{d_2^*} F_2^* \to \cdots$$

Clearly $\mathrm{Ext}^j(M,A) = \ker d^*_{j+1}/\mathrm{im}\, d^*_j$ is finitely generated for all $j \geq 0$. In order to equip $\mathrm{Ext}^j(M,A)$ with a filtration we need to define a filtration on $F^* = \mathrm{Hom}(F,A)$ where F is finitely generated free module over A. Note that if M is equipped with a good filtration $\{\Gamma_j\}_{j\geq 0}$ and N is a submodule of M then N and M/N can be equipped with filtrations $\{N \cap \Gamma_j\}_{j\geq 0}$ and $\{(\Gamma_j + N)/N\}_{j\geq 0}$ respectively which gives rise to the following exact sequence

$$0 \to \mathrm{gr}(N) \to \mathrm{gr}(M) \to \mathrm{gr}(M/N) \to 0$$

Since $\mathrm{gr}(M)$ is a finitely generated $\mathrm{gr}(A)-$module, so are $\mathrm{gr}(N)$ and $\mathrm{gr}(M/N)$ whence $\{N \cap \Gamma_j\}_{j\geq 0}$ and $\{(\Gamma_j + N)/N\}_{j\geq 0}$ are good filtrations of N and M/N respectively.

Proposition 2.1.9. *Let M be a finitely generated $A-$module generated by $m_1, \ldots, m_t$, let $\Gamma = \{\Gamma_j\}_{j\geq 0}$ be a good filtration on M, let $F = \bigoplus_{i=1}^t A\varepsilon_i$ be a free $A-$module of rank t and let $d : F \to M$ be the map which sends ε_i to m_i for $1 \leq i \leq t$. Then:*

1. *There exists a good filtration* $\Omega = \Omega_0 \subseteq \Omega_1 \subseteq \Omega_2 \subseteq \cdots$ *with* $d(\Omega_j) \subseteq \Gamma_j$ *for all* $j \geq 0$ *and* $\mathrm{gr}_\Omega(F)$ *is a finitely generated graded free* $\mathrm{gr}(A)-$*module.*

2. *The module* $F^* = \mathrm{Hom}(F,A)$ *has a good filtration* $\Omega^* = \{\Omega^*_\nu\}_{\nu \geq 0}$ *with*

$$\mathrm{gr}_{\Omega^*}(F^*) \cong \mathrm{Hom}^*_{\mathrm{gr}_\Sigma(A)}(\mathrm{gr}_\Omega(F), \mathrm{gr}_\Sigma(A)).$$

3. *If* $G = \bigoplus_{i=1}^{t'} A\varepsilon'_i$ *is another finitely generated free* $A-$*module and* $\delta : G \to F$ *is an* $A-$*homomorphism, then there exists a filtration* $\Omega' = \Omega'_0 \subseteq \Omega'_1 \subseteq \Omega'_2 \subseteq \cdots$ *with* $\delta(\Omega'_j) \subseteq \Omega_j$ *for all* $j \geq 0$ *and we have the following commutative diagram where the horizontal maps are natural and vertical maps are isomorphims of graded* $\mathrm{gr}_\Sigma(A)-$*modules.*

$$\begin{array}{ccc} \mathrm{gr}_{\Omega^*}(F^*) & \xrightarrow{\quad\overline{\delta^*}\quad} & \mathrm{gr}_{\Omega'^*}(G^*) \\ \downarrow \Theta_F & & \downarrow \Theta_G \\ \mathrm{Hom}^*_{\mathrm{gr}_\Sigma(A)}(\mathrm{gr}_\Omega(F), \mathrm{gr}_\Sigma(A)) & \xrightarrow{\mathrm{gr}(\delta)^*} & \mathrm{Hom}^*_{\mathrm{gr}_\Sigma(A)}(\mathrm{gr}_{\Omega'}(G), \mathrm{gr}_\Sigma(A)) \end{array}$$

4. *Let* $\cdots \to F_1 \xrightarrow{d_1} F_0 \xrightarrow{\mu} M \to 0$ *be a free resolution of a finitely generated* $A-$*module* M *equipped with a good filtration* Γ. *Each free module* F_j *can be taken to be finitely generated with* $d_j(\Omega_i^{(j)}) \subseteq \Omega_i^{(j-1)}$ *where* $\Omega^{(j)} = \{\Omega_i^{(j)}\}_{i \geq 0}$ *is a good filtration on* F_j. *Then the complexes*

$$0 \to \mathrm{gr}_{\Omega^*}(F_0^*) \to \mathrm{gr}_{\Omega^*}(F_1^*) \to \cdots$$

and

$$0 \to \mathrm{Hom}_{\mathrm{gr}(A)}(\mathrm{gr}_\Omega(F_0), \mathrm{gr}(A)) \to \mathrm{Hom}_{\mathrm{gr}(A)}(\mathrm{gr}_\Omega(F_1), \mathrm{gr}(A)) \to \cdots$$

are isomorphic and the Ω^*-*filtrations induce a good filtration on* $\{\mathrm{Ext}^j(M,A)\}$ *which are the cohomology groups of the complex* $0 \to F_0^* \xrightarrow{d_1^*} F_1^* \xrightarrow{d_2^*} F_2^* \cdots$.

Proof. (1) For each $1 \leq i \leq t$ there exists an integer u_i with $m_i \in \Gamma_{u_i} \setminus \Gamma_{u_i - 1}$. We define $\Omega_n = \bigoplus_{i=1}^t \Sigma_{n-u_i}\varepsilon_i$. Clearly $\mathrm{gr}_\Omega(F)$ is generated by $\varepsilon_1, \ldots, \varepsilon_t$ as a free graded $\mathrm{gr}(A)-$module and $d(\Omega_j) \subseteq \Gamma_j$ for all $j \geq 0$.

(2) We define the filtration

$$\Omega^*_\nu = \{\Phi \in F^* : \Phi(\Omega_k) \subseteq \Sigma_{k+\nu} \;\text{ for all k}\}$$

Let $i, v \geq 0$ be given. We need to show that $\Sigma_i \Omega^*_v \subseteq \Omega_{v+i}$. Let $\Phi \in \Omega^*_v$ then we have $\Phi(\Omega_k(F)) \subseteq \Sigma_{k+v}$ for all $k \geq 0$. Therefore $\Sigma_i \Phi(\Omega_k(F)) \subseteq \Sigma_i \Sigma_{k+v} \subseteq \Sigma_{k+v+i}$ for all $k \geq 0$. Hence $\Sigma_i \Phi \in \Omega^*_{v+i}$. Next we verify that $\bigcup_{v=0}^{\infty} \Omega^*_v = F^*$. Let $\theta : \bigoplus_{i=1}^{s} A\varepsilon_i \to A$ be given. For each $i \geq 0$ there exists $\alpha_i \geq 0$ with $\theta(\varepsilon_i) \in \Sigma_{\alpha_i}$. Let $\alpha^* = \max(\alpha_1, \ldots, \alpha_s)$. Clearly $\theta(\Omega_k) = \theta(\oplus \Sigma_{k-u_i}\varepsilon_i) \subseteq \Sigma_{k+\alpha^*}$ therefore $\theta \in \Omega^*_{\alpha^*}$ whence $\bigcup_{v=0}^{\infty} \Omega^*_v = F^*$ and Ω^* is a filtration on F^*.

Now we define a graded homomorphism of degree zero $\Theta_F : \mathrm{gr}_{\Omega^*}(F^*) \to \mathrm{Hom}^*_{\mathrm{gr}(A)}(\mathrm{gr}_{\Omega}(F), \mathrm{gr}(A))$ with $\Theta_F(\overline{\Phi})(\overline{\alpha}) = \overline{\Phi(\alpha)}$ where $\Phi \in \Omega^*_v$, $\alpha \in \Omega_n$, $^- : \Omega_n \to \Omega_n/\Omega_{n-1}$ and $^- : \Omega^*_v \to \Omega^*_v/\Omega^*_{v-1}$ indicate natural homomorphisms for all $v, n \geq 0$. Note that if $\overline{\Phi} = 0$ then $\Phi \in \Omega^*_{v-1}$ hence $\Phi(\alpha) \in \Sigma_{n+v-1}$ and therefore Θ_F is well-defined. We claim that Θ_F is a graded isomorphism between graded modules. If $\theta_F(\overline{\Phi}) = 0$ then $\Phi(\alpha) \in \Sigma_{n+v-1}$ for all $\alpha \in \Omega_n$ hence $\Phi \in \Omega^*_{v-1}$ thus Θ_F is $1-1$. Now let $\mu : \mathrm{gr}_{\Omega}(F) \to \mathrm{gr}(A)$ be a graded homomorphism of degree v. Assume that $\mu(\overline{\varepsilon_i}) = \overline{r_{u_i+v}}$ where $r_{u_i+v} \in \Sigma_{u_i+v}$. Now define $\Phi : F \to A$ by mapping ε_i to r_{u_i+v}. Clearly $\Phi \in \Omega^*_v$ and $\Theta_F(\overline{\Phi}) = \mu$.

(3) Let $\delta(\varepsilon'_i) = f_i \in \Omega_{w_i} \setminus \Omega_{w_i-1}$ for $1 \leq i \leq t'$. Similar to the construction in (1) let $\Omega'_n = \oplus_{i=1}^{t'} \Sigma_{n-w_i}\varepsilon'_i$. We define $\overline{\delta^*} : \Omega^*_v/\Omega^*_{v-1} \to \Omega'^*_v/\Omega'^*_{v-1}$ with $\overline{\delta^*}(\overline{\Phi}) = \overline{\Phi(\delta)}$ where $\overline{\Phi}$ is the image of Φ in $\Omega^*_v/\Omega^*_{v-1}$. On the other hand, we have the graded map of degree 0, $\mathrm{gr}(\delta) : \mathrm{gr}_{\Omega'}(G) \to \mathrm{gr}_{\Omega}(F)$ with $\mathrm{gr}(\delta)(\overline{\alpha'}) = \overline{\delta(\alpha')}$ where $\alpha' \in \Omega'_n$. Clearly the diagram is commutative.

(4) Since M is finitely generated, there exists a resolution of finitely generated free $A-$modules for M. That each F_j can be equipped with filtrations $\Omega^{(j)} = \{\Omega^{(j)}_i\}_{i \geq 0}$ with $d_j(\Omega^{(j)}_i) \subseteq \Omega^{(j-1)}_i$ for $i, j \geq 0$ is a direct result of (3). We have $\mathrm{Ext}^j(M,A) = \ker d^*_{j+1}/\operatorname{im} d^*_j$. Note that for each $j \geq 0$ we have the exact sequence $0 \to \ker d^*_{j+1} \hookrightarrow F^*_j$. It can be easily seen that $\mathrm{gr}_{\Omega^*}(F^*_j)$ is generated by $p_i : F \to A$ where p_i is the $i-$th projection. Hence Ω^* induces a good filtration on $\ker d^*_{j+1}$ given by $\Gamma_v = (\Omega^*_v(F^*_j) \cap \ker d^*_{j+1}$. Similarly the natural map $\ker d^*_{j+1} \to \mathrm{Ext}^j(M,A)$ induces a good filtration on $\mathrm{Ext}^j(M,A)$ given by $\Omega^*_v(\mathrm{Ext}^j(M,A)) = (\Omega^*_v(F^*_j) \cap \ker d^*_{j+1} + \operatorname{im} d^*_j)/\operatorname{im} d^*_j)$. This completes the proof. □

Let $\mathscr{B}$ be an Abelian group and let $\cdots \subseteq \Upsilon_{j-1} \subseteq \Upsilon_j \subseteq \Upsilon_{j+1} \subseteq \cdots$ be a filtration of subgroups of $\mathscr{B}$ with $\bigcup \Upsilon_j = \mathscr{B}$. Assume that $d : \mathscr{B} \to \mathscr{B}$ is a map with $d^2 = 0$ and $d(\Upsilon_j) \subseteq \Upsilon_j$. We set $Z_j^\infty = \ker d \cap \Upsilon_j$ and $B_j^\infty = \operatorname{im} d \cap \Upsilon_j$. Let $\mathscr{H} = \ker d / \operatorname{im} d$ and let $\Upsilon_j(\mathscr{H}) = Z_j^\infty + \operatorname{im} d / \operatorname{im} d$. We have $\operatorname{gr}(\mathscr{H}) = \oplus \frac{Z_j^\infty + \operatorname{im} d}{Z_{j-1}^\infty + \operatorname{im} d}$. On the other hand, the map d gives rise to the natural map $\operatorname{gr}(d) : \bigoplus_{j \geq 0} \Upsilon_j / \Upsilon_{j-1} \to \bigoplus_{j \geq 0} \Upsilon_j / \Upsilon_{j-1}$. Set $H = \ker(\operatorname{gr}(d)) / \operatorname{im}(\operatorname{gr}(d))$. For $k \geq 0$ and all j we set $Z_j^k = \{x \in \Upsilon_j : dx \in \Upsilon_{j-k}\}$ and $B_j^k = \Upsilon_j \cap d(\Upsilon_{j+k-1})$. Clearly we have $\Upsilon_j = Z_j^0 \supseteq Z_j^1 \supseteq Z_j^2 \supseteq \cdots \supseteq Z_j^\infty$ and $B_j^0 \subseteq B_j^1 \subseteq \cdots \subseteq B_j^\infty$. Since $d^2 = 0$ we have $B_j^k \subseteq Z_j^k$ whence the quotient $(Z_j^k + \Upsilon_{j-1}) / (B_j^k + \Upsilon_{j-1})$ makes sense. For each $k \geq 0$ we define
$H^k = \bigoplus_{j \geq 0} (Z_j^k + \Upsilon_{j-1}) / (B_j^k + \Upsilon_{j-1})$.

Definition 2.1.10. An Abelian group H is a subfactor of another Abelian group G if $H \cong G_0 / K$ where $K \subseteq G_0 \subseteq G$.

Proposition 2.1.11. *[3, Proposition 4.9] Let w be an integer with $Z_j^\infty + \Upsilon_{j-1} = Z_j^w + \Upsilon_{j-1}$ then $\operatorname{gr}(\mathscr{H})$ is isomorphic to a subfactor of H.*

Proposition 2.1.12. *The $\operatorname{gr}(A)-$module $\operatorname{gr}_{\Omega^*}(\operatorname{Ext}^j(M,A))$ is a subfactor of the $\operatorname{gr}_\Sigma(A)-$module $\operatorname{Ext}^j_{\operatorname{gr}(A)}(\operatorname{gr}_\Gamma(M), \operatorname{gr}(A)))$ for each j.*

Proof. Since M is finitely generated there exists a free resolution

$$\cdots \to F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{\mu} M \to 0$$

Let $^* = \operatorname{Hom}(-,A)$ stand for the dual functor. We have the deleted dual complex

$$0 \to F_0^* \xrightarrow{d_1^*} F_1^* \xrightarrow{d_2^*} F_2^* \to \cdots$$

Set $\mathscr{B} = \bigoplus F_j^*$, $d = \bigoplus d_j^*$ and $\Omega_v(\mathscr{B}) = \bigoplus \Omega_v^*(F_j^*)$ where Ω_v^* has the meaning of 2.1.9. Clearly

$$\cdots \subseteq \Omega_{v-1}(\mathscr{B}) \subseteq \Omega_v(\mathscr{B}) \subseteq \Omega_{v+1}(\mathscr{B}) \subseteq \cdots$$

and $d^2 = 0$. Therefore the map d gives rise to the map $\operatorname{gr}(d) : \operatorname{gr}(\mathscr{B}) \to \operatorname{gr}(\mathscr{B})$. We can simplify $\operatorname{gr}(\mathscr{B})$ as follows:

$$\operatorname{gr}(\mathscr{B}) = \bigoplus_v \frac{\Omega_v(\mathscr{B})}{\Omega_{v-1}(\mathscr{B})} = \bigoplus_v \frac{\bigoplus_j \Omega_v^*(F_j^*))}{\bigoplus_j \Omega_{v-1}^*(F_j^*))} = \bigoplus_v \bigoplus_j \frac{\Omega_v^*(F_j^*)}{\Omega_{v-1}^*(F_j^*)} = \bigoplus_j \operatorname{gr}_{\Omega^*}(F_j^*)$$

Therefore $\mathrm{gr}(d): \bigoplus_j \mathrm{gr}_{\Omega^*}(F_j^*) \to \bigoplus_j \mathrm{gr}_{\Omega^*}(F_j^*)$. By 2.1.9 we have

$$\mathrm{gr}(d): \bigoplus_j \mathrm{Hom}_{\mathrm{gr}(A)}(\mathrm{gr}(F_j), \mathrm{gr}(A)) \to \bigoplus_j \mathrm{Hom}_{\mathrm{gr}(A)}(\mathrm{gr}(F_j), \mathrm{gr}(A))$$

Note that the complex

$$\cdots \to \mathrm{gr}(F_2) \xrightarrow{\mathrm{gr}(d_2)} \mathrm{gr}(F_1) \xrightarrow{\mathrm{gr}(d_1)} \mathrm{gr}(F_0) \xrightarrow{\mathrm{gr}(\mu)} \mathrm{gr}_\Gamma(M) \to 0$$

is a free resolution for $\mathrm{gr}_\Gamma(M)$. Therefore we have

$$H = \ker \mathrm{gr}(d))/\operatorname{im}\mathrm{gr}(d) = \bigoplus_j \mathrm{Ext}^j(\mathrm{gr}_\Gamma(M), \mathrm{gr}(A))$$

On the other hand

$$\mathrm{gr}(\mathscr{H}) = \bigoplus_v (Z_v^\infty + \operatorname{im} d)/(Z_{v-1}^\infty + \operatorname{im} d) = \bigoplus_v \frac{\ker d \cap \Omega_v(\mathscr{B}) + \operatorname{im} d}{\ker d \cap \Omega_{v-1}(\mathscr{B}) + \operatorname{im} d}$$

We have $\ker(d : \bigoplus F_j^* \to \bigoplus F_j^*) \cap \Omega_v(\mathscr{B}) = \bigoplus_j \ker d_{j+1}^* \cap \Omega_v^*(F_j^*)$. Therefore we have

$$\mathrm{gr}(\mathscr{H}) = \bigoplus_j \bigoplus_v \frac{\ker d_{j+1}^* \cap \Omega_v^*(F_j^*) + d_j^*(F_{j-1}^*)}{\ker d_{j+1}^* \cap \Omega_{v-1}^*(F_j^*) + d_j^*(F_{j-1}^*)} = \bigoplus_j \mathrm{gr}_{\Omega^*}(\mathrm{Ext}^j(M,A))$$

By 2.1.11 $\mathrm{gr}(\mathscr{H})$ is a subfactor of H therefore the proof is complete. □

Theorem 2.1.13. *Let Γ be some good filtration on a finitely generated $A-$module M. Then $d(\mathrm{Ext}^j(M,A)) \leq \dim_{\mathrm{gr}(A)}(\mathrm{Ext}^j_{\mathrm{gr}(A)}(\mathrm{gr}_\Gamma(M), \mathrm{gr}(A))$ for all $j \geq 0$.*

Proof. Immediate by 2.1.9 and 2.1.12. □

Lemma 2.1.14. *Let $F \xrightarrow{d} G \xrightarrow{e} H$ be a complex of filtered $A-$modules with filtrations $\Gamma(F) = \{\Gamma_v(F)\}_{v\geq 0}$, $\Gamma(G) = \{\Gamma_v(G)\}_{v\geq 0}$ and $\Gamma(H) = \{\Gamma_v(H)\}_{v\geq 0}$, let $d(\Gamma_v(F)) \subseteq \Gamma_v(G)$ and $e(\Gamma_v(G)) \subseteq \Gamma_v(F)$ and let $\mathrm{gr}(F) \xrightarrow{\mathrm{gr}(d)} \mathrm{gr}(G) \xrightarrow{\mathrm{gr}(e)} \mathrm{gr}(H)$ be and exact sequence of homogeneous maps of degree 0 then $F \xrightarrow{d} G \xrightarrow{e} H$ is exact.*

Proof. Let $g \in \ker e$. There exists $v \geq 0$ with $g \in \Gamma_v(G) \setminus \Gamma_{v-1}(G)$ with $\mathrm{gr}(e)(\bar{g}) = 0$. Therefore there exists $f_v \in \Gamma_v(F)$ with $\bar{g} = \mathrm{gr}(d)(\bar{f}_v)$. We have $g - d(f_v) \in \Gamma_{v-1}(G)$ hence there exists

$g_{v-1} \in \Gamma_{v-1}(G)$ with $g_{v-1} = g - d(f_v)$. Since $ed = 0$ we have $e(g_{v-1}) = 0$ therefore by the same method we can show that there exists $g_{v-2} \in \Gamma_{v-2}(F)$ with $g_{v-2} = g - d(f_{v-1}) \in \Gamma_{v-2}(G)$. Since $\Gamma_{-1}(G) = 0$ after v steps we have $g = d(f_1 + \cdots + f_v)$ for some $f_1, \ldots, f_v \in F$. Thus $F \xrightarrow{d} G \xrightarrow{e} H$ is exact. □

Lemma 2.1.15. *Let N be a filtered right $A-$module with filtration $\Omega(N) = \{\Omega_j\}_{j\geq 0}$. Let $F = \bigoplus_{i=1}^{s} A$ be a finitely generated free $A-$module with filtration $\Gamma_j(F) = \oplus_{i=1}^{s}\Sigma_j$. Then:*

1. *Each element of $N \otimes_A F$ is of the form $\sum_{i=1}^{s} n_i \otimes \varepsilon_i$ where ε_i is the i^{th} injection at $1 \in A$.*

2. *Let $\sum_{i=1}^{s} n_i \otimes \varepsilon_i = 0$ if and only if $n_i = 0$ for all $1 \leq i \leq s$.*

3. *We define τ_v to be the subgroup of $N \otimes_A F$ generated by $n \otimes \varepsilon_i$ where $n \in \Omega_t$ and $t \leq v$. Then the graded abelian groups $\mathrm{gr}_\tau(N \otimes_A F)$ and $\mathrm{gr}_\Omega(N) \otimes_{\mathrm{gr}(A)} \mathrm{gr}(F)$ are isomorphic.*

4. *Let N' be another filtered right $A-$module with filtration $\Omega'(N') = \{\Omega'_j\}_{j\geq 0}$ and let $\phi : N' \to N$ be an $A-$homomorphism with $\phi(\Omega'_j) \subseteq \Omega_j$ for all $j \geq 0$ and let $\tau' = \{\tau'_v\}_{v\geq 0}$ have the meaning of (3) for $N' \otimes F$. Then we have the following commutative digram of graded $\mathrm{gr}(A)-$modules whose vertical maps are isomorphisms.*

$$\begin{array}{ccc}
\mathrm{gr}_{\tau'}(N' \otimes_A F) & \xrightarrow{\mathrm{gr}(\phi\otimes 1_F)} & \mathrm{gr}_\tau(N \otimes_A F) \\
\big\downarrow \Theta_{N'} & & \big\downarrow \Theta_N \\
\mathrm{gr}_{\Omega'}(N') \otimes_{\mathrm{gr}(A)} \mathrm{gr}(F) & \xrightarrow{\mathrm{gr}(\phi)\otimes 1_{\mathrm{gr}(A)}} & \mathrm{gr}_\Omega(N) \otimes_{\mathrm{gr}(A)} \mathrm{gr}(F)
\end{array}$$

Proof. We just need to prove (3) as (1), (2) and (4) are clear. We define

$$\Theta_N : \mathrm{gr}_\tau(N \otimes_A F) \to \mathrm{gr}_\Omega(N) \otimes_{\mathrm{gr}(A)} \mathrm{gr}(F)$$

by $\Theta(\overline{\sum n_i \otimes \varepsilon_i}) = \sum \pi_v(n_i) \otimes \varepsilon_i$. Where $\sum n_i \otimes \varepsilon_i \in \tau_v, \bar{\ } : \tau_v \to \tau_v/\tau_{v-1}$ and $\pi_t : \Omega_t \to \Omega_t/\Omega_{t-1}$ are natural maps for all integers of v and t.

Let $\overline{\sum n_i \otimes \varepsilon_i} = 0$ where $\sum n_i \otimes \varepsilon_i \in \tau_v$. Since $\sum n_i \otimes \varepsilon_i \in \tau_{v-1}$ we have $\sum n_i \otimes \varepsilon_i = \sum n'_i \otimes \varepsilon_i$ for some $n'_i \in \Omega_t$ with $t \leq v-1$. Therefore we have $\sum (n_i - n'_i) \otimes \varepsilon_i = 0$. By (2), $n_i = n'_i$ for all i. Hence $\pi_v(n_i) = 0$. Thus Θ is well-defined. Now assume that $\Theta(\overline{\sum n_i \otimes \varepsilon_i}) = 0$ where

$\sum n_i \otimes \varepsilon_i \in \tau_v$. We have $\sum \pi_v(n_i) \otimes \overline{\varepsilon_i} = 0$. Since $\mathrm{gr}(F)$ is a free module generated by $\overline{\varepsilon_i} \in \Gamma_0$ we have $\pi_v(n_i) = 0$ therefore $n_i \in \Omega_{v-1}$. This shows that $\sum n_i \otimes \varepsilon_i \in \tau_{v-1}$ hence Θ is injective. By the definition of Θ it is clear that it is also surjective. This completes the proof. □

Proposition 2.1.16. *Let M be a left $A-$module and let Γ be a filtration on M. Then* $\mathrm{fd}_A(M) \leq \mathrm{fd}_{\mathrm{gr}(A)}(\mathrm{gr}_\Gamma(M))$. *Therefore* $\mathrm{w.dim}(A) \leq \mathrm{w.dim}(\mathrm{gr}(A))$.

Proof. Let $w = \mathrm{fd}_{\mathrm{gr}(A)}(\mathrm{gr}_\Gamma(M))$ and let

$$\cdots \to F_1 \to F_0 \to M \to 0$$

be a free resolution of M. Let N be a right $A-$module with filtration $\Omega(N) = \{\Omega_j(N)\}_{j\geq 0}$. We have the following complex whose j^{th} homology group is $\mathrm{Tor}_j(\mathrm{gr}_\Omega(N), \mathrm{gr}_\Gamma(M))$.

$$0 \to \mathrm{gr}_\Omega(N) \otimes \mathrm{gr}(F_0) \to \mathrm{gr}_\Omega(N) \otimes \mathrm{gr}(F_1) \to \mathrm{gr}_\Omega(N) \otimes \mathrm{gr}(F_2) \to \cdots \tag{2.1}$$

By 2.1.15, each of the modules $N \otimes_A F_i$ has a filtration $\tau_i = \{\tau_{ij}(N \otimes F_j)\}_{j\geq 0}$ with

$$\mathrm{gr}_{\tau_i}(N \otimes F_i) \cong \mathrm{gr}_\Omega(N) \otimes \mathrm{gr}(F_i)$$

Since $w = \mathrm{fd}_{\mathrm{gr}(A)}(\mathrm{gr}_\Gamma(M))$ the w^{th} homology in (1.4) is zero we have the following exact sequence

$$\mathrm{gr}_{\tau_{w-1}}(N \otimes F_{w-1}) \to \mathrm{gr}_{\tau_w}(N \otimes F_w) \to \mathrm{gr}_{\tau_{w+1}}(N \otimes F_{w+1})$$

by 2.1.14, the following sequence is exact

$$N \otimes F_{w-1} \to N \otimes F_w \to N \otimes F_{w+1}$$

Therefore $\mathrm{Tor}_w(N,M) = 0$ and $\mathrm{fd}(M) \leq w = \mathrm{fd}_{\mathrm{gr}(A)}(\mathrm{gr}_\Gamma(M))$. □

Proposition 2.1.17. *[3, Proposition 4.14] If M is a left $A-$module such that* $\mathrm{Ext}^s(M,A) = 0$ *for all $s \neq j$ then* $\mathrm{Ext}^s(M,N) = 0$ *for all $s > j$ and all right $A-$modules N.*

2.2 Preliminaries on regular rings

Let R be a ring and let M be an $R-$module. It is well known that M does have a flat resolution. In the other words, there exist a exact sequence of the form

$$\mathscr{F} = \cdots \to F_{n+1} \xrightarrow{d_{n+1}} F_n \to \cdots \to F_1 \xrightarrow{d_1} F_0 \xrightarrow{\sigma} M \to 0$$

where each of the F_ns are flat. We define its n^{th} *syzygy* to be $Y_0 = \ker \sigma$ and $Y_n = \ker d_n$.
Let $T : {}_R\mathscr{M} \to {}_R\mathscr{M}$ be an additive covariant functor and let $\mathscr{F}_M$ denote the resolution $\mathscr{F}$ with the module M deleted. We define the the n^{th} *left derived functors* of T by

$$L_nT(M) = H_n(T\mathscr{F}_M) = \frac{\ker(Td_n)}{\operatorname{im}(Td_{n+1})}$$

It is shown that the definition of left derived functors of T is independent of flat resolutions chosen.
Similary for the case that $T : {}_R\mathscr{M} \to {}_R\mathscr{M}$ is a contravariant additive functor we define the n^{th} *right derived functors* of T by

$$R^nT(M) = H^n(T\mathscr{F}_M) = \frac{\ker(Td_{n+1})}{\operatorname{im}(Td_n)}$$

and the definition is independent of the flat resolution chosen. When $T = \operatorname{Hom}(-,A)$ then the n^{th} right derived functor is denoted by $\operatorname{Ext}^n_R(-,A)$ and when $T = -\otimes_R A$ then n^{th} left derived functor is denoted by $\operatorname{Tor}^R_n(-,A)$.
Given an exact sequence of modules $0 \to A' \to A \to A'' \to 0$ the *Horseshoe lemma* states that given two flat resolutions $\mathscr{F}'$ and $\mathscr{F}''$ for A' and A'' respectively, there exists a flat resolution $\mathscr{F}$ for A such that $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence of complexes. Also, given module B there are long exact sequences

$$0 \to \operatorname{Hom}(A'',B) \to \operatorname{Hom}(A,B) \to \operatorname{Hom}(A',B) \to \operatorname{Ext}^1(A'',B) \to \cdots$$

$$0 \to A' \otimes_R B \to A \otimes_R B \to A'' \otimes_R B \to \operatorname{Tor}^R_1(A',B) \to \cdots$$

Similarly, an *injective resolution* of an $R-$module N is an exact sequence

$$0 \to N \xrightarrow{\mu} E^0 \xrightarrow{\delta^0} E^1 \xrightarrow{\delta^1} \cdots \to E^n \xrightarrow{\delta^n} E^{n+1} \to \cdots$$

where each E^n is injective. We define the n^{th} cosyzygy V^n of N to be $V^0 = \text{coker}(\mu)$ and $V^n = \text{coker}(\delta^n)$. Obviously the cosyzygy of an $R-$module N depend of the injective resolution chosen.

Definition 2.2.1. An injective resolution $0 \to N \to E^0 \to E^1 \to E^2 \to \cdots \to E^n \to 0$ of an $R-$module N, is said to be of length n. The smallest such n is called the injective dimension of N and is denoted by $\text{id}_R(N)$.

Theorem 2.2.2. *The following statements are equivalent for a left $R-$module N.*

1. $\text{id}(N) \leq n$.
2. $\text{Ext}^k(M,N) = 0$ *for all left $R-$modules M and $k \geq n+1$.*
3. $\text{Ext}^{n+1}(M,N) = 0$ *for all left $R-$modules M.*
4. *Every injective resolution of N has its $(n-1)$st cosyzygy injective.*

Definition 2.2.3. Let R be a ring. We define its *left injective dimension* $\text{liD}(R)$ to be

$$\text{liD}(R) = \sup\{\text{id}(N) : N \in {}_R\mathfrak{M}\}.$$

Definition 2.2.4. A *flat resolution* $0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$ of an $R-$module M is said to be of *length n*. The smallest such n is called the *flat dimension* of M and is denoted by $\text{fd}_R(M)$.

Theorem 2.2.5. *[9, Theorem 9.13] The following are equivalent for a left $R-$module N:*

1. $\text{fd}(N) \leq n$;
2. $\text{Tor}_k(M,N) = 0$ *for all modules M and $k \geq n+1$;*
3. $\text{Tor}_{n+1}(M,N) = 0$ *for all modules M;*

4. *every flat resolution of N has its $n-1$st syzygy flat.*

Proposition 2.2.6. *Let R be a ring and* $0 \to N \xrightarrow{\sigma} F \xrightarrow{\delta} M \to 0$ *be an exact sequence of* $R-$*modules with F flat and* $\mathrm{fd}(N) = n$ *then* $\mathrm{fd}(M) = n+1$.

Proof. Let $\mathscr{F} = 0 \to F_n \xrightarrow{d_n} F_{n-1} \to \cdots \to F_1 \xrightarrow{d_1} F_0 \xrightarrow{\phi} N \to 0$ be flat resolution of length n for N. Clearly

$$0 \to F_n \xrightarrow{d_n} F_{n-1} \to \cdots \to F_1 \xrightarrow{d_1} F_0 \xrightarrow{\sigma\phi} F \xrightarrow{\delta} M \to 0$$

is a flat resolution of length $n+1$ for M. Therefore $\mathrm{fd}(M) \leq n+1$. Let $w = \mathrm{fd}(M)$. If $w < n+1$ then by 2.2.5 $\ker d_{n-2}$ is flat which is a contradiction since it implies that $n-2^{\text{nd}}$ syzygy of $\mathscr{F}$ is flat thus $\mathrm{fd}(N) \leq n-1$. Therefore $\mathrm{fd}(M) = n+1$. □

Theorem 2.2.7. *Let* $0 \to N' \to N \to N'' \to 0$ *be an exact sequence of* $R-$*modules with* $\mathrm{fd}(N') = n < \infty$ *and* $\mathrm{fd}(N'') \leq n$ *then* $\mathrm{fd}(N) = n$.

Proof. We proceed by induction on n. Given a module M we have the long exact sequence

$$\cdots \to \mathrm{Tor}_1(M,N') \to \mathrm{Tor}_1(M,N) \to \mathrm{Tor}_1(M,N'') \to \cdots$$

If $n = 0$ then by 2.2.5 $\mathrm{Tor}_1(M,N') = \mathrm{Tor}_1(M,N'') = 0$ whence $\mathrm{Tor}_1(M,N) = 0$ therefore $\mathrm{fd}(N) = 0$. Now assume that $n > 0$ and we are done for all integers less than n. Let $w = \mathrm{fd}(N'')$. We have the following flat resolutions for N and N'' respectively.

$$\mathscr{F}' = 0 \to F'_n \xrightarrow{d'_{n-1}} \cdots \to F'_1 \xrightarrow{d'_1} F'_0 \xrightarrow{\sigma'} N' \to 0$$

$$\mathscr{F}'' = 0 \to F''_w \xrightarrow{d''_{w-1}} \cdots \to F''_1 \xrightarrow{d''_1} F''_0 \xrightarrow{\sigma''} N'' \to 0$$

By the Horseshoe lemma there exists a flat resolution $\mathscr{F} = \cdots \to F_1 \xrightarrow{d_1} F_0 \xrightarrow{\sigma} N \to 0$ such that $0 \to \mathscr{F}'' \to \mathscr{F} \to \mathscr{F} \to 0$ is an exact sequence of complexes. The construction of $\mathscr{F}$ in Horseshoe lemma shows that $0 \to \ker\sigma' \to \ker\sigma \to \ker\sigma'' \to 0$ is exact moreover $\mathrm{fd}(\ker\sigma') = n-1$ and $\mathrm{fd}(\ker\sigma'') \leq n-1$ by flat resolutions $0 \to F'_n \xrightarrow{d'_n} \cdots \to F'_1 \xrightarrow{d'_1} \ker\sigma' \to 0$ and

$0 \to F''_w \xrightarrow{d''_{w-1}} \cdots \to F''_1 \xrightarrow{d''_1} \ker\sigma'' \to 0$ therefore by the hypothesis of the induction we have $\mathrm{fd}(\ker\sigma) = n-1$. Let $u = \mathrm{fd}(N)$. If $u < n$ then by 2.2.5 $\ker d_{u-1}$ has to be flat whence we have the flat resolution

$$0 \to \ker d_{u-1} \to F_{u-1} \to \cdots \to F_1 \to \ker\sigma \to 0$$

for $\ker\sigma$ which shows that $\mathrm{fd}(\ker\sigma) \leq u-1$ which is a contradiction whence $\mathrm{fd}(N) = n$. □

Definition 2.2.8. Let A be a subring of ring B. B is called flat over A when B is flat both as a left and as a right $A-$module.

Using the above definitions, we have the following proposition.

Proposition 2.2.9. *Let A be subring of a ring B and assume that B is flat over A. Then:*

1. *If K is a left $B-$module then* $\mathrm{Tor}_v^A(N,K) \cong \mathrm{Tor}_v^B(N \otimes_A B, K)$ *for all right $A-$modules N and all v.*

2. *If L is a left $A-$module, then* $\mathrm{fd}_A(B \otimes_A L) \leq \mathrm{fd}_B(B \otimes_A L)$.

Proof. 1. Let $\cdots \to F_2 \xrightarrow{\delta_2} F_1 \xrightarrow{\delta_1} F_0 \xrightarrow{\delta_0} K \to 0$ be a free resolution for the $B-$module K.
We have the following commutative diagram of complexes whose vertical maps are natural isomorphisms with $\mu : N \to N \otimes_A B$ mapping n to $n \otimes 1$.

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & N \otimes_A F_2 & \xrightarrow{1_N \otimes \delta_2} & N \otimes_A F_1 & \xrightarrow{1_N \otimes \delta_1} & N \otimes_A F_0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mu \otimes 1_{F_2}} & & \downarrow{\scriptstyle \mu \otimes 1_{F_1}} & & \downarrow{\scriptstyle \mu \otimes 1_{F_0}} & & \\
\cdots & \longrightarrow & (N \otimes_A B) \otimes_B F_2 & \xrightarrow{1_{N \otimes_A B} \otimes \delta_2} & (N \otimes_A B) \otimes_B F_1 & \xrightarrow{1_{N \otimes_A B} \otimes \delta_1} & (N \otimes_A B) \otimes_B F_0 & \longrightarrow & 0
\end{array}$$

Since B is flat over A, F_vs are flat $A-$modules for all $v \geq 0$. Now the above diagram gives the result.
2. Let $w = \mathrm{fd}_B(B \otimes_A L)$, let $v > w$ and let N be a right $A-$module. By the first part and 2.2.5 we have

$$\mathrm{Tor}_v^A(N, B \otimes_A L) \cong \mathrm{Tor}_v^B(N \otimes_A B, B \otimes_A L) = 0$$

thus $\mathrm{fd}_A(B \otimes_A L) \leq w$. □

Definition 2.2.10. Let R be a ring. We define its *weak global dimension* $\mathrm{w.dim}(R)$ to be

$$\mathrm{w.dim}(R) = \sup\{\mathrm{fd}(B) : B \in {}_R\mathfrak{M}\}.$$

Theorem 2.2.11. *[9, Theorem 9.19] For any ring R,*

$$\mathrm{w.dim}(R) = \sup\{\mathrm{fd}(R/I) : I \textit{ is a left ideal in } R\}$$

$$= \sup\{\mathrm{fd}(R/I) : I \textit{ is a right ideal in } R\}.$$

Theorem 2.2.12. *[9, Theorem 9.22] Let R be a left Noetherian ring then* $\mathrm{w.dim}(R) = \mathrm{liD}(R)$.

Theorem 2.2.13. *[10, Proposition 9.3] Let* $(R, \mathfrak{m})$ *be a local ring and let* $k = R/\mathfrak{m}$ *be its residue field. Let G be a finitely generated* $R-$*module, let* $g_1, \ldots, g_n \in G$ *and let* $\bar{} : G \to G/\mathfrak{m}G$ *be the natural map. Then the following statements are equivalent:*

1. *G is generated by* $g_1, \ldots, g_n$.
2. *the* $R-$*module* $G/\mathfrak{m}G$ *is generated by* $\bar{g}_1, \ldots, \bar{g}_n$.
3. *the* $k-$*vector space* $G/\mathfrak{m}G$ *is generated by* $\bar{g}_1, \ldots, \bar{g}_n$.

Furthermore, the number of elements in each minimal generating set for the $R-$*module G is equal to* $\mathrm{Vdim}_k(G/\mathfrak{m}G)$.

Theorem 2.2.14. *[10, Corollary 15.18] Let* $(R, \mathfrak{m})$ *be a local ring. Then* $\dim(R)$ *is equal to the least number of elements of R that are needed to generate an* $\mathfrak{m}-$*primary ideal.*

Definition 2.2.15. Let $(R, \mathfrak{m})$ be a local ring of dimension d. A *system of parameters* for R is a set of d elements which generate an $\mathfrak{m}-$primary ideal.

In the light of 2.2.14 every local ring $(R, \mathfrak{m})$ does have a system of parameters. This fact together with 2.2.13 imply that $\dim(R) \leq \mathrm{Vdim}_k(\mathfrak{m}/\mathfrak{m}^2)$. When the equality $\dim(R) = \mathrm{Vdim}_k(\mathfrak{m}/\mathfrak{m}^2)$ happens, the ring R is of great importance. We have the following definition.

Definition 2.2.16. Let $(R, \mathfrak{m})$ be a local ring. Then R is said to be *regular* if $\dim(R) = \mathrm{Vdim}_k(\mathfrak{m}/\mathfrak{m}^2)$.

Theorem 2.2.17. *[9, Theorem 9.57] A local ring* $(R,\mathfrak{m})$ *is regular iff* $\mathrm{w.dim}(R) < \infty$ *in this case we have* $\mathrm{w.dim}(R) = \dim(R)$.

Theorem 2.2.18. *[9, Theorem 9.33] Let R be a commutative Noetherian ring and* $x \in R$ *be an element which is neither a unit nor a zero divisor. Let* $R^* = R/Rx$ *and asssume that* $\mathrm{w.dim}(R^*) < \infty$ *then*

$$\mathrm{w.dim}(R) \geq \mathrm{w.dim}(R^*) + 1.$$

Theorem 2.2.19. *[9, Theorem 9.34] Let R be a commutative Noetherian ring then*

$$\mathrm{w.dim}(R[t]) = \mathrm{w.dim}(R) + 1.$$

Theorem 2.2.20. *[1, Theorem 11.22] Let* $(R,\mathfrak{m})$ *be a local ring of dimension d and let* $k = R/\mathfrak{m}$. *Then the following are equivalent:*

1. $\mathrm{gr}_{\mathfrak{m}}(R) = \bigoplus_{i=0}^{\infty} \mathfrak{m}^i/\mathfrak{m}^{i+1} \cong k[x_1,\ldots,x_d]$ *where the* x_i *are independent indeterminates.*
2. $\mathrm{Vdim}_k(\mathfrak{m}/\mathfrak{m}^2) = d$.
3. $\mathfrak{m}$ *can be generated by d elements.*

Lemma 2.2.21. *[8, Theorem 18.1] Let* $(R,\mathfrak{m})$ *be a regular local ring of dimension n and let K be a zero dimensional module then* $\mathrm{Ext}^j(K,R) = 0$ *if* $j \neq n$ *and is isomorphic to* $R/\mathfrak{m}$ *if* $j = n$.

The last two results of this section is highly dependent on the following lemma which has an elementary proof.

Lemma 2.2.22. *Let M be a finitely generated module over a local ring* $(R,\mathfrak{m})$ *with* $\dim(M) > 0$. *Then there exists a zero-dimensional submodule K of M and an element* $x \in \mathfrak{m}$ *such that x is a non-zero divisor on* $\frac{M}{K}$.

Proof. Since M is finitely generated over R, $M \cong \frac{F}{S}$ for some finitely generated free $R-$module F. Let

$$S = S_1 \cap \cdots \cap S_n$$

be a minimal primary decomposition of S in R. If all of the $S_i s$ are $\mathfrak{m}-$primary in F then for each $1 \leq i \leq n$ there exists $m_i \geq 1$ with $\mathfrak{m}^{m_i} F \subseteq S_i$ whence $\mathfrak{m}^{m^*} F \subseteq S_1 \cap \cdots \cap S_n = S$ where $m^* = m_1 + \cdots + m_n$ hence $\dim(M) = \dim(\frac{F}{S}) = 0$ which is a contradiction. Hence there exists a $1 \leq i \leq n$ with $\dim(\frac{F}{S_i}) \neq 0$. Take $S' = S_{i_1} \cap \cdots \cap S_{i_l}$ where $\dim(\frac{F}{S_{i_j}}) \neq 0$. Put $N = \frac{F}{S'}$ and $K = \frac{S'}{S}$. We have the following exact sequence

$$0 \to K \to M \to N \to 0$$

We claim the $\frac{S'}{S}$ is of zero dimension. To see this, using the same argument in the very beginning of the proof, it can be shown that there exists a number m with $\mathfrak{m}^m S' \subseteq S$ hence $\dim(K) = \dim(\frac{S'}{S}) = 0$. Now we have

$$\mathrm{Zdv}_R(\frac{M}{K}) = \mathrm{Zdv}_R(\frac{F}{S'}) = \bigcup_{\mathfrak{p} \in \mathrm{Ass}(\frac{F}{S'})} \mathfrak{p}$$

Since for each $\mathfrak{p} \in \mathrm{Ass}(\frac{F}{S'})$ we have $\mathfrak{p} \neq \mathfrak{m}$ the prime avoidance lemma gives the result. □

Proposition 2.2.23. *Let $(R, \mathfrak{m})$ be a regular local ring of dimension n and let M be a finitely generated $R-$module then* $\dim(\mathrm{Ext}^j_R(M,R)) \leq n - j$ *for all* $j \geq 0$.

Proof. We proceed by induction on $\dim(M)$. Let $\dim(M) = 0$. Since $l(M/\mathfrak{m}^{v+1}M)$ is a polynomial of degree $\dim(M)$ for large v we have $\mathrm{Vdim}_{R/\mathfrak{m}}(\mathfrak{m}^v M/\mathfrak{m}^{v+1}M) = 0$ whence $\mathfrak{m}^{v+1}M = \mathfrak{m}^v M$. By NAK $\mathfrak{m}^v M = 0$. There exists an integer s and a filtration

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_{s-1} \subsetneq M_s = M$$

with $M_v/M_{v-1} \cong R/\mathfrak{m}$ for all $1 \leq v \leq s$ whence we have the exact sequence

$$0 \to M_{v-1} \to M_v \to R/\mathfrak{m} \to 0$$

which gives rise to the following long exact sequence

$$\begin{array}{l} \cdots \longrightarrow \operatorname{Ext}^{j-1}(M_{v-1},R) \\ \hookrightarrow \operatorname{Ext}^{j}(R/\mathfrak{m},R) \longrightarrow \operatorname{Ext}^{j}(M_{v},R) \longrightarrow \operatorname{Ext}^{j}(M_{v-1},R) \\ \hookrightarrow \operatorname{Ext}^{j+1}(R/\mathfrak{m},R) \longrightarrow \cdots \end{array} \tag{2.2}$$

We proceed by induction on s. If $s=1$ then $M \cong R/\mathfrak{m}$. Whence by 2.2.21 $\dim(\operatorname{Ext}^{j}(M,R)) = 0 < n-j$. Now let $s>1$ and the hypothesis of the induction holds for all integers less than s. If $j<n-1$ then $\operatorname{Ext}^{j}(R/\mathfrak{m},R) \cong 0 \cong \operatorname{Ext}^{j+1}(R/\mathfrak{m},R)$ thus $\operatorname{Ext}^{j}(M,R) \cong \operatorname{Ext}^{j}(M_{s-1},R)$. Therefore by the hypothesis of the induction we have $\dim(\operatorname{Ext}^{j}(M,R)) = \dim(\operatorname{Ext}^{j}(M_{s-1},R)) \leq n-j$.
If $j=n-1$ then by 2.2.21 and (2.2) we have the exact sequence

$$0 \to \operatorname{Ext}^{n-1}(M,R) \to \operatorname{Ext}^{n-1}(M_{s-1},R) \to R/\mathfrak{m}$$

therefore by the hypothesis of the induction $\dim(\operatorname{Ext}^{n-1}(M,R)) \leq \dim(\operatorname{Ext}^{n-1}(M_{s-1},R)) \leq 1$.
If $j>n-1$ then by 2.2.21 amnd (2.2) we have the exact sequence

$$\cdots \to \operatorname{Ext}^{j}(M,R) \to \operatorname{Ext}^{j}(M_{s-1},R) \to 0$$

which leads to the exact sequence $0 \to T \to \operatorname{Ext}^{n}(M,R) \to \operatorname{Ext}^{n}(M_{s-1},R) \to 0$ where T is either zero or $R/\mathfrak{m}$ whence $\dim(\operatorname{Ext}^{n}(M,R)) = 0$.
Now assume that $l = \dim(M) \geq 1$ and the hypothesis of the induction holds for every finitely generated $R-$module of dimension less than l. By 2.2.22 there exists a zero dimensional module K and $x \in \mathfrak{m}$ such that x is a non-zero divisor on M/K. We have the exact sequence $0 \to K \to M \to M/K \to 0$. Since $\dim(K) = 0$ by 2.2.21 $\operatorname{Ext}^{j}(K,R) = 0$ if $j<n$ and $\operatorname{Ext}^{n}(K,R)$ is a submodule of $R/\mathfrak{m}$. Using the long exact sequence we have

$$\operatorname{Ext}^{j}(M/K,R) \to \operatorname{Ext}^{j}(M,R) \to \operatorname{Ext}^{j}(K,R)$$

it is enough to show that $\dim(\operatorname{Ext}^{j}(M/K,R)) \leq n-j$.

Set $N = M/K$. Since $x \in \mathfrak{m}$ is a non-zero divisor on N we have the following exact sequence

$$0 \to N \xrightarrow{x} N \to N/xN \to 0$$

which gives rise the following long exact sequence

$$\cdots \to \operatorname{Ext}^j(N/xN,R) \to \operatorname{Ext}^j(N,R) \xrightarrow{x} \operatorname{Ext}^j(N,R) \to \operatorname{Ext}^{j+1}(N/xN,R) \to \cdots$$

Set $Q = \operatorname{Ext}^j(N,R)$. Since Q/xQ is isomorphic with a submodule of $\operatorname{Ext}^{j+1}(N/xN,R)$ we have $\dim(Q/xQ) \leq n-j-1$ therefore $\dim(Q) \leq n-j$. □

Proposition 2.2.24. *Let $(R,\mathfrak{m})$ be regular local ring of dimension n and let M be a finitely generated $R-$module then $\operatorname{Ext}^v(M,R) = 0$ for all $v < n - \dim(M)$.*

Proof. We proceed by induction on $l = \dim(M)$. If $\dim(M) = 0$ then by 2.2.21 $\operatorname{Ext}^j(M,R) = 0$ if $j \neq n$ and $\operatorname{Ext}^n(M,R)$ is a submodule of $R/\mathfrak{m}$ and we are done. Now assume that $l = \dim(M) > 0$ and $\operatorname{Ext}^v(S,R) = 0$ for all $v < n - \dim(S)$ if $\dim(S) < l$. By 2.2.22 M contains a submdule K of M and $x \in \mathfrak{m}$ which is a non-zero divisor on M/K. We have the following exact sequence

$$0 \to K \to M \to M/K \to 0$$

thus we have

$$\operatorname{Ext}^j(M/K,R) \to \operatorname{Ext}^j(M,R) \to \operatorname{Ext}^j(K,R)$$

It is enough to show that $\dim(\operatorname{Ext}^v(N,R)) = 0$ if $v < n - \dim(M)$ where $N = M/K$. The exact sequence $0 \to N \xrightarrow{x} N \to N/xN \to 0$ leads to the exact sequence

$$\operatorname{Ext}^v(N/xN,R) \to \operatorname{Ext}^v(N,R) \xrightarrow{x} \operatorname{Ext}^v(N,R) \to \operatorname{Ext}^{v+1}(N/xN,R)$$

Since $\dim(N/xN) \leq \dim(N) - 1$ we have $v+1 < n - \dim(N/xN)$. Since by the hypothesis of the induction we have $\operatorname{Ext}^v(N/xN,R) = \operatorname{Ext}^{v+1}(N/xN,R) = 0$ we have $\operatorname{Ext}^v(N,R) = \mathfrak{m}\operatorname{Ext}^v(N,R)$. By NAK $\operatorname{Ext}^v(N,R) = 0$. This completes the induction. □

Theorem 2.2.25. *[8, Theorem, 19.3] Let $(R,\mathfrak{m})$ be a regular local ring and $\mathfrak{p}$ a prime ideal then $R_{\mathfrak{p}}$ is again regular.*

Using the above theorem we can extend the definition of local rings to arbitrary commutative rings.

Definition 2.2.26. A Noetherian ring R is a *regular ring* if its localization on any prime ideal is a regular local ring.

Theorem 2.2.27. *Let R be regular ring of dimension n and M a finitely generated $R-$module then:*

1. $\dim(\operatorname{Ext}^j_R(M,R)) \leq n-j$ *for all* $j \geq 0$.
2. $\operatorname{Ext}^v(M,R) = 0$ *for all* $v < n - \dim(M)$.

Proof. The results are immediate from 2.2.23, 2.2.24, $S^{-1}\operatorname{Ext}^i_R(M,N) \cong \operatorname{Ext}^i_{S^{-1}R}(S^{-1}M, S^{-1}N)$ for every multiplicatively closed subset S of R, $\dim(M) = \sup\{\dim(M_\mathfrak{p}) : \mathfrak{p}$ is a prime ideal in $R\}$ and $M = 0$ iff $M_\mathfrak{p} = 0$ for all prime ideals $\mathfrak{p}$ of R. □

Theorem 2.2.28. *[3, Theorem, 4.15] Let A be a left and right Noetherian ring, M an $A-$module and let $\mu = \operatorname{w.dim}(A) < \infty$. If $\operatorname{Ext}^v(\operatorname{Ext}^j(M,A),A) = 0$ for all $v < j$ then M has a filtration $\mathscr{R}_0(M) \subseteq \cdots \subseteq \mathscr{R}_\mu(M) = M$ such that*

$$\mathscr{R}_v(M)/\mathscr{R}_{v-1}(M) \hookrightarrow \operatorname{Ext}^{\mu-v}(\operatorname{Ext}^{\mu-v}(M,A),A)$$

for all $v \geq 1$.

Theorem 2.2.29. *[8, Theorem, 19.5] If R is regular then so are $R[X]$ and $R[[X]]$.*

Theorem 2.2.30. *Let A be a left Noetherian filtered ring, let $\mu = \operatorname{w.dim}(A)$ and let M be a non-zero finitely generated $A-$module then there exists a number $0 \leq j \leq \mu$ with $\operatorname{Ext}^j(M,A) \neq 0$.*

Proof. Suppose to the contrary that $\operatorname{Ext}^i(M,A) = 0$ for all integers $i \geq 0$. By 2.1.17, $\operatorname{Ext}^i(M,N) = 0$ for all $i > 0$ and all right $A-$modules N. Therefore M is a projective $A-$module and hence is a summand of a free module. Thus, there exists an $A-$module M' with $M \bigoplus M' = \bigoplus_\alpha A$. Clearly $\operatorname{Hom}(M,A) \neq 0$ which is a contradiction. Since $\mu = \operatorname{liD}(A)$ by 2.2.12, $0 \leq j \leq \mu$. □

Let $w = \dim(\mathrm{gr}_\Sigma(A))$ and let $\mathrm{gr}_\Sigma(A)$ be a regular ring. By 2.2.17 and 2.1.16 we have $\mathrm{w.dim}(A) \leq w$. Let $\mu = \mathrm{w.dim}(A)$ and let M be a finitely generated $A-$module then by 2.2.5 there exists a finite number $j(M)$ with $\mathrm{Ext}_A^{j(M)}(M,A) \neq 0$. By the definition of global injective dimension, we have $0 \leq j_A(M) \leq \mu$. The number $j_A(M)$ is called the *extent* of the module M as an $A-$module.

2.3 Preliminaries on the ring of differentials over the ring of formal power series

Definition 2.3.1. Let R be a ring containing a field k and M an $R-$module. A map $\delta : R \to M$ is called a $k-$linear derivation from R to M if it is additive, $k-$linear and $\delta(ab) = a\delta(b) + b\delta(a)$ for all $a, b \in R$.

Definition 2.3.2. Let $(R, \mathfrak{m}, k)$ be local and δ a $k-$linear derivation from R to R. Clearly $\delta(\mathfrak{m}^2) \subseteq \mathfrak{m}$. Hence δ gives rise to an $R/\mathfrak{m}-$homomorphism $\mathfrak{m}/\mathfrak{m}^2 \to R/\mathfrak{m}$ which is called the *tangent* map. Let $\mathrm{Der}_k(R)$ denote the set generated by of all $k-$linear derivations on R with multiplication operators defined on elements of R. We say that $\mathrm{Der}_k(R)$ has maximum rank if every $R/\mathfrak{m}-$homomorphism $\mathfrak{m}/\mathfrak{m}^2 \to R/\mathfrak{m}$ is a tangent map of some derivation.

Theorem 2.3.3. *Let $(R, \mathfrak{m})$ be a local ring which contains the field of rational numbers and let $\mathrm{Der}_k(R)$ be of maximum rank then R is regular.*

Proof. Let $n = \mathrm{Vdim}_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2)$. By 2.2.13 there exists a minimal set of generators $x_1, \ldots, x_n$ for $\mathfrak{m}$. We define $\delta_i : \mathfrak{m}/\mathfrak{m}^2 \to R/\mathfrak{m}$ with

$$\delta_i(x_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Let $t_1, \ldots, t_n$ be independent indeterminates over $R/\mathfrak{m}$. We define $\Phi : (R/\mathfrak{m})[t_1, \ldots, t_n] \to \mathrm{gr}_\mathfrak{m}(R)$ with $\Phi(\sum \overline{r_\alpha} t^\alpha) = \sum \overline{r_\alpha} x^\alpha$. Clearly Φ is a surjective ring homomorphism. We show that Φ is injective. Let $\sum \overline{r_\alpha} x_1^{\alpha_1} \cdots x_n^{\alpha_n} = 0$ where $^- : R \to R/\mathfrak{m}$ is the natural map from R to $R/\mathfrak{m}$. Let $s = |\alpha^*| = \alpha_1^* + \cdots + \alpha_n^*$ be maximum in $\sum \overline{r_\alpha} x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. We have $\sum_{|\alpha|=s} r_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in \mathfrak{m}^{s+1}$.

Set $\delta^* = \delta_1^{\alpha_1^*} \cdots \delta_n^{\alpha_n^*}$. Since $\delta_i(\mathfrak{m}^{l+1}) \subseteq \mathfrak{m}^l$ for all $l \geq 0$ we have $r_{\alpha^*} \in \mathfrak{m}$. Whence $\ker(\Phi) = 0$. By 2.2.20 R is regular. □

Now let $R = k[[x_1, \cdots, x_n]]$ be the ring of power series on n variables over a field of characteristic zero k. Let $D_n(R)$ stand for the ring of $k-$linear derivations on R. To be more precise $D_n(R)$ is generated by $d_i = \partial/\partial x_i : R \to R$ and multiplications on R therefore each element of R is a finite sum of elements of the form $f_\alpha d_1^{\alpha_1} \cdots d_n^{\alpha_n}$ where f_α is an element of R. $D_n(R)$ is filtered ring with the $\Sigma_i = \{\sum f_\alpha d_1^{\alpha_1} \cdots d_n^{\alpha_n} : \sum \alpha_j = i\}$. Clearly

$$\mathrm{gr}_\Sigma(D_n(R)) = \Sigma_0 \oplus (\Sigma_1/\Sigma_0) \oplus (\Sigma_2/\Sigma_1) \oplus \cdots = R[\bar{d}_1, \ldots, \bar{d}_n]$$

Therefore $\mathrm{gr}_\Sigma(D_n(R))$ is regular by 2.2.29.

Lemma 2.3.4. *[3, Lemma, 4.2] Let M be a left $\mathscr{D}_1-$module such that* $\mathrm{Vdim}_k(M) < \infty$ *then* $M = 0$.

Lemma 2.3.5. *Let M be a left $\mathscr{D}_n(k)-$module. Then the $\mathscr{D}_n(k)-$linear map $\Delta : M \to (B \otimes_{\mathscr{D}_n} M) \oplus (E \otimes_{\mathscr{D}_n} M)$ which maps m to $(1 \otimes m, 1 \otimes m)$ is injective where $B = k(x_n) \otimes_{k[x_n]} \mathscr{D}_n(k)$ and $E = k(\partial_n) < x_1, \ldots, x_n; \partial_1, \ldots, \partial_{n-1} >$.*

Proof. If $\Delta(m) = 0$ then $p(x_n)m = q(\partial_n)m = 0$. We have $\mathscr{D}_1(k) = k < x_n, \partial_n >$. Clearly $\mathscr{D}_1 m$ is finite dimension whence by 2.3.4 $\mathscr{D}_1 m = 0$ and therefore $m = 0$. □

Theorem 2.3.6. *Let k be a field of characteristic zero then* $\mathrm{w.dim}(\mathscr{D}_n(k)) = n$.

Proof. Clearly $\mathscr{D}_n(k)/L \cong k[x_1, \ldots, x_n]$ where L is the ideal generated by $\partial_1, \ldots, \partial_n$. Since $\mathscr{D}_n(k)$ is a flat $k[x_1, \ldots, x_n]-$module and $\mathrm{w.dim}(k[x_1, \ldots, x_n]) = n$ therefore $\mathrm{w.dim}(\mathscr{D}_n(k)) \geq n$.
By 2.1.16 and 2.2.19 we have $\mathrm{w.dim}(\mathscr{D}_n) \leq \mathrm{w.dim}(\mathrm{gr}(\mathscr{D}_n)) = 2n$. Thus the weak dimension of $\mathscr{D}_n(k)$ is finite. We use induction on n to prove the theorem. We have $\mathscr{D}_0(k) = k$ and clearly $\mathrm{w.dim}(k) = 0$. Assume that $n > 0$ and we are done for all integers less than n. We have $\mathscr{D}_n(k) = k < x_1, \ldots, x_n; \partial_1, \ldots, \partial_n >$. Let $B = k(x_n) \otimes_{k[x_n]} \mathscr{D}_n(k)$. Each element of B is a finite sum of elements of the form $f_\beta \partial_1^{\beta_1} \cdots \partial_n^{\beta_n}$ where $f_\beta \in k(x_n)[x_1, \ldots, x_{n-1}]$. If $b \in B$ then there exists

$q(x_n) \in k[x_n]$ with $q(x_n)b \in \mathscr{D}_n(k)$. We filter B with $\Sigma_v(B) = \{b : b = \Sigma_{j\leq v} b_j(x,\partial')\partial_n^j\}$ where $b_j(x,\partial') \in k(x_n) < x_1,\ldots,x_{n-1};\partial_1,\ldots,\partial_{n-1} >= \mathscr{D}_{n-1}(k(x_n))$. Clearly $\Sigma(B)$ is filtration on B with $\mathrm{gr}_{\Sigma(B)}(B) \cong \mathscr{D}_{n-1}(k(x_n))[t]$. The ring $\mathrm{gr}(B)$ is a polynomial ring in one variable with coefficients in Weyle algebra $\mathscr{D}_{n-1}(k(x_n))$. By the induction hypothesis, $\mathrm{w.dim}(\mathscr{D}_{n-1}(k(x_n)) = n-1$ therefore by 2.2.19 $\mathrm{w.dim}(\mathrm{gr}(B)) = \mathrm{w.dim}(\mathscr{D}_{n-1}(k(x_n))[t]) = n$ hence by 2.1.16, $\mathrm{w.dim}(B) \leq n$. Replacing the procedure with ∂_n instead of x_n we set $E = k(\partial_n) < x_1,\ldots,x_n;\partial_1,\ldots,\partial_{n-1} >$ and we have $\mathrm{w.dim}(E) \leq n$.

Let M be a $\mathscr{D}_n(k)-$module. Since B is localization of $\mathscr{D}_n$ with respect the multiplicatively closed set $k[x_n]\setminus 0$, it is flat over $\mathscr{D}_n(k)$. By 2.2.9 we have $\mathrm{fd}_{\mathscr{D}_n}(B\otimes_{\mathscr{D}_n} M) \leq \mathrm{fd}_B(B\otimes_{\mathscr{D}_n} M) \leq n$. Since by the induction $\mathrm{w.dim}(B) \leq n$. Similarly we have $\mathrm{fd}_{\mathscr{D}_n}(E\otimes_{\mathscr{D}_n} M) \leq n$.

In order to complete the proof we assume that $\mathrm{w.dim}(\mathscr{D}_n(k)) = w > n$ and look for a contradiction. Let M be a $\mathscr{D}_n(k)-$module with $\mathrm{fd}_{\mathscr{D}_n(k)}(M) = w$. Using 2.3.5 we have the following exact sequence

$$0 \to M \xrightarrow{\Delta} (B\otimes M)\oplus(E\otimes M) \to \overline{M} \to 0$$

Since $\mathrm{w.dim}(\overline{M}) \leq w$ by 2.2.7 we have $\mathrm{w.dim}((B\otimes M)\oplus(E\otimes M)) = w$ which is a contradiction. This completes the induction and the proof of this theorem. □

Lemma 2.3.7. *Let $\mathscr{K}$ be the quotient field of $R = k[[x_1,\ldots,x_n]]$ and let $B = \mathscr{K}\otimes_R D_n \otimes_k \mathscr{D}_m(k)$. Then* $\mathrm{w.dim}(B) \leq n+m$*.*

Proof. We have $\mathscr{D}_m(\mathscr{K}) = \mathscr{K} < y_1,\ldots,y_m;\partial/\partial y_1,\ldots,\partial/\partial y_m >$. Each element of B is of the from $\sum Q_\alpha d^\alpha$ where $Q_\alpha \in \mathscr{D}_m(\mathscr{K})$. We filter B with

$$\Sigma_v(B) = \{\sum_{|\alpha|\leq v} Q_\alpha d^\alpha : Q_\alpha \in \mathscr{D}_m(\mathscr{K})\}$$

Clearly $\mathrm{gr}_{\Sigma(B)}(B)$ is isomorphic to the ring of polynomials in n variables over $\mathscr{D}_m(\mathscr{K})$ therefore by 2.1.16, 2.3.6 and 2.2.19 we have $\mathrm{w.dim}(B) \leq n+m$. □

Lemma 2.3.8. *Let $D_{n,m} = D_n(k)\otimes_k \mathscr{D}_m(k)$. If w is finite and* $\mathrm{w.dim}(D_{n,m}) = w > n+m$ *then there exists a left ideal L of $D_{n,m}$ which contains a non-zero element $f \in R$ with* $\mathrm{fd}(D_{n,m}/L) = w-1$.

Proof. By 2.2.11 we have

$$\mathrm{w.dim}(D_{n,m}) = \sup\{\mathrm{fd}(D_{n,m}/L) : L \text{ is a left ideal of } D_{n,m}\}$$

there exists a left ideal L^* of $D_{n,m}$ with $\mathrm{w.dim}(D_{n,m}/L^*) = w$. We have the exact sequence $0 \to L^* \to D_{n,m} \to D_{n,m}/L^* \to 0$. By 2.2.6 we have $\mathrm{fd}(L^*) = w-1$. Since $D_{n,m}$ is Noetherian, we may assume that L^* is maximal with this property. We claim that if L is an ideal of $D_{n,m}$ with $L^* \subsetneq L$ then $\mathrm{fd}(L) \leq w-2$. Suppose $\mathrm{fd}(L) > w-2$. By the maximality of L^* we can only have $\mathrm{fd}(L) = w$. By 2.2.6 we have $\mathrm{fd}(L) = \mathrm{fd}(D_{n,m}/L) - 1 \leq w-1$ which is a contradiction. Therefore $\mathrm{fd}(L) \leq w-2$.

Let $M = D_{n,m}/L^*$ and let $B = \mathscr{K} \otimes_R D_{n,m}$ where $\mathscr{K}$ is the fraction field of $R = k[[x_1,\ldots,x_n]]$. Clearly B is flat over $D_{n,m}$. Let $M_1 = B \otimes_{D_{n,m}} M$ by 2.2.9 and 2.3.7 we have $\mathrm{fd}_{D_{n,m}}(M_1) \leq \mathrm{fd}_B(M_1) \leq n+m$. Let $\Delta : M \to M_1$ be a map from M to M_1 which maps $m \in M$ to $1 \otimes m \in M_1$. If Δ is injective we have the following exact sequence

$$0 \to M \xrightarrow{\Delta} M_1 \to M_1/\Delta(M) \to 0$$

Since $\mathrm{fd}(M) = w$ and $\mathrm{fd}(M_1/\Delta(M)) \leq w$ therefore by 2.2.7 we have $\mathrm{fd}(M_1) = w$. But $\mathrm{fd}(M_1) \leq n+m < w$ which is a contradiction. Therefore Δ is not injective, hence there exits a non-zero element $t \in M$ with $1 \otimes t = 0$ in M_1. Whence there exists $f \in R$ with $ft = 0$. Let Q be a lift of t in $D_{n,m}$. Clearly $fQ \in L^*$. Let $L = L^* + D_{n,m}Q$. By the first part of the proof we have $\mathrm{fd}(L) \leq w-2$. By 2.2.6 the exact sequence $0 \to L^* \to L \to L/L^* \to 0$ with $\mathrm{fd}(L/L^*) \leq w-1$ and $\mathrm{fd}(L^*) = w-1$ gives $\mathrm{fd}(L) = w-1$ thus $\mathrm{fd}(L/L^*) = w$. Let $L_0 = \{R \in D_{n,m} : RQ \in L^*\}$. Clearly $L/L^* \cong D_{n,m}/L_0$ whence $\mathrm{fd}(L_0) = w-1$ and clearly $f \in L_0$. □

Remark 2.3.9. Let $GL(n,k)$ stand for the general linear group of degree n with entries in the field k. The group $GL(n,k)$ acts on D_n naturally. To be more precise if $\mathscr{A} = (k_{iv}) \in GL(n,k)$ and $\mathscr{B} = (q_{vi})$ is its inverse we define $\widetilde{x_i} = \sum_{v=1}^{n} k_{vi}x_v$ and $\widetilde{d_i} = \sum q_{iv}d_v$. Clearly $\widetilde{\ }: D_n \to D_n$ is an isomorphism of D_n.

Definition 2.3.10. An element $f \in k[[x_1,\ldots,x_n]]$ is said to be regular in x_n if $f(0,\ldots,0,x_n) \neq 0$.

Theorem 2.3.11. *[11, Lemma3. ChVII] If $0 \neq f \in R = k[[x_1, \cdots, x_n]]$ then there exists an invertible linear transformation $\Phi : R \to R$ such that $\Phi(f)$ is regular in x_n.*

Proposition 2.3.12. *[11, Corollary1. ChVII] Let $f \in k[[x_1, \ldots, x_n]]$ be regular in x_n and let t be the order of $f(0, \ldots, 0, x_n)$. If $t \geq 1$ then there exist power series $g \in k[[x_1, \ldots, x_n]]$ and $s_i \in k[[x_1, \ldots, x_{n-1}]]$ $(i = 0, 1, \ldots, t-1)$ such that*

$$f = g(x_n^t + s_{t-1}x_n^{t-1} + \cdots + s_0)$$

where g and s_i are uniquely determined, g is invertible and none of the s_i is invertible.

Theorem 2.3.13. *[11, Theorem 5. ChVII] Let $f \in k[[x_1, \ldots, x_n]]$ be a non-invertible power series containing a term of them from ax_n^h where $0 \neq a \in k$ and $h \geq 1$. Let $t \geq 1$ be the smallest integer of this property. Then for every $g \in k[[x_1, \ldots, x_n]]$ there exist power series u and $s_i \in k[[x_1, \ldots, x_{n-1}]]$ $(i = 0, \ldots, s-1)$ such that*

$$g = uf + \sum_{i=0}^{t-1} s_i x_n^i$$

The power series u and s_i are uniquely determined by g and f.

Proposition 2.3.14. $\mathrm{w.dim}(D_n(k) \otimes_k \mathscr{D}_m(k)) = n + m$.

Proof. Let $D_{n,m} = D_n(k) \otimes \mathscr{D}_m(k)$. We proceed by induction on n. If $n = 0$ then $D_{n,m} = \mathscr{D}_m(k)$ and we are done by 2.3.6. Let $n > 0$ and we are done for all integers less than n. First we show that $\mathrm{w.dim}(D_{n,m}) \leq n + m$. Let $w = \mathrm{w.dim}(D_{n,m})$. We claim that w is finite. To see this consider the filtration

$$\eta = \Sigma_0 \otimes_k \mathscr{D}_m(k) \subseteq \Sigma_1 \otimes_k \mathscr{D}_m(k) \subseteq \Sigma_2 \otimes_k \mathscr{D}_m(k) \subseteq \cdots$$

which gives the associated graded ring

$$\mathrm{gr}_\eta(D_{n,m}) = R[\zeta_1, \ldots, \zeta_n] \otimes_k \mathscr{D}_m(k)$$

where $\zeta_1, \ldots, \zeta_n$ are the images of $d_1, \ldots, d_n$ in Σ_1/Σ_0. Now filtering the above ring with the filtration

$$\tau = R[\zeta_1, \ldots, \zeta_n] \otimes_k \Xi_0 \subseteq R[\zeta_1, \ldots, \zeta_n] \otimes_k \Xi_1 \subseteq R[\zeta_1, \ldots, \zeta_n] \otimes_k \Xi_2 \subseteq \cdots$$

gives

$$\mathrm{gr}_\tau(\mathrm{gr}_\eta(D_{n,m})) = R[\zeta_1,\dots,\zeta_n,y_1,\dots,y_n,\rho_1,\dots,\rho_m]$$

where $\mathscr{D}_m(k) = k < y_1,\dots,y_m;\partial_1,\dots,\partial_m >$ and $\rho_1,\dots,\rho_m$ are images of $\partial_1,\dots,\partial_m$ in Ξ_1/Ξ_0. Using 2.1.16 two times we have $w \le 2n+2m$ and hence is finite. If $w > n+m$ then by 2.3.8, there exists a left ideal L of $D_{n,m}$ which contains a non-zero element $f \in R_n = k[[x_1,\cdots,x_n]]$ with $\mathrm{fd}(L) = w-1$. By 2.3.11 and 2.3.12, we may assume that $f = x_n^t + r_1(x')x_n^{t-1} + \cdots + r_t(x')$ where $r_i(x') \in R_{n-1} = k[[x_1,\cdots,x_{n-1}]]$. We have $\mathscr{D}_{m+1}(k) = k < y_1,\cdots,y_m,x_n;\partial/\partial y_1,\cdots,\partial/\partial y_m,\partial/\partial x_n >$ and the inclusion map $D_{n-1,m+1} = D_{n-1}\otimes\mathscr{D}_{m+1} \hookrightarrow D_{n,m}$. Set $L_0 = L \cap D_{n-1,m+1}$. By 2.3.13 L is generated by L_0 in $D_{n,m}$. Since $D_{n,m}$ is flat over $D_{n-1,m+1}$ we have $L \cong D_{n,m}\otimes_{D_{n-1,m+1}} L_0$. By the induction hypothesis $\mathrm{w.dim}(D_{n-1,m+1}) = n+m$. Using the exact sequence

$$0 \to L_0 \to D_{n-1,m+1} \to D_{n-1,m+1}/L_0 \to 0$$

and the fact that $D_{n,m}$ is flat over $D_{n-1,m+1}$ we have $D_{n-1,m+1}/L_0 \otimes_{D_{n-1,m+1}} D_{n,m} \cong D_{n,m}/L$. Now $\mathrm{fd}_{D_{n-1,m+1}}(D_{n-1,m+1}/L_0) = \mathrm{fd}_{D_{n-1,m+1}}(L_0)+1$ by 2.2.6 whence $\mathrm{fd}_{D_{n-1,m+1}}(L_0) \le n+m-1$. On the other hand since $D_{n,m}$ is flat over $D_{n-1,m+1}$ thus

$$\mathrm{fd}_{D_{n-1,m+1}}(L) = \mathrm{fd}_{D_{n-1,m+1}}(L_0\otimes_{D_{n-1,m+1}} D_{n,m}) \le \mathrm{fd}_{D_{n-1,m+1}}(L_0) \le n+m-1$$

whence $\mathrm{fd}_{D_{n,m}}(L) \le \mathrm{fd}_{D_{n-1,m+1}}(L) \le n+m-1$ which is a contradiction. Therefore $\mathrm{w.dim}(D_{n,m}) \le n+m$.

It remains to show that $\mathrm{w.dim}(D_{n,m}) \ge n+m$. By 2.2.17 and 2.2.19 we have

$$\mathrm{w.dim}(R\otimes_k k[y_1,\dots,y_m]) \ge n+m$$

Since $D_{n,m}$ is flat over $R\otimes_k k[y_1,\dots,y_m]$, every flat $D_{n,m}-$module is a flat $R\otimes k[y_1,\dots,y_m]-$module, therefore $\mathrm{fd}_{D_{n,m}}(R\otimes k[y_1,\dots,y_m]) \ge n+m$ and hence $\mathrm{w.dim}(D_{n,m}) \ge n+m$. This completes the induction. □

Theorem 2.3.15. $\mathrm{w.dim}(D_n) = n$.

Proof. Immediate from 2.3.14 by setting $m = 0$. □

2.4 Björk's proof of the Bernstein inequality for power series

Now we are in the position to give Björk's proof of the Bernstein inequality. Though it was a long journey, it gave us a lot of splendid results. Our main reference is [3].

Proposition 2.4.1. *Let M be finitely generated D_n-module then $d(\mathrm{Ext}^j(M,D_n)) \leq 2n-j$ for all $j \geq 0$.*

Proof. By 2.2.29 $\mathrm{gr}(D_n) = R[\overline{d_1},\ldots,\overline{d_n}]$ is a regular ring where $R = k[[x_1,\ldots,x_n]]$. Let Γ be a good filtration for M. By 2.1.12 and 2.2.27 we have

$$d(\mathrm{Ext}^j(M,D_n)) \leq \dim(\mathrm{Ext}^j(\mathrm{gr}_\Gamma(M),\mathrm{gr}(D_n)) \leq 2n-j.$$

□

Let M be a finitely generated $D-$module. Lets recall that the extent of M is the largest integer $j(M)$ with $\mathrm{Ext}^{j(M)}(M,D_n) \neq 0$.

Lemma 2.4.2. *Let M be a finitely generated $D-$module. Then $\mathrm{Ext}^v(M,D_n) = 0$ if $v < 2n-d(M)$ which means that $j(M)+d(M) \geq 2n$.*

Proof. Let Γ be a good filtration on M. By 2.1.9 and 2.1.12 $\mathrm{Ext}^j(M,D_n)$ does have a filtration Γ^* such that the $\mathrm{gr}(D)-$module $\mathrm{gr}_{\Gamma^*}(\mathrm{Ext}^j(M,D_n))$ is a subfactor of the module $\mathrm{Ext}^j_{\mathrm{gr}(D)}(\mathrm{gr}_\Gamma(M),\mathrm{gr}(D_n)))$ for each j. Since by 2.2.29, $\mathrm{gr}(D)$ is a regular ring therefore by 2.2.27 $\mathrm{Ext}^j_{\mathrm{gr}(D)}(\mathrm{gr}_\Gamma(M),\mathrm{gr}(D_n))) =$ 0 when $j < 2n-d(M)$. Hence $\mathrm{Ext}^j(M,D_n) = 0$ when $j < 2n-d(M)$ whence $j(M) \geq 2n-d(M)$. □

Corollary 2.4.3. $\mathrm{Ext}^v(\mathrm{Ext}^j(M,D_n),D_n) = 0$ *if $v < j$.*

Proof. Immediate from 2.4.2 and 2.4.1. □

Proposition 2.4.4. *Let M be finitely generated $D-$module then $d(M)+j(M) = 2n$.*

Proof. By 2.3.15, $\mathrm{w.dim}(D_n) = n$. Therefore by 2.3.4, the D_n-module M has a filtration $\mathscr{R}_0(M) \subseteq \mathscr{R}_1(M) \subseteq \cdots \subseteq \mathscr{R}_n(M) = M$ with $M(v) = \mathscr{R}_v(M)/\mathscr{R}_{v-1}(M) \hookrightarrow \mathrm{Ext}^{n-v}(\mathrm{Ext}^{n-v}(M,D_n),D_n)$

for $1 \leq v \leq n$. By 2.4.1, $d(M(v)) \leq d(\mathrm{Ext}^{n-v}(\mathrm{Ext}^{n-v}(M,D_n),D_n)) \leq n+v$. The exact sequence $0 \to \mathscr{R}_{v-1}(M) \to \mathscr{R}_v(M) \to M(v) \to 0$ gives $d(\mathscr{R}_v) = \sup\{d(\mathscr{R}_{v-1}(M)), d(M(v))\}$. Induction on v shows that $d(\mathscr{R}_v(M)) \leq n+v$.

By 2.4.2, $d(M) + j(M) \geq 2n$. In order to complete the proof we need to show that $d(M) \leq 2n - j(M)$. By the definition of $j(M)$ we have $\mathrm{Ext}^{n-v}(M,D_n) = 0$ if $n-v < j(M)$ and hence $\mathrm{Ext}^{n-v}(\mathrm{Ext}^{n-v}(M,D_n),D_n) = 0$ if $n-v < j(M)$ therefore $M(v) = 0$ if $n-v < j(M)$. This implies that $\mathscr{R}_{n-j(M)}(M) = M$. By the previous paragraph we have

$$d(M) = d(\mathscr{R}_{n-j(M)}(M)) \leq n+n-j(M) = 2n - j(M)$$

This completes the proof. □

The next theorem is a well-known inequality in the theory of $D-$modules which is called the Bernstein inequality. All of the materials we presented so far, led to the following result.

Theorem 2.4.5. *Let M be a finitely generated $D-$module then $d(M) \geq n$.*

Proof. We have $d(M) = 2n - j(M)$. Note that since D_n is left Noetherian, $j(M) \leq n$ by 2.2.30 hence $d(M) \geq n$. □

Chapter 3

A Short Proof of the Bernstein Inequality for the ring of power series

In the previous chapter we went over Björk's proof of the Bernstein ineqaulity. In this chapter we shall give a short proof of the inequality. While the original proof of the Bernstein inequality is elegant in its own right, it is indeed lengthy and as we saw it uses a lot of sophisiticated mathematicial tools. Our proof is inspired by [6] and has an elemantary nature. The main simplification compared to Bjork's proof is that our proof does not use homological algebra.

3.1 Preliminaries

Before we present a short proof for Bernstein inequality for the ring of power series, we cover some theorems needed for our proof. Our main references will be [3] and [11].

Lemma 3.1.1. *[3, 3.3.22] Let $\mathfrak{p}$ be a prime ideal in $k[[x_1,\ldots,x_n]]$ and put $h = \mathrm{ht}(\mathfrak{p})$. Then there exists a $k-$linear transformation which sends x_i to $y_i = \sum k_{vi}x_v$ where $(k_{vi}) \in GL(n,k)$ such that $k[[x_1,\ldots,x_n]]$ is identified with $k[[y_1,\ldots,y_n]]$. Let $S = k[[y_{h+1},\ldots,y_n]]$. Then:*

1. $\mathfrak{p} \cap S = 0$.

2. *There exists an irreducible polynomial* $t(y',y_1) = y_1^e + r_1(y')y_1^{e-1} + \cdots + r_e(y') \in S[y_1] \cap \mathfrak{p}$ *where* y' *indicates the variables* $y_{h+1},\ldots,y_n$.

Lemma 3.1.2. *Let* $\mathfrak{p}$ *be a prime ideal of height* h *in* $R = k[[x_1,\ldots,x_n]]$ *then the variables* $x_1,\ldots x_n$ *can be chosen so that* $\mathfrak{p} \cap k[[x_{h+1},\ldots,x_n]] = 0$ *and* $R/\mathfrak{p}$ *is finite over* $k[[x_{h+1},\ldots,x_n]]$.

Proof. By 3.1.1 there exist a change of variables for R such that $k[[x_{h+1},\ldots,x_n]] \cap \mathfrak{p} = 0$ and for each $1 \le i \le h$ there exists an irreducible polynomial

$$t_i(x',x_i) = x_i^{e_i} + r_{1i}(x')x_i^{e_i-1} + \cdots + r_{e_i i}(x') \in S[x_i] \cap \mathfrak{p}$$

where x' indicates the variables $x_{h+1},\ldots,x_n$ and $S = k[[x_{h+1},\ldots,x_n]]$. For each $1 \le i \le h$ let $s_i \ge 1$ denote the smallest integer such that t_i contains a non-zero term of the form $ax_i^{s_i}$. The existence of the term $x_i^{e_i}$ guarantees the existence of s_i. We claim that $R/\mathfrak{p}$ is generated by elements of the form $\overline{x_1^{j_1}\cdots x_h^{j_h}}$ over S where $^- : R \to R/\mathfrak{p}$ indicates the natural map and $0 \le j_l \le s_l - 1$ for all $1 \le l \le h$. Let $f \in R$ be non-invertible. By 2.3.12 there exists $U_1 \in k[[x_1,\ldots,x_n]]$ and $R_1,\ldots,R_{s_1-1} \in k[[x_2,\ldots,x_n]]$ such that

$$f = U_1t_1 + \sum_{i=0}^{s_1-1} R_i x_1^i$$

Clearly $U_1t_1 \in \mathfrak{p}$ since $t_1 \in \mathfrak{p}$. Using 2.3.12 for R_is and t_2 we have

$$R_i = V_it_2 + \sum_{j=0}^{s_2-1} R_j^{(i)} x_2^j$$

Where $V_i \in k[[x_2,\ldots,x_n]]$ and $R_j^{(i)} \in k[[x_3,\ldots,x_n]]$. This process can proceed and after h steps f can be written as the sum of an element in $\mathfrak{p}$ and sum of elements of the form $R_{j_1\cdots j_h}x_1^{j_1}\cdots x_h^{j_h}$ where $R_{j_1\cdots j_h} \in S$ and $0 \le j_l \le s_l - 1$ for all $1 \le l \le h$. □

3.2 A Short Proof for Theorem 2.4.5

Let M be a finitely generated $D-$module. Clearly M is an $R-$module. Since R is Noetherian, there exists some $z \in M$ with $\mathrm{Ann}_R(z)$ is a prime ideal. Let $\mathfrak{p} = \mathrm{Ann}_R(z)$ and let $N = Dz$. Note that by 2.1.8 we have $d(N) \le d(M)$. Therefore, in order to show the Bernstein inequality, it is enough to show that $d(N) \ge n$.

Consider the filtration $\Sigma_0 z \subseteq \Sigma_1 z \subseteq \Sigma_2 z \subseteq \cdots$. We have

$$\mathrm{gr}(N) = \Sigma_0 z \oplus (\Sigma_1 z/\Sigma_0 z) \oplus (\Sigma_2 z/\Sigma_1 z) \oplus \cdots$$

Clearly the $\mathrm{gr}(D)-$module $\mathrm{gr}(N)$ is generated by $z \in \Sigma_0 z$ whence $\mathrm{gr}(N) \cong \mathrm{gr}(D)/J$ where J is the annihilator of $\mathrm{gr}(N)$ as a $\mathrm{gr}(D)-$module. Since $\mathrm{gr}(N)$ is a graded $\mathrm{gr}(D)-$module, J has to be a graded ideal in $\mathrm{gr}(D)$ and clearly $J \cap R = \mathfrak{p}$. We have

$$d(N) = \dim(\mathrm{gr}(N)) = \dim(\mathrm{gr}(D)/J)$$

We need to show that the dimension of the ring $\mathrm{gr}(D)/J$ is at least n. Let $\widetilde{D} = \mathrm{gr}(D)/J$ and let $\widetilde{D}_i$ stand for the i^{th} graded piece of $\widetilde{D}$ whence $\widetilde{D} = \widetilde{D}_0 \oplus \widetilde{D}_1 \oplus \widetilde{D}_2 \oplus \cdots$. Let $\widetilde{D}_+ \subseteq \widetilde{D}$ be the ideal generated by elements of positive degree. Clearly $\widetilde{D}_+$ is a homogeneous prime ideal of $\widetilde{D}$, hence $\dim(\widetilde{D}) \ge \dim(\widetilde{D}/\widetilde{D}_+) + \mathrm{ht}(\widetilde{D}_+)$. We have $\widetilde{D}/\widetilde{D}_+ \cong R/\mathfrak{p}$ whence $\dim(\widetilde{D}/\widetilde{D}_+) = \dim(R/\mathfrak{p}) = n - h$. It follows that it is enough to show that $\mathrm{ht}(\widetilde{D}_+) \ge h$.

Let $S \subseteq \widetilde{D}_0$ be consisted of non-zero elements of $\widetilde{D}_0$. Since $\widetilde{D}_0 \cong R/\mathfrak{p}$ and $\mathfrak{p}$ is a prime ideal in R, S is a multiplicatively closed set. Let K stand for the fraction field of $R/\mathfrak{p}$ and let $S^{-1}\widetilde{D} = S^{-1}\widetilde{D}_0 \oplus S^{-1}\widetilde{D}_1 \oplus \cdots$ be the ring obtained from $\widetilde{D}$ by inverting every element of S. Clearly $\mathrm{ht}(\widetilde{D}_+) \ge \mathrm{ht}(S^{-1}\widetilde{D}_+)$ whence it is enough to show that

$$\mathrm{ht}(S^{-1}\widetilde{D}_+) \ge h.$$

Clearly

$$S^{-1}\widetilde{D}/S^{-1}\widetilde{D}_+ \cong S^{-1}(\widetilde{D}/\widetilde{D}_+) \cong S^{-1}(R/\mathfrak{p}) = K$$

therefore $S^{-1}\widetilde{D}_+$ is a maximal ideal of $S^{-1}\widetilde{D}$ whence $\dim(S^{-1}\widetilde{D}) = \mathrm{ht}(S^{-1}\widetilde{D}_+)$. By 1.2.15 the function $\widetilde{p(t)} = \mathrm{Vdim}_K(S^{-1}\widetilde{D}_0 \oplus S^{-1}\widetilde{D}_1 \oplus \cdots \oplus S^{-1}\widetilde{D}_t)$ is, for sufficiently large t, a polynomial in t, which we denote $\widetilde{p(t)}$, and $\deg \widetilde{p(t)} = \dim(S^{-1}\widetilde{D}) = \mathrm{ht}(S^{-1}\widetilde{D_+})$. It is enough to prove that

$$\deg \widetilde{p(t)} \ge h.$$

Let $T = k[[x_{h+1},\dots,x_n]]$, and let $\mathscr{K}$ be the field of fractions of T. By 3.1.2, K is finite extension of $\mathscr{K}$. Let d be the degree of the extension. Since $\mathrm{Vdim}_{\mathscr{K}}(L) = d(\mathrm{Vdim}_K(L))$ for every $K-$vector space L, we conclude that

$$q(t) = \mathrm{Vdim}_{\mathscr{K}}(S^{-1}\widetilde{D_0} \oplus S^{-1}\widetilde{D_1} \oplus \cdots \oplus S^{-1}\widetilde{D_t}) = dp(t)$$

Hence $q(t)$, for sufficiently high t, is a polynomial in t of the same degree as $\widetilde{p(t)}$. We denote this polynomial $\widetilde{q(t)}$. It is enough to show that

$$\deg \widetilde{q(t)} \geq h.$$

Since K is finite field extension of $\mathscr{K}$, $S^{-1}L = \mathscr{K} \otimes_T L$ for every $(\widetilde{D_0} = R/\mathfrak{p})-$module L. In particular, $S^{-1}\widetilde{D_0} \oplus S^{-1}\widetilde{D_1} \oplus \cdots \oplus S^{-1}\widetilde{D_t} = (\mathscr{K} \otimes_T \widetilde{D_0}) \oplus (\mathscr{K} \otimes_T \widetilde{D_1}) \oplus \cdots \oplus (\mathscr{K} \otimes_T \widetilde{D_t})$. But $\widetilde{D}_i \cong \Sigma_i z/\Sigma_{i-1}z$, hence

$$\begin{aligned}\widetilde{q(t)} &= \mathrm{Vdim}_{\mathscr{K}}((\mathscr{K} \otimes_T \widetilde{D_0}) \oplus (\mathscr{K} \otimes_T \widetilde{D_1}) \oplus \cdots \oplus (\mathscr{K} \otimes_T \widetilde{D_t}))\\ &= \mathrm{Vdim}_{\mathscr{K}}(\mathscr{K} \otimes_T (\widetilde{D_0} \oplus \widetilde{D_1} \oplus \cdots \oplus \widetilde{D_t}))\\ &= \mathrm{Vdim}_{\mathscr{K}}(\mathscr{K} \otimes_T (\Sigma_t z))\end{aligned}$$

The last equality and the next lemma use the crucial fact that $\Sigma_t z$ and N are $T-$modules, hence $\mathscr{K} \otimes_T (\Sigma_t z)$ and $\mathscr{K} \otimes_T N$ exist (in contrast, $\Sigma_t z$ and N are not $(\widetilde{D_0} = R/\mathfrak{p})-$modules).

Lemma 3.2.1. *The set $\{d_1^{t_1} \cdots d_h^{t_h} z\} \subseteq \mathscr{K} \otimes_T N$ as $t_1,\dots,t_h$ range over all non-negative integers is linearly independent over $\mathscr{K}$ (by slight abuse of notation we identify the elements $d_1^{t_1} \cdots d_h^{t_h} z$ of N with their images in $\mathscr{K} \otimes_T N$ under localization map $N \to \mathscr{K} \otimes_T N$ that sends every $n \in N$ to $1 \otimes n$).*

Proof. Since K is a finite extension of $\mathscr{K}$, let f_i, for very i with $1 \leq i \leq h$, be the monic monomial polynomial of $\bar{x}_i$ over $\mathscr{K}$, where $\overline{x_i}$ is the image of x_i in K. Clearly, $f_i(x_i) \in \mathfrak{p}$ and therefore $f_i(x_i)z = 0$ while $f_i'(x_i)$, where f_i' is the derivative of f_i, in non-zero in K and therefore $f_i'(x_i)z \neq 0$. We claim that if $s > t$ then

$$f_i(x_i)^s d_i^t z = 0 \tag{3.1}$$

If $t = 0$ (hence $s \geq 1$), there is nothing to prove since $f_i(x_i)z = 0$. Clearly

$$f_i(x_i)^s d_i = d_i f_i(x_i)^s - s f_i'(x_i) f_i(x_i)^{s-1}$$

therefore for $t > 0$ we have

$$f_i(x_i)^s d_i^t z = d_i f_i(x_i)^s d_i^{t-1} z - s f_i'(x_i) f_i(x_i)^{s-1} d_i^{t-1} z = 0, \tag{3.2}$$

where both summands in the middle vanish by induction on t. This proves the claim.

Equalities (3.1) and (3.2) imply by induction on t that

$$f_i(x_i)^t d_i^t z = (-1)^t t! f_i'(x_i)^t z \neq 0 \tag{3.3}$$

Now let

$$\gamma = \sum_{t_1,\ldots,t_h} c_{t_1,\ldots,t_h} d_1^{t_1} \cdots d_h^{t_h} z,$$

Where $c_{t_1,\ldots,t_h} \in \mathscr{K}$ be a linear combination of finitely many elements of the set $\{d_1^{t_1} \cdots d_h^{t_h} z\}$. Let $\{\tau_1, \ldots, \tau_h\}$ be an index of highest total degree $\tau_1 + \cdots + \tau_h$. Every other $c_{t_1,\ldots,t_h} d_1^{t_1} \cdots d_h^{t_h} z$ in this linear combination has some t_j with $t_j < \tau_j$, hence $f_j(x_j) c_{i_1,\ldots,i_n} d_1^{i_1} \cdots d_n^{i_n} z = 0$ and

$$\begin{aligned} f_1(x_1)^{\tau_1} \cdots f_h(x_h)^{\tau_h} \gamma &= c_{\tau_1,\ldots,\tau_h} f_1(x_1)^{\tau_1} \cdots f_h(x_h)^{\tau_h} d_1^{\tau_1} \cdots d_h^{\tau_h} z \\ &= (-1)^{\tau_1+\cdots+\tau_h} t_1! \cdots t_h! c_{\tau_1,\ldots,\tau_h} f_1'(x_1)^{\tau_1} \cdots f_h'(x_h)^{\tau_h} z \neq 0 \end{aligned}$$

where we use (3.3) and the fact that $f_i(x_i)^{\tau_i}$ and $f_i'(x_i)^{\tau_i}$ commute with every $d_j^{\tau_j}$ with $j \neq i$. Therefore $\gamma \neq 0$. □

The number of elements $\{d_1^{t_1} \cdots d_h^{t_h} z\}$, as $d_1 + \cdots + d_h \leq t$, is the number of monomials in h variables of total degree at most t, which equals $\binom{t+h}{h}$. Since these elements are in $\Sigma_t z$ and are linearly independent, $\widetilde{q(t)} \geq \binom{t+h}{h}$ for sufficiently high t. But $\widetilde{q(t)}$ is a polynomial in t and $\binom{t+h}{h}$ is a polynomial in t of degree h. Hence the degree of $\widetilde{q(t)}$ is at least h. This completes the proof of 2.4.5.

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