The Möbius Function of the Rook Monoid

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Summary

Let R_n be the rook monoid, a partially ordered set (poset) with the Bruhat order. For a given element $a \in R_n$, we observe the properties of the set $I(a) = \{b \in R_n \,|\, \mu(a,b) \neq 0\}$ where μ is the Möbius function. We then show that $I(a)$ is a closed interval of R_n that is isomorphic to a closed interval of a symmetric group, S_m for some m, with the Bruhat order. This property characterizes $I(a)$ and its elements, providing a way to check whether an element $b \in R_n$ satisfies $\mu(a,b) \neq 0$.

Rook Monoid, Deodhar Order, and Bruhat Order

We first introduce the Deodhar order on $\mathbb{R}^n$, the set of all n-tuples of real numbers. This order turns $\mathbb{R}^n$ into a partially ordered set. Note that for $x \in \mathbb{R}^n$, we put subscripts on the components of x according to their relative positions in order to distinguish them from one another. For instance, instead of writing $x = (1, 0, 2.8, 1, 0) \in \mathbb{R}^5$, we write $x = (1_1, 0_1, 2.8_1, 1_2, 0_2)$ to indicate that 1_1 is the first component of x that is equal to 1, 1_2 is the second component of x that is equal to 1, and so on. We can omit putting subscript on the component that does not repeat to write $x = (1_1, 0_1, 2.8, 1_2, 0_2)$. As a component of x, 1_1 is different from 1_2, although their values are the same (equal to 1).

Definition 1. Let $a = (a_1, \dots, a_n) \in \mathbb{R}^n$. For $i \in \{1, \dots, n\}$, we write $[a]_i = a_i$, the $i-$component of a.

Definition 2. Let $a = (a_1, \dots, a_n) \in \mathbb{R}^n$. We write $\mathrm{Comp}(a) = \{a_1, a_2, \dots, a_n\}$.

Definition 3. Let $k \in \{1, \dots, n\}$ and $a = (a_1, \dots, a_n) \in \mathbb{R}^n$. The $k-$*truncation* of a is $a(k) = (a_1, \dots, a_k) \in \mathbb{R}^k$.

Definition 4. Let $a = (a_1, \dots, a_n) \in \mathbb{R}^n$. We write $\overline{a}$ to denote the rearrangement of the components $a_1, \dots, a_n$ of a in a non-increasing fashion.

Definition 5 (Deodhar Order)**.** Let a and b be elements of $\mathbb{R}^n$. We say that a is less than or equal to b if and only if for each $k \in \{1, 2, \dots, n\}$

$$\left[\overline{a(k)}\right]_i \leq \left[\overline{b(k)}\right]_i$$

for all $i = 1, 2, \dots, k$. We write $a \leq b$ to denote that a is less than or equal to b.

The rook monoid R_n is a subset of $\mathbb{R}^n$ that satisfies

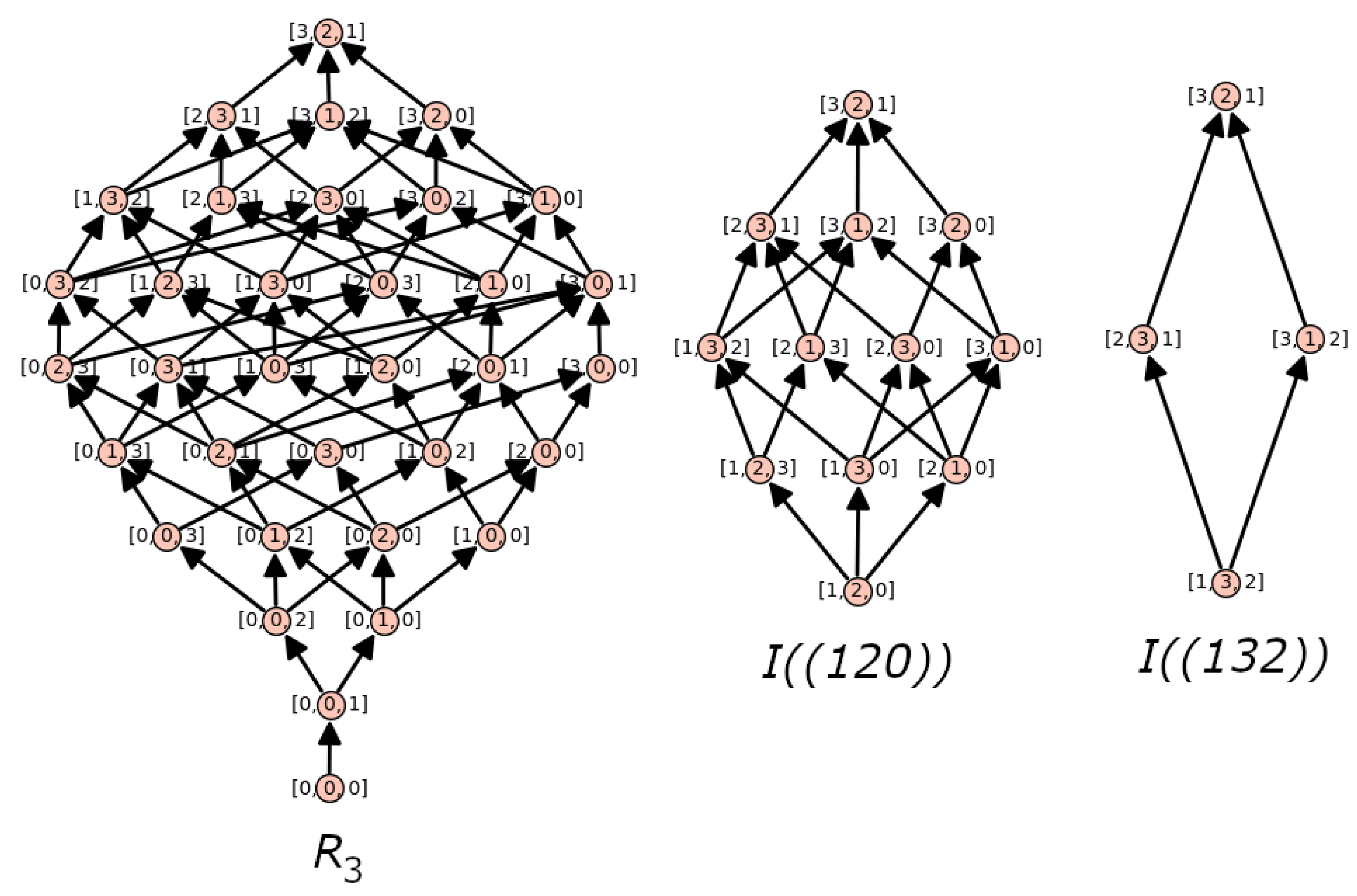
$$R_n = \left\{ \begin{array}{l} a \in \mathbb{R}^n \,|\, [a]_i \in \{0, 1, \dots, n\} \text{ for all } i = 1, \dots, n \\ \text{and the positive components of } a \text{ are not repeated} \end{array} \right\}.$$

Each element of the rook monoid R_n corresponds to a non-attacking rook placement in an $n \times n$ board.

The rook monoid is a partially order set under the Bruhat order. Because the general description of the Bruhat order is abstract, we use the following result from [1] to describe it instead.

Lemma 6. *The Deodhar order is the same as the Bruhat order on R_n.*

Remark 7. The symmetric group with the Bruhat order S_n is a subposet of R_n and so is a subposet of $\mathbb{R}^n$. Consequently, the Deodhar order is also equivalent to the Bruhat order in S_n.



The Möbius Function

For a poset P, the Möbius function μ is the integer valued function defined on $P \times P$ by

$$\mu(x,y) \begin{cases} 1 & \text{if } x = y \\ -\sum_{x \leq z < y} \mu(x,z) & \text{if } x < y \\ 0 & \text{otherwise} \end{cases}.$$

For the rook monoid R_n, there is a formula for computing its Möbius function. The following two results from [2] state what the possible values of the Möbius functions on R_n and S_n are.

Lemma 8. *Let $x, y \in R_n$ such that $x \leq y$. Then $\mu(x,y) \in \{-1, 0, 1\}$.*

Lemma 9. *Let $x, y \in S_n$ such that $x \leq y$. Then $\mu(x,y) \in \{-1, 1\}$.*

Main Result

Definition 10. Let $a \in R_n$. The element $b \in R_n$ is said to be a *unit increment* of a, denoted by $a \lessdot_1 b$, if and only if there exists an index i such that $[a]_i + 1 = [b]_i$ and $a \lessdot^1 b$.

Definition 11. Let $a \in R_n$. We define $\hat{a}$ to be an element of R_n that is obtained from a by rearranging the nonzero components of a in a non-increasing fashion.

Definition 12. Let $a \in R_n$. We define $\langle a \rangle$ to be an element of R_n that is obtained from a by moving the first zero component of a to its last component, if a contains a zero component. If a does not contain any zero component, then we let $\langle a \rangle = a$.

Definition 13. Let $a \in R_n$. We define $\lceil a \rceil$ to be an element of R_n that satisfies the chain of unit increments $a = c_0 \lessdot_1 c_1 \lessdot_1 c_2 \lessdot_1 \cdots \lessdot_1 c_k = \lceil a \rceil$ where c_1 is the unit increment of c_0 by its greatest incrementable component, for all $2 \leq i \leq k$, c_i is the unit increment of c_{i-1} by its greatest incrementable component that is less than the incremented component of c_{i-2}, and $c_k = \lceil a \rceil$ is the unit increment of c_{k-1} by its least incrementable component.

Definition 14. Let a be an element of R_n. We define $f(a) = \lceil \langle \hat{a} \rangle \rceil$.

Definition 15. Let $a \in \mathbb{R}^n$, $x \in \mathbb{R}$, and $k \in \mathbb{Z}_+$. We define

$$a[x_k] = \begin{cases} i, & \text{if the } [a]_i \text{ is the } k^{\text{th}} \text{component of } a \text{ that is equal to } x \\ 0, & \text{otherwise} \end{cases}.$$

Definition 16. Let $a \in R_n$. We write $\mathcal{R}(a) = \mathrm{Comp}(a) \cup \mathrm{Comp}(f(a))$.

Definition 17. Let $a \in R_n$. For $m = |\mathcal{R}(a)|$, let $c_1, c_2, \dots, c_m$ be all of the elements of $\mathcal{R}(a)$ that are written in non-decreasing fashion (we arrange 0_p before 0_q if $p < q$). We define the *canonical assignment* of the elements in $\mathcal{R}(a)$ to be the bijective map

$$\sigma_a = \begin{pmatrix} c_1 & c_2 & \cdots & c_m \\ 1 & 2 & \cdots & m \end{pmatrix}.$$

Definition 18. Let $a = (a_1, \dots, a_n) \in R_n$ and σ_a be its canonical assignment. With respect to σ_a, we define the *congruence* of a to be the element

$$a_{\mathrm{co}} = (\sigma_a(a_1), \sigma_a(a_2), \dots, \sigma_a(a_n), t_{n+1}, t_{n+2}, \dots, t_m)$$

of the symmetric group of degree $m = |\mathcal{R}(a)|$, S_m, where $t_j \in S_m / \{\sigma(a_1), \sigma(a_2), \dots, \sigma(a_n)\}$ and $t_j \leq t_k$ for all $n+1 \leq j \leq k \leq m$.

The congruence of $f(a) = (b_1, b_2, \dots, b_n)$ with respect to the canonical assignment σ_a is defined in the same way.

Theorem 19. *Let a, b be elements of R_n such that $a \leq b$. Suppose that the number of zero components of a is $p > 1$. Then $b \in [a, f(x)]$ if and only if all of the following two conditions hold.*

1. $\left[\overline{b}\right]_i \leq [\overline{a}]_i + 1$ for all $i = 1, \dots, n$, and

2. $b[0_j] \leq a[0_{j+1}] - 1$ for all $j = 1, \dots, p-1$.

Theorem 20. *Let $a \in R_n$ and its congruence a_{co} be an element of the symmetric group S_m with Bruhat order. Then $I(a)$ is isomorphic to the interval $[a, f(a)] \subseteq \mathbb{R}_n$ and is also isomorphic to the interval $[a_{\mathrm{co}}, f(a)_{\mathrm{co}}] \subseteq S_m$. That is, $I(a) \cong [a, f(a)] \cong [a_{\mathrm{co}}, f(a)_{\mathrm{co}}]$.*

References

[1] M.B. Can, L.E. Renner. The Bruhat-Chevalley ordering on the rook monoid. *Turkish Journal of Math*, 36(4):499–519, 2012.

[2] J.R. Stembridge. A Short Derivation of the Möbius Function for the Bruhat Order. *Journal of Algebraic Combinatorics*, 25(2):141–148, 2007. https://doi.org/10.1007/s10801-006-0027-2.