On the Local Theory of Certain Global Zeta Integrals and Related Problems

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Dedication

To those who held me up over the years.

Abstract

In this thesis, we are going to develop some local theory for the global integral constructed by D. Ginzburg [Gin] in 1991, which represents the partial L-function of the adjoint representation of GL_3. We are going to show the absolute convergence of local integrals and establish their meromorphic continuations. To accomplish this, we have to use the asymptotic expansion of the Whittaker functions. Such asymptotic expansions are developed in N. Matringe's work [Mat1] in 2011 when the local field F is non-archimedean. When F is archimedean, we will follow the method suggested by D. Soudry [Sou] to obtain the asymptotic expansion of Whittaker functions for an irreducible admissible generic representation π of $\mathrm{GL}_3(F)$. When F is non-archimedean, we can also show that the local integrals form a fractional ideal, hence they admit a g.c.d, that defines the local adjoint L-function for GL_3. When F is archimedean, by applying the asymptotic expansion, we are able to show that the local integrals are continuous bilinear forms on the projective tensor product of the representation spaces. Following the Bruhat Theory, we can obtain the Uniqueness Theorem, which leads to the functional equation of the local archimedean integrals and the definition of local gamma factors $\Gamma(s,\pi,\mathrm{Ad},\psi)$. To compute $\Gamma(s,\pi,\mathrm{Ad},\psi)$ explicitly, we will provide a useful lemma which helps us to exchange the order of Mellin transform and Fourier transform. With the help of such a lemma, we are finally able to compute the local Gamma factors explicitly.

To complete the local theory for Ginzburg's global integral, we have to establish the local functional equation at the non-archimedean places where the local representation π of GL_3 is ramified. The existence of the g.c.d. will be proved in this thesis, yet it still remains to show that the g.c.d. provides the local adjoint L-function for GL_3, according to local Langlands correspondence. At the archimedean places, we have to show that the local integrals must be a holomorphic multiple of the local adjoint L-function. Moreover, we also expect that the local L-function can be written as a finite linear combination of the local integrals $Z(W_v, f_s)$ with W_v, f_s chosen to be K-finite. We will consider these unsolved problems in a future work.

Contents

Chapter 1

Introduction and Main Results

In 1859, B. Riemann studied the distribution of prime numbers via the Riemann zeta function $\zeta(s)$,

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}.$$

The zeta series $\zeta(s)$ converges absolutely when $\mathrm{Re} s > 1$, and it admits an Euler product

$$\zeta(s) = \prod_{\text{prime } p} \frac{1}{1-p^{-s}}.$$

The complete zeta function $\xi(s)$ defined by

$$\xi(s) := \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$$

has an integral representation

$$\xi(s) = \frac{1}{2}\int_0^{+\infty} y^{\frac{s}{2}}(\vartheta(y)-1)d^{\times}y,$$

where $\vartheta(y)$ is the theta series

$$\vartheta(y) = \sum_{n=-\infty}^{+\infty} e^{-\pi n^2 y}.$$

We can show that $\xi(s)$ (hence $\zeta(s)$) admits a meromorphic continuation to the whole complex plane, and satisfies a functional equation

$$\xi(s) = \xi(1-s).$$

In 1950, J. Tate recast such a theory by considering the Mellin transform of a Bruhat-Schwartz function in the adelic setting. Tate's thesis is now regarded as a part of the theory of the automorphic forms of GL_1. The complete Riemman zeta function $\xi(s)$ can be re-interpreted as the complete standard L-function for GL_1.

In general, we can consider any linear reductive group G defined over a number field F. Let ${}^L\mathrm{G}^0$ be the connected component of its L-group, i.e the complex reductive group with the dual root datum. The L-group is a semidirect product of ${}^L\mathrm{G}^0$ and Galois group Γ_F:

$${}^L\mathrm{G} = {}^L\mathrm{G}^0 \rtimes \Gamma_F.$$

Let (π, V_π) be an irreducible automorphic cuspidal representation of $\mathrm{G}(\mathbb{A})$, where $\mathbb{A}$ is the ring of adeles of F. Then by the Tensor product Theorem (see [F]), we can write π as a restricted tensor product

$$\pi = \otimes_v' \pi_v,$$

where each π_v is a local representation of $\mathrm{G}(F_v)$. Let r be a finite dimensional representation of ${}^L\mathrm{G}$. Then we can attach each local representation π_v a local L-function $L(s, \pi_v, r)$ via a conjectural local Langlands correspondence. The local Langlands correspondence is already known for all archimedean places (due to the work of R. Langlands [L]) and for unramified representations at non-archimedean places via Satake isomorphisms (see [Sha, Chapter 2]). In the special case of $\mathrm{G} = \mathrm{GL}_n$, the local Langlands Correspondence is established for all places (see [He], [H-T], [Sch]). We can define the complete L-function

$$L(s, \pi, r) := \prod_v L(s, \pi_v, r).$$

It is a theorem of Langlands that $L(s, \pi, r)$ converges on some right half plane. Langlands also conjectured that

Conjecture 1.0.1. *The L-function $L(s, \pi, r)$ has a meromorphic continuation to the* *whole complex plane which has only finitely many poles. It also satisfies a functional equation relating s to $1-s$.*

Since Riemann's time, studying the L-function via integral representations has provided fruitful results. A common technique to construct integral representations of L-functions is the Rankin-Selberg method, which was first developed by R. Rankin and A. Selberg independently around 1940. Using the modern adelic language, the integral constructed by Rankin and Selberg provides the tensor product L-function for $\mathrm{GL}_2 \times \mathrm{GL}_2$. To study L-function via Rankin-Selberg method, we will follow the following steps (see [Ge-Sh, Section 1]):

1. Find suitable global integrals: we expect that the global integrals admit an Euler product, and the local integrals should be related to the local L-functions via the local Langlands correspondence. Unfortunately, there is no general theory to establish an global integral for any data (π, G) and r. We also expect that the global integrals satisfy a functional equation. If an Eisenstein series is involved in the construction of the global integrals, then the global functional equation is usually inherited from the functional equation of the Eisenstein series.

2. Prove that the local integrals have a meromorphic continuation, and satisfy a local functional equation. Check that the local gamma factors are exactly the ones predicted by Langlands.

3. Prove that the local integrals have a g.c.d., which means the poles of the local integrals are controlled by the local L-functions. Moreover, for unramified non-archimedean places v, there exists finitely many local data such that the linear combination of the local integrals associated with these data obtains the local L-function predicted by Langlands. For archimedean local places, things are more complicated: for technical reasons, sometimes we solve the g.c.d. problem in the Casselman-Wallach completion of the local representation first, and then we try to prove that such g.c.d. can be obtained by vectors in the $(\mathfrak{g}, K)-$module.

4. Patch up all local results and prove the functional equation of the global L-function.

In this thesis, we specialize to the case of GL_3 and aim to develop the basic local theory of the global zeta integral constructed by D. Ginzburg [Gin] in 1991 that represents the adjoint L-function of GL_3. Let us first review D. Ginzburg's work on this topic. Let F be a number field, and $\mathbb{A}$ be its ring of adeles. Ginzburg constructed a global zeta integral via the embedding of SL_3 into the split simply connected group G_2. From now on, we will identify SL_3 with such a subgroup of G_2. The group G_2 only has two simple roots α (the short simple root) and β (the long simple root). Then we can see that

1. there are three positive short roots: α, $\alpha+\beta$ and $2\alpha+\beta$;

2. there are three positive long roots: β, $3\alpha+\beta$ and $3\alpha+2\beta$.

All the six long roots $\pm\beta, \pm(3\alpha+\beta), \pm(3\alpha+2\beta)$ generate a subgroup of G_2 isomorphic to SL_3. For each simple root $\gamma \in \{\alpha, \beta\}$, there is a homomorphism $\phi_\gamma : \mathrm{SL}_2 \to \mathrm{G}_2$ such that

$$\phi_\gamma(\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}) = x_{-\gamma}(t), \quad \phi_\gamma(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}) = x_\gamma(t).$$

Set

$$w_\gamma = \phi_\gamma(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}).$$

Then it is easy to check that

$$w_\gamma = x_\gamma(1)x_{-\gamma}(-1)x_\gamma(1),$$

and w_γ is a representative for the simple reflection relative to simple root γ. When no confusion arises, we will not distinguish an element in the Weyl group from its representative.

Let $\mathfrak{g}_2$ be the Lie algebra of G_2. Then for any root γ, let $\mathfrak{g}_{2,\gamma}$ be the root space of $\mathfrak{g}_2$ corresponding to γ. Denote by $x_\gamma(t)$ the one parameter unipotent subgroup of G_2 with Lie algebra $\mathfrak{g}_{2,\gamma}$. Following [SGA3, Exposé XXIII, Section 3.4], for any root γ, since

$\dim \mathfrak{g}_{2,\gamma} = 1$, we can choose generators $X_\gamma \in \mathfrak{g}_{2,\gamma}$ in the following way:

$$\begin{aligned} &X_\alpha = dx_\alpha(1), X_{-\alpha} = dx_{-\alpha}(1), X_\beta = dx_\beta(1), X_{-\beta} = dx_{-\beta}(1), \\ &X_{\alpha+\beta} = \mathrm{Ad}(w_\beta)(X_\alpha), X_{2\alpha+\beta} = \mathrm{Ad}(w_\alpha)(X_{\alpha+\beta}), \\ &X_{3\alpha+\beta} = -\mathrm{Ad}(w_\alpha)(X_\beta), X_{3\alpha+2\beta} = \mathrm{Ad}(w_\beta)(X_{3\alpha+\beta}). \end{aligned} \tag{1.1}$$

For any negative root γ, we can construct generators $X_\gamma \in \mathfrak{g}_{2,\gamma}$ from $X_{-\alpha}, X_{-\beta}$ and simple reflections using the same recipe. Finally, for any root γ, the one dimensional subgroup $x_\gamma(t)$ is parameterized such that $X_\gamma = dx_\gamma(1)$. Then the following commutator relations hold:

Lemma 1.0.2. *[SGA3, Exposé XXIII, Section 3.4] We adopt the following convention of commutator $(x, y) = x^{-1}y^{-1}xy$, and have*

$$\begin{aligned} (x_\beta(s), x_\alpha(t)) &= x_{\alpha+\beta}(st)x_{2\alpha+\beta}(st^2)x_{3\alpha+\beta}(st^3)x_{3\alpha+2\beta}(s^2t^3), \\ (x_{\alpha+\beta}(s), x_\alpha(t)) &= x_{2\alpha+\beta}(2st)x_{3\alpha+\beta}(3st^2)x_{3\alpha+2\beta}(3s^2t), \\ (x_{2\alpha+\beta}(s), x_\alpha(t)) &= x_{3\alpha+\beta}(3st), \\ (x_{3\alpha+\beta}(s), x_\beta(t)) &= x_{3\alpha+2\beta}(-st), \\ (x_{2\alpha+\beta}(s), x_{\alpha+\beta}(t)) &= x_{3\alpha+2\beta}(3st). \end{aligned} \tag{1.2}$$

For any other pair (γ, δ) of positive roots, $(x_\gamma(s), x_\delta(t)) = 1$.

Also, we have

Lemma 1.0.3. *[SGA3, Exposé XXIII, Section 3.4]*

$$\begin{aligned} &\mathrm{Ad}(w_\alpha)(X_{2\alpha+\beta}) = -X_{\alpha+\beta}, \mathrm{Ad}(w_\alpha)(X_{3\alpha+\beta}) = X_\beta, \mathrm{Ad}(w_\alpha)(X_{3\alpha+2\beta}) = X_{3\alpha+2\beta}, \\ &\mathrm{Ad}(w_\beta)(X_{\alpha+\beta}) = -X_\alpha, \mathrm{Ad}(w_\beta)(X_{2\alpha+\beta}) = X_{2\alpha+\beta}, \mathrm{Ad}(w_\beta)(X_{3\alpha+2\beta}) = -X_{3\alpha+\beta}. \end{aligned} \tag{1.3}$$

Let P be the maximal parabolic subgroup of G_2 with Levi decomposition is $P = MU$ where U is generated by the following one parameter subgroups

$$x_\beta(t), x_{\alpha+\beta}(t), x_{2\alpha+\beta}(t), x_{3\alpha+\beta}(t), x_{3\alpha+2\beta}(t).$$

Now we quote two results in [Gin].

Lemma 1.0.4. *1.* $|P\backslash \mathrm{G}_2/\mathrm{SL}_3| = 2$*, with representatives* e *and* $\gamma = x_{-(\alpha+\beta)}(-1)w_\beta$*.*

2. Denote by N_2 *the subgroup of* SL_3 *consisting of matrices of the form*

$$\begin{pmatrix} 1 & x & z \\ & 1 & -x \\ & & 1 \end{pmatrix}.$$

Then the stabilizer of γ

$$\mathrm{SL}_3^\gamma := \mathrm{SL}_3 \cap \gamma^{-1} P \gamma = \left\{ n_2 \begin{pmatrix} a & & \\ & 1 & \\ & & a^{-1} \end{pmatrix} \middle| n_2 \in N_2 \right\}.$$

The first statement of Lemma 1.0.4 is exactly the Lemma in [Gin, Section 1]. The second statement can be found at the beginning of the proof of [Gin, Theorem 1].

Remark 1.0.5. *The choices of root vectors here are slightly different from those of [Gin]. For example, [Gin, Formula 1.1] implies that* $\mathrm{Ad}(w_\beta)(X_{\alpha+\beta}) = X_\alpha$ *under Ginzburg's choice of* $X_{\alpha+\beta}$*, but this contradicts to Lemma 1.0.3. We note that our choice and Ginzburg's choice only differ by a sign.*

We also write N' for the one dimensional unipotent subgroup x_β, and clearly $N = N'N_2$. Let δ_P be the modular character of $P(\mathbb{A})$, which is explicitly given by

$$\delta_P(\begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix}) = |t_1^3 t_3^{-3}|_{\mathbb{A}}.$$

Ginzburg considered the normalized parabolically induced representation

$$\rho_s := \mathrm{Ind}_{P(\mathbb{A})}^{\mathrm{G}_2(\mathbb{A})} \delta_P^{s-\frac{1}{2}}$$

on the space V_{ρ_s}. Clearly, every function $f_s(g) \in V_{\rho_s}$ satisfies

$$f_s(nmg) = \delta_P^s(m) f_s(g).$$

Lemma 1.0.6. *For any $n_2 \in N_2(\mathbb{A})$ and $f_s \in V_{\rho_s}$, $f_s(\gamma n_2 g) = f_s(\gamma g)$.*

Proof. This is obvious from Lemma 1.0.4. □

Define the Eisenstein series

$$E(g, f_s) := \sum_{\gamma \in P(F)\backslash G_2(F)} f_s(\gamma g).$$

From Langlands' theory of Eisenstein series (for example see [Sha]), the Eisenstein series $E(g, f_s)$ can be extended to a meromorphic function of $s \in \mathbb{C}$. It satisfies a functional equation

$$E(g, f_s) = E(g, M(w)f_s),$$

where $M(w)$ is the intertwining operator relative to the Weyl element $w = w_{3\alpha+2\beta}$. Let (π, V_π) be an irreducible cuspidal automorphic representation of $\mathrm{GL}_3(\mathbb{A})$ whose central character is trivial. Let ψ be a nontrivial unitary character of $F\backslash\mathbb{A}$. Then ψ defines a character ψ_N of $N(\mathbb{A})$ as follows

$$\psi_N(\begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix}) = \psi(x + y), \tag{1.4}$$

where N is the maximal unipotent subgroup of SL_3. Then π is generic and has a nonzero Whittaker model $\mathcal{W}(\pi, \psi)$. The global representation π can be written as a restricted tensor product $\otimes'_\nu \pi_\nu$, and each local representation π_ν also admits a non-zero Whittaker model $\mathcal{W}(\pi_\nu, \psi_\nu)$. We take a cusp form $\phi \in V_\pi$. The global zeta integral which Ginzburg constructed is

$$Z(\phi, f_s) := \int_{\mathrm{SL}_3(F)\backslash \mathrm{SL}_3(\mathbb{A})} \phi(g) E(g, f_s) dg,$$

and he proved that

Theorem 1.0.7 ([Gin] Theorem 1)**.** *Except for a finite number of s, where the Eisenstein series has poles, $Z(\phi, f_s)$ converges and satisfies the functional equation*

$$Z(\phi, f_s) = Z(\phi, M(w)f_s). \tag{1.5}$$

Moreover, the global integral admits an Euler product

$$Z(\phi, f_s) = \prod_{\nu} Z_\nu(W_{\phi_\nu}, f_{s,\nu}). \tag{1.6}$$

Here in (1.6), *the local integral* $Z_\nu(W_{\phi_\nu}, f_{s,\nu})$ *is defined as follows:*

$$Z_\nu(W_{\phi_\nu}, f_{s,\nu}) := \int_{N_2(F_\nu)\backslash SL_3(F_\nu)} W_{\phi_\nu}(g) f_{s,\nu}(\gamma g) dg, \tag{1.7}$$

where $\gamma = x_{-(\alpha+\beta)}(-1)w_\beta$ *and* $W_{\phi_\nu} \in \mathcal{W}(\pi_\nu, \psi_\nu)$.

Ginzburg also computes the local integral when ν is a non-archimedean local place where π_ν and ψ_v are unramified:

Theorem 1.0.8 ([Gin] Theorem 2)**.** *Let* ν *be a non-archimedean local place where* π_ν *and* ψ_ν *are unramified. We take a spherical Whittaker function* W_{ϕ_v}*, and a spherical function* $f_{s,\nu} \in V_{\rho_s,\nu}$ *such that*

$$W_{\phi_v}(I) = f_{s,\nu}(I) = 1.$$

Then $Z_\nu(W_{\phi_\nu}, f_{s,\nu})$ *converges absolutely when* $Re(s)$ *is sufficiently large, and*

$$Z_\nu(W_{\phi_\nu}, f_{s,\nu}) = \frac{L(3s-1, \pi_\nu, \mathrm{Ad})}{\zeta_\nu(3s)\zeta_\nu(6s-2)\zeta_\nu(9s-3)} \tag{1.8}$$

in the sense of meromorphic continuation. Here $\zeta_\nu(s) = (1 - q_\nu^{-s})^{-1}$*, and* q_v *is the cardinality of the residue field.*

The objective of this thesis is to develop the local theory for such local integrals defined in (1.7). Since we only focus on local theory, to simplify notation, we will reformulate our setup here. From now on, F is a local field of characteristic zero. We will assume that (π, V_π) is an irreducible generic admissible smooth representation of $\mathrm{GL}_3(F)$ with a trivial central character. When F is archimedean, we further assume that all representations considered in this thesis are Casselman-Wallach representations (smooth admissible Fréchet representations of finite length which satisfies the moderate growth condition). We take a non-zero unitary character ψ of F and define the character ψ_N on $N(F)$ in the exact same way as (1.4). Then (π, V_π) has a non-zero Whittaker model $\mathcal{W}(\pi,\psi)$. The induced representation we consider is $\rho_s = \mathrm{Ind}_{P(F)}^{\mathrm{G}_2(F)}\delta_P^{s-\frac{1}{2}}$. We rewrite the local integral as

$$Z(W_v, f_s) = \int_{N_2(F)\backslash \mathrm{SL}_3(F)} W_v(g) f_s(\gamma g) dg, \tag{1.9}$$

where $\gamma = x_{-(\alpha+\beta)}(-1)w_\beta$, $W_v \in \mathcal{W}(\pi,\psi)$ and $f_s \in V_{\rho_s}$. The analogue of Lemma 1.0.6 also holds:

Lemma 1.0.9. *For any $n_2 \in N_2(F)$ and $f_s \in V_{\rho_s}$, $f_s(\gamma n_2 g) = f_s(\gamma g)$.*

Hence the integrand in (1.9) is $N_2(F)$-invariant.

We first prove that

Theorem 1.0.10. *There is a sufficiently large s_0 such that when $Re(s) > s_0$, the local integrals $Z(W_v, f_s)$ defined in (1.9) converge absolutely.*

From the definition, the Whittaker functions W_v in (1.9) depend on the nontrivial character ψ of F. We will write $W_v^\psi \in \mathcal{W}(\pi,\psi)$ to emphasize the dependence on the additive character ψ. By Pontryagin Duality, once we fix one nontrivial character ψ of F, all the other nontrivial characters are of the form

$$\psi_c(x) = \psi(cx)$$

for some $c \in F^\times$. We will show in Section 4.2 that

$$Z(W_v^{\psi_c}, f_s) = |c|_F^{-3s+3} Z(W_v^\psi, f_s). \tag{1.10}$$

Therefore, once we prove the meromorphic continuation of $Z(W_v^\psi, f_s)$ for one fixed nontrivial character ψ, same assertions also hold for $Z(W_v^{\psi_c}, f_s)$. Thus it suffices to fix one nontrivial character ψ. In particular, when F is archimedean, we fix the character ψ to be

$$\psi(x) = \begin{cases} e^{2\pi i x} & \text{if } F = \mathbb{R}; \\ e^{2\pi i(x+\overline{x})} & \text{if } F = \mathbb{C}. \end{cases} \tag{1.11}$$

The next step is to establish a meromorphic continuation of the local integrals. To achieve this, we first need to establish an asymptotic expansion of the Whittaker function

W_v along the torus of $\mathrm{GL}_3(F)$. In the non-archimedean case, this is done by N. Matringe (see Proposition 3.2.2). Then meromorphic continuation of the local integrals follow directly. When F is archimedean, there are two asymptotic expansions of this nature given in this thesis. The first one is a coarse asymptotic expansion due to H. Jacquet and J. Shalika (see [J-S]) in Chapter 3. However, this coarse asymptotic expansion can not help us to detect the dependence of the each summand in the expansion on the choice of vector v. A refined asymptotic expansion is suggested by the work of Soudry (see [Sou]). Moreover, if the representation π is a principal series

$$\pi = \pi_u = \mathrm{Ind}_B^{\mathrm{GL}_3(F)} |\quad|_F^{u_1} \chi_1 \otimes |\quad|_F^{u_2} \chi_2 \otimes |\quad|_F^{u_3} \chi_3,$$

we write $W_{v,u}$ for the analytic continuation of the Jacquet integral (see Chapter 10). We can also keep track on the dependence of the asymptotic expansion of $W_{v,u}$ on the complex parameters $u = (u_1, u_2, u_3)$. Soudry proved these results for orthogonal groups. He also claimed that his approach works for all split real reductive groups. Thus, in Chapter 5, we follow the method in [Sou] closely and carry out the detailed proof for $\mathrm{GL}_3(F)$.

In Chapter 6, we will use the asymptotic expansions of the Whittaker functions W_v along the torus to establish the meromorphic continuation of the local integrals. For any two Fréchet spaces V and W, denote by $V \hat{\otimes} W$ their projective tensor product. Now we state the theorem.

Theorem 1.0.11. *Let F be a local field of characteristic zero. The local integral $Z(W_v, f_s)$ defined in* (1.9) *extends to a meromorphic function of s on the whole complex plane. Moreover, if $\pi = \pi_u$ is a principal series, then $Z(W_{v,u}, f_s)$ is also meromorphic in u. If we further assume that F is archimedean, then under the projective tensor product topology, $Z(W_v, f_s)$ is a continuous bilinear form on $V_\pi \hat{\otimes} V_{\rho_s}$.*

Moreover, when F is non-archimedean, we can say more. In fact, we can show that the local integrals form a fractional ideal in $\mathbb{C}(q^{-3s})$, thus it admits a g.c.d.,with which one defines the local adjoint L-function of GL_3 through the Rankin-Selberg method. At the moment, we can only define this g.c.d. We expect that this g.c.d is essentially the local L-function for the adjoint representation of $\mathrm{GL}_3(F)$, in the sense of Langlands. We may discuss this in a future work.

To establish the functional equation when F is archimedean, we will also consider a local integral $\widetilde{Z}(W_v, f_s)$ obtained by applying the intertwining operator $M(w_{3\alpha+2\beta})$ to f_s. Here, the intertwining operator is defined by

$$(M(w_{3\alpha+2\beta})f_s)(g) = \int_U f_s(w_{3\alpha+2\beta}^{-1}ug)du$$

when $\mathrm{Re}(s) > \frac{2}{3}$ (see Chapter 9) and has a meromorphic continuation in s. The local integrals $\widetilde{Z}(W_v, f_s)$ on the other side of the local functional equations are

$$\widetilde{Z}(W_v, f_s) = \int_{N_2(F)\backslash \mathrm{SL}_3(F)} W_v(g) \cdot (M(w_{3\alpha+2\beta})f_s)(\gamma \cdot g)dg.$$

$\widetilde{Z}(W_v, f_s)$ also has a meromorphic continuation in s and is continuous on $V_\pi \hat{\otimes} V_{\rho_s}$ for all parameter s. Now we state the Uniqueness Theorem.

Theorem 1.0.12. *When F is archimedean, given any irreducible Casselman-Wallach representation π whose central character is trivial, there is a discrete, at most countable subset S of $\mathbb{C}$ depending on π, such that whenever $s \notin S$, the space of bilinear forms $Z(W_v, f_s)$ satisfying the following equivariant property*

$$Z(\pi(g)W_v, \rho_s(g)f_s) = Z(W_v, f_s), \quad \forall g \in \mathrm{SL}_3(F), \tag{1.12}$$

is at most one dimensional.

We will prove the above theorem in Chapter 7. Now that $Z(W_v, f_s)$ and $\widetilde{Z}(W_v, f_s)$ are both meromorphic functions in s satisfying the same equivariant property (1.12), hence there exists a meromorphic function $\Gamma(s, \pi, \mathrm{Ad}, \psi)$ such that

$$\Gamma(s, \pi, \mathrm{Ad}, \psi)Z(W_v, f_s) = \widetilde{Z}(W_v, f_s). \tag{1.13}$$

We also note that if $\pi = \pi_u$ is a principal series, then $\Gamma(s, \pi_u, \mathrm{Ad}, \psi)$ is also meromorphic in u.

From Chapter 9 to Chapter 11, we will compute $\Gamma(s, \pi, \mathrm{Ad}, \psi)$ explicitly when F is archimedean. In view of (1.10), it suffices to compute $\Gamma(s, \pi, \mathrm{Ad}, \psi)$ explicitly when ψ is the character given in (1.11). Let us first point out the difficulties in computing the Gamma function in general. There are two aspects:

1. Unlike the unramified computation in the non-archimedean places where we can use the Casselman-Shalika formula, we lack of a good understanding of special functions on groups of higher rank. For example, the spherical Whittaker functions on $\mathrm{GL}_2(\mathbb{R})$ is essentially the classical Bessel function. The spherical Whittaker functions for $\mathrm{GL}_n(\mathbb{R})$ has been studied by E. Stade, T. Ishii, D. Bump, S. Friedberg, and J. Hoffstein, etc. (for example see [St]). The explicit formulae for these Whittaker functions do help in some special cases, such as the archimedean integrals for $\mathrm{GL}_n \times \mathrm{GL}_{n-1+l}$ and $\mathrm{SO}_{2n+1} \times \mathrm{GL}_{n+l}$, for $l = -1, 0, 1$ (see [I-St]). In these cases, the local integrals can be reduced to some known integrals of Mellin-Barnes type. But in general, we do not know how to evaluate an arbitrary integral involving these transcendental functions on a group of higher rank.

2. Some known archimedean functional equations can be reduced to computations on GL_2 or GL_1, such as the Rankin-Selberg integrals for the standard $L-$function for $\mathrm{GL}_n \times \mathrm{GL}_m$ (see [Jac]), the triple product L-function for GL_2 (see [I]), the standard $L-$function for $\mathrm{SO}_{2l+1} \times \mathrm{GL}_n$ (see [Sou]). Yet when the local integrals are not easily related to some known examples, things may get complicated.

To get around the above difficulties, we note that by Casselman's Subrepresentation Theorem (see [Wal1, Section 3.8.3]), it suffices to compute $\Gamma(s, \pi, \mathrm{Ad}, \psi)$ explicitly for all principal series π. We assume that the principal series are induced from the lower Borel subgroup, i.e. $\pi = \pi_u := \mathrm{Ind}_{B^-}^{\mathrm{GL}_3(F)} \eta$ with η being the character defined by

$$\eta(\begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix}) := \prod_{i=1}^{3} |t_i|_F^{u_i} \chi_i(t_i), \tag{1.14}$$

where each $u_i \in \mathbb{C}$ and each χ_i is a unitary character of $F^\times$. Since $\Gamma(s, \pi_u, \mathrm{Ad}, \psi)$ is meromorphic in both u and s, it suffices to compute $\Gamma(s, \pi_u, \mathrm{Ad}, \psi)$ for a special domain in $\mathbb{C}^4$.

Let us further assume that $\mathrm{Re}(u_1) < \mathrm{Re}(u_2) < \mathrm{Re}(u_3)$. We take a function $\varphi \in V_{\pi_u}$. Then the Jacquet integral

$$W_\varphi(g) = \int_{N(F)} \varphi(ng)\psi^{-1}(n)dn \tag{1.15}$$

converges absolutely (see [Wal2, 15.4.1]). After replacing the Whittaker function by its Jacquet integral, we realize that the local integrals can be obtained by first taking a Fourier transform of a function $F_s(a,z)$, then taking the Mellin transform (see Chapter 10). Thus intuitively, we wish to interchange the order of the Fourier transform and the Mellin transform. In Section 8.3, we will investigate a condition under which we can switch the order of the Fourier transform and the Mellin transform (see Lemma 8.3.1). In Chapter 10, we will check the four conditions stated in Lemma 8.3.1. Then we can factorize $Z(W_\varphi, f_s)$ into a simple integral $Z_1(W_\varphi, f_s)$ and a gamma factor γ_F in Tate's thesis in the sense of meromorphic continuation. In the following Theorem, T is the split torus contained in the standard Borel subgroup B of $\mathrm{GL}_3(F)$, N^- is the unipotent radical of the lower Borel subgroup, and K is the maximal compact subgroup of $\mathrm{GL}_3(F)$ such that the following Iwasawa decomposition holds.

$$\mathrm{GL}_3(F) = BK.$$

Now we state the Theorem.

Theorem 1.0.13. *Suppose u_1, u_2, u_3 are complex numbers with sufficiently small real parts (in terms of absolute value) such that $Re(u_1) < Re(u_2) < Re(u_3)$, $u_1+u_2+u_3 = 0$, and the domain D defined by*

$$\begin{aligned} &Re(s) > \frac{2}{3}, && Re(3s-2+u_2-u_3) \in (-1,0),\\ &Re(3s-2+u_1-u_2) \in (-1,0), && Re(3s-2+u_1-u_3) \in (-1,0), \end{aligned}$$

is non-empty. Let F be an archimedean local field and $\pi = \pi_u = \mathrm{Ind}_{B^-}^{\mathrm{GL}_3(F)}\eta$ be the principal series with η defined as in (1.14). For every $\varphi \in V_\pi$, the local integral $Z(W_\varphi, f_s)$ converges for every $s \in D$. It is a product of

$$Z_1(W_\varphi, f_s) = \int_{T\times N^-\times K} (\delta_{B^-}^{\frac{1}{2}}\eta)(a)\varphi(k) f_s(\gamma \begin{pmatrix} 1 & 1 & \\ & 1 & \\ & & 1 \end{pmatrix} a\overline{n}k) da d\overline{n} dk$$

and

$$\int_F |z|_F^{3s-2+u_1-u_3}(\chi_1\chi_3^{-1})(z)\psi(z)dz.$$

Moreover, the second factor is equal to $\gamma_F(2-3s-u_1+u_3,\chi_1^{-1}\chi_3,\psi)$*. In particular,* $Z_1(W_v,f_s)$ *also has a meromorphic continuation in* s,u_1,u_2,u_3*.*

Thus it suffices to establish the functional equation for $Z_1(W_v,f_s)$. Integrating over the compact group K will only provide us with a new function

$$\int_K \varphi(k)\rho_s(k)f_s dk$$

in V_{ρ_s}. Hence it is enough to establish the functional equation for $I(f_s)$, where

$$I(f_s)=\int_{T\times N^-}(\delta_{B^-}^{\frac{1}{2}}\eta)(a)f_s(\gamma\begin{pmatrix}1&1&\\&1&\\&&1\end{pmatrix}a\overline{n})dad\overline{n}$$

In fact, we can rewrite $I(f_s)$ and regard it as a Mellin transform of the restriction of f_s on the lower unipotent radical (see Lemma 10.2.1). Finally, the functional equation of $I(f_s)$ can be established by realizing the intertwining operator $M(w_{3\alpha+2\beta})$ as a convolution operator on the unipotent radical of the parabolic subgroup P^- of G_2 opposite to P (see Chapter 9 and Chapter 11). From this, we can establish the functional equation for $I(f_s)$.

For any unitary character χ of $F^\times$, denote by $L_F(s,\chi)$ the local L-functions. We will omit the subscript F when no confusion arises. Also, when χ is trivial, we will also write $L_F(s)$ for $L_F(s,1)$ to simplify notation. When F is archimedean, the exact formulae of $L_F(s,\chi)$ will be listed in the Appendix B. Together with Theorem 1.0.13, we can obtain that

Theorem 1.0.14. *Let* F *be an archimedean local field and* $\pi=\pi_u=\mathrm{Ind}_{B^-}^{\mathrm{GL}_3(F)}\eta$ *be the principal series with* η *defined as in* (1.14)*. We fix the character* ψ *as in* (1.11)*. Then the Gamma function defined in (1.13) is*

$$\Gamma(s,\pi_u,Ad,\psi)=\frac{L(3s-2)}{L(3-3s)}\cdot\frac{L(6s-3)}{L(4-6s)}\cdot\frac{L(9s-5)}{L(6-9s)}\cdot\frac{L(2-3s,\pi_u,Ad)}{L(3s-1,\pi_u,Ad)},$$

where $L(s,\pi_u,Ad)$ *is the Adjoint L-function defined by Local Langlands Correspondence.*

Chapter 2

Preliminaries

2.1 Preliminaries on Analysis

In this Section, we will collect some well known facts on analysis that will be useful in this thesis. We start with some facts on Fréchet spaces. Our main reference is [Tr]. No proofs will be given here.

A topological vector space (TVS) is a vector space over the field of complex numbers $\mathbb{C}$ such that addition and scalar multiplication are continuous. A TVS is called a locally convex space (LCS) if there is a basis of neighbourhoods consisting of convex sets. In practice, we usually use a family of continuous seminorms $\{p_\alpha\}$ to describe the topology of a LCS. More precisely, in an LCS V, there exists a basis of neighbourhoods of zero consisting of barrels (closed absorbing balanced convex sets). For each barrel U in V, we can find a unique seminorm p such that

$$\{x \in V \,|\, p(x) \leq 1\} = U.$$

Conversely, suppose $\mathcal{P} = \{p_\alpha\}$ is a family of seminorms on a vector space V. Then we declare a subset $U \subseteq V$ to be open if for every $x \in U$, we can find finitely many seminorms $p_1, p_2, \cdots, p_n \in \mathcal{P}$ and $\epsilon_1, \epsilon_2, \cdots, \epsilon_n > 0$ such that

$$\bigcap_{k=1}^{n} \{y \in V \,|\, p_k(y - x) < \epsilon_k\} \subseteq U.$$

In this way, the family of seminorms defines a topology on V under which addition and scalar multiplication are continuous. A Fréchet space is a complete metrizable LCS. If V is a Fréchet space, then we can find a countable family of seminorms $\mathcal{P} = \{p_n \mid n = 1, 2, 3, \cdots\}$ defining the topology of V. Moreover, we can define a translation invariant metric $d(x, y)$ as follows:

$$d(x,y) = \sum_{n=1}^{+\infty} 2^{-n} \frac{p_n(x-y)}{1+p_n(x-y)}. \tag{2.1}$$

The topology induced by the translation invariant metric $d(x, y)$ coincides with the original topology defined by the family $\mathcal{P}$ of seminorms. For example, the space of Schwartz functions on an Euclidean space is a Fréchet space.

Later, we will also work with continuous linear functions on the product of two Fréchet spaces. Let U, V and W be three LCS. The canonical bilinear map from $V \times W$ to $V \otimes W$ is denoted by ι. There exists one and only one topology called the projective (inductive resp.) topology on $V \otimes W$, such that for every bilinear map $\phi : V \otimes W \to U$, ϕ is continuous if and only if $\phi \circ \iota$ is continuous (separately continuous resp.). The completion of $V \otimes W$ under projective (inductive resp.) topology, denoted by $V \hat{\otimes} W$ ($V \overline{\otimes} W$ resp.), is called the projective (inductive resp.) tensor product of V and W. Moreover, when V and W are both Fréchet spaces, the projective tensor product and inductive tensor product of V and W coincide. In this case, every separately continuous bilinear function on $V \times W$ must also be continuous.

2.2 Preliminaries on Representations of Real Reductive Groups

In this Section, we will review some notions and facts of representations of real reductive groups. Our main reference is [Wal1] and [Wal2].

Let G be a real reductive group with Lie algebra $\mathfrak{g}$ and K be a compact subgroup of G. We write $\hat{K}$ for the unitary dual of K. A $\mathfrak{g}$-module V that is also a K-module is called a $(\mathfrak{g}, K)$-module if every $v \in V$ is K-finite, and the $\mathfrak{g}$-action and K-action are compatible. A $(\mathfrak{g}, K)$-module V is called admissible if for each $\gamma \in \hat{K}$, the isotypic component $V(\gamma)$ is finite dimensional. A smooth Fréchet representation (π, V) of G is a representation of G on a Fréchet space V where each vector $v \in V$ is smooth, i.e. the map $g \mapsto \pi(g)v$ is smooth. A smooth Fréchet representation (π, V) of G is said to be of moderate growth if for every continuous seminorm p on V, there exists a $d \in \mathbb{R}$ and a continuous seminorm q on V such that

$$p(\pi(g)v) \leq \|g\|^d q(v),$$

where $\| \quad \|$ is a norm defined on G. In particular, when $\mathrm{G} = \mathrm{GL}_n(\mathbb{R})$ or $\mathrm{GL}_n(\mathbb{C})$, we can take $\| \quad \|$ to be the Harish-Chandra norm:

$$\|g\| := \mathrm{Tr}(g \cdot \overline{g^t}) + \mathrm{Tr}(g^{-1} \cdot \overline{(g^{-1})^t}).$$

A fundamental result of Casselman and Wallach confirms that every finitely generated admissible $(\mathfrak{g}, K)$-module V_0 corresponds to, up to isomorphism, a unique smooth Fréchet representation (π, V) of G having moderate growth such that the underlying $(\mathfrak{g}, K)$-module of V is isomorphic to V_0. We call such a (π, V) the Casselman-Wallach completion of V_0. For convenience, we also call such (π, V) a Casselman-Wallach representation.

2.3 Matrix realizations of the Lie algebra $\mathfrak{g}_2$

Let F be an archimedean local field. For explicit computations with functions on $\mathrm{G}_2(F)$, we provide a matrix realization of the Lie algebra $\mathfrak{g}_2(F)$. We set

$$X_\alpha = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad X_\beta = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$X_{-\alpha} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \qquad X_{-\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$w_\alpha = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \quad w_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.2}$$

We can cook up all the other matrices for root vectors X_γ via (1.1) and Lemma 1.0.3. More precisely,

$$
\begin{aligned}
X_{\alpha+\beta} &= \begin{pmatrix} 0&0&1&0&0&0&0\\ 0&0&0&-2&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&-1&0\\ 0&0&0&0&0&0&-1\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix}, \quad X_{-(\alpha+\beta)} = \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0\\ 0&-1&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&-2&0&0&0\\ 0&0&0&0&-1&0&0 \end{pmatrix}\\
X_{2\alpha+\beta} &= \begin{pmatrix} 0&0&0&2&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&-1\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix}, \qquad X_{-(2\alpha+\beta)} = \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&-2&0&0&0 \end{pmatrix}\\
X_{3\alpha+\beta} &= \begin{pmatrix} 0&0&0&0&-1&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix}, \qquad X_{-(3\alpha+\beta)} = \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ -1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0 \end{pmatrix}\\
X_{3\alpha+2\beta} &= \begin{pmatrix} 0&0&0&0&0&-1&0\\ 0&0&0&0&0&0&-1\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix}, \quad X_{-(3\alpha+2\beta)} = \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ -1&0&0&0&0&0&0\\ 0&-1&0&0&0&0&0 \end{pmatrix}
\end{aligned}
\tag{2.3}
$$

The above provides us an embedding of $\mathfrak{g}_2(F)$ into $\mathfrak{gl}_7(F)$ (or more precisely the split orthogonal Lie algebra $\mathfrak{so}_7(F)$).

Remark 2.3.1. *We believe that there is a small sign error in [B]. If we follow [SGA3] closely, the root vectors $X_{3\alpha+\beta}$ and $X_{3\alpha+2\beta}$ should be as above. The sign error most likely comes from the sign in the equation $X_{3\alpha+\beta} = -\mathrm{Ad}(w_\alpha)(X_\beta)$.*

Chapter 3

Convergence of Local Integrals

In this Chapter, we will prove the convergence of local integrals (both in archimedean and non-archimedean cases) and the meromorphic continuation of local integrals in the non-archimedean cases.

3.1 Convergence (The Archimedean Case)

Let us first start with the archimedean case. We have to distinguish two absolute values $|\ \ |$ and $|\ \ |_F$ when $F = \mathbb{C}$. The absolute value with the subscript is defined by $|z|_{\mathbb{C}} = z\overline{z}$, while the absolute value without the subscript is the ordinary one. The absolute value $|\ \ |$ will mainly appear in the Iwasawa decomposition of the elements in the lower unipotent subgroups of $\mathrm{G}_2(F)$. We fix a maximal compact subgroup K_{G_2} such that the following Iwasawa decomposition holds:

$$\mathrm{G}_2(F) = N_{\mathrm{G}_2} A_{\mathrm{G}_2} K_{\mathrm{G}_2}, \tag{3.1}$$

where N_{G_2} is the standard maximal unipotent subgroup of $\mathrm{G}_2(F)$ generated by all positive roots. Then $K_{\mathrm{SL}_3} = \mathrm{SL}_3(F) \bigcap K_{\mathrm{G}_2}(F)$ is a maximal compact subgroup of $\mathrm{SL}_3(F)$. Hence

$$K_{\mathrm{SL}_3} = \begin{cases} \mathrm{SO}_3(\mathbb{R}) & \text{if } F = \mathbb{R}, \\ \mathrm{SU}_3 & \text{if } F = \mathbb{C}. \end{cases} \tag{3.2}$$

Let us prove a simple lemma first.

Lemma 3.1.1. *For any function $f_s \in V_{\rho_s}$,*

$$
\begin{aligned}
&f_s(\gamma \begin{pmatrix} 1 & z & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} t_1t_2 & & \\ & t_2 & \\ & & t_1^{-1}t_2^{-2} \end{pmatrix} g) \\
=&|t_1t_2^3|_F^{3s} \cdot \left| |t_1^{-1}z|^2 + (1+|t_2|^2)^3 \right|_F^{-\frac{3}{2}s} \cdot f_s(k''(-\frac{-t_1^{-1}z}{(1+|t_2|^2)^{\frac{3}{2}}})k'(t_2)w_\beta g)
\end{aligned}
\tag{3.3}
$$

where k' and k'' are smooth functions taking values in K_{G_2}.

Proof. By Lemma 1.0.2, $x_{-\beta}$ and $x_{-(\alpha+\beta)}$ commutes. The following simple computation holds:

$$
\gamma \cdot \begin{pmatrix} 1 & z & \\ & 1 & \\ & & 1 \end{pmatrix} \cdot \begin{pmatrix} t_1t_2 & & \\ & t_2 & \\ & & t_1^{-1}t_2^{-2} \end{pmatrix} = \begin{pmatrix} t_2 & & \\ & t_1t_2 & \\ & & t_1^{-1}t_2^{-2} \end{pmatrix} x_{-\beta}(-t_1^{-1}z)x_{-(\alpha+\beta)}(-t_2)w_\beta .
\tag{3.4}
$$

The Iwasawa decomposition of $\mathrm{SL}_2(F)$ provides us a decomposition of $x_{-(\alpha+\beta)}(-t_2)$ in $\mathrm{G}_2(F)$

$$
x_{-(\alpha+\beta)}(-t_2) = \begin{pmatrix} (1+|t_2|^2)^{-1} & & \\ & (1+|t_2|^2)^{\frac{1}{2}} & \\ & & (1+|t_2|^2)^{\frac{1}{2}} \end{pmatrix} x_{\alpha+\beta}(-\overline{t_2})k'(t_2),
\tag{3.5}
$$

where k' is a smooth function taking values in the maximal compact subgroup K_{G_2}, and $\overline{t_2}$ is the complex conjugate of t_2. Hence

$$
\begin{aligned}
&f_s(\gamma \begin{pmatrix} 1 & z & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} t_1t_2 & & \\ & t_2 & \\ & & t_1^{-1}t_2^{-2} \end{pmatrix} g) \\
=&|t_1t_2^3|_F^{3s} f_s(x_{-\beta}(-t_1^{-1}z)x_{-(\alpha+\beta)}(-t_2)w_\beta g) \\
=&|t_1t_2^3|_F^{3s} f_s(x_{-\beta}(-t_1^{-1}z) \begin{pmatrix} (1+|t_2|^2)^{-1} & & \\ & (1+|t_2|^2)^{\frac{1}{2}} & \\ & & (1+|t_2|^2)^{\frac{1}{2}} \end{pmatrix} x_{\alpha+\beta}(-\overline{t_2})k'(t_2)w_\beta g) \\
=&|t_1t_2^3|_F^{3s} \cdot |1+|t_2|^2|_F^{-\frac{9}{2}s} \cdot f_s(x_{-\beta}(\frac{-t_1^{-1}z}{(1+|t_2|^2)^{\frac{3}{2}}})x_{\alpha+\beta}(-\overline{t_2})k'(t_2)w_\beta g).
\end{aligned}
$$

By Lemma 1.0.2, $x_{-\beta}(-\frac{t_1^{-1}z}{(1+|t_2|^2)^{\frac{3}{2}}})x_{\alpha+\beta}(-\overline{t_2})x_{-\beta}(\frac{t_1^{-1}z}{(1+|t_2|^2)^{\frac{3}{2}}})$ belongs to the maximal unipotent subgroup of $\mathrm{G}_2(F)$. Hence

$$
f_s(x_{-\beta}(-\frac{t_1^{-1}z}{(1+|t_2|^2)^{\frac{3}{2}}})x_{\alpha+\beta}(-\overline{t_2})k'(t_2)w_\beta g) = f_s(x_{-\beta}(-\frac{t_1^{-1}z}{(1+|t_2|^2)^{\frac{3}{2}}})k'(t_2)w_\beta g).
$$

Now we use the Iwasawa decomposition of $x_{-\beta}(-z)$:

$$
x_{-\beta}(-z) = \begin{pmatrix} (|z|^2+1)^{-\frac{1}{2}} & & \\ & (|z|^2+1)^{\frac{1}{2}} & \\ & & 1 \end{pmatrix} x_\beta(-\overline{z})k''(z),
$$

where $k''(z)$ is a smooth function taking values in K_{G_2}, and $\overline{z}$ is the complex conjugate of z. Then the invariant property of f_s yields

$$
\begin{aligned}
&f_s(x_{-\beta}(-\frac{t_1^{-1}z}{(1+|t_2|^2)^{\frac{3}{2}}})k'(t_2)w_\beta g) \\
=&\Big||-\frac{t_1^{-1}z}{(1+|t_2|^2)^{\frac{3}{2}}}|^2+1\Big|_F^{-\frac{3}{2}s} f_s(k''(-\frac{t_1^{-1}z}{(1+|t_2|^2)^{\frac{3}{2}}})k'(t_2)w_\beta j(k)).
\end{aligned}
$$

Combining all of the above, we get (3.3). $\square$

Now we return to the proof of Theorem 1.0.10.

Proof of Theorem 1.0.10, the Archimedean Case. We parameterize the torus of $\mathrm{SL}_3(F)$ in the following way,

$$a = \begin{pmatrix} t_1t_2 & & \\ & t_2 & \\ & & t_1^{-1}t_2^{-2} \end{pmatrix}.$$

Using the Iwasawa decomposition for $\mathrm{SL}_3(F)$, we obtain that

$$Z(W_v, f_s) = \int_F \int_{A_{\mathrm{SL}_3}} \int_{K_{\mathrm{SL}_3}} W_v(\begin{pmatrix} 1 & z & \\ & 1 & \\ & & 1 \end{pmatrix} ak) f_s(\gamma \cdot \begin{pmatrix} 1 & z & \\ & 1 & \\ & & 1 \end{pmatrix} ak) \delta_{B_{\mathrm{SL}_3}}^{-1}(a) dkdadz. \tag{3.6}$$

Here $A_{\mathrm{SL}_3} = \left\{ \begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_1^{-1}t_2^{-1} \end{pmatrix} \middle| t_1, t_2 > 0 \right\}$, K_{SL_3} is the maximal compact subgroup of $\mathrm{SL}_3(F)$ as in (3.2). Thus, by Lemma 3.1.1, we get

$$\begin{aligned} Z(W_v, f_s) = \int_F \int_{(\mathbb{R}_+^\times)^2} \int_{K_{\mathrm{SL}_3}} W_v(\begin{pmatrix} t_1t_2 & & \\ & t_2 & \\ & & t_1^{-1}t_2^{-2} \end{pmatrix} k) \psi(z) |t_1^{3s-4} t_2^{9s-6}|_F \cdot \\ \left| |t_1^{-1}z|^2 + (1+t_2^2)^3 \right|_F^{-\frac{3}{2}s} \cdot f_s(k''(-\frac{-t_1^{-1}z}{(1+t_2^2)^{\frac{3}{2}}}) k'(t_2) w_\beta k) dk d^\times t_1 d^\times t_2 dz. \end{aligned} \tag{3.7}$$

We change variable $z \mapsto t_1(1+t_2^2)^{\frac{3}{2}} z$ and reduce (3.7) to

$$\begin{aligned} Z(W_v, f_s) = \int_F \int_{(\mathbb{R}_+^\times)^2} \int_{K_{\mathrm{SL}_3}} W_v(\begin{pmatrix} t_1t_2 & & \\ & t_2 & \\ & & t_1^{-1}t_2^{-2} \end{pmatrix} k) \psi(t_1(1+t_2^2)^{\frac{3}{2}} z) |t_1|_F^{3s-3} |t_2|_F^{9s-6} \\ \cdot \left| |z|^2 + 1 \right|_F^{-\frac{3}{2}s} \cdot \left| 1 + t_2^2 \right|_F^{-\frac{9}{2}s + \frac{3}{2}} f_s(k''(-z) k'(t_2) w_\beta k) dk d^\times t_1 d^\times t_2 dz. \end{aligned} \tag{3.8}$$

By our assumption, π has a trivial central character, so

$$W_v(\begin{pmatrix} t_1t_2 & & \\ & t_2 & \\ & & t_1^{-1}t_2^{-2} \end{pmatrix} k) = W_v(\begin{pmatrix} t_1 \cdot t_1t_2^3 & & \\ & t_1t_2^3 & \\ & & 1 \end{pmatrix} k).$$

We change the variable $t_2' = t_1t_2^3$ and rewrite (3.8) as

$$\begin{aligned} Z(W_v, f_s) = \int_F \int_{(\mathbb{R}_+^\times)^2} \int_{K_{SL_3}} & W_v(\begin{pmatrix} t_1t_2' & & \\ & t_2' & \\ & & 1 \end{pmatrix} k)\psi(t_1(1+t_1^{-\frac{2}{3}}t_2'^{\frac{2}{3}})^{\frac{3}{2}}z) \\ & \cdot \Big|\frac{t_1^{3s}t_2'^{3s}}{(t_1^{\frac{2}{3}}+t_2'^{\frac{2}{3}})^{\frac{9}{2}s}}\Big|_F \cdot |t_1^{-2}t_2'^{-2}|_F \cdot ||z|^2+1|_F^{-\frac{3}{2}s} \cdot |t_1^{\frac{2}{3}}+t_2'^{\frac{2}{3}}|_F^{\frac{3}{2}} \\ & \cdot \frac{1}{3} f_s(k''(-z)k'(t_1^{-\frac{1}{3}}t_2'^{\frac{1}{3}})w_\beta j(k))dkd^\times t_1 d^\times t_2 dz \end{aligned} \tag{3.9}$$

To save notations, we change $t_2' \mapsto t_2$ in (3.9). Proposition 2 in [J-S, Section 4.3] provides a coarse asymptotic expansion of Whittaker functions: there exists a finite set X of finite functions such that

$$W_v(\begin{pmatrix} t_1t_2 & & \\ & t_2 & \\ & & 1 \end{pmatrix} k) = \sum_{\xi \in X} \varphi_\xi(t_1,t_2,k)\xi(t_1,t_2), \tag{3.10}$$

where $\varphi_\xi \in S(F^2 \times K_{\mathrm{GL}_3})$ are Schwartz functions. After integrating over K_{GL_3} in (3.9), we can see that $Z(W_v, f_s)$ is majorized by a finite linear combination of some integrals of the following type

$$\int_F \int_{(\mathbb{R}_+^\times)^2} \varphi(t_1,t_2)\xi(t_1,t_2) \cdot \Bigg|\left(\frac{t_1^{\frac{2}{3}}t_2^{\frac{2}{3}}}{t_1^{\frac{2}{3}}+t_2^{\frac{2}{3}}}\right)^{\frac{9}{2}s-\frac{3}{2}}\Bigg|_F \cdot |t_1^{-1}t_2^{-1}|_F \cdot \big|(|z|^2+1)^{-\frac{3}{2}s}\big|_F d^\times t_1 d^\times t_2 dz,$$

where φ is a nonnegative Schwartz function on F^2. We first assume that $\mathrm{Re}(s) > \frac{1}{3}$, then clearly the dz-integral converges. By Cauchy inequality, we have

$$\left|\left(\frac{t_1^{\frac{2}{3}}t_2^{\frac{2}{3}}}{t_1^{\frac{2}{3}}+t_2^{\frac{2}{3}}}\right)^{\frac{9}{2}s-\frac{3}{2}}\right|_F \leq |t_1^{\frac{1}{3}}t_2^{\frac{1}{3}}|_F^{\frac{9}{2}\mathrm{Re}(s)-\frac{3}{2}}.$$

So $Z(W_v, f_s)$ is majorized by a finite linear combination of integrals of the following type

$$\int_{(\mathbb{R}_+^\times)^2} \varphi(t_1,t_2)\xi(t_1,t_2)\cdot|t_1t_2|_F^{\frac{3}{2}\mathrm{Re}(s)-\frac{3}{2}}d^\times t_1 d^\times t_2.$$

Therefore, when $\mathrm{Re}(s)$ is sufficiently large, $Z(W_v, f_s)$ converges absolutely. □

3.2 Convergence and Meromorphic Continuation (The Non-archimedean Case)

Now we turn to the case when F is non-archimedean. We can prove Theorem 1.0.10 using the same method. In fact, we can modify the above proof and prove convergence of the local integral and its meromorphic continuation together by using the K-finiteness conditions of W_v and f_s. Let us reformulate the theorem that we want to show in this Section as:

Theorem 3.2.1. *Suppose that F is non-archimedean. The local integral $Z(W_v, f_s)$ converges absolutely when Re(s) is sufficiently large. They are rational functions of q^{-3s}, where q is the cardinality of the residue field. Hence they automatically enjoy a meromorphic continuation to the whole complex plane. When π is a principal series (not necessarily unramified) $\pi = \mathrm{Ind}_{B^-}^{GL_3}|\quad|^{u_1}\chi_1\otimes|\quad|^{u_2}\chi_2\otimes|\quad|^{u_3}\chi_3$, the local integrals are also meromorphic in u_1, u_2, u_3.*

Let us first briefly review Bernstein-Zelevinsky derivative and the asymptotic expansion of the Whittaker functions which are essential for the proof of Theorem 3.2.1. Let $P_n = \mathrm{GL}_{n-1}(F)\ltimes U_n$ be the mirabolic subgroup of $\mathrm{GL}_n(F)$. Given a non-trivial unitary character ψ of F. Then ψ defines a character ψ_n of U_n:

$$\psi_n(u) = \psi(u_{n-1,n}).$$

For any smooth representation τ of P_n, set

$$V_\tau(U_n,\psi)=\langle \tau(u)v-\psi_n(u)v|u\in U_n, v\in V_\tau\rangle.$$

Then $V_\tau/V_\tau(U_n,\psi)$ forms a representation of the Levi subgroup $\mathrm{GL}_{n-1}(F)$, where $m\in \mathrm{GL}_{n-1}(F)$ acts by

$$m\cdot(v+V_\tau(U_n,\psi))=|\det m|_F^{-\frac{1}{2}}(\tau(m)+V_\tau(U_n,\psi)).$$

This $\mathrm{GL}_{n-1}(F)$-module is called the normalized twisted Jacquet module, which is denoted by $J_{U_n,\psi}(\tau)$. When ψ is trivial, we will write $J_{U_n}(\tau)$ instead of $J_{U_n,1}(\tau)$ for simplicity. The functor Ψ^- defined in [B-Z, Section 3] sends a smooth representation τ of P_n to its normalized Jacquet module $J_{U_n}(\tau)$. The functor Φ^- defined in [B-Z, Section 3] sends a smooth representation τ of P_n to the smooth representation $J_{U_n}(\tau,\theta)|_{P_{n-1}}$ of P_{n-1}. Given a smooth $\mathrm{GL}_n(F)$-module π, the k-th Bernstein-Zelevinsky derivative of π, which is denoted by $\pi^{(k)}$, is the $\mathrm{GL}_{n-k}(F)$-module $\Psi^-(\Phi^-)^{k-1}(\pi|_{P_n})$.

Suppose that π is an irreducible admissible generic smooth representation of $\mathrm{GL}_n(F)$. Then by [Z, Theorem 9.7], we can write π as

$$\pi\simeq \mathrm{Ind}_P^{\mathrm{GL}_n(F)}\sigma_1\otimes\sigma_2\otimes\cdots\otimes\sigma_r, \tag{3.11}$$

where P is the standard parabolic subgroup of $\mathrm{GL}_n(F)$ corresponding to a partition $n=n_1+n_2+\cdots+n_r$, and each σ_j is an essentially discrete series of $\mathrm{GL}_{n_j}(F)$, i.e. a twist of σ by a character is square-integrable modulo center. For simplicity, we also write $\sigma_1\times\sigma_2\times\cdots\times\sigma_r$ for the parabolic induction (3.11). For any integer k such that

1. $0\le k\le n$, $\pi^{(k)}\neq 0$,
2. there exists a partition of $k=k_1+k_2+\cdots+k_r$ such that $\sigma_j^{(k_j)}\neq 0$ for $j=1,2,\cdots,r$,

then the central character of $\sigma_1^{(k_1)}\times\sigma_2^{(k_2)}\times\cdots\times\sigma_r^{(k_r)}$ is called an exponent of π.

Now we recall the asymptotic expansion of the Whittaker function of $\mathrm{GL}_n(F)$ along the torus. This following result is [Mat2, Proposition 2.4], whose proof can be found in [Mat1, Theorem 2.1].

Proposition 3.2.2. *Let π be the irreducible admissible generic smooth representation given in* (3.11)*. Then for any Whittaker function $W \in \mathcal{W}(\pi, \psi)$, the function*

$$W(\mathrm{diag}(t_1 t_2 \cdots t_{n-1}, t_2 t_3 \cdots t_{n-1}, \cdots, t_{n-1}, 1)$$

is a finite linear combination of functions of the following form:

$$\prod_{k=1}^{n-1} c_k(t_k)|t_k|_F^{\frac{n-k}{2}} \mathrm{val}(t_k)^{m_k} \phi_k(t_k), \tag{3.12}$$

where val *is the valuation of the local field, ϕ_k are compactly supported functions on F, c_k are exponents of π and m_k are non-negative integers for $k = 1, 2, \cdots, n-1$.*

Now we are ready to give the proof of Theorem 3.2.1.

Proof of Theorem 3.2.1. We use the Iwasawa decomposition of $\mathrm{SL}_3(F) = NT_{\mathrm{SL}_3}K_{\mathrm{SL}_3}$, where we still parameterize the torus T_{SL_3} as

$$a = \begin{pmatrix} t_1 t_2 & & \\ & t_2 & \\ & & t_1^{-1}t_2^{-2} \end{pmatrix}.$$

The local integral can be written as

$$Z(W_v, f_s) = \int_F \int_{T_{\mathrm{SL}_3}} \int_{K_{\mathrm{SL}_3}} W_v(\begin{pmatrix} 1 & z & \\ & 1 & \\ & & 1 \end{pmatrix} ak) f_s(\gamma \cdot \begin{pmatrix} 1 & z & \\ & 1 & \\ & & 1 \end{pmatrix} ak) \delta_{B_{\mathrm{SL}_3}}^{-1}(a) dk da dz. \tag{3.13}$$

Equation (3.4) still holds in the non-archimedean case. Thus, the the RHS of (3.13) becomes

$$\int_F \int_{T_{\mathrm{SL}_3}} \int_{K_{\mathrm{SL}_3}} W_v(\begin{pmatrix} 1 & z & \\ & 1 & \\ & & 1 \end{pmatrix} ak) \cdot |t_1 t_2^3|_F^{3s}$$
$$\cdot f_s(x_{-\beta}(-t_1^{-1}z) x_{-(\alpha+\beta)}(-t_2) w_\beta k) \delta_{B_{\mathrm{SL}_3}}^{-1}(a) dk da dz.$$

Since W_v is K_{SL_3}-finite and f_s is K_{G_2}-finite, $Z(W_v, f_s)$ must be a finite linear combination of the integral of the following type

$$\int_F \int_{(F^\times)^2} W_v(a)\psi(t_1 z) \cdot |t_1|_F^{3s-3}|t_2|_F^{9s-6} f_s(x_{-\beta}(-z)x_{-(\alpha+\beta)}(-t_2))d^\times t_1 d^\times t_2 dz. \quad (3.14)$$

where we use the change of variable $z \mapsto t_1 z$ in (3.14).

To show the absolute convergence of (3.14), we break it into four integrals:

$$\begin{aligned} &\int_{|z|_F\le 1}\int_{|t_2|_F\le 1}\int_{t_1\in F^\times}, \qquad \int_{|z|_F>1}\int_{|t_2|_F\le 1}\int_{t_1\in F^\times}, \\ &\int_{|z|_F\le 1}\int_{|t_2|_F>1}\int_{t_1\in F^\times}, \qquad \int_{|z|_F>1}\int_{|t_2|_F>1}\int_{t_1\in F^\times}. \end{aligned} \quad (3.15)$$

1) Clearly when $|z|_F, |t_2|_F \le 1$, we have $x_{-\beta}(-z)x_{-(\alpha+\beta)}(-t_2) \in K_{\mathrm{G}_2}$. Thus the integral

$$\int_{|z|_F\le 1}\int_{|t_2|_F\le 1}\int_{t_1\in F^\times} W_v(a)\psi(t_1 z)\cdot|t_1|_F^{3s-3}|t_2|_F^{9s-6} f_s(x_{-\beta}(-z)x_{-(\alpha+\beta)}(-t_2))d^\times t_1 d^\times t_2 dz$$

is majorized by a constant multiple of

$$\int_{|t_2|_F\le 1}\int_{t_1\in F^\times} |W_v(a)|_F \cdot |t_1|_F^{3s-3}|t_2|_F^{9s-6} d^\times t_1 d^\times t_2,$$

which, by Proposition 3.2.2, converges absolutely when $\mathrm{Re}(s)$ is sufficiently large.

2) When $|z|_F > 1, |t_2|_F \le 1$, we use the Iwasawa decomposition of $x_{-\beta}(-z)$:

$$x_{-\beta}(-z) = \mathrm{diag}(-z^{-1}, -z, 1)x_\beta(-z)k''(z) \quad (3.16)$$

for some $k''(z) \in K_{\mathrm{G}_2}$. Then the integral

$$\int_{|z|_F>1}\int_{|t_2|_F\le 1}\int_{t_1\in F^\times} W_v(a)\psi(t_1 z)\cdot|t_1|_F^{3s-3}|t_2|_F^{9s-6} f_s(x_{-\beta}(-z)x_{-(\alpha+\beta)}(-t_2))d^\times t_1 d^\times t_2 dz$$

is majorized by

$$\int_{|z|_F>1}\int_{|t_2|_F\le 1}\int_{t_1\in F^\times} |W_v(a)|_F \cdot |t_1|_F^{3s-3}|t_2|_F^{9s-6}|z|_F^{-3s} d^\times t_1 d^\times t_2 dz.$$

Again, by Proposition 3.2.2, the above integral converges absolutely when $\mathrm{Re}(s)$ is sufficiently large.

3) When $|z|_F \leq 1, |t_2|_F > 1$, we use the Iwasawa decomposition of $x_{-(\alpha+\beta)}(-t_2)$:

$$x_{-(\alpha+\beta)}(-t_2) = \mathrm{diag}(t_2^{-2}, -t_2, -t_2)x_{\alpha+\beta}(-t_2)k'(t_2) \tag{3.17}$$

for some $k'(t_2) \in K_{\mathrm{G}_2}$. Since $x_{-\beta}$ and $x_{-(\alpha+\beta)}$ commutes (Lemma 1.0.2), we can rewrite the integral

$$\int_{|z|_F\leq 1}\int_{|t_2|_F>1}\int_{t_1\in F^\times} W_v(a)\psi(t_1 z)\cdot|t_1|_F^{3s-3}|t_2|_F^{9s-6} f_s(x_{-\beta}(-z)x_{-(\alpha+\beta)}(-t_2))d^\times t_1 d^\times t_2 dz$$

as

$$\int_{|z|_F\leq 1}\int_{|t_2|_F>1}\int_{t_1\in F^\times} W_v(a)\psi(t_1 z)\cdot|t_1|_F^{3s-3}|t_2|_F^{9s-6}\cdot|t_2|_F^{-9s} f_s(k'(t_2)x_{-\beta}(-z))d^\times t_1 d^\times t_2 dz.$$

The above integral is majorized by a constant multiple of

$$\int_{|t_2|_F>1}\int_{t_1\in F^\times} |W_v(a)|_F|t_1|_F^{3s-3}|t_2|_F^{-6} d^\times t_1 d^\times t_2.$$

Again, the absolute convergence follows from Proposition 3.2.2.

4) When $|z|_F > 1, |t_2|_F > 1$, we have to do the Iwasawa decomposition for both $x_{-(\alpha+\beta)}(-t_2)$ and $x_{-\beta}(-z)$. This resembles the archimedean computation. By the Iwasawa decomposition of $x_{-(\alpha+\beta)}(-t_2)$ (see (3.17)),

$$\begin{aligned} f_s(x_{-\beta}(-z)x_{-\alpha+\beta}(-t_2)) =& f_s(x_{-\beta}(-z)\cdot \mathrm{diag}(t_2^{-2}, -t_2, -t_2)x_{\alpha+\beta}(-t_2)k'(t_2)) \\ =& |t_2|_F^{-9s} f_s(x_{-\beta}(-zt_2^3)x_{\alpha+\beta}(-t_2)k'(t_2)). \end{aligned}$$

By Lemma 1.0.2, the above is equal to $|t_2|_F^{-9s} f_s(x_{-\beta}(-zt_2^3)k'(t_2))$. By our assumption in this current case, $|zt_2^3|_F > 1$, hence we have to use the Iwasawa decomposition of $x_{-\beta}(-zt_2^3)$ (just replace z by zt_2^3 in (3.16)). Thus,

$$\begin{aligned} f_s(x_{-\beta}(-z)x_{-\alpha+\beta}(-t_2)) =& |t_2|_F^{-9s} f_s(x_{-\beta}(-zt_2^3)k'(t_2)) \\ =& |t_2|_F^{-9s}\cdot|z|_F^{-3s}|t_2|_F^{-9s} f_s(k'(zt_2^3)k'(t_2)). \end{aligned}$$

Therefore, we can rewrite the integral

$$\int_{|z|_F>1}\int_{|t_2|_F>1}\int_{t_1\in F^\times} W_v(a)\psi(t_1z)\cdot|t_1|_F^{3s-3}|t_2|_F^{9s-6}f_s(x_{-\beta}(-z)x_{-(\alpha+\beta)}(-t_2))d^\times t_1d^\times t_2dz$$

as

$$\begin{aligned}\int_{|z|_F>1}\int_{|t_2|_F>1}\int_{t_1\in F^\times} & W_v(a)\psi(t_1z)\cdot|t_1|_F^{3s-3}|t_2|_F^{9s-6}\cdot|t_2|_F^{-9s}\cdot|z|_F^{-3s}|t_2|_F^{-9s}\\ &\cdot f_s(k'(zt_2^3)k'(t_2))d^\times t_1d^\times t_2dz.\end{aligned}\tag{3.18}$$

The integral (3.18) is majorized by a constant multiple of

$$\int_{|z|_F>1}\int_{|t_2|_F>1}\int_{t_1\in F^\times}|W_v(a)|_F\cdot|t_1|_F^{3s-3}|t_2|_F^{-9s-6}\cdot|z|_F^{-3s}d^\times t_1d^\times t_2dz.$$

Again, the absolute convergence follows from Proposition 3.2.2. Combining all the above four cases, we immediately get the absolute convergence of (3.14). Thus we finish the proof of the first statement of Theorem 3.2.1.

Next, we are going to show that the integral (3.14) must be a rational function of q^{-3s}. As we have shown the absolute convergence of (3.14), we can switch the order of integration in whichever way we want. We break the integral (3.14) into two parts according to the condition $|t_2|_F\le 1$ and $|t_2|_F>1$.

1) When $|t_2|_F\le 1$, we first look at the dz-integral in (3.14). By changing $z\mapsto t_1^{-1}z$,

we find

$$\begin{aligned}
&|t_1|_F^{3s+1}\int_F f_s(x_{-\beta}(-z)x_{-(\alpha+\beta)}(-t_2))\psi(t_1z)dz\\
=&|t_1|_F^{3s}\int_F f_s(x_{-\beta}(-t_1^{-1}z)x_{-(\alpha+\beta)}(-t_2))\psi(z)dz\\
=&|t_1|_F^{3s}\int_F f_s(\begin{pmatrix}1&&\\&t_1^{-1}&\\&&t_1\end{pmatrix}x_{-\beta}(-z)\begin{pmatrix}1&&\\&t_1&\\&&t_1^{-1}\end{pmatrix}x_{-(\alpha+\beta)}(-t_2))\psi(z)dz\\
=&\int_F f_s(x_{-\beta}(-z)\begin{pmatrix}1&&\\&t_1&\\&&t_1^{-1}\end{pmatrix}x_{-(\alpha+\beta)}(-t_2))\psi(z)dz\\
=&\int_F f_s(w_\beta x_\beta(z)\begin{pmatrix}t_1&&\\&1&\\&&t_1^{-1}\end{pmatrix}w_\beta x_{-(\alpha+\beta)}(-t_2))\psi(z)dz.
\end{aligned} \tag{3.19}$$

Note that the group $\mathrm{GL}_2(F)$ embeds into $\mathrm{SL}_3(F)$ via the map: $g\mapsto\begin{pmatrix}g&\\&(\det g)^{-1}\end{pmatrix}$. Set H be the subgroup of $\mathrm{SL}_3(F)$ isomorphic to that $\mathrm{GL}_2(F)$. As we regard $\mathrm{SL}_3(F)$ as a subgroup of $\mathrm{G}_2(F)$, H naturally becomes a subgroup of $\mathrm{G}_2(F)$. For any $f_s\in V_{\rho_s}$, the restriction of f_s on H satisfies the following equivariant property:

$$f_s(\begin{pmatrix}t_1&\star&\\&t_2&\\&&t_1^{-1}t_2^{-1}\end{pmatrix}\begin{pmatrix}g&\\&(\det g)^{-1}\end{pmatrix})=|t_1^2t_2|_F^{3s}f_s(\begin{pmatrix}g&\\&(\det g)^{-1}\end{pmatrix}).$$

Thus, $f_s|_H$ lives in the induced representation $\mathrm{Ind}_{B_{GL_2}}^{\mathrm{GL}_2(F)}|\quad|^{6s-\frac{1}{2}}\otimes|\quad|^{3s+\frac{1}{2}}$. We apply the above observation to $\rho_s(w_\beta x_{-(\alpha+\beta)}(-t_2))f_s\in V_{\rho_s}$ and conclude that the dz-integral in the LHS of (3.19) must be a Whittaker function for $\mathrm{GL}_2(F)$, denoted by $W(g;f_s,t_2)$ along the torus $\begin{pmatrix}t_1&\\&1\end{pmatrix}$, as the RHS is exactly the Jacquet integral associated to the function

$$\rho_s(w_\beta x_{-(\alpha+\beta)}(-t_2))f_s|_H\in\mathrm{Ind}_{B_{GL_2}}^{\mathrm{GL}_2(F)}|\quad|^{6s-\frac{1}{2}}\otimes|\quad|^{3s+\frac{1}{2}}.$$

By Proposition 3.2.2, for each fixed t_2, $W(\begin{pmatrix} t_1 & \\ & 1 \end{pmatrix}; f_s, t_2)$ is a finite linear combination of the functions of the following form

$$|t_1|_F^{P(s)}\phi(t_1;t_2),$$

where $P(s)$ is either $6s$ or $3s+1$, and ϕ_{t_1,t_2} is smooth function on $F\times F$, compactly supported on the first variable t_1. Applying Proposition 3.2.2 to π, we know that $W_v(a)$ is also a finite linear combination of the functions of the following form

$$c_1(t_1)c_2(t_1t_2^3)\mathrm{val}(t_1)^{m_1}\mathrm{val}(t_1t_2^3)^{m_2}\phi_\pi(t_1,t_1t_2^3).$$

where c_1, c_2 are exponents of π, ϕ_π is a smooth compactly supported function on $F\times F$. Thus, the integral

$$\begin{aligned} &\int_F\int_{|t_2|_F\le 1}\int_{F^\times} W_v(a)\psi(t_1z)\cdot|t_1|_F^{3s-3}|t_2|_F^{9s-6}f_s(x_{-\beta}(-z)x_{-(\alpha+\beta)}(-t_2))d^\times t_1 d^\times t_2 dz \\ =&\int_{|t_2|_F\le 1}\int_{F^\times} W_v(a)W(\begin{pmatrix} t_1 & \\ & 1 \end{pmatrix}; f_s, t_2)|t_1|_F^{-4}|t_2|_F^{9s-6}d^\times t_1 d^\times t_2 \end{aligned} \tag{3.20}$$

is a finite linear combination of integrals of the following form

$$\int_{|t_2|_F\le 1}\int_{F^\times}|t_1|_F^{P(s)-4}|t_2|_F^{9s-6}\phi(t_1;t_2)\cdot c_1(t_1)c_2(t_1t_2^3)\mathrm{val}(t_1)^{m_1}\mathrm{val}(t_1t_2^3)^{m_2}\phi_\pi(t_1,t_1t_2^3)d^\times t_1 d^\times t_2.$$

Hence (3.20) is a rational function of q^{-3s}.

2) When $|t_2|_F>1$, by applying the Iwasawa decomposition of $x_{-(\alpha+\beta)}(-t_2)$, we rewrite the dz-integral as

$$\begin{aligned} &\int_F f_s(x_{-\beta}(-z)x_{-(\alpha+\beta)}(-t_2))\psi(t_1z)dz \\ =&|t_2|_F^{-9s}\int_F f_s(x_{-\beta}(-zt_2^3)x_{\alpha+\beta}(-t_2)k'(t_2))\psi(t_1z)dz \\ =&|t_2|_F^{-9s}|t_1|_F^{-1}\int_F f_s(x_{-\beta}(-zt_1^{-1}t_2^3)k'(t_2))\psi(z)dz. \end{aligned}$$

Thus,

$$\int_F \int_{|t_2|_F>1} \int_{F^\times} W_v(a)\psi(t_1 z) \cdot |t_1|_F^{3s-3} |t_2|_F^{9s-6} f_s(x_{-\beta}(-z)x_{-(\alpha+\beta)}(-t_2)) d^\times t_1 d^\times t_2 dz$$

$$= \int_F \int_{|t_2|_F>1} \int_{F^\times} W_v(\begin{pmatrix} t_1 \cdot t_1 t_2^3 & & \\ & t_1 t_2^3 & \\ & & 1 \end{pmatrix}) |t_1|_F^{3s-4} |t_2|_F^{-6}$$

$$\cdot f_s(x_{-\beta}(-z t_1^{-1} t_2^3) k'(t_2)) \psi(z) d^\times t_1 d^\times t_2 dz.$$

By changing the variable $t_1 \mapsto t_1 t_2^3$, we can rewrite the above integral as

$$\begin{aligned} &\int_F \int_{|t_2|_F>1} \int_{F^\times} W_v(\begin{pmatrix} t_1 t_2^3 \cdot t_1 t_2^6 & & \\ & t_1 t_2^6 & \\ & & 1 \end{pmatrix}) |t_1|_F^{3s-4} |t_2|_F^{9s-18} \\ &\cdot f_s(x_{-\beta}(-z t_1^{-1}) k'(t_2)) \psi(z) d^\times t_1 d^\times t_2 dz. \end{aligned} \tag{3.21}$$

Same as the case when $|t_2|_F \leq 1$, now the dz-integral becomes a Whittaker function $W(\begin{pmatrix} t_1 & \\ & 1 \end{pmatrix}; f_s, t_2)$ for $\rho_s(k'(t_2)) f_s|_H$ in the induced representation

$$\mathrm{Ind}_{B_{GL_2}}^{\mathrm{GL}_2(F)} |\quad|^{6s-\frac{1}{2}} \otimes |\quad|^{3s+\frac{1}{2}}.$$

The integral (3.21) can be rewritten as

$$\int_{|t_2|_F>1} \int_{F^\times} W_v(\begin{pmatrix} t_1 t_2^3 \cdot t_1 t_2^6 & & \\ & t_1 t_2^6 & \\ & & 1 \end{pmatrix}) |t_1|_F^{-4} |t_2|_F^{9s-18} W(\begin{pmatrix} t_1 & \\ & 1 \end{pmatrix}; f_s, t_2) d^\times t_1 d^\times t_2. \tag{3.22}$$

By applying the asymptotic expansion of W_v and $W(\begin{pmatrix} t_1 & \\ & 1 \end{pmatrix}; f_s, t_2)$, the above integral is a finite linear combination of

$$\begin{aligned} &\int_{F^\times} \int_{|t_2|_F>1} |t_1|_F^{P(s)-4} |t_2|^{9s-18} \phi(t_1; t_2) \cdot c_1(t_1 t_2^3) c_2(t_1 t_2^6) \\ &\cdot \mathrm{val}(t_1 t_2^3)^{m_1} \mathrm{val}(t_1 t_2^6)^{m_2} \phi_\pi(t_1 t_2^3, t_1 t_2^6) d^\times t_2 d^\times t_1, \end{aligned} \tag{3.23}$$

where c_1, c_2 are exponents of π, ϕ_π is a smooth compactly supported function on $F \times F$ and ϕ is a smooth function on $F \times F$ that is compactly supported on the first variable and bounded on the second variable. For each fixed t_1, as a function of t_2, $\phi_\pi(t_1t_2^3, t_1t_2^6)$ is also compactly supported. Thus the inner $d^\times t_2$-integral in (3.23) is in fact a finite summation, hence it is a polynomial function in q^{3s} whose coefficients are compactly supported smooth functions of t_1 (because ϕ is compactly supported on t_1). It follows that the integral (3.23) is a finite linear combination of

$$q^{3ks}\int_{F^\times}|t_1|_F^{P(s)-4}\chi'(t_1)\mathrm{val}(t_1)^m\varphi(t_1)d^\times t_1, \tag{3.24}$$

where k is an integer, $\varphi(t_1)$ is a smooth compactly supported function, χ' is a character of $F^\times$. Therefore, (3.24) and hence the integral

$$\int_F\int_{|t_2|_F>1}\int_{F^\times} W_v(a)\psi(t_1z)\cdot|t_1|_F^{3s-3}|t_2|_F^{9s-6}f_s(x_{-\beta}(-z)x_{-(\alpha+\beta)}(-t_2))d^\times t_1 d^\times t_2 dz$$

is a rational function of q^{-3s}. This finishes the proof of the second statement of Theorem 3.2.1.

Moreover, when π is the principal series $\mathrm{Ind}_{B^-}^{GL_3}|\ \ |^{u_1}\chi_1\otimes|\ \ |^{u_2}\chi_2\otimes|\ \ |^{u_3}\chi_3$, the exponents in the asymptotic expansion of $W_v\in\mathcal{W}(\pi,\psi)$ are degree 1 polynomials in u_1, u_2, u_3. This shows that the integral (3.14) (hence the local integral $Z(W_v, f_s)$) is meromorphic in u_1, u_2, u_3. $\square$

Chapter 4

Existence of Local G.C.D (Non-archimedean Case)

In this Chapter, we aim to show the existence of local g.c.d in the non-archimedean case. From Theorem 3.2.1, we know that all local integrals $Z(W_v, f_s)$ live in a subspace of $\mathbb{C}(q^{-3s})$. We first show that the constant function 1 in fact lives in that subspace.

4.1 Local Integral can be Made Constant (Non-archimedean Case)

Theorem 4.1.1. *When F is non-archimedean, there exists a Whittaker function $W_v \in \mathcal{W}(\pi, \psi)$ and a function $f_s \in V_{\rho_s}$ such that $Z(W_v, f_s) = 1$ for every $s \in \mathbb{C}$.*

Proof. Let U_ψ be the conductor of ψ. Then ψ is trivial on U_ψ. We first show that given a Whittaker function $W_{\widetilde{v}} \in \mathcal{W}(\pi, \psi)$, there exists a Whittaker function $W_v \in \mathcal{W}(\pi, \psi)$ and a small neighbourhood V_0 of 1, such that

$$W_v(\begin{pmatrix} t_1 t_2 & & \\ & t_2 & \\ & & 1 \end{pmatrix}) = \begin{cases} W_{\widetilde{v}}(I) & \text{if } t_1, t_2 \text{ lie in a small neighbourhood } V_0 \text{ of } 1; \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

Indeed, given a Whittaker function $W_{\widetilde{v}}(g)$, since it is locally constant, we can find a small neighbourhood V_0 of 1 such that

$$W_{\widetilde{v}}(\begin{pmatrix} t_1t_2 & & \\ & t_2 & \\ & & 1 \end{pmatrix}) = W_{\widetilde{v}}(I) \tag{4.2}$$

when $t_1, t_2 \in V_0$. Without loss of generality, we may assume that $V_0 = 1 + \mathfrak{p}^n$ for a very large positive integer n. We set

$$W_{v_1}(g) = \int_{U_2} W_{\widetilde{v}}(g \begin{pmatrix} 1 & & \\ & 1 & y_2 \\ & & 1 \end{pmatrix})\psi^{-1}(y_2)dy_2,$$

where y_2 ranges over some open compact subgroup U_2 of F. Then $W_{v_1} \in \mathcal{W}(\pi, \psi)$ and

$$W_{v_1}(\begin{pmatrix} t_1t_2 & & \\ & t_2 & \\ & & 1 \end{pmatrix}) = \int_{U_2} \psi(t_2y_2 - y_2)dy_2 \cdot W_{\widetilde{v}}(\begin{pmatrix} t_1t_2 & & \\ & t_2 & \\ & & 1 \end{pmatrix})$$

The dy_2-integral is equal to the measure of U_2 if $\psi((t_2 - 1)y_2) = 1$ for all $y_2 \in U_2$, otherwise it is zero. We can choose U_2 to be a very large open compact subgroup such that the condition $(t_2 - 1)y_2 \in U_\psi$ for all $y_2 \in U_2$ implies that $t_2 \in V_0$. Thus,

$$W_{v_1}(\begin{pmatrix} t_1t_2 & & \\ & t_2 & \\ & & 1 \end{pmatrix}) = \begin{cases} C_2 \cdot W_{\widetilde{v}}(\begin{pmatrix} t_1t_2 & & \\ & t_2 & \\ & & 1 \end{pmatrix}) & \text{if } t_2 \in V_0; \\ 0 \quad \text{otherwise,} & \end{cases}$$

where C_2 is the measure of U_2 (hence non-zero).

Similarly, we consider

$$W_{v_2}(g) = \int_{U_1} W_{\widetilde{v}}(g \begin{pmatrix} 1 & y_1 & \\ & 1 & \\ & & 1 \end{pmatrix})\psi^{-1}(y_1)dy_1,$$

where U_1 is a large open compact subgroup of F. Using the same trick, we can show that

$$W_{v_2}(\begin{pmatrix} t_1t_2 & & \\ & t_2 & \\ & & 1 \end{pmatrix}) = \begin{cases} C_1C_2 \cdot W_{\widetilde{v}}(\begin{pmatrix} t_1t_2 & & \\ & t_2 & \\ & & 1 \end{pmatrix}) & \text{if } t_1, t_2 \in V_0; \\ 0 & \text{otherwise.} \end{cases}$$

Taking (4.2) into account and rescaling W_{v_2}, we finally find a Whittaker function $W_v \in \mathcal{W}(\pi, \psi)$ such that

$$W_v(\begin{pmatrix} t_1t_2 & & \\ & t_2 & \\ & & 1 \end{pmatrix}) = \begin{cases} W_{\widetilde{v}}(I) & \text{if } t_1, t_2 \in V_0; \\ 0 & \text{otherwise.} \end{cases} \tag{4.3}$$

Now we apply the above claim to our scenario. We fix a Whittaker function $W_{\widetilde{v}}$ with the property that $W_{\widetilde{v}}(I) = C$ for some constant C. Then according to (4.3), we can choose a Whittaker function W_v and two small neighbourhoods V_1 and V_2 of 1 such that

$$W_v(\begin{pmatrix} t_1t_2^3 & & \\ & t_2^3 & \\ & & 1 \end{pmatrix}) = \begin{cases} C & \text{if } t_1 \in V_1, t_2 \in V_2; \\ 0 & \text{otherwise.} \end{cases} \tag{4.4}$$

For such a Whittaker function W_v, we can find a small open compact subgroup K_0 of $\mathrm{GL}_3(F)$ such that W_v is right K_0-invariant.

By applying the Bruhat decomposition of $\mathrm{SL}_3(F)$, we rewrite the local integral

$Z(W_v, f_s)$ as

$$\begin{aligned}
Z(W_v, f_s) &= \int_F \int_{T_{\mathrm{SL}_3}} \int_{N^-(F)} W_v(\begin{pmatrix} 1 & z & \\ & 1 & \\ & & 1 \end{pmatrix} a\overline{n}) f_s(\gamma \cdot \begin{pmatrix} 1 & z & \\ & 1 & \\ & & 1 \end{pmatrix} a\overline{n}) \delta^{-1}_{B_{\mathrm{SL}_3}}(a) d\overline{n} da dz \\
&= \int_F \int_{(F^\times)^2} \int_{N^-(F)} W_v(\begin{pmatrix} t_1 t_2^3 & & \\ & t_2^3 & \\ & & 1 \end{pmatrix} \overline{n}) \psi(z) \\
&\cdot f_s(\gamma \cdot \begin{pmatrix} 1 & z & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} t_1 t_2^3 & & \\ & t_2^3 & \\ & & 1 \end{pmatrix} \overline{n}) \cdot |t_1|_F^{-4} |t_2|_F^{-6} d\overline{n} d^\times t_1 d^\times t_2 dz
\end{aligned} \tag{4.5}$$

Next we are going to choose a function f_s such that a right translation of f_s will have a very small support modulo the parabolic subgroup $P(F)$. As we have shown in (3.4) that

$$\gamma \cdot \begin{pmatrix} 1 & z & \\ & 1 & \\ & & 1 \end{pmatrix} \cdot \begin{pmatrix} t_1 t_2 & & \\ & t_2 & \\ & & t_1^{-1} t_2^{-2} \end{pmatrix} = \begin{pmatrix} t_2 & & \\ & t_1 t_2 & \\ & & t_1^{-1} t_2^{-2} \end{pmatrix} x_{-(\alpha+\beta)}(-t_2) x_{-\beta}(-t_1^{-1} z) w_\beta.$$

Thus for $\overline{n} \in N^-(F)$, we have that

$$\begin{aligned}
&\gamma \cdot \begin{pmatrix} 1 & z & \\ & 1 & \\ & & 1 \end{pmatrix} \cdot \begin{pmatrix} t_1 t_2 & & \\ & t_2 & \\ & & t_1^{-1} t_2^{-2} \end{pmatrix} \overline{n} \\
&= \begin{pmatrix} t_2 & & \\ & t_1 t_2 & \\ & & t_1^{-1} t_2^{-2} \end{pmatrix} x_{-(\alpha+\beta)}(-t_2) x_{-\beta}(-t_1^{-1} z) w_\beta \overline{n} w_\beta^{-1} \cdot w_\beta \\
&= \begin{pmatrix} t_2 & & \\ & t_1 t_2 & \\ & & t_1^{-1} t_2^{-2} \end{pmatrix} x_{-(\alpha+\beta)}(-t_2) x_{-\beta}(-t_1^{-1} z) x_\beta(x_1) x_{-(3\alpha+\beta)}(x_2) x_{-(3\alpha+2\beta)}(x_3) w_\beta.
\end{aligned} \tag{4.6}$$

Here we parameterize the unipotent subgroup $w_\beta N^-(F)w_\beta^{-1}$ as

$$w_\beta \overline{n} w_\beta^{-1} = x_\beta(x_1)x_{-(3\alpha+\beta)}(x_2)x_{-(3\alpha+2\beta)}(x_3). \tag{4.7}$$

It is easy to check that whenever $xy+1 \neq 0$, the following matrix computation holds:

$$\begin{pmatrix} 1 & \\ x & 1 \end{pmatrix} \begin{pmatrix} 1 & y \\ & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{xy+1} & \\ & xy+1 \end{pmatrix} \begin{pmatrix} 1 & (xy+1)y \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ \frac{x}{xy+1} & 1 \end{pmatrix}.$$

Thus, whenever $-t^{-1}zx_1 + 1 \neq 0$, we have

$$\begin{aligned} x_{-\beta}(-t_1^{-1}z)x_\beta(x_1) = &\begin{pmatrix} (-t^{-1}zx_1+1)^{-1} & & \\ & -t^{-1}zx_1+1 & \\ & & 1 \end{pmatrix} x_\beta((-t^{-1}zx_1+1)x_1) \\ &\cdot x_{-\beta}(\frac{-t_1^{-1}z}{-t^{-1}zx_1+1}). \end{aligned} \tag{4.8}$$

Combining (4.6) and (4.8), we find that

$$\begin{aligned} &\gamma \cdot \begin{pmatrix} 1 & z & \\ & 1 & \\ & & 1 \end{pmatrix} \cdot \begin{pmatrix} t_1t_2 & & \\ & t_2 & \\ & & t_1^{-1}t_2^{-2} \end{pmatrix} \overline{n} \cdot \gamma^{-1} \\ = &\begin{pmatrix} t_2 & & \\ & t_1t_2 & \\ & & t_1^{-1}t_2^{-2} \end{pmatrix} \cdot \begin{pmatrix} (-t^{-1}zx_1+1)^{-1} & & \\ & -t^{-1}zx_1+1 & \\ & & 1 \end{pmatrix} \cdot x_{-(\alpha+\beta)}(\frac{-t_2}{-t^{-1}zx_1+1}) \\ &\cdot x_\beta((-t^{-1}zx_1+1)x_1) \cdot x_{-\beta}(\frac{-t_1^{-1}z}{-t^{-1}zx_1+1}) \cdot x_{-(3\alpha+\beta)}(x_2) \cdot x_{-(3\alpha+2\beta)}(x_3) \cdot x_{-(\alpha+\beta)}(1). \end{aligned} \tag{4.9}$$

Recall the commutator relation in Lemma 1.0.2

$$x_\beta(-s)x_\alpha(-t)x_\beta(s) = x_{\alpha+\beta}(st)x_{2\alpha+\beta}(st^2)x_{3\alpha+\beta}(st^3)x_{3\alpha+2\beta}(s^2t^3)x_\alpha(-t).$$

Applying $w_{2\alpha+\beta} = w_\alpha w_\beta w_\alpha w_\beta w_\alpha$, we find that

$$\begin{aligned}&x_\beta(-s)x_{-(\alpha+\beta)}(-t)x_\beta(s)\\&= x_{-\alpha}(st)x_{-(2\alpha+\beta)}(st^2)x_{-(3\alpha+2\beta)}(st^3)x_{-(3\alpha+\beta)}(s^2t^3)x_{-(\alpha+\beta)}(-t).\end{aligned}$$

Applying the above identity to $s = (-t_1^{-1}zx_1+1)x_1$ and $t = \frac{t_2}{-t^{-1}zx_1+1}$, we find that

$$\begin{aligned}&x_\beta(-(-t_1^{-1}zx_1+1)x_1)\cdot x_{-(\alpha+\beta)}(-\frac{t_2}{-t^{-1}zx_1+1})\cdot x_\beta((-t_1^{-1}zx_1+1)x_1)\\=&x_{-\alpha}(t_2x_1)\cdot x_{-(2\alpha+\beta)}(\frac{t_2^2x_1}{-t_1^{-1}zx_1+1})\cdot x_{-(3\alpha+2\beta)}(\frac{t_2^3x_1}{(-t_1^{-1}zx_1+1)^2})\\&\cdot x_{-(3\alpha+\beta)}(\frac{t_2^3x_1^2}{-t_1^{-1}zx_1+1})\cdot x_{-(\alpha+\beta)}(-\frac{t_2}{-t^{-1}zx_1+1}).\end{aligned} \tag{4.10}$$

Thus, by (4.9), we find that for every $f_s \in V_{\rho_s}$,

$$\begin{aligned}&f_s(\gamma\cdot\begin{pmatrix}1 & z & \\ & 1 & \\ & & 1\end{pmatrix}\cdot\begin{pmatrix}t_1t_2 & & \\ & t_2 & \\ & & t_1^{-1}t_2^{-2}\end{pmatrix}\overline{n})\\=&(\rho_s(\gamma)f_s)(\gamma\cdot\begin{pmatrix}1 & z & \\ & 1 & \\ & & 1\end{pmatrix}\cdot\begin{pmatrix}t_1t_2 & & \\ & t_2 & \\ & & t_1^{-1}t_2^{-2}\end{pmatrix}\overline{n}\cdot\gamma^{-1})\\=&|t_1t_2^3|_F^{3s}\cdot|-t_1^{-1}zx_1+1|_F^{-3s}\cdot(\rho_s(\gamma)f_s)(x_{-(\alpha+\beta)}(\frac{-t_2}{-t^{-1}zx_1+1})\\&\cdot x_\beta((-t^{-1}zx_1+1)x_1)\cdot x_{-\beta}(\frac{-t_1^{-1}z}{-t^{-1}zx_1+1})\cdot x_{-(3\alpha+\beta)}(x_2)\cdot x_{-(3\alpha+2\beta)}(x_3)\cdot x_{-(\alpha+\beta)}(1)).\end{aligned}$$

The RHS of the above can be simplified via (4.10) as

$$\begin{aligned}&|t_1t_2^3|_F^{3s}\cdot|-t_1^{-1}zx_1+1|_F^{-3s}\cdot(\rho_s(\gamma)f_s)(x_{-(2\alpha+\beta)}(\frac{t_2^2x_1}{-t_1^{-1}zx_1+1})\\&\cdot x_{-(3\alpha+2\beta)}(\frac{t_2^3x_1}{(-t_1^{-1}zx_1+1)^2})\cdot x_{-(3\alpha+\beta)}(\frac{t_2^3x_1^2}{-t_1^{-1}zx_1+1})\cdot x_{-(\alpha+\beta)}(-\frac{t_2}{-t^{-1}zx_1+1})\\&\cdot x_{-\beta}(\frac{-t_1^{-1}z}{-t^{-1}zx_1+1})\cdot x_{-(3\alpha+\beta)}(x_2)\cdot x_{-(3\alpha+2\beta)}(x_3)\cdot x_{-(\alpha+\beta)}(1)).\end{aligned}$$

As the short negative roots commutes with the long negative roots, we can summarize the above computation as follows:

$$\begin{aligned}
&f_s(\gamma\cdot\begin{pmatrix}1 & z & \\ & 1 & \\ & & 1\end{pmatrix}\cdot\begin{pmatrix}t_1t_2 & & \\ & t_2 & \\ & & t_1^{-1}t_2^{-2}\end{pmatrix}\overline{n}) \\
=&|t_1t_2^3|_F^{3s}\cdot|-t_1^{-1}zx_1+1|_F^{-3s}\cdot(\rho_s(\gamma)f_s)(x_{-(\alpha+\beta)}(\frac{-t_2}{-t_1^{-1}zx_1+1}+1) \\
&x_{-(2\alpha+\beta)}(\frac{t_2^2x_1}{-t_1^{-1}zx_1+1})x_{-(3\alpha+\beta)}(\frac{t_2^3x_1^2}{-t_1^{-1}zx_1+1}+x_2)x_{-\beta}(\frac{-t_1^{-1}z}{-t^{-1}zx_1+1}) \\
&x_{-(3\alpha+2\beta)}(\frac{t_2^3x_1}{(-t_1^{-1}zx_1+1)^2}+x_3-\frac{-t_1^{-1}zx_2}{-t_1^{-1}zx_1+1})).
\end{aligned} \tag{4.11}$$

We choose a function $f_s\in V_{\rho_s}$ so that the restriction of $\rho_s(\gamma)f_s$ on $U^-(F)$ has a small compact support. In other words, if we write the restriction of $\rho_s(\gamma)f_s$ on $U^-(F)$ as

$$(\rho_s(\gamma)f_s)(x_{-(\alpha+\beta)}(u_1)x_{-(2\alpha+\beta)}(u_2)x_{-(3\alpha+\beta)}(u_3)x_{-\beta}(u_4)x_{-(3\alpha+2\beta)}(u_5),$$

then each u_i live in a small compact subgroup U_i of $0\in F$. We can further assume that

$$(\rho_s(\gamma)f_s)(x_{-(\alpha+\beta)}(u_1)x_{-(2\alpha+\beta)}(u_2)x_{-(3\alpha+\beta)}(u_3)x_{-\beta}(u_4)x_{-(3\alpha+2\beta)}(u_5)$$

is identical to 1 when all $u_i\in U_i$. Thus, if we choose such a $f_s\in V_{\rho_s}$, then RHS of (4.11) is zero unless all the following five properties hold:

1. $\frac{-t_2}{-t_1^{-1}zx_1+1}+1\in U_1$;

2. $\frac{t_2^2x_1}{-t_1^{-1}zx_1+1}\in U_2$;

3. $\frac{t_2^3x_1^2}{-t_1^{-1}zx_1+1}+x_2\in U_3$;

4. $\frac{-t_1^{-1}z}{-t^{-1}zx_1+1}\in U_4$;

5. $\frac{t_2^3x_1}{(-t_1^{-1}zx_1+1)^2}+x_3-\frac{-t_1^{-1}zx_2}{-t_1^{-1}zx_1+1})\in U_5$.

We will show that the above five properties will imply that x_1,x_2,x_3 must live in some small neighbourhoods X_1,X_2,X_3 of 0 respectively. Indeed, the first property implies that $\frac{t_2}{-t_1^{-1}zx_1+1}$ live in $1+U_1$, a small neighbourhood of 1. Hence, by the second property,

$$t_2x_1 = \frac{t_2^2x_1}{-t_1^{-1}zx_1+1} \cdot (\frac{t_2}{-t_1^{-1}zx_1+1})^{-1}$$

must live in a small neighbourhood $U_2 \cdot (1+U_1)^{-1}$ of 0. Hence

$$\frac{t_2^3x_1^2}{-t_1^{-1}zx_1+1} = \frac{t_2}{-t_1^{-1}zx_1+1} \cdot (t_2x_1)^2$$

must live in a small neighbourhood $U_2^2 \cdot (1+U_1)^{-1}$ of 0. Combining the third property, we find x_2 must live in a small neighbourhood

$$X_2 := U_3 - U_2^2 \cdot (1+U_1)^{-1}$$

of 0. This also implies that

$$\frac{-t_1^{-1}zx_2}{-t_1^{-1}zx_1+1} = \frac{-t_1^{-1}z}{-t^{-1}zx_1+1} \cdot x_2$$

must belong to a small neighbourhood U_4X_2 of 0. Also, as the first and second property suggests,

$$\frac{t_2^3x_1}{(-t_1^{-1}zx_1+1)^2} = \frac{t_2}{-t_1^{-1}zx_1+1} \cdot \frac{t_2^2x_1}{-t_1^{-1}zx_1+1}$$

belongs to a small neighbourhood $U_2 \cdot (1+U_1)$ of 0. Taking the fifth property into account, we conclude that x_3 must live in a small neighbourhood X_3 of 0, where

$$X_3 = U_5 - U_2 \cdot (1+U_1) + U_4X_2.$$

To show that x_1 also belongs to a small neighbourhood of 0, we use the first and fourth property and conclude that

$$-t_1^{-1}t_2^{-1}z = \frac{-t_1^{-1}z}{-t^{-1}zx_1+1} \cdot (\frac{t_2}{-t_1^{-1}zx_1+1})^{-1}$$

lives in a small neighbourhood $U_4 \cdot (1+U_1)^{-1}$ of 0. Then it follows that

$$-t_1^{-1} z x_1 = -t_1^{-1} t_2^{-1} z \cdot t_2 x_1$$

lives in a small neighbourhood $U_2 U_4 \cdot (1+U_1)^{-2}$ of 0. Therefore, t_2 belongs to $(1+U_1) \cdot (1+U_2U_4 \cdot (1+U_1)^{-2})$, a small neighbourhood of 1. Finally, it follows that $x_1 = t_2 x_1 \cdot t_2^{-1}$ belongs to a small neighbourhood X_1 of 0, where

$$X_1 = U_2 \cdot (1+U_1)^{-2} \cdot (1+U_2U_4(1+U_1)^{-2})^{-1}.$$

We may choose a f_s satisfying the following property: each U_i is so small such that each $x_i \in X_i$ implies $\overline{n} \in K_0$ (recall that we chose W_v to be right K_0-invariant). With such a choice of f_s and W_v, the integrand in the local integral $Z(W_v, f_s)$

$$\begin{aligned} & W_v(\begin{pmatrix} t_1t_2^3 & & \\ & t_2^3 & \\ & & 1 \end{pmatrix} \overline{n})\psi(z) f_s(\gamma \cdot \begin{pmatrix} 1 & z & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} t_1t_2^3 & & \\ & t_2^3 & \\ & & 1 \end{pmatrix} \overline{n}) \cdot |t_1|_F^{-4} |t_2|_F^{-6} \\ =& W_v(\begin{pmatrix} t_1t_2^3 & & \\ & t_2^3 & \\ & & 1 \end{pmatrix})\psi(z) f_s(\gamma \cdot \begin{pmatrix} 1 & z & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} t_1t_2^3 & & \\ & t_2^3 & \\ & & 1 \end{pmatrix} \overline{n}) \cdot |t_1|_F^{-4} |t_2|_F^{-6} \end{aligned}$$

is non-zero unless each $t_i \in V_i$ (a small neighbourhood of 1) and each $x_i \in X_i$ (a small neighbourhood of 0). Then $|t_1|_F = |t_2|_F = 1$. Thus, the local integral becomes

$$\begin{aligned} Z(W_v, f_s) =& \int_F \int_{V_1 \times V_2} \int_{X_1 \times X_2 \times X_3} W_v(\begin{pmatrix} t_1t_2^3 & & \\ & t_2^3 & \\ & & 1 \end{pmatrix})\psi(z) \\ & f_s(\gamma \cdot \begin{pmatrix} 1 & z & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} t_1t_2^3 & & \\ & t_2^3 & \\ & & 1 \end{pmatrix} \overline{n}) \cdot |t_1|_F^{-4} |t_2|_F^{-6} dx_1 dx_2 dx_3 d^\times t_1 d^\times t_2 dz \\ =& \int_{U_z} \int_{V_1 \times V_2} \int_{X_1 \times X_2 \times X_3} C \cdot \psi(z) |-t_1^{-1} z x_1 + 1|_F^{-3s} dx_1 dx_2 dx_3 d^\times t_1 d^\times t_2 dz, \end{aligned} \tag{4.12}$$

where $z \in U_z$ is a subset of F such that all the five properties of f_s hold for $t_1 \in V_1, t_2 \in$

V_2. Then U_z is a small neighbourhood of 0. (Indeed, as we have shown that $-t^{-1}zx_1+1$ lives in a small neighbourhood of 1, the fourth properties of f_s implies that z must live in a small neighbourhood of 0.) We may further assume that the choices U_i are so small such that U_z is contained in the conductor U_ψ of ψ. Thus, the local integral becomes

$$Z(W_v, f_s) = \int_{U_z} \int_{V_1 \times V_2} \int_{X_1 \times X_2 \times X_3} C dx_1 dx_2 dx_3 d^\times t_1 d^\times t_2 dz.$$

We finally can choose a constant C such that the above integral is equal to 1. $\square$

4.2 Local G.C.D (Non-archimedean Case)

In this Section, F can be archimedean or non-archimedean. We first show that if any qualitative analytic property holds for one given non-trivial additive character ψ, then the same property also holds for all the other nontrivial additive characters.

We fix one non-trivial additive character ψ of F. By Pontryagin Duality, any non-trivial additive character of F is of the form

$$\psi_c(x) = \psi(cx) \quad \text{for some } c \in F^\times.$$

To emphasize the dependence of Whittaker functions on the additive character, we write W_v^ψ for a Whittaker function in $\mathcal{W}(\pi, \psi)$. Note that if $W_v^\psi \in \mathcal{W}(\pi, \psi)$, then the function $W_v^{\psi_c}$ defined by

$$W_v^{\psi_c}(g) = W_v^\psi(\begin{pmatrix} c & & \\ & 1 & \\ & & c^{-1} \end{pmatrix} g)$$

satisfies

$$W_v^{\psi_c}(\begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} g) = \psi(cx+cy) W_v^\psi(\begin{pmatrix} c & & \\ & 1 & \\ & & c^{-1} \end{pmatrix} g) = \psi_c(x+y) W_v^{\psi_c}(g).$$

Hence $W_v^{\psi_c} \in \mathcal{W}(\pi, \psi_c)$.

We can also relate the local integrals $Z(W_v^{\psi}, f_s)$ and $Z(W_v^{\psi_c}, f_s)$. In the non-archimedean case,

$$Z(W_v^{\psi}, f_s) = \int_F \int_{(F^\times)^2} \int_{K_{SL_3}} W_v^{\psi}(\begin{pmatrix} 1 & z & \\ & 1 & \\ & & 1 \end{pmatrix} ak)$$
$$f_s(\gamma \cdot \begin{pmatrix} 1 & z & \\ & 1 & \\ & & 1 \end{pmatrix} ak)\delta_{B_{SL_3}}^{-1}(a)dkd^\times t_1 d^\times t_2 dz,$$

hence

$$Z(W_v^{\psi_c}, f_s) = \int_F \int_{(F^\times)^2} \int_{K_{SL_3}} W_v^{\psi}(\begin{pmatrix} c & & \\ & 1 & \\ & & c^{-1} \end{pmatrix} \begin{pmatrix} 1 & z & \\ & 1 & \\ & & 1 \end{pmatrix} ak)$$
$$f_s(\gamma \cdot \begin{pmatrix} 1 & z & \\ & 1 & \\ & & 1 \end{pmatrix} ak)\delta_{B_{SL_3}}^{-1}(a)dkd^\times t_1 d^\times t_2 dz.$$

By changing variables $z \mapsto c^{-1}z$, $a \mapsto \begin{pmatrix} c^{-1} & & \\ & 1 & \\ & & c \end{pmatrix} a$, and using the invariant property of f_s:

$$f_s(\gamma \cdot \begin{pmatrix} c^{-1} & & \\ & 1 & \\ & & c \end{pmatrix} \begin{pmatrix} 1 & z & \\ & 1 & \\ & & 1 \end{pmatrix} ak) = |c|_F^{-3s} f_s(\gamma \cdot \begin{pmatrix} 1 & z & \\ & 1 & \\ & & 1 \end{pmatrix} ak),$$

we can obtain that

$$Z(W_v^{\psi_c}, f_s) = |c|_F^{-3s+3} Z(W_v^{\psi}, f_s). \tag{4.13}$$

Therefore, if some qualitative assertions hold for $Z(W_v^{\psi}, f_s)$ (for example, convergence issue, meromorphic continuation), then similar assertions also hold for $Z(W_v^{\psi_c}, f_s)$.

When we vary the Whittaker models of π and consider all such local integrals $Z(W_v, f_s)$, all of these $Z(W_v, f_s)$ span a subspace of $\mathbb{C}(q^{-3s})$ which is closed under multiplication by q^{-3s} and q^{3s}. It is a fractional ideal containing 1 by Theorem 4.1.1. Thus, it is generated by a function $\frac{1}{P(q^{-3s})}$ for some polynomial P. We expect that

$\frac{1}{P(q^{-3s})}$ is essentially the adjoint L-function for π, in the sense of Langlands. We will consider this problem in a future work.

When F is archimedean, we can apply the same trick. The only difference comes from (3.6). For any $c \in F^\times$, we write $c = |c|c_0$, then c_0 lies inside the abelian compact group S^1, hence $\begin{pmatrix} c_0 & & \\ & 1 & \\ & & c_0^{-1} \end{pmatrix}$ lies inside the maximal compact subgroup. In (3.6), we assume that t_1, t_2 are both positive real numbers. To apply the trick, we only need to change variables $z \mapsto c^{-1}z$, $a \mapsto \begin{pmatrix} |c|^{-1} & & \\ & 1 & \\ & & |c| \end{pmatrix} a$ and $k \mapsto \begin{pmatrix} |c_0|^{-1} & & \\ & 1 & \\ & & |c_0| \end{pmatrix} k$. Therefore, (4.13) also holds in the archimedean cases. Thus, we can conclude from (4.13) that the set of poles of the local integrals is independent on the choice of ψ.

From now on, without further emphasis, we will always assume when F is archimedean that

$$\psi(x) = \begin{cases} e^{2\pi i x} & \text{if } F = \mathbb{R}; \\ e^{2\pi i (x+\overline{x})} & \text{if } F = \mathbb{C}. \end{cases} \tag{4.14}$$

Chapter 5

Asymptotic Expansion of the Whittaker Functions (Archimedean Cases)

In this Chapter, we assume that F is either $\mathbb{R}$ or $\mathbb{C}$. Because the local representation π satisfies the moderate growth property, for any a in the torus of $\mathrm{SL}_3(F)$, the Whittaker function $W_v(a)$ behave like a Schwartz function when a approaches to infinity. Thus, the asymptotic behaviour of $W_v(a)$ near 0 should be responsible for the poles of the local integrals $A(W_v, f_s)$.

Recall that in Chapter 3, we use the coarse asymptotic expansion (3.10) to prove the convergence theorem. The Schwartz functions φ_ξ in (3.10) depend on the vector $v \in V_\pi$. Unfortunately, when we vary $v \in V_\pi$, we can not see how all the φ_ξ change accordingly. In this Chapter, we will follow Soudry's method (see [Sou]) closely and obtain an asymptotic expansion of $W_v(a)$ near 0. We will see that each term in the desired asymptotic expansion in this Chapter is continuous in v. This result is the archimedean analogue of Proposition 3.2.2.

5.1 First Step

We first fix a norm on $\mathrm{GL}_3(F)$ (see Section 2.2): for any real or complex matrix g, define its Euclidean norm

$$\|g\|_e = \mathrm{Tr}(g \cdot \overline{g}^t)^{\frac{1}{2}} = (\sum |g_{ij}|^2)^{\frac{1}{2}}.$$

Moreover, when $g \in \mathrm{GL}_3(F)$, we define its Harish-Chandra norm by

$$\|g\|_H = \|g\|_e^2 + \|g^{-1}\|_e^2.$$

In particular, when $g = \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix}$, $\|g\|_H = \sum_{i=1}^3 |a_i|^2 + |a_i|^{-2}$. It is easy to see that

$$\|\begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix}\begin{pmatrix} b_1 & & \\ & b_2 & \\ & & b_3 \end{pmatrix}\|_H \leq \|\begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix}\|_H \cdot \|\begin{pmatrix} b_1 & & \\ & b_2 & \\ & & b_3 \end{pmatrix}\|_H.$$

Clearly the Harish-Chandra norm is bi-K_{GL_3} invariant. In the following, we will always use Harish-Chandra norm and drop subscript H. As in the Introduction, we assume that π is an irreducible generic representation of $\mathrm{GL}_3(F)$. If $\pi = \mathrm{Ind}_{B_{\mathrm{GL}_3}}^{\mathrm{GL}_3(F)} \eta$ is a principal series, where B_{GL_3} is the standard Borel subgroup of $\mathrm{GL}_3(F)$ and η a character of $B^-_{\mathrm{GL}_3}$ defined by

$$\eta(\begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix}) := \prod_{i=1}^{3} |t_i|_F^{u_i} \chi_i(t_i),$$

then we also write $\pi = \pi_u, W_v = W_{v,u}$ to emphasize the dependence on the complex parameters $u = (u_1, u_2, u_3) \in \mathbb{C}^3$.

By the continuity of the Whittaker functional, there exist a $\mu > 0$ and a continuous seminorm q on V_π such that

$$|W_v(g)| \leq \|g\|^\mu \cdot q(v), \tag{5.1}$$

for any $v \in V_\pi$. If a compact set $\Omega \in \mathbb{C}^3$ is given, then a similar inequality

$$|W_{v,u}(g)| \leq \|g\|^\mu \cdot q(v) \tag{5.2}$$

also holds for all $v \in V_{\pi_u}$, $u \in \Omega$. The constant μ and seminorm q in (5.2) only depends on Ω.

Put

$$H_0 = -\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad H_1 = -\begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}, \quad H_2 = -\begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix},$$

$$X_1 = \begin{pmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & & \\ & 0 & 1 \\ & & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & & 1 \\ & 0 & \\ & & 0 \end{pmatrix},$$

and set $\alpha_1 = \epsilon_1 - \epsilon_2$, $\alpha_2 = \epsilon_2 - \epsilon_3$. Then

$$\alpha_1(H_0) = \alpha_2(H_0) = 0, \alpha_1(H_1) = \alpha_2(H_2) = -1, \alpha_1(H_2) = \alpha_2(H_1) = 0.$$

For $m = 1, 2$, let P_m be the standard parabolic subgroup of $\mathrm{GL}_3(F)$ of type $(m, 3-m)$ with the Levi decomposition $P_m = M_m U_m$. Denote by V_0 ($V_{0,u}$ resp.) the underlying $(\mathfrak{gl}_3, K_{\mathrm{GL}_3})$-module of V_π (V_{π_u} resp.). Set $K_m = M_m \cap K_{\mathrm{GL}_3}$. By [Wal1, Section 4.3.1], for any positive integer k, $V_0/\mathrm{Lie}(U_m)^k V_0$ is a finitely generated admissible $(\mathrm{Lie}(M_m), K_m)$-module, and it decomposes into finitely many generalized H_m-eigenspaces

$$V_0/\mathrm{Lie}(U_m)^k V_0 = \bigoplus_\xi \big(V_0/\mathrm{Lie}(U_m)^k V_0\big)_\xi,$$

where

$$\big(V_0/\mathrm{Lie}(U_m)^k V_0\big)_\xi := \{\overline{v} \in V_0/\mathrm{Lie}(U_m)^k V_0 \,|\, (\pi(H_m) - \xi)^d \overline{v} = 0 \text{ for some } d > 0\}.$$

Denote by $E_k^{(m)}$ ($E_{k,u}^{(m)}$ resp.) the finite set of generalized eigenvalues of H_m on $V_0/\mathrm{Lie}(U_m)^k V_0$ ($V_{0,u}/\mathrm{Lie}(U_m)^k V_{0,u}$ resp.). In particular, $E_1^{(m)}$ ($E_{1,u}^{(m)}$ resp.) is the finite set of generalized eigenvalues of H_m on $V_0/\mathrm{Lie}(U_m)V_0$ ($V_{0,u}/\mathrm{Lie}(U_m)V_{0,u}$ resp.). Define

$$E^{(m)} := \bigcup_{k=1}^{+\infty} E_k^{(m)}, \quad E_u^{(m)} := \bigcup_{k=1}^{+\infty} E_{k,u}^{(m)}.$$

By [Wal1, Section 4.4.2],

$$E^{(m)} \subseteq \{\xi - n \,|\, \xi \in E_1^{(m)}, n \in \mathbb{N}\}, \quad E_u^{(m)} \subseteq \{\xi - n \,|\, \xi \in E_{1,u}^{(m)}, n \in \mathbb{N}\}.$$

Moreover, by [Wal2, Section 12.4.6 and 12.4.7], $E_{1,u}^{(m)}$ consists of polynomials of u_1, u_2, u_3 of degree 1.

In [Wal2, Section 15.2.4], Wallach proves the following lemma (we only state the lemma in the case of $\mathfrak{gl}_3$).

Lemma 5.1.1. *For each $m \in \{1, 2\}$, and $k_m \in \mathbb{N}$, there exists an integer $N(m) \in \mathbb{N}$, a finite set $\{e_i^{(m)}\}_{i=1}^{N(m)} \subseteq U(\mathfrak{gl}_3(\mathbb{C}))$, a finite set $\{D_{r,i}^{(m)}\} \subseteq U(\mathfrak{gl}_3(\mathbb{C}))$, where r indexes the basis of monomials $X_r^{(m)}$ in $\mathrm{Lie}(U_m)^{k_m}$, (for example, if $m = 1$, $X_r^{(1)}$ is of the form $X_3^{l_3} X_2^{l_2}$) and $i = 1, 2, \cdots, N(m)$, such that for any $(\mathfrak{gl}_3, K_{\mathrm{GL}_3})$-module V_0, there exists an $N(m) \times N(m)$ matrix $B^{(m,k_m)} = (B_{ij}^{(m,k_m)})$ depending on V_0 such that*

1. $e_1^{(m)} = 1$,
2. *For any $v \in V_0$,*

$$\pi(H_m)\pi(e_i^{(m)})v = \sum_j B_{ij}^{(m,k_m)} \pi(e_j^{(m)})v + \sum_r \pi(X_r^{(m)})\pi(D_{r,i}^{(m)})v. \tag{5.3}$$

If we take V_0 to be the $(\mathfrak{gl}_3, K_{GL_3})$-module of π_u in the above lemma, then we write

$$B^{(m,k_m)} = B_u^{(m,k_m)} = (B_{ij,u}^{(m,k_m)})$$

to emphasize the dependence on the complex parameter u. If we fix one k_m, we also omit the superscript k_m and write $B^{(m)}$($B_u^{(m)}$ resp.) for $B^{(m,k_m)}$ ($B_u^{(m,k_m)}$ resp.).

It is clear from Lemma 5.1.1 that the projection of $\mathrm{Span}\{\pi(e_i^{(m)})v\}$ in $V_0/\mathrm{Lie}(U_m)^{k_m}V_0$ is a finite dimensional vector space and H_m acts on it according to the matrix $B^{(m,k_m)}$. All of the generalized eigenvalues of $B_u^{(m,k_m)}$ are contained in $E_{k_m,u}^{(m)}$. In particular, all of the generalized eigenvalues of $B_u^{(m,k_m)}$ are polynomials of u of degree 1. Moreover by [Wal2, Section 15.2.4], $B_u^{(m,k_m)}$ are rational in u_1, u_2, u_3.

We label all elements in $E^{(m)} = \{\xi_j^{(m)} \,|\, j = 1, 2, \cdots\}$ in the descending order, i.e.

$$\mathrm{Re}(\xi_1^{(m)}) \geq \mathrm{Re}(\xi_2^{(m)}) \geq \cdots .$$

We also write $\xi_{j,u}^{(m)}$ when we emphasize the dependence on parameters. In this Chapter, we fix two numbers ξ_m for $m = 1, 2$. Later, when we prove Theorem 1.0.11, we will choose ξ_1, ξ_2 as negative as we want to cover enough generalized eigenvalues.

Theorem 5.1.2. *Fix $\xi_2 \in \mathbb{C}$. Suppose $V_0 = V_{0,u}$, where u runs in a fixed closed ball $\Omega = B(u_0, r_0) \in \mathbb{C}^3$ (u_0 is the center) with a radius $r_0 > 0$. Let k_2 be a sufficiently large positive integers such that*

$$-Re(\xi_2) - k_2 + 2\mu < -1.$$

There is a finite set $C_u^{(2)} \subseteq \bigcup_{j=1}^{k_2} E_{j,u}^{(2)}$, and a finite set $\mathcal{L}$ of nonnegative integers, a finite subset $\mathcal{P}_u \subseteq \mathbb{C}(u)[t]$ and a finite subset $\mathcal{D} \subseteq U(\mathfrak{gl}_3(\mathbb{C}))$ such that for all $v \in V_{0,u}$, $x_2 \geq 0$, $W_{v,u}(e^{x_1 H_1 + x_2 H_2})$ is a finite linear combination of terms of the following types:

1. $e^{x_2 \xi_u} p_u(x_2) W_{\pi(e)v,u}(e^{x_1 H_1})$,
2. $e^{x_2 \xi_u} h_u(x_2) \int_0^{x_2} e^{(-\xi_u - k_2)t} t^l W_{\pi(D)v,u}(e^{x_1 H_1 + t H_2}) dt := h_u(x_2) \widetilde{\phi}_{l,e,D}(x_1, x_2, v, u)$,
3. $e^{x_2 \xi_u} h_u(x_2) \int_0^{+\infty} e^{(-\xi_u - k_2)t} t^l W_{\pi(D)v,u}(e^{x_1 H_1 + t H_2}) dt := e^{x_2 \xi_u} h_u(x_2) \phi_{l,e,D}(x_1, x_2, v, u)$,
4. $e^{x_2 \xi_u} h_u(x_2) \int_{x_2}^{+\infty} e^{(-\xi_u - k_2)t} t^l W_{\pi(D)v,u}(e^{x_1 H_1 + t H_2}) dt := h_u(x_2) \widetilde{\phi}_{l,e,D}(x_1, x_2, v, u)$,

where $p_u, h_u \in \mathcal{P}_u$, $e, D \in \mathcal{D}$, and $\xi_u \in C_u^{(2)}$ are polynomials of u of degree 1. For each $\xi_u \in C_u^{(2)}$, we decompose $\Omega = \Omega_{\xi_u,1} \bigcup \Omega_{\xi_u,2}$ where

$$\Omega_{\xi_u,1} := \{u \in \Omega \,|\, -k_2 - Re(\xi_u) + 2\mu \geq -\frac{1}{2}\},$$
$$\Omega_{\xi_u,2} := \{u \in \Omega \,|\, -k_2 - Re(\xi_u) + 2\mu \leq -\frac{1}{3}\}.$$

If $u \in \Omega_{\xi_u,1}$, then we only need terms of type (1) and type (2). The function $\phi_{l,e,D}(x_1, x_2, v, u)$ defined in type (2) is holomorphic in u and uniformly continuous *in x_1, x_2, v when u runs in $\Omega_{\xi_u,1}$. It satisfies the following estimate*

$$\left|\widetilde{\phi}_{l,e,D}(x_1, x_2, v, u)\right| \leq e^{Re(\xi_2)x_2} \cdot \widetilde{\delta}_l(x_2) \cdot \|e^{x_1 H_1}\|^{\mu} \cdot q'(v) \tag{5.4}$$

where q' is a continuous seminorm and $\widetilde{\delta}_l(t) = \frac{t^{l+1}}{l+1}$.

If $u \in \Omega_{\xi_u,2}$, we only need terms of type (1), (3) and (4). The functions $\phi_{l,e,D}(x_1, x_2, v, u)$ and $\widetilde{\phi}_{l,e,D}(x_1, x_2, v, u)$ are also holomorphic in $u \in \Omega_{\xi_u,2}$ and uniformly continuous in x_1, x_2, v when u runs in $\Omega_{\xi_u,2}$. They satisfy the following estimates:

$$\left|\phi_{l,e,D}(x_1, x_2, v, u)\right| \leq \delta_l(x_2) \cdot \|e^{x_1 H_1}\|^{\mu} \cdot q'(v); \tag{5.5}$$

$$\left|\widetilde{\phi}_{l,e,D}(x_1, x_2, v, u)\right| \leq e^{Re(\xi_2)x_2}\widetilde{\delta}_l(x_2) \cdot \|e^{x_1 H_1}\|^{\mu} \cdot q'(v), \tag{5.6}$$

where q' is a continuous seminorm, $\delta_l(t)$ is the constant function

$$\delta_l(t) = \int_0^{+\infty} e^{-\frac{1}{3}t} t^l dt$$

and $\widetilde{\delta}_l(t)$ is a polynomial of t of degree l with constant coefficients coming from integration by parts.

Same expansions and estimates also hold if we drop the subscript u and consider the underlying $(\mathfrak{gl}_3, K_{\mathrm{GL}_3})$-module for all irreducible generic Casselman-Wallach representations.

Proof of Theorem 5.1.2. We only look at the case when $\pi = \pi_u$ and keep track on the dependence on the parameters u. A word by word argument also works for all irreducible admissible generic Casselman-Wallach representations π when we drop the subscript u.

For k_2, we choose $N(2) \in \mathbb{N}$, $\{e_i^{(2)}\}_{i=1}^{N(2)}$, $\{D_{r,i}^{(2)}\}$, $B_u^{(2)}$ as in Lemma 5.1.1. Put

$$\vec{F}_u(x_1, t, v) := \begin{pmatrix} W_{\pi(e_1^{(2)})v,u}(e^{x_1 H_1 + tH_2}) \\ \cdots \\ W_{\pi(e_{N(2)}^{(2)})v,u}(e^{x_1 H_1 + tH_2}) \end{pmatrix},$$

and

$$\vec{G}_u(x_1,t,v) := \begin{pmatrix} \sum_r W_{\pi(X_r^{(2)})\pi(D_{r,1}^{(2)})v,u}(e^{x_1H_1+tH_2}) \\ \cdots \\ \sum_r W_{\pi(X_r^{(2)})\pi(D_{r,N(2)}^{(2)})v,u}(e^{x_1H_1+tH_2}) \end{pmatrix}.$$

According to (5.3), it is clear that

$$\begin{aligned} &\frac{d}{dt}W_{\pi(e_i^{(2)})v,u}(e^{x_1H_1+tH_2}) \\ =&\sum_j B_{ij,u}^{(2)} \cdot W_{\pi(e_j^{(2)})v,u}(e^{x_1H_1+tH_2}) + \sum_r W_{\pi(X_r^{(2)})\pi(D_{r,i}^{(2)})v,u}(e^{x_1H_1+tH_2}), \end{aligned}$$

i.e.

$$\frac{d}{dt}\vec{F}_u(x_1,t,v) = B_u^{(2)}\vec{F}_u(x_1,t,v) + \vec{G}_u(x_1,t,v). \tag{5.7}$$

Note that $\mathrm{Lie}(U_2)$ is spanned by X_3 and X_2, so every $X_r^{(2)}$ is of the form $X_3^{l_3}X_2^{l_2}$ with $l_3+l_2=k_2$. If $l_3=0$, in other words $l_2=k_2$ and $X_r^{(2)}=X_2^{k_2}$, we define $\widetilde{D}_{r,i}^{(2)} = D_{r,i}^{(2)}$. For simplicity, we write $\frac{d}{d\vec{s}}\Big|_{\vec{s}=\vec{0}}$ for the differential operator $\frac{d}{ds_1}\frac{d}{ds_2}\cdots\frac{d}{ds_{k_2}}\Big|_{s_1=s_2=\cdots=s_{k_2}=0}$. Then We have

$$\begin{aligned} &W_{\pi(X_2^{k_2})\pi(D_{r,i}^{(2)})v,u}(e^{x_1H_1+tH_2}) \\ =&\frac{d}{d\vec{s}}\Big|_{\vec{s}=\vec{0}} W_{\pi(D_{r,i}^{(2)})v,u}(e^{x_1H_1+tH_2}\begin{pmatrix}1&&\\&1&s_1\\&&1\end{pmatrix}\begin{pmatrix}1&&\\&1&s_2\\&&1\end{pmatrix}\cdots\begin{pmatrix}1&&\\&1&s_{k_2}\\&&1\end{pmatrix}) \\ =&\frac{d}{d\vec{s}}\Big|_{\vec{s}=\vec{0}} W_{\pi(D_{r,i}^{(2)})v,u}(\begin{pmatrix}1&&\\&1&e^{-t}s_1\\&&1\end{pmatrix}\begin{pmatrix}1&&\\&1&e^{-t}s_2\\&&1\end{pmatrix}\cdots\begin{pmatrix}1&&\\&1&e^{-t}s_{k_2}\\&&1\end{pmatrix}e^{x_1H_1+tH_2}) \\ =&\frac{d}{d\vec{s}}\Big|_{\vec{s}=\vec{0}} \psi(e^{-t}s_1)\psi(e^{-t}s_2)\cdots\psi(e^{-t}s_{k_2})W_{\pi(D_{r,i}^{(2)})v,u}(e^{x_1H_1+tH_2}). \end{aligned}$$

When $F=\mathbb{R}$, $\psi(x)=e^{icx}$ for some $c\in\mathbb{R}$, then

$$\frac{d}{ds}\Big|_{s=0}\psi(e^{-t}s) = ic\cdot e^{-t};$$

when $F=\mathbb{C}$, similar things happen. Hence in both real and complex cases, there exists $C_\psi \in \mathbb{C}$, only depending on ψ and F, such that

$$W_{\pi(X_r^{(2)})\pi(D_{r,i}^{(2)})v,u}(e^{x_1H_1+tH_2}) = C_\psi e^{-k_2t} W_{\pi(\widetilde{D}_{r,i}^{(2)})v,u}(e^{x_1H_1+tH_2})$$

By rescaling $\widetilde{D}_{r,i}^{(2)}$, we rewrite

$$W_{\pi(X_r^{(2)})\pi(D_{r,i}^{(2)})v,u}(e^{x_1H_1+tH_2}) = e^{-k_2t} W_{\pi(\widetilde{D}_{r,i}^{(2)})v,u}(e^{x_1H_1+tH_2}) \tag{5.8}$$

If $l_3>0$, we define $\widetilde{D}_{r,i}^{(2)}=0$. We have

$$\begin{aligned}
&W_{\pi(X_r^{(2)})\pi(D_{r,i}^{(2)})v,u}(e^{x_1H_1+tH_2})\\
=&\frac{d}{ds}\bigg|_{s=0} W_{\pi(X_3^{l_3-1}X_2^{l_2})\pi(D_{r,i}^{(2)})v,u}(e^{x_1H_1+tH_2}e^{sX_3})\\
=&\frac{d}{ds}\bigg|_{s=0} W_{\pi(X_3^{l_3-1}X_2^{l_2})\pi(D_{r,i}^{(2)})v,u}(\begin{pmatrix}1 & & e^{-x_1-t}s\\ & 1 & \\ & & 1\end{pmatrix} e^{x_1H_1+tH_2})\\
=&0.
\end{aligned}$$

Hence (5.8) also hold in this case. Set

$$\vec{R}_u(x_1,t,v) := \begin{pmatrix} W_{\pi(\widetilde{D}_{r,1}^{(2)})v,u}(e^{x_1H_1+tH_2}) \\ \cdots \\ W_{\pi(\widetilde{D}_{r,N(2)}^{(2)})v,u}(e^{x_1H_1+tH_2}) \end{pmatrix}.$$

Thus

$$\vec{G}_u(x_1,t,v) = e^{-k_2t}\vec{R}_u(x_1,t,v). \tag{5.9}$$

The solution to the differential equation (5.7) is

$$\vec{F}_u(x_1,x_2,v) = e^{x_2B_u^{(2)}}\vec{F}_u(x_1,0,v) + \int_0^{x_2} e^{(x_2-t)B_u^{(2)}}\vec{G}_u(x_1,t,v)dt \tag{5.10}$$

We compare the first coordinates of both sides in (5.10). The first coordinate on the

LHS of (5.10) is $W_v(e^{x_1H_1+x_2H_2})$. Since $B_u^{(2)}$ is rational in u, each entry of the matrix $e^{x_2B_u^{(2)}}$ is a linear combination

$$e^{x_2\xi_u}p_u(x_2),$$

where $p_u(x_2)$ belongs to a finite subset of $\mathbb{C}(u)[x_2]$, and ξ_u lies in a finite subset of $\bigcup_{i=1}^{k_2} E_{i,u}^{(2)}$. Thus, the first coordinate of the first term on the right hand side of (5.10) is a finite linear combination of

$$e^{x_2\xi}p_u(x_2)W_{\pi(e_i)}(e^{x_1H_1}), \tag{5.11}$$

where ξ belongs to a finite subset of $\bigcup_{i=1}^{k_2} E_{i,u}^{(2)}$ consisting of polynomials of degree 1, and $p_u(t)$ belongs to a finite subset of $\mathbb{C}(u)[t]$.

By (5.9), the first coordinate of the second term on the RHS of (5.10) is a finite linear combination of

$$\int_0^{x_2} e^{(x_2-t)\xi_u}p_u(x_2-t)\cdot e^{-k_2t}W_{\pi(\widetilde{D}_{r,i}^{(2)})v,u}(e^{x_1H_1+tH_2})dt,$$

where ξ_u, $p_u(t)$ satisfy the same condition as above. After expanding the polynomials $p_u(x_2-t)$ into monomials, we can see that the first coordinate of the second term on the RHS of (5.10) is a finite linear combination of

$$e^{x_2\xi_u}h_u(x_2)\int_0^{x_2} e^{(-\xi_u-k_2)t}t^lW_{\pi(D)v,u}(e^{x_1H_1+tH_2})dt, \tag{5.12}$$

where $h_u(t)$ belongs to a finite subset of $\mathbb{C}(u)[t]$, ξ_u lies in a finite subset of $\bigcup_{i=1}^{k_2} E_{i,u}^{(2)}$, l runs in a finite set $\mathcal{L}$ of nonnegative integers, and D belongs to a finite set $\{\widetilde{D}_{r,i}^{(2)}\}$. We also note that $\mathcal{L}$ only depends on the size of $B_u^{(2)}$, which is independent on the parameter u.

Since the continuous function $-\mathrm{Re}(\xi_u)$ is bounded on the compact set Ω, we may assume that $-\mathrm{Re}(\xi_u) < C$ for some constant C. Then the integral in (5.12) admits the following estimate

$$
\begin{aligned}
&|\int_0^{x_2} e^{(-\xi_u-k_2)t}t^l W_{\pi(D)v}(e^{x_1H_1+tH_2})dt| \\
\leq &\int_0^{x_2} e^{(-\mathrm{Re}(\xi_u)-k_2)t}t^l\|e^{x_1H_1+tH_2}\|^\mu dt\cdot q(\pi(D)v) \\
\leq &\int_0^{x_2} e^{(C-k_2)t}t^l\|e^{x_1H_1}\|^\mu(2e^{2t}+2e^{-2t}+2)^\mu dt\cdot q(\pi(D)v) \\
\leq &6^\mu\int_0^{x_2} e^{(C-k_2+2\mu)t}t^l\|e^{x_1H_1}\|^\mu dt\cdot q(\pi(D)v) \\
= &\int_0^{x_2} e^{(C-k_2+2\mu)t}t^l dt\cdot\|e^{x_1H_1}\|^\mu\cdot q'(v)
\end{aligned}
\tag{5.13}
$$

for some continuous seminorm q'. Thus as a function of x_1, x_2, u, v, the integral

$$\int_0^{x_2} e^{(-\xi_u-k_2)t}t^l W_{\pi(D)v,u}(e^{x_1H_1+tH_2})dt$$

is holomorphic in $u\in\Omega$ and continuous in x_1, x_2, v (also uniformly continuous when u runs in Ω).

If $u\in\Omega_{\xi_u,1}$, we can use a similar estimate and get

$$
\begin{aligned}
&|\int_0^{x_2} e^{(-\xi_u-k_2)t}t^l W_{\pi(D)v}(e^{x_1H_1+tH_2})dt| \\
\leq &6^\mu\int_0^{x_2} e^{(-\mathrm{Re}(\xi_u)-k_2+2\mu)t}t^l\|e^{x_1H_1}\|^\mu dt\cdot q(\pi(D)v) \\
= &\int_0^{x_2} e^{(-\mathrm{Re}(\xi_u)-k_2+2\mu+\frac{1}{2})t}\cdot e^{-\frac{1}{2}t}t^l dt\cdot\|e^{x_1H_1}\|^\mu\cdot q'(v) \\
\leq &\int_0^{x_2} e^{(-\mathrm{Re}(\xi_u)-k_2+2\mu+\frac{1}{2})x_2}\cdot t^l dt\cdot\|e^{x_1H_1}\|^\mu\cdot q'(v) \\
= &e^{(-\mathrm{Re}(\xi_u)-k_2+2\mu+\frac{1}{2})x_2}\cdot\widetilde{\delta}_l(x_2)\cdot\|e^{x_1H_1}\|^\mu\cdot q'(v) \\
\leq &e^{(\mathrm{Re}(\xi_2)-\mathrm{Re}(\xi_u))x_2}\cdot\widetilde{\delta}_l(x_2)\cdot\|e^{x_1H_1}\|^\mu\cdot q'(v)
\end{aligned}
\tag{5.14}
$$

for some continuous seminorm q' and $\widetilde{\delta}_l(t)=\frac{t^{l+1}}{l+1}$.

We now claim that the integral

$$\int_0^{+\infty} e^{(-\xi_u-k_2)t}t^l W_{\pi(D)v}(e^{x_1H_1+tH_2})dt \tag{5.15}$$

converges absolutely when $u \in \Omega_{\xi_u,2}$. Indeed, by imitating the estimate (5.14), we can show that the integrand in (5.15) is dominated by

$$e^{(-\mathrm{Re}(\xi_u)-k_2)t}t^l \cdot \|e^{x_1H_1}\| \cdot 6^\mu e^{2\mu t}q_1(v)$$

for some continuous seminorm q_1. Thus,

$$\begin{aligned}|\int_0^{+\infty} e^{(-\xi_u-k_2)t}t^l W_{\pi(D)v}(e^{x_1H_1+tH_2})dt| &\leq \int_0^{+\infty} e^{(-\mathrm{Re}(\xi_u)-k_2+2\mu)t}t^l dt \cdot \|e^{x_1H_1}\|^\mu \cdot q'(v)\\ &\leq \delta_l(x_2) \cdot \|e^{x_1H_1}\|^\mu \cdot q'(v),\end{aligned} \tag{5.16}$$

where q' is a continuous seminorm and $\delta_l(t)$ is the constant function

$$\int_0^{+\infty} e^{-\frac{1}{3}t}t^l dt.$$

Hence our claim follows. Moreover, the estimate (5.16) also shows that as a function of x_1, u, v, the integral (5.15) is holomorphic in u and uniformly continuous in x_1, v when u runs in $\Omega_{\xi_u,2}$.

For $u \in \Omega_{\xi_u,2}$, now we can write

$$\begin{aligned}&\int_0^{x_2} e^{(-\xi_u-k_2)t}t^l W_{\pi(D)v}(e^{x_1H_1+tH_2})dt\\ =&\int_0^{+\infty} e^{(-\xi_u-k_2)t}t^l W_{\pi(D)v}(e^{x_1H_1+tH_2})dt - \int_{x_2}^{+\infty} e^{(-\xi_u-k_2)t}t^l W_{\pi(D)v}(e^{x_1H_1+tH_2})dt.\end{aligned}$$

By (5.16), each summand on the RHS of the above equation converges absolutely and defines a holomorphic function in u which is also uniformly continuous in x_1, x_2, v when u runs in $\Omega_{\xi_u,2}$. By using the same trick, we can check that the second summand above admits the following estimate

$$\begin{aligned}|\int_{x_2}^{+\infty} e^{(-\xi_u-k_2)t}t^l W_{\pi(D)v}(e^{x_1H_1+tH_2})dt| &\leq \int_{x_2}^{+\infty} e^{(-\mathrm{Re}(\xi_u)-k_2+2\mu)t}t^l dt \cdot \|e^{x_1H_1}\|^\mu \cdot q'(v)\\ &\leq e^{(-\mathrm{Re}(\xi_u)-k_2+2\mu)x_2}\widetilde{\delta}_l(x_2) \cdot \|e^{x_1H_1}\|^\mu \cdot q'(v)\end{aligned} \tag{5.17}$$

where $\widetilde{\delta}_l(t)$ is a polynomial of t of degree l coming from integration by parts. Thus,

$$\begin{aligned}\big|e^{x_2\xi_u}\int_{x_2}^{+\infty}e^{(-\xi_u-k_2)t}t^lW_{\pi(D)v}(e^{x_1H_1+tH_2})dt\big| \leq & e^{(-k_2+2\mu)x_2}\widetilde{\delta}_l(x_2)\cdot\|e^{x_1H_1}\|^{\mu}\cdot q'(v)\\ \leq & e^{\mathrm{Re}(\xi_2)x_2}\widetilde{\delta}_l(x_2)\cdot\|e^{x_1H_1}\|^{\mu}\cdot q'(v)\end{aligned} \tag{5.18}$$

□

We can also fix ξ_1 and prove a similar theorem by switching the role of x_1 and x_2. We only state the result. The proof is exactly the same as that of Theorem 5.1.2.

Theorem 5.1.3. *Fix $\xi_1\in\mathbb{C}$. Suppose $V_0=V_{0,u}$, where u runs in a fixed closed ball $\Omega=B(u_0,r_0)\in\mathbb{C}^3$ (u_0 is the center) with a radius $r_0>0$. Let k_1 be a sufficiently large positive integers such that*

$$-Re(\xi_1)-k_1+2\mu<-1.$$

There is a finite set $C_u^{(1)}\subseteq\bigcup_{j=1}^{k_1}E_{j,u}^{(2)}$, and a finite set $\mathcal{L}$ of nonnegative integers, a finite subset $\mathcal{P}_u\in\mathbb{C}(u)[t]$ and a finite subset $\mathcal{D}\in U(\mathfrak{gl}_3(\mathbb{C}))$ such that for all $v\in V_{0,u}$, $x_1\geq 0$, $W_{v,u}(e^{x_1H_1+x_2H_2})$ is a linear combination of terms of the following types:

1. $e^{x_1\eta_u}p_u(x_1)W_{\pi(e)v,u}(e^{x_2H_2})$,
2. $e^{x_1\eta_u}h_u(x_1)\int_0^{x_1}e^{(-\eta_u-k_1)r}r^lW_{\pi(D)v,u}(e^{x_2H_2+rH_1})dr:=h_u(x_1)\widetilde{\phi}_{l,e,D}(x_1,x_2,v,u)$,
3. $e^{x_1\eta_u}h_u(x_1)\int_0^{+\infty}e^{(-\eta_u-k_1)r}r^lW_{\pi(D)v,u}(e^{x_2H_2+rH_1})dr:=e^{x_1\eta_u}h_u(x_1)\phi_{l,e,D}(x_1,x_2,v,u)$,
4. $e^{x_1\eta_u}h_u(x_1)\int_{x_1}^{+\infty}e^{(-\eta_u-k_1)r}r^lW_{\pi(D)v,u}(e^{rH_1+x_2H_2})dr:=h_u(x_1)\widetilde{\phi}_{l,e,D}(x_1,x_2,v,u)$,

where $p_u,h_u\in\mathcal{P}_u$, $e,D\in\mathcal{D}$, and $\eta_u\in C_u^{(1)}$ are polynomials of u of degree 1. For each $\eta_u\in C_u^{(1)}$, we decompose $\Omega=\Omega_{\eta_u,1}\bigcup\Omega_{\eta_u,2}$ where

$$\Omega_{\eta_u,1}:=\{u\in\Omega\,|\,-k_1-Re(\eta_u)+2\mu\geq-\frac{1}{2}\};$$
$$\Omega_{\eta_u,2}:=\{u\in\Omega\,|\,-k_1-Re(\eta_u)+2\mu\leq-\frac{1}{3}\}.$$

If $u\in\Omega_{\eta_u,1}$, then we only need terms of type (1) and type (2). The function $\phi_{l,e,D}(x_1,x_2,v,u)$ defined in type (2) is holomorphic in u and uniformly continuous in

x_1, x_2, v when u runs in $\Omega_{\eta_u,1}$. It satisfies the following estimate

$$\left|\widetilde{\phi}_{l,e,D}(x_1,x_2,v,u)\right| \leq e^{Re(\xi_1)x_1} \cdot \widetilde{\delta}_l(x_1) \cdot \|e^{x_2 H_2}\|^{\mu} \cdot q'(v) \tag{5.19}$$

where q' is a continuous seminorm and $\widetilde{\delta}_l(t) = \frac{t^{l+1}}{l+1}$.

If $u \in \Omega_{\eta_u,2}$, we only need terms of type (1), (3) and (4). The functions $\phi_{l,e,D}(x_1,x_2,v,u)$ and $\widetilde{\phi}_{l,e,D}(x_1,x_2,v,u)$ are also holomorphic in $u \in \Omega_{\eta_u,2}$ and uniformly continuous in x_1, x_2, v when u runs in $\Omega_{\eta_u,2}$. They satisfy the following estimates:

$$\left|\phi_{l,e,D}(x_1,x_2,v,u)\right| \leq \delta_l(x_2) \cdot \|e^{x_1 H_1}\|^{\mu} \cdot q'(v); \tag{5.20}$$

$$\left|\widetilde{\phi}_{l,e,D}(x_1,x_2,v,u)\right| \leq e^{Re(\xi_2)x_2}\widetilde{\delta}_l(x_2) \cdot \|e^{x_1 H_1}\|^{\mu} \cdot q'(v); \tag{5.21}$$

where q' is a continuous seminorm, $\delta_l(t)$ is the constant function

$$\delta_l(t) = \int_0^{+\infty} e^{-\frac{1}{3}t} t^l dt$$

and $\widetilde{\delta}_l(t)$ is a polynomial of t of degree l with constant coefficients coming from integration by parts.

Same expansions and estimates also hold if we drop the subscript u and consider the underlying $(\mathfrak{gl}_3, K_{GL_3})$-module for all irreducible generic Casselman-Wallach representations.

5.2 Second Step

From Theorem 5.2.2, we can see that certain summand in the asymptotic expansion will only appear for some certain u (not all $u \in \Omega$). Hence to shorten the statement of our theorem, we say

Definition 5.2.1. *A summand S in the asymptotic expansion of the Whittaker function is holomorphic in $u \in \Omega$ and uniformly continuous with respect to the other variables when u runs in Ω, if S appears when u lies inside a subset $\Omega_0 \subseteq \Omega$, and it is holomorphic in $u \in \Omega_0$ and uniformly continuous with respect to the other variables when u runs in Ω_0.*

The following is the main theorem of this Section.

Theorem 5.2.2. *Fix ξ_1, ξ_2. Suppose $V_0 = V_{0,u}$ where u runs in a fixed closed ball $\Omega = B(u_0, r_0) \in \mathbb{C}^3$ (u_0 is the center) with a radius $r_0 > 0$. Let k_1, k_2 be two sufficiently large positive integers such that*

$$-Re(\xi_1) - k_1 + 2\mu < -1,$$
$$-Re(\xi_2) - k_2 + 2\mu < -1.$$

Then there are finite sets $C_u^{(1)} \subseteq \bigcup_{j=1}^{k_1} E_j^{(1)}$ and $C_u^{(2)} \subseteq \bigcup_{j=1}^{k_2} E_j^{(2)}$, and a finite set $\mathcal{P}_u \in \mathbb{C}(u)[t_1, t_2]$, a finite subset $\mathcal{D} \in U(\mathfrak{gl}_3(\mathbb{C}))$ and a finite subset $\mathcal{L}$ of nonnegative integers such that for all $v \in V_{0,u}$, $x_1, x_2 \geq 0$, $W_v(e^{x_1 H_1 + x_2 H_2})$ is a linear combination of terms of the following types:

1. $e^{x_1 \eta_u + x_2 \xi_u} P_u(x_1, x_2) f_0(v, u)$,
2. $e^{x_1 \eta_u} P_u(x_1, x_2) f_2(x_2, v, u)$,
3. $e^{x_2 \xi_u} P_u(x_1, x_2) f_1(x_1, v, u)$,
4. $P_u(x_1, x_2) f_3(x_1, x_2, v, u)$,

where $\eta_u \in C_u^{(1)}$, $\xi_u \in C_u^{(2)}$ are polynomials of u of degree 1, $P \in \mathcal{P}_u$, f_0, f_1, f_2, f_3 are holomorphic in u and uniformly continuous in x_1, x_2, v when u runs in Ω. They satisfy the following estimates

1. $|f_0(v, u)| \leq q'(v)$,
2. $|f_1(x_1, v, u)| \leq e^{Re(\xi_1) \cdot x_1} h_1(x_1) q'(v)$,
3. $|f_2(x_2, v, u)| \leq e^{Re(\xi_2) \cdot x_2} h_2(x_2) q'(v)$,
4. $|f_3(x_1, x_2, v, u)| \leq e^{Re(\xi_1) \cdot x_1 + Re(\xi_2) \cdot x_2} h_3(x_1, x_2) q'(v)$.

Here in the above, q' is a continuous seminorm on V_π and h_1, h_2, h_3 are polynomials with complex coefficients. Same statements also hold when we drop the subscript u and consider the underlying $(\mathfrak{gl}_3, K_{GL_3})$-module for all irreducible generic Casselman-Wallach representations.

Proof. For each $k_m (m = 1, 2)$, we choose $N(m) \in \mathbb{N}$, $\{e_i^{(m)}\}_{i=1}^{N(m)}$, $\{D_{r,i}^{(m)}\}$, $B_u^{(m)}$ as in Lemma 5.1.1, So far, we have proved that there exist finite sets $\mathcal{D}, \mathcal{P}_u, \mathcal{L}, C_u^{(2)}$ such that $W_{v,u}(e^{x_1H_1+x_2H_2})$ is a finite linear combination of

$$e^{x_2\xi_u} p_u(x_2) W_{\pi(e)v,u}(e^{x_1H_1})$$

and

$$e^{x_2\xi_u} h_u(x_2) \int_0^{x_2} e^{(-\xi_u - k_2)t} t^l W_{\pi(D)v,u}(e^{x_1H_1+tH_2}) dt.$$

Because $x_1 \geq 0$, by Theorem 5.1.3, we can expand $W_{v,u}(e^{x_1H_1+tH_2})$ with respect to x_1. To save notations, we combine the finite set of non-negative integers, the finite set of the elements in the Lie algebra etc. Then by Theorem 5.1.3, $W_{v,u}(e^{x_1H_1+tH_2})$ is a finite linear combination of terms of the following types:

1.
$$e^{x_1\eta_u} r_u(x_1) W_{\pi(e)v,u}(e^{tH_2}) \tag{5.22}$$

2.
$$e^{x_1\eta_u} r_u(x_1) \int_0^{x_1} e^{(-\eta_u - k_1)r} r^{l'} W_{\pi(D)v,u}(e^{rH_1+tH_2}) dr \tag{5.23}$$

3.
$$e^{x_1\eta_u} r_u(x_1) \int_0^{+\infty} e^{(-\eta_u - k_1)r} r^{l'} W_{\pi(D)v,u}(e^{rH_1+tH_2}) dr \tag{5.24}$$

4.
$$e^{x_1\eta_u} r_u(x_1) \int_{x_1}^{\infty} e^{(-\eta_u - k_1)r} r^{l'} W_{\pi(D)v,u}(e^{rH_1+tH_2}) dr \tag{5.25}$$

where $\eta_u \in C_u^{(1)}$, r_u belongs to a finite subset Q_u of $\mathbb{C}(u)[t]$, $l' \in \mathcal{L}$, $D, e \in \mathcal{D}$. In particular, if we set $t = 0$, we get an expansion of $W_{v,u}(e^{x_1H_1})$. Each term appears if u

satisfying the corresponding conditions in Theorem 5.1.3. Thus

$$e^{x_2\xi_u}p_u(x_2)W_{\pi(e)v,u}(e^{x_1H_1})$$

is a finite linear combination of

$$e^{x_2\xi_u+x_1\eta_u}p_u(x_2)r_u(x_1)\phi_{l,e,D}(x_1,u,v)$$

and

$$e^{x_2\xi_u}p_u(x_2)r_u(x_1)\widetilde{\phi}_{l,e,D}(x_1,u,v),$$

where $\phi_{l,e,D}(x_1,u,v), \widetilde{\phi}_{l,e,D}(u,v)$ are holomorphic in u and uniformly continuous in x_1, x_2, v when u runs in Ω. By the estimates in Theorem 5.1.3,

$$|\widetilde{\phi}_{l,e,D}(x_1,u,v)| \leq e^{\mathrm{Re}(\xi_1)x_1}h_1(x_1)q'(v)$$

and

$$|\phi_{l,e,D}(u,v)| \leq q'(v)$$

for some polynomial h_1 with constant coefficients and continuous seminorm q' (h_1, q' may be different in different inequalities). Thus all summands in the expansion of

$$e^{x_2\xi_u}p_u(x_2)W_{\pi(e)v,u}(e^{x_1H_1})$$

satisfy the properties in the theorem.

As in the proof of Theorem 5.1.2, for all $u \in \Omega$, $W_{v,u}(e^{x_1H_1+tH_2})$ is a finite linear combination of terms of the following types:

1. $$e^{x_1\eta_u}r_u(x_1)W_{\pi(e)v,u}(e^{tH_2})$$

2. $$e^{x_1\eta_u}r_u(x_1)\int_0^{x_1} e^{(-\eta_u-k_1)r}r^{l'}W_{\pi(D)v,u}(e^{rH_1+tH_2})dr.$$

Thus,

$$e^{x_2\xi_u}h_u(x_2)\int_0^{x_2} e^{(-\xi_u-k_2)t}t^lW_{\pi(D)v,u}(e^{x_1H_1+tH_2})dt$$

is a finite linear combination of terms of follow types

1.
$$e^{x_1\eta_u+x_2\xi_u}r_u(x_1)h_u(x_2)\int_0^{x_2}e^{(-\xi_u-k_2)t}t^l W_{\pi(e)v,u}(e^{tH_2})dt; \tag{5.26}$$

2.
$$e^{x_1\eta_u+x_2\xi_u}r_u(x_1)h_u(x_2)\int_0^{x_2}\int_0^{x_1}e^{(-\xi_u-k_2)t}t^l e^{(-\eta_u-k_1)r}r^{l'}W_{\pi(D)v,u}(e^{rH_1+tH_2})drdt. \tag{5.27}$$

The summands in the expansion of (5.26) can be analyzed in the exact same way as in (5.12). Thus those summands satisfy the properties in Theorem 5.2.2. It remains to deal with the integral (5.27). By using the same estimate method as in Theorem 5.1.2, it is easy to see that the integral (5.27) defines a holomorphic function in $u \in \Omega$ which is uniformly continuous in x_1, x_2, v when u runs in Ω. Let us define

$$\begin{aligned}
\Omega_{\xi_u,1} &:= \{u \in \Omega \,|\, -k_2 - \mathrm{Re}(\xi_u) + 2\mu \geq -\frac{1}{2}\};\\
\Omega_{\xi_u,2} &:= \{u \in \Omega \,|\, -k_2 - \mathrm{Re}(\xi_u) + 2\mu \leq -\frac{1}{3}\};\\
\Omega_{\eta_u,1} &:= \{u \in \Omega \,|\, -k_1 - \mathrm{Re}(\eta_u) + 2\mu \geq -\frac{1}{2}\};\\
\Omega_{\eta_u,2} &:= \{u \in \Omega \,|\, -k_1 - \mathrm{Re}(\eta_u) + 2\mu \leq -\frac{1}{3}\}.
\end{aligned}$$

If $u \in \Omega_{\xi_u,1}$, we leave the dt-integral unchanged; if $u \in \Omega_{\xi_u,2}$, then we rewrite the dt-integral as $\int_0^{+\infty} - \int_{x_2}^{+\infty}$. We can do the similar operation for the $dr-$integral. Then we can imitate the method in Theorem 5.1.2 and finish the proof. Here we only deal one case, the others can be analyzed in the same way.

Let us assume that $u \in \Omega_{\xi_u,1} \bigcap \Omega_{\eta_u,2}$, then (5.27) can be rewritten as

$$e^{x_1\eta_u+x_2\xi_u}r_u(x_1)h_u(x_2)\int_0^{x_2}\int_0^{+\infty} - e^{x_1\eta_u+x_2\xi_u}r_u(x_1)h_u(x_2)\int_0^{x_2}\int_{x_1}^{+\infty}. \tag{5.28}$$

Using a similar estimate as (5.16), we can show that the integral

$$\int_0^{+\infty} e^{(-\eta_u-k_1)r}r^{l'}W_{\pi(D)v,u}(e^{rH_1+tH_2})dr$$

converges absolutely and is bounded by $\|e^{tH_2}\|^{\mu} \cdot q'(v)$. Thus, both summands in (5.28) define a holomorphic function in u which is uniformly continuous in the other variables when u runs in Ω. It suffices to prove that they satisfy the desired estimates. Indeed,

$$\begin{aligned}&\big|e^{x_2\xi_u}\int_0^{x_2}\int_0^{+\infty} e^{(-\xi_u-k_2)t}t^l e^{(-\eta_u-k_1)r}r^{l'}W_{\pi(D)v,u}(e^{rH_1+tH_2})drdt\big|\\ \leq&\big|e^{x_2\xi_u}\cdot\int_0^{x_2} e^{(-\xi_u-k_2)t}t^l\cdot\|e^{tH_2}\|^{\mu}q'(v)dt\big|.\end{aligned}$$

Thus, by the method in (5.14), the above is bounded by

$$e^{\mathrm{Re}(\xi_2)x_2}h(x_2)q''(v)$$

for some polynomial h with constant coefficients and a continuous seminorm q''. Therefore the first summand in (5.28) contributes a term of the form $e^{x_1\eta_u}p_u(x_1,x_2)f_2(x_2,v,u)$ in Theorem 5.2.2.

By using a similar estimate as (5.17), we can show that the integral

$$\big|e^{x_1\eta_u}\int_{x_1}^{+\infty} e^{(-\eta_u-k_1)r}r^{l'}W_{\pi(D)v,u}(e^{rH_1+tH_2})dr\big|.$$

is bounded by $e^{\mathrm{Re}(\xi_1)x_1}h_1(x_1)q'(v)$ for some polynomial h_1 and continuous seminorm q'. Then using the method in (5.14), we can obtain that

$$\big|e^{x_1\eta_u+x_2\xi_u}\int_0^{x_2}\int_{x_1}^{+\infty} e^{(-\xi_u-k_2)t}t^l e^{(-\eta_u-k_1)r}r^{l'}W_{\pi(D)v,u}(e^{rH_1+tH_2})drdt\big|$$

is bounded by

$$e^{\mathrm{Re}(\xi_1)\cdot x_1+\mathrm{Re}(\xi_2)\cdot x_2}h_3(x_1,x_2)q''(v)$$

for some continuous seminorm q'' and polynomial h_3 with complex coefficients. Thus the second summand in (5.28) contributes a term of the form $p_u(x_1,x_2)f_3(x_1,x_2,v,u)$ in Theorem 5.2.2. $\square$

Remark 5.2.3. *Since the space of K-finite vectors in V_π is dense, by the continuity of Whittaker functional, the asymptotic expansion in Theorem 5.2.2 also holds for all $v \in V_\pi$.*

Chapter 6

Meromorphic Continuation of the Local Integrals (Archimedean Case)

In this Chapter, we will use Theorem 5.2.2 to prove Theorem 1.0.11 by analysing the asymptotic behaviour of the integrand of $Z(W_v, f_s)$ near 0. Throughout this Chapter, whenever an integral parameterized by s is a bilinear form on the projective tensor product space $V_\pi \hat{\otimes} V_{\rho_s}$ (see Section 2.1), we say that it satisfies property $\mathcal{M}$, if

1. it converges absolutely when $\mathrm{Re}(s)$ is sufficiently large,
2. it extends to a meromorphic function of s to the whole complex plane,
3. its meromorphic continuation is a continuous on $V_\pi \hat{\otimes} V_{\rho_s}$.

We start from some reductions to get rid of the integral over the maximal compact subgroup.

6.1 Some Reductions

We define

$$B(W_v, f_s) := \int_F \int_{A_{\mathrm{SL}_3}} W_v(\begin{pmatrix} 1 & z & \\ & 1 & \\ & & 1 \end{pmatrix} a) f_s(\gamma \cdot \begin{pmatrix} 1 & z & \\ & 1 & \\ & & 1 \end{pmatrix} a) \delta^{-1}_{B_{\mathrm{SL}_3}}(a) da dz.$$

Then we can rewrite $Z(W_v, f_s)$ as

$$Z(W_v, f_s) = \int_{K_{\mathrm{SL}_3}} B(\pi(k)W_v, \rho_s(k) f_s) dk. \tag{6.1}$$

Lemma 6.1.1. *If $B(W_v, f_s)$ satisfies property $\mathcal{M}$, then $Z(W_v, f_s)$ satisfies property $\mathcal{M}$. Moreover, if $\pi = \pi_u$ is a principal series and $B(W_{v,u}, f_s)$ is meromorphic in u, then $Z(W_{v,u}, f_s)$ is also meromorphic in u.*

Proof. If $B(W_v, f_s)$ satisfies property $\mathcal{M}$, then whenever $s = s_0$ belongs to a compact set away from the poles of $Z(W_v, f_s)$, the function

$$k \quad \mapsto \quad B(\pi(k)W_v, \rho_s(k) f_s),$$

with both $v \in V_\pi$ and $f_s \in V_{\rho_s}$ fixed, is a continuous function on a compact group, hence it is bounded. Moreover, the integral

$$Z(W_v, f_s) = \int_{K_{\mathrm{SL}_3}} B(\pi(k)W_v, \rho_s(k) f_s) dk \tag{6.2}$$

converges absolutely and uniformly in s when s runs in that compact set. This proves the meromorphic continuation of $Z(W_v, f_s)$. Similarly, when $\pi = \pi_u$, the convergence of (6.2) is also uniform in u when u runs in a compact set Ω away from the poles. Thus, if $B(W_{v,u}, f_s)$ is meromorphic in u, then $Z(W_{v,u}, f_s)$ is also meromorphic in u. For each fixed $k \in K_{\mathrm{SL}_3}$, the function

$$(v, f_s) \quad \mapsto \quad B(\pi(k)W_v, \rho_s(k) f_s)$$

is a bounded continuous bilinear form on $V_\pi \hat{\otimes} V_{\rho_s}$. Set

$$B_k(v, f_s) = B(\pi(k)W_v, \rho_s(k)f_s).$$

Then by the Uniform Boundedness Principle (see [Tr]), the family of bounded continuous bilinear form B_k indexed by $k \in K_{\mathrm{SL}_3}$ is equicontinuous. In other words, let $d_{\pi,\rho}$ be the metric describing the topology of the Fréchet space $V_\pi \hat{\otimes} V_{\rho_s}$ (see Section 2.1). Then for any $\epsilon > 0$, there exists $\delta > 0$, such that for any two pairs $(v_1, f_{1,s}), (v_2, f_{2,s}) \in V_\pi \hat{\otimes} V_{\rho_s}$ satisfying

$$d_{\pi,\rho}((v_1, f_{1,s}), (v_2, f_{2,s})) < \delta,$$

we have

$$|B(\pi(k)W_{v_1}, \rho_s(k)f_{1,s}) - B(\pi(k)W_{v_2}, \rho_s(k)f_{2,s})| < \epsilon.$$

Therefore

$$\begin{aligned} |Z(W_{v_1}, f_{1,s}) - Z(W_{v_2}, f_{2,s})| &\leq \int_{K_{\mathrm{SL}_3}} |B(\pi(k)W_{v_1}, \rho_s(k)f_{1,s}) - B(\pi(k)W_{v_2}, \rho_s(k)f_{2,s})| dk \\ &\leq \epsilon. \end{aligned}$$

This shows that $Z(W_v, f_s)$ is a continuous bilinear form. $\square$

The following Dixmier-Malliavin lemma (see [D-M]) is well known to experts.

Lemma 6.1.2. *Let (π, V) be a continuous representation of a Lie group G on a Fréchet space V. Then every smooth vector $v \in V$ can be represented by a finite linear combination*

$$v = \sum_i \pi(f_i)v_i = \sum_i \int_{\mathrm{G}} \pi(x)f_i(x)v_i dx,$$

where $f_i(x) \in C_c^\infty(\mathrm{G})$, and v_i are smooth vectors in V.

We apply the Dixmier-Malliavin lemma to the subgroup $U_1 = \left\{ \begin{pmatrix} 1 & x & \\ & 1 & \\ & & 1 \end{pmatrix} \right\}$.

Every $v \in V_\pi$ is a finite linear combination of $\pi(\varphi_i^{(1)})v_i$:

$$v = \sum_i \pi(\varphi_i^{(1)})v_i = \sum_i \int_F \varphi_i^{(1)}(x)\pi(\begin{pmatrix} 1 & x & \\ & 1 & \\ & & 1 \end{pmatrix})v_i dx,$$

where $\varphi_i^{(1)} \in C_c^\infty(F)$ and each $v_i \in V_\pi$. Thus,

$$W_{\pi(\varphi_i^{(1)})v_i}(\begin{pmatrix} t_1^3 t_2^3 & & \\ & t_2^3 & \\ & & 1 \end{pmatrix}) = W_{v_i}(\begin{pmatrix} t_1^3 t_2^3 & & \\ & t_2^3 & \\ & & 1 \end{pmatrix})\widehat{\varphi_i^{(1)}}(t_1^3).$$

We apply the Dixmier-Malliavin lemma again to the subgroup $U_2 = \left\{ \begin{pmatrix} 1 & & \\ & 1 & x \\ & & 1 \end{pmatrix} \right\}$. Then every above $v_i \in V_\pi$ can be represented as a finite linear combination

$$v_i = \sum_j \pi(\varphi_j^{(2)})v_j' = \sum_j \int_F \varphi_j^{(2)}(x)\pi(\begin{pmatrix} 1 & & \\ & 1 & x \\ & & 1 \end{pmatrix})v_j' dx,$$

where $\varphi_j^{(2)} \in C_c^\infty(F)$ and each $v_j' \in V_\pi$. Then,

$$W_{\pi(\varphi_j^{(2)})v_j'}(\begin{pmatrix} t_1^3 t_2^3 & & \\ & t_2^3 & \\ & & 1 \end{pmatrix}) = W_{v_j'}(\begin{pmatrix} t_1^3 t_2^3 & & \\ & t_2^3 & \\ & & 1 \end{pmatrix})\widehat{\varphi_j^{(2)}}(t_2^3).$$

Here in the above, $\widehat{\varphi_i^{(1)}}$ and $\widehat{\varphi_j^{(2)}}$ are the Fourier transforms of $\varphi_i^{(1)}$ and $\varphi_j^{(2)}$, resp.

It is more convenient to regard all above test functions $\varphi_i^{(1)}, \varphi_j^{(2)}$ as Schwartz functions. This is because the space of test functions is not metrizable, while the space of Schwartz functions $\mathcal{S}(F)$ is a Fréchet space. Because both $\mathcal{S}(F)$ and V_π are Fréchet spaces and the bilinear map $\mathcal{S}(F) \times V_\pi \to V_\pi$:

$$(\varphi, v) \mapsto \pi(\varphi)v$$

is separately continuous, the linear map $\mathcal{S}(F)\hat{\otimes}V_\pi \mapsto V_\pi$:

$$\varphi\hat{\otimes}v \mapsto \pi(\varphi)v$$

is continuous and surjective (by Deximier-Malliavin lemma). Hence it is also an open map by the Open Mapping Theorem.

Using the same calculation as in the proof of Theorem 1.0.10 (see Section 3.1), we can obtain that

$$\begin{aligned} B(W_v, f_s) = \int_F \int_{(\mathbb{R}_+^\times)^2} W_v(\begin{pmatrix} t_1^3t_2^3 & & \\ & t_2^3 & \\ & & 1 \end{pmatrix})\psi((t_1^2+t_2^2)^{\frac{3}{2}}z) \cdot \Big|\frac{t_1^{9s}t_2^{9s}}{(t_1^2+t_2^2)^{\frac{9}{2}s}}\Big|_F \cdot |t_1^{-6}t_2^{-6}|_F \\ \cdot ||z|^2+1|_F^{-\frac{3}{2}s} \cdot |t_1^2+t_2^2|_F^{\frac{3}{2}} \cdot 3f_s(k''(z)k'(t_1^{-1}t_2)w_\beta)d^\times t_1 d^\times t_2 dz. \end{aligned} \tag{6.3}$$

Therefore to prove that $B(W_{v,u}, f_s)$ satisfies the property $\mathcal{M}$ and is meromorphic in u, it suffices to prove that the following integral satisfies property $\mathcal{M}$ and is meromorphic in u:

$$\begin{aligned} C(W_{v,u}, f_s) := \int_F \int_0^{+\infty} \int_0^{+\infty} W_{v,u}(\begin{pmatrix} t_1^3t_2^3 & & \\ & t_2^3 & \\ & & 1 \end{pmatrix})\psi(z) \cdot |t_1^{9s-6}t_2^{9s-6}|_F \cdot \varphi_1(t_1)\varphi_2(t_2) \\ \cdot ||z|^2+(t_1^2+t_2^2)^3|_F^{-\frac{3}{2}s} \cdot 3f_s(k''(\frac{z}{(t_1^2+t_2^2)^{\frac{3}{2}}})k'(t_1^{-1}t_2)w_\beta)d^\times t_1 d^\times t_2 dz, \end{aligned} \tag{6.4}$$

where φ_1 and φ_2 are Schwartz functions on F (the Fourier transforms of test functions are Schwartz functions).

6.2 Proof of Theorem 1.0.11

We only focus on the real case ($F = \mathbb{R}$), the proof of the complex case is similar. Without loss of generality, we can assume that our additive character $\psi(x) = e^{2\pi ix}$ as in (1.11). Note that by the Iwasawa decomposition of $\mathrm{SL}_2(\mathbb{R})$,

$$\begin{pmatrix} 1 & \\ t_1^{-1}t_2 & 1 \end{pmatrix} = \begin{pmatrix} \frac{t_1}{\sqrt{t_1^2+t_2^2}} & \\ & \frac{\sqrt{t_1^2+t_2^2}}{t_1} \end{pmatrix} \begin{pmatrix} 1 & t_1^{-1}t_2 \\ & 1 \end{pmatrix} \begin{pmatrix} \frac{t_1}{\sqrt{t_1^2+t_2^2}} & \frac{-t_2}{\sqrt{t_1^2+t_2^2}} \\ \frac{t_2}{\sqrt{t_1^2+t_2^2}} & \frac{t_1}{\sqrt{t_1^2+t_2^2}} \end{pmatrix}.$$

Hence as a function of two variables t_1, t_2, k' is a smooth bounded function on $[0,+\infty)\times[0,+\infty)-\{(0,0)\}$. If we set $t_1 = r\cos\theta, t_2 = r\sin\theta$, then variable r doesn't appear in k'. As a function of θ, k' can be extended to a smooth function on $[0, 2\pi]$.

Let us first deal with the dz-integral first.

Set

$$F(t_1,t_2,s) := 3(t_1^2+t_2^2)^{\frac{9}{4}s} \int_{\mathbb{R}} \psi(z)\cdot(|z|^2+(t_1^2+t_2^2)^3)^{-\frac{3}{2}s} f_s(k''(\frac{z}{(t_1^2+t_2^2)^{\frac{3}{2}}})k'(t_1^{-1}t_2)w_\beta)dz. \tag{6.5}$$

Lemma 6.2.1. *For any $(t_1,t_2)\neq(0,0)$, $F(t_1,t_2,s)$ converges absolutely when $Re(s) > \frac{1}{3}$, and it has a holomorphic continuation in the whole complex plane. Moreover, if we use polar coordinate*

$$t_1 = r\cos\theta, t_2 = r\sin\theta,$$

then as a function of r and θ, $F(t_1(r,\theta), t_2(r,\theta), s)$ (simply denoted by $F(r,\theta,s)$ in the following) is a bounded smooth function in θ and behaves like a Schwartz function when r tends to infinity. $F(r,\theta,s)$ also has an asymptotic expansion

$$F(r,\theta,s) \sim \sum_{k=0}^{+\infty} a_k(\theta) r^{\frac{9}{2}s+6k} + \sum_{k=0}^{+\infty} b_k(\theta) r^{3-\frac{9}{2}s+6k}.$$

when r tends to zero, where $a_k(\theta), b_k(\theta)$ are bounded smooth functions of θ.

Proof. Clearly, for every fixed $(t_1,t_2)\neq(0,0)$, $F(t_1,t_2,s)$ converges absolutely when $\mathrm{Re}(s) > \frac{1}{3}$, hence it defines a holomorphic function in the right half plane $\mathrm{Re}(s) > \frac{1}{3}$.

If we set $t_1 = r\cos\theta, t_2 = r\sin\theta$, then

$$F(r,\theta,s) = 3r^{\frac{9}{2}s} \int_{\mathbb{R}} \psi(z)\cdot(z^2+r^6)^{-\frac{3}{2}s} f_s(k''(\frac{z}{r^3})k'(\theta)w_\beta)dz.$$

$\mathrm{SL}_3(\mathbb{R})$ contains a subgroup H that is isomorphic to $\mathrm{SL}_2(\mathbb{R})$ lying on the upper left corner:

$$g \mapsto \begin{pmatrix} g & \\ & 1 \end{pmatrix}.$$

When restricted on this subgroup, any $f_s \in V_{\rho_s}$ satisfies the following invariant property:

$$f_s(\begin{pmatrix} 1 & z & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} t & & \\ & t^{-1} & \\ & & 1 \end{pmatrix} x) = |t|^{3s} f_s(x),$$

hence the restriction of f_s on H lies inside a principal series of $\mathrm{SL}_2(\mathbb{R})$: $\mathrm{Ind}_{B_{SL_2}}^{SL_2} |\quad|^{\frac{3s-1}{2}} \otimes |\quad|^{-\frac{3s-1}{2}}$. When we fix θ, then $F(r, \theta, s)$ is exactly the Jacquet integral of a right translation of f_s. Hence by [Wal2, Section 15.4], the above integral has a holomorphic continuation to the whole complex plane, and it behaves like a Schwartz function when r tends to infinity. By the asymptotic expansion of the generalized matrix coefficients of $\mathrm{SL}_2(\mathbb{R})$, $F(r, \theta, s)$ has the following asymptotic expansion when r tends to zero:

$$F(r, \theta, s) \sim \sum_{k=0}^{+\infty} a_k(\theta) r^{\frac{9}{2}s + 6k} + \sum_{k=0}^{+\infty} b_k(\theta) r^{3 - \frac{9}{2}s + 6k}.$$

Because all the asymptotic coefficients $a_k(\theta), b_k(\theta)$ can be obtained by limit process recursively (see [Bl-H]), they are bounded smooth functions of θ. $\square$

Remark 6.2.2. *A function that behaves like a Schwartz function when $r \to +\infty$ doesn't necessarily behave like a Schwartz function for each variable. The Schwartz functions φ_1, φ_2 coming from the Dixmier-Malliavin lemma will compensate this defect.*

Now we begin to prove the desired meromorphic continuation of

$$C(W_{v,u}, f_s) = \int_0^{+\infty} \int_0^{+\infty} W_{v,u}(\begin{pmatrix} t_1^3 t_2^3 & & \\ & t_2^3 & \\ & & 1 \end{pmatrix}) \cdot \frac{t_1^{9s-6} t_2^{9s-6}}{(t_1^2 + t_2^2)^{\frac{9}{4}s}}$$
$$\cdot \varphi_1(t_1) \varphi_2(t_2) \cdot F(t_1, t_2, s) d^\times t_1 d^\times t_2$$

We note that $F(t_1, t_2, s)$ is smooth on $[0, +\infty) \times [0, \infty) - \{(0, 0)\}$. At the origin $(0, 0)$, we can only expect an asymptotic expansion as above.

Proof of Theorem 1.0.11, real case. Still, because we want to keep track on the parameters u, we only focus on the case $\pi = \pi_u$. A word by word repetition will also work for any irreducible generic Casselman-Wallach representation π. Let u run in a fixed closed ball $\Omega = B(u_0, r_0)$. Since both V_{π_u} and V_{ρ_s} are Fréchet spaces, the notions of continuity and separate continuity on $V_{\pi_u}\hat{\otimes}V_{\rho_s}$ coincide. So we can fix f_s first and prove the continuity in v. We fix a vertical strip $\mathrm{Re}(s) \in (a, b)$. For $m = 1, 2$, we choose two negative numbers ξ_1, ξ_2 which has sufficiently large absolute values. The exact conditions that they satisfy will be clear from the following proof.

We choose k_1, k_2 so large such that

$$
\begin{aligned}
&-\mathrm{Re}(\xi_1) - k_1 + 2\mu < -1,\\
&-\mathrm{Re}(\xi_2) - k_2 + 2\mu < -1.
\end{aligned}
$$

Then there exist two finite subsets $C_u^{(1)}, C_u^{(2)}$, a finite set of polynomials $\mathcal{P}_u$, a finite subset of non-negative integers $\mathcal{L}$ and a finite subset $\mathcal{D} \subseteq U(\mathfrak{gl}_3(\mathbb{C}))$ such that $W_{v,u}(e^{x_1H_1+x_2H_2})$ has an asymptotic expansion as in Theorem 5.1.2 when $x_2 \geq 0$, and $W_v(e^{x_1H_1+x_2H_2})$ has an asymptotic expansion as in Theorem 5.2.2 when $x_1, x_2 \geq 0$.

We fix a very small positive number $1 > \epsilon > 0$ and split the integral $C(W_{v,u}, f_s)$ into four parts:

$$
\int_\epsilon^{+\infty}\int_\epsilon^{+\infty}, \qquad \int_0^\epsilon\int_\epsilon^{+\infty}, \qquad \int_\epsilon^{+\infty}\int_0^\epsilon, \qquad \int_0^\epsilon\int_0^\epsilon. \tag{6.6}
$$

Case 1: $t_1, t_2 \geq \epsilon$. Since both $\varphi_1(t_1)$ and $\varphi_2(t_2)$ are Schwartz functions and $F(t_1, t_2, s)$ behaves like a Schwartz function when $t_1^2 + t_2^2 \to +\infty$, the integral

$$
\int_\epsilon^{+\infty}\int_\epsilon^{+\infty} W_{v,u}(\begin{pmatrix} t_1^3t_2^3 & & \\ & t_2^3 & \\ & & 1 \end{pmatrix}) \cdot \frac{t_1^{9s-6}t_2^{9s-6}}{(t_1^2+t_2^2)^{\frac{9}{4}s}} \cdot \cdot \varphi_1(t_1)\varphi_2(t_2) \cdot F(t_1, t_2, s) d^\times t_1 d^\times t_2
$$

converges absolutely and defines a holomorphic function of s. Moreover, by the estimate (5.1), the above integral defines a holomorphic function in $u \in \Omega$ which is continuous in v.

Case 2: $t_1 \geq \epsilon, t_2 \leq \epsilon$. We apply the asymptotic expansion of $W_{v,u}$ in variable

t_2. Then by Theorem 5.1.2, $W_{v,u}(\begin{pmatrix} t_1^3t_2^3 & & \\ & t_2^3 & \\ & & 1 \end{pmatrix})$ is a finite $\mathbb{C}(u)$-linear combination of terms of the form

1. type 1: $t_2^{-3\xi}(\ln t_2)^r \cdot W_{\pi(e)v,u}(\begin{pmatrix} t_1^3 & & \\ & 1 & \\ & & 1 \end{pmatrix})$,
2. type 2: $t_2^{-3\xi}(\ln t_2)^r \cdot \int_0^{+\infty} e^{(-\xi-k_2)t}t^l W_{\pi(D)v,u}(\begin{pmatrix} t_1^3 & & \\ & 1 & \\ & & 1 \end{pmatrix} \cdot e^{tH_2})dt$,
3. type 3: $t_2^{-3\xi}(\ln t_2)^r \cdot \int_{-3\ln t_2}^{+\infty} e^{(-\xi-k_2)t}t^l W_{\pi(D)v,u}(\begin{pmatrix} t_1^3 & & \\ & 1 & \\ & & 1 \end{pmatrix} \cdot e^{tH_2})dt$,
4. type 4: $t_2^{-3\xi}(\ln t_2)^r \cdot \int_0^{-3\ln t_2} e^{(-\xi-k_2)t}t^l W_{\pi(D)v,u}(\begin{pmatrix} t_1^3 & & \\ & 1 & \\ & & 1 \end{pmatrix} \cdot e^{tH_2})dt$,

where all $e, D \in \mathcal{D}$, $r, l \in \mathcal{L}$, $\xi_u \in C_u^{(2)}$. To unify our notations, each term above can be written as

$$t_2^{-3\xi}(\ln t_2)^r \cdot H(t_1, t_2, v, u),$$

where $H(t_1, t_2, v, u)$ is holomorphic in $u \in \Omega$ and uniformly continuous in t_1, t_2, v when u runs in Ω. It also satisfies the estimates in Theorem 5.1.2. Put

$$G(t_2, v, u, s) := \int_\epsilon^{+\infty} H(t_1, t_2, v, u)\frac{t_1^{9s-6}}{(t_1^2+t_2^2)^{\frac{9}{4}s}}\varphi_1(t_1)F(t_1, t_2, s)d^\times t_1.$$

Then for each fixed t_2, the integral $G(t_2, v, u, s)$ converges absolutely for all s and $u \in \Omega$, hence it defines a holomorphic function in s and u. It is also a continuous in t_2 and linear in v. The second part of $C(W_{v,u}, f_s)$ in (6.6) is a finite linear combination of integrals of the form

$$\int_0^\epsilon t_2^{9s-6-3\xi}(\ln t_2)^r\varphi_2(t_2)G(t_2, v, u, s)d^\times t_2. \tag{6.7}$$

If $H(t_1, t_2, u, v)$ is of type 1 or 2, then it is in fact a continuous function in t_1 (t_2 does not appear in H) bounded by $\|e^{x_1 H_1}\|^\mu q'(v)$ for some continuous seminorm q' (by Theorem 5.1.2). In these two cases, from the expression

$$G(t_2, v, u, s) = \int_\epsilon^{+\infty} H(t_1, v, u) \frac{t_1^{9s-6}}{(t_1^2 + t_2^2)^{\frac{9}{4}s}} \varphi_1(t_1) F(t_1, t_2, s) d^\times t_1,$$

$G(t_2, v, u, s)$ and all its partial derivatives $\frac{\partial^l G}{\partial t_2^l}(t_2, v, u, s)$ are smooth functions in t_2, holomorphic in u, and continuous and linear in v. Therefore, the Taylor expansion of $\varphi_2(t_2) G(t_2, v, s)$ with Peano remainder provides a meromorphic continuation of

$$\int_0^\epsilon t_2^{9s-6-3\xi} (\ln t_2)^r \varphi_2(t_2) G(t_2, v, u, s) d^\times t_2$$

by the standard method in Tate's thesis. This meromorphic continuation is also holomorphic in u and continuous in v.

If $H(t_1, t_2, u, v)$ is of type 3, then by the estimates in Theorem 5.1.2,

$$\begin{aligned}
&|t_2^{-3\xi} (\ln t_2)^r G(t_2, v, u, s)| \\
\leq &\int_\epsilon^{+\infty} \left| t_2^{-3\xi} (\ln t_2)^r H(t_1, t_2, u, v) \frac{t_1^{9s-6}}{(t_1^2 + t_2^2)^{\frac{9}{4}s}} \varphi_1(t_1) F(t_1, t_2, s) \right| d^\times t_1 \\
\leq &\int_\epsilon^{+\infty} \left| t_2^{-3\mathrm{Re}(\xi_2)} \cdot t_1^{6\mu} h(\ln t_2) q'(v) \cdot \frac{t_1^{9s-6}}{(t_1^2 + t_2^2)^{\frac{9}{4}s}} \varphi_1(t_1) F(t_1, t_2, s) \right| d^\times t_1 \\
\leq &t_2^{-3\mathrm{Re}(\xi_2)} h(\ln t_2) q''(v)
\end{aligned}$$

for some polynomial h and continuous seminorm q''. Therefore

$$|t_2^{9s-6-3\xi} (\ln t_2)^r \varphi_2(t_2) G(t_2, v, s)| \leq t_2^{9\mathrm{Re}(s)-6-3\mathrm{Re}(\xi_2)} h(\ln t_2) |\varphi_2(t_2)| q''(v)$$

for some polynomial h and continuous seminorm q''. Since we can choose $\mathrm{Re}(\xi_2)$ as negative as we want, if we assume that the exponent $9\mathrm{Re}(s) - 6 - 3\mathrm{Re}(\xi_2) > 1$ in the vertical strip $\mathrm{Re}(s) \in (a, b)$, then (6.7) is holomorphic in s and u, and defines a continuous linear function in v.

If $H(t_1, t_2, u, v)$ is of type 4, the estimate for type 3 also holds in this case, since the estimate for 2) and 4) in Theorem 5.1.2 have the same pattern. Thus, the integral

$$\int_0^\epsilon t_2^{9s-6-3\xi}(\ln t_2)^r \varphi_2(t_2) G(t_2, v, u, s) d^\times t_2$$

also defines a holomorphic function in s and u which is continuous in v.

Case 3: $t_1 \leq \epsilon, t_2 \geq \epsilon$. Since t_1 and t_2 play a symmetric role in the integral, the proof is exactly the same as case 2.

Case 4: $t_1 < \epsilon, t_2 < \epsilon$. We apply Theorem 5.2.2 again. When $t_1 < \epsilon, t_2 < \epsilon$, $W_v(\begin{pmatrix} t_1^3 t_2^3 & & \\ & t_2^3 & \\ & & 1 \end{pmatrix})$ is a finite $\mathbb{C}(u)$-linear combination of terms of the form

1. $t_1^{-3\eta_u} t_2^{-3\xi_u} (\ln t_1)^{r_1} (\ln t_2)^{r_2} f_0(v, u)$,
2. $t_1^{-3\eta_u} (\ln t_1)^{r_1} (\ln t_2)^{r_2} f_2(\ln t_2, v, u)$,
3. $t_2^{-3\xi_u} (\ln t_1)^{r_1} (\ln t_2)^{r_2} f_1(\ln t_1, v, u)$,
4. $(\ln t_1)^{r_1} (\ln t_2)^{r_2} f_3(\ln t_1, \ln t_2, v, u)$,

where $r_1, r_2 \in \mathcal{L}$, $\eta_u \in C_u^{(1)}$, $\xi_u \in C_u^{(2)}$. Moreover, f_0, f_1, f_2 admit estimates as in Theorem 5.2.2, i.e

1. $|f_0(v, u)| \leq q'(v)$,
2. $|f_1(\ln t_1, v, u)| \leq t_1^{-3\mathrm{Re}(\xi_1)} h_1(\ln t_1) q'(v)$,
3. $|f_2(\ln t_2, v, u)| \leq t_2^{-3\mathrm{Re}(\xi_2)} h_2(\ln t_2) q'(v)$,
4. $|f_3(\ln t_1, \ln t_2, v, u)| \leq t_1^{-3\mathrm{Re}(\xi_1)} t_2^{-3\mathrm{Re}(\xi_2)} h_3(\ln t_1, \ln t_2) q'(v)$.

We need to study

$$\int_0^\epsilon \int_0^\epsilon W_{v,u}(\begin{pmatrix} t_1^3 t_2^3 & & \\ & t_2^3 & \\ & & 1 \end{pmatrix}) \cdot \frac{t_1^{9s-6} t_2^{9s-6}}{(t_1^2 + t_2^2)^{\frac{9}{4}s}} \cdot \varphi_1(t_1) \varphi_2(t_2) F(t_1, t_2, s) d^\times t_1 d^\times t_2. \tag{6.8}$$

We can choose ϵ so small that $F(t_1, t_2, s)$ (or equivalently $F(r, \theta, s)$) can be approximated by its asymptotic expansion (see Lemma 6.2.1).

$$F(r,\theta,s) \sim \sum_{k=0}^{+\infty} a_k(\theta) r^{\frac{9}{2}s+6k} + \sum_{k=0}^{+\infty} b_k(\theta) r^{3-\frac{9}{2}s+6k}.$$

After we cut off the first finite terms $F_q(r, \theta, s)$ in the asymptotic expansion of $F(r, \theta, s)$, the remainder

$$F(r,\theta,s) - F_q(r,\theta,s) = O(r^p).$$

Here p can be as large as we want if we choose a sufficiently large q. Now we use the estimate

$$|W_{v,u}(\begin{pmatrix} t_1^3 t_2^3 & & \\ & t_2^3 & \\ & & 1 \end{pmatrix})| \leq 6^{\mu} t_1^{-6\mu} t_2^{-6\mu} q(v).$$

If we choose a large p such that p can beat any exponent appearing in the integrand of (6.8), then

$$\int_0^{\epsilon}\int_0^{\epsilon} W_{v,u}(\begin{pmatrix} t_1^3 t_2^3 & & \\ & t_2^3 & \\ & & 1 \end{pmatrix}) \cdot \frac{t_1^{9s-6} t_2^{9s-6}}{(t_1^2+t_2^2)^{\frac{9}{4}s}} \cdot \varphi_1(t_1)\varphi_2(t_2)(F(t_1,t_2,s) - F_q(t_1,t_2,s)) d^{\times}t_1 d^{\times}t_2$$

is a holomorphic function in s and u, and continuous in v. We note that the choice of p (hence q) only depends on the vertical strip (a, b). Similarly, by applying Taylor expansion of φ_1 and φ_2 (with remainder of Peano form), the remainder contributes a holomorphic function in s, which is continuous in v. Therefore it suffices to prove the meromorphic continuation of

$$\int_0^{\epsilon}\int_0^{\epsilon} W_{v,u}(\begin{pmatrix} t_1^3 t_2^3 & & \\ & t_2^3 & \\ & & 1 \end{pmatrix}) \cdot \frac{t_1^{9s-6+l_1} t_2^{9s-6+l_2}}{(t_1^2+t_2^2)^{\frac{9}{4}s}} \cdot F_q(t_1,t_2,s), d^{\times}t_1 d^{\times}t_2.$$

Clearly the above is a linear combination of

$$\int_0^\epsilon \int_0^\epsilon W_{v,u}(\begin{pmatrix} t_1^3 t_2^3 & & \\ & t_2^3 & \\ & & 1 \end{pmatrix}) \cdot \frac{t_1^{9s-6+l_1} t_2^{9s-6+l_2}}{(t_1^2+t_2^2)^{\frac{9}{4}s}} \cdot r^{\frac{9}{2}s+6k} a_k(\theta) d^\times t_1 d^\times t_2.$$

and

$$\int_0^\epsilon \int_0^\epsilon W_{v,u}(\begin{pmatrix} t_1^3 t_2^3 & & \\ & t_2^3 & \\ & & 1 \end{pmatrix}) \cdot \frac{t_1^{9s-6+l_1} t_2^{9s-6+l_2}}{(t_1^2+t_2^2)^{\frac{9}{4}s}} \cdot r^{-\frac{9}{2}s+6k+3} b_k(\theta), d^\times t_1 d^\times t_2. \tag{6.9}$$

The proofs for above two integrals are almost the same, so we only do the second one in detail. (The second one is slightly more complicated because the denominator $(t_1^2+t_2^2)^{\frac{9}{4}s} = r^{\frac{9}{2}s}$ can not be canceled out.) We note that there are only finitely many l_1, l_2, k. They only depends on the vertical strip Re(s)$\in (a,b)$, not on the choice of ξ_1, ξ_2.

The term $t_1^{-3\eta} t_2^{-3\xi} (\ln t_1)^{r_1} (\ln t_2)^{r_2} f_0(v,u)$ contributes an integral

$$\int_0^\epsilon \int_0^\epsilon t_1^{-3\eta} t_2^{-3\xi} (\ln t_1)^{r_1} (\ln t_2)^{r_2} \frac{t_1^{9s-6+l_1} t_2^{9s-6+l_2}}{(t_1^2+t_2^2)^{\frac{9}{2}s}} \cdot r^{6k+3} b_k(\theta) d^\times t_1 d^\times t_2 \cdot f_0(v,u) \tag{6.10}$$

in (6.9). We aim to show that the above integral has a meromorphic continuation for nonnegative integers l_1, l_2, k and any bounded smooth functions $b_k(\theta)$. Because the integrand is not factorizable in the Cartesian coordinate, we are facing a situation slightly more complicated than that in [Sou], yet this complication still can be resolved in the polar coordinate.

$$t_1 = r\cos\theta, t_2 = r\sin\theta.$$

After expanding polynomials $(\ln t_1)^{r_1} = (\ln r + \ln\cos\theta)^{r_1}$ and $(\ln t_2)^{r_2} = (\ln r + \ln\sin\theta)^{r_2}$, we only need to prove that the following integral has a meromorphic continuation in s

$$\iint_D r^{9s+a_1} (\ln r)^{a_2} \cdot (\cos\theta)^{9s+b_1} (\ln\cos\theta)^{b_2} \cdot (\sin\theta)^{9s+c_1} (\ln\cos\theta)^{c_2} f(\theta) dr d\theta,$$

where D is a square $[0,\epsilon]\times[0,\epsilon]$, a_1, b_1, c_1 are complex numbers, a_2, b_2, c_2 are nonnegative integers, and $f(\theta)$ is a bounded smooth function in θ. But this is very elementary and the proof is based on integration by parts. For details, see Section 6.3.

The term $t_1^{-3\eta_u}(\ln t_1)^{r_1}(\ln t_2)^{r_2} f_2(\ln t_2, v, u)$ contributes an integral

$$\int_0^\epsilon \int_0^\epsilon (t_1)^{-3\eta_u}(\ln t_1)^{r_1}(\ln t_2)^{r_2} f_2(\ln t_2, v, u) \frac{t_1^{9s-6} t_2^{9s-6}}{(t_1^2+t_2^2)^{\frac{9}{2}s}} \cdot t_1^{l_1} t_2^{l_2} r^{6k+3} b_k(\theta) d^\times t_1 d^\times t_2 \quad (6.11)$$

in (6.9). We change variable $t_2 \mapsto t_1 t_2$, then (6.11) is equal to a finite linear combination of

$$\begin{aligned} \int_0^\epsilon \int_0^{\epsilon t_1} & t_1^{9s-9+l_1+l_2+6k-3\eta_u}(\ln t_1)^{r'_1}(\ln t_2)^{r'_2} f_2(\ln t_1 t_2, v, u) \\ & \cdot \frac{t_2^{9s-6+l_2}}{(t_2^2+1)^{\frac{1}{2}(9s-6k-3)}} \cdot b_k(\arctan t_2) d^\times t_2 d^\times t_1. \end{aligned} \quad (6.12)$$

By the estimate of f_2,

$$|f_2(\ln t_1 t_2, v, u)| \leq (t_1 t_2)^{-3\mathrm{Re}(\xi_2)} h_2(\ln t_1 t_2) q'(v),$$

for some polynomial h_2 and a continuous seminorm q'. As we mentioned before, we only have finitely many l_1, l_2, k which only depend on (a, b). If we require the exponent $\mathrm{Re}(\xi_2)$ to be sufficiently negative so that it will beat the exponents of t_1, t_2 in (6.12), then the integral (6.12) converges absolutely in the vertical strip $\mathrm{Re}(s) \in (a, b)$. Thus it defines a holomorphic function in s and u which is continuous in v. The contribution of the term $t_2^{-3\xi_u}(\ln t_1)^{r_1}(\ln t_2)^{r_2} f_1(\ln t_1, v, u)$ can be analyzed in the same way.

The term $(\ln t_1)^{r_1}(\ln t_2)^{r_2} f_3(\ln t_1, \ln t_2, v)$ contributes an integral

$$\int_0^\epsilon \int_0^\epsilon (\ln t_1)^{r_1}(\ln t_2)^{r_2} f_3(\ln t_1, \ln t_2, v, u) \frac{t_1^{9s-6+l_1} t_2^{9s-6+l_2}}{(t_1^2+t_2^2)^{\frac{9}{2}s}} \cdot r^{6k+3} b_k(\theta) d^\times t_1 d^\times t_2 \quad (6.13)$$

in (6.9). By the estimate of f_3, if we choose $\mathrm{Re}(\xi_1)$, $\mathrm{Re}(\xi_2)$ so negative that the exponent -3$\mathrm{Re}(\xi_1)$ and -3$\mathrm{Re}(\xi_2)$ in the estimate of f_3 can beat all the exponents in the integrand of (6.13), then the integral (6.13) is holomorphic in s, u and continuous in v. In the end, after taking the $\mathbb{C}(u)$-linear combination, we finally prove that the local integrals are meromorphic in s, u.

To summarize all of the above, we proved that $C(W_{v,u}, f_s)$ has a meromorphic continuation in s, u and continuous in v under the Fréchet topology of V_{π_u}. The continuity on the second variable is straight forward. Because f_s only affects $F(t_1, t_2, s)$. The coefficients $a_k(\theta)$, $b_k(\theta)$ depend on f_s. They are continuous with respect to f_s when f_s runs in the Fréchet space V_{ρ_s}. So, if a sequence $f_{s,k} \to 0$ in V_{ρ_s}, then all $a_k(\theta)$, $b_k(\theta)$ tend to zero. This implies the meromorphic continuation of $C(W_{v,u}, f_s)$ is continuous on V_{ρ_s}. $\square$

Remark 6.2.3. *To end this Section, we remark that the proof of the complex case proceeds almost in the same way as the real case, the only difference comes from the dz-integral. In the complex case, the dz-integral is the Jacquet integral of a principal series of* $\mathrm{SL}_2(\mathbb{C})$.

6.3 Some Elementary Integrals

In this Section, we will study some elementary integrals and their meromorphic continuations. These integrals appears in the proof of Theorem 1.0.11. Let D_ϵ be the unit square $[0,\epsilon] \times [0,\epsilon]$ on the xy-plane. We use polar coordinate

$$x = r\cos\theta, y = r\sin\theta.$$

Our goal is to prove

Proposition 6.3.1. *Let $f(\theta, s)$ be a smooth periodic function of θ on $[0, 2\pi]$, which is holomorphic in s in some right half plane $Re(s) \geq s_0$, then for any complex numbers a_1, b_1, c_1, and any nonnegative integers a_2, b_2, c_2, the following integral*

$$\iint_{D_\epsilon} r^{s+a_1}(\ln r)^{a_2} \cdot (\cos\theta)^{s+b_1}(\ln\cos\theta)^{b_2} \cdot (\sin\theta)^{s+c_1}(\ln\cos\theta)^{c_2} f(\theta, s) dr d\theta$$

converges absolutely when $Re(s)$ is sufficiently large, and it has a meromorphic continuation to the same right half plane $Re(s) \geq s_0$.

Lemma 6.3.2. *Let $f(\theta, s_1, s_2)$ be a smooth periodic function of θ on $[0, 2\pi]$, holomorphic in some right half space $\{(s_1, s_2) \in \mathbb{C}^2 \mid Re(s_1), Re(s_2) \geq s_0\}$. Then for any complex numbers b_1, c_1, and nonnegative integers b_2, c_2 the following integral*

$$\int_0^{\frac{\pi}{2}} (\cos\theta)^{s_1+b_1}(\ln\cos\theta)^{b_2} \cdot (\sin\theta)^{s_2+c_1}(\ln\cos\theta)^{c_2} f(\theta, s_1, s_2) d\theta \tag{6.14}$$

converges absolutely when $Re(s_1), Re(s_2)$ are sufficiently large, and it has a meromorphic continuation to the same right half space $\{(s_1, s_2) \in \mathbb{C}^2 \mid Re(s_1), Re(s_2) \geq s_0\}$.

Proof. It is clear that the integral converges when $\mathrm{Re}(s_1), \mathrm{Re}(s_2)$ are sufficiently large. When $\mathrm{Re}(s_1), \mathrm{Re}(s_2)$ are large, by integration by parts, we have

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} (\cos\theta)^{s_1+b_1+2} (\ln\cos\theta)^{b_2} \cdot (\sin\theta)^{s_2+c_1} (\ln\cos\theta)^{c_2} f(\theta, s_1, s_2) d\theta \\
=& \frac{1}{s_2+c_1+1} \int_0^{\frac{\pi}{2}} (s_1+b_1+1)(\sin\theta)^{s_2+c_1+2} (\cos\theta)^{s_1+b_1} (\ln\cos\theta)^{b_2} (\ln\sin\theta)^{c_2} f(\theta, s_1, s_2) d\theta \\
&+ \frac{1}{s_2+c_1+1} \int_0^{\frac{\pi}{2}} (\sin\theta)^{s_2+c_1+2} (\cos\theta)^{s_1+b_1} (\ln\cos\theta)^{b_2-1} (\ln\sin\theta)^{c_2} f(\theta, s_1, s_2) d\theta \\
&- \frac{1}{s_2+c_1+1} \int_0^{\frac{\pi}{2}} (\sin\theta)^{s_2+c_1} (\cos\theta)^{s_1+b_1+2} (\ln\cos\theta)^{b_2} (\ln\sin\theta)^{c_2-1} f(\theta, s_1, s_2) d\theta \\
&- \frac{1}{s_2+c_1+1} \int_0^{\frac{\pi}{2}} (\sin\theta)^{s_2+c_1+1} (\cos\theta)^{s_1+b_1+1} (\ln\cos\theta)^{b_2} (\ln\sin\theta)^{c_2} \frac{\partial f}{\partial\theta}(\theta, s_1, s_2) d\theta.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\frac{s_1+b_1+1}{s_2+c_1+1} \int_0^{\frac{\pi}{2}} (\cos\theta)^{s_1+b_1} (\ln\cos\theta)^{b_2} \cdot (\sin\theta)^{s_2+c_1} (\ln\cos\theta)^{c_2} f(\theta, s_1, s_2) d\theta \\
&= (1 + \frac{s_1+b_1+1}{s_2+c_1+1}) \int_0^{\frac{\pi}{2}} (\cos\theta)^{s_1+b_1+2} (\ln\cos\theta)^{b_2} \cdot (\sin\theta)^{s_2+c_1} (\ln\cos\theta)^{c_2} f(\theta, s_1, s_2) d\theta \\
&- \frac{1}{s_2+c_1+1} \int_0^{\frac{\pi}{2}} (\sin\theta)^{s_2+c_1+2} (\cos\theta)^{s_1+b_1} (\ln\cos\theta)^{b_2-1} (\ln\sin\theta)^{c_2} f(\theta, s_1, s_2) d\theta \\
&+ \frac{1}{s_2+c_1+1} \int_0^{\frac{\pi}{2}} (\sin\theta)^{s_2+c_1} (\cos\theta)^{s_1+b_1+2} (\ln\cos\theta)^{b_2} (\ln\sin\theta)^{c_2-1} f(\theta, s_1, s_2) d\theta \\
&+ \frac{1}{s_2+c_1+1} \int_0^{\frac{\pi}{2}} (\sin\theta)^{s_2+c_1+1} (\cos\theta)^{s_1+b_1+1} (\ln\cos\theta)^{b_2} (\ln\sin\theta)^{c_2} \frac{\partial f}{\partial\theta}(\theta, s_1, s_2) d\theta.
\end{aligned}$$

Similarly, there is another integration by parts which raises the exponents of sin by 2. So, whenever we apply integration by parts, either the exponents of sin or cos increase, or the exponents of $\ln\sin$ or $\ln\cos$ decrease. Hence it suffices to prove that the following integral has a meromorphic continuation

$$\int_0^{\frac{\pi}{2}} (\cos\theta)^{s_1+b_1} (\sin\theta)^{s_2+c_1} f(\theta, s_1, s_2) d\theta. \tag{6.15}$$

As above, by integration by parts,

$$\begin{aligned}&\frac{s_1+b_1+1}{s_2+c_1+1}\int_0^{\frac{\pi}{2}}(\cos\theta)^{s_1+b_1}(\sin\theta)^{s_2+c_1}f(\theta,s_1,s_2)d\theta\\=&(1+\frac{s_1+b_1+1}{s_2+c_1+1})\int_0^{\frac{\pi}{2}}(\cos\theta)^{s_1+b_1+2}(\sin\theta)^{s_2+c_1}f(\theta,s_1,s_2)d\theta\\+&(\frac{1}{s_2+c_1+1})\int_0^{\frac{\pi}{2}}(\cos\theta)^{s_1+b_1+1}(\sin\theta)^{s_2+c_1+1}\frac{\partial f}{\partial\theta}(\theta,s_1,s_2)d\theta.\end{aligned}$$

After doing integration by parts as many times as we need, the exponents of $\sin\theta$ and $\cos\theta$ are nonnegative when $\mathrm{Re}(s_1), \mathrm{Re}(s_2) \geq s_0$, this shows (6.15), hence (6.14) has a meromorphic continuation. □

Remark 6.3.3. *We can also show that the following integrals*

$$\int_0^{\frac{\pi}{4}}(\cos\theta)^{s+b_1}(\ln\cos\theta)^{b_2}\cdot(\sin\theta)^{s+c_1}(\ln\cos\theta)^{c_2}f(\theta,s)d\theta$$

and

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(\cos\theta)^{s+b_1}(\ln\cos\theta)^{b_2}\cdot(\sin\theta)^{s+c_1}(\ln\cos\theta)^{c_2}f(\theta,s)d\theta$$

also have meromorphic continuation. The proof is the same as the proof of Lemma 6.3.2. We only need to change the upper and lower bounds of the integrals.

Now we return to prove Proposition 6.3.1. For simplicity, we only prove the Proposition for $\epsilon = 1$, the proof for general ϵ is exactly the same (only notationally more complicated). In the following proof, to simplify notations, we write D for the unit square in the first quadrant instead of D_1.

Proof of Proposition 6.3.1. Set D_1 to be the quarter of unit disk in the first quadrant, $D_2 = D - D_1$. Then

$$\begin{aligned}&\iint_{D_1} r^{s+a_1}(\ln r)^{a_2}\cdot(\cos\theta)^{s+b_1}(\ln\cos\theta)^{b_2}\cdot(\sin\theta)^{s+c_1}(\ln\cos\theta)^{c_2}f(\theta,s)drd\theta\\=&\int_0^1 r^{s+a_1}(\ln r)^{a_2}dr\cdot\int_0^{\frac{\pi}{2}}(\cos\theta)^{s+b_1}(\ln\cos\theta)^{b_2}\cdot(\sin\theta)^{s+c_1}(\ln\cos\theta)^{c_2}f(\theta,s)d\theta,\end{aligned}$$

hence by Lemma 6.3.2, the above integral has a meromorphic continuation. On the other hand,

$$\iint_{D_2} r^{s+a_1}(\ln r)^{a_2}\cdot(\cos\theta)^{s+b_1}(\ln\cos\theta)^{b_2}\cdot(\sin\theta)^{s+c_1}(\ln\cos\theta)^{c_2}f(\theta,s)drd\theta$$

$$= \int_0^{\frac{\pi}{4}}\Big(\int_1^{\frac{1}{\cos\theta}} r^{s+a_1}(\ln r)^{a_2}dr\Big)(\cos\theta)^{s+b_1}(\ln\cos\theta)^{b_2}\cdot(\sin\theta)^{s+c_1}(\ln\cos\theta)^{c_2}f(\theta,s)d\theta$$

$$+ \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\Big(\int_1^{\frac{1}{\sin\theta}} r^{s+a_1}(\ln r)^{a_2}dr\Big)(\cos\theta)^{s+b_1}(\ln\cos\theta)^{b_2}\cdot(\sin\theta)^{s+c_1}(\ln\cos\theta)^{c_2}f(\theta,s)d\theta.$$

Applying integration by parts to dr-integral successively, we can show that the dr-integral in the first term on the right hand side is a linear combination of $(\cos\theta)^{b_1'}(\ln\cos\theta)^{b_2'}$, with coefficients in the field of rational functions $\mathbb{C}(s)$. So by Lemma 6.3.2 (and the remark below it), the first term on the right hand side has a meromorphic continuation. Similarly, the second term also has a meromorphic continuation. These conclude the Proposition. $\square$

Chapter 7

The Uniqueness Theorem (Archimedean Case)

Now we prove Theorem 1.0.12.

Proof of Theorem 1.0.12. We will follow Chap 5 of [War] closely. Roughly speaking, the Uniqueness Theorem is a direct consequence of Bruhat's theory. Given two Fréchet representations of π_1 and π_2 of a Lie group G, we denote by $\mathrm{Bil}_G(\pi_1, \pi_2)$ the space of G-equivariant continuous bilinear form on $V_{\pi_1} \hat{\otimes} V_{\pi_2}$. The dimension of $\mathrm{Bil}_G(\pi_1, \pi_2)$ is called the intertwining number between representations π_1 and π_2. When $F = \mathbb{C}$ or $\mathbb{R}$, the restriction of π on $\mathrm{SL}_3(F)$ is also irreducible. So in this Chapter, we can also regard π as a representation of $\mathrm{SL}_3(F)$. By the Reciprocity Law ([War, Theorem 5.3.3.1]),

$$\dim \mathrm{Bil}_{\mathrm{SL}_3(F)}(\pi, \quad \mathrm{Ind}_{P(F)}^{\mathrm{G}_2(F)} \delta_P^{s-\frac{1}{2}}\Big|_{\mathrm{SL}_3(F)}) = \dim \mathrm{Bil}_{\mathrm{G}_2(F)}(\mathrm{Ind}_{\mathrm{SL}_3(F)}^{\mathrm{G}_2(F)} \pi, \quad \mathrm{Ind}_{P(F)}^{\mathrm{G}_2(F)} \delta_P^{s-\frac{1}{2}}).$$

Let $\mathrm{SL}_3(F)$ act on the flag variety $P(F)\backslash \mathrm{G}_2(F)$ on the right. By Lemma 1.0.4, there are exactly two orbits with representatives e and $\gamma = x_{-(\alpha+\beta)}(-1)w_\beta$. Let Ω_e (Ω_γ resp.) be orbit containing e (γ resp.). Then by [War, Lemma 5.2.4.4],

$$\dim \mathrm{Bil}_{\mathrm{G}_2(F)}(\mathrm{Ind}_{\mathrm{SL}_3(F)}^{\mathrm{G}_2(F)} \pi, \mathrm{Ind}_{P(F)}^{\mathrm{G}_2(F)} \delta_P^{s-\frac{1}{2}}) \leq \dim\, [\Omega_e] + \dim\, [\Omega_\gamma],$$

where $[\Omega_e]$ ($[\Omega_\gamma]$ resp.) is the vector space of $V_\pi \hat{\otimes} V_{\rho_s}$-distribution T on $\mathrm{G}_2(F)$ supported on Ω_e (Ω_γ resp.) satisfying a certain equivariant property. For the explicit formula of this equivariant property, see [War, Theorem 5.3.2.1], but we do not need it here. The dimensions of $[\Omega_e]$ and $[\Omega_\gamma]$ can be estimated (by [War, Section 5.2.3]) in the following way.

We first analyze the small orbit Ω_e, put $H_e = P(F) \cap \mathrm{SL}_3(F)$, then clearly H_e contains the standard maximal unipotent subgroup of $\mathrm{SL}_3(F)$. By [War, Section 5.2.3],

$$\dim\, [\Omega_e] \leq \sum_{n=0}^{+\infty} i(\delta_p^{s-\frac{1}{2}}, \pi, \Omega_e, n).$$

Here $i(\delta_p^{s-\frac{1}{2}}, \pi, \Omega_e, n)$ is the intertwining number between the representation of H_e

$$\delta_P^{s-\frac{1}{2}} \hat{\otimes} \pi$$

and some finite dimensional representation Λ_n of H_e coming from transversal derivatives. We know that any eigenvalue of a unipotent matrix on a finite dimensional vector space must be 1. Suppose that $i(\delta_p^{s-\frac{1}{2}}, \pi, \Omega_e, n) \neq 0$ for some n. Since H_e contains all standard upper triangular unipotent matrices in $\mathrm{SL}_3(F)$, the generic character must be trivial. We get a contradiction.

For the big open orbit Ω_γ, put

$$H_\gamma = \mathrm{SL}_3(F) \cap \gamma^{-1} P(F) \gamma = \left\{ \begin{pmatrix} 1 & -x & z \\ & 1 & x \\ & & 1 \end{pmatrix} \begin{pmatrix} a & & \\ & 1 & \\ & & a^{-1} \end{pmatrix} \right\}.$$

Then by [War, Section 5.2.3], $\dim[\Omega_\gamma]$ is bounded by $i(\delta_P^{s-\frac{1}{2}}, \pi, \Omega, 0)$, where $i(\delta_P^{s-\frac{1}{2}}, \pi, \Omega, 0)$ is the intertwining number between two representations of H_γ:

$$\delta_\gamma : h \mapsto \delta_P^{s-\frac{1}{2}}(\gamma h \gamma^{-1}) \qquad \text{and} \qquad \pi\big|_{H_\gamma}.$$

By Casselman's Subrepresentation Theorem [Wal1, 3.8.3], we may assume that π is a quotient of a principal series $\mathrm{Ind}_{B_{\mathrm{SL}_3}}^{\mathrm{SL}_3(F)}\sigma$, then

$$\begin{aligned}\dim\,\mathrm{Bil}_{H_\gamma}(\pi\big|_{H_\gamma},\delta_\gamma) &= \dim\,\mathrm{Bil}_{\mathrm{SL}_3(F)}(\pi,\quad \mathrm{Ind}_{H_\gamma}^{\mathrm{SL}_3(F)}\delta_\gamma)\\ &\leq \dim\,\mathrm{Bil}_{\mathrm{SL}_3(F)}(\mathrm{Ind}_{B_{SL_3}}^{\mathrm{SL}_3(F)}\sigma,\quad \mathrm{Ind}_{H_\gamma}^{\mathrm{SL}_3(F)}\delta_\gamma).\end{aligned}$$

So by Bruhat's theory, the orbit space which we consider this time is $B_{\mathrm{SL}_3}\backslash\mathrm{SL}_3(F)/H_\gamma$. There are twelve orbits. We can choose the representatives $y_j = w_j \begin{pmatrix}1 & 1 & \\ & 1 & \\ & & 1\end{pmatrix}$ and w_j, where w_j are Weyl elements of $\mathrm{SL}_3(F)$ listed below:

$$\begin{aligned} w_1 &= I, & w_2 &= \begin{pmatrix}1 & & \\ & & 1\\ & -1 & \end{pmatrix}, w_3 = \begin{pmatrix} & 1 & \\ -1 & & \\ & & 1\end{pmatrix},\\ w_4 &= \begin{pmatrix} & 1 & \\ & & 1\\ 1 & & \end{pmatrix}, & w_5 &= \begin{pmatrix} & & 1\\ 1 & & \\ & -1 & \end{pmatrix}, w_6 = \begin{pmatrix} & & 1\\ & 1 & \\ -1 & & \end{pmatrix}.\end{aligned}$$

Then

$$\dim\,\mathrm{Bil}_{\mathrm{SL}_3(F)}(\mathrm{Ind}_{B_{SL_3}}^{\mathrm{SL}_3(F)}\sigma,\quad \mathrm{Ind}_{H_\gamma}^{\mathrm{SL}_3(F)}\delta_\gamma)$$

is bounded by

$$\sum_n\sum_{j=0}^{6} i(\sigma,\delta_\gamma,y_j,n) + \sum_n\sum_{i=0}^{6} i(\sigma,\delta_\gamma,w_i,n).$$

Here, for $i = 1,2,3,4,5,6$, clearly $H_i = B_{\mathrm{SL}_3}\cap w_i^{-1}H_\gamma w_i$ contains a nontrivial diagonal subgroup. Similarly, for $j = 1,2,3,4,5$, $H_j = B_{\mathrm{SL}_3}\cap y_j^{-1}H_\gamma y_j$ still contains a nontrivial one dimensional abelian subgroup $A(i)$. To be more precise,

$$A(1) = \left\{ y_1^{-1} \begin{pmatrix} a & a-1 & \\ & 1 & \\ & & a^{-1} \end{pmatrix} y_1 \,\middle|\, a \in F^\times \right\};$$

$$A(2) = \left\{ y_2^{-1} \begin{pmatrix} a & a-1 & \\ & 1 & \\ & & a^{-1} \end{pmatrix} y_2 \,\middle|\, a \in F^\times \right\};$$

$$A(3) = \left\{ y_3^{-1} \begin{pmatrix} 1 & & z \\ & 1 & \\ & & 1 \end{pmatrix} y_3 \,\middle|\, z \in F \right\};$$

$$A(4) = \left\{ \begin{pmatrix} 1 & \frac{a-1}{a} & \\ & a^{-1} & \\ & & a \end{pmatrix} \,\middle|\, a \in F^\times \right\};$$

$$A(5) = \left\{ \begin{pmatrix} a^{-1} & & \\ & a & -a+1 \\ & & 1 \end{pmatrix} \,\middle|\, a \in F^\times \right\}.$$

For each above abelian subgroup $A(i)$, we can choose a generator d_i. Recall $i(\sigma, \delta_\gamma, y_j, n)$ is the intertwining number between the representation

$$\sigma \otimes (\delta_\gamma)^{y_j} \tag{7.1}$$

of H_j and some finite dimensional representation $\widetilde{\Lambda}_n$ of H_j coming from transversal derivatives. Here $(\delta_\gamma)^{y_j}$ is the representation obtained from twisting δ_γ by y_j. If $i(\sigma, \delta_\gamma, y_j, n)$ is nonzero, then $\sigma \otimes (\delta_\gamma)^{y_j}$ is a one dimensional subrepresentation of $\widetilde{\Lambda}_n$. Hence

$$\sigma \otimes (\delta_\gamma)^{y_j}(d_j) = \chi(d_j) \tag{7.2}$$

for some quasicharacter χ. The above is in fact an equation of s which only has at most countably many solutions. These solutions form a discrete set. The same argument applies to $i(\sigma, \delta_\gamma, w_i, n)$.

Thus so far, we have at most countably many finite dimensional representations

$\widetilde{\Lambda_n}$. For each $\widetilde{\Lambda}_n$, we only have finitely many one-dimensional subrepresentations. Thus there exists a discrete, at most countable subset S of $\mathbb{C}$ such that whenever $s \notin S$,

$$i(\sigma, \delta_\gamma, y_j, n) = i(\sigma, \delta_\gamma, w_i, n) = 0,$$

where $j = 1, 2, 3, 4, 5$, and $i = 1, 2, 3, 4, 5, 6$. As for H_6, it is in fact a trivial subgroup, so the above equation (7.2) is in fact an identity. The big orbit $B_{\mathrm{SL}_3} y_6 H_\gamma$ contributes one in

$$\dim \mathrm{Bil}_{\mathrm{SL}_3(F)}(\pi, \quad \mathrm{Ind}_{P(F)}^{\mathrm{G}_2(F)} \delta_P^{s-\frac{1}{2}}\Big|_{\mathrm{SL}_3(F)}).$$

$\square$

Chapter 8

Gamma and Beta Function and their Integral Representations

In this Chapter, we collect some facts on the Gamma functions and Beta functions over local fields that will be intensively use in the rest of this thesis. Our main reference are [V1], [V2]. We will also prove a useful Lemma on Fourier transform and Mellin transform.

8.1 Definitions of Gamma and Beta Function over Local Field

When F is non-archimedean, we set ψ to be a nontrivial additive character of F and r to be the conductor of ψ. As usual, let $\mathcal{O}_F$ be the ring of integers and q be the cardinality of the residual field. The group $F^\times \simeq \mathbb{Z} \times \mathcal{O}_F^\times$. In this case, we assume that χ, χ_1, χ_2 are multiplicative characters of $F^\times$ trivial on $\mathbb{Z}$. When F is archimedean, let ψ also be an additive character of F. Clearly $F^\times \simeq \mathbb{R}_+^\times \times K_F$, where K_F is a compact abelian subgroup of $F^\times$. In this case, we set $r = 0, q = 1$, and assume that χ, χ_1, χ_2 are multiplicative characters of $F^\times$ which are trivial on $\mathbb{R}_+^\times$. In both cases, we define, for any $\mathrm{Re}(s) \in (0, 1)$, the Gamma function over F by

$$\Gamma_F(s, \chi) = q^{-\frac{r}{2}} \int_F \chi(x)|x|_F^{s-1}\psi(x)dx \tag{8.1}$$

The RHS of (8.1) is a convergent integral when $\mathrm{Re}(s) \in (0,1)$. For the other s, $\Gamma_F(s,\chi)$ is defined by its meromorphic continuation. We emphasize that when F is archimedean, the RHS of (8.1) is only conditionally convergent (not absolutely convergent)! We also need to consider the following Beta function over F,

$$B_F(s_1,s_2,\chi_1,\chi_2) = \chi_1(-1)\chi_2(-1)\int_F \chi_1(x)\chi_2(1-x)|x|_F^{s_1-1}|1-x|_F^{s_2-1}dx. \tag{8.2}$$

We require that $\mathrm{Re}(s_1) > 0, \mathrm{Re}(s_2) > 0, \mathrm{Re}(s_1+s_2) < 1$ in order for the above integral to converge absolutely.

Our goal is to relate the Gamma function $\Gamma_F(s,\chi)$ and Beta function $B_F(s_1,s_2,\chi_1,\chi_2)$ with the local gamma factors in the Tate thesis when F is archimedean. This can be done by switching order of the Fourier transform and Mellin transform in the local zeta integral considered in the Tate thesis. For simplicity, for the rest of this Chapter, we assume that

1. if F is archimedean, then ψ is the unitary character of F as in (1.11);
2. if F is non-archimedean, then ψ is unramified.

8.2 Local Integral in Tate's Thesis

In the theory of integral representations of L-functions, we often encounter the situation where we have to first apply the Fourier transform and then apply the Mellin transform. Sometimes, it is very convenient if we can interchange these two integral transforms. We will use Tate's thesis to illustrate this idea. In this Section, F can be either archimedean or non-archimedean.

Let us assume that Φ is a Schwartz function on F. The local Zeta-integrals considered in Tate's thesis are

$$Z(s,\chi,\Phi) = \int_F \Phi(a)\chi(a)|a|_F^s d^\times a.$$

They enjoy a functional equation

$$Z(1-s,\chi^{-1},\hat{\Phi}) = \gamma_F(s,\chi,\psi)Z(s,\chi,\Phi), \tag{8.3}$$

where $\hat{\Phi}$ is the Fourier transform of Φ. We can rewrite the LHS as

$$Z(1-s,\chi^{-1},\hat{\Phi}) = \int_F \Big(\int_F \Phi(x)\psi(ax)dx\Big)\chi^{-1}(a)|a|_F^{-s}da. \tag{8.4}$$

We note that the RHS of (8.4) should be interpreted as an iterated integral. Unfortunately it does not converge absolutely as a double integral. Yet, formally, we want to change the order of integration and rewrite the RHS of (8.4) as

$$\int_F \int_F \Phi(x)\psi(ax)\chi^{-1}(a)|a|_F^{-s}dadx. \tag{8.5}$$

The above is also interpreted as an iterated integral (da-integral converges when $\mathrm{Re}(s) \in (0,1)$). Switching the role of x and a in (8.5), we rewrite the functional equation (8.3) as

$$\int_F \Phi(a)\Big(\int_F \psi(ax)\chi^{-1}(x)|x|_F^{-s}dx\Big)da = \gamma_F(s,\chi,\psi)\int_F \Phi(a)\chi(a)|a|_F^{s-1}da.$$

Both sides of the above equation can be interpreted as an integral of a Schwartz function against a distribution. Therefore, when the integral

$$\int_F \psi(ax)\chi^{-1}(x)|x|_F^{-s}dx$$

converges (i.e the distribution is defined by a function),

$$\int_F \psi(ax)\chi^{-1}(x)|x|_F^{-s}dx = \gamma_F(s,\chi,\psi)\chi(a)|a|_F^{s-1}.$$

Changing the variable $x \mapsto xa^{-1}$ on the LHS of the above, we obtain that

$$\int_F \psi(x)\chi^{-1}(x)|x|_F^{-s}dx \cdot \chi(a)|a|_F^{s-1} = \gamma_F(s,\chi,\psi)\chi(a)|a|_F^{s-1}.$$

Hence,

$$\gamma_F(s,\chi,\psi) = \int_F \psi(x)\chi^{-1}(x)|x|_F^{-s}dx = \Gamma_F(1-s,\chi^{-1}). \tag{8.6}$$

Thus it remains to justify the change of order of integration in the iterated integral (8.4). In fact, this change of order of integration fits in a more general situation (Lemma 8.3.1). It helps us to change the order of Mellin transform and Fourier transform. We will discuss this technical Lemma 8.3.1 in the next Section.

8.3 The Lemma on Fourier Transform and Mellin Transform

In this Section, F can be archimedean or non-archimedean. For any F^m and $r > 0$, we define a ball

$$B_{r_1,r_2,\cdots,r_m} = \{(x_1, x_2, \cdots, x_m) \in F^m \mid |x_i|_F \leq r_i, i = 1, 2, \cdots, m\}.$$

When no confusion rises, to simplify notations, we also write

$$B_{r,m} = \{(x_1, x_2, \cdots, x_m) \in F^m \mid |x_i|_F \leq r, i = 1, 2, \cdots, m\}.$$

When no confusion arises, we will drop the dimension subscript m and only emphasize the size of the ball. In the following lemma, we use dx and $d^\times x$ to emphasize the difference between the additive Haar measure and the multiplicative Haar measure. Now we state the lemma.

Lemma 8.3.1. *Let $X = F^m$, $A = (F^\times)^n$. We assume that $m \leq n$. The group $X_0 = (F^\times)^m$ form a dense subspace of X. We assume that there is an embedding $j : X_0 \to A$ whose image $j(X_0) = A_2$. We write $A = A_1 \times A_2$. Suppose that a smooth function $F_s(a, x)$ on $A \times X$ satisfies*

$$F_s(a, x) = \sigma(j(x))F_s(a \cdot j(x), 1), \quad \text{for every } a \in A, x \in X_0, \tag{8.7}$$

where σ is a quasicharacter of A which may depend on s. Let χ be another quasicharacter of A and ψ be a nontrivial additive character of X. We assume that there exists a domain D such that for any $s \in D$

1. *$\int_A F_s(a, 1)\chi(a)d^\times a$ converges absolutely,*

2. *$\int_{A_1 \times X} F_s(a_1, x)\chi(a_1)dxd^\times a$ converges absolutely,*

3. *$\int_X (\sigma\chi^{-1})(j(x))\psi(x)dx$ converges (conditionally),*

4. *$\int_{j^{-1}(a_2) \in B_r} (\sigma\chi^{-1})(a_2) \cdot |j^{-1}(a_2)|d^\times a_2$ converges absolutely for every $r > 0$.*

Then the following two iterated integrals

1. $\int_A \int_X F_s(a,x)\psi(x)\chi(a)dxd^\times a$

2. $\int_X \int_A F_s(a,x)\psi(x)\chi(a)d^\times adx$

both converge when $s \in D$ and they are equal. Moreover, they are also equal to either of the following two integrals

1. $\int_{A_1\times A_2}\int_X F_s(a_1,x)\sigma(a_2)\psi(x\cdot j^{-1}(a_2))\chi(a_1a_2^{-1})|j^{-1}(a_2)|_F dxd^\times a_1 d^\times a_2$ *(interpreted as an iterated integral),*

2. $\int_A F_s(a,1)\chi(a)d^\times a \cdot \int_X (\sigma\chi^{-1})(j(x))\psi(x)dx.$

Proof. 1) We first prove that in the range of convergence,

$$\begin{aligned}&\int_A\int_X F_s(a,x)\psi(x)\chi(a)dxd^\times a\\ =&\int_{A_1\times A_2}\int_X F_s(a_1,x)\sigma(a_2)\psi(x\cdot j^{-1}(a_2))\chi(a_1a_2^{-1})|j^{-1}(a_2)|_F dxd^\times a_1d^\times a_2.\end{aligned} \tag{8.8}$$

Indeed, by (8.7), it is easy to prove that

$$F_s(a,x_1x_2)=\sigma(j(x_1))F_s(a\cdot j(x_1),x_2)$$

for any $x_1,x_2\in X_0$. Hence,

$$\begin{aligned}&\int_A\int_X F_s(a,x)\psi(x)\chi(a)dxd^\times a\\ =&\int_{A_1\times A_2}\int_X F_s(a_1a_2,x)\psi(x)\chi(a_1a_2)dxd^\times a_1d^\times a_2\\ =&\int_{A_1\times A_2}\int_X F_s(a_1,x\cdot j^{-1}(a_2))\sigma^{-1}(a_2)\psi(x)\chi(a_1a_2)dxd^\times a_1d^\times a_2.\end{aligned}$$

We change $a_2\mapsto a_2^{-1}$, then the above is equal to

$$\int_{A_1\times A_2}\int_X F_s(a_1,x\cdot j^{-1}(a_2^{-1}))\sigma(a_2)\psi(x)\chi(a_1a_2^{-1})dxd^\times a_1d^\times a_2.$$

Then we only need to change the variable $x\mapsto x\cdot j^{-1}(a_2)$ to prove (8.8).

2) Now we prove that the iterated integral

$$\int_X \int_A F_s(a,x)\psi(x)\chi(a)d^\times a dx$$

converges when $s \in D$ and it is equal to

$$\int_A F_s(a,1)\chi(a)d^\times a \cdot \int_X (\sigma\chi^{-1})(j(x))\psi(x)dx.$$

This part is easy. We use (8.7). In the range of convergence,

$$\int_X \int_A F_s(a,x)\psi(x)\chi(a)d^\times a dx = \int_{X_0} \int_A \sigma(j(x))F_s(a \cdot j(x),1)\psi(x)\chi(a)d^\times a dx.$$

Here we ignore a measure-zero set $X - X_0$ since it will not change the value of the convergent integral. Now we only need to change the variable $a \mapsto a \cdot j(x)^{-1}$ to complete the proof.

3) By using the isomorphism between A_2 and X_0, we can rewrite

$$\begin{aligned}
&\int_A F_s(a,1)\chi(a)d^\times a \cdot \int_X (\sigma\chi^{-1})(j(x))\psi(x)dx \\
=&\int_{A_1\times A_2} F_s(a_1a_2,1)\chi(a_1a_2)d^\times a_1 d^\times a_2 \cdot \int_{X_0} (\sigma\chi^{-1})(j(x))\psi(x)dx \\
=&\int_{A_1\times X_0} F_s(a_1 \cdot j(x),1)\chi(a_1 \cdot j(x))d^\times a_1 d^\times j(x) \\
&\cdot \int_{A_2} (\sigma\chi^{-1})(a_2)\psi(j^{-1}(a_2))|j^{-1}(a_2)|_F d^\times j^{-1}(a_2).
\end{aligned}$$

The two Haar measure $d^\times a_2$ and $d^\times j^{-1}(a_2)$ only differ by a constant C_1, Similarly, the Haar measure $d^\times x$ and $d^\times j(x)$ differ by a constant C_1^{-1}, because inverse map have inverse Jacobian. Hence

$$d^\times x d^\times a_2 = d^\times j(x) d^\times j^{-1}(a_2).$$

Thus,

$$\begin{aligned}
&\int_A F_s(a,1)\chi(a)d^\times a \cdot \int_X (\sigma\chi^{-1})(j(x))\psi(x)dx \\
=&\int_X \int_{A_1\times A_2} F_s(a_1 \cdot j(x),1)\chi(a_1 \cdot j(x))(\sigma\chi^{-1})(a_2)\psi(j^{-1}(a_2))|j^{-1}(a_2)|_F d^\times a_1 d^\times a_2 d^\times x.
\end{aligned}$$

Using (8.7) and then changing $a_2 \mapsto a_2 \cdot j(x)$, we can obtain that

$$\begin{aligned} &\int_X \int_A F_s(a,x)\psi(x)\chi(a)d^\times a dx \\ =&\int_X \int_{A_1\times A_2} F_s(a_1,x)\sigma(a_2)\psi(x\cdot j^{-1}(a_2))\chi(a_1a_2^{-1})|j^{-1}(a_2)|_F d^\times a_1 d^\times a_2 dx. \end{aligned} \tag{8.9}$$

Note that by the assumption 2) and 4), the integral

$$\int_{A_1}\int_{j^{-1}(a_2)\in B_r}\int_X F_s(a_1,x)\sigma(a_2)\psi(x\cdot j^{-1}(a_2))\chi(a_1a_2^{-1})|j^{-1}(a_2)|_F dx d^\times a_2 d^\times a_1$$

converges absolutely for any $r > 0$. Thus,

$$\begin{aligned} &\int_{A_1}\int_{j^{-1}(a_2)\in B_r}\int_X F_s(a_1,x)\sigma(a_2)\psi(x\cdot j^{-1}(a_2))\chi(a_1a_2^{-1})|j^{-1}(a_2)|_F dx d^\times a_2 d^\times a_1 \\ =&\int_X\int_{A_1}\int_{j^{-1}(a_2)\in B_r} F_s(a_1,x)\sigma(a_2)\psi(x\cdot j^{-1}(a_2))\chi(a_1a_2^{-1})|j^{-1}(a_2)|_F d^\times a_2 d^\times a_1 dx. \end{aligned} \tag{8.10}$$

To complete the proof of the lemma, it suffices to prove the following statement: Given any $\epsilon > 0$, we can choose a sufficiently large C, such that whenever $r > C$, the difference between

$$\int_{A_1}\int_{j^{-1}(a_2)\in B_r}\int_X F_s(a_1,x)\sigma(a_2)\psi(x\cdot j^{-1}(a_2))\chi(a_1a_2^{-1})|j^{-1}(a_2)|_F dx d^\times a_1 d^\times a_2$$

and

$$\int_X\int_{A_1\times A_2} F_s(a_1,x)\sigma(a_2)\psi(x\cdot j^{-1}(a_2))\chi(a_1a_2^{-1})|j^{-1}(a_2)|_F d^\times a_1 d^\times a_2 dx$$

is bounded by ϵ.

Indeed, if the above statement holds, then the following equation follows

$$\begin{aligned} &\int_X\int_{A_1\times A_2} F_s(a_1,x)\sigma(a_2)\psi(x\cdot j^{-1}(a_2))\chi(a_1a_2^{-1})|j^{-1}(a_2)|_F d^\times a_1 d^\times a_2 dx \\ =&\int_{A_1\times A_2}\int_X F_s(a_1,x)\sigma(a_2)\psi(x\cdot j^{-1}(a_2))\chi(a_1a_2^{-1})|j^{-1}(a_2)|_F dx d^\times a_1 d^\times a_2, \end{aligned} \tag{8.11}$$

and we finish proving of the lemma.

4) We only need to prove the statement in 3). We observe that by (8.10),

$$\begin{aligned}
&\Big|\int_{A_1}\int_{j^{-1}(a_2)\in B_r}\int_X F_s(a_1,x)\sigma(a_2)\psi(x\cdot j^{-1}(a_2))\chi(a_1a_2^{-1})|j^{-1}(a_2)|_F dx d^\times a_1 d^\times a_2\\
&-\int_X\int_{A_1\times A_2} F_s(a_1,x)\sigma(a_2)\psi(x\cdot j^{-1}(a_2))\chi(a_1a_2^{-1})|j^{-1}(a_2)|_F d^\times a_1 d^\times a_2 dx\Big|\\
=&\Big|\int_X\int_{A_1}\int_{j^{-1}(a_2)\notin B_r} F_s(a_1,x)\sigma(a_2)\psi(x\cdot j^{-1}(a_2))\chi(a_1a_2^{-1})|j^{-1}(a_2)|_F d^\times a_1 d^\times a_2 dx\Big|.
\end{aligned} \tag{8.12}$$

By the assumption 1) that

$$\int_A F_s(a,1)\chi(a)d^\times a=\int_{A_1\times X_0} F_s(a_1\cdot j(x),1)\chi(a_1)\chi(j(x))d^\times a_1 d^\times j(x)$$

converges absolutely, the function of t defined by

$$\int_{A_1}\int_{x\in B_t}|F_s(a_1\cdot j(x),1)\chi(a_1)\chi(j(x))|d^\times a_1 d^\times j(x)$$

is a bounded continuous function. Let us say that it is bounded by L, i.e.,

$$\int_{A_1}\int_{x\in B_t}\big|F_s(a_1\cdot j(x),1)\chi(a_1)\chi(j(x))\big|d^\times a_1 d^\times j(x)\le L. \tag{8.13}$$

Also, given any $\epsilon_0>0$, there exists a sufficiently large $t_0>0$ such that for any $t>t_0$,

$$\int_{A_1}\int_{x\notin B_t}\big|F_s(a_1\cdot j(x),1)\chi(a_1)\chi(j(x))\big|d^\times a_1 d^\times j(x)\le \epsilon_0. \tag{8.14}$$

Similarly, by the assumption 3), the function of $r_1,r_2,\cdots,r_m$ defined by

$$\int_{x\notin B_{r_1,r_2,\cdots,r_m}}(\sigma\chi^{-1})(j(x))\psi(x)dx$$

is also a bounded continuous function. We assume that it is bounded by M, i.e.,

$$\Big|\int_{x\notin B_{r_1,r_2,\cdots,r_m}}(\sigma\chi^{-1})(j(x))\psi(x)dx\Big|\le M. \tag{8.15}$$

We can also choose a sufficiently large $r_0 > 0$ such that whenever $r_1, r_2, \cdots, r_m > r_0$,

$$|\int_{x\notin B_{r_1,r_2,\cdots,r_m}} (\sigma\chi^{-1})(j(x))\psi(x)dx\Big| \leq \epsilon_0. \tag{8.16}$$

We fix one $t > t_0$ satisfying (8.14), and choose a sufficiently large C such that $C > r_0 t$. Then for any $r > C$, we break the integral on the RHS of (8.12) into two parts:

$$\int_X \int_{A_1} \int_{j^{-1}(a_2)\notin B_r} = \int_{x\in B_t} \int_{A_1} \int_{j^{-1}(a_2)\notin B_r} + \int_{x\notin B_t} \int_{A_1} \int_{j^{-1}(a_2)\notin B_r}.$$

We change the variable $a_2 \mapsto a_2 \cdot j(x)^{-1}$ and use (8.7). Then

$$\begin{aligned}
&\int_{x\in B_t} \int_{A_1} \int_{j^{-1}(a_2)\notin B_r} F_s(a_1, x)\sigma(a_2)\psi(x \cdot j^{-1}(a_2))\chi(a_1 a_2^{-1})|j^{-1}(a_2)|_F d^\times a_2 d^\times a_1 dx \\
= &\int_{x\in B_t} \int_{A_1} \int_{j^{-1}(a_2)\cdot x\notin B_r} F_s(a_1 \cdot j(x), 1)\chi(a_1 \cdot j(x))(\sigma\chi^{-1})(a_2) \\
&\quad \psi(j^{-1}(a_2))|j^{-1}(a_2)|_F d^\times a_2 d^\times a_1 d^\times x \\
= &\int_{x\in B_t} \int_{A_1} \int_{j^{-1}(a_2)\cdot x\notin B_r} F_s(a_1 \cdot j(x), 1)\chi(a_1 \cdot j(x))(\sigma\chi^{-1})(a_2) \\
&\quad \psi(j^{-1}(a_2))|j^{-1}(a_2)|_F d^\times j^{-1}(a_2) d^\times a_1 d^\times j(x)
\end{aligned}$$

Let us change the variable $a_2' = j(x)$ and $y = j^{-1}(a_2)$, then the above is equal to

$$\int_{j^{-1}(a_2')\in B_t} \int_{A_1} \int_{j^{-1}(a_2')\cdot y\notin B_r} F_s(a_1 a_2', 1)\chi(a_1 a_2')(\sigma\chi^{-1})(j(y))\psi(y)dy d^\times a_1 d^\times a_2'. \tag{8.17}$$

The condition $j^{-1}(a_2') \in B_t$ implies that the absolute value of each coordinate of $j^{-1}(a_2')$ is bounded by t. Thus together with the condition that $j^{-1}(a_2')\cdot y \notin B_r$, we can conclude that each coordinate of y is greater than $rt^{-1} > r_0$. Thus the inner dy-integral is bounded by ϵ_0. Therefore,

$$\begin{aligned}
&|\int_{j^{-1}(a_2')\in B_t}\int_{A_1}\int_{j^{-1}(a_2')\cdot y\notin B_r} F_s(a_1a_2',1)\chi(a_1a_2')(\sigma\chi^{-1})(j(y))\psi(y)dyd^\times a_1d^\times a_2'\big|\\
&\leq \int_{j^{-1}(a_2')\in B_t}\int_{A_1}\big|F_s(a_1a_2',1)\chi(a_1a_2')\int_{j^{-1}(a_2')\cdot y\notin B_r}(\sigma\chi^{-1})(j(y))\psi(y)dy\big|d^\times a_1d^\times a_2'\\
&\leq \epsilon_0\int_{j^{-1}(a_2')\in B_t}\int_{A_1}\big|F_s(a_1a_2',1)\chi(a_1a_2')\big|d^\times a_1d^\times a_2'\\
&\leq \epsilon_0 L
\end{aligned}$$

by (8.13). Thus, we have proved that

$$|\int_{x\in B_t}\int_{A_1}\int_{j^{-1}(a_2)\notin B_r} F_s(a_1,x)\sigma(a_2)\psi(x\cdot j^{-1}(a_2))\chi(a_1a_2^{-1})|j^{-1}(a_2)|_F d^\times a_2d^\times a_1dx|\leq \epsilon_0 L. \tag{8.18}$$

Applying the same trick, we obtain that

$$\begin{aligned}
&\int_{x\notin B_t}\int_{A_1}\int_{j^{-1}(a_2)\notin B_r} F_s(a_1,x)\sigma(a_2)\psi(x\cdot j^{-1}(a_2))\chi(a_1a_2^{-1})|j^{-1}(a_2)|_F d^\times a_2d^\times a_1dx\\
=&\int_{j^{-1}(a_2')\notin B_t}\int_{A_1}\int_{j^{-1}(a_2')\cdot y\notin B_r} F_s(a_1a_2',1)\chi(a_1a_2')(\sigma\chi^{-1})(j(y))\psi(y)dyd^\times a_1d^\times a_2'.
\end{aligned}$$

In this case, the inner dy-integral is bounded by M by (8.15). Thus,

$$\begin{aligned}
&|\int_{x\notin B_t}\int_{A_1}\int_{j^{-1}(a_2)\notin B_r} F_s(a_1,x)\sigma(a_2)\psi(x\cdot j^{-1}(a_2))\chi(a_1a_2^{-1})|j^{-1}(a_2)|_F d^\times a_2d^\times a_1dx\big|\\
\leq&\int_{j^{-1}(a_2')\notin B_t}\int_{A_1}\big|F_s(a_1a_2',1)\chi(a_1a_2')\int_{j^{-1}(a_2')\cdot y\notin B_r}(\sigma\chi^{-1})(j(y))\psi(y)dy\big|d^\times a_1d^\times a_2'\\
\leq& M\int_{j^{-1}(a_2')\notin B_t}\int_{A_1}\big|F_s(a_1a_2',1)\chi(a_1a_2')\big|d^\times a_1d^\times a_2'\\
\leq&\epsilon_0 M
\end{aligned}$$

by (8.14). Combining the above with (8.18), we obtain that

$$\begin{aligned}
&|\int_X\int_{A_1}\int_{j^{-1}(a_2)\notin B_r} F_s(a_1,x)\sigma(a_2)\psi(x\cdot j^{-1}(a_2))\chi(a_1a_2^{-1})|j^{-1}(a_2)|_F d^\times a_2d^\times a_1dx\big|\\
\leq&\epsilon_0(L+M).
\end{aligned}$$

Hence the lemma is clear once we set $\epsilon_0 = \epsilon \cdot (L+M)^{-1}$. $\square$

8.4 An Application of Lemma 8.3.1 in Tate's Local Integral

Now we go back to see how to apply the Lemma 8.3.1 in Tate's thesis.

Corollary 8.4.1. *When* $Re(s) \in (0,1)$*, the following two iterated integrals are equal.*

$$\int_F \int_F \Phi(x)\psi(ax)\chi^{-1}(a)|a|_F^{-s} dx da = \int_F \int_F \Phi(x)\psi(ax)\chi^{-1}(a)|a|_F^{-s} da dx.$$

Proof. In this simple case, we take $A = X_0 = F^\times$, $X = F$, j to be the identity map. Take

$$F_s(a,x) = \Phi(ax)|a|_F^s.$$

Clearly

$$F_s(a,x) = F_s(ax,1)\sigma(x),$$

where $\sigma(x) = |x|_F^{-s}$. By Lemma 8.3.1, in the range of convergence,

$$\int_F \int_F \Phi(ax)\psi(x)\chi(a)|a|_F^s dx d^\times a = \int_F \int_F \Phi(x)\psi(ax)\chi^{-1}(a)|a|_F^{1-s} dx d^\times a$$

Now we check all four assumptions in Lemma 8.3.1.

Assumption 1): In this case,

$$\int_A F_s(a,1)\chi(a) d^\times a = \int_{F^\times} \Phi(a)|a|_F^s \chi(a) d^\times a.$$

The above integral clearly converges absolutely when $\mathrm{Re}(s) \in (0,1)$.

Assumption 2): In this case, A_1 is trivial and we only need to integrate over $X = F$.

$$\int_X F_s(1,x) dx = \int_F \Phi(x) dx$$

converges absolutely.

Assumption 3): This is also straightforward, since the range of convergence for the conditionally convergent integral

$$\int_F (\sigma\chi^{-1})(x)\psi(x)dx = \int_F |x|_F^{-s}\chi^{-1}(x)\psi(x)dx$$

is exactly $\mathrm{Re}(s) \in (0,1)$.

Assumption 4):

$$\int_{j^{-1}(a_2)\in B_r} (\sigma\chi^{-1})(a_2)\cdot|j^{-1}(a_2)|d^\times a_2 = \int_{|a|_F\le r} |a|_F^{1-s}\chi^{-1}(a)d^\times a.$$

is absolutely convergent for every $r>0$. □

Hence we complete our justification of (8.6) when $\mathrm{Re}(s) \in (0,1)$. For the other s, (8.6) is established by meromorphic continuation.

We close this Section by relating the Beta function $B_F(s_1,s_2,\chi_1,\chi_2)$ with the local gamma factors $\gamma_F(s,\chi,\psi)$.

Theorem 8.4.2. *Let F be a local field. We assume that the additive character ψ is unramified when F is non-archimedean and ψ is given as in* (1.11) *when F is archimedean. Then the Gamma Function Γ_F and Beta function B_F defined in (8.1) and (8.2) resp. satisfy*

$$\begin{aligned}\Gamma_F(s,\chi) &= \gamma_F(1-s,\chi^{-1},\psi)\\ B_F(s_1,s_2,\chi_1,\chi_2) &= \gamma_F(1-s_1,\chi_1^{-1},\psi)\gamma_F(1-s_2,\chi_2^{-1},\psi)\gamma_F(s_1+s_2,\chi_1\chi_2,\psi).\end{aligned}$$

where $\gamma_F(s,\chi,\psi)$ is the local gamma factor defined via the functional equation (8.3) *in Tate's thesis.*

Proof. The identity of $\Gamma_F(s,\chi)$ is already proved in (8.6). We only need to prove the identity of $B_F(s_1,s_2,\chi_1,\chi_2)$. In fact, in [V1], Vladimirov proves the Theorem when χ,χ_1,χ_2 are trivial. His argument can be generalized to all cases.

Set $f_i(x) = \chi_i(x)|x|_F^{s_i-1}, (i = 1, 2)$. It is easy to compute the convolution

$$\begin{aligned}(f_1 * f_2)(x) &= \int_F \chi_1(y)\chi_2(x-y)|y|_F^{s_1-1}|x-y|_F^{s_2-1}dy \\ &= \chi_1(x)\chi_2(x)|x|_F^{s_1+s_2-1}\int_F \chi_1(y)\chi_2(1-y)|y|_F^{s_1-1}|1-y|_F^{s_2-1}dy \\ &= \chi_1(x)\chi_2(x)|x|_F^{s_1+s_2-1}B_F(s_1, s_2, \chi_1, \chi_2)\chi_1(-1)\chi_2(-1).\end{aligned}$$

By taking the Fourier transform on both sides, we obtain that

$$\Gamma_F(s_1, \chi_1)\Gamma_F(s_2, \chi_2) = \Gamma_F(s_1 + s_2, \chi_1\chi_2)B(s_1, s_2, \chi_1, \chi_2)\chi_1(-1)\chi_2(-1). \tag{8.19}$$

The local gamma factors $\gamma_F(s, \chi, \psi)$ satisfy

$$\gamma_F(s, \chi, \psi)\gamma_F(1-s, \chi^{-1}, \psi) = \chi(-1). \tag{8.20}$$

Hence by (8.6), Γ_F also enjoys a functional equation

$$\Gamma_F(s, \chi)\Gamma_F(1-s, \chi^{-1}) = \chi(-1). \tag{8.21}$$

Plugging (8.21) into (8.19), we get

$$B_F(s_1, s_2, \chi_1, \chi_2) = \Gamma_F(s_1, \chi_1)\Gamma_F(s_2, \chi_2)\Gamma_F(1 - s_1 - s_2, \chi_1^{-1}\chi_2^{-1}). \tag{8.22}$$

In terms of the local gamma factors $\gamma_F(s, \chi, \psi)$, we can write

$$B_F(s_1, s_2, \chi_1, \chi_2) = \gamma_F(1 - s_1, \chi_1^{-1}, \psi)\gamma_F(1 - s_2, \chi_2^{-1}, \psi)\gamma_F(s_1 + s_2, \chi_1\chi_2, \psi). \tag{8.23}$$

$\square$

Chapter 9

The Intertwining Operator

9.1 Range of Convergence

The intertwining operator $M(w_{3\alpha+2\beta})$ which we consider in the local integrals intertwines two spherical representations:

$$M(w_{3\alpha+2\beta}) : \mathrm{Ind}_{P(F)}^{\mathrm{G}_2(F)} \delta_P^{s-\frac{1}{2}} \to \mathrm{Ind}_{P(F)}^{\mathrm{G}_2(F)} \delta_P^{-s+\frac{1}{2}}$$

When $\mathrm{Re}(s)$ is sufficiently large, $M(w_{3\alpha+2\beta})$ is defined by the following integral operator

$$(M(w_{3\alpha+2\beta})f_s)(g) = \int_{U(F)} f_s(w_{3\alpha+2\beta}^{-1} ug)du. \tag{9.1}$$

Let f_s^0 be the unique spherical function in V_{ρ_s} such that $f_s^0(I) = 1$. Then above integral converges absolutely when

$$\int_{U^-(F)} f_s(\overline{u})d\overline{u} \tag{9.2}$$

converges absolutely, where U^- is the unipotent radical of the parabolic subgroup of G_2 opposite to P. The integral (9.2) is majorized by a constant multiple of

$$\int_{U^-(F)} f_s^0(\overline{u})d\overline{u}. \tag{9.3}$$

Then by Gindikin-Karpelevich formula (also see [Hun]), the integral (9.3) converges absolutely when $\mathrm{Re}(s) > \frac{2}{3}$, and it is equal to

$$\frac{L(3s-1)}{L(3s)} \cdot \frac{L(9s-4)}{L(9s-3)} \cdot \frac{L(6s-3)}{L(6s-2)} \cdot \frac{L(9s-5)}{L(9s-4)} \cdot \frac{L(3s-2)}{L(3s-1)}.$$

Thus, we have shown that the intertwining integral (9.1) converges absolutely when $\mathrm{Re}(s) > \frac{2}{3}$.

9.2 The Intertwining Operator and Convolution Operator

Any smooth function f_s in the induced space V_{ρ_s} is determined by its value on the lower unipotent radical $U^-(F)$, since $P(F)U^-(F)$ forms a dense subset of $\mathrm{G}_2(F)$. Thus to detect the action of the intertwining operator on any f_s, it suffices to study the restriction of $M(w_{3\alpha+2\beta})f_s$ on $U^-(F)$. Now we are going to make full use of the matrix realization of G_2 described Section 2.3.

Using a simple matrix computation, we can obtain that

$$w_{3\alpha+2\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} & & -I_2 \\ & I_3 & \\ I_2 & & \end{pmatrix}.$$

We can parameterize the upper unipotent radical U in the following way

$$
\begin{aligned}
u &= x_{\beta}(a)x_{\alpha+\beta}(x)x_{3\alpha+2\beta}(b+xy)x_{2\alpha+\beta}(y)x_{3\alpha+\beta}(c)\\
&= \begin{pmatrix}
1 & 0 & x & 2y & -c & -b & cx-y^2\\
0 & 1 & -a & -2x & y & x^2-ay & -b-ac+xy\\
0 & 0 & 1 & 0 & 0 & y & c\\
0 & 0 & 0 & 1 & 0 & -x & -y\\
0 & 0 & 0 & 0 & 1 & -a & -x\\
0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\end{aligned}
$$

For simplicity, we set

$$
X = \begin{pmatrix} x & 2y & -c\\ -a & -2x & y \end{pmatrix}, \quad Y = \begin{pmatrix} -b & cx-y^2\\ x^2-ay & -b-ac+xy \end{pmatrix},
$$

and

$$
J_1 = \begin{pmatrix} 0 & 1\\ -1 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & 1\\ 0 & \frac{1}{2} & 0\\ 1 & 0 & 0 \end{pmatrix},
$$

then we can write

$$
u = \begin{pmatrix} I_2 & X & Y\\ & I_3 & J_2^T X^T J_1^T\\ & & I_2 \end{pmatrix}.
$$

We aim to decompose $w_{3\alpha+2\beta}^{-1}u$ into $p(w_{3\alpha+2\beta}^{-1}u)\cdot\overline{u}(w_{3\alpha+2\beta}^{-1}u)$, where $p(w_{3\alpha+2\beta}^{-1}u)\in P$ and $\overline{u}(w_{3\alpha+2\beta}^{-1}u)\in U^{-}$. This can be done whenever Y is invertible. To be more precise,

$$
\begin{aligned}
w_{3\alpha+2\beta}^{-1}u &= \begin{pmatrix} & & I_2\\ & I_3 & \\ -I_2 & & \end{pmatrix}\begin{pmatrix} I_2 & X & Y\\ & I_3 & J_2^T X^T J_1^T\\ & & I_2 \end{pmatrix} = \begin{pmatrix} & & I_2\\ & I_3 & J_2^T X^T J_1^T\\ -I_2 & -X & -Y \end{pmatrix}\\
&= \begin{pmatrix} \star & \star & \star\\ 0 & \star & \star\\ 0 & 0 & -Y \end{pmatrix}\begin{pmatrix} I_2 & & \\ \star & I_3 & \\ Y^{-1} & Y^{-1}X & I_2 \end{pmatrix},
\end{aligned}
$$

Denote by $p(w_{3\alpha+2\beta}^{-1}u)$ the first factor above, and by $\overline{u}(w_{3\alpha+2\beta}^{-1}u)$ the second factor. As we will see in a moment, there is no need to determine the explicit formulae for the 'stars' in the above decomposition. The Levi subgroup M of P is isomorphic to GL_2, and any element $m \in M$ is realized as a block diagonal matrix

$$m = \begin{pmatrix} \text{2 by 2 matrix } D_1 & & \\ & \text{3 by 3 matrix } D_2 & \\ & & \text{2 by 2 matrix } D_3 \end{pmatrix}.$$

In particular, the diagonal subgroup of M is embedded in G_2 via

$$\begin{pmatrix} t_3^{-1} & \\ & t_1 \end{pmatrix} \mapsto \begin{pmatrix} t_3^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t_1^{-1}t_3^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & t_1t_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & t_1^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & t_3 \end{pmatrix}.$$

Thus the modular character $\delta_P(m) = |t_1t_3^{-1}|_F^3 = |\det D_3|_F^{-3}$. This implies that

$$\delta_P(p(w_{3\alpha+2\beta}^{-1}u)) = |\det Y|_F^{-3}.$$

We can conjugate u by $w_{3\alpha+2\beta}$ and obtain

$$\begin{aligned} & x_{-(3\alpha+\beta)}(-a')x_{-(2\alpha+\beta)}(x')x_{-(3\alpha+2\beta)}(-b'-x'y')x_{-(\alpha+\beta)}(-y')x_{-\beta}(c') \\ = & \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -y' & -c' & 1 & 0 & 0 & 0 & 0 \\ x' & y' & 0 & 1 & 0 & 0 & 0 \\ a' & x' & 0 & 0 & 1 & 0 & 0 \\ b' & y'^2 - c'x' & x' & 2y' & -c' & 1 & 0 \\ a'y' - x'^2 & b' + a'c' - x'y' & -a' & -2x' & y' & 0 & 1 \end{pmatrix} \end{aligned}$$

Similarly, we set

$$X' = \begin{pmatrix} x' & 2y' & -c' \\ -a' & -2x' & y' \end{pmatrix}, \quad Y' = \begin{pmatrix} b' & y'^2 - c'x' \\ a'y' - x'^2 & b' + a'c' - x'y' \end{pmatrix}.$$

Hence if we set

$$\overline{u}(w_{3\alpha+2\beta}^{-1}u) = \begin{pmatrix} I_2 & & \\ \star & I_3 & \\ Y^{-1} & Y^{-1}X & I_2 \end{pmatrix} = \begin{pmatrix} I_2 & & \\ \star & I_3 & \\ Y' & X' & I_2 \end{pmatrix},$$

then

$$Y = Y'^{-1}, \quad X = Y'^{-1}X'.$$

If we compare each entry of the above equations, it is easy to obtain that

$$\begin{cases} x & = \frac{1}{\det Y'}(b'x' - x'^2y' + a'y'^2), \\ a & = -\frac{1}{\det Y'}(x'^3 - a'x'y' - a'b'), \\ c & = -\frac{1}{\det Y'}(-b'c' - a'c'^2 + 2x'y'c' - y'^3), \\ y & = \frac{1}{\det Y'}(-x'^2c' + a'y'c + b'y') \\ b & = \frac{1}{\det Y'}(b' + a'c' - x'y'), \end{cases} \tag{9.4}$$

where

$$\det Y' = b'^2 - b'x'y' + a'b'c' - c'x'^3 + x'^2y'^2 + a'c'x'y' - a'y'^3. \tag{9.5}$$

Except for a measure zero set, the map from (x', y', a', b', c') to (x, y, a, b, c) is a bijection, as Y is uniquely determined by Y'. The Jacobian between (x, y, a, b, c) and (x', y', a', b', c') is

$$-1/(b'^2 - b'x'y' + a'c'b - c'x'^3 + x'^2y'^2 + a'c'x'y' - a'y'^3)^3,$$

which is $-(\det Y')^{-3}$ by (9.5). In terms of the Haar measure of U, we can write

$$du = dxdydadbdc = |\det Y'|_F^{-3} dx'dy'da'db'dc'. \tag{9.6}$$

Thus, when $\mathrm{Re}(s) > \frac{2}{3}$,

$$\begin{aligned}(M(w_{3\alpha+2\beta})f_s)(\overline{u}) &= \int_{U(F)} f_s(w_{3\alpha+2\beta}^{-1}u\overline{u})du \\ &= \int_{U(F)} |\det Y|_F^{-3s} f_s(\overline{u}(w_{3\alpha+2\beta}^{-1}u)\overline{u})du\end{aligned} \tag{9.7}$$

In the above, u is a function of x, y, a, b, c and $du = dxdydadbdc$. Thus if we change the variable $(x, y, a, b, c) \mapsto (x', y', a', b', c')$ as in (9.4), then (9.7) can be written as

$$(M(w_{3\alpha+2\beta})f_s)(\overline{u}) = \int_{x',y',a',b',c'\in F} |\det Y'|_F^{3s-3} f_s(\begin{pmatrix} I_2 & & \\ \star & I_3 & \\ Y' & X' & I_2 \end{pmatrix}\overline{u})dx'dy'da'db'dc' \tag{9.8}$$

If we set

$$\begin{aligned}\overline{u} &= x_{-(3\alpha+\beta)}(t_3)x_{-(2\alpha+\beta)}(t_1)x_{-(3\alpha+2\beta)}(t_4)x_{-(\alpha+\beta)}(t_2)x_{-\beta}(t_5) \\ &= x_{-(2\alpha+\beta)}(t_1)x_{-(\alpha+\beta)}(t_2)\overline{n}(t_3, t_4, t_5),\end{aligned}$$

then by a simple matrix computation, we can get the following

$$\begin{aligned}&\overline{u}(w_{3\alpha+2\beta}^{-1}u)\overline{u} \\ =&x_{-(3\alpha+\beta)}(-a')x_{-(2\alpha+\beta)}(x')x_{-(3\alpha+2\beta)}(-b' - x'y')x_{-(\alpha+\beta)}(-y')x_{-\beta}(c')\cdot \\ &x_{-(3\alpha+\beta)}(t_3)x_{-(2\alpha+\beta)}(t_1)x_{-(3\alpha+2\beta)}(t_4)x_{-(\alpha+\beta)}(t_2)x_{-\beta}(t_5) \\ =&x_{-(3\alpha+\beta)}(t_3 - a')x_{-(2\alpha+\beta)}(t_1 + x')x_{-(3\alpha+2\beta)}(t_4 + \widetilde{b})x_{-(\alpha+\beta)}(t_2 - y')x_{-\beta}(t_5 + c'),\end{aligned}$$

where

$$\widetilde{b} = -b' + (t_1 + x')(t_2 - y') - t_1t_2 - c't_3 - t_2x' - 2t_1y'. \tag{9.9}$$

Thus,

$$\begin{aligned}
&(M(w_{3\alpha+2\beta})f_s)(\overline{u})\\
=&M(w_{3\alpha+2\beta})f_s)(x_{-(2\alpha+\beta)}(t_1)x_{-(\alpha+\beta)}(t_2)\overline{n}(t_3,t_4,t_5))\\
=&\int_{x',y',a',b',c'\in F}|\det Y'|_F^{3s-3}f_s(\begin{pmatrix} I_2 & & \\ \star & I_3 & \\ Y' & X' & I_2\end{pmatrix}\overline{u})dx'dy'da'db'dc' \qquad (9.10)\\
=&\int_{x',y',a',b',c'\in F}|\det Y'|_F^{3s-3}f_s(x_{-(2\alpha+\beta)}(t_1+x')x_{-(\alpha+\beta)}(t_2-y')\\
&\quad \overline{n}(t_3-a',t_4+\widetilde{b},t_5+c'))dx'dy'da'db'dc'.
\end{aligned}$$

We change the variables $b'\mapsto b'x'y'$, $c'\mapsto x'^{-1}y'^{2}c'$, $a'\mapsto x'^{2}y'^{-1}a'$, then

$$\begin{aligned}
&(M(w_{3\alpha+2\beta})f_s)(\overline{u})\\
=&\int_{x',y',a',b',c'\in F}|x'y'|_F^{6s-4}|\det\begin{pmatrix} b' & 1-c'\\ a'-1 & b'+a'c'-1\end{pmatrix}|_F^{3s-3}\cdot f_s(x_{-(2\alpha+\beta)}(t_1+x') \qquad (9.11)\\
&\quad x_{-(\alpha+\beta)}(t_2-y')\overline{n}(t_3-x'^{2}y'^{-1}a',t_4+\widetilde{b}',t_5+x'^{-1}y'^{2}c'))dx'dy'da'db'dc',
\end{aligned}$$

where

$$\widetilde{b}'=-b'x'y'+(t_1+x')(t_2-y')-t_1t_2-c'x'^{-1}y'^{2}t_3-t_2x'-2t_1y' \qquad (9.12)$$

As we explained, the function f_s is determined by its value on $U^-(F)$ which is diffeomorphic to an Euclidean space. From (9.11), the image of the intertwining operator, i.e. $M(w_{3\alpha+2\beta})f_s$, which is also regarded as a function on $U^-(F)$, can be obtained by 'convolving' f_s with the function

$$|x'y'|_F^{6s-4}|\det\begin{pmatrix} b' & 1-c'\\ a'-1 & b'+a'c'-1\end{pmatrix}|_F^{3s-3}\cdot.$$

We put an quote mark above the word convolving to indicate that this convolution is not standard, as we can see that the $\overline{n}$-term is very complicated in the integrand of the RHS of (9.11). Still, by the standard theory of intertwining operators, the above integral (9.11) converges when $\mathrm{Re}(s)>\frac{2}{3}$ and admits a meromorphic continuation in s.

Chapter 10

Another Formula for the Local Integral $Z(W_v, f_s)$

10.1 The Function $F_s(a, z)$

In this Chapter, we will assume that F is archimedean. We aim to prove some preliminary results that will help us compute $\Gamma(s, \pi, \mathrm{Ad}, \psi)$ in the next Chapter when F is archimedean. By Casselman's Subrepresentation theorem (see [Wal1, Section 3.8.3]), any representation can be embedded into a principal series, thus it suffices to compute $\Gamma(s, \pi, \mathrm{Ad}, \psi)$ explicitly for all principal series. To simplify notations, we will write N for $N(F)$, N^- for $N^-(F)$ etc. in this and the next Chapter.

We assume that $\pi = \pi_u = \mathrm{Ind}_{B^-}^{\mathrm{GL}_3(F)}\eta$, where $B^- := B^-_{\mathrm{GL}}$ is the standard lower Borel subgroup of $\mathrm{GL}_3(F)$ and η a character of B^- defined by

$$\eta(\begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix}) = \prod_{i=1}^{3} |t_i|_F^{u_i} \chi_i(t_i),$$

where χ_i are all unitary characters of F. We assume that $\eta(aI) = 1$ for all $a \in F^\times$, in order for π to have a trivial central character. Since we have already proved that the local integrals $Z(W_v, f_s)$ are meromorphic functions in s and all u_i, we only need to compute

$\Gamma(s,\pi,\mathrm{Ad},\psi)$ on some special open subset in $\mathbb{C}^4$. Once this is done, for the other s and parameter u_i, $\Gamma(s,\pi,\mathrm{Ad},\psi)$ can be obtained by its meromorphic continuation.

We also assume that the character ψ is given in (1.11). Let us further assume that $\mathrm{Re}(u_1) < \mathrm{Re}(u_2) < \mathrm{Re}(u_3)$. Then the Jacquet integral for $\varphi \in V_\pi$

$$W_\varphi(g) = \int_N \varphi(ng)\psi^{-1}(n)dn \tag{10.1}$$

converges absolutely (see [Wal2, 15.4.1]). The local integral

$$\begin{aligned} & Z(W_\varphi, f_s) \\ =& \int_{N'\times T\times K} W_\varphi(ak)\psi(n')\cdot f_s(\gamma n'ak)\delta_B^{-1}(a)dn'dadk \\ =& \int_{T\times K}\Big(\int_{N''\times N_2}\varphi(n''n_2ak)\psi^{-1}(n'')dn_2dn''\Big)\cdot\Big(\int_{N'} f_s(\gamma n'ak)\psi(n')dn'\Big)\delta_B^{-1}(a)dkda, \end{aligned} \tag{10.2}$$

where $N'' = N'$. By Lemma 3.1.1, the dn'-integral

$$\int_{N'} f_s(\gamma n'ak)\psi(n')dn'$$

converges absolutely when $\mathrm{Re}(s) > \frac{1}{3}$. Thus under the assumption that $\mathrm{Re}(u_1) < \mathrm{Re}(u_2) < \mathrm{Re}(u_3)$ and $\mathrm{Re}(s) > \frac{1}{3}$, the integral

$$\int_K\Big(\int_{N''\times N_2}\varphi(n''n_2ak)\psi^{-1}(n'')dn_2dn''\Big)\cdot\Big(\int_{N'} f_s(\gamma n'ak)\psi(n')dn'\Big)\delta_B^{-1}(a)dk \tag{10.3}$$

converges absolutely. Thus, we can switch the order of integration and change the variable $n' \mapsto n''n'$. Then (10.3) can be rewritten as

$$\int_{N'}\Big(\int_{K\times N''\times N_2}\varphi(n_2n''ak)f_s(\gamma n'n''ak)\delta_B^{-1}(a)dn''dn_2dk\Big)\psi(n')dn'.$$

We write $n' = \begin{pmatrix} 1 & z & \\ & 1 & \\ & & 1 \end{pmatrix}$ explicitly. Define a function

$$F_s(a,z) := \int_{K\times N''\times N_2} \varphi(n_2 n'' ak) f_s(\gamma \begin{pmatrix} 1 & z & \\ & 1 & \\ & & 1 \end{pmatrix} n''ak)\delta_B^{-1}(a) dn'' dn_2 dk. \tag{10.4}$$

By Lemma 1.0.9 and the observation that N' normalizes N_2, $F_s(a,z)$ can be written as

$$\begin{aligned} F_s(a,z) &= \int_{K\times N} \varphi(nak) f_s(\gamma \begin{pmatrix} 1 & z & \\ & 1 & \\ & & 1 \end{pmatrix} nak)\delta_B^{-1}(a) dndk \\ &= \int_{K\times N} \varphi(ank) f_s(\gamma \begin{pmatrix} 1 & z & \\ & 1 & \\ & & 1 \end{pmatrix} ank) dndk. \end{aligned} \tag{10.5}$$

The local integral $Z(W_\varphi, f_s)$ can be obtained by first taking the Fourier transform of F_s, then by taking the Mellin transform with respect to trivial character $\chi = id$. Intuitively, we wish to switch the order of the Mellin transform and the Fourier transform. To accomplish the change of order of two integral transforms, we have to check all conditions in Lemma 8.3.1. We start from checking (8.7) in this case.

We set $A = T = (F^\times)^2$, $X = F$, $X_0 = F^\times$. Define the embedding $j : X_0 \to A$ via

$$j(z) = \begin{pmatrix} z^{-1} & & \\ & 1 & \\ & & z \end{pmatrix}.$$

Then the invariant properties of f_s and φ yield

$$\begin{aligned}
F_s(a,z) &= \int_{K\times N} \varphi(ank) f_s(\gamma \begin{pmatrix} 1 & z & \\ & 1 & \\ & & 1 \end{pmatrix} ank)dndk \\
&= \int_{K\times N} \varphi(ank) f_s(\gamma \begin{pmatrix} z & & \\ & 1 & \\ & & z^{-1} \end{pmatrix} \begin{pmatrix} 1 & 1 & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} z^{-1} & & \\ & 1 & \\ & & z \end{pmatrix} ank)dndk \\
&= \int_{K\times N} \varphi(ank)|z|_F^{3s} f_s(\gamma \begin{pmatrix} 1 & 1 & \\ & 1 & \\ & & 1 \end{pmatrix} j(z)ank)dndk \\
&= \int_{K\times N} |z|_F^{3s}(\delta_{B^-}^{\frac{1}{2}}\eta)(j(z)^{-1})\varphi(aj(z)nk) f_s(\gamma \begin{pmatrix} 1 & 1 & \\ & 1 & \\ & & 1 \end{pmatrix} j(z)ank)dndk \\
&= |z|_F^{3s}(\delta_{B^-}^{\frac{1}{2}}\eta)(j(z)^{-1})F_s(aj(z),1).
\end{aligned} \tag{10.6}$$

Thus, we only need to set

$$\sigma(\begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix}) = |t_1|_F^{-\frac{3s-2}{2}-u_1}\chi_1^{-1}(t_1)\cdot|t_2|_F^{-u_2}\chi_2^{-1}(t_2)\cdot|t_3|_F^{\frac{3s-2}{2}-u_3}\chi_3^{-1}(t_3).$$

Then

$$\sigma(j(z)) = |z|_F^{3s}(\delta_{B^-}^{\frac{1}{2}}\eta)(j(z)^{-1}) = |z|_F^{3s-2+u_1-u_3}(\chi_1\chi_3^{-1})(z),$$

and $F_s(a,z)$ satisfies the invariant property (8.7).

10.2 Checking Assumption 1)

The character χ in Lemma 8.3.1 is trivial in this case. We aim to detect a range of absolute convergence for the integral

$$\begin{aligned}
\int_T F_s(a,1)da &= \int_{T\times N\times K} \varphi(ank) f_s(\gamma \begin{pmatrix} 1 & 1 & \\ & 1 & \\ & & 1 \end{pmatrix} ank) da dn dk \\
&= \int_{SL_3} \varphi(g) f_s(\gamma \begin{pmatrix} 1 & 1 & \\ & 1 & \\ & & 1 \end{pmatrix} g) dg \\
&= \int_{T\times N^-\times K} \varphi(a\overline{n}k) f_s(\gamma \begin{pmatrix} 1 & 1 & \\ & 1 & \\ & & 1 \end{pmatrix} a\overline{n}k) da d\overline{n} dk \\
&= \int_{T\times N^-\times K} (\delta_{B^-}^{\frac{1}{2}}\eta)(a)\varphi(k) f_s(\gamma \begin{pmatrix} 1 & 1 & \\ & 1 & \\ & & 1 \end{pmatrix} a\overline{n}k) da d\overline{n} dk
\end{aligned}$$

Integrating over K will provide us another smooth function in the induced space V_{ρ_s}, hence it suffices to study the integral

$$I(f_s) := \int_{T\times N^-} \delta_{B^-}^{\frac{1}{2}}(a)\eta(a) f_s(\gamma \begin{pmatrix} 1 & 1 & \\ & 1 & \\ & & 1 \end{pmatrix} a\overline{n}) da d\overline{n}. \tag{10.7}$$

It is not easy to see a range of convergence for $I(f_s)$ from the above formula. We plan to rewrite $I(f_s)$ as a Mellin transform of the restriction on f_s on the lower unipotent radical U^-. Then we can obtain a range of convergence by comparing the new formula for $I(f_s)$ with the intertwining operator $M(w_{3\alpha+2\beta})$. An intuitive computation in the case of $\mathrm{GL}_2(F)$ is outlined in Appendix A.

Lemma 10.2.1. *In the range of absolute convergence,*

$$\begin{aligned}
I(f_s) = \int_{F^2}\int_{N^-} &|t_1t_2|_F^{3s-2} f_s(x_{-(2\alpha+\beta)}(-t_1)x_{-(\alpha+\beta)}(t_2)\overline{n}) \\
&|t_1|_F^{u_2-u_3}(\chi_2\chi_3^{-1})(t_1)\cdot|t_2|_F^{u_1-u_2}(\chi_1\chi_2^{-1})(t_2) d\overline{n} dt_1 dt_2.
\end{aligned} \tag{10.8}$$

Proof. We parameterize T as we did in Chapter 3, i.e.

$$a = \begin{pmatrix} t_1t_2 & & \\ & t_2 & \\ & & t_1^{-1}t_2^{-2} \end{pmatrix}.$$

The following decomposition in SL_2 holds when $z \neq 0$:

$$\begin{aligned} &\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -z \end{pmatrix} \\ = &\begin{pmatrix} -z^{-1} & 1 \\ 0 & -z \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z^{-1} & 1 \end{pmatrix} = \begin{pmatrix} -z^{-1} & 0 \\ 0 & -z \end{pmatrix} \begin{pmatrix} 1 & -z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z^{-1} & 1 \end{pmatrix}. \end{aligned} \tag{10.9}$$

Therefore, by (10.9),

$$\begin{aligned} \gamma \begin{pmatrix} 1 & 1 & \\ & 1 & \\ & & 1 \end{pmatrix} a &= x_{-(\alpha+\beta)}(-1) w_\beta x_\beta(1) \cdot \begin{pmatrix} t_1t_2 & & \\ & t_2 & \\ & & t_1^{-1}t_2^{-2} \end{pmatrix} \\ &= x_{-(\alpha+\beta)}(-1) \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix} x_\beta(-1) x_{-\beta}(1) \begin{pmatrix} t_1t_2 & & \\ & t_2 & \\ & & t_1^{-1}t_2^{-2} \end{pmatrix} \\ &= \begin{pmatrix} -t_1t_2 & & \\ & -t_2 & \\ & & t_1^{-1}t_2^{-2} \end{pmatrix} x_{-(\alpha+\beta)}(t_1t_2) x_\beta(-t_1^{-1}) x_{-\beta}(t_1). \end{aligned} \tag{10.10}$$

Hence,

$$f_s(\gamma \begin{pmatrix} 1 & 1 & \\ & 1 & \\ & & 1 \end{pmatrix} a\overline{n}) = |t_1^2 t_2^3|_F^{3s} f_s(x_{-(\alpha+\beta)}(t_1t_2) x_\beta(-t_1^{-1}) x_{-\beta}(t_1)\overline{n}). \tag{10.11}$$

Plugging (10.11) into (10.7),

$$\begin{aligned}I(f_s) = \int_{(F^\times)^2}\int_{N^-} & |t_1^2t_2^3|_F^{3s} f_s(x_{-(\alpha+\beta)}(t_1t_2)x_\beta(-t_1^{-1})x_{-\beta}(t_1)\overline{n}) \\ & \eta(a)\delta_{B^-}^{\frac{1}{2}}(a)d\overline{n}d^\times t_1 d^\times t_2 dz.\end{aligned} \tag{10.12}$$

Recall the commutator relation in Lemma 1.0.2

$$x_\beta(-s)x_\alpha(-t)x_\beta(s) = x_{\alpha+\beta}(st)x_{2\alpha+\beta}(st^2)x_{3\alpha+\beta}(st^3)x_{3\alpha+2\beta}(s^2t^3)x_\alpha(-t).$$

Applying $w_{2\alpha+\beta} = w_\alpha w_\beta w_\alpha w_\beta w_\alpha$ in the above identity and setting $s = -t_1^{-1}, t = -t_1t_2$, we get

$$\begin{aligned} & x_\beta(t_1^{-1})x_{-(\alpha+\beta)}(t_1t_2)x_\beta(-t_1^{-1}) \\ =& x_{-\alpha}(t_2)x_{-(2\alpha+\beta)}(-t_1t_2^2)x_{-(3\alpha+2\beta)}(-t_1^2t_2^3)x_{-(3\alpha+\beta)}(-t_1t_2^3)x_{-(\alpha+\beta)}(t_1t_2) \\ =& x_{-\alpha}(t_2)x_{-(2\alpha+\beta)}(-t_1t_2^2)x_{-(\alpha+\beta)}(t_1t_2)x_{-(3\alpha+2\beta)}(-t_1^2t_2^3)x_{-(3\alpha+\beta)}(-t_1t_2^3).\end{aligned} \tag{10.13}$$

Here in the last identity above, we use the fact that the short root $x_{-(\alpha+\beta)}$ commutes with all the long negative roots of G_2. Hence

$$\begin{aligned} & f_s(x_{-(\alpha+\beta)}(t_1t_2)x_\beta(-t_1^{-1})x_{-\beta}(t_1)\overline{n}) = \\ & f_s(x_{-(2\alpha+\beta)}(-t_1t_2^2)x_{-(\alpha+\beta)}(t_1t_2)x_{-(3\alpha+2\beta)}(-t_1^2t_2^3)x_{-(3\alpha+\beta)}(-t_1t_2^3)x_{-\beta}(t_1)\overline{n})\end{aligned} \tag{10.14}$$

Plugging (10.14) into (10.12) and changing the variable

$$\overline{n} \mapsto (x_{-(3\alpha+2\beta)}(-t_1^2t_2^3)x_{-(3\alpha+\beta)}(-t_1t_2^3)x_{-\beta}(t_1))^{-1}\overline{n},$$

we obtain that $I(f_s)$ is equal to

$$\int_{(F^\times)^2}\int_{N^-} |t_1^2t_2^3|_F^{3s} f_s(x_{-(2\alpha+\beta)}(-t_1t_2^2)x_{-(\alpha+\beta)}(t_1t_2)\overline{n})\eta(a)\delta_{B^-}^{\frac{1}{2}}(a)d\overline{n}d^\times t_1 d^\times t_2. \tag{10.15}$$

Now we set $t'_1 = t_1 t_2^2, t'_2 = t_1 t_2$, then we have

$$\eta(a) = \eta(\begin{pmatrix} t'_2 & & \\ & t'_1 t_2'^{-1} & \\ & & t_1'^{-1} \end{pmatrix}) = |t'_1|_F^{u_2-u_3}(\chi_2\chi_3^{-1})(t'_1) \cdot |t'_2|_F^{u_1-u_2}(\chi_1\chi_2^{-1})(t'_2), \quad (10.16)$$

and

$$\delta_{B^-}^{\frac{1}{2}}(a) = \delta_{B^-}^{\frac{1}{2}}(\begin{pmatrix} t'_2 & & \\ & t'_1 t_2'^{-1} & \\ & & t_1'^{-1} \end{pmatrix}) = |t'_1 t'_2|_F^{-1}. \quad (10.17)$$

Plugging (10.16) and (10.17) into (10.15), we obtain that $I(f_s)$ becomes

$$\begin{aligned} &\int_{(F^\times)^2} \int_{N^-} |t'_1 t'_2|_F^{3s-1} f_s(x_{-(2\alpha+\beta)}(-t'_1) x_{-(\alpha+\beta)}(t'_2)\overline{n}) \\ &|t'_1|_F^{u_2-u_3}(\chi_2\chi_3^{-1})(t'_1) \cdot |t'_2|_F^{u_1-u_2}(\chi_1\chi_2^{-1})(t'_2) d\overline{n} d^\times t'_1 d^\times t'_2 \\ &= \int_{F^2} \int_{N^-} |t_1 t_2|_F^{3s-2} f_s(x_{-(2\alpha+\beta)}(-t_1) x_{-(\alpha+\beta)}(t_2)\overline{n}) \\ &|t_1|_F^{u_2-u_3}(\chi_2\chi_3^{-1})(t_1) \cdot |t_2|_F^{u_1-u_2}(\chi_1\chi_2^{-1})(t_2) d\overline{n} dt_1 dt_2. \end{aligned} \quad (10.18)$$

Here in the last identity of the above, we switch $t'_1 \mapsto t_1$ and $t'_2 \mapsto t_2$ to simplify notations. $\square$

Now we return to the convergence of $I(f_s)$.

Proposition 10.2.2. *If we assume that $Re(s) > \frac{2}{3}$, $Re(3s-2+u_2-u_3) \in (-1,0)$, $Re(3s-2+u_1-u_2) \in (-1,0)$, then the integral (10.8) (hence (10.7)) converges absolutely.*

Proof. Let f_s^0 be the spherical function in V_{ρ_s} as in Section 9.1. Then $I(f_s)$ is majorized by a constant multiple of

$$\begin{aligned} I_0 := \int_{F^2} \int_{N^-} &|t_1 t_2|_F^{\mathrm{Re}(3s-2)} f_s^0(x_{-(2\alpha+\beta)}(-t_1) x_{-(\alpha+\beta)}(t_2)\overline{n}) \\ &|t_1|_F^{\mathrm{Re}(u_2-u_3)} |t_2|_F^{\mathrm{Re}(u_1-u_2)} d\overline{n} dt_1 dt_2. \end{aligned} \quad (10.19)$$

We rewrite I_0 as

$$I_0 = \int_{F^2} \int_{N^-} f_s^0(x_{-(2\alpha+\beta)}(t_1) x_{-(\alpha+\beta)}(t_2)\overline{n}) |t_1|_F^{\mathrm{Re}(3s-2+u_2-u_3)} |t_2|_F^{\mathrm{Re}(3s-2+u_1-u_2)} d\overline{n} dt_1 dt_2. \tag{10.20}$$

By the range of convergence of the intertwining operator (see (9.3)), the integral

$$I_0' := \int_{F^2} \int_{N^-} f_s^0(x_{-(2\alpha+\beta)}(t_1) x_{-(\alpha+\beta)}(t_2)\overline{n}) d\overline{n} dt_1 dt_2.$$

converges absolutely when $\mathrm{Re}(s) > \frac{2}{3}$.

Therefore, the $d\overline{n}$-integral in (10.20) converges absolutely when $\mathrm{Re}(s) > \frac{2}{3}$. We break the integral in (10.20) into four parts, i.e.

$$\int_{|t_1|_F \le 1} \int_{|t_1|_F \le 1} \int_{N^-}, \qquad \int_{|t_1|_F > 1} \int_{|t_1|_F \le 1} \int_{N^-},$$
$$\int_{|t_1|_F \le 1} \int_{|t_1|_F > 1} \int_{N^-}, \qquad \int_{|t_1|_F > 1} \int_{|t_1|_F > 1} \int_{N^-}.$$

By our assumptions that $\mathrm{Re}(3s-2+u_2-u_3) > -1$, $\mathrm{Re}(3s-2+u_1-u_2) > -1$, the integral

$$\int_{|t_1|_F \le 1} \int_{|t_1|_F \le 1} \int_{N^-}$$

converges absolutely, since after integrating over N^-, the integrand

$$\int_{N^-} f_s^0(x_{-(2\alpha+\beta)}(t_1) x_{-(\alpha+\beta)}(t_2)\overline{n}) d\overline{n}$$

is a smooth function in t_1, t_2 near $(0,0)$. Now we look at

$$\int_{|t_1|_F > 1} \int_{|t_1|_F \le 1} \int_{N^-}.$$

Also, because $\mathrm{Re}(3s-2+u_2-u_3) < 0$, the integral

$$\int_{|t_1|_F > 1} \int_{|t_1|_F \le 1} \int_{N^-} f_s^0(x_{-(2\alpha+\beta)}(t_1) x_{-(\alpha+\beta)}(t_2)\overline{n})$$
$$|t_1|_F^{\mathrm{Re}(3s-2+u_2-u_3)} |t_2|_F^{\mathrm{Re}(3s-2+u_1-u_2)} d\overline{n} dt_1 dt_2$$

is dominated by

$$\int_{|t_1|_F>1}\int_{|t_1|_F\le 1}\int_{N^-} f_s^0(x_{-(2\alpha+\beta)}(t_1)x_{-(\alpha+\beta)}(t_2)\overline{n})|t_2|_F^{\mathrm{Re}(3s-2+u_1-u_2)}.d\overline{n}dt_1dt_2.$$

The $d\overline{n}dt_1$-integral converges absolutely when $\mathrm{Re}(s)>\frac{2}{3}$ (by the absolute convergence of I_0'). It defines a smooth function in t_2. Thus the whole integral

$$\int_{|t_1|_F>1}\int_{|t_1|_F\le 1}\int_{N^-} f_s^0(x_{-(2\alpha+\beta)}(t_1)x_{-(\alpha+\beta)}(t_2)\overline{n})$$
$$|t_1|_F^{\mathrm{Re}(3s-2+u_2-u_3)}|t_2|_F^{\mathrm{Re}(3s-2+u_1-u_2)}d\overline{n}dt_1dt_2$$

converges absolutely under our assumption that $\mathrm{Re}(3s-2+u_2-u_3)>-1$. The absolute convergence of the remaining two integrals

$$\int_{|t_1|_F\le 1}\int_{|t_1|_F>1}\int_{N^-},\quad \int_{|t_1|_F>1}\int_{|t_1|_F>1}\int_{N^-}$$

can be proved in a similar fashion. $\square$

Therefore, under the assumptions that

$$\mathrm{Re}(s)>\frac{2}{3},\quad \mathrm{Re}(3s-2+u_2-u_3)\in(-1,0),\quad \mathrm{Re}(3s-2+u_1-u_2)\in(-1,0),$$

the integral

$$\int_T F_s(a,1)da$$

converges absolutely.

10.3 Checking the other three Assumptions

The assumption 3) and 4) are easy to check.

Assumption 3): It is straight forward. The range of convergence for the conditionally convergent integral

$$\int_F \sigma(j(z))\psi(z)dz=\int_F |z|_F^{3s-2+u_1-u_3}(\chi_1\chi_3^{-1})(z)\psi(z)dz$$

is exactly $\mathrm{Re}(3s-2+u_1-u_3) \in (-1,0)$.

Assumption 4): We require that

$$\int_{j^{-1}(a_2)\in B_r} \sigma(a_2)|j^{-1}(a_2)|_F d^\times a_2 = \int_{|z|_F\le r} |z|_F^{3s-2+u_1-u_3}(\chi_1\chi_3^{-1})(z)\cdot |z|_F d^\times z$$

converges absolutely for any $r>0$. This is true when $\mathrm{Re}(3s-2+u_1-u_3) > -1$.

To check the second assumption, it is more convenient to work with (10.4). We aim to detect a range of absolute convergence for

$$\begin{aligned}&\int_{A_1\times F} F_s(a_1,z)dzd^\times a_1\\ &=\int_{A_1\times F}\int_{K\times N''\times N_2} \varphi(n_2n''a_1k)f_s(\gamma\begin{pmatrix}1 & z & \\ & 1 & \\ & & 1\end{pmatrix}n''a_1k)\delta_B^{-1}(a_1)dn''dn_2dkdzda_1.\end{aligned} \tag{10.21}$$

After changing $\begin{pmatrix}1 & z & \\ & 1 & \\ & & 1\end{pmatrix} \mapsto \begin{pmatrix}1 & z & \\ & 1 & \\ & & 1\end{pmatrix}\cdot (n'')^{-1}$, we can rewrite the above as

$$\begin{aligned}&\int_{A_1\times F} F_s(a_1,z)dzd^\times a_1\\ &=\int_{A_1\times F}\int_{K\times N} \varphi(na_1k)f_s(\gamma\begin{pmatrix}1 & z & \\ & 1 & \\ & & 1\end{pmatrix}a_1k)\delta_B^{-1}(a_1)dndkdzda_1\\ &=\int_{A_1\times F}\int_{K\times N} (\delta_{B^-}^{\frac{1}{2}}\eta)(a_1)\varphi(nk)f_s(\gamma\begin{pmatrix}1 & z & \\ & 1 & \\ & & 1\end{pmatrix}a_1k)dndkdzda_1\end{aligned}$$

The dn-integral converges absolutely when $\mathrm{Re}(u_1)<\mathrm{Re}(u_2)<\mathrm{Re}(u_3)$. Integrating over the compact group K will not affect the range of convergence. Thus it suffices to study the integral

$$\int_{A_1\times F} |(\delta_{B^-}^{\frac{1}{2}}\eta)(a_1)f_s(\gamma\begin{pmatrix}1 & z & \\ & 1 & \\ & & 1\end{pmatrix}a_1)|dzda_1. \tag{10.22}$$

We parameterize

$$a_1 = \begin{pmatrix} t_2 & & \\ & t_2 & \\ & & t_2^{-2} \end{pmatrix},$$

then by Lemma 3.1.1,

$$f_s(\gamma \begin{pmatrix} 1 & z & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} t_2 & & \\ & t_2 & \\ & & t_2^{-2} \end{pmatrix}) = |t_2|_F^{9s} \cdot \left| |z|^2 + (1+|t_2|^2)^3 \right|_F^{-\frac{3}{2}s} Q(t_2, z)$$

for some smooth bounded function Q in t_2, z. The character

$$(\delta_{B^-}^{\frac{1}{2}} \eta)(\begin{pmatrix} t_2 & & \\ & t_2 & \\ & & t_2^{-2} \end{pmatrix}) = |t_2|_F^{u_1+u_2-2u_3-3} (\chi_1\chi_2\chi_3^{-2})(t_2).$$

Thus the integral (10.22) is majorized by a constant multiple of

$$\int_{F^\times} \int_F |t_2|_F^{\mathrm{Re}(9s+u_1+u_2-2u_3-3)} \left| |z|^2 + (1+|t_2|^2)^3 \right|_F^{-\frac{3}{2}s} dz d^\times t_2. \tag{10.23}$$

After we change the variable $z \mapsto z \cdot (1+|t_2|^2)^{\frac{3}{2}}$, the integral (10.23) becomes a product of

$$\int_F \left| |z|^2 + 1 \right|_F^{-\frac{3}{2}s} dz \tag{10.24}$$

and

$$\int_{F^\times} |t_2|_F^{\mathrm{Re}(9s+u_1+u_2-2u_3-3)} \left| 1 + |t_2|^2 \right|_F^{-\frac{9}{2}s+\frac{3}{2}} d^\times t_2. \tag{10.25}$$

The integral (10.24) converges absolutely when $\mathrm{Re}(s) > \frac{1}{3}$. Under the assumptions that

$$\mathrm{Re}(s) > \frac{2}{3}, \qquad \mathrm{Re}(3s-2+u_2-u_3) \in (-1,0),$$
$$\mathrm{Re}(3s-2+u_1-u_2) \in (-1,0), \qquad \mathrm{Re}(3s-2+u_1-u_3) \in (-1,0),$$

$\mathrm{Re}(9s + u_1 + u_2 - 2u_3 - 3)$ is positive, thus,

$$|t_2|_F^{\mathrm{Re}(9s+u_1+u_2-2u_3-3)} \left|1 + |t_2|^2\right|_F^{-\frac{9}{2}s+\frac{3}{2}} \tag{10.26}$$

is integrable near 0. Also, (10.26) behaves like $|t_2|_F^{\mathrm{Re}(u_1+u_2-2u_3)}$ when $|t_2|_F$ is sufficiently large, and the exponent $\mathrm{Re}(u_1 + u_2 - 2u_3) < 0$ under our assumption that

$$\mathrm{Re}(u_1) < \mathrm{Re}(u_2) < \mathrm{Re}(u_3).$$

Thus, (10.26) is also integrable near infinity. Therefore, (10.25) (hence (10.23)) converges absolutely, which implies that (10.21) converges absolutely.

Thus we have checked all four conditions and Theorem 1.0.13 is a direct consequence of Lemma 8.3.1. To establish the functional equation for $Z(W_v, f_s)$, it suffices to establish a functional equation for $Z_1(W_v, f_s)$. As we remarked before, integrating over K will only provide us another smooth function in V_{ρ_s}. Thus $I(f_s)$ (defined in (10.7)) also has a meromorphic continuation, and the functional equation satisfied by $I(f_s)$ is the same as that of $Z_1(W_v, f_s)$.

Chapter 11

The Functional Equations

We will finish computing the local gamma factor $\Gamma(s, \pi, \mathrm{Ad}, \psi)$ in this Chapter.

11.1 The Functional Equation for $I(f_s)$

In the previous Chapter, we have shown that

$$Z(W_\varphi, f_s) = Z_1(W_\varphi, f_s) \cdot \gamma_F(2 - 3s - u_1 + u_3, \chi_1^{-1}\chi_3, \psi).$$

In this section, we always regard f_s, $M(w_{3\alpha+2\beta})f_s$, etc. as functions on U^-, i.e. we consider their restrictions on U^-. Recall that in (10.8),

$$\begin{aligned} I(f_s) = &\int_{F^2} \int_{N^-} |t_1|_F^{3s-2+u_2-u_3} (\chi_2\chi_3^{-1})(t_1) |t_2|_F^{3s-2+u_1-u_2} (\chi_1\chi_2^{-1})(t_2) \\ &\cdot f_s(x_{-(2\alpha+\beta)}(-t_1) x_{-(\alpha+\beta)}(t_2)\overline{n}) d\overline{n} dt_1 dt_2. \end{aligned} \tag{11.1}$$

and $I(f_s)$ is also meromorphic in all u_i. We aim to establish a functional equation between $I(f_s)$ and

$$\begin{aligned} \widetilde{I}(f_s) = &\int_{F^2} \int_{N^-} |t_1|_F^{1-3s+u_2-u_3} (\chi_2\chi_3^{-1})(t_1) |t_2|_F^{1-3s+u_1-u_2} (\chi_1\chi_2^{-1})(t_2) \\ &\cdot (M(w_{3\alpha+2\beta})f_s)(x_{-(2\alpha+\beta)}(-t_1) x_{-(\alpha+\beta)}(t_2)\overline{n}) d\overline{n} dt_1 dt_2. \end{aligned} \tag{11.2}$$

By Theorem 1.0.12, it suffices to establish the functional equation for one $f_s \in V_{\rho_s}$. We expect that for such f_s, we can drop the assumption

$$\mathrm{Re}(u_1) < \mathrm{Re}(u_2) < \mathrm{Re}(u_3).$$

Purely formally, we want to plug (9.11) into (11.2), yet we will face fatal analytic difficulties that the integral does not converge absolutely! So, we can not switch the order of integration and prove the functional equation. The obstruction comes from the $d\overline{n}$-integral in (11.2) as well as the exponents $1-3s+u_2-u_3$, $1-3s+u_1-u_2$. Indeed, we can rewrite the $d\overline{n}$-integral as

$$\begin{aligned} &\int_{N^-} (M(w_{3\alpha+2\beta})f_s)(x_{-(2\alpha+\beta)}(-t_1)x_{-(\alpha+\beta)}(t_2)\overline{n})d\overline{n} \\ =&\int_{N^-} (M(w_{3\alpha+2\beta})\rho_s(x_{-(2\alpha+\beta)}(-t_1)x_{-(\alpha+\beta)}(t_2))f_s)(\overline{n})d\overline{n}. \end{aligned} \tag{11.3}$$

The restriction of any function $f_{1-s} \in V_{\rho_{1-s}}$ on $\mathrm{SL}_3(F)$ belongs to a principal series

$$\mathrm{Ind}_B^{\mathrm{SL}_3(F)} |\quad|_F^{2-3s} \otimes \mathrm{id} \otimes |\quad|_F^{3s-2}.$$

Hence, the integral (11.3) converges absolutely when $\mathrm{Re}(s) < \frac{2}{3}$. There is no s such that both (9.11) and (11.3) converge absolutely. The obstruction arising from the two exponents may not be seen at first glance, yet it will become clearer if we keep track the formal computation in the below.

To resolve these problems, we choose an $f_s \in V_{\rho_s}$ whose restriction on U^- has compact support and its support is away from the identity element I. Thus the integral defining $I(f_s)$ (i.e. (11.1)) converges absolutely for all s, u. Consider another operator

$M(w_{3\alpha+2\beta}, s_1, s_2, s_3)$ by the rule

$$
\begin{aligned}
&(M(w_{3\alpha+2\beta}, s_1, s_2, s_3)f_s)(\overline{u}) \\
= &\int_{x',y',a',b',c' \in F} |x'y'|_F^{6s-4+s_3} |\det \begin{pmatrix} b' & 1-c' \\ a'-1 & b'+a'c'-1 \end{pmatrix}|_F^{3s-3} |a'|_F^{s_1} |c' - \frac{3a'-2}{a'^2}|_F^{s_2} \cdot \\
&\quad f_s(x_{-(2\alpha+\beta)}(t_1+x')x_{-(\alpha+\beta)}(t_2-y')\overline{n}(t_3 - x'^2y'^{-1}a', t_4 + \widetilde{b'}, t_5 + x'^{-1}y'^2c')) \\
&\quad dx'dy'da'db'dc',
\end{aligned}
\tag{11.4}
$$

where $\widetilde{b'}$ is given by (9.12), and s_1, s_2, s_3 are three complex parameters having small negative real parts (in terms of absolute value). Then $M(w_{3\alpha+2\beta}, s_1, s_2, s_3)f_s$ is also meromorphic in s_1, s_2, s_3 when they vary in a small vertical strip

$$
-\epsilon_0 \le \mathrm{Re}(s_1), \mathrm{Re}(s_2), \mathrm{Re}(s_3) \le 0.
$$

We note that when $\mathrm{Re}(s) > \frac{2}{3}$,

$$
M(w_{3\alpha+2\beta}, 0, 0, 0)f_s = M(w_{3\alpha+2\beta})f_s, \tag{11.5}
$$

Thus, (11.5) also holds for all s in the sense of meromorphic continuation.

Now we proceed with a formal computation. Define

$$
\begin{aligned}
I(f_s, \lambda_1, \lambda_2) = \int_{F^2} \int_{N^-} &|t_1|_F^{\lambda_1} (\chi_2\chi_3^{-1})(t_1) |t_2|_F^{\lambda_2} (\chi_1\chi_2^{-1})(t_2) \\
&\cdot f_s(x_{-(2\alpha+\beta)}(-t_1)x_{-(\alpha+\beta)}(t_2)\overline{n}) d\overline{n} dt_1 dt_2,
\end{aligned}
\tag{11.6}
$$

and

$$
\begin{aligned}
\widetilde{I}(f_s, s_1, s_2, s_3, \lambda_1, \lambda_2) = \int_{F^2} \int_{N^-} &|t_1|_F^{\lambda_1} (\chi_2\chi_3^{-1})(t_1) |t_2|_F^{\lambda_2} (\chi_1\chi_2^{-1})(t_2) \\
&\cdot (M(w_{3\alpha+2\beta}, s_1, s_2, s_3)f_s)(x_{-(2\alpha+\beta)}(-t_1)x_{-(\alpha+\beta)}(t_2)\overline{n}) d\overline{n} dt_1 dt_2.
\end{aligned}
\tag{11.7}
$$

Since we assume that $f_s|_{U^-}$ has compact support away from I, the integral (11.6) converges absolutely for all s, λ_1, λ_2. The following formal computation for

$\widetilde{I}(f_s, s_1, s_2, s_3, \lambda_1, \lambda_2)$ holds in the range of convergence.

$$
\begin{aligned}
&\widetilde{I}(f_s, s_1, s_2, s_3, \lambda_1, \lambda_2) \\
=&\int_{t_1,t_2,t_3,t_4,t_5\in F} |t_1|_F^{\lambda_1}(\chi_2\chi_3^{-1})(t_1)|t_2|_F^{\lambda_2}(\chi_1\chi_2^{-1})(t_2) \\
&\cdot \int_{x',y',a',b',c'\in F} |x'y'|_F^{6s-4+s_3}|\det\begin{pmatrix} b' & 1-c' \\ a'-1 & b'+a'c'-1\end{pmatrix}|_F^{3s-3}|a'|_F^{s_1}|c'-\frac{3a'-2}{a'^2}|_F^{s_2}\cdot \\
&\quad f_s(x_{-(2\alpha+\beta)}(t_1+x')x_{-(\alpha+\beta)}(t_2-y')\overline{n}(t_3-x'^2y'^{-1}a', t_4+\widetilde{b}', t_5+x'^{-1}y'^2c')) \\
&\quad dx'dy'da'db'dc'dt_1dt_2dt_3dt_4dt_5.
\end{aligned}
\tag{11.8}
$$

After switching the order of integration and changing the variable $t_3 \mapsto t_3 + x'^2y'^{-1}a'$, $t_4 \mapsto t_4 - \widetilde{b}', t_5 \mapsto t_5 - x'^{-1}y'^2c'$, we obtain that

$$
\begin{aligned}
&\widetilde{I}(f_s, s_1, s_2, s_3, \lambda_1, \lambda_2) \\
=&\int_{t_1,t_2,t_3,t_4,t_5\in F} |t_1|_F^{\lambda_1}(\chi_2\chi_3^{-1})(t_1)|t_2|_F^{\lambda_2}(\chi_1\chi_2^{-1})(t_2) \\
&\cdot \int_{x',y',a',b',c'\in F} |x'y'|_F^{6s-4+s_3}|\det\begin{pmatrix} b' & 1-c' \\ a'-1 & b'+a'c'-1\end{pmatrix}|_F^{3s-3}|a'|_F^{s_1}|c'-\frac{3a'-2}{a'^2}|_F^{s_2}\cdot \\
&f_s(x_{-(2\alpha+\beta)}(t_1+x')x_{-(\alpha+\beta)}(t_2-y')\overline{n}(t_3, t_4, t_5))dx'dy'da'db'dc'dt_1dt_2dt_3dt_4dt_5.
\end{aligned}
$$

We change the variable $t_1 \mapsto t_1 - x'$, $t_2 \mapsto t_2 + y'$, then $x' \mapsto x't_1$, $y' \mapsto y't_2$. Then

$$
\begin{aligned}
&\widetilde{I}(f_s, s_1, s_2, s_3, \lambda_1, \lambda_2) \\
=&\int_{t_1,t_2,t_3,t_4,t_5\in F} |t_1 - x't_1|_F^{\lambda_1}(\chi_2\chi_3^{-1})(t_1 - x't_1)|t_2 + y't_2|_F^{\lambda_2}(\chi_1\chi_2^{-1})(t_2 + y't_2) \\
&\cdot \int_{x',y',a',b',c'\in F} |t_1t_2x'y'|_F^{6s-4+s_3}|\det\begin{pmatrix} b' & 1-c' \\ a'-1 & b'+a'c'-1\end{pmatrix}|_F^{3s-3}|a'|_F^{s_1}|c'-\frac{3a'-2}{a'^2}|_F^{s_2}\cdot \\
&\quad f_s(x_{-(2\alpha+\beta)}(t_1)x_{-(\alpha+\beta)}(t_2)\overline{n}(t_3, t_4, t_5)) \cdot |t_1t_2|_F dx'dy'da'db'dc'dt_1dt_2dt_3dt_4dt_5.
\end{aligned}
$$

Clearly, the above integral is factorizable. Thus $\widetilde{I}(f_s, s_1, s_2, s_3\lambda_1, \lambda_2)$ is a product of three integrals:

$$\begin{aligned}&\int_{t_1,t_2,t_3,t_4,t_5\in F}|t_1|_F^{6s-3+\lambda_1+s_3}(\chi_2\chi_3^{-1})(t_1)|t_2|_F^{6s-3+\lambda_2+s_3}(\chi_1\chi_2^{-1})(t_2)\\&\cdot f_s(x_{-(2\alpha+\beta)}(-t_1)x_{-(\alpha+\beta)}(t_2)\overline{n}(t_3,t_4,t_5))dt_1dt_2dt_3dt_4dt_5;\end{aligned} \tag{11.9}$$

$$\begin{aligned}&\int_F |x'|_F^{6s-4+s_3}|1-x'|_F^{\lambda_1}(\chi_2\chi_3^{-1})(1-x')dx'\\&\cdot\int_F |y'|_F^{6s-4+s_3}|1+y'|_F^{\lambda_2}(\chi_1\chi_2^{-1})(1+y')dy';\end{aligned} \tag{11.10}$$

and

$$J(s,s_1,s_2):=\int_{a',b',c'\in F}|\det\begin{pmatrix} b' & 1-c' \\ a'-1 & b'+a'c'-1\end{pmatrix}|_F^{3s-3}|a'|_F^{s_1}|c'-\frac{3a'-2}{a'^2}|_F^{s_2}da'db'dc'. \tag{11.11}$$

The integral (11.9) is exactly $I(f_s,6s-3+\lambda_1+s_3,6s-3+\lambda_2+s_3)$, which converges absolutely for all $s,s_3,\lambda_1,\lambda_2$ by our assumption on f_s. The integral (11.10) converges absolutely when

$$\mathrm{Re}(6s-3+s_3)>0,\mathrm{Re}(\lambda_i)>-1,\mathrm{Re}(6s-3+\lambda_i+s_3)<0, \qquad (i=1,2), \tag{11.12}$$

and it is equal to

$$(\chi_1\chi_3^{-1})(-1)B_F(6s-3+s_3,\lambda_1+1,id,\chi_2\chi_3^{-1})B_F(6s-3+s_3,\lambda_2+1,id,\chi_1\chi_2^{-1}).$$

We should note that when s_3 varies in a small vertical strip $\epsilon\le\mathrm{Re}(s_3)\le 0$, we can choose small λ_i such that the domain defined by the conditions (11.12) has non-empty intersection with the vertical strip $\mathrm{Re}(s)>\frac{2}{3}$. Also, when $s_3=0,\lambda_1=1-3s+u_2-u_3,\lambda_2=1-3s+u_1-u_2$, by (8.22),

$$\begin{aligned}&B_F(6s-3+s_3,\lambda_1+1,id,\chi_2\chi_3^{-1})B_F(6s-3+s_3,\lambda_2+1,id,\chi_1\chi_2^{-1})\\=&B_F(6s-3,2-3s+u_2-u_3,id,\chi_2\chi_3^{-1})B_F(6s-3,2-3s+u_1-u_2,id,\chi_1\chi_2^{-1})\\=&\Gamma_F(6s-3)\Gamma_F(2-3s+u_2-u_3,\chi_2\chi_3^{-1})\Gamma_F(2-3s-u_2+u_3,\chi_2^{-1}\chi_3)\\&\cdot\Gamma_F(6s-3)\Gamma_F(2-3s+u_1-u_2,\chi_1\chi_2^{-1})\Gamma_F(2-3s-u_1+u_2,\chi_1^{-1}\chi_2)\end{aligned} \tag{11.13}$$

In the next Section, we will compute $J(s,s_1,s_2)$ explicitly.

11.2 Explicit Formula for $J(s,s_1,s_2)$

We will finish our computation for $J(s,s_1,s_2)$ in this Section. To simplify our notations, in the following, we write a for a' etc. After changing the variable $b\mapsto b-\frac{ac-1}{2}$, $c\mapsto c+\frac{3a-2}{a^2}$, $c\mapsto c\cdot\frac{2}{a}$, successively, we can obtain that the integral $J(s,s_1,s_2)$ defined in (11.11) is equal to

$$\int |a|_F^{s_1-1}|c|_F^{s_2}|b^2-c^2-\frac{(a-1)^3}{a^2}|_F^{3s-3}dadbdc.$$

We change the variable $b\mapsto b\cdot\frac{a-1}{a}$ and $c\mapsto c\cdot\frac{a-1}{a}$, and then set $x_1=b^2-c^2-(a-1)$. Then $J(s,s_1,s_2)$ is equal to

$$\int |a|_F^{s_1-1}|c|_F^{s_2}|\frac{a-1}{a}|_F^{6s-4+\lambda_2}|x_1|_F^{3s-3}|x_1+c^2+(a-1)|_F^{-\frac{1}{2}}dx_1dadc. \tag{11.14}$$

We change the variable $x_1\mapsto x_1\cdot(c^2+(a-1))$, then the integral (11.14) can be factorized into a product of

$$\int |a|_F^{s_1-1}|c|_F^{s_2}|\frac{a-1}{a}|_F^{6s-4+\lambda_2}|c^2+(a-1)|_F^{3s-\frac{5}{2}}dbdc. \tag{11.15}$$

and

$$\int |x_1|_F^{3s-3}\cdot|x_1+1|_F^{-\frac{1}{2}}dx_1. \tag{11.16}$$

Clearly the integral (11.16) converges absolutely when $\mathrm{Re}(s)\in(\frac{2}{3},\frac{5}{6})$, and it is equal to $B_F(3s-2,\frac{1}{2},id,id)$.

Similarly, we set $x_2=c^2+(a-1)$ in (11.15), then change the variable $x_2\mapsto x_2\cdot(a-1)$. Then (11.15) can be factorized into a product of

$$\int |a|_F^{s_1-6s+3-s_2}|a-1|_F^{9s-6+\frac{3}{2}s_2}da \tag{11.17}$$

and

$$\int |x_2-1|_F^{\frac{1}{2}s_2-\frac{1}{2}}|x_2|_F^{3s-\frac{5}{2}}dx_2. \tag{11.18}$$

The integral (11.17) converges absolutely when

$$\mathrm{Re}(s_1 - 6s + 4 - s_2) > 0, \mathrm{Re}(9s - 5 - \frac{3}{2}s_2) > 0, \mathrm{Re}(3s - 2 + s_1 + \frac{1}{2}s_2) < 0.$$

and it is equal to $B_F(s_1 - 6s + 4 - s_2, 9s - 5 - \frac{3}{2}s_2, id, id)$. The integral (11.18) converges absolutely when

$$Re(3s - 2 + s_2) < 0.$$

and it is equal to $B_F(3s - \frac{3}{2}, \frac{s_2}{2} + \frac{1}{2}, id, id)$. (The other two convergent conditions for $B_F(3s - \frac{3}{2}, \frac{s_2}{2} + \frac{1}{2}, id, id)$ are automatic, since we have already assumed that $\mathrm{Re}(s) > \frac{2}{3}$ and $\mathrm{Re}(s_1), \mathrm{Re}(s_2)$ are small negative numbers.)

Thus when s_1 and s_2 have sufficiently small negative real parts (in absolute value), the domain D defined by all the above convergent conditions is nonempty. Thus whenever $s \in D$, we establish the equation

$$\begin{aligned} &J(s, s_1, s_2) \\ =&B_F(3s - 2, \frac{1}{2}, id, id)B_F(s_1 - 6s + 4 - s_2, 9s - 5 - \frac{3}{2}s_2, id, id) \\ &\cdot B_F(3s - \frac{3}{2}, \frac{s_2}{2} + \frac{1}{2}, id, id) \end{aligned}$$

The RHS is meromorphic in s, s_1, s_2, thus $J(s, s_1, s_2)$ has a meromorphic continuation and

$$J(s, 0, 0) = B_F(3s - 2, \frac{1}{2}, id, id)B_F(-6s + 4, 9s - 5, id, id)B_F(3s - \frac{3}{2}, \frac{1}{2}, id, id) \quad (11.19)$$

in the sense of meromorphic continuation. By (8.22) and the obvious observation that $\Gamma_F(\frac{1}{2}) = 1$, the RHS of (11.19) is equal to

$$\Gamma_F(3s - 2)\Gamma_F(\frac{5}{2} - 3s)\Gamma_F(3s - \frac{3}{2})\Gamma_F(2 - 3s)\Gamma_F(9s - 5)\Gamma_F(-6s + 4)\Gamma_F(-3s + 2).$$

Then we apply the identity $\Gamma_F(s)\Gamma_F(1 - s) = 1$, the above leads to

$$J(s, 0, 0) = \Gamma_F(3s - 2)\Gamma_F(2 - 3s)\Gamma_F(9s - 5)\Gamma_F(6s - 3)^{-1}\Gamma_F(2 - 3s) \quad (11.20)$$

in the sense of meromorphic continuation.

To sum up, when $s_1, s_2, s_3, \lambda_1, \lambda_2$ varies in a certain open set in the negative half space $\mathrm{Re}(s_i), \mathrm{Re}(\lambda_i) \leq 0$, the domains of convergence for $J(s, s_1, s_2)$ and

$$B_F(6s-3+s_3, \lambda_1+1, id, \chi_2\chi_3^{-1})B_F(6s-3+s_3, \lambda_2+1, id, \chi_1\chi_2^{-1})$$

have non-empty intersection (in fact, the intersection lies in the half space $\mathrm{Re}(s) > \frac{2}{3}$). Moreover, we establish the functional equation

$$\begin{aligned}
&\widetilde{I}(f_s, s_1, s_2, s_3, \lambda_1, \lambda_2) \\
=&I(f_s, 6s-3+\lambda_1+s_3, 6s-3+\lambda_2+s_3) \cdot J(s, s_1, s_2) \cdot \\
&(\chi_1\chi_3^{-1})(-1) \cdot B_F(6s-3+s_3, \lambda_1+1, id, \chi_2\chi_3^{-1})B_F(6s-3+s_3, \lambda_2+1, id, \chi_1\chi_2^{-1}).
\end{aligned}$$

Therefore by (11.5),

$$\begin{aligned}
\widetilde{I}(f_s) &= \widetilde{I}(f_s, 0, 0, 0, 1-3s+u_2-u-3, 1-3s+u_1-u_2) \\
&= I(f_s) \cdot J(s, 0, 0) \cdot (\chi_1\chi_3^{-1})(-1) \cdot \\
&\quad B_F(6s-3, 2-3s+u_2-u_3, id, \chi_2\chi_3^{-1})B_F(6s-3, 2-3s+u_1-u_2, id, \chi_1\chi_2^{-1}).
\end{aligned}$$

Combining (11.13) and (11.20), we have

$$\begin{aligned}
\widetilde{I}(f_s) &= I(f_s) \cdot J(s, 0, 0) \cdot (\chi_1\chi_3^{-1})(-1) \cdot \\
&\quad B_F(6s-3, 2-3s+u_2-u_3, id, \chi_2\chi_3^{-1})B_F(6s-3, 2-3s+u_1-u_2, id, \chi_1\chi_2^{-1}) \\
&= (\chi_1\chi_3^{-1})(-1)I(f_s) \cdot \Gamma_F(3s-2)\Gamma_F(6s-3)\Gamma_F(9s-5) \\
&\quad \cdot \Gamma_F(2-3s)^2\Gamma_F(2-3s+u_2-u_3, \chi_2\chi_3^{-1})\Gamma_F(2-3s-u_2+u_3, \chi_2^{-1}\chi_3) \\
&\cdot \Gamma_F(2-3s+u_1-u_2, \chi_1\chi_2^{-1})\Gamma_F(2-3s-u_1+u_2, \chi_1^{-1}\chi_2).
\end{aligned}$$

We finally remark that the above is also the functional equation satisfied by $Z_1(W_v, f_s)$.

11.3 The Functional Equation for $Z(W_v, f_s)$

We will finish establishing the functional equation for the local integrals in this Section. Recall that in Theorem 1.0.13, we have proved that

$$\widetilde{Z}(W_v, f_s) = \widetilde{Z}_1(W_v, f_s) \cdot \gamma_F(3s - 1 - u_1 + u_3, \chi_1^{-1}\chi_3, \psi).$$

Combining the functional equation for $Z_1(W_v, f_s)$, we find that

$$\begin{aligned}\widetilde{Z}(W_v, f_s) = (\chi_1\chi_3^{-1})(-1)Z_1(W_v, f_s) &\cdot \Gamma_F(3s-2)\Gamma_F(6s-3)\Gamma_F(9s-5) \\ &\cdot \Gamma_F(2-3s)^2\Gamma_F(2-3s+u_2-u_3, \chi_2\chi_3^{-1})\Gamma_F(2-3s-u_2+u_3, \chi_2^{-1}\chi_3) \\ &\cdot \Gamma_F(2-3s+u_1-u_2, \chi_1\chi_2^{-1})\Gamma_F(2-3s-u_1+u_2, \chi_1^{-1}\chi_2) \\ &\cdot \gamma_F(3s-1-u_1+u_3, \chi_1^{-1}\chi_3, \psi)\end{aligned} \tag{11.21}$$

Again by Theorem 1.0.13,

$$Z(W_v, f_s) = Z_1(W_v, f_s) \cdot \gamma_F(2 - 3s - u_1 + u_3, \chi_1^{-1}\chi_3, \psi).$$

Hence by the functional equation (8.20), we get

$$(\chi_1\chi_3^{-1})(-1)Z_1(W_v, f_s) = Z(W_v, f_s) \cdot \gamma_F(3s - 1 + u_1 - u_3, \chi_1\chi_3^{-1}, \psi). \tag{11.22}$$

Combining (11.21) and (11.22), we obtain that

$$\begin{aligned}\widetilde{Z}(W_v, f_s) = Z(W_v, f_s) &\cdot \Gamma_F(3s-2)\Gamma_F(6s-3)\Gamma_F(9s-5) \\ &\cdot \Gamma_F(2-3s)^2\Gamma_F(2-3s+u_2-u_3, \chi_2\chi_3^{-1})\Gamma_F(2-3s-u_2+u_3, \chi_2^{-1}\chi_3) \\ &\cdot \Gamma_F(2-3s+u_1-u_2, \chi_1\chi_2^{-1})\Gamma_F(2-3s-u_1+u_2, \chi_1^{-1}\chi_2) \\ &\cdot \gamma_F(3s-1-u_1+u_3, \chi_1^{-1}\chi_3, \psi)\gamma_F(3s-1+u_1-u_3, \chi_1\chi_3^{-1}, \psi).\end{aligned}$$

Therefore,

$$\begin{aligned}\Gamma(s,\pi_u,\mathrm{Ad},\psi)=&\Gamma_F(3s-2)\Gamma_F(6s-3)\Gamma_F(9s-5)\\ &\cdot\Gamma_F(2-3s)^2\Gamma_F(2-3s+u_2-u_3,\chi_2\chi_3^{-1})\Gamma_F(2-3s-u_2+u_3,\chi_2^{-1}\chi_3)\\ &\cdot\Gamma_F(2-3s+u_1-u_2,\chi_1\chi_2^{-1})\Gamma_F(2-3s-u_1+u_2,\chi_1^{-1}\chi_2)\\ &\cdot\gamma_F(3s-1-u_1+u_3,\chi_1^{-1}\chi_3,\psi)\gamma_F(3s-1+u_1-u_3,\chi_1\chi_3^{-1},\psi).\end{aligned}$$

By the Local Langlands Correspondence at the archimedean places (see [Kn]), we consider the adjoint representation of the L-group of GL_3 (which is $\mathrm{GL}_3(\mathbb{C})$) on the Lie algebra of PGL_3. When $\pi=\pi_u$ is a principal series, the adjoint $L-$function is

$$L(s,\pi_u,\mathrm{Ad})=L(s)^2\prod_{i\neq j,1\leq i,j\leq 3}L(s+u_i-u_j,\chi_i\chi_j^{-1})$$

It is easy to check that

$$\begin{aligned}&\frac{L(2-3s,\pi_u,\mathrm{Ad})}{L(3s-1,\pi_u,\mathrm{Ad})}\\ =&\frac{L(2-3s)^2\prod_{i\neq j,1\leq i,j\leq 3}L(2-3s+u_i-u_j,\chi_i\chi_j^{-1})}{L(3s-1)^2\prod_{i\neq j,1\leq i,j\leq 3}L(3s-1+u_i-u_j,\chi_i\chi_j^{-1})}\\ =&\Gamma_F(2-3s)^2\Gamma_F(2-3s+u_2-u_3,\chi_2\chi_3^{-1})\Gamma_F(2-3s-u_2+u_3,\chi_2^{-1}\chi_3)\\ &\cdot\Gamma_F(2-3s+u_1-u_2,\chi_1\chi_2^{-1})\Gamma_F(2-3s-u_1+u_2,\chi_1^{-1}\chi_2)\\ &\cdot\gamma_F(3s-1-u_1+u_3,\chi_1^{-1}\chi_3,\psi)\gamma_F(3s-1+u_1-u_3,\chi_1\chi_3^{-1},\psi)\end{aligned}$$

Thus,

$$\Gamma(s,\pi_u,\mathrm{Ad},\psi)=\Gamma_F(3s-2)\Gamma_F(6s-3)\Gamma_F(9s-5)\cdot\frac{L(2-3s,\pi_u,\mathrm{Ad})}{L(3s-1,\pi_u,\mathrm{Ad})},$$

where

$$\Gamma_F(3s-2)\Gamma_F(6s-3)\Gamma_F(9s-5)=\frac{L(3s-2)}{L(3-3s)}\cdot\frac{L(6s-3)}{L(4-6s)}\cdot\frac{L(9s-5)}{L(6-9s)}$$

comes from the normalizing factor of the intertwining operator $M(w_{3\alpha+2\beta})$ (see Chapter 9). This finishes our proof of Theorem 1.0.14.

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Appendix A

An Intuitive Computation of GL_2

In this Appendix, we briefly discuss an application of Lemma 8.3.1 in the case of standard L-function for GL_2. It also provides us some insights for the computation on the adjoint L-function for GL_3. For simplicity, we only confine ourselves in a simple case.

Let us take a principal series $\tau_s = \mathrm{Ind}_B^{\mathrm{GL}_2(\mathbb{R})} |\quad|^{s-\frac{1}{2}} \otimes |\quad|^{-s+\frac{1}{2}}$. Then every function $f_s \in V_{\tau_s}$ satisfies

$$f_s(\begin{pmatrix} a & \star \\ 0 & d \end{pmatrix} x) = |a|^s |d|^{-s} f_s(x). \tag{A.1}$$

If we assume that $\mathrm{Re}(s) > \frac{1}{2}$, then the Jacquet integral

$$\int_{\mathbb{R}} f_s(w^{-1} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g)\psi(x)dx$$

converges absolutely, and defines a Whittaker function $W(g)$ of $\mathrm{GL}_2(\mathbb{R})$. The Mellin transform of W along the torus is related to the standard L-function for $\mathrm{GL}_2(\mathbb{R})$. When $\mathrm{Re}(s) > \frac{1}{2}$, we can replace the Whittaker function by the Jacquet integral. Thus, we need to study the iterated integral

$$\int_{\mathbb{R}^\times} \int_{\mathbb{R}} f_s(w^{-1} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} a & \\ & 1 \end{pmatrix})\psi(x)|a|_F^{\mu} dx d^\times a. \tag{A.2}$$

We note that this integral does not converges absolutely as a double integral. Now we follow the notations in Lemma 8.3.1 and check all four assumptions.

Let $A = X_0 = \mathbb{R}^\times, X = \mathbb{R}$, j be the isomorphism defined by inverse, i.e. $j(x) = x^{-1}$. We take $\chi(a) = |a|_{\mathbb{R}}^{\mu}$ and define

$$F_s(a,x) = f_s(w^{-1}\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\begin{pmatrix}a & \\ & 1\end{pmatrix}) = f_s(w^{-1}\begin{pmatrix}1 & a^{-1}x\\ & 1\end{pmatrix})|a|^{-s}.$$

Then $F_s(a,x)$ satisfies (8.7), where σ is the character defined by $\sigma(x) = |x|^s$. We aim to find a domain D (may depend on μ) such that whenever $s \in D$, all four assumptions in Lemma 8.3.1 are satisfied.

Assumption 1): We need to check that

$$\int_{\mathbb{R}^\times} f_s(w^{-1}\begin{pmatrix}1 & a^{-1}\\ & 1\end{pmatrix})|a|^{-s+\mu}d^\times a \tag{A.3}$$

converges absolutely. Changing the variable $a \mapsto a^{-1}$, we can rewrite the above integral as

$$\int_{\mathbb{R}}(\tau_s(w^{-1})f_s)(\begin{pmatrix}1 & \\ a & 1\end{pmatrix})|a|^{s-\mu-1}da. \tag{A.4}$$

We can regard the above integral as a Mellin transform of the restriction of $\tau_s(w^{-1})f_s$ on the lower unipotent radical. In this simple case, we can use the Iwasawa decomposition of $\mathrm{GL}_2(\mathbb{R})$ to detect the exact range of convergence of (A.4). Yet, when we deal with groups of higher rank, it becomes much harder. Instead, we can get a weaker result by comparing (A.4) with the intertwining operator. It is easy to see that the intertwining operator converges absolutely when $\mathrm{Re}(s) > \frac{1}{2}$. If we assume that $\mathrm{Re}(s-\mu-1) \in (-1,0)$, then $(\tau_s(w^{-1})f_s)(\begin{pmatrix}1 & \\ a & 1\end{pmatrix})|a|^{s-\mu-1}$ is integrable near 0, since $(\tau_s(w^{-1})f_s)(\begin{pmatrix}1 & \\ a & 1\end{pmatrix})$ is a smooth function of a. Moreover, when $|a|_F > 1$, then the integrand is dominated by $|(\tau_s(w^{-1})f_s)(\begin{pmatrix}1 & \\ a & 1\end{pmatrix})|$. Hence $(\tau_s(w^{-1})f_s)(\begin{pmatrix}1 & \\ a & 1\end{pmatrix})$ is also integrable near infinity. Thus, (A.3) converges absolutely when $\mathrm{Re}(s) > \frac{1}{2}$ and $\mathrm{Re}(s-\mu-1) \in (-1,0)$.

Assumption 2): In this case, A_1 is still trivial. Thus we only need to consider

$$\int_{\mathbb{R}} f_s(w^{-1}\begin{pmatrix}1 & x\\ & 1\end{pmatrix})dx,$$

which converges absolutely when $\mathrm{Re}(s) > \frac{1}{2}$.

Assumption 3): In this case,

$$\int_{\mathbb{R}} (\sigma\chi^{-1})(j(x))\psi(x)dx = \int_{\mathbb{R}} |x|^{-s+\mu}\psi(x)dx, \tag{A.5}$$

which converges conditionally when $\mathrm{Re}(-s+\mu) \in (-1, 0)$.

Assumption 4): In this case,

$$\int_{j^{-1}(a_2)\in B_r} (\sigma\chi^{-1})(a_2) \cdot |j^{-1}(a_2)| d^\times a_2 = \int_{|a^{-1}|\le r} |a|^{s-\mu-1} d^\times a,$$

which converges when $\mathrm{Re}(s-\mu-1) < 0$.

Therefore, if we assume that $\mathrm{Re}(\mu) \in (0, \frac{1}{2})$, then we can take

$$D = \{s \in \mathbb{C} \,|\, \mathrm{Re}(s) \in (\frac{1}{2}, \mathrm{Re}(\mu)+1)\}.$$

Then by Lemma 8.3.1, for any $\mathrm{Re}(\mu) \in (0, \frac{1}{2})$, and $s \in D$, the integral (A.2) converges and it is equal to the product of (A.3) and (A.5). Clearly, (A.5) is a Gamma function. Thus, to establish the functional equation (A.2), it suffices to establish a functional equation for (A.3), or equivalently for (A.4). Indeed, this can be done once we realize the intertwining operator as a convolution operator in the lower unipotent subgroup. We omit the details here.

Appendix B

Some Explicit Computation Gamma Function and Beta Function

The following explicit computations are clear from Theorem 8.4.2.

Case 1: $F = \mathbb{R}$.

Denote by

$$L_{\mathbb{R}}(s, (\mathrm{sgn})^{\epsilon}) = \pi^{-\frac{s+\epsilon}{2}} \Gamma(\frac{s+\epsilon}{2}), \quad \epsilon = 0, 1,$$

the local L-functions at the real place, where Γ is the classical Gamma function (different from $\Gamma_{\mathbb{R}}$). When $\epsilon = 0$, we will omit the trivial character and write $L_{\mathbb{R}}(s)$ rather than $L_{\mathbb{R}}(s, id)$. The Gamma function $\Gamma_{\mathbb{R}}$ in this case is

$$\Gamma_{\mathbb{R}}(s, (\mathrm{sgn})^{\epsilon}) = \int_{\mathbb{R}} (\mathrm{sgn} x)^{\epsilon} \psi(x) |x|^{s-1} dx = i^{\epsilon} \frac{L_{\mathbb{R}}(s, (sgn)^{\epsilon})}{L_{\mathbb{R}}(1-s, (sgn)^{\epsilon})}. \tag{B.1}$$

The Beta function $B_{\mathbb{R}}$ at the real place is equal to

$$\begin{aligned}
&B_{\mathbb{R}}(s_1, s_2, (sgn)^{\epsilon_1}, (sgn)^{\epsilon_2})\\
&= (-1)^{\epsilon_1+\epsilon_2} \int_{\mathbb{R}} (sgnx)^{\epsilon_1} |x|^{s_1-1} (sgn(1-x))^{\epsilon_2} |1-x|^{s_2-1} dx\\
&= \Gamma_{\mathbb{R}}(s_1, (sgn)^{\epsilon_1}) \Gamma_{\mathbb{R}}(s_2, (sgn)^{\epsilon_2}) \Gamma_{\mathbb{R}}(1-s_1-s_2, (sgn)^{\epsilon_1+\epsilon_2})\\
&= i^{(\epsilon_1+\epsilon_2)} \cdot i^{(\epsilon_1+\epsilon_2) \mod 2} \frac{L_{\mathbb{R}}(s_1, (sgn)^{\epsilon_1})}{L_{\mathbb{R}}(1-s_1, (sgn)^{\epsilon_1})}\\
&\quad \cdot \frac{L_{\mathbb{R}}(s_2, (sgn)^{\epsilon_2})}{L_{\mathbb{R}}(1-s_2, (sgn)^{\epsilon_2})} \cdot \frac{L_{\mathbb{R}}(1-s_1-s_2, (sgn)^{\epsilon_1+\epsilon_2})}{L_{\mathbb{R}}(s_1+s_2, (sgn)^{\epsilon_1+\epsilon_2})}
\end{aligned} \tag{B.2}$$

Case 2: $F = \mathbb{C}$.

Let χ be the unitary multiplicative character of $\mathbb{C}^\times$ defined by

$$\chi(z) = z \cdot (z\overline{z})^{-\frac{1}{2}}.$$

Denote by

$$L_{\mathbb{C}}(s, \chi^k) = 2(2\pi)^{-s-\frac{|k|}{2}} \Gamma(s + \frac{|k|}{2}), \quad k \in \mathbb{Z},$$

the local L-functions at the complex place. The complex norm $|x|_{\mathbb{C}} = |x\overline{x}|$. The Gamma function $\Gamma_{\mathbb{C}}$ in this case is

$$\Gamma_{\mathbb{C}}(s, \chi^k) = \int_{\mathbb{C}} \chi^k(x) \psi(x) |x|_{\mathbb{C}}^{s-1} dx = i^{|k|} \frac{L_{\mathbb{C}}(s, \chi^k)}{L_{\mathbb{C}}(1-s, \chi^{-k})}. \tag{B.3}$$

The Beta function $B_{\mathbb{C}}$ at the complex place is equal to

$$\begin{aligned}
&B_{\mathbb{C}}(s_1, s_2, \chi^{k_1}, \chi^{k_2})\\
=&\chi^{k_1}(-1)\chi^{k_2}(-1) \int_{\mathbb{C}} \chi^{k_1}(x) |x|_{\mathbb{C}}^{m-1} \chi^{k_2}(1-x) |1-x|_{\mathbb{C}}^{n-1} dx\\
=&\Gamma_{\mathbb{C}}(s_1, \chi^{k_1}) \Gamma_{\mathbb{C}}(s_2, \chi^{k_2}) \Gamma_{\mathbb{C}}(1-s_1-s_2, \chi^{-k_1-k_2})\\
=&i^{|k_1|+|k_2|+|k_1+k_2|} \frac{L_{\mathbb{C}}(s_1, \chi^{k_1})}{L_{\mathbb{R}}(1-s_1, \chi^{-k_1})} \cdot \frac{L_{\mathbb{C}}(s_2, \chi^{k_2})}{L_{\mathbb{C}}(1-s_2, \chi^{-k_2})} \cdot \frac{L_{\mathbb{C}}(1-s_1-s_2, \chi^{-k_1-k_2})}{L_{\mathbb{C}}(s_1+s_2, \chi^{k_1+k_2})}
\end{aligned} \tag{B.4}$$

Remark B.0.1. *Because the difference of choice of ψ in this paper, our $\Gamma_{\mathbb{R}}(s, (sgn)^\epsilon)$ and the $\Gamma_{\mathbb{R}}(s, (sgn)^\epsilon)$ in [V2] differ by $(-1)^\epsilon$; our $\Gamma_{\mathbb{C}}(s, \chi^k)$ and the $\Gamma_{\mathbb{C}}(s, \chi^k)$ in [V2] differ by $\chi^k(-1)$. But these slight differences will not affect the formulae of $B_{\mathbb{R}}$ or $B_{\mathbb{C}}$.*

Case 3: F is non-archimedean.

Let us assume that χ, χ_1, χ_2 are unramified (i.e they are trivial). Again, denote by

$$L_F(s, id) = (1 - q^{-s})^{-1}$$

the local L-function. Then

$$\Gamma_F(s, id) = \int_F |x|_F^{s-1} \psi(x) dx = \frac{1 - q^{s-1}}{1 - q^{-s}} = \frac{L_F(s, id)}{L_F(1 - s, id)}, \tag{B.5}$$

and

$$B_F(s_1, s_2, id, id) = \int_F |x|_F^{s_1 - 1} |1 - x|^{s_2 - 1} dx = \Gamma_F(s_1, id) \Gamma_F(s_2, id) \Gamma_F(-s_1 - s_2, id). \tag{B.6}$$