New Exotic Symplectic 4-Manifolds with Nonnegative Signatures and Exotic Smooth Structures on Small 4-Manifolds

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Abstract

The focus of this thesis is twofold. First one is the geography problem of symplectic and smooth 4-manifolds with nonnegative signatures. We construct new non-spin, irreducible, symplectic and smooth 4-manifolds with nonnegative signatures, with more than one smooth structures and small topology. These manifolds are interesting with respect to the symplectic and smooth geography problems. More specifically, we construct infinite families of smooth, closed, simply-connected, minimal, symplectic and non-symplectic 4-manifolds with nonnegative signatures that have the smallest Euler characteristics among the all known such manifolds, and with more than one smooth structures.

The second focus of this thesis is the study of fibrations of complex curves of genus two and constructing exotic 4-manifolds with small Euler characteristics. In [93, 94] Namikawa and Ueno gave complete classification of all singular fibers in pencils of genus two curves, where each pencil is a family of complex curves of genus two over the 2-disc with one singular curve over the origin. They gave the list of all singular fibers arising in such families. In the constructions of singularities they used algebro-geometric techniques. In this thesis, we topologically construct certain singularity types in the Namikawa-Ueno's list. More precisely, we find pencils of genus two curves in Hirzebruch surfaces and from which we obtain specific types of Namikawa-Ueno's genus two singular fibers and sections, precisely. In addition to constructing these singularities topologically, we also introduce a deformation technique of the singular fibers of certain types Lefschetz fibrations over the 2-sphere. Then by using them and via symplectic surgeries, we build new exotic minimal symplectic 4-manifolds with small topology.

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Chapter 1

Introduction

In this chapter we will remind the basic definitions and theorems to be used in the sequel. An *n-dimensional topological manifold* is a separable Hausdorff topological space X such that every point $p \in X$ has a neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n_+ = \{(x_1, ..., x_n) \in \mathbb{R}^n | x_n \geq 0\}$. A pair (U_p, ϕ_p) of such a neighborhood and a homeomorphism is called a *chart*. A collection of charts covering X is called an *atlas*. The composition $\phi_q \circ \phi_p^{-1}$ on $\phi_p(U_p \cap U_q)$ is the *transition function* between the charts (U_p, ϕ_p) and (U_q, ϕ_q). The points of X that have neighborhoods homeomorphic to open subsets of $\{(x_1, ..., x_n) \in \mathbb{R}^n | x_n = 0\}$ form the *boundary* of X which is an $(n-1)$ dimensional submanifold of X denoted by ∂X. On this thesis we will work with manifolds with no boundaries.

A *smooth manifold* is a topological manifold with an atlas such that all the transition functions are of class C^∞. Let us also remind that for a given smooth manifold X, by an *exotic copy of X* we mean a smooth manifold that is homeomorphic but not diffeomorphic to X.

A *complex n-manifold* is a manifold whose coordinate charts are open subsets of $\mathbb{C}^n$ and the transition functions between charts are holomorphic functions. For a compact, complex manifold X of dimension n, if the Kodaira dimension of X is equal to n, then X is called *of general type* [63]. *An algebraic n-fold* is an algebraic variety over $\mathbb{C}$ of complex dimension n. When an algebraic n-fold is non-singular, then it is called a complex algebraic manifold.

In this thesis we will work with 4-manifolds and complex surfaces. In the remaining sections of this chapter we will give a recap of basic results in the 4-manifolds (and complex surfaces) theory.

1.1 Classification of 4-Manifolds up to Homeomorphism, Freedman's Theorem

The homeomorphism type of simply connected, smooth 4-manifolds are determined by their *intersection forms* by Freedman's theorem [42]. Thus we begin with defining the intersection form and related invariants.

Definition 1.1.1. *Let X be a compact, oriented, topological 4-manifold, and $[X] \in H_4(X, \partial X; \mathbb{Z})$ be its fundamental class induced from the orientation. The symmetric bilinear form*

$$Q_X : H^2(X, \partial X; \mathbb{Z}) \times H^2(X, \partial X; \mathbb{Z}) \to \mathbb{Z}$$

defined by $Q_X(a, b) = < a \cup b, [X] > = a \cdot b \in \mathbb{Z}$ is called the intersection form of X. By Poincaré duality it is also defined on $H_2(X, \mathbb{Z}) \times H_2(X, \mathbb{Z})$.

Geometrically, Q_X can be interpreted as follows. First, for X a closed (compact without boundary), oriented, smooth 4-manifold, every element of $H_2(X, \mathbb{Z})$ can be represented by an embedded surface [48]. That is to say, for $\alpha \in H_2(X, \mathbb{Z})$ there is a closed oriented surface Σ and an embedding $i : \Sigma \hookrightarrow X$ such that $i_*([\Sigma]) = \alpha$, where $[\Sigma]$ is the fundamental class of Σ. Now, let $a, b \in H^2(X, \mathbb{Z})$; $\alpha, \beta \in H_2(X, \mathbb{Z})$ be their Poincaré duals and $\Sigma_\alpha, \Sigma_\beta$ be their surface representatives. Then $Q_X(a, b)$ is the number of points in $\Sigma_\alpha \cap \Sigma_\beta$ counted with sign.

For a given symmetric, bilinear form Q on the finitely generated free abelian group A, we diagonalize Q over $A \otimes_{\mathbb{Z}} \mathbb{R}$. The number of $+1$'s and -1's are denoted by b_2^+ and b_2^- respectively. Then rank (rk) and signature (σ) of Q are defined as follows; $rk(Q) = b_2^+ + b_2^-$, and $\sigma(Q) = b_2^+ - b_2^-$. The form Q is even if $Q(\alpha, \alpha) \equiv 0 \pmod 2$, for every $\alpha \in A$; Q is odd otherwise.

Theorem 1.1.2. *(Freedman [42]) For every symmetric bilinear form Q with $\det Q = \pm 1$, there exists a simply connected, closed, topological 4-manifold X such that $Q_X \cong Q$.*

If Q is even, the manifold is unique up to homeomorphism. If Q is odd, there corresponds exactly two different homeomorphism types of manifolds and at most one of them admits smooth structure. Consequently, simply connected, smooth 4-manifolds are determined up to homeomorphism by their intersection forms.

A 4-manifold is called *minimal* if there is no 2 dimensional sphere S^2 satisfying $[S^2]^2 = -1$, where $[S^2] \in H_2(X, \mathbb{Z})$ is the homology class. We recall that a 4-manifold X is called *spin* if the second Stiefel-Whitney class $w_2(X)$ is zero. If X is spin then its intersection form Q_X is even. Hence a spin 4-manifold cannot contain any surface with odd self intersection, so it is minimal. In addition, a **simply connected** 4-manifold is spin if and only if its intersection form is even [48]. Let us also recall

Theorem 1.1.3. *(Rohlin [111]) If X is a smooth, closed, spin 4-manifold, then* $\sigma(X) \equiv 0$ *(mod 16).*

Lastly, a smooth 4-manifold is *irreducible* if for every smooth connected sum decomposition $X = X_1 \# X_2$, either X_1 or X_2 is homeomorphic to S^4.

Let us note that in the sequel $\mathbb{CP}^2$ denotes the complex projective plane with the standard orientation and $\overline{\mathbb{CP}}^2$ denotes the underlying smooth 4-manifold equipped with the opposite orientation.

1.2 Complex and Symplectic Geography Problems

For a smooth compact oriented manifold X, an *almost-complex structure* on X means an endomorphism $J : TX \to TX$ of the tangent bundle of X with $J^2 = -1$ which determines the given orientation of X. Such a structure makes TX into a complex vector bundle, so that one can speak of the Chern classes and Chern numbers of X [81]. Let X be an (almost) complex n-manifold, then tangent bundle TX is an n-complex vector bundle. The i-th Chern class $\mathbf{c_i(TX)}$ of TX is an obstruction to the existence of $(n - i + 1)$ everywhere complex linearly independent vector fields on TX. The class $\mathbf{c_i(TX)}$ is in the $2i$-th cohomology of the base space and thus the *Chern classes* $\mathbf{c_i(X)}$ of an (almost) complex n-manifold X are defined as

$$\mathbf{c_i(X)} := \mathbf{c_i(TX)} \in H^{2i}(X, \mathbb{Z}),\ i = 0, \cdots, n \tag{1.1}$$

where $\mathbf{c_0(X)} := 1$ and the top Chern class $\mathbf{c_n(X)}$ is the Euler class $\mathbf{e(X)}$.

The *Chern numbers* of a compact, (almost) complex n-manifold X^n are defined in terms of its Chern classes. For each partition $I = i_1, \cdots, i_r$ of n, we define the I-th Chern number as

$$c_I(X) := c_{i_1} \cdots c_{i_r}[X] = < \mathbf{c_{i_1}(X)} \cup \cdots \cup \mathbf{c_{i_r}(X)}, \mu > \in \mathbb{Z} \tag{1.2}$$

where the cup product $\mathbf{c_{i_1}(X)} \cup \cdots \cup \mathbf{c_{i_r}(X)}$ is in $H^{2n}(X, \mathbb{Z})$ and the fundamental class μ is in $H_{2n}(X, \mathbb{Z})$. Hence $c_I(X)$ is an integer. To illustrate, a complex 3-manifold X has 3 Chern numbers $c_1^3(X)$, $c_1c_2(X)$ and the Euler number $c_3(X) = < \mathbf{e(X)}, \mu > := e(X)$. Moreover, $e(X)$ equals the Euler characteristic $\chi(X) = \sum_{i=0}^{n}(-1)^i b_i(X)$ as X is smooth, compact and oriented [91, 48].

In the sequel we will work with complex surfaces and (real) 4-manifolds. Thus, let us recall a significant theorem for complex surfaces.

Theorem 1.2.1. *(Adjunction Formula) Let S be a complex surface with $i : C \hookrightarrow S$ a smooth (nonsingular), connected complex curve in it. Let us denote the genus of C by $g(C)$ and the self intersection by $[C]^2$, then*

$$2g(C) - 2 = [C]^2 - < \mathbf{c_1(S)}, [C] > = [C]^2 + < K_S, [C] > \tag{1.3}$$

where K_S is the canonical class of S.

Now let us discuss the complex geography problem first. For X a compact complex surface of general type, it is well known that $c_1^2(X) \leq 9\chi_h(X)$ where $\chi_h = p_g - q + 1$ is the holomorphic Euler characteristic and c_1^2 is the square of the first Chern class. If equality holds then X is a quotient of a ball in $\mathbb{C}^2$ and $c_1^2 = 9\chi_h$ is called the *Bogomolov-Miyaoka-Yau (BMY) line*. The *geography problem* for complex surfaces asks the following. For which ordered pairs of integers (a, b), there corresponds a minimal complex surface X of general type such that $(\chi_h(X), c_1^2(X)) = (a, b)$? This problem was introduced and studied in [107], then further progress has been made ([92, 115, 31, 109, 112]). However, it is still hard to determine all such pairs (a, b) that can be realized even if one considers the complex surfaces with $c_1^2 < 8\chi_h$ ([24]). Thus it is still a challenging problem in the theory of complex surfaces. Moreover, since all simply connected complex surfaces are

Kähler, thus symplectic, this problem naturally gave rise to the symplectic geography problem.

We recall that a *symplectic manifold* (X^{2n}, w) is a smooth $2n$-dimensional manifold, equipped with a closed nondegenerate differential 2-form w. Any symplectic manifold has compatible almost-complex structures [48, 53], thus the Chern numbers are defined for (X^{2n}, w). To a closed simply connected symplectic 4-manifold, we associate two invariants χ and c_1^2 defined below and study the *symplectic geography problem* in dimension four.

Definition 1.2.2. *Let X be a closed simply connected symplectic 4-manifold, and $e(X)$ and $\sigma(X)$ denote the Euler characteristic and the signature of X, respectively. We define the following two invariants of X*

$$\chi(X) := (e(X) + \sigma(X))/4 \quad \text{and} \quad c_1^2(X) := 2e(X) + 3\sigma(X)$$

(We note that when X is complex, χ and c_1^2 are the holomorphic Euler characteristic and the square of the first Chern class, respectively.)

The symplectic geography problem *is the problem of determining which ordered pairs of non-negative integers (a, b) are realized as $(\chi(X),\ c_1^2(X))$ for some simply connected, minimal, symplectic 4-manifold X.*

There is also a related problem called the *symplectic botany problem* which asks how many diffeomorphism classes do there exist for the simply connected, minimal, symplectic 4-manifold constructed with the given topological invariants (χ, c_1^2)?

Thus, the geography and botany problems in 4-manifold topology refer to the existence and uniqueness, respectively, of a simply connected, minimal, symplectic 4-manifold with given c_1^2 and χ. The botany problem is more challenging, however, it is known that most ordered pairs are realized by infinitely many pairwise nondiffeomorphic simply connected minimal symplectic 4-manifolds (see [48]).

The geography problem has been first systematically studied in [46] and since then many studies have been done (e.g. [38, 106, 102]). In [106], the coordinates (χ, c_1^2) where $0 \leq c_1^2 < 8\chi$, i.e., $\sigma < 0$, have been realized by spin, simply connected, symplectic 4-manifolds. More recently, it was shown in [7] and [11], that all the lattice points (χ,

c_1^2) with $\sigma < 0$ can be realized with simply connected, minimal, symplectic, nonspin 4-manifolds. More precisely, it was shown that there exists an irreducible symplectic 4-manifold and infinitely many irreducible non-symplectic 4-manifolds with odd intersection forms that realize these coordinates. That is to say, the work in [106, 7, 11] completed the symplectic geography problem for the negative signature case.

Nevertheless, the geography problem is not complete for the nonnegative signature case; not all the points (χ, c_1^2) with $c_1^2 \geq 8\chi$ have been realized by simply connected, symplectic 4-manifolds. In addition, constructing small, simply connected, symplectic 4-manifolds with positive signatures, especially close to the BMY line, is also interesting in terms of the followings. As of today, the complex projective plane $\mathbb{CP}^2$ is the only known simply connected symplectic 4-manifold lying on the BMY line. In addition, it is not known whether there exists any symplectic 4-manifold lying strictly above the BMY line ([13, 14])). In [12, 8], new symplectic 4-manifolds with $\sigma \geq 0$ have been constructed. Later, in [19] we have improved these results further. Namely, we have constructed the smallest known irreducible, nonspin, symplectic and non-symplectic, pairwise non-diffeomorphic 4-manifolds with non-negative signatures [19]. In Section 2 we will discuss the techniques and building blocks of our constructions.

Before ending this section let us remind a substantial theorem. First we note that for a closed, oriented 4-manifold X we have $\sigma(X) + \chi(X) = b_2^+(X) - b_2^-(X) + (b_2^+(X) + b_2^-(X) - 2b_1(X) + 2) = 2(1 - b_1(X) + b_2^+(X))$. Next, we recall the Noether's formula.

Theorem 1.2.3. *(Noether's formula, [24] p.167, 168, [48]) For any (almost) complex surface S, $c_1^2(S) + c_2(S) = 3(\sigma(S) + \chi(S))$ is divisible by 12, or equivalently, $1 - b_1(X) + b_2^+(X)$ is even. In particular, if S is a simply connected, complex surface (hence Kähler and thus symplectic) then $b_2^+(S)$ is odd.*

(Here we would like to note that there are simply connected, symplectic 4-manifolds which are not Kähler [46, 48]). Moreover we know that any symplectic manifold admits an almost complex structure ([48]). From these results we have the following theorem:

Theorem 1.2.4. *([48]) Let X be a closed symplectic 4-manifold. Then, $1 - b_1(X) + b_2^+(X)$ is even. In particular, for a simply connected, symplectic 4-manifold X, $b_2^+(X)$ is odd.*

1.3 Mapping Class Groups and Lefschetz Pencils

Let us now review some definitions and results on the topology of smooth and symplectic 4-manifolds. We begin with the mapping class groups by following [16, 17].

Definition 1.3.1. *(Mapping Class Groups) Let $\Sigma_{g,n}$ be a 2-dimensional, compact, oriented, and connected surface of genus g with n boundary components. Let $Diff^+(\Sigma_{g,n})$ be the group of all orientation-preserving self-diffeomorphisms of $\Sigma_{g,n}$ which are the identity on the boundary and $Diff_0^+(\Sigma_{g,n})$ be the subgroup of $Diff^+(\Sigma_{g,n})$ consisting of all orientation-preserving self-diffeomorphisms that are isotopic to the identity. The isotopies are also assumed to fix the points on the boundary.* The mapping class group *$MCG(\Sigma_{g,n})$ of $\Sigma_{g,n}$ is defined to be the group of isotopy classes of orientation-preserving diffeomorphisms of $\Sigma_{g,n}$, i.e.,*

$$MCG(\Sigma_{g,n}) = Diff^+(\Sigma_{g,n})/Diff_0^+(\Sigma_{g,n}).$$

For simplicity, we write $\Sigma_g = \Sigma_{g,0}$. The hyperelliptic mapping class group H_g of Σ_g is defined as the subgroup of $MCG(\Sigma_g)$ consisting of all isotopy classes commuting with the isotopy class of the hyperelliptic involution $\iota : \Sigma_g \to \Sigma_g$.

Definition 1.3.2. *Let a be a simple closed curve on $\Sigma_{g,n}$. A* right handed *(or positive)* Dehn twist *about a is a diffeomorphism of $t_a : \Sigma_{g,n} \to \Sigma_{g,n}$ obtained by cutting the surface $\Sigma_{g,n}$ along a and gluing the ends back after rotating one of the ends 2π to the right.*

It is well-known that the mapping class group $MCG(\Sigma_{g,n})$ is generated by Dehn twists. Another fact is that the conjugate of a Dehn twist is again a Dehn twist, i.e., if $\phi : \Sigma_{g,n} \to \Sigma_{g,n}$ is an orientation-preserving diffeomorphism, then we have $\phi \circ t_a \circ \phi^{-1} = t_{\phi(a)}$.

Lemma 1.3.3. *([71]) Let a and b be two simple closed curves on $\Sigma_{g,n}$. If a and b are disjoint, then their corresponding Dehn twists satisfy the commutativity relation: $t_a t_b = t_b t_a$. If a and b transversely intersect at a single point, then their corresponding Dehn twists satisfy the braid relation: $t_a t_b t_a = t_b t_a t_b$.*

Next, we will review the notions called Lefschetz pencils and fibrations which characterize symplectic 4-manifolds:

Definition 1.3.4. *(Lefschetz pencil [47]) A* Lefschetz pencil *on a smooth, closed, oriented, 4-manifold X is (B, f) where B is a finite subset $B \subset X$ called the base locus, and f is a smooth map $f : X - B \to \mathbb{CP}^1$ such that*

(1) *each $b \in B$ is mapped to $0 \in \mathbb{C}^2$ by an orientation-preserving local coordinate map under which f corresponds to projectivization $\mathbb{C}^2 - \{0\} \to \mathbb{CP}^1$,*

(2) *each critical point of f has an orientation-preserving local coordinate chart in which $f(z, w) = z^2 + w^2$ (or $f(z, w) = zw$ after a linear change of coordinates) for some holomorphic local chart in $\mathbb{CP}^1$.*

By blowing up the base locus, we obtain a Lefschetz fibration $X \# n\overline{\mathbb{CP}}^2 \to \mathbb{CP}^1$, $n \in \mathbb{N}$, where each exceptional sphere is a section. (We will define the blow-up process in the next chapter.) More generally we have

Definition 1.3.5. *(Lefschetz fibration) Let X be a compact, connected, oriented, smooth 4-manifold. A* Lefschetz fibration *on X is a smooth map $f : X \to \Sigma$, where Σ is a compact, oriented, smooth 2-manifold, such that each critical point of f has an orientation-preserving chart on which $f : \mathbb{C}^2 \to \mathbb{C}$ is given by $f(w, z) = wz$.*

For a Lefschetz fibration as above, the generic fibers are compact 2-manifolds Σ_g and the singular fibers are immersed surfaces each of which can be assumed to have a single transverse self-intersection (by perturbing the critical points if necessary). Indeed, since $f(w, z) = wz$ in a neighborhood of a critical point, we see that the singular fiber (corresponding to $f^{-1}(0)$ locally) is an immersed surface with a single transverse self-intersection. Therefore, for intuition, a Lefschetz fibration should be pictured as a smooth fibration of X by surfaces Σ_g, with finitely many singular fibers, each of which has a single transverse self-intersection.

We refer to a Lefschetz fibration, according to its fiber genus, as a genus g Lefschetz fibration. If no singular fiber of a Lefschetz fibration $\mathcal{F}$ contains an embedded sphere of self-intersection -1, then $\mathcal{F}$ is called a relatively minimal Lefschetz fibration. Any sphere of this kind can be blown down in a way which preserves the fibration, thus this condition can always be arranged [44]. In our definition we allow manifolds with boundary, with the singular fibers necessarily in the interior of X.

A singular fiber Σ^0 in a Lefschetz fibration is obtained by taking a simple closed curve γ in a nearby regular fiber and gradually shrinking it to a point as we approach Σ^0. The curve γ is called the *vanishing cycle* for that fiber. Thus each critical point corresponds to a vanishing cycle in a nearby generic fiber.

Lefschetz fibrations can be described by the genus of the generic fiber, and the *monodromies* of the singular fibers which are elements of the mapping class group of the generic fiber. The monodromy is a right-handed Dehn twist about the vanishing cycle for that fiber and the *global monodromy* of the fibration $\mathcal{F}$ is the composition of such Dehn twists by considering all the singular fibers of $\mathcal{F}$.

The (monodromy or) vanishing cycle of a singular fiber Σ^0 completely determines the topology of a neighborhood of Σ^0, up to diffeomorphism. Indeed, the boundary of a neighborhood of a singular fiber in a Lefschetz fibration is a Σ_g-bundle over S^1 and the monodromy demonstrates how a fiber behaves as we traverse the boundary once along its S^1 factor. So it plays a crucial role in understanding the neighborhood of a singular fiber [44].

Moreover, a Lefschetz fibration over a sphere canonically determines a Lefschetz fibration over a disc by removing a neighborhood of a generic fiber. Conversely, a Lefschetz fibration over a disc D^2 extends to one over a sphere if and only if the monodromy around ∂D^2 is trivial. For more details we refer the reader to [48, 44].

Let us give an example of Lefschetz fibrations.

Example 1.3.6. *([44, 1]) Let a_1 and a_2 be the standard simple closed curves on the torus (see Figure 3.3 for the genus 2 case). Let us denote the Dehn twists about them by the same letters. The mapping class group Γ_1 is $SL(2,\mathbb{Z})$ and is generated by a_1 and a_2, where*

$$a_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad a_2 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

and they satisfy the relation $(a_1a_2)^6 = 1$. This relation defines an elliptic ($g = 1$)

fibration over the 2-disk D^2*, which can be extended to an elliptic fibration* $E(1) := \mathbb{CP}^2\#9\overline{\mathbb{CP}}^2 \to S^2$*. Then we form the n-fold fiber sum* $E(n) = \#_F(nE(1))$ *by using the identity homeomorphism on generic fibers (e.g.* $E(2)$ *is the* $K3$ *surface). This gives us an example of a genus one Lefschetz fibration whose global monodromy is* $(a_1a_2)^{6n}$*. It is known that* $E(n)$ *is a simply connected elliptic surface* $T^2 \to E(n) \to S^2$*, with a sphere section of self-intersection* $-n$*, and it has holomorphic Euler characteristic* $\chi_h = n$*.*

Moreover, it was proven by Moishezon that the global monodromy of any elliptic Lefschetz fibration is equivalent to the relation $(a_1a_2)^{6n} = 1$*. Thereby the family of* $E(n)$*'s is a complete classification of genus one Lefschetz fibrations with at least one singular fiber.*

Let us also note the following fact. In addition to the torus fibration, $E(n)$ *also admits a genus* $(n-1)$ *Lefschetz fibration over* S^2*.*

Finally, we have the following theorems which are due to Donaldson and Gompf:

Theorem 1.3.7. *[34] Any symplectic 4-manifold* X *admits a Lefschetz pencil.*

Theorem 1.3.8. *[48] Assume that a closed 4-manifold* X *admits a Lefschetz fibration* $X \to \Sigma$*, and let* $[F]$ *denote the homology class of the fiber. Then* X *admits a symplectic structure with symplectic fibers if and only if* $[F] \neq 0$ *in* $H_2(X;\mathbb{R})$*. If* $e_1, \cdots e_n$ *is a finite set of sections of the Lefschetz fibration, the symplectic form* w *can be chosen in such a way that all these sections are symplectic.*

We note that in the above theorem the condition $[F] \neq 0$ in $H_2(X;\mathbb{R})$ is essential. For example, we take the Hopf bundle $S^1 \to S^3 \to S^2$ and by multiplying it by S^1 we obtain a torus bundle $T^2 \to S^1 \times S^3 \to S^2$. But since $H_2(S^1 \times S^3;\mathbb{R}) = 0$, $S^1 \times S^3$ is not symplectic.

From these two theorems we have

Theorem 1.3.9. *([34, 48]) A 4-manifold* X *admits a symplectic structure if and only if it admits a Lefschetz pencil.*

Therefore, Lefschetz pencils/fibrations give a topological characterization of symplectic 4-manifolds.

1.4 Seiberg-Witten Invariants

In this section we will briefly review Seiberg-Witten invariants for smooth, closed, oriented, simply connected 4 manifolds with $b_2^+ > 1$ and odd. Let us recall $Spin^c$ structures first, as the Seiberg-Witten equations depend on the choice of a $Spin^c$ structure on a 4-manifold X. In dimension four, the $Spin^c$ group is $(U(1)Spin(4))/(\mathbb{Z}/2\mathbb{Z})$, and there is a homomorphism from it to the special orthogonal group $SO(4)$ of order four. On the other hand, we know that the tangent bundle TX of X has the natural $SO(4)$ structure (given by the Riemannian metric and orientation). A $Spin^c$ structure on X is a lift of this $SO(4)$ structure on TX to the $Spin^c$ group. In fact, every smooth compact 4-manifold X has $Spin^c$ structures.

Now let us recall Seiberg-Witten equations. Let X be a smooth, closed, oriented, simply connected 4 manifold with $b_2^+ > 1$ and odd. We choose a $Spin^c$ structure s on X, and let W^+, W^- be the associated spinor bundles, and L be the determinant line bundle. For ϕ is a self-dual spinor field (a section of W^+) and A is a $U(1)$ connection on L; the Seiberg-Witten equations for (ϕ, A) are

$$D^A\phi = 0 \text{ and } F_A^+ = \sigma(\phi) + i\omega$$

where D^A is the Dirac operator of A, F_A is the curvature 2-form of A, and F_A^+ is its self-dual part, and σ is the squaring map from W^+ to imaginary self-dual 2-forms and ω is a real self dual two form. The solutions (ϕ, A) to the Seiberg–Witten equations are called *monopoles*. The gauge group acts on the space of solutions, and the quotient by this action is called the *moduli space of solutions*. This moduli space is a closed, orientable manifold of dimension $(c_1(s)^2 - 2\chi(X) - 3\sigma(X))/4$ for a generic metric g and perturbation δ ([48]). We denote it by $\mathcal{M}_K^\delta(g)$.

Now for X a 4-manifold as above, let $\mathcal{C}_X := \{K \in H^2(X, \mathbb{Z}) \,|\, K \equiv w_2(X) \pmod 2\}$ be the set of characteristic elements. (Recall that for $K \in \mathcal{C}_X$ and $\alpha \in H_2(X, \mathbb{Z})$, $< K, \alpha > \equiv \alpha^2 = Q_X(\alpha, \alpha) \pmod 2$.) The *Seiberg-Witten invariant* SW_X of X is an integer valued function defined on $\mathcal{C}_X$. (Indeed, $SW_X(K) :=< \mu^m, [\mathcal{M}_K^\delta(g)] >$ where $dim(\mathcal{M}_K^\delta(g)) = 2m$; and $SW_X(K) := 0$ if $dim(\mathcal{M}_K^\delta(g)) < 0$. However, for the sake of brevity we skip the details and refer the reader to [48] and references therein.) Alternatively, SW_X is a map from the $Spin^c$ structures on X to $\mathbb{Z}$.

Theorem 1.4.1. *([48]) The Seiberg-Witten function $SW_X : \mathcal{C}_X \to \mathbb{Z}$ is a diffeomorphism invariant of the smooth 4-manifold X, it does not depend on the chosen metric or perturbation. For an orientation preserving diffeomorphism $f : X \to X'$, we have $SW_{X'}(K) = \pm SW_X(f^*K)$.*

Definition 1.4.2. *The cohomology class $K \in \mathcal{C}_X \subset H^2(X, \mathbb{Z})$ is called a "Seiberg-Witten basic class" of X, if $SW_X(K) \neq 0$. The simply connected 4-manifold is of "simple type" if each basic class satisfies $K^2 = c_1^2(X) = 3\sigma(X) + 2\chi(X)$.*

Moreover, if the 4-manifold is symplectic then it has simple type ([48]).

Now we will state the two main theorems that will be used in the sequel.

Theorem 1.4.3. *(Vanishing Theorems [48]) Let X be a smooth, closed, oriented, simply connected 4-manifold with $b_2^+(X) > 1$ and odd.*

1. *If $X = X_1 \# X_2$ and $b_2^+(X_i) > 0$ for $i = 1, 2$, then $SW_X \equiv 0$.*

2. *If X admits a metric with positive scalar curvature, then $SW_X \equiv 0$.*

3. *If X has an embedded sphere S^2 with $[S^2]^2 \geq 0$ and $[S^2] \neq 0$ in $H_2(X, \mathbb{Z})$, then $SW_X \equiv 0$.*

Theorem 1.4.4. *(Nonvanishing Theorems [48])*

1. *If S is a simply connected, complex surface (hence Kähler and thus symplectic, in addition, $b_2^+(S)$ is odd by the Theorem 1.2.4 above), and $b_2^+(S) > 1$, then $SW_S(\pm c_1(S)) \neq 0$.*

2. *([120]) More generally, if (X, w) is a simply connected, symplectic 4-manifold and $b_2^+(X) > 1$, then $SW_X(\pm c_1(X, w)) = \pm 1$.*

The following theorems describe the effects of the blow-up and connected sum on the Seiberg-Witten basic classes.

Theorem 1.4.5. *(The blow-up formula [48]) Let X be a simply connected 4-manifold of simple type with the set of basic classes $Bas_X = \{K_i \,|\, i = 1, \cdots, s\}$. If $X' = X \# \overline{\mathbb{CP}}^2$ is the blow-up of X and $E \in H^2(X', \mathbb{Z})$ denotes the Poincaré dual of the homology class $e \in H_2(X', \mathbb{Z})$ of the exceptional sphere, then the set of basic classes of X' is $Bas_{X'} = \{K_i \pm E \,|\, i = 1, \cdots, s\}$.*

Theorem 1.4.6. *([48]) Assume that the simply connected 4-manifold X' decomposes as $X' = X\#N$, where X is of simple type. If $b_2^+(N) = 0$, thus $H^2(N,\mathbb{Z})$ has an orthogonal basis $\{E_i \in H^2(N,\mathbb{Z}) \,|\, i = 1,\cdots,b_2(N)\}$ with $E_i^2 = -1$, then $Bas_{X'} = \{K_i \pm E_1 \pm \cdots \pm E_{b_2(N)} \,|\, K_i \in Bas_X\}$.*

Theorem 1.4.7. *(Generalized Adjunction Formula [80, 99]) Assume that $\Sigma \subset X$ is an embedded, oriented, connected surface of genus $g(\Sigma)$ with self-intersection $[\Sigma]^2 \geq 0$ and $[\Sigma] \neq 0$. Then, for every SW basic class K of X, $2g(\Sigma) - 2 \geq [\Sigma]^2 + |K \cdot [\Sigma]|$.*

If X is of simple type and $g(\Sigma) > 0$, then the same inequality holds for $\Sigma \subset X$ with arbitrary square $[\Sigma]^2$.

In addition the generalized adjunction formula was proven for the $b_2^+ = 1$ case:

Theorem 1.4.8. *(Generalized adjunction formula, $b_2^+ = 1$ case [83]) Suppose M is a symplectic four-manifold with $b_2^+ = 1$ and w is a symplectic form. Let C be a smooth, connected, embedded surface with nonnegative self-intersection. If $[C] \cdot w > 0$, then $2g(C) - 2 \geq [C]^2 + K \cdot [C]$.*

1.5 Some Symplectic Surgeries

To construct symplectic manifolds and also complex 3-folds, there are many fascinating methods. By taking branched coverings along the hyperplane arrangements in projective spaces, by applying finite group actions or via symplectic surgeries, one can attain new interesting manifolds. In this part, let us review some symplectic operations which are called symplectic connected sum, generalized rational blow-down, Luttinger surgery and knot surgery.

1.5.1 Symplectic Connected Sum and Symplectic Minimality

Definition 1.5.1. *(Symplectic Connected Sum [46]) Let $(X_1,\ \omega_1)$ and $(X_2,\ \omega_2)$ be closed symplectic 4-dimensional manifolds containing closed embedded surfaces F_1 and F_2 of genus g, with normal bundles ν_1 and ν_2, respectively. Assume that the Euler class of ν_i satisfy $e(\nu_1) + e(\nu_2) = 0$. Then for any choice of an orientation reversing bundle* *isomorphism* $\psi : \nu_1 \cong \nu_2$, *the* symplectic connected sum *of* X_1 *and* X_2 *along* F_1 *and* F_2 *is the smooth manifold* $X_1 \#_\psi X_2 = (X_1 - \nu_1) \cup_\psi (X_2 - \nu_2)$.

Note that the diffeomorphism type of $X_1 \#_\psi X_2$ depends on the choice of the embeddings and isomorphism ψ.

Theorem 1.5.2. *([52, 89, 46]) The* 4*-manifold* $X_1 \#_\psi X_2$ *admits a canonical symplectic structure* ω *induced by* ω_1 *and* ω_2.

The Euler characteristic and the signature of the symplectic connected sum $X_1 \#_\psi X_2$ are given by the following formulas:

$$\begin{aligned} &e(X_1 \#_\psi X_2) = e(X_1) + e(X_2) - 2e(\Sigma_g) = e(X_1) + e(X_2) + 4(g-1), \\ &\sigma(X_1 \#_\psi X_2) = \sigma(X_1) + \sigma(X_2) \end{aligned} \tag{1.4}$$

where the signature formula follows from the Mayer-Vietoris sequence. These equalities, in turn, imply the followings:

$$\begin{aligned} &\chi(X_1 \#_\psi X_2) = \chi(X_1) + \chi(X_2) + (g-1), \\ &c_1^2(X_1 \#_\psi X_2) = c_1^2(X_1) + c_1^2(X_2) + 8(g-1). \end{aligned} \tag{1.5}$$

Next, we state a proposition which will be useful in the fundamental group computations of our examples obtained via the symplectic connected sum operation. The proof of this proposition can be found in [46] and [53].

Proposition 1.5.3. *Let* X *be closed, smooth* 4*-manifold, and* Σ *be closed submanifold of dimension* 2*. Suppose that there exist a sphere* S *in* X *that intersects* Σ *transversally in exactly one point, then the homomorphism* $j_* : \pi_1(X \setminus \Sigma) \to \pi_1(X)$ *induced by inclusion is an isomorphism. In particular, if* X *is simply connected, then so is* $X \setminus \Sigma$.

Before passing to the next operation, let us discuss minimality of symplectic 4-manifolds. We call a symplectic 4-manifold X *minimal* if there is no 2 dimensional **symplectic** sphere Σ satisfying $[\Sigma]^2 = -1$, where $[\Sigma] \in H_2(X, \mathbb{Z})$ is the homology class. This is equivalent to the condition that X does not contain any smoothly embedded spheres of square -1 ([121, 82]). Moreover we have

Theorem 1.5.4. *(Minimality of Sympletic Sums, [124]) Let $Z = X_1 \#_{F_1=F_2} X_2$ be sympletic connected sum of manifolds X_1 and X_2 and assume that the F_i have positive genus g. Then:*

(i) If either $X_1 - F_1$ or $X_2 - F_2$ contains an embedded sympletic sphere of square -1, then Z is not minimal.

(ii) If one of the summands X_i (say X_1) admits the structure of an S^2-bundle over a surface of genus g such that F_i is a section of this fiber bundle, then Z is minimal if and only if X_2 is minimal.

(iii) In all other cases, Z is minimal.

1.5.2 Rational blow-downs and Its Generalizations

Generalized rational blow-down is the next surgery we would like to present here. Rational blow-down surgery was introduced in [39]. The basic idea of the surgery is that if a smooth 4-manifold X contains a particular configuration C_p of transversally intersecting 2-spheres whose boundary is the lens space $L(p^2, 1-p)$ ([27]), then one can replace C_p with rational homology ball B_p to construct a new manifold X_p. If one knows the Seiberg-Witten invariants of the original manifold X, then one can determine the Seiberg-Witten invariants of X_p. The rational blow-down surgery technique was generalized in [101]. Since we will be also using the generalized rational blow-down in our construction, let us review the generalized rational blow-down below.

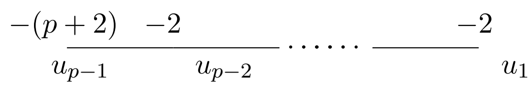
Definition 1.5.5. *(Generalized Rational blow-down [101]) Let $p \geq q \geq 1$ and p, q are relatively prime integers. Let $C_{p,q}$ denote the smooth 4-manifold obtained by plumbing disk bundles over the 2-spheres according to the following linear diagram*

$$\underset{u_k}{\overset{-r_k}{\bullet}} \text{———} \underset{u_{k-1}}{\overset{-r_{k-1}}{\bullet}} \cdots\cdots \text{———} \underset{u_1}{\overset{-r_1}{\bullet}}$$

where $p^2/(pq-1) = [r_k, r_{k-1}, ..., r_1]$ is the unique continued linear fraction with all $r_i \geq 2$ and each vertex u_i of the linear diagram represents a disk bundle over 2-sphere with Euler number $-r_i$. The boundary of $C_{p,q}$ is the lens space $L(p^2, 1-pq)$ ([27]), which *also bounds a rational ball $B_{p,q}$ with $\pi_1(B_{p,q}) = \mathbb{Z}_p$ and $\pi_1(\partial B_{p,q}) \to \pi_1(B_{p,q})$ surjective. If $C_{p,q}$ is embedded in a 4-manifold X then the generalized rational blow-down manifold $X_{p,q}$ is defined as $X_{p,q} := (X - C_{p,q}) \cup B_{p,q}$.*

Moreover, this operation can be performed also symplectically ([118]).

The case when $q = 1$ is the construction of Fintushel-Stern with $C_p = C_{p,1}$ given by



In this case the operation is called the rational blow-down. Furthermore, when $q = 1$ and $p = 2$, the configuration consists of only one -4 sphere whose boundary is $L(4, 1)$ and the corresponding surgery is the usual blow-down.

Lemma 1.5.6. *Let $X_{p,q}$ be the smooth 4-manifold obtained from X by a rational blow-down of the configuration $C_{p,q}$. Then ${b_2}^+(X_{p,q}) = {b_2}^+(X)$, ${b_2}^-(X_{p,q}) = {b_2}^-(X) - k$, $e(X_{p,q}) = e(X) - k$, and ${c_1}^2(X_{p,q}) = {c_1}^2(X) + k$.*

Proof. Since $C_{p,q}$ is negative definite plumbing of length k, we have ${b_2}^+(X_{p,q}) = {b_2}^+(X)$, ${b_2}^-(X_{p,q}) = {b_2}^-(X) - k$, and consequently $e(X_{p,q}) = e(X) - k$. Using the formula ${c_1}^2 := 3\sigma + 2e$, we compute ${c_1}^2(X_{p,q}) = 3\sigma(X_{p,q}) + 2e(X_{p,q}) = 3(\sigma(X)+k)+2(e(X)-k) = {c_1}^2(X) + k$. □

The following theorem gives a way to compute the Seiberg-Witten invariants of $X_{p,q}$ using the Seiberg-Witten invariants of X.

Theorem 1.5.7. *[101]. Suppose X is a smooth 4-manifold with $b_2^+(X) > 1$ which contains a configuration $C_{p,q}$. If L is a characteristic line bundle on X such that, $SW_X(L) \neq 0$, $(L|_{C_{p,q}})^2 = -b_2(C_{p,q})$ and $c_1(L|_{L(p^2,1-pq)}) = mp \in \mathbb{Z}_{p^2} \cong H^2(L(p^2, 1 - pq); \mathbb{Z})$ with $m \equiv (p-1) \mod 2$, then L induces a SW basic class $\bar{L}$ of $X_{p,q}$ such that $SW_{X_{p,q}}(\bar{L}) = SW_X(L)$.*

Corollary 1.5.8. *[101]. Suppose X is a smooth 4-manifold with $b_2^+(X) > 1$ which contains a configuration $C_{p,q}$. If L is a SW basic class of X satisfying $L \cdot u_i = (r_i - 2)$*

for any i with $1 \le i \le k$ (or $L \cdot u_i = -(r_i - 2)$, then L induces a SW basic class $\bar{L}$ of $X_{p,q}$ such that $SW_{X_{p,q}}(\bar{L}) = SW_X(L)$.

1.5.3 Knot Surgery and Luttinger Surgery

Lastly, let us present knot and Luttinger surgeries.

Definition 1.5.9. *(Knot Surgery [40, 37]) Let X be a smooth 4-manifold which contains a smooth homologically essential torus T of self-intersection 0, and let K be a knot in S^3. Let $N(K)$ be a tubular neighborhood of K in S^3, and let $T \times D^2$ be a tubular neighborhood of T in X. Then the knot surgery manifold X_K is defined by*

$$X_K = (X - (T \times D^2)) \cup (S^1 \times (S^3 - N(K))) \tag{1.6}$$

The two pieces are glued together in such a way that the homology class $[pt \times \partial D^2]$ is identified with $[pt \times \lambda]$ where λ is the class of a longitude of K. If the complement of T in X is simply connected, then X_K is homeomorphic to X.

This operation can be done symplectically under certain conditions. Before discussing it, we will recap some basic information on knot theory and then give an equivalent description of this surgery.

Let us take a knot $K \in S^3$. We say that K is *fibered* if the complement $S^3 - K$ admits a fibration

$$(\Sigma_g)^0 \to S^3 - K \to S^1$$

where $(\Sigma_g)^0$ denotes a punctured genus g surface. Roughly speaking, the *Alexander polynomial* is a knot invariant which assigns a polynomial with integer coefficients to each knot type. We denote the Alexander polynomial of K as $\Delta_K(t)$. It is known that K is fibered if and only if $\Delta_K(t)$ is monic ([40]).

Now, let m denote a meridional circle to K, and M_K be the 3-manifold obtained by performing 0-framed surgery on K. That is to say, we perform p/q surgery,

$$\begin{aligned} M_K = (S^3 - (K \times D^2)) \quad &\cup \quad (S^1 \times D^2) \\ p[m] + q[l] \quad &\hookleftarrow \quad [\partial D^2] \end{aligned}$$

with $p = 0$, where, we denote the homology classes of the meridian and the longitude of K by $[m]$ and $[l]$, respectively. Since $p = 0$, m can be viewed as a circle in M_K.

Remark 1.5.10. *We would like to remark that there are different notations in the literature. In the above equality, note that $\partial(S^3 - (K \times D^2)) = T^2$ and m and l are* **longitude** *and* **meridian** *of T^2 respectively. Thus, sometimes in the above definition of p/q surgery, $q[m] + p[l]$ is used with the roles of m and l switched.*

Next, we take $T_m := m \times S^1 \subset M_K \times S^1$ which is a smooth torus of self-intersection 0. A neighborhood of T_m can be canonically identified with $T_m \times D^2$.

Then, the knot surgered manifold X_K above is defined as the normal connected sum

$$\begin{aligned} X_K &= X\#_{T=T_m}(M_K \times S^1) \\ &= (X - (T \times D^2)) \cup ((M_K \times S^1) - (T_m \times D^2)) \end{aligned} \quad (1.7)$$

where $T \times D^2$ is a tubular neighborhood of the homologically essential, square zero torus T in X with $\pi_1(X) = \pi_1(X - T^2) = 1$.

We note that $((M_K \times S^1) - (T_m \times D^2))$ is diffeomorphic to $(S^1 \times (S^3 - N(K)))$. Hence two descriptions 1.6 and 1.7 are equivalent.

Next, we assume that K is a fibered knot in S^3 (i.e., $\Delta_K(t)$ is monic). By definition we have

$$(\Sigma_g)^0 \to (S^3 - K) \to S^1.$$

Therefore we have the fibration

$$\Sigma_g \to M_K \to S^1.$$

Then by crossing with S^1 we obtain

$$\Sigma_g \to (M_K \times S^1) \to T^2,$$

with $T_m = m \times S^1$ as section. (The fact that $T_m^2 = 0$ can be seen as follows. We take a parallel copy m' of m. Then $T_{m'} = m' \times S^1$ is the push-off of T_m. Since $m \cdot m' = 0$, we have $T_m \cdot T_{m'} = 0$. Therefore a tubular neighborhood of T_m can be identified with $T_m \times D^2$.) Now by the following theorem, both $(M_K \times S^1)$ and the section T_m are symplectic:

Theorem 1.5.11. *([122]) Let $\Sigma_g \to X \to \Sigma_h$ be a bundle where Σ_g and Σ_h are closed, oriented, 2-dimensional surfaces. If the homology class $[\Sigma_g]$ of the fiber is nonzero in $H_2(X, \mathbb{R})$, then X has a symplectic structure with symplectic section.*

Indeed, since there is a section T_m in the fibration $\Sigma_g \to (M_K \times S^1) \to T^2$, we have $[\Sigma_g] \cdot [T_m] = 1$ implying that the homology class $[\Sigma_g]$ of the fiber is nonzero in $H_2(M_K \times S^1, \mathbb{R})$. Hence the theorem applies.

In addition to $(M_K \times S^1)$ and T_m being symplectic, if X is also a symplectic 4-manifold and the torus T is symplectically embedded in X with self-intersection 0, then the knot surgered manifold X_K is symplectic since it is given as a symplectic connected sum $X_K = X \#_{T=T_m} (M_K \times S^1)$ (see Theorem 1.5.2 above).

Finally, from the SW-computations ([40]) we know that the SW-invariants of X_K depends on SW_X and $\Delta_K(t)$. Therefore, using knots with different (nontrivial) Alexander polynomials produces infinitely many symplectic and nonsymplectic, pairwise nondiffeomorphic manifolds, each of which are homeomorphic to the manifold X. In fact, to obtain nonsymplectic manifolds we take the torus T in X lying in a cusp neighborhood and a non-fibered knot K, i.e., $\Delta_K(t)$ is not monic. Then X_K does not admit a symplectic structure ([120, 40]).

Now let us recall Luttinger surgery. First we retrieve

Definition 1.5.12. *A submanifold Y of a symplectic manifold (X, w) is said to be* Lagrangian*, if at each $p \in Y$, the restriction of w_p to the subspace T_pY is trivial and dim $Y = 1/2$ dim X.*

Let (X, ω) be a symplectic 4-manifold, and Λ be a Lagrangian torus embedded in (X, ω). Then, since Λ is Lagrangian, from the adjunction formula the self-intersection number of Λ is 0, thus it has a trivial normal bundle. By Weinstein's Lagrangian neighborhood theorem, a tubular neighborhood $\nu\Lambda$ of Λ in X can be identified symplectically with a neighborhood of the zero-section in the cotangent bundle $T^*\Lambda \simeq \Lambda \times \mathbb{R}^2$ with its standard symplectic structure. Let γ be any simple closed curve on Λ. The Lagrangian framing described above determines, up to homotopy, a push-off of γ in $\partial(\nu\Lambda)$. Let γ' is a simple loop on $\partial(\nu\Lambda)$ that is parallel to γ under the Lagrangian framing.

Definition 1.5.13. *(Luttinger surgery) For any integer m, the $(\Lambda, \gamma, 1/m)$* Luttinger surgery *on X is defined as $X_{\Lambda,\gamma}(1/m) = (X \setminus \nu(\Lambda)) \cup_\phi (S^1 \times S^1 \times \mathbb{D}^2)$, where, for a meridian μ_Λ of Λ, the gluing map $\phi : S^1 \times S^1 \times \partial\mathbb{D}^2 \to \partial(X \setminus \nu(\Lambda))$ satisfies $\phi([\partial\mathbb{D}^2]) = m[\gamma'] + [\mu_\Lambda]$ in $H_1(\partial(X \setminus \nu(\Lambda))$.*

It is shown in [23] that $X_{\Lambda,\gamma}(1/m)$ possesses a symplectic form which agrees with the original symplectic form ω on $X \setminus \nu\Lambda$. The following lemma is not hard to verify, the proof will be omitted.

Lemma 1.5.14. *We have $\pi_1(X_{\Lambda,\gamma}(1/m)) = \pi_1(X - \nu\Lambda)/N(\mu_\Lambda \gamma'^m)$, where $N(\mu_\Lambda \gamma'^m)$ denotes the normal subgroup of $\pi_1(X - \nu\Lambda)$ generated by $\mu_\Lambda \gamma'^m$. Moreover, we have $\sigma(X) = \sigma(X_{\Lambda,\gamma}(1/m))$, and $e(X) = e(X_{\Lambda,\gamma}(1/m))$, where σ and χ denote the signature and the Euler characteristic, respectively.*

In addition the symplectic Kodaira dimension is also preserved by the Luttinger surgery [62].

Luttinger surgeries on product manifolds $\Sigma_n \times \Sigma_2$ and $\Sigma_n \times T^2$

In the following, we recall the construction of symplectic 4-manifolds in [11], obtained from $\Sigma_n \times \Sigma_2$ and $\Sigma_n \times T^2$ by performing a sequence of Luttinger surgeries along the Lagrangian tori. We use the same notations as in [11]. The following two families of symplectic 4-manifolds will be used as the building blocks in our constructions in the next chapter.

The first family of examples have the same cohomology ring as $(2n-3)(S^2 \times S^2)$, and are constructed as follows. We fix integer $n \geq 2$, and denote by Y_n the symplectic 4-manifold obtained by performing $2n+4$ Luttinger surgeries on $\Sigma_n \times \Sigma_2$, which consist of the following 8 surgeries

$$\begin{aligned}
&(a_1' \times c_1', a_1', -1), \quad (b_1' \times c_1'', b_1', -1),\\
&(a_2' \times c_2', a_2', -1), \quad (b_2' \times c_2'', b_2', -1),\\
&(a_2' \times c_1', c_1', +1), \quad (a_2'' \times d_1', d_1', +1),\\
&(a_1' \times c_2', c_2', +1), \quad (a_1'' \times d_2', d_2', +1),
\end{aligned}$$

followed by the set of additional $2(n-2)$ Luttinger surgeries

$$\begin{array}{cc} (b_1' \times c_3', c_3', -1), & (b_2' \times d_3', d_3', -1), \\ \dots, & \dots, \\ (b_1' \times c_n', c_n', -1), & (b_2' \times d_n', d_n', -1). \end{array}$$

In the notation above, a_i, b_i $(i = 1, 2)$ and c_j, d_j $(j = 1, \dots, n)$ denote the standard loops that generate $\pi_1(\Sigma_2)$ and $\pi_1(\Sigma_n)$, respectively. To see typical Lagrangian tori along which the Luttinger surgeries are performed, we refer the reader to [19], Figure 3 or [11].

Here we note that the tori $a \times c$, $b \times c$, $a \times d$ and $b \times d$ with the appropriate indices are Lagrangian in the product manifold $\Sigma_2 \times \Sigma_n$. This can be seen as follows. To simplify the notation, let us consider $T^2 \times T^2$ case. Let $(T_\alpha := a \times b, w_1)$ and $(T_\beta := c \times d, w_2)$ be the two symplectic tori with the symplectic forms $w_1,\ w_2$. Then $T_\alpha \times T_\beta$ is also symplectic with the product symplectic form; $w = \pi_1^*(w_1) + \pi_2^*(w_2)$, where π_i is projection onto the i-th component. Let us show that the torus $T_\gamma := a \times c$ is Lagrangian. We take a point p in the torus T_γ and let X_a, Y_c be the vectors on the tangent space T_pT_γ in the direction of a, c respectively. Then

$$\begin{aligned} w_p(X_a, Y_c) &= (\pi_1^*(w_1) + \pi_2^*(w_2))_p(X_a, Y_c) \\ &= \pi_1^*(w_1)_p(X_a, Y_c) + \pi_2^*(w_2)_p(X_a, Y_c) \\ &= (w_1)_{\pi_1(p)}((\pi_1)_* X_a, (\pi_1)_* Y_c) + (w_2)_{\pi_2(p)}((\pi_2)_* X_a, (\pi_2)_* Y_c) \\ &= 0. \end{aligned}$$

The third equality above follows from the definition of the pull-back map. This shows that $T_\gamma := a \times c$ is Lagrangian. Similarly, the tori $b \times c$, $a \times d$ and $b \times d$ are Lagrangian, too.

Now let us discuss some properties of Y_n; the symplectic 4-manifold obtained by performing $2n+4$ Luttinger surgeries above. By Lemma 1.5.14, we see that the Euler characteristic of Y_n is $4n-4$ and the signature is 0. Furthermore, the Lemma 1.5.14 implies that the fundamental group $\pi_1(Y_n)$ is generated by loops a_i, b_i, c_j, d_j $(i = 1, 2$

and $j = 1, \ldots, n$) and the following relations hold in $\pi_1(Y_n)$:

$$
\begin{gathered}
[b_1^{-1}, d_1^{-1}] = a_1, \quad [a_1^{-1}, d_1] = b_1, \quad [b_2^{-1}, d_2^{-1}] = a_2, \quad [a_2^{-1}, d_2] = b_2, \\
[d_1^{-1}, b_2^{-1}] = c_1, \quad [c_1^{-1}, b_2] = d_1, \quad [d_2^{-1}, b_1^{-1}] = c_2, \quad [c_2^{-1}, b_1] = d_2, \\
[a_1, c_1] = 1, \quad [a_1, c_2] = 1, \quad [a_1, d_2] = 1, \quad [b_1, c_1] = 1, \\
[a_2, c_1] = 1, \quad [a_2, c_2] = 1, \quad [a_2, d_1] = 1, \quad [b_2, c_2] = 1, \\
[a_1, b_1][a_2, b_2] = 1, \quad \prod_{j=1}^{n} [c_j, d_j] = 1, \\
[a_1^{-1}, d_3^{-1}] = c_3, \quad [a_2^{-1}, c_3^{-1}] = d_3, \ \ldots, \ [a_1^{-1}, d_n^{-1}] = c_n, \quad [a_2^{-1}, c_n^{-1}] = d_n, \\
[b_1, c_3] = 1, \quad [b_2, d_3] = 1, \ \ldots, \ [b_1, c_n] = 1, \quad [b_2, d_n] = 1.
\end{gathered} \tag{1.8}
$$

Note that the surfaces $\Sigma_2 \times \{\mathrm{pt}\}$ and $\{\mathrm{pt}\} \times \Sigma_n$ in $\Sigma_2 \times \Sigma_n$ are not affected by the above Luttinger surgeries, thus they descend to surfaces in Y_n. We will denote these symplectic submanifolds by Σ_2 and Σ_n. Notice that we have $[\Sigma_2]^2 = [\Sigma_n]^2 = 0$ and $[\Sigma_2] \cdot [\Sigma_n] = 1$. Moreover, when $n \geq 3$, the symplectic 4-manifold Y_n contains $2n - 4$ pairs of geometrically dual Lagrangian tori. These Lagrangian tori together with Σ_2 and Σ_n generates the second homology group $H_2(Y_n) \cong \mathbb{Z}^{4n-6}$.

Now we will consider a different family. Let us fix integers $n \geq 2$, $m \geq 1$, $p \geq 1$ and $q \geq 1$. Let $Y_n(1/p, m/q)$ denote smooth 4-manifold obtained by performing the following $2n$ torus surgeries on $\Sigma_n \times T^2$:

$$
\begin{aligned}
&(a_1' \times c', a_1', -1), \quad (b_1' \times c'', b_1', -1), \\
&(a_2' \times c', a_2', -1), \quad (b_2' \times c'', b_2', -1), \\
&\cdots, \quad \cdots \\
&(a_{n-1}' \times c', a_{n-1}', -1), \quad (b_{n-1}' \times c'', b_{n-1}', -1), \\
&(a_n' \times c', c', +1/p), \quad (a_n'' \times d', d', +m/q).
\end{aligned} \tag{1.9}
$$

Let a_i, b_i $(i = 1, 2, \cdots, n)$ and c, d denote the standard generators of $\pi_1(\Sigma_n)$ and $\pi_1(T^2)$, respectively. Note that all the torus surgeries listed above are Luttinger surgeries when $m = 1$. In addition, the Luttinger surgery preserves minimality ([62]). Therefore, $Y_n(1/p, 1/q)$ is a minimal symplectic 4-manifold. The fundamental group of

$Y_n(1/p, m/q)$ is generated by a_i, b_i $(i = 1, 2, 3 \cdots, n)$ and c, d, and the Lemma 1.5.14 implies that the following relations hold in $\pi_1(Y_n(1/p, m/q))$:

$$
\begin{gathered}
[b_1^{-1}, d^{-1}] = a_1, \quad [a_1^{-1}, d] = b_1, \quad [b_2^{-1}, d^{-1}] = a_2, \quad [a_2^{-1}, d] = b_2, \\
\cdots, \quad \cdots, \\
[b_{n-1}^{-1}, d^{-1}] = a_{n-1}, \quad [a_{n-1}^{-1}, d] = b_{n-1}, \quad [d^{-1}, b_n^{-1}] = c^p, \quad [c^{-1}, b_n]^{-m} = d^q, \\
[a_1, c] = 1, \quad [b_1, c] = 1, \quad [a_2, c] = 1, \quad [b_2, c] = 1, \\
[a_3, c] = 1, \quad [b_3, c] = 1, \\
\cdots, \quad \cdots, \\
[a_{n-1}, c] = 1, \quad [b_{n-1}, c] = 1, \\
[a_n, c] = 1, \quad [a_n, d] = 1, \\
[a_1, b_1][a_2, b_2] \cdots [a_n, b_n] = 1, \quad [c, d] = 1.
\end{gathered} \tag{1.10}
$$

On this thesis we will only consider the case $p = q = 1$. Let us denote by $\Sigma_n', \Sigma_1' \subset Y_n(1, l)$ a genus n surface and a torus that descend from the surfaces $\Sigma_n \times \{\text{pt}\}$ and $\{\text{pt}\} \times T^2$ in $\Sigma_n \times T^2$. The surfaces Σ_1' and Σ_n' generates the second homology group $H_2(Y_n(1, l)) \cong \mathbb{Z}^2$.

These two families Y_n and $Y_n(1, l)$ will be used as building blocks in the following chapter.

Before we end this chapter let us state two theorems and their corollary ([12] and [8], see also [7], Theorem 23; [11], Theorem 2). Their proofs involve symplectic surgeries, e.g. Luttinger surgeries and symplectic connected sum, in addition to other techniques. They are useful to obtain infinitely many homeomorphic but pairwise nondiffeomorphic 4-manifolds.

Theorem 1.5.15. *([12, 8]) Let X be a closed symplectic 4-manifold that contains a symplectic torus T of self-intersection 0. Let νT be a tubular neighborhood of T and $\partial(\nu T)$ its boundary. Suppose that the homomorphism $\pi_1(\partial(\nu T)) \to \pi_1(X \setminus \nu T)$ induced by the inclusion is trivial. Then for any pair of integers (χ, c) satisfying*

$$\chi \geq 1 \textit{ and } 0 \leq c \leq 8\chi \tag{1.11}$$

there exist a symplectic 4-manifold Y *with* $\pi_1(Y) = \pi_1(X)$,

$$\chi_h(Y) = \chi_h(X) + \chi \text{ and } {c_1}^2(Y) = {c_1}^2(X) + c \tag{1.12}$$

Moreover, if X *is minimal then* Y *is minimal as well. If* $c < 8\chi$*, or* $c = 8\chi$ *and* X *has an odd intersection form, then the corresponding* Y *has an odd indefinite intersection form.*

Moreover in the simply connected case we have

Theorem 1.5.16. *([12, 8]) Let* Y *be a closed simply connected minimal symplectic 4-manifold with* $b_2^+(Y) > 1$*. Assume that* Y *contains a symplectic torus* T *of self-intersection* 0 *such that* $\pi_1(Y \setminus T) = 1$*. Then there exist an infinite family of pairwise nondiffemorphic irreducible symplectic* 4*-manifolds and an infinite family of pairwise nondiffemorphic irreducible nonsymplectic 4-manifolds, all of which are homemorphic to* Y*.*

The following corollary follows from the above theorems.

Corollary 1.5.17. *([8]) Let* X *be a closed simply connected nonspin minimal symplectic* 4*-manifold with* $b_2^+(X) > 1$ *and* $\sigma(X) \geq 0$*. Assume that* X *contains disjoint symplectic tori* T_1 *and* T_2 *of self-intersections* 0 *such that* $\pi_1(X \setminus (T_1 \cup T_2)) = 1$*. Suppose that* σ *is a fixed integer satisfying* $0 \leq \sigma \leq \sigma(X)$*. If* $\lceil x \rceil = min\{k \in \mathbb{Z} | k \geq x\}$ *and we define*

$$l(\sigma) = \left\lceil \frac{\sigma(X) - \sigma}{8} - 1 \right\rceil \tag{1.13}$$

Next, if k *is any odd integer satisfying* $k \geq {b_2}^+(X) + 2l(\sigma) + 2$*, then there exist an infinite family of pairwise nondiffemorphic irreducible symplectic* 4*-manifolds and an infinite family of pairwise nondiffemorphic irreducible nonsymplectic 4-manifolds, all of which are homemorphic to* $k\mathbb{CP}^2 \# (k - \sigma)\overline{\mathbb{CP}}^2$*.*

In the next chapter we use these theorems to obtain infinitely many homeomorphic but pairwise nondiffeomorphic, simply connected 4-manifolds.

The outline of the remaining chapters is as follows. In Chapter 2 we construct symplectic and smooth 4-manifolds with nonnegative signatures, with more than one smooth structures and with small Euler characteristics. For this purpose in Section 2.2 we discuss abelian Galois coverings and complex surfaces of Hirzebruch and Bauer-Catanese on Bogomolov-Miyaoka-Yau line (with $c_1^2 = 45$ and $\chi_h = 5$), obtained as an abelian covering of $\mathbb{CP}^2$ branched along a complete quadrangle [57, 24, 25]. From these we obtain our first building block which is presented in Section 2.3. Next, we provide our second building blocks; exotic symplectic 4-manifolds constructed in [6, 10, 7, 11, 18], obtained via combinations of symplectic connected sum and Luttinger surgery operations in Section 2.4. Then, by using these building blocks, in Sections 2.5 and 2.6 we construct new non-spin irreducible symplectic and smooth 4-manifolds with nonnegative signatures that are interesting with respect to the symplectic and smooth geography problems. More specifically, we prove our main theorems, Theorem 2.1.1 in Section 2.5 and Theorem 2.1.2 in Section 2.6.

In Chapter 3 our foci are fibrations of genus two complex curves and constructions of exotic 4-manifolds with small Euler characteristics. In Section 3.1 we overview Hirzebruch surfaces [58]. Section 3.2 presents the classification of the singular fibers of genus two fibrations due to Namikawa and Ueno [93, 94], and then genus two pencils in the $K3$ surface. In Section 3.3 we introduce a deformation technique of the singular fibers of certain types Lefschetz fibrations over the 2-sphere S^2. Finally, in Section 3.4 we prove our main theorems; Theorem 3.4.2, Theorem 3.4.5, Theorem 3.4.6, Theorem 3.4.7, Theorem 3.4.9, and Theorem 3.4.10.

More detailed outlines of Chapter 2 and Chapter 3 could also be found at the end of their introductions, i.e., before Section 2.1 and 3.1, respectively.

Chapter 2

New Exotic Symplectic 4-Manifolds with Nonnegative Signatures via Abelian Galois Ramified Coverings and Symplectic Surgeries

In [12, 8], irreducible symplectic 4-manifolds which are exotic copies of $(2n-1)\mathbb{CP}^2\#(2n-1)\overline{\mathbb{CP}}^2$ for each integer $n \geq 25$, and families of simply connected irreducible nonspin symplectic 4-manifolds with positive signatures were constructed. These constructions are interesting in terms of the symplectic geography problem (see the previous chapter for the description of this problem). In [19], we improved the main results in [12, 8]. In particular, we constructed

(i) infinitely many irreducible symplectic and non-symplectic 4-manifolds that are homeomorphic but not diffeomorphic to $(2n-1)\mathbb{CP}^2\#(2n-1)\overline{\mathbb{CP}}^2$ for each integer $n \geq 12$,

and

(ii) families of simply connected irreducible nonspin symplectic 4-manifolds that have the smallest Euler characteristics among the all known simply connected 4-manifolds with positive signature and with more than one smooth structure.

We have two building blocks for our constructions. These are the complex surfaces of Hirzebruch and Bauer-Catanese on Bogomolov-Miyaoka-Yau line with $c_1^2 = 9\chi_h = 45$, obtained as $(\mathbb{Z}/5\mathbb{Z})^2$ covering of $\mathbb{CP}^2$ branched along a complete quadrangle [57, 24, 25] (and their generalization in [29]), and exotic symplectic 4-manifolds constructed in [6, 10, 7, 11, 18], obtained via the combinations of symplectic connected sum and Luttinger surgery operations.

In this chapter we will present our constructions ((**i**) and (**ii**) above) of new non-spin irreducible symplectic and smooth 4-manifolds with nonnegative signatures that are interesting with respect to the symplectic and smooth geography problems (cf. Introduction). Outline of this chapter is as follows. In Section 2.1 we will state our main theorems in [19] (Theorems 2.1.1 and 2.1.2 below). In Sections 2.2 and 2.3 we provide the first algebraic building blocks that are the complex surfaces of Hirzebruch and Bauer-Catanese. In Section 2.4 we will give our second symplectic building blocks which are the exotic non-spin symplectic and smooth 4-manifolds with negative signatures constructed in [6, 7, 11, 18]. Then we will prove Theorem 2.1.1 in Section 2.5, while Section 2.6 is devoted to the proof of Theorem 2.1.2.

2.1 Statements of Main Results

In [19], we have constructed new nonspin irreducible symplectic and non-symplectic (but smooth), pairwise nondiffeomorphic 4-manifolds with nonnegative signatures that are interesting with respect to the symplectic and smooth geography problems. Our main results are as follows.

Theorem 2.1.1. *([19]) Let M be $(2n-1)\mathbb{CP}^2\#(2n-1)\overline{\mathbb{CP}}^2$ for any integer $n \geq 12$. Then there exist an infinite family of nonspin irreducible symplectic 4-manifolds and an infinite family of irreducible non-symplectic 4-manifolds that all are homeomorphic but not diffeomorphic to M.*

The above theorem improves one of the main results in [12] (see page 11) where exotic irreducible smooth structures on $(2n-1)\mathbb{CP}^2\#(2n-1)\overline{\mathbb{CP}}^2$ for $n \geq 25$ were constructed.

Besides, in the positive signature case exotic copies of

- $(2n-1)\mathbb{CP}^2\#(2n-2)\overline{\mathbb{CP}}^2$ for any integer $n \geq 25$.
- $(2n-1)\mathbb{CP}^2\#(2n-3)\overline{\mathbb{CP}}^2$ for any integer $n \geq 24$.
- $(2n-1)\mathbb{CP}^2\#(2n-4)\overline{\mathbb{CP}}^2$ for any integer $n \geq 27$.

were constructed in [12, 8]. Our next result is an advancement of these results:

Theorem 2.1.2. *([19]) Let M be one of the following 4-manifolds.*

(i) $(2n-1)\mathbb{CP}^2\#(2n-2)\overline{\mathbb{CP}}^2$ *for any integer* $n \geq 14$.

(ii) $(2n-1)\mathbb{CP}^2\#(2n-3)\overline{\mathbb{CP}}^2$ *for any integer* $n \geq 13$.

(iii) $(2n-1)\mathbb{CP}^2\#(2n-4)\overline{\mathbb{CP}}^2$ *for any integer* $n \geq 15$.

Then there exist an infinite family of irreducible symplectic 4-manifolds and an infinite family of irreducible non-symplectic 4-manifolds that are homeomorphic but not diffeomorphic to M.

We have two main building blocks in the construction as mentioned above. In the next two sections we will present our first building blocks which are the complex surfaces of Hirzebruch and Bauer-Catanese on Bogomolov-Miyaoka-Yau line with $c_1^2 = 9\chi_h = 45$, obtained as $(\mathbb{Z}/5\mathbb{Z})^2$ coverings of $\mathbb{CP}^2$ branched along a complete quadrangle [57, 24, 25] (and their generalization in [29]). Let us give some background information first on the abelian Galois coverings.

2.2 Construction of Complex 2-Manifolds via Line Arrangements in the Complex Projective Plane

In constructing complex surfaces, taking branched coverings over the line arrangements in the complex projective plane $\mathbb{CP}^2$ is an efficient method. Using this method, in [57]

Hirzebruch constructed three ball quotients, i.e. complex surfaces Y_i with $c_1^2(Y_i) = 3c_2(Y_i)$, where c_1^2 is the square of the first Chern class and $c_2(Y_i) = e(Y_i)$ is the Euler class of Y_i as defined above. Below we will briefly go over his construction. But let us first recall branched coverings.

Definition 2.2.1. *([48]) A d-fold* branched covering *is a smooth, proper map $f : X^n \to Y^n$ with critical set $B \subset Y$ called the* branch locus*, such that away from the branch locus $f|_{X-f^{-1}(B)} : X - f^{-1}(B) \to Y - B$ is a covering map of degree d, and for each $p \in f^{-1}(B)$ there are local coordinate charts $U, V \to \mathbb{C}\times\mathbb{R}_+^{n-2}$ around $p, f(p)$ respectively, on which f is given by $(z, x) \mapsto (z^m, x)$. Here m is a positive integer called the* branching / ramification index *of f at p and also denoted by e_p.*

As we see from the definition, there exists a small neighborhood U of $p \in f^{-1}(B)$ such that $f(p)$ has exactly one preimage in U, but the image of any other point in U has exactly m preimages in U. In addition, the sum of the indices of all points in $f^{-1}(f(p))$ is the degree d of f.

Definition 2.2.2. *(Riemann-Hurwitz Formula) Now let us consider branched coverings between two complex curves (Riemann surfaces). Let $\pi : S' \to S$ be a complex analytic, surjective map of degree d, and S', S be Riemann surfaces of genus g', g, respectively. Then we have*

$$\begin{aligned} e(S') &= d\, e(S) - \sum_{p\in S'} (e_p - 1) \quad \textit{equivalently}, \\ 2g' - 2 &= d\,(2g - 2) + \sum_{p\in S'} (e_p - 1) \end{aligned} \tag{2.1}$$

where all but finitely many p have $e_p = 1$, so the sum is finite.

2.2.1 Kummer Extensions and Abelian Galois Coverings

Let $\mathbb{CP}^2$ be the complex projective plane with coordinates z_0, z_1, z_2. *An arrangement of k lines*, $\mathcal{L}$, is a set of distinct lines in $\mathbb{CP}^2$. If the lines are given by linear forms $l_1, \cdots, l_k$, then $\mathcal{L}$ is the zero set of the product $l_1 \cdots l_k$. We call a point p an r-fold point, if it lies on exactly r lines of the arrangement. In the sequel we will assume $k \geq 3$ and there is no k-fold point in $\mathcal{L}$, i.e., it is not allowed that all the lines of $\mathcal{L}$ pass through a single point.

Kummer Extensions

For an arrangement $\mathcal{L}$ of k lines as above and a natural number $n \geq 2$, we consider the function field

$$\mathbb{C}(z_1/z_0, z_2/z_0)((l_2/l_1)^{1/n}, \cdots, (l_k/l_1)^{1/n})$$

which is an extension of the function field $\mathbb{C}(z_1/z_0, z_2/z_0)$ of $\mathbb{CP}^2$ of degree n^{k-1} and Galois group $(\mathbb{Z}/n\mathbb{Z})^{k-1}$. This function field is called the Kummer extension and determines an algebraic surface X with normal singularities which ramifies over $\mathbb{CP}^2$ with the arrangement as the branch divisor. Hence we have a degree n^{k-1} branched cover $\pi : X \to \mathbb{CP}^2$, branched of degree n along all of the k lines of $\mathcal{L}$. If a point $p \in \mathbb{CP}^2$ lies on r lines of the arrangement, with $r \geq 0$, then $\pi^{-1}(p)$ consists of n^{k-1-r} points which are an orbit of the Galois group acting on X.

If p is not a normal crossing, i.e., $r > 2$, then there are singular points over p. These singularities of X are resolved by resolving the branch locus. Namely, we blow-up $\mathbb{CP}^2$ at all r-fold points where $r > 2$. Denoting the blown up complex projective space by $\widehat{\mathbb{CP}^2}$ we obtain a commutative diagram

$$\begin{array}{ccc} Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ \widehat{\mathbb{CP}^2} & \longrightarrow & \mathbb{CP}^2 \end{array}$$

where $Y \to \widehat{\mathbb{CP}^2}$ is branched along the proper transforms of the lines of $\mathcal{L}$ and along the exceptional curves of the blow-ups. The degree of this map is n^{k-1} and the branching degree is n along the branch locus. The smooth algebraic surface Y is said to be "associated to $\mathcal{L}$" and its Chern numbers can also be computed from the data of the arrangement [57].

In [57], Hirzebruch constructed three smooth algebraic surfaces Y_1, Y_2, Y_3 associated to three special line arrangements; complete quadrangle, Hesse configuration and the dual Hesse configuration. Then he showed that they are ball quotients, which is to say they satisfy $c_1^2 = 3c_2$. Their Euler numbers are:

$$c_2(Y_1) = 15 \times 5^3, c_2(Y_2) = 48 \times 3^9, c_2(Y_3) = 111 \times 5^6. \tag{2.2}$$

Moreover, in [69] it was shown that Y_1 admits an action of $A = (\mathbb{Z}/5\mathbb{Z})^5$ and there are subgroups A_2 of A of order 5^2 acting freely on Y_1. Taking the quotient, Ishida obtained a surface Y_1/A_2 with $c_2 = 75$ which is again a ball quotient.

In [25], the ball quotient Y_1 was restudied and smooth algebraic surfaces with $c_1^2 = 3c_2 = 45$ were constructed. The main tool used was the theory of abelian Galois coverings, developed by Pardini in [100]. Let us introduce the abelian coverings now.

Abelian Galois Coverings

In this section, we recall basic definitions and properties of Abelian Galois ramified coverings. The proofs will be omitted, and the reader is referred to [100, 24] for the details.

Definition 2.2.3. *Let Y be a variety. An* abelian Galois ramified cover *of Y with abelian Galois group G is a finite map $p : X \to Y$ with a faithful action of G on X such that p exhibits Y as the quotient of X by G.*

We call such coverings *abelian G-covers*. We will assume that X and Y are normal, projective varieties and Y is smooth. Let R denote the ramification divisor of p which consists of the points of X that have nontrivial stabilizer. Indeed, R is the critical set of p, and $p(R)$ is the branch divisor denoted by D. The map $p : X \to Y$ is determined by the surjective homomorphism $\phi : \pi_1(Y - D) \to G$ which factors through $\varphi : H_1(Y - D, \mathbb{Z}) \to G$, as G is assumed to be abelian. Moreover, if $H_1(Y, \mathbb{Z}) = 0$ then $H_1(Y - D, \mathbb{Z})$ is:

$$H_1(Y - D, \mathbb{Z}) \cong (H^{2n-2}(D, \mathbb{Z}) \cong \bigoplus_{i=1}^{k} \mathbb{Z}[D_i])/H^{2n-2}(Y, \mathbb{Z}) \tag{2.3}$$

where n is the complex dimension of Y and D_i are the components of D. This follows from the long exact sequence of cohomology [25], [28].

It is also known that to every component of D, we can associate a cyclic subgroup H of G and a generator ψ of H^*, the group of characters of H ([100], p195). We let $D_{H,\psi}$ be the sum of all components of D which have the same group H and character ψ. For the abelian G-cover $p : X \to Y$ as above, and for any cyclic subgroup H of G, let g and m_H denote the orders of G and H, respectively. Then, the canonical classes of X and Y satisfy

$$K_X{}^2 = g\Big(K_Y + \sum_{H,\psi} \frac{m_H - 1}{m_H} D_{H,\psi}\Big)^2 \tag{2.4}$$

where the sum is taken over the set $\mathcal{C}$ of cyclic subgroups of G and for each H in $\mathcal{C}$, the set of generators ψ of H^* (cf. [100], Prop 4.2).

Let us consider an abelian G-cover and let $D = \bigcup_{i=1}^{k} D_i$ be its branch divisor with smooth irreducible components. Let $\chi : G \to \mathbb{Z}/d$ be a character of G and L_χ be a divisor associated to the eigensheaf $\mathcal{O}(L_\chi)$. Then we have (cf. [25])

$$dL_\chi = \sum_{i=1}^{k} \delta_i D_i,\ \delta_i \in \mathbb{Z}/d\mathbb{Z} \simeq \{0, 1, \ldots, d-1\}. \tag{2.5}$$

2.2.2 Construction of Smooth Algebraic Surfaces with $c_1^2 = 45$ and $\chi_h = 5$

In this section we will recall the construction of smooth algebraic surfaces with $c_1^2 = K^2 = 45$ and $\chi_h = 5$, by following [25]. These complex surfaces of general type are obtained as abelian covering of the complex plane branched over an arrangement of six lines shown as in Figure 2.1, and were initially studied by Hirzebruch (cf. [57], p.134).

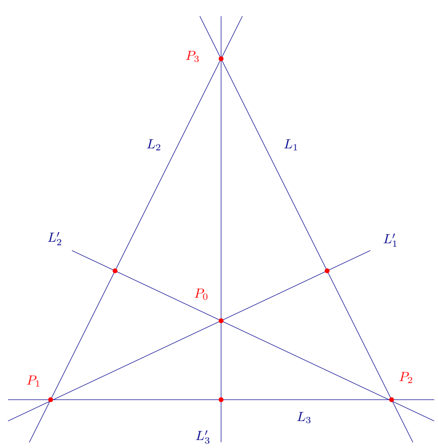


Figure 2.1: Complete Quadrangle in $\mathbb{CP}^2$

In complex projective plane $\mathbb{CP}^2$ we take a complete quadrangle Δ, which consist of the union of 6 lines through 4 points $P_0, \cdots, P_3$ in the general positions (see Figure 2.1).

Here let us remind the well-known "blow-up" process and some teminology.

Definition 2.2.4. *For a smooth, oriented 4-manifold X, the connected sum $X' = X\#\overline{\mathbb{CP}}^2$ is called the* blow-up *of X. Indeed, a point P in X is replaced by a sphere $\overline{\mathbb{CP}}^1$ in $\overline{\mathbb{CP}}^2$. The sphere $\overline{\mathbb{CP}}^1$ has self intersection -1 and it is called the* exceptional sphere*. The blow-up map $\pi : X' \to X$ is a diffeomorphism between $X' - \overline{\mathbb{CP}}^1$ and $X - \{P\}$, and $\pi^{-1}(P) = \overline{\mathbb{CP}}^1$.*

If Σ is a smooth surface in X and $P \in \Sigma$, then after the blow-up at P, the inverse image $\pi^{-1}(\Sigma)$ of Σ is called the total transform *of Σ and the closure $cl(\pi^{-1}(\Sigma - \{P\}))$ is said to be the* proper/strict transform *of Σ.*

Moreover, the blow-up operation can be generalized to the symplectic setting. If (X, w) is a symplectic manifold, then $X' = X\#\overline{\mathbb{CP}}^2$ also carries a symplectic structure. That is to say, a symplectic manifold can always be symplectically blown up. ([48])

Now let us go back to the quadrangle picture and blow-up $\mathbb{CP}^2$ at the points $P_0, \cdots, P_3$. Let $\pi : Y := \widehat{\mathbb{CP}^2} \to \mathbb{CP}^2$ be the blow-up map and E_i be the exceptional divisor corresponding to the blow-up at the point P_i for $i = 0, \cdots, 3$. We introduce some notations now. In the sequel, i, j, k denote distinct elements of the set $\{1, 2, 3\}$. Let H be the total transform in Y of a line in $\mathbb{CP}^2$, and let $\widetilde{L_j}$ and $\widetilde{L'_j}$ be the strict transform of the lines L_j and L'_j in $\mathbb{CP}^2$. In other words, we have

$$\widetilde{L_j} = H - E_i - E_k,\ \widetilde{L'_j} = H - E_0 - E_j. \tag{2.6}$$

Now let us take a divisor that consists of the union of 10 lines; arising from the six lines of the quadrangle Δ and four exceptional divisors coming from the blow-ups:

$$D = \widetilde{L_1} + \widetilde{L_2} + \widetilde{L_3} + \widetilde{L'_1} + \widetilde{L'_2} + \widetilde{L'_3} + E_0 + \cdots + E_3. \tag{2.7}$$

D is a divisor on Y having simple normal crossings. Notice that $H, E_0, \cdots, E_3$ are generators of $H^2(Y, \mathbb{Z})$, and $H_1(Y - D, \mathbb{Z})$ is generated by $e_0, \cdots, e_3, l_1, l_2, l_3, l'_1, l'_2, l'_3$ with the relations

$$e_0 = l'_1 + l'_2 + l'_3,\ e_i = l_i + l'_j + l'_k,\ \sum l_i + \sum l'_i = 0$$

where e_i, l_i, l_i' denote simple closed loops around E_i, $\tilde{L}_i$ and $\tilde{L}_i'$ respectively. Hence $H_1(Y - D, \mathbb{Z})$ is free group of rank 5. We know from above that a surjective homomorphism $\varphi : \mathbb{Z}^5 \simeq H_1(Y - D, \mathbb{Z}) \to (\mathbb{Z}/5\mathbb{Z})^2$ determines an abelian $(\mathbb{Z}/5\mathbb{Z})^2-$cover $p : S \to Y = \widehat{\mathbb{CP}^2}$. It can be shown that p is branched exactly in D given by (2.7). Since D has simple normal crossings, the total space S is smooth.

Invariants of the surface S

Now for the total space S of an abelian $(\mathbb{Z}/5\mathbb{Z})^2-$cover p over Y, branched at D, we will compute that $c_1^2(S) = K_S^2 = 45$ and $\chi_h(S) = 5$. Since the canonical class K_Y of Y is $-3H + \sum_{i=0}^{3} E_i$, using the equation (2.4), we compute

$$K_S^2 = 5^2\Big((-3H + \sum_{i=0}^{3} E_i) + \frac{4}{5}\sum_{i=0}^{3} E_i + \frac{4}{5}\sum_{i=1}^{3}(L_i + L_i')\Big)^2$$

Next, using the equalities in (2.6), we obtain

$$\begin{aligned} K_S^2 &= 5^2\Big((-3H + \sum_{i=0}^{3} E_i) + \frac{4}{5}(6H - 2E_0 - 2E_1 - 2E_2 - 2E_3)\Big)^2 \\ &= 5^2\Big(\frac{9}{5}H - \frac{3}{5}\sum_{i=0}^{3} E_i\Big)^2 \end{aligned}$$

Since $H \cdot E_i = 0$, $H^2 = 1$ and $E_i^2 = -1$, the above equation simplifies to: $K_S^2 = 9^2 - 4 \cdot 3^2 = 45$.

The Euler number $e(S)$ of S can be found as follows. We use the inclusion-exclusion principle. In fact, if the degree 25 cover was unramified, we would have the Euler number $e = 25e(\widehat{\mathbb{CP}^2})$. Since for the lines in D the cover is of degree 5, their contribution to $e(S)$ is $10 \cdot 5e(\mathbb{CP}^1)$. Therefore, we subtract $10 \cdot 20e(\mathbb{CP}^1)$ from $e = 25e(\widehat{\mathbb{CP}^2})$. But then for the points at the intersection of the lines in D, we need to add back 16 times the Euler number of 15 points. Hence

$$e(S) = 25e(\widehat{\mathbb{CP}^2} = \mathbb{CP}^2\#4\overline{\mathbb{CP}^2}) - 20 \cdot 10e(\mathbb{CP}^1) + 16 \cdot 15 = 15.$$

Finally, as $12\chi_h(S) - c_1^2(S) = e(S)$, we find that $\chi_h(S) = 5$.

In [25], the existence of four nonisomorphic surfaces S_1, S_2, S_3, S_4 was shown. These surfaces were obtained from abelian $(\mathbb{Z}/5\mathbb{Z})^2-$covers over Y, branched at D, with invariants $K^2 = 45$ and $\chi_h = 5$. For S_3, $H^0(S, \mathcal{O}_S(K_S)) \simeq \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$ which is found by using (2.5). Hence the geometric genus $p_g = \dim H^0(S, \mathcal{O}_S(K_S)) = 4$. Furthermore, from the formula

$$\chi_h = p_g - q + 1$$

where $q = p_g - p_a$ is the irregularity of the surface, and the fact that $\chi_h(S) = 5$, we find that q for S_3 is zero, hence S_3 is regular. Similarly, the irregularity of S_i, $i \in \{1, 2, 4\}$, is 2. Therefore, only one of them is a regular surface. Let S denote one of the surfaces S_i, for $i \in \{1, 2, 4\}$. This will give one of our building blocks in our constructions of new, symplectic 4-manifolds in [19] discussed in the previous section.

We would like to note that there is a more general construction given in [57], p.134 and [24], p.240. In [57, 24], using the arrangements of k lines in $\mathbb{CP}^2$ and taking their associated abelian $(\mathbb{Z}/n\mathbb{Z})^{k-1}$-covers, various algebraic surfaces were constructed. For the total spaces $X(m)$ of $(\mathbb{Z}/5\mathbb{Z})^m$-covers over the above configuration of 6 lines (where $m \geq 2$), the followings hold:

$$c_1^2(X(m)) = 45 \times 5^{m-2} \quad \text{and} \quad e(X(m)) = 15 \times 5^{m-2}, for\ m \geq 2. \tag{2.8}$$

It would also be interesting to work with these general constructions.

2.3 The First Building Block $\widehat{S}$

From the complex surface S mentioned above with $q = 2, c_1^2 = 3c_2 = 45$, we will obtain one ingredient $\widehat{S} := S\#\overline{\mathbb{CP}}^2$ of our construction. We will first discuss the fibration structure on the surface S.

2.3.1 Genus four fibration on the surface S with three singular fibers

Indeed S admits four genus 4 fibrations with three singular fibers each. We show this as follows. Let $R_1, \cdots, R_{10}$ be the ramification divisors of $p : S \to Y$ lying over the lines $\widetilde{L_1'}$, $\widetilde{L_2'}$, $\widetilde{L_3'}$, $\widetilde{L_1}$, $\widetilde{L_2}$, $\widetilde{L_3}$, $E_0, \cdots, E_3$, respectively. Since $R_i^2 = -1$ and $K_S \cdot R_i = 3$, by the adjunction formula (Theorem 1.2.1 above),

$$
\begin{aligned}
K_S \cdot R_i + R_i^2 &= 2g_{R_i} - 2, \\
g_{R_i} &= 2
\end{aligned}
\tag{2.9}
$$

we see that the genus of the complex curve R_i is two, for $i = 1, \cdots, 10$. Now let us consider the composition $p \circ \pi : S \to \mathbb{CP}^2$, where π is the blow-up map. Let P be one of the four vertices of the complete quadrangle Δ in $\mathbb{CP}^2$ (see Figure 2.1). The pencil of lines in $\mathbb{CP}^2$ passing through the point P lifts to a fibration on S. Indeed, we note that there are four distinct fibrations in S coming from the four vertices P_i's of the complete quadrangle.

Let us take one such point say, P_3 which is the intersection point of L_2, L_3' and L_1 in $\Delta \subset \mathbb{CP}^2$. The pencil of lines in $\mathbb{CP}^2$ passing through P_3 gives a fibration on S. Now we will find the genus of the generic fiber of this fibration. Let us take a line K passing through P_3 that is different than L_2, L_3' and L_1 (see Figure 2.2). We observe that on K there are 4 branch points. Furthermore, above each point on K where no two lines intersect, there are 25/5 points (cf. [24], p.241).

Now we consider the proper transform $K - E_3$ of the line K after the blow-ups. $K - E_3$ lies in $Y = \mathbb{CP}^2 \# 4\overline{\mathbb{CP}}^2$ and its preimage $p^{-1}(K - E_3)$ in the surface S will be the generic fiber of the given fibration. Note that $p^{-1}(K - E_3)$ is a degree 5 cover of $K - E_3$ branched at 4 points. Hence, to determine the genus g of the surface $p^{-1}(K - E_3)$, lying above the 2-sphere $K - E_3$, we apply the Riemann-Hurwitz ramification formula

$$
\begin{aligned}
2g - 2 &= 5(-2) + 4 \cdot 4 \\
g &= 4.
\end{aligned}
\tag{2.10}
$$

It shows that generic fibers are of genus 4 surfaces.

2.3.2 Singular fibers of the fibration on the surface S

Now we will determine the topological types of the singular fibers of the fibration on S given above. Let us state some well-known results in the theory of complex surfaces.

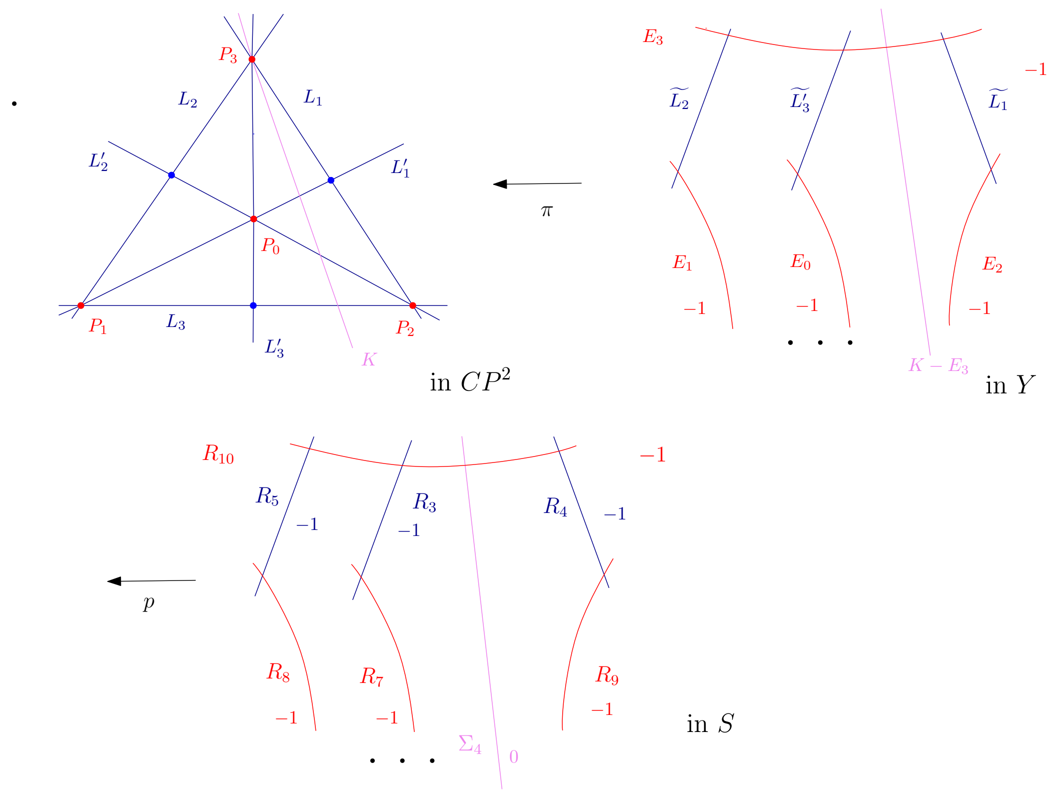


Figure 2.2: Genus 4 fibration on S with three singular fibers

Proposition 2.3.1. *([24], Proposition 11.4) Let X be a compact connected smooth surface, and C be a smooth connected curve. Let $f : X \to C$ be a fibration with $g > 0$, where g is the genus of the general fiber X_s, and X_{gen} a nonsingular fiber. Then*

1. $e(X_s) \geq e(X_{gen})$ *for all fibers.*

2. $e(X_s) > e(X_{gen})$ *for all singular fibers X_s, unless X_s is a multiple fiber with $(X_s)_{red}$ nonsingular elliptic curve.*

3. $e(X) = e(X_{gen}) \cdot e(C) + \sum(e(X_s) - e(X_{gen}))$.

This proposition is quite useful to detect the topological type of the singular fibers of the fibration on S that we discussed above.

Before stating our main proposition of this section, let us introduce some notations and facts. We will follow [126, 124]. Let $\varphi : X \to C$ be a fibration, and $F = \varphi^{-1}(c)$ denote a regular fiber of φ. The inclusion map $i : F \hookrightarrow X$ induces the homomorphism $i_* : \pi_1(F) \to \pi_1(X)$. We denote the image of $\pi_1(F)$ under i_* by $\mathcal{V}_\varphi$ and call it the *vertical part* of $\pi_1(X)$. Then we have

Lemma 2.3.2. *([124], pages 13-14) $\mathcal{V}_\varphi$ is a normal subgroup of $\pi_1(X)$, and is idependent of the choice of F.*

Let us define the *horizontal part* of $\pi_1(X)$ as $\mathcal{H}_\varphi := \pi_1(X)/\mathcal{V}_\varphi$. Thus, we have $1 \to \mathcal{V}_\varphi \to \pi_1(X) \to \mathcal{H}_\varphi \to 1$.

Next, let $\{r_1, \cdots, r_s\}$ stand for the images of all the multiple fibers of φ (which maybe empty) and $\{m_1, \cdots, m_s\}$ are their corresponding multiplicities. Let $C' = C \setminus \{r_1, \cdots, r_s\}$ and γ_i be a small loop around the point r_i.

Lemma 2.3.3. *([124], pages 13-14) The horizontal part $\mathcal{H}_\varphi$ of $\pi_1(X)$ is the quotient of the fundamental group $\pi_1(C')$ by the normal subgroup generated by the conjugates of ${\gamma_i}^{m_i}$ for all $i = 1, \cdots s$.*

In addition we have,

Proposition 2.3.4. *([124, 126]) Let F' be any fiber of φ with multiplicity m. Then the image of $\pi_1(F')$ in $\pi_1(X)$ contains $\mathcal{V}_\varphi$ as a normal subgroup, whose quotient group is cyclic of order m, which maps isomorphically onto the subgroup of $\mathcal{H}_\varphi$ generated by the class of a small loop around the image of F' in X.*

As a consequence,

Corollary 2.3.5. *If the fibration φ has a section, then $1 \to \mathcal{V}_\varphi \to \pi_1(X) \to C \to 1$.*

Lastly, we will state the following proposition (which is Corollary 2.4 B in [96]). It follows from Nori's work on Zariski's conjecture.

Proposition 2.3.6. *([96]) Let C be an embedded algebraic curve with $C^2 > 0$ in an algebraic surface X. Then the inclusion-induced group homomorphism $\pi_1(C) \to \pi_1(X)$ is surjective.*

By using the discussion above and Propositions 2.3.1, 2.3.4, and 2.3.6, we will prove our following result which allows us to apply symplectic connected sum to $\widehat{S} := S\#\overline{\mathbb{CP}}^2$. We recall that S is the total space of an abelian $(\mathbb{Z}/5\mathbb{Z})^2-$ branched cover over $Y = \mathbb{CP}^2\#4\overline{\mathbb{CP}}^2$, with $c_1^2(S) = K_S^2 = 45$ and $\chi_h(S) = 5$.

Proposition 2.3.7. *([19]) Let S be the surface with irregularity $q = 2$ given as above. Then the followings hold:*

1. *S admits a genus 4 fibration over genus 2 surface with three singular fibers.*
2. *S contains an embedded symplectic genus 6 submanifold (of real dimension 2) R such that $\pi_1(R) \to \pi_1(S)$ is surjective.*
3. *$\widehat{S} = S\#\overline{\mathbb{CP}}^2$ contains an embedded symplectic genus 6 submanifold $\widetilde{R}$ with self-intersection zero such that $\pi_1(\widetilde{R}) \to \pi_1(S\#\mathbb{CP}^2)$ is surjective.*

Proof. **1.** We consider the fibration given above, arising from the pencil of lines in $\mathbb{CP}^2$ passing through one of the vertices of the quadrangle Δ. As we have shown above, the ramification curves $\widetilde{L_1'}$, $\widetilde{L_2'}$, $\widetilde{L_3'}$, $\widetilde{L_1}$, $\widetilde{L_2}$, $\widetilde{L_3}$, E_0, $\cdots$, E_3 lifts to -1 complex curves R_1, $\cdots$, R_{10} in S, respectively where R_i's are genus 2 curves (see Equation 2.9). We have also shown that the generic fiber S_{gen} of this fibration has genus 4 (see Equation 2.10). We will determine the number and topology of the singular fibers S_s now.

Let us take the exceptional sphere E_3 whose preimage is the square -1, genus 2 curve R_{10} in S. Thus, we have a fibration $f : S \to R_{10}$. Furthemore, using the facts that $e(S) = 15$, $e(R_{10}) = e(\Sigma_2) = -2$, $e(S_{gen}) = e(\Sigma_4) = -6$ and Proposition 2.3.1, we find

$$\begin{aligned} e(S) &= e(S_{gen}) \cdot e(R_{10}) + \sum(e(S_s) - e(S_{gen})) \\ e(S) &= e(\Sigma_4) \cdot e(\Sigma_2) + \sum(e(S_s) - e(\Sigma_4) \\ 15 &= (-6. - 2) + \sum(e(S_s) + 6) \\ 3 &= m(e(S_s) + 6) \end{aligned} \tag{2.11}$$

where m is the number of singular fibers. Evidently $m > 1$ (from the branched cover description of S). This enforces that $m = 3$ and $e(S_s) = -5$.

Hence we have shown that there is a fibration $f : S \to \Sigma_2$ which has three singular fibers and each singular fiber has Euler characteristic -5. Furthemore, this in turn shows that each singular fiber has two irreducible components, where each component is genus two curve of square -1. In fact the singular fibers are arising from the curves $\widetilde{L_2} \cup E_1$, $\widetilde{L_3'} \cup E_0$, and $\widetilde{L_1} \cup E_2$ in Y (see the Figure 2.2).

2. To construct an embedded symplectic genus 6 submanifold R in S we will *resolve* intersection points of some of the curves in S. Thus, let us first recall the well known operation called *symplectic resolution* ([46, 48]). Assume that closed, smooth, oriented surfaces Σ_1 and Σ_2 be symplectic submanifolds of a symplectic 4-manifold (X, w) and that they intersect each other transversally at a point p. Evidently $\Sigma_1 \cup \Sigma_2$ is not a manifold but it defines a homology class $[\Sigma_1] + [\Sigma_2] \in H_2(X, \mathbb{Z})$. We take a 4-ball neighborhood D of the point p, in which $\Sigma_1 \cup \Sigma_2$ is modeled on F; two symplectic 2-disks intersecting each other at one point in D. Then the pair (D, F) is cut out from the ambient 4-manifold X and replaced with (D, A), where A is a symplectic annulus. (In fact, A can be obtained from perturbing the graph of $z_2 = \epsilon/z_1$, $z_i \in \mathbb{C}, |z_1|^2 + |z_2|^2 \leq 1, 0 < \epsilon << 1$.) Replacement eliminates the singularity, however the ambient manifold does not change. The following figure illustrates this operation.

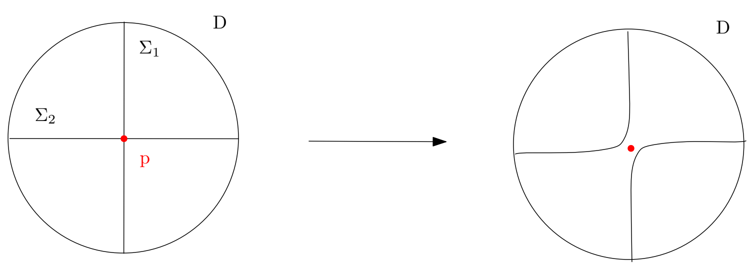


Figure 2.3: Symplectic Resolution

Now let us go back to our construction of the symplectic genus 6 submanifold R. We take one copy of a singular fiber S_{sing}, say $R_3 \cup R_7$ of the fibration $f : S \to R_{10}$, and the base curve R_{10} which is of square -1. We resolve the transverse intersection point of $R_3 \cup R_7$ and R_{10} symplectically. In addition, we also symplectically resolve the single intersection point of the irreducible components R_3 and R_7 of S_{sing}, which are smooth genus two curves of square -1. The resulting curve R has the self-intersection one:

$$\begin{aligned} R^2 &= (S_{sing} + R_{10})^2 \\ &= (R_3 + R_7 + R_{10})^2 \\ &= R_3^2 + R_7^2 + R_{10}^2 + 2R_3R_7 + 2R_3R_{10} \\ &= (-3) + 2 + 2 \\ &= 1. \end{aligned} \tag{2.12}$$

Above we have $R_7 \cdot R_{10} = 0$, since R_7 and R_{10} are disjoint (see the Figure 2.2). In addition, as we have $R^2 > 0$, using the lemmas above or Proposition 2.3.6, we deduce that $\pi_1(R) \to \pi_1(S)$ is surjective.

In fact, we can construct more than one such symplectic submanifolds having positive self intersections by using various R_i curves.

Furthermore, the explicit computation of the fundamental group of S is also known ([25], Proposition 5.2. It relies on the work of Terada (see Theorem 5.1)). However we will not need it in our computations.

3. We take $\widetilde{R}$ to be the symplectic genus six submanifold in $S\#\overline{\mathbb{CP}}^2$ obtained by blowing up R at a point. Therefore, $\widetilde{R}^2 = 0$. Since we showed that $\pi_1(R) \to \pi_1(S)$ is surjective in part **2.**, after blow-up $\pi_1(\widetilde{R}) \to \pi_1(S\#\mathbb{CP}^2)$ is still a surjection. □

Hence we have obtained one piece $\widehat{S} = S\#\overline{\mathbb{CP}}^2$ of our construction which contains an embedded symplectic genus 6 submanifold $\widetilde{R}$ with self-intersection zero. The second building blocks are exotic symplectic 4-manifolds constructed in [6, 10, 7, 11, 18], obtained via the combinations of symplectic connected sum and Luttinger surgery operations. Let us go into details of these constructions.

2.4 Symplectic Building Blocks

Above we have built one of our building blocks which is a complex (and Kähler) surface $\widehat{S} = S\#\overline{\mathbb{CP}}^2$. In this section we will provide the symplectic building blocks that will be used in our constructions of exotic 4-manifolds with nonnegative signatures in [19].

These symplectic building blocks have negative signatures and were constructed in [6, 11, 18]).

Our first family of symplectic building blocks comes from [6, 18] (see Theorem 5.1 in the latter). Let us state this theorem in a special case that we will need.

Theorem 2.4.1. *([6, 18]) Let M be $(2k-1)\mathbb{CP}^2\#(2k+3)\overline{\mathbb{CP}}^2$ for any $k \geq 1$. There exist a family of smooth closed simply-connected minimal symplectic 4-manifold and an infinite family of non-symplectic 4-manifolds that are homeomorphic but not diffeomorphic to M and that can obtained by a sequence of Luttinger surgeries and a single generalized torus surgery on Lefschetz fibrations.*

For the convenience of the reader, we will go over the construction of the manifold M (in a special case $n = 1$), and direct the reader to the references for full details. In constructing the exotic manifold M, given as in the statement of the theorem above, two building blocks were used.

First one is the symplectic 4-manifold $Y(k) = \Sigma_k \times \mathbb{S}^2\#4\overline{\mathbb{CP}}^2$ which admits a genus $2k$ Lefschetz fibration over $\mathbb{S}^2$ with $2k+2$ vanishing cycles [78] shown as in Figure 2.4 (which is borrowed from [19] and references therein [78, 18]). We endow $Y(k) = \Sigma_k \times \mathbb{S}^2\#4\overline{\mathbb{CP}}^2$ with the symplectic structure induced from the given Lefschetz fibration. Thus, $Y(k)$ has a genus $2k$ symplectic submanifold $\Sigma_{2k} \subset Y(k)$, which is a regular fiber of the Lefschetz fibration.

The other building block of the exotic M is the smooth 4-manifold $Y_g(1, m)$, along the submanifold Σ'_g of genus g. The manifold $Y_g(1, m)$ was obtained from the product 4-manifold $\Sigma_g \times T^2$ by performing appropriate $2g-1$ Luttinger surgeries, and one generalized torus surgery ([18], pages 14-15). Let us set $g = 2k$.

Let $X(k, m)$ denote the smooth 4-manifold obtained by forming the smooth fiber sum of $Y(k)$ and $Y_{2k}(1, m)$ along the surfaces Σ_{2k} and Σ'_{2k}.

$$X(k, m) := (\Sigma_k \times \mathbb{S}^2\#4\overline{\mathbb{CP}}^2)\#_{\Sigma_{2k}=\Sigma'_{2k}} Y_{2k}(1, m) \tag{2.13}$$

We shall need the following Theorem ([18] Theorem 5.1, pages 14-18), which summarizes topological properties of the manifold $X(k, m)$.

Theorem 2.4.2. *(i) $X(k, m)$ is simply connected*

(ii) $e(X(k, m)) = 4k+4$, $\sigma(X(k, m)) = -4$, $c_1^2(X(n, k, m)) = 8k-4$, and $\chi(X(k, m)) = k$.

(iii) $X(k, m)$ is minimal and symplectic for $m = \pm 1$ and non-symplectic for $|m| > 1$.

(iv) $X(k, m)$ contains the smooth surface Σ of genus $2k$ with self-intersection 0, and 4 tori T_i of self-intersection -1 intersecting Σ positively and transversally. Moreover, if $m = \pm 1$, these submanifolds all are symplectic.

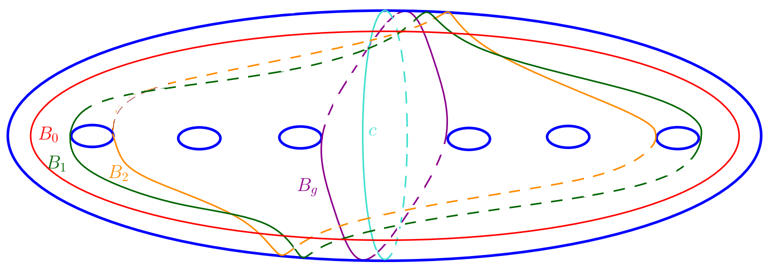


Figure 2.4: Vanishing cycles of a genus $2k$ Lefschetz fibration on $Y(k)$

Our next symplectic building blocks comes from [7] (see Theorem 5.1, page 14)

Theorem 2.4.3. *For any integer $g \geq 1$, there exist a minimal symplectic 4-manifold $X_{g,g+2}$ obtained via Luttinger surgery such that*

(i) $X_{g,g+2}$ is simply connected

(ii) $e(X_{g,g+2}) = 4g + 2$, $\sigma(X_{g,g+2}) = -2$, $c_1^2(X_{g,g+2}) = 8g - 2$, and $\chi(X_{g,g+2}) = g$.

(iii) $X_{g,g+2}$ contains the symplectic surface Σ of genus 2 with self-intersection 0 and 2 genus g surfaces with self-intersection -1 intersecting Σ positively and transversally.

Our third symplectic building blocks comes from [11].

Theorem 2.4.4. *There exist a minimal symplectic 4-manifold $X_{g,g+1}$ obtained via Luttinger surgery such that* *(i)* $X_{g,g+1}$ *is simply connected*

(ii) $e(X_{g,g+1}) = 4g+1$, $\sigma(X_{g,g+1}) = -1$, $c_1^2(X_{g,g+2}) = 8g-1$, *and* $\chi(X_{g,g+1}) = g$.

(iii) $X_{g,g+1}$ *contains the symplectic surface* Σ *of genus* 2 *with self-intersection* 0*, genus* Σ_{g+1} *symplectic surface with self-intersection* 0 *intersecting* Σ *positively and transversally.*

We will give the details of the constructions of $X_{g,g+2}$ and $X_{g,g+1}$ in Section 2.6 for convenience of the reader.

2.5 Construction of Exotic 4-Manifolds with Zero Signatures

In this section we will prove our first main theorem in [19] (Theorem 2.1.1 above), which improves the main result obtained in [12]. That is to say, we will construct nonspin simply connected symplectic and smooth 4-manifolds with signature zero which realize new points on the (χ, c_1^2) geography chart. We will split the Theorem 2.1.1 above, into two theorems and prove them separately. The first theorem (Theorem 2.5.1) deals with the case $n \geq 13$, and the second theorem (Theorem 2.5.4) addresses the case $n = 12$, for which the construction is slightly different than $n \geq 13$ case.

We will prove Theorems 2.5.1 and 2.5.4 in several steps. First, we construct our manifolds using the symplectic connected sum of the complex surface $\widehat{S}$, and the symplectic building blocks given in Section 2.4 obtained via Luttinger surgery. In the second step, we show that the fundamental groups of our manifolds are trivial, and determine their homeomorphism types. Next, using the Seiberg-Witten invariants and Usher's Minimality Theorem [124] (Theorem 1.5.4 above), we distinguish the diffeomorphism types of our 4-manifolds from the standard $(2n-1)\mathbb{CP}^2\#(2n-1)\overline{\mathbb{CP}}^2$. Finally, we obtain an infinite family of pairwise non-diffeomorphic irreducible symplectic and non-symplectic exotic copies of $(2n-1)\mathbb{CP}^2\#(2n-1)\overline{\mathbb{CP}}^2$, by performing the knot surgery operation along a homologically essential torus on these symplectic 4-manifolds.

2.5.1 Exotic copies of $(2n-1)\mathbb{CP}^2\#(2n-1)\overline{\mathbb{CP}}^2$, for $n \geq 13$

Let us begin with Theorem 2.5.1 which is the $n \geq 13$ case of Theorem 2.1.1 as explained above.

Theorem 2.5.1. *([19]) Let M be $(2n-1)\mathbb{CP}^2\#(2n-1)\overline{\mathbb{CP}}^2$ for any $n \geq 13$. There exists an infinite family of smooth closed simply-connected minimal symplectic 4-manifolds and an infinite family of non-symplectic 4-manifolds that all are homeomorphic but not diffeomorphic to M.*

Proof. ([19]) The first building block is the complex surface $S\#\overline{\mathbb{CP}}^2$ containing the genus 6 symplectic surface $\widetilde{R} \subset S\#\overline{\mathbb{CP}}^2$, which we constructed in Section 2.3. We endowed $S\#\overline{\mathbb{CP}}^2$ with the symplectic structure induced from the Kähler structure.

Our second building block will be the symplectic 4-manifold $X(3,1)$ that contains the symplectic submanifold Σ_6' (see Section 2.4).

Let $Z(3)$ be the symplectic 4-manifold obtained by forming the symplectic connected sum of $S\#\overline{\mathbb{CP}}^2$ and $X(3,1)$ along the surfaces $\widetilde{R}$ and Σ_6'.

$$Z(3) = (S\#\overline{\mathbb{CP}}^2)\#_{\widetilde{R}=\Sigma_6'}X(3,1).$$

It follows from Gompf's theorem in [46] that $Z(3)$ is symplectic.

Lemma 2.5.2. $Z(3)$ *is simply-connected.*

Proof. By applying the Seifert-Van Kampen theorem, we see that

$$\pi_1(Z(3)) \;=\; \frac{\pi_1(S\#\overline{\mathbb{CP}}^2 \setminus \nu\widetilde{R}) * \pi_1(X(3,1) \setminus \nu\Sigma_6')}{\langle a_1 = a_1',\, b_1 = b_1',\, \cdots,\, a_6 = a_6',\, b_6 = b_6',\, \mu = \mu' = 1\rangle}.$$

where a_i, b_i, and a_i', b_i' (for $i = 1, \cdots, 6$) denote the standard generators of the fundamental group of the genus 6 Riemann surfaces $\widetilde{R}$ and Σ_6' in $S\#\overline{\mathbb{CP}}^2$ and in $X(3,1)$, and μ and μ' denote their meridians in $S\#\overline{\mathbb{CP}}^2 \setminus \nu\widetilde{R}$ and in $X(3,1) \setminus \nu\Sigma_6'$ respectively. Using the surjection in Proposition 2.3.7 **3.**, and the facts that the normal circle $\mu = \{\mathrm{pt}\} \times S^1$ of $\widetilde{R}$ in $\pi_1(S\#\overline{\mathbb{CP}}^2 \setminus \nu(\widetilde{R}))$ and the loops a_1', b_1', $\cdots$, a_6', b_6' in $\pi_1(X(3,1) \setminus \nu(\Sigma_6'))$ are all trivial, we see that the fundamental group of $Z(3)$ is the trivial group.

□

Lemma 2.5.3. *$e(Z(3)) = 52$, $\sigma(Z(3)) = 0$, $c_1^2(Z(3)) = 104$, and $\chi(Z(3)) = 13$.*

Proof. By applying the formulas 1.4 and 1.5, we have

$$\begin{aligned} e(Z(3)) &= e(S\#\overline{\mathbb{CP}}^2) + e(X(3,1)) + 4(6-1), \\ \sigma(Z(3)) &= \sigma(S\#\overline{\mathbb{CP}}^2) + \sigma(X(3,1)), \\ c_1^2(Z(3)) &= c_1^2(S\#\overline{\mathbb{CP}}^2) + c_1^2(X(3,1)) + 8(6-1), \\ \chi(Z(3)) &= \chi(S\#\overline{\mathbb{CP}}^2) + \chi(X(3,1)) + (6-1). \end{aligned}$$

Since we have

$$e(X(3,1)) = 16,\ \sigma(X(3,1)) = -4,\ c_1^2(X(3,1) = 20,\ \chi(X(3,1)) = 3,$$

and

$$e(S\#\overline{\mathbb{CP}}^2) = 16,\ \sigma(S\#\overline{\mathbb{CP}}^2) = 4,\ c_1^2(S\#\overline{\mathbb{CP}}^2) = 44,\ \chi(S\#\overline{\mathbb{CP}}^2) = 5$$

the proof of lemma follows.

□

Using Freedman's classification theorem for simply-connected 4-manifolds [42] (see the Introduction), the lemma above and the fact that $S\#\overline{\mathbb{CP}}^2$ contains genus two surface of self-intersection -1 disjoint from $\widetilde{R}$, we conclude that $Z(3)$ is homeomorphic to $(2n-1)\mathbb{CP}^2\#(2n-1)\overline{\mathbb{CP}}^2$ for $n = 13$.

Since $Z(3)$ is symplectic, by Taubes' theorem in [120], $Z(3)$ has non-trivial Seiberg-Witten invariant (see the Nonvanishing Theorems 1.4.4 above). Next, using the connected sum theorem for the Seiberg-Witten invariant (cf. Theorem 1.4.3), we deduce that the Seiberg-Witten invariant of $25\mathbb{CP}^2\#25\overline{\mathbb{CP}}^2$ is trivial. Since the Seiberg-Witten invariant is a diffeomorphism invariant, $Z(3)$ is not diffeomorphic to $25\mathbb{CP}^2\#25\overline{\mathbb{CP}}^2$.

Furthermore, $Z(3)$ is a minimal symplectic 4-manifold by Usher's Minimality Theorem [124] (Theorem 1.5.4 above). Since symplectic minimality implies smooth minimality (cf. [82]), $Z(3)$ is also smoothly minimal, and thus is smoothly irreducible.

Infinitely many copies

Now we will produce an infinite family of exotic $25\mathbb{CP}^2\#25\overline{\mathbb{CP}}^2$'s. We replace the building block $Y_6(1,1)$ used in our construction of $X(3,1)$ above with $Y_6(1,m)$ (see Section 2.4), where $|m|>1$. Let us denote the resulting smooth 4-manifold as $Z(3,m)$. In the presentation of the fundamental group, the above surgery amounts to replacing a single relation $[c^{-1},b_n]=d$ in $\pi_1(X(3,1))$, corresponding to the Luttinger surgery $(a''_n\times d',d',1)$, with $[c^{-1},b_n]^{-m}=d$. Notice that changing this relation has no affect on our proof of $\pi_1(Z(3))=1$; all the fundamental group calculations follow the same lines of arguments, and thus $\pi_1(Z(3,m))$ is trivial group.

Let us denote by $Z(3)_0$ the symplectic 4-manifold obtained by performing the following Luttinger surgery on: $(a''_n\times d',d',0/1)$ instead of $(a''_n\times d',d',1)$ in the construction of $Z(3)$. It is easy to check that $\pi_1(Z(3)_0)=\mathbb{Z}$ and the canonical class of $Z(3)_0$ is given by the formula $K_{Z(3)_0}=K_{S\#\overline{\mathbb{CP}}^2}+2[\Sigma_6]+\sum_{j=1}^{4}[\bar{R}_j]+\Sigma'_6+\widetilde{R}+\dots$, where $\bar{R}_j$ are tori of self-intersection -1. Moreover, the Seiberg-Witten invariants of the basic class β_m of $Z(3,m)$ corresponding to the canonical class $K_{Z(3)_0}$ evaluates as $SW_{Z(3)}(\beta_m)=SW_{Z(3)}(K_{Z(3)})+(m-1)SW_{Z(3)_0}(K_{Z(3)_0})=1+(m-1)=m$. Thus, we conclude that $Z(3,m)$ is nonsymplectic for any $m\geq 2$.

Furthemore, by applying Theorem 1.5.15, and then Theorem 1.5.16 to symplectic 4-manifold $Z(3)$, we obtain infinitely many minimal symplectic 4-manifolds and infinitely many non-symplectic 4-manifolds that is homeomorphic but not diffeomorphic to $(2n-1)\mathbb{CP}^2\#(2n-2)\overline{\mathbb{CP}}^2$ for any integer $n\geq 14$. This concludes the proof of our theorem.

$\square$

2.5.2 Exotic copies of $23\mathbb{CP}^2\#23\overline{\mathbb{CP}}^2$

Now we will prove the following theorem which is the case $n=12$ of Theorem 2.1.1 above. The proof is similar to the proof of previous theorem, thereby we will omit some details.

Theorem 2.5.4. *([19]) Let M be $23\mathbb{CP}^2\#23\overline{\mathbb{CP}}^2$. There exists an irreducible symplectic 4-manifold and an infinite family of pairwise non-diffemorphic irreducible non-symplectic 4-manifolds that all of which are homeomorphic to M.*

Proof. ([19]) The first building block again will be the complex surface $S\#\overline{\mathbb{CP}}^2$ endowed $S\#\overline{\mathbb{CP}}^2$ with the symplectic structure induced from the Kähler structure, along with the genus 6 complex submanifold $\widetilde{R} \subset S\#\overline{\mathbb{CP}}^2$, that was constructed in Section 2.3.

Our second building block will be obtained from the symplectic 4-manifold $X_{2,4}$ by blowing up a symplectic submanifold of it twice. Indeed, we will build a symplectic genus 6 surface of self intersection 0 inside $X_{2,4}\#2\overline{\mathbb{CP}}^2$. Recall from Theorem 2.4.3 that $X_{2,4}$ contains symplectic surface Σ_2 with self intersection 0 and two genus 2 surfaces, say S_1 and S_2, with self intersections -1. Moreover, S_1 and S_2 intersect with Σ_2 positively and transversally.

By symplectically resolving the intersections of Σ_2 with S_1 and Σ_2 with S_2, we obtain the genus six symplectic surface Σ_6' of square $+2$ in $X_{2,4}$. We symplectically blow-up Σ_6' at two points and hence obtain a symplectic surface Σ_6'' of self intersection 0 in $X_{2,4}\#2\overline{\mathbb{CP}}^2$ (see Figure 2.5).

By forming the symplectic connected sum of $S\#\overline{\mathbb{CP}}^2$ and $X_{2,4}\#2\overline{\mathbb{CP}}^2$ along the surfaces $\widetilde{R}$ and Σ_6'', we obtain a symplectic 4-manifold that we denote by $Z(2)$:

$$Z(2) := (S\#\overline{\mathbb{CP}}^2)\#_{\widetilde{R}=\Sigma_6''}X_{2,4}\#2\overline{\mathbb{CP}}^2$$

It follows from Gompf's theorem in [46] that $Z(2)$ is symplectic.

Lemma 2.5.5. *$Z(2)$ is simply-connected.*

Proof. This follows from Van Kampen's Theorem. Notice that we have

$$\pi_1(Z(2)) \;=\; \frac{\pi_1(S\#\overline{\mathbb{CP}}^2 \setminus \nu\widetilde{R}) * \pi_1(X_{2,4}\#2\overline{\mathbb{CP}}^2 \setminus \nu\Sigma_6'')}{\langle a_1 = a_1'',\, b_1 = b_1'',\, \cdots,\, a_6 = a_6'',\, b_6 = b_6'',\, \mu = \mu'' = 1\rangle}.$$

where a_i, b_i, and a_i'', b_i'' (for $i = 1, 2, 3$) denote the standard generators of the fundamental group of the genus 6 Riemann surfaces $\widetilde{R}$ and Σ_6'' in $S\#\overline{\mathbb{CP}}^2$ and in $X_{2,4}\#2\overline{\mathbb{CP}}^2$, and μ and μ'' denote their meridians respectively.

By applying the Proposition 2.3.7 **3.**, and the facts that the normal circle μ of $\widetilde{R}$ in $\pi_1(S\#\overline{\mathbb{CP}}^2 \setminus \nu\widetilde{R})$ and the loops a_1'', b_1'', $\cdots$, a_6'', b_6'', and μ'' in $\pi_1(X_{2,4}\#2\overline{\mathbb{CP}}^2 \setminus \nu\Sigma_6'')$ are all trivial, we conclude that the fundamental group of $Z(2)$ is trivial.

□

Lemma 2.5.6. *$e(Z(2)) = 48$, $\sigma(Z(2)) = 0$, $c_1^2(Z(2)) = 96$, and $\chi(Z(2)) = 12$.*

Proof. Using the formulas 1.4 and 1.5, we have

$$\begin{aligned} e(Z(2)) &= e(S\#\overline{\mathbb{CP}}^2) + e(X_{2,4}\#2\overline{\mathbb{CP}}^2) + 4(6-1), \\ \sigma(Z(2)) &= \sigma(S\#\overline{\mathbb{CP}}^2) + \sigma(X_{2,4}\#2\overline{\mathbb{CP}}^2), \\ c_1^2(Z(2)) &= c_1^2(S\#\overline{\mathbb{CP}}^2) + c_1^2(X_{2,4}\#2\overline{\mathbb{CP}}^2) + 8(6-1), \\ \chi(Z(2)) &= \chi(S\#\overline{\mathbb{CP}}^2) + \chi(X_{2,4}\#2\overline{\mathbb{CP}}^2) + (6-1). \end{aligned}$$

Since we have

$$e(X_{2,4}\#2\overline{\mathbb{CP}}^2) = 12,\ \sigma(X_{2,4}\#2\overline{\mathbb{CP}}^2) = -4,\ c_1^2(X_{2,4}\#2\overline{\mathbb{CP}}^2) = 16,\ \chi(X_{2,4}\#2\overline{\mathbb{CP}}^2) = 2$$

and

$$e(S\#\overline{\mathbb{CP}}^2) = 16,\ \sigma(S\#\overline{\mathbb{CP}}^2) = 4,\ c_1^2(S\#\overline{\mathbb{CP}}^2) = 44,\ \chi(S\#\overline{\mathbb{CP}}^2) = 5,$$

the proof of lemma follows.

□

Now by the lemmas above, Freedman's classification theorem for simply-connected 4-manifolds [42], and the fact that $Z(2)$ contains -1 genus two surface resulting from internal sum, we see that $Z(2)$ is homeomorphic to $23\mathbb{CP}^2\#23\overline{\mathbb{CP}}^2$.

To show that $Z(2)$ is not diffeomorphic to $23\mathbb{CP}^2\#23\overline{\mathbb{CP}}^2$, we again use Taubes' theorem. Since $Z(2)$ is symplectic and thus it has non-trivial Seiberg-Witten invariants by Taubes' theorem [120] (cf. Nonvanishing Theorems 1.4.4 above); $Z(2)$ is an exotic copy of $23\mathbb{CP}^2\#23\overline{\mathbb{CP}}^2$.

To produce an infinite family of exotic $23\mathbb{CP}^2\#23\overline{\mathbb{CP}}^2$'s, we need to replace the building block $Y_2(1,1)$ used in our construction of $X_{2,4}$ above with $Y_2(1,m)$, where $|m| > 1$. The proof of the rest of the theorem is identical to that of Theorem 2.5.1, and therefore we will skip the details.

□

2.6 Constructions of Exotic 4-Manifolds with Positive Signatures

In this section, by following [19], we will construct the families of simply connected non-spin symplectic and nonsymplectic (but smooth) 4-manifolds with positive signatures such that they have small Euler characteristics and small signatures. Our construction will prove the second main theorem (Theorem 2.1.2) stated in Section 2.1. We will first prove the Theorem 2.1.2 in special cases of (i)-(iii), and then derive the general cases using the Theorems 1.5.15, 1.5.16, and Corollary 1.5.17.

2.6.1 Exotic copies of $(2n-1)\mathbb{CP}^2\#(2n-2)\overline{\mathbb{CP}}^2$, for $n \geq 14$

We will prove the existence of an infinite family of irreducible symplectic 4-manifolds and an infinite family of irreducible non-symplectic 4-manifolds that are exotic copies of $(2n-1)\mathbb{CP}^2\#(2n-2)\overline{\mathbb{CP}}^2$ for any integer $n \geq 14$ (i.e. the case (i) of Theorem 2.1.2). We note that the signature of these manifolds is one. Let us begin with the construction of an exotic copy of $27\mathbb{CP}^2\#26\overline{\mathbb{CP}}^2$, which is the first step ($n = 14$) of the signature equal to one case.

The first building block is the complex surface $S\#\overline{\mathbb{CP}}^2$ which contains the genus 6 symplectic surface $\widetilde{R}$ constructed in Section 2.3.

The second building block is obtained from the symplectic 4-manifold $X_{4,6}$, in the notation of Theorem 2.4.3. The manifold $X_{4,6}$ contains a symplectic genus two surface Σ_2 with self-intersection 0 and two genus 4 symplectic surfaces with self intersections -1 intersecting Σ_2 positively and transversally. For the convenience of the reader, we will review the construction of $X_{4,6}$ (see [7] for the details). We take symplectic $T^2 \times T^2$ that is equipped with the product symplectic form. In $T^2 \times T^2$, we take a copy of $T^2 \times \{pt\}$ and $\{pt\} \times T^2$, and symplectically resolve the intersection point of these dual symplectic tori. As a reult of the resolution we attain a symplectic genus two surface of self intersection $+2$ in $T^2 \times T^2$. Next, we symplectically blow up this surface twice. Thus, we obtain a symplectic genus 2 surface Σ_2 with self-intersection 0, with two -1 spheres intersecting it positively and transversally in $T^4\#2\overline{\mathbb{CP}}^2$. The two -1 spheres are the exceptional spheres resulting from the blow-ups.

Then, we take the symplectic connected sum of $T^4\#2\overline{\mathbb{CP}}^2$ with $\Sigma_2 \times \Sigma_4$ along the genus two surfaces Σ_2 and $\Sigma_2 \times \{pt\}$. By performing a sequence of appropriate ± 1 Luttinger surgeries on $(T^4\#2\overline{\mathbb{CP}}^2)\#_{\Sigma_2=\Sigma_2\times\{pt\}}(\Sigma_2 \times \Sigma_4)$, we obtain the symplectic 4-manifold $X_{4,6}$ constructed in [7] (see Theorem 5.1, page 14). $X_{4,6}$ is an exotic copy of $7\mathbb{CP}^2\#9\overline{\mathbb{CP}}^2$ (see the Figure 2.5).

From the internal sum of the punctured exceptional spheres in $T^4\#2\overline{\mathbb{CP}}^2 \setminus \nu(\Sigma_2)$ and the punctured genus four surfaces in $\Sigma_2 \times \Sigma_4 \setminus \nu(\Sigma_2 \times \{pt\})$, there are two genus 4 surfaces S_1 and S_2 with self intersections -1, in $X_{4,6}$. Moreover, from the construction we see that $X_{4,6}$ contains symplectic surface Σ_2 with self intersection 0, and the surfaces S_1 and S_2 have positive and transverse intersections with Σ_2 (see the Figure 2.5).

Now we resolve the intersection of Σ_2 and one of the genus 4 surfaces, say S_1 in $X_{4,6}$ symplectically. This produces a genus six surface Σ_6' of square $+1$ and it intersects the other genus 4 surface S_2 of self-intersection -1. We then blow-up Σ_6' at a point to obtain a symplectic surface Σ_6 of self intersection 0 in $X_{4,6}\#\overline{\mathbb{CP}}^2$ (see Figure 2.5).

As we have that each of our two symplectic building blocks $S\#\overline{\mathbb{CP}}^2$ and $X_{4,6}\#\overline{\mathbb{CP}}^2$ contain symplectic genus 6 surfaces of self intersections 0, we can form their symplectic connected sum along these surfaces $\widetilde{R}$ and Σ_6. Let

$$M_{1,4} = (S\#\overline{\mathbb{CP}}^2)\#_{\widetilde{R}=\Sigma_6}(X_{4,6}\#\overline{\mathbb{CP}}^2).$$

Lemma 2.6.1. *$e(M_{1,4}) = 55$, $\sigma(M_{1,4}) = 1$, $c_1^2(M_{1,4}) = 113$, $\chi(M_{1,4}) = 14$.*

Proof. We will use the topological invariants of $X_{4,6}$ and $S\#\overline{\mathbb{CP}}^2$ to compute the topological invariants of $M_{1,4}$. Since

$$e(S) = 15,\ \sigma(S) = 5,\ c_1^2(S) = 45,\ \chi(S) = 5, \tag{2.14}$$

we have

$$e(S\#\overline{\mathbb{CP}}^2) = 16,\ \sigma(S\#\overline{\mathbb{CP}}^2) = 4,\ c_1^2(S\#\overline{\mathbb{CP}}^2) = 44,\ \chi(S\#\overline{\mathbb{CP}}^2) = 5. \tag{2.15}$$

On the other hand, by Theorem 2.4.3, we have

$$e(X_{4,6}) = 18,\ \sigma(X_{4,6}) = -2,\ c_1^2(X_{4,6}) = 30,\ \chi(X_{4,6}) = 4. \tag{2.16}$$

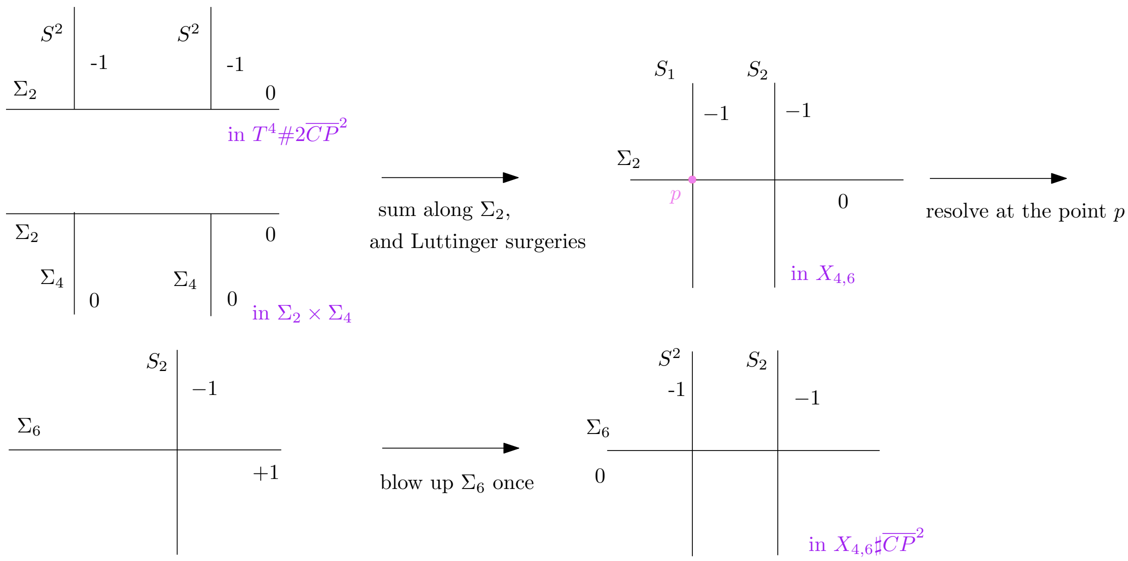


Figure 2.5: Illustration showing steps for the signature equals one case.

Thus, we have

$$e(X_{4,6}\#\overline{\mathbb{CP}}^2) = 19,\ \sigma(X_{4,6}\#\overline{\mathbb{CP}}^2) = -3,\ c_1^2(X_{4,6}\#\overline{\mathbb{CP}}^2) = 29,\ \chi(X_{4,6}\#\overline{\mathbb{CP}}^2) = 4. \tag{2.17}$$

Now using the formulas 1.4 and 1.5 for symplectic connected sum, we compute the topological invariants of $M_{1,4}$ as given above.

□

Similary as in the signature zero case in Section 2.5, we have that $M_{1,4}$ is symplectic by Gompf's Theorem 1.5.2, and simply connected by Van Kampen's Theorem. Using the same lines of arguments as in Section 2.5, we see that $M_{1,4}$ is an exotic copy of $27\mathbb{CP}^2\#26\overline{\mathbb{CP}}^2$.

Infinitely many copies

Next let us show that there are infinitely many pairwise non-diffeomorphic 4-manifolds, either symplectic or nonsymplectic and all are homeomorphic to $M_{1,4}$.

First, we note that in one of our building blocks $X_{4,6}$ there are at least two pairs of Lagrangian tori in $\Sigma_2 \times \Sigma_4$ that were away from the standard symplectic surfaces $\Sigma_2 \times \{pt\}$ and $\{pt\} \times \Sigma_4$ used in the construction, and the Lagrangian tori that were used for Luttinger surgeries (for an explanation, see Section 1.5.3). Thereby, $X_{4,6}$ contains a pair of disjoint Lagrangian tori T_1 and T_2 which descend from $\Sigma_2 \times \Sigma_4$, and survive in $X_{4,6}$ after symplectic connected sum and the Luttinger surgeries. In addition we have that $\pi_1(X_{4,6} \setminus (T_1 \cup T_2)) = 1$ ([12], Theorem 8).

Hence in turn $M_{1,4}$ also contains a pair of disjoint Lagrangian tori T_1 and T_2 of self-intersection 0 such that $\pi_1(M_{1,4} \setminus (T_1 \cup T_2)) = 1$, satisfying the properties of the Corollary 1.5.17.

We can perturb the symplectic form on $M_{1,4}$ in such a way that one of the tori, say T_1, becomes symplectically embedded. The reader is refered to [46] for the existence of such perturbation. Indeed, $M_{1,4}$ is closed, simply connected, minimal, symplectic manifold and $b_2^+ > 1$, hence such perturbation is possible by the Lemma 1.6 in [46]. Now we perform a knot surgery, (using a knot K with non-trivial Alexander polynomial) on $M_{1,4}$ along the symplectically embedded T_1 to obtain irreducible 4-manifold $(M_{1,4})_K$ that is homeomorphic but not diffemorphic to $M_{1,4}$:

$$(M_{1,4})_K = (M_{1,4} \setminus (T_1 \times D^2)) \cup (S^1 \times (S^3 \setminus N(K))$$

where $N(K)$ is the tubular neighborhood of the knot K in S^3, and $T_1 \times D^2$ is the tubular neighborhood of T_1 in $M_{1,4}$. By varying our choice of K, we can realize infinitely many pairwise non-diffeomorphic 4-manifolds, either symplectic or non-symplectic (see Theorem 1.5.16). In fact, choosing knots with non-monic Alexander polynomials gives non-symplectic 4-manifolds. This finishes $n = 14$ case for $(2n-1)\mathbb{CP}^2\#(2n-2)\overline{\mathbb{CP}}^2$.

Finally, for the $n > 14$ case, we apply Theorems 1.5.15, 1.5.16, and Corollary 1.5.17, and build infinitely many irreducible symplectic and infinitely many irreducible non-symplectic 4-manifolds that is homeomorphic but not diffeomorphic to $(2n-1)\mathbb{CP}^2\#(2n-$

$2)\overline{\mathbb{CP}}^2$ for any integer $n > 14$.

$\square$

2.6.2 Exotic copies of $(2n-1)\mathbb{CP}^2\#(2n-3)\overline{\mathbb{CP}}^2$, for $n \geq 13$

Now we will construct an infinite family of irreducible symplectic 4-manifolds and an infinite family of irreducible non-symplectic 4-manifolds that are homeomorphic but not diffeomorphic to $(2n-1)\mathbb{CP}^2\#(2n-3)\overline{\mathbb{CP}}^2$ for any integer $n \geq 13$ (part (ii) of Theorem 2.1.2). We note that these manifolds have signature two. The construction in this case is similar to that of the previous case ($\sigma = 1$) above, therefore we will omit some of the familiar details discussed above. We will first construct an exotic copy of $25\mathbb{CP}^2\#23\overline{\mathbb{CP}}^2$, and use the Theorems 1.5.15 and 1.5.16 and Corollary 1.5.17 to deduce the general case.

As the first building block we again take $S\#\overline{\mathbb{CP}}^2$, containing genus 6 surface $\widetilde{R}$ of square 0.

To obtain the second symplectic building block, we form the symplectic connected sum of $T^4\#2\overline{\mathbb{CP}}^2$ with $\Sigma_2 \times \Sigma_5$ along the genus two surfaces Σ_2 and $\Sigma_2 \times \{pt\}$. Let

$$X_{5,7} = (T^4\#2\overline{\mathbb{CP}}^2)\#_{\Sigma_2=\Sigma_2\times\{pt\}}(\Sigma_2 \times \Sigma_5).$$

The manifold $X_{5,7}$ is an exotic copy of $9\mathbb{CP}^2\#11\overline{\mathbb{CP}}^2$ ([7], Theorem 5.1). What is more, in the construction of $X_{5,7}$ we take the internal sum of a punctured genus one surface in $T^4\#2\overline{\mathbb{CP}}^2 \setminus \nu(\Sigma_2)$ and a punctured genus five surface Σ_5 in $\Sigma_2 \times \Sigma_5 \setminus \nu(\Sigma_2 \times \{pt\})$. Therefore we easily see that $X_{5,7}$ contains a symplectic genus 6 surface Σ_6 of square 0 (see Figure 2.6).

Next, let us take the symplectic connected sum of $S\#\overline{\mathbb{CP}}^2$ and $X_{5,7}$ along the genus six surfaces $\widetilde{R}$ and Σ_6

$$M_{2,5} = (S\#\overline{\mathbb{CP}}^2)\#_{\widetilde{R}=\Sigma_6}X_{5,7}.$$

along the copies of Σ_6 in both of the 4-manifolds. It is easy to check as above that

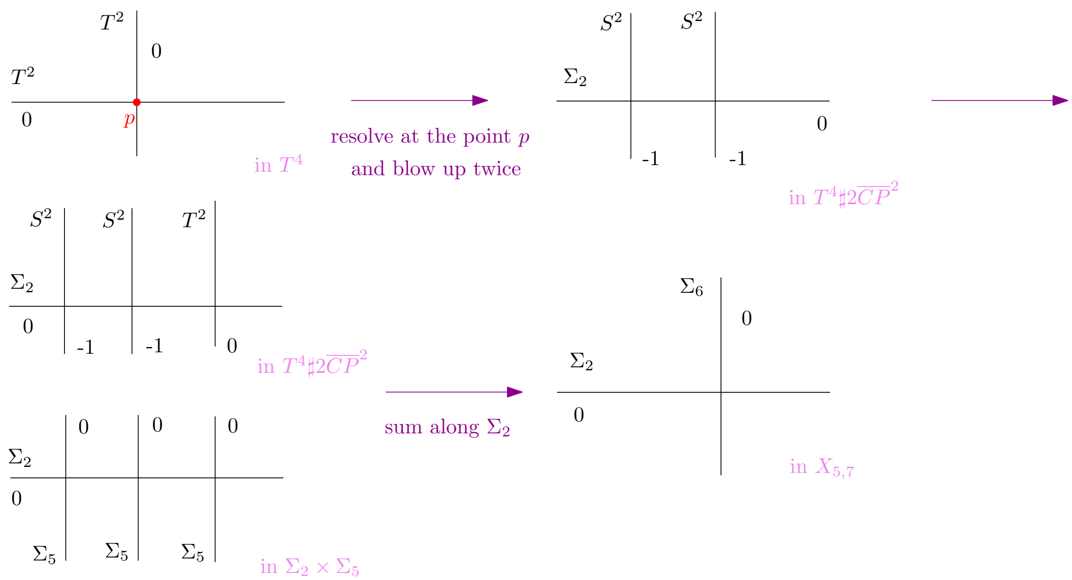


Figure 2.6: Illustration showing steps for the signature equals two case.

the following lemma holds

Lemma 2.6.2. $e(M_{2,5}) = 50$, $\sigma(M_{2,5}) = 2$, $c_1^2(M_{2,5}) = 106$, $\chi(M_{2,5}) = 13$.

We conclude, as above, $M_{2,5}$ is symplectic and simply connected and an exotic copy of $25\mathbb{CP}^2\#23\overline{\mathbb{CP}}^2$.

Lastly, once again by applying Theorems 1.5.15 and 1.5.16, and Corollary 1.5.17, we obtain infinitely many minimal symplectic 4-manifolds and an infinitely many non-symplectic 4-manifolds that is homeomorphic but not diffeomorphic to $(2n-1)\mathbb{CP}^2\#(2n-3)\overline{\mathbb{CP}}^2$ for any integer $n \geq 13$.

□

2.6.3 Exotic copies of $(2n-1)\mathbb{CP}^2\#(2n-4)\overline{\mathbb{CP}}^2$, for $n \geq 15$

Lastly, let us prove the existence of an infinite family of irreducible symplectic 4-manifolds and an infinite family of irreducible non-symplectic 4-manifolds that are homeomorphic but not diffeomorphic to $(2n-1)\mathbb{CP}^2\#(2n-4)\overline{\mathbb{CP}}^2$ for any integer $n \geq 15$ (part (iii) of Theorem 2.1.2). Thus, in what follows, we will construct simply connected nonspin irreducible symplectic and smooth 4-manifolds with signature 3. We will first consider the case $n = 15$, thus we will construct infinitely many exotic copies of $29\mathbb{CP}^2\#26\overline{\mathbb{CP}}^2$. The general case again will be proved by appealing to Theorems 1.5.15, 1.5.16, and Corollary 1.5.17.

The first building block is the surface $S\#\overline{\mathbb{CP}}^2$ presented above (in Section 2.3), and the second building block is the symplectic 4-manifold $X_{5,6}$, an exotic $9\mathbb{CP}^2\#10\overline{\mathbb{CP}}^2$ constructed in [11]. Let us describe our second building block $X_{5,6}$ here. In fact $X_{5,6}$ also contains an embedded symplectic genus 6 submanifold $\widetilde{R}$ with self-intersection zero, therefore we will be able to take the symplectic connected sum of $\widehat{S}$ and $X_{5,6}$ to obtain our new exotic symplectic and smooth 4-manifolds.

Before going into details of constructions of $X_{5,6}$, we will recall the following fact from Chapter 1. For X a closed, oriented, smooth 4-manifold, every element of $H_2(X,\mathbb{Z})$ can be represented by an embedded surface which means that for $\alpha \in H_2(X,\mathbb{Z})$ there is a closed oriented surface Σ and an embedding $i : \Sigma \hookrightarrow X$ such that $i_*([\Sigma]) = \alpha$, where $[\Sigma]$ is the fundamental class of Σ. ([48])

Now let us take $T^2 \times T^2$ equipped with the product symplectic form. Next, take a copy of $T^2 \times \{pt\}$ and the braided torus T_β which are symplectically embedded in $T^2 \times T^2$. Indeed, the braided torus T_β is also a symplectic submanifold ([11]) and represents the homology class $2[\{pt\} \times T^2]$ in $H_2(T^2 \times T^2, \mathbb{Z})$. Let us summarize the construction of T_β.

We take a smooth, simple closed curve $\beta \subset T^2 \times S^1$ whose homology class is $2[pt \times S^1] \in H_1(T^2 \times S^1, \mathbb{Z})$. Then we take $\beta \times S^1 \subset (T^2 \times S^1) \times S^1 = T^4$. The torus $\beta \times S^1$ is embedded in T^4 and the former is a symplectic submanifold of the latter ([11]). We call this torus T_β.

The symplectic tori $T^2 \times \{pt\}$ and T_β intersect at two points. We symplectically blow-up one of these intersection points, and symplectically resolve the other intersection point to obtain the symplectic genus two surface of self intersection 0 in $T^4\#\overline{\mathbb{CP}}^2$. The symplectic genus 2 surface Σ_2 has a dual symplectic torus T^2 of self intersections zero intersecting Σ_2 positively and transversally at one point. We form the symplectic connected sum of $T^4 \# \overline{\mathbb{CP}}^2$ with $\Sigma_2 \times \Sigma_5$ along the genus two surfaces Σ_2 and $\Sigma_2 \times \{pt\}$. By performing the sequence of appropriate ± 1 Luttinger surgeries on $(T^4 \# \overline{\mathbb{CP}}^2) \#_{\Sigma_2 = \Sigma_2 \times \{pt\}} (\Sigma_2 \times \Sigma_5)$, we obtain the symplectic 4-manifold $X_{5,6}$ constructed in [11] (see the Figure 2.7).

In the construction, resulting from the internal sum of the punctured torus in $T^4 \# \overline{\mathbb{CP}}^2 \setminus \nu(\Sigma_2)$ and one of the punctured genus five surfaces in $\Sigma_2 \times \Sigma_5 \setminus \nu(\Sigma_2 \times \{pt\})$, we see that $X_{5,6}$ contains a symplectic surface Σ_6 with self intersection zero (Figure 2.7).

Furthemore, we note that $X_{5,6}$ contains a pair of disjoint Lagrangian tori T_1 and T_2 of self-intersections 0 such that $\pi_1(X_{5,6} \setminus (T_1 \cup T_2)) = 1$. These Lagrangian tori descend from $\Sigma_2 \times \Sigma_5$ and survive in $X_{5,6}$ after symplectic connected sum and the Luttinger surgeries and will be used to obtain infinitely many irreducible symplectic and non-symplectic 4-manifolds at the end (cf. Corollary 1.5.17).

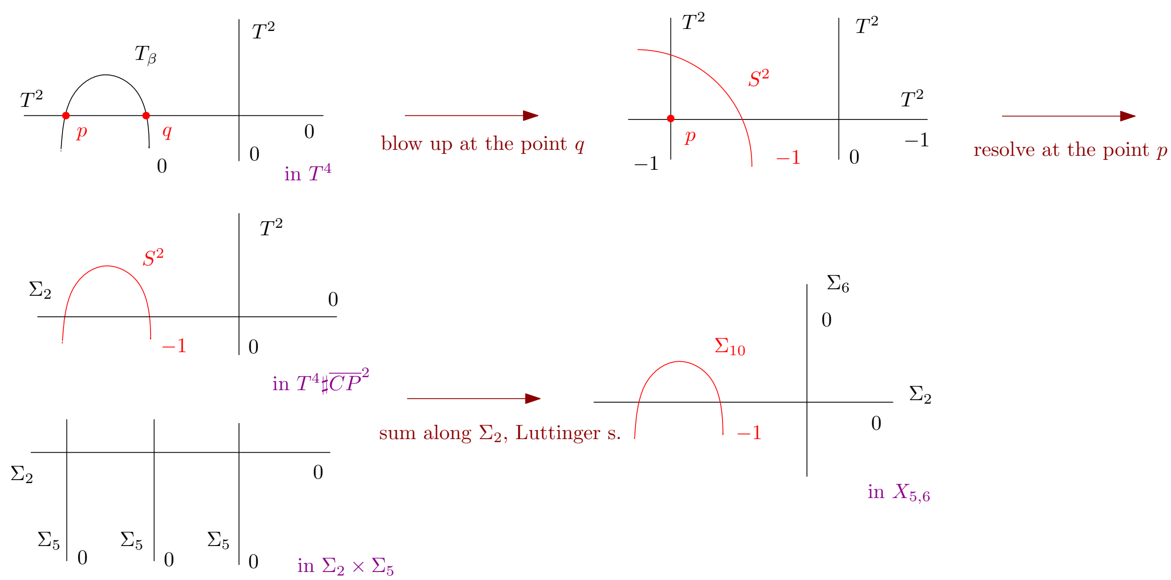


Figure 2.7: Illustration showing steps for the signature equals three case.

Hence, as we have the two pieces, we can form their symplectic connected sum along the genus 6 surfaces they contain. Let us define

$$M_{3,5} = (\widehat{S})\#_{\widetilde{R}=\Sigma_6}(X_{5,6}).$$

Now we will prove that $M_{3,5}$ is an exotic copy of $29\mathbb{CP}^2\#26\overline{\mathbb{CP}}^2$, which is smoothly minimal. First, as the two components of $M_{3,5}$ are symplectic, $M_{3,5}$ is also symplectic [46]. Next, we compute the topological invariants of $X_{5,6}$ and $M_{3,5}$ using the formulas 1.4 and 1.5 above. We obtain

Lemma 2.6.3. *$e(M_{3,5}) = 57$, $\sigma(M_{3,5}) = 3$, $c_1^2(M_{3,5}) = 123$, $\chi(M_{3,5}) = 15$.*

□

Proof. Firstly, we compute the topological invariants $X_{5,6}$. Notice that

$$e(T^4\#\overline{\mathbb{CP}}^2) = 1,\ \sigma(T^4\#\overline{\mathbb{CP}}^2) = -1,\ c_1^2(T^4\#\overline{\mathbb{CP}}^2) = -1,\ \chi(T^4\#\overline{\mathbb{CP}}^2) = 0. \tag{2.18}$$

For $\Sigma_2 \times \Sigma_5$, we have

$$e(\Sigma_2 \times \Sigma_5) = 16,\ \sigma(\Sigma_2 \times \Sigma_5) = 0,\ c_1^2(\Sigma_2 \times \Sigma_5) = 32,\ \chi(\Sigma_2 \times \Sigma_5) = 4. \tag{2.19}$$

Therefore, for the symplectic connected sum manifold $X_{5,6}$, we have

$$e(X_{5,6}) = 21,\ \sigma(X_{5,6}) = -1,\ c_1^2(X_{5,6}) = 39,\ \chi(X_{5,6}) = 5. \tag{2.20}$$

Now, as we know the invariants of $S\#\overline{\mathbb{CP}}^2$ and $X_{5,6}$, we compute the topological invariants of $M_{3,5}$ as above using the formulas 1.4 and 1.5.

□

In addition, by Seifert-Van Kampen Theorem we find that

Lemma 2.6.4. *$M_{3,5}$ is simply-connected.*

Proof. By applying the Seifert-Van Kampen theorem, we see that

$$\pi_1(M_{3,5}) \;=\; \frac{\pi_1(\widehat{S} \setminus \nu\widetilde{R}) * \pi_1(X_{5,6} \setminus \nu\Sigma_6)}{\langle a_1 = a_1',\, b_1 = b_1',\, \cdots,\, a_6 = a_6',\, b_6 = b_6',\, \mu = \mu' = 1\rangle}.$$

where a_i, b_i, and a'_i, b'_i (for $i = 1, \cdots, 6$) denote the standard generators of the fundamental group of the genus 6 Riemann surfaces $\widetilde{R}$ and Σ_6 in $S\#\overline{\mathbb{CP}}^2$ and in $X_{5,6}$, and μ and μ' denote their meridians in $S\#\overline{\mathbb{CP}}^2 \setminus \nu\widetilde{R}$ and in $X_{5,6} \setminus \nu\Sigma_6$ respectively. Using the Proposition 2.3.7 (iii), and the facts that the normal circle $\mu = \{\mathrm{pt}\} \times S^1$ of $\widetilde{R}$ in $\pi_1(S\#\overline{\mathbb{CP}}^2 \setminus \nu(\widetilde{R}))$ and the loops a'_1, b'_1, $\cdots$, a'_6, b'_6 in $\pi_1(X_{5,6} \setminus \nu(\Sigma_6))$ are all trivial, we see that the fundamental group of $M_{3,5}$ is the trivial group.

□

Using Freedman's classification theorem for simply-connected 4-manifolds [42], the lemma above and the fact that $S\#\overline{\mathbb{CP}}^2$ contains genus two surface of self-intersection -1 disjoint from $\widetilde{R}$, we conclude that $M_{3,5}$ is homeomorphic to $(2n-1)\mathbb{CP}^2\#(2n-4)\overline{\mathbb{CP}}^2$ for $n = 15$. Since $M_{3,5}$ is symplectic, by Taubes' theorem [120] $M_{3,5}$ has non-trivial Seiberg-Witten invariant (see the Nonvanishing Theorems 1.4.4 above). Next, using the connected sum theorem for the Seiberg-Witten invariant (see Theorem 1.4.3), we deduce that the Seiberg-Witten invariant of $29\mathbb{CP}^2\#26\overline{\mathbb{CP}}^2$ is trivial. Since the Seiberg-Witten invariant is a diffeomorphism invariant, $M_{3,5}$ is not diffeomorphic to $29\mathbb{CP}^2\#26\overline{\mathbb{CP}}^2$. Furthermore, $M_{3,5}$ is a minimal symplectic 4-manifold by Usher's Minimality Theorem ([124], Theorem 1.1, iii)) (Theorem 1.5.4 above). Since symplectic minimality implies smooth minimality (cf. [82]), $M_{3,5}$ is smoothly minimal, too. In addition, $M_{3,5}$ is simply connected, thus we conclude that $M_{3,5}$ is smoothly irreducible.

Infinitely many copies

To obtain infinitely many copies of $M_{3,5}$, we proceed as follows. As explained above, $M_{3,5}$ contains a pair of disjoint Lagrangian tori T_1 and T_2 of self-intersection 0 such that $\pi_1(M_{3,5} \setminus (T_1 \cup T_2)) = 1$. We can perturb the symplectic form on $M_{3,5}$ in such a way that one of the tori, say T_1, becomes symplectically embedded. Indeed, $M_{3,5}$ is closed, simply connected, minimal, symplectic and $b_2^+ > 1$, hence such perturbation is possible by the Lemma 1.6 in [46]. Therefore we can perform a knot surgery, (using a knot K with non-trivial Alexander polynomial) on $M_{3,5}$ along the symplectically embedded T_1 to obtain an irreducible 4-manifold $(M_{3,5})_K$ that is homeomorphic but not diffemorphic to $M_{3,5}$:

$$(M_{3,5})_K = (M_{3,5} \setminus (T_1 \times D^2)) \cup (S^1 \times (S^3 \setminus N(K))$$

where $N(K)$ is the tubular neighborhood of the knot K in S^3, and $T_1 \times D^2$ is the tubular neighborhood of T_1 in $M_{3,5}$. By varying our choice of K, we can realize infinitely many pairwise non-diffeomorphic 4-manifolds, either symplectic or non-symplectic (see Theorem 1.5.16). In fact, choosing knots with non-monic Alexander polynomials gives non-symplectic 4-manifolds. This gives us infinitely many exotic $(2n-1)\mathbb{CP}^2\#(2n-4)\overline{\mathbb{CP}}^2$ for $n = 15$.

What is more, by applying Theorems 1.5.15, 1.5.16, and Corollary 1.5.17 above we also obtain infinitely many irreducible symplectic and infinitely many irreducible non-symplectic 4-manifolds that is homeomorphic but not diffeomorphic to $(2n-1)\mathbb{CP}^2\#(2n-4)\overline{\mathbb{CP}}^2$ for any integer $n > 15$. This finishes the proof of part (iii) of Theorem 2.1.2.

□

■

Chapter 3

Exotic Smooth Structures on Small 4-Manifolds via Deformation of Singular Fibers of Genus Two Fibrations

This chapter is devoted to the study of exotic smooth structures on 4-manifolds with small Euler characteristics. Let us give a brief history first. The first smooth exotic 4-manifolds were constructed by Donaldson in [33], where he proved that the Dolgachev surface $E(1)_{2,3}$, as a smooth manifold, is an exotic copy of the elliptic surface $E(1) = \mathbb{CP}^2 \# 9\,\overline{\mathbb{CP}}^2$. Then, in [43] infinitely many irreducible smooth structures on $E(1)$ were constructed. In 1989, Kotschick showed that the Barlow's surface is homemorphic but not diffemorphic to $\mathbb{CP}^2 \# 8\,\overline{\mathbb{CP}}^2$ ([79]).

In the last 15 years, constructions of exotic simply connected 4-manifolds with small Euler characteristics has been an active research area. The following papers are a few examples on this subject [103, 116, 41, 105, 117, 104, 3, 2, 10, 9, 7, 11, 90, 15, 72]. Let us summarize what are already known in this direction. The first known exotic smooth structure on $\mathbb{CP}^2 \# 7\,\overline{\mathbb{CP}}^2$ i.e. smooth 4-manifold homeomorphic but not diffeomophic to $\mathbb{CP}^2 \# 7\overline{\mathbb{CP}}^2$, was constructed in 2004 ([103]). In the construction, the elliptic surface

$E(1)$ with a certain elliptic fibration structure, and blow-up and rational blow-down operations were used. Subsequently, using the generalized rational blow-down, exotic copies of $\mathbb{CP}^2 \# 6\overline{\mathbb{CP}}^2$ were built in [116]. Then, in [41] a new technique, the knot surgery in double nodes, was introduced which gave rise to infinitely many distinct smooth structures on $\mathbb{CP}^2 \# k\,\overline{\mathbb{CP}}^2$, for $k = 6, 7, 8$. Also, by using [41] and the rational blow-down surgery, infinitely many exotic smooth structures on $\mathbb{CP}^2 \# 5\,\overline{\mathbb{CP}}^2$ were constructed ([105]). By using similar ideas, exotic smooth structures on $3\,\mathbb{CP}^2 \# k\,\overline{\mathbb{CP}}^2$ for $k = 9$, and $k = 8$ were given in [117] and [104], respectively. All these infinite family of exotic 4-manifolds were obtained from the elliptic surface $E(1)$ and $E(2) = E(1) \# E(1)$ (the fiber sum of 2 copies of $E(1)$), with certain elliptic fibration structures on them, by applying the combination of blow-ups, rational blow-down and knot surgery in double nodes. Similar results from the elliptic surfaces $E(n)$ for $n \geq 3$ were obtained in [1]. In addition, related recent works in [90, 72] again used elliptic fibrations on $E(1)$. One of the key ingredients in the above mentioned articles was the use of Kodaira's classification of the singular fibers in elliptic fibrations.

By using different methods, Akhmedov constructed the first symplectic exotic irreducible smooth structure on $\mathbb{CP}^2 \# 5\,\overline{\mathbb{CP}}^2$ and the first exotic irreducible smooth structure on $3\mathbb{CP}^2 \# 7\,\overline{\mathbb{CP}}^2$ in 2006 ([2]). In 2007, Akhmedov and Park built irreducible symplectic 4-manifolds that are exotic copies of $\mathbb{CP}^2 \# 3\,\overline{\mathbb{CP}}^2$, $3\mathbb{CP}^2 \# 5\,\overline{\mathbb{CP}}^2$ and $(2n-1)\mathbb{CP}^2 \# (2n+1)\,\overline{\mathbb{CP}}^2$ for any integer $n \geq 3$ ([10]). Then Akhmedov and Park constructed irreducible symplectic and infinitely many pairwise non-diffeomorphic irreducible non-symplectic 4-manifolds that are exotic copies of $\mathbb{CP}^2 \# 2\,\overline{\mathbb{CP}}^2$, $\mathbb{CP}^2 \# 4\,\overline{\mathbb{CP}}^2$, $3\mathbb{CP}^2 \# k\,\overline{\mathbb{CP}}^2$, $k = 4, 6, 8, 10$, and $(2n-1)\mathbb{CP}^2 \# (2n)\,\overline{\mathbb{CP}}^2$ for any integer $n \geq 3$ (2007, [11]).

In [20], we have constructed smooth and symplectic, simply connected 4-manifolds with small Euler characteristics, and with exotic structures. Our manifolds are exotic, minimal, symplectic 4-manifolds which are homeomorphic but not diffeomorphic to $\mathbb{CP}^2 \# 6\overline{\mathbb{CP}}^2$, $\mathbb{CP}^2 \# 7\overline{\mathbb{CP}}^2$, and $3\mathbb{CP}^2 \# k\overline{\mathbb{CP}}^2$ for $k = 16, 17, 18, 19$. The main differences of our work from the previous ones are as follows. We have used pencils of genus two curves on Hirzebruch surfaces and thus certain genus two Lefschetz fibration structures on $\mathbb{CP}^2 \# 13\,\overline{\mathbb{CP}}^2$ and also on $E(2) \# 2\,\overline{\mathbb{CP}}^2$. In addition, our work is the first which uses the complete classification of the singular fibers in fibrations of genus two curves over the 2-disk, given by Namikawa and Ueno in [93, 94]. In constructing such singular fibers, Namikawa and Ueno have used algebro-geometric and analytical methods. In [20] we have found specific Lefschetz pencils and from which we obtained certain types of the singular fibers given in [93, 94]. In other words, we have reconstructed the singular fibers from Lefschetz pencils, thus we have given topological constructions of the singular fibers of [93, 94]. In addition, in [20] we have introduced 2-nodal spherical deformation of certain singular fibers of genus 2 fibrations. Hence, by using the singular fibers we have reconstructed, 2-nodal spherical deformations, symplectic blow-ups, and (generalized) rational blow-down surgeries, we have constructed above mentioned exotic 4-manifolds with small topology.

Below we will provide minimal, symplectic 4-manifolds which are exotic copies of $\mathbb{CP}^2\#6\overline{\mathbb{CP}}^2$, $\mathbb{CP}^2\#7\overline{\mathbb{CP}}^2$, and $3\mathbb{CP}^2\#k\overline{\mathbb{CP}}^2$ for $k = 16, 17, 18, 19$. The outline for the remaining sections is as follows. In the next section we will review some background information on Hirzebruch surfaces [58]. In Section 3.2, we will discuss the classification of the singular fibers of genus two fibrations due to Namikawa and Ueno [93, 94], and then genus two pencils in the $K3$ surface, which is a simply connected complex surface with $c_1 = 0$, and diffeomorphic to $E(2)$. In Section 3.3 we will introduce a useful technique which we call *g-nodal spherical deformation* of the singular fibers of a genus $g \geq 2$ Lefschetz fibration over the 2-sphere S^2. Finally, in Section 3.4 we will prove our main theorems; Theorem 3.4.2, Theorem 3.4.5, Theorem 3.4.6, Theorem 3.4.7, Theorem 3.4.9, and Theorem 3.4.10.

3.1 Hirzebruch Surfaces

In this section, we review some basic facts and properties of the Hirzebruch surfaces $\mathbb{F}_n$ which will be needed in the sequel. More detail could be found in [58]. The complex surfaces $\mathbb{F}_n$, where $n \geq 0$, are the holomorphic $\mathbb{CP}^1$-bundles over $\mathbb{CP}^1$ with holomorphic sections of self-intersections $\pm n$. Hence for any $n \geq 0$, $\mathbb{F}_n$ has a structure of $\mathbb{CP}^1$ bundle (i.e., they are geometrically ruled complex surfaces). Conversely, any $\mathbb{CP}^1$ bundle over $\mathbb{CP}^1$ is isomorphic to $\mathbb{F}_n$ for some n. Let us remind that $\mathbb{F}_n$ is a minimal complex surface if and only if $n \neq 1$. Moreover we have $\mathbb{F}_0 = \mathbb{CP}^1 \times \mathbb{CP}^1 = S^2 \times S^2$, and $\mathbb{F}_1 = \mathbb{CP}^2\#\overline{\mathbb{CP}}^2 = S^2\tilde{\times}S^2$.

As smooth 4-manifolds, $\mathbb{F}_n$ is diffeomorphic to $\mathbb{F}_m$ if and only if $n \equiv m$ $(mod\ 2)$. However, the smooth 4-manifolds $S^2 \times S^2$ and $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ admit infinitely many inequivalent complex strucutres. Indeed, as complex manifolds $\mathbb{F}_n$ is complex diffeomorphic to $\mathbb{F}_m$ if and only if $n = m$ ([58]).

The n-th Hirzebruch surface $\mathbb{F}_n$ admits two disjoint holomorphic sections of self intersections n and $-n$. We denote them by C_∞ and C_0 respectively, and the fiber class of $\mathbb{F}_n$ by F. It is easy to verify that $C_0 = C_\infty - nF$. To determine the canonical class $K_{\mathbb{F}_n}$ of $\mathbb{F}_n$, we write $K_{\mathbb{F}_n}$ as $K_{\mathbb{F}_n} = aC_\infty + bF$. By applying the adjunction formula (Theorem 1.2.1 above) to the classes F and then C_∞, we find

$$K_{\mathbb{F}_n} = -2C_\infty + (n-2)F. \tag{3.1}$$

In particular, $K_{\mathbb{F}_2} = -2C_\infty$ and $K_{\mathbb{F}_3} = -2C_\infty + F$.

We know that $\mathbb{F}_2 \# \overline{\mathbb{CP}}^2 = S^2 \times S^2 \# \overline{\mathbb{CP}}^2$ is diffeomorphic to $S^2 \tilde{\times} S^2 \# \overline{\mathbb{CP}}^2 = \mathbb{CP}^2 \# 2\overline{\mathbb{CP}}^2$, which can be verified by applying the sequence of 2-handle moves as in Figure 3.1.

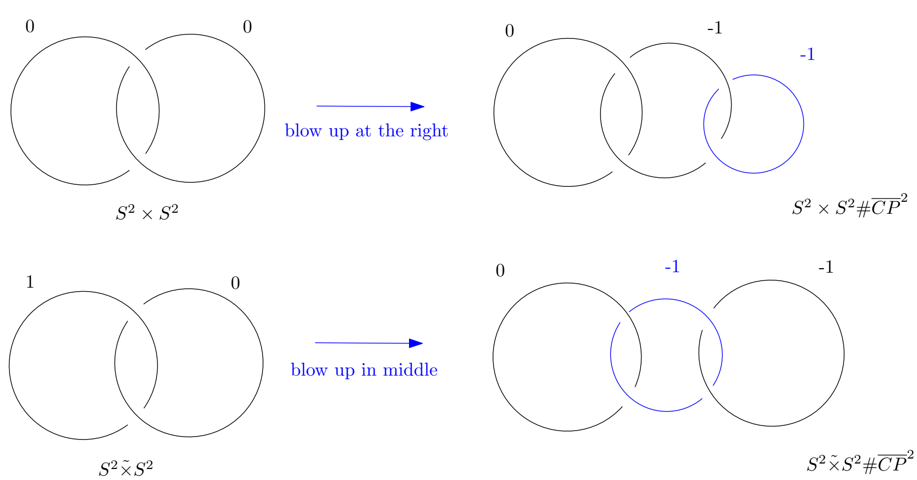


Figure 3.1: 2-handle moves

In what follows, we will write down some explicit classes that will be needed later on in our computations. In $\mathbb{F}_2\#\overline{\mathbb{CP}}^2$ let us take the classes F, C_∞, C_0 and the class e of the exceptional divisor coming from the blow-up, where $F^2 = 0, C_\infty^2 = 2, C_0^2 = -2, e^2 = -1$ and $C_\infty \cdot C_0 = 0, C_\infty \cdot F = 1, C_0 \cdot F = 1$. For computational purposes, we will write them in terms of the classes h, e_1 and e_2 of squares 1, -1 and -1 in the diffeomorphic manifold $\mathbb{CP}^2\#2\overline{\mathbb{CP}}^2$. First, the canonical class K of $\mathbb{CP}^2\#2\overline{\mathbb{CP}}^2$ is $-3h+e_1+e_2$, which follows from the blow-up formula (see Introduction). Let $F = ah+be_1+ce_2$. By solving the equations $F^2 = 0$, and the one coming from the adjunction equality for F (Theorem 1.2.1 above), we find a, b and c, which gives

$$F = h - e_1. \tag{3.2}$$

In the same way we find

$$C_\infty = 2h - e_1 - e_2, \;\; C_0 = e_1 - e_2, \;\; e = h - e_1 - e_2. \tag{3.3}$$

See also Figure 3.2 for obtaining these classes in $\mathbb{CP}^2\#2\overline{\mathbb{CP}}^2$. There we take a conic $2h$ in $\mathbb{CP}^2$ and a point p lying on $2h$. Next we consider the pencil of lines passing through p. After two consecutive blow ups we obtain above mentioned classes. Moreover, by blowing down $h - e_1 - e_2$ gives the Hirzebruch surface $\mathbb{F}_2$.

In the sequel, we will also use $\mathbb{F}_3 \cong \mathbb{CP}^2\#\overline{\mathbb{CP}}^2$. Similarly, we take the classes F, C_∞, C_0, where $F^2 = 0, C_\infty^2 = 3, C_0^2 = -3$ and $C_\infty \cdot C_0 = 0, C_\infty \cdot F = 1, C_0 \cdot F = 1$, and the classes h, e_1 of squares 1 and -1 in $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$. As above we find that

$$F = h - e_1, \;\; C_\infty = 2h - e_1, \;\; C_0 = 2e_1 - h. \tag{3.4}$$

3.1.1 More on Hirzebruch Surfaces

We will now discuss linear systems on the Hirzebruch surfaces and state two propositions that will be used in the following sections which are indeed one of the main ingredients in the proofs of our main theorems. The proofs of these propositions can be found in [54, 55] (see V - Corollary 2.18 in [54], and Lemma on page 865 in [113]).

First we recall some basic notions in algebraic geometry. A few definitions are in order. Let X be a noetherian integral separated scheme which is regular in codimension one.

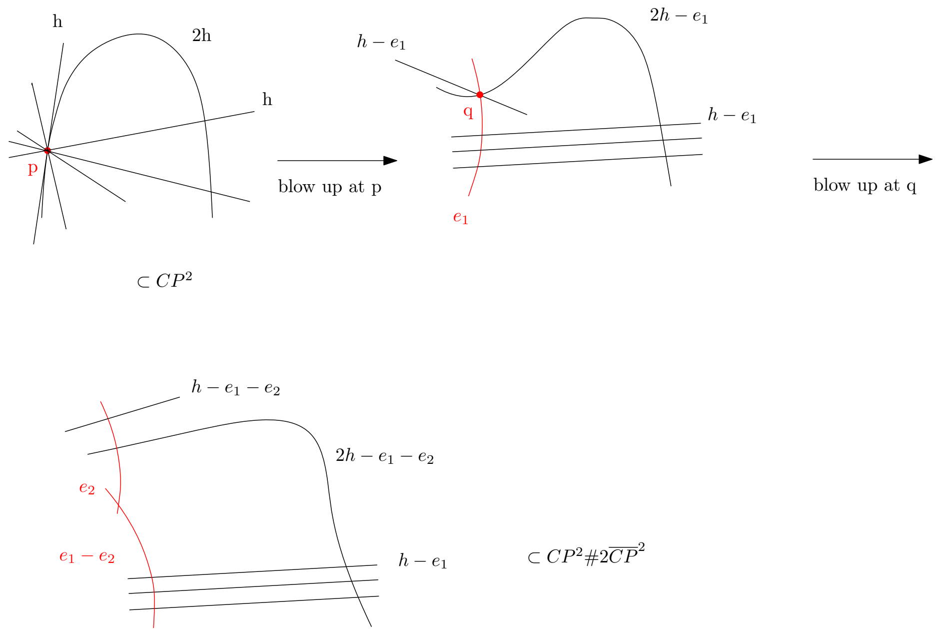


Figure 3.2: Configuration of spheres on the blow up of Hirzebruch surface

Definition 3.1.1. *A* prime divisor *on X is a closed integral subscheme Y of codimension one. The free abelian group generated by the prime divisors is denoted by $Div\, X$. A* Weil divisor $D = \sum n_i Y_i$ *is an element of $Div\, X$, where n_i are integers finitely many of which are nonzero, and Y_i are prime divisors. If all the $n_i \geq 0$, D is called* effective.

Two divisors D and D' are said to be linearly equivalent *if $D - D'$ is a principal divisor, i.e., $D - D'$ is the divisor of a nonzero rational function.*

Now let's let X be a nonsingular projective variety over an algebraically closed field k.

Definition 3.1.2. *A* complete linear system *on X is the (possibly empty) set of all effective divisors linearly equivalent to some given divisor D, and denoted by $|D|$. Indeed $|D|$ is a projective space ([54]).*

A linear system *on X is a subset of a complete linear system $|D|$ which is a projective subspace of $|D|$.*

Let us also recall very ample and ample divisors.

Definition 3.1.3. *Let X be a scheme over a scheme Y. An invertible sheaf $\mathscr{L}$ on X is* very ample *relative to Y, if there is an immersion $i : X \to \mathbb{P}^r_Y$ for some r such that $i^*(\mathscr{O}(1)) \cong \mathscr{L}$.*

Definition 3.1.4. *An invertible sheaf $\mathscr{L}$ on a noetherian scheme X is called* ample *if for every coherent sheaf $\mathscr{F}$ on X, there is an integer $n_{\mathscr{F}} > 0$ such that for every $n \geq n_{\mathscr{F}}$, the sheaf $\mathscr{F} \otimes \mathscr{L}^{\otimes n}$ is generated by its global sections.*

The relation between very ample and ample sheaves is as follows. If $\mathscr{L}$ is a very ample sheaf on a projective scheme over a noetherian ring, then $\mathscr{L}$ is ample. On the other hand, let X be a scheme of finite type over a noetherian ring A, and $\mathscr{L}$ be an invertible sheaf on X. If $\mathscr{L}$ is ample, then there is an $n_0 > 0$ such that $\mathscr{L}^{\otimes n}$ is very ample for all $n \geq n_0$ ([54]).

For any divisor D on a scheme X, there is an associated invertible sheaf $\mathscr{L}(D)$ on X (for more details see e.g. [54]). We say that the divisor D is *ample* or *very ample*, if the corresponding sheaf $\mathscr{L}(D)$ is.

Now we will consider linear systems on the Hirzebruch surfaces. Let $\pi : \mathbb{F}_e \to \mathbb{P}^1$ be the Hirzebruch surface of degree e with $0 \leq e \leq g$, where g is a genus of a regular fiber F of π. We recall that ${C_\infty}^2 = e$, $(C_\infty \cdot F) = 1$, and $F^2 = 0$. Also, the section C_0 of $\mathbb{F}_e$ is equal to $C_\infty - eF$, hence ${C_0}^2 = -e$.

Proposition 3.1.5. *([54]) Let D be the divisor $aC_0 + bF$ on the rational ruled surface $\mathbb{F}_e$, and $e \geq 0$. Then:*

1. *D is very ample $\iff$ D is ample $\iff$ $a > 0$ and $b > ae$.*
2. *The linear system $|D|$ contains an irreducible nonsingular curve $\iff$ it contains an irreducible curve $\iff$ $a = 0$, $b = 1$ (namely F); $a = 1$, $b = 0$ (namely C_0); or $a > 0$, $b > ae$; or $e > 0$, $a > 0$, $b = ae$.*

Proposition 3.1.6. *([55, 113]) Let $a = g + 1 - e > 0$. Then*

1. *The linear system $2C_\infty + aF = 2C_0 + (2e + a)F$ on the rational ruled surface $\mathbb{F}_e$ is very ample.*

2. *A general member D of $|2C_\infty + aF|$ is a non-singular irreducible hyperelliptic curve of genus g.*

Proof. Since $a = g + 1 - e > 0$, the first part follows from the Proposition 3.1.5. The Proposition 3.1.5 also implies that there exists a nonsingular irreducible member of the linear system $|D|$. Since a natural projection $D \to \mathbb{P}^1$ is a $2:1$ map, D is a hyperelliptic curve. Using the canonical class formula $K_{\mathbb{F}_e} = -2C_\infty + (e-2)F$, we compute $g(D) = \frac{(K_{\mathbb{F}_e}+D)\cdot D}{2} + 1 = \frac{(g-1)F\cdot D}{2} + 1 = g$. Moreover, using the very ampleness of the linear system $|2C_\infty + aF|$, we see that its generic smooth irreducible members D_0 and D_1 determine a Lefschetz pencil on $\mathbb{F}_e$. This means that a generic member $\{D_t\}_{t\in\mathbb{P}^1}$ of the pencil $\{D_t\}_{t\in\mathbb{P}^1}$, given by D_0 and D_1, is smooth and every member in the pencil is irreducible and has at most one node as its singularity.

□

3.2 Singular fibers in genus two pencils

3.2.1 Classification of singular fibers in pencils of curves of genus two

In [75], Kodaira classified all singular fibers in pencils of elliptic curves, and showed that in a pencil of elliptic curves, each fiber is either an irreducible curve of arithmetic genus one, i.e. an elliptic curve, or a rational curve with a node or a cusp, or a sum of rational curves of self-intersection -2 which fall into seven different types. Later, in [97], Ogg applied Kodaira's argument to pencils of curves of genus two. He classified all possible numerical types of fibers in pencils of genus two curves, and showed that there are 44 types. Iitaka [70] also gave such a classification independently. These fibers were shown actually to arise by Winters ([125]).

Later Namikawa and Ueno gave geometrical classification of all fibres in pencils of genus two curves [93, 94]. To be more precise, let $\pi : X \to \mathbb{D}$ be a family of (complete) curves of genus two over a disc $\mathbb{D} = \{t \in \mathbb{C}, |t| < \epsilon\}$, where X is a minimal, non-singular (complex analytic) surface, and π is smooth over the punctured disc $D' = \mathbb{D} - \{0\}$. Thus, for every $t \in D'$ the fiber $\pi^{-1}(t)$ is a compact non-singular curve (Riemann surface) of genus two and the restriction of π to D' is a topological fiber bundle. For such a family, Namikawa and Ueno gave the complete list of all singular fibers.

Before stating a recent result let us remind the followings. A fibration is called relatively minimal if no fiber contains an exceptional curve. It is called isotrivial if all smooth fibers are isomorphic to each other. Moreover, a non-trivial family $f : S \to \mathbb{CP}^1$ of complex curves of genus $g \geq 1$ admits at least two singular fibers. Recently, Gong, Lu and Tan, by using Namikawa and Ueno's classification, have studied the relatively minimal, isotrivial fibrations of genus $g \geq 2$ ([49]). Let $f : S \to \mathbb{CP}^1$ be a fibration with two singular fibers F_1 and F_2. In this case, f is isotrivial [49]. In addition, we have

Theorem 3.2.1. *(Theorem 1.2 [49]) Let $f : S \to \mathbb{CP}^1$ be a relatively minimal fibration of genus $g = 2$ with two singular fibers F_1 and F_2. Then $F_1 - F_2$ are one of the following 11 types I*-I*, II-II, III-III, IV-IV, V-V*, VI-VI, VII-VII*, (VIII-1)-(VIII-4), (VIII-2)-(VIII-3), (IX-1)-(IX-4), (IX-2)-(IX-3) where each number denotes a singular fiber in [94].*

In the above theorem and in the sequel roman numerals denote the types of the singular fibers as in [94], which are configurations of spheres and tori with various self intersections.

3.2.2 Pencils of genus two curves in the $K3$ surface

In this subsection, we go over some constructions of genus two pencils in the $K3$ surface (a simply connected complex surface with $c_1 = 0$, and diffeomorphic to $E(2)$), and state a result from [77]. In the final section, we will use these pencils to construct minimal symplectic 4-manifolds which are homeomorphic but not diffeomorphic to $3\mathbb{CP}^2\#k\overline{\mathbb{CP}}^2$ for $k = 16, \cdots, 19$. Here we closely follow [77] and refer the reader to [77] for further details.

Let $f_5(x_1, x_2)$ denote a homogeneous polynomial of degree 5 in two variables, x_1 and x_2, and let C be the plane quintic curve defined by the equation

$$x_0{}^5 = f_5(x_1, x_2) = \prod_{i=1}^{5}(x_1 - \lambda_i x_2) \tag{3.5}$$

We will denote by E_0 and L_i $(1 \leq i \leq 5)$ the lines defined by the equations

$$E_0 : x_0 = 0,\ L_i : x_1 = \lambda_i x_2. \tag{3.6}$$

We note that all L_i are members of the pencil of lines through $(1:0:0)$ and L_i meets C at $(0:\lambda_i:1)$ with multiplicity 5.

Then we take the minimal resolution of the double cover of $\mathbb{CP}^2$ branched along the sextic curve E_0+C, and denote the resulting surface by S. Indeed, S is a $K3$ surface. We denote the inverse image of the line E_0 in S by the same symbol E_0.

Next we consider two cases:

1. The equation $f_5=0$ has no multiple roots.

In this case there are five (-2) curves E_i $(1\le i\le 5)$ in S, which are the exceptional curves of the minimal resolution of singularities corresponding to the intersection of C and E_0 in $\mathbb{CP}^2$. On the other hand, the inverse image of L_i is the union of two smooth rational curves F_i, G_i such that F_i is tangent to G_i at one point, at which they intersect E_i, and both F_i and G_i have the self-intersection -3. Moreover, for p and q the inverse images of $(1:0:0)$, we may assume that all F_i (resp. G_i) pass through p (resp. q).

2. $f_5=0$ has a multiple root.

In this case the double cover S has a rational double point of type D_7. Hence S contains seven smooth rational curves E'_j , $(1\le j\le 7)$ whose dual graph is of type D_7. We assume that E'_1 meets E_0, and $E'_1\cdot E'_2=E'_2\cdot E'_3=E'_3\cdot E'_4=E'_4\cdot E'_5=E'_5\cdot E'_6=E'_5\cdot E'_7=1$. If λ_i is a multiple root, then F_i and G_i are disjoint and each of them meets one componet of D_7, for example, F_i meets E'_6 and G_i meets E'_7.

Hence, the pencil of lines on $\mathbb{CP}^2$ through $(1:0:0)$ gives rise to a pencil of curves of genus two on S. A general member is a smooth curve of genus two. Such a curve is unique up to isomorphism and is given by $y^2=x(x^5+1)$ (see [26]). If λ_i is a simple root of the equation $f_5=0$, then the line L_i defines a singular member of this pencil consisting of three smooth rational curves $E_i+F_i+G_i$. This singular member is called a singular member of type I. If λ_i is a multiple root of $f_5=0$, then the line L_i defines a singular fiber consisting of nine smooth rational curves $E'_1,\cdots,E'_7$, F_i, G_i, which is called a singular fiber of type II. Consequently, we have the following lemma.

Lemma 3.2.2. *([77]) The pencil of lines on* $\mathbb{CP}^2$ *through* $(1:0:0)$ *gives rise to a pencil of curves of genus two on the* $K3$ *surface. A general member of this pencil is a smooth curve of genus two. In case that* $f_5=0$ *has no multiple roots, it has five singular* *members of type I. In case that $f_5 = 0$ has a multiple root (resp. two multiple roots), it has three singular members of type I and one singular member of type II (resp. one of type I and two of type II).*

The two points p, q are the base points of the pencil. After blowing up at p and q, we obtain a base point free pencil of curves of genus two in $K3\#2\overline{\mathbb{CP}}^2$. The singular fibers of such pencils are completely classified by Namikawa and Ueno [94] as we discussed above. The type I (resp. type II) singular fiber corresponds to the fiber of type IX-2 (resp. IX-4) in [94]. Hence on $K3\#2\overline{\mathbb{CP}}^2$ there exist pencils of genus 2 curves with

i) five singular members of type IX-2,

ii) three singular members of type IX-2, and one singular member of type IX-4

iii) one singular member of type IX-2, and two singular members of type IX-4.

In the last section we will work with these pencils and construct the above mentioned exotic, minimal, symplectic 4-manifolds.

3.3 Nodal spherical deformation of the singular fibers of Lefschetz fibration

In this section, we will introduce a technique that we call *g-nodal spherical deformation* of the singular fibers of a genus $g \geq 2$ Lefschetz fibration over S^2, and prove a few lemmas that will be used in the proofs of our main theorems. Let us first recall some fundamental facts concerning the Lefschetz fibrations, and list some examples of Lefschetz fibrations for which our nodal spherical deformation technique will be applied. In this section we will work with Lefschetz fibrations over the 2-sphere and we assume they are relatively minimal, i.e, we define

Definition 3.3.1. Let X be a closed, oriented smooth 4-manifold. A smooth map $f : X \to S^2$ is a genus-g *Lefschetz fibration* if it satisfies the following conditions:
(i) f has finitely many critical values $b_1, \ldots, b_m \in S^2$, and f is a smooth Σ_g-bundle over $S^2 - \{b_1, \ldots, b_m\}$,
(ii) for each i ($i = 1, \ldots, m$), there exists a unique critical point p_i in the *singular fiber* $f^{-1}(b_i)$ such that about each p_i and b_i there are local complex coordinate charts agreeing

with the orientations of X and S^2 on which f is of the form $f(z_1, z_2) = z_1^2 + z_2^2$,
(iii) f is relatively minimal (i.e. no fiber contains a (-1)-sphere.)

Example 3.3.2. *([44]) In addition to the examples of Lefschetz fibrations of genus one that we discussed in Chapter 1, let us also give additional ones which are of higher genera.*

There are three families of hyperelliptic Lefschetz fibrations, which are building blocks in many constructions of new Lefschetz fibrations. Let a_1, a_2, ... , a_{2g}, a_{2g+1} denote the collection of the standard simple closed curves on Σ_g (see Figure 3.3 for the genus 2 case), and a_i denote the right handed Dehn twists t_{a_i} along the curve a_i by abuse of notation. Then, the following relations hold in the mapping class group $MCG(\Sigma_g)$:

$$\begin{aligned}
&(a_{2g+1} \cdots a_3 a_2 a_1^2 a_2 a_3 \cdots a_{2g+1})^2 = 1, \\
&(a_1 a_2 \cdots a_{2g} a_{2g+1})^{2g+2} = 1, \\
&(a_1 a_2 \cdots a_{2g-1} a_{2g})^{2(2g+1)} = 1.
\end{aligned} \tag{3.7}$$

These equations give rise to Lefschetz fibrations with total spaces $X(g)$, $Y(g)$ and $Z(g)$, respectively. These examples are complex, and for $g = 2$ any holomorphic Lefschetz fibration with only nonseparating vanishing cycles is a fiber sum of one of these three ([30, 44]).

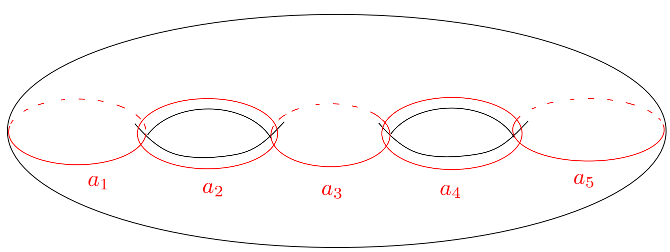


Figure 3.3: Standard Simple Closed Curves on Σ_2

We will apply our nodal deformation technique to the families $X(g)$, $Y(g)$ above.

Thus we consider the following two relations

$$\begin{aligned} H(g) &:= (a_1 a_2 \cdots a_{2g-1} a_{2g} a_{2g+1}{}^2 a_{2g} a_{2g-1} \cdots a_2 a_1)^2 = 1, \\ I(g) &:= (a_1 a_2 \cdots a_{2g} a_{2g+1})^{2g+2} = 1, \end{aligned} \tag{3.8}$$

The total spaces of the above two genus g hyperelliptic Lefschetz fibrations over S^2 given by the above monodromies $H(g) = 1$, and $I(g) = 1$ in the mapping class group Γ_g are $X(g)$ and $Y(g)$ respectively. In fact, the first monodromy relation corresponds to the genus g Lefschetz fibrations over S^2 with total space $X(g) = \mathbb{CP}^2 \# (4g+5)\overline{\mathbb{CP}}^2$, the complex projective plane blown up at $4g+5$ points. In the second case, the total spaces of genus g Lefschetz fibrations over S^2, corresponding to relations $I(g) = 1$, are also well-known families of complex surfaces. For example, $Y(2) = K3 \# 2\overline{\mathbb{CP}}^2$, and the Lefschetz fibration structure arises from a well-known pencil in the $K3$ surface with two base points (see for example the references [45, 16]).

Now we consider the Lefschetz fibrations $X(g)$ and $Y(g)$ when $g = 2$ and we describe the deformation technique by using the monodromies $W_1 = a_1 a_2 a_3 a_4$ and $W_2 = a_1 a_2 a_3 a_4 a_5$ respectively. We also discuss the case $g \geq 3$.

Lemma 3.3.3. *Let $f_1 : X \to \mathbb{D}^2$ denote a Lefschetz fibration given by the monodromy $(a_1 a_2 a_3 a_4)$ in Γ_2. Then it can be deformed to contain two disjoint spherical 2-nodal singular fibers given by the word below*

$$(a_1 a_2 a_3 a_4) = (a_4^{-1} a_1 a_3 a_4)(a_4^{-1} a_3^{-1} a_2 a_4 a_3 a_4) \tag{3.9}$$

Proof. At first, for a word $a_1 a_2 \cdots a_n$ in the Γ_g, by Hurwitz moves we mean either one of the following two equalities:

$$a_1 a_2 \cdots a_i a_{i+1} \cdots a_n = a_1 a_2 \cdots (a_i a_{i+1} a_i^{-1})(a_i) \cdots a_n \tag{3.10}$$

$$a_1 a_2 \cdots a_i a_{i+1} \cdots a_n = a_1 a_2 \cdots (a_{i+1})(a_{i+1}^{-1} a_i a_{i+1}) \cdots a_n. \tag{3.11}$$

By applying these moves, the braid relation $a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1}$, and the commutativity relation of disjoint curves, we compute

$$\begin{aligned} a_1a_2a_3a_4 &= a_1a_2(a_3a_4a_3^{-1})a_3 \\ &= a_1a_2(a_4^{-1}a_3a_4)a_3 \\ &= a_4^{-1}a_1a_2a_3a_4a_3 \\ &= a_4^{-1}a_1a_3(a_3^{-1}a_2a_3)a_4a_3 \\ &= a_4^{-1}a_1a_3(a_2a_3a_2^{-1})a_4a_3 \\ &= a_4^{-1}a_1a_3a_2a_3a_4a_2^{-1}a_3 \\ &= a_4^{-1}a_1a_3a_2(a_4)(a_4^{-1}a_3a_4)a_2^{-1}a_3 \\ &= (a_4^{-1}a_1a_3a_4)(a_4^{-1}a_2a_3a_4a_2^{-1}a_3) \\ &= (a_4^{-1}a_1a_3a_4)(a_4^{-1}a_2a_3a_2^{-1}a_4a_3) \\ &= (a_4^{-1}a_1a_3a_4)(a_4^{-1}a_3^{-1}a_2a_3a_4a_3) \\ &= (a_4^{-1}a_1a_3a_4)(a_4^{-1}a_3^{-1}a_2a_4a_3a_4). \end{aligned}$$

Geometrically we can view the above process as in Figure 3.4. The resulting two singular fibers corresponding to $(a_4^{-1}a_1a_3a_4)$ and $(a_4^{-1}a_3^{-1}a_2a_4a_3a_4)$, are two disjoint spherical fibers with 2 nodes on each.

□

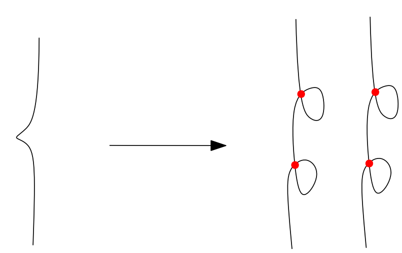


Figure 3.4: Deforming $(2, 5)$ cusp into two disjoint 2-nodal fibers

Lemma 3.3.4. *Let $f_2 : X \to \mathbb{D}^2$ denote a Lefschetz fibration given by the monodromy $(a_1a_2a_3a_4a_5)$ in Γ_2. Then it can be deformed to contain two disjoint spherical 2-nodal* *singular fibers given by the word below*

$$(a_1a_2a_3a_4a_5) = (a_1a_4)(a_2a_5)a_5^{-1}a_3a_4a_3^{-1}a_5. \tag{3.12}$$

Proof. By applying Hurwitz moves, the braid relation $a_ia_{i+1}a_i = a_{i+1}a_ia_{i+1}$, and the commutativity relation of disjoint curves, we compute

$$\begin{aligned} a_1a_2a_3a_4a_5 &= a_1a_2(a_4a_4^{-1})a_3a_4a_5 \\ &= a_1a_2a_4(a_3a_4a_3^{-1})a_5 \\ &= a_1a_4a_2a_3a_4a_3^{-1}a_5 \\ &= a_1a_4a_2a_5a_5^{-1}a_3a_4a_3^{-1}a_5. \end{aligned}$$

Geometrically we again sketch this as in Figure 3.4. The resulting two singular fibers corresponding to a_1a_4 and a_2a_5, are two disjoint spherical fibers with 2 nodes on each, and the third singular fiber corresponding to $a_5^{-1}a_3a_4a_3^{-1}a_5$ is the Lefschetz type nodal fiber.

□

Now we apply our technique to a genus three Lefschetz fibration.

Lemma 3.3.5. *Let $f_3 : X \to \mathbb{D}^2$ denote a Lefschetz fibration given by the monodromy $(a_1a_2a_3a_4a_5a_6a_7)$ in Γ_3. Then it can be deformed to contain two disjoint spherical* 3-*nodal singular fibers given by the word below*

$$a_1a_2a_3a_4a_5a_6a_7 = a_1a_2a_1^{-1}(a_1a_3a_5)(a_1a_2a_1^{-1})^{-1}(a_1a_2a_1^{-1})(a_5^{-1}a_4a_5)a_7a_7^{-1}a_6a_7.$$

Proof. By applying Hurwitz moves, the braid relation $a_ia_{i+1}a_i = a_{i+1}a_ia_{i+1}$, and the commutativity relation of disjoint curves, we compute

$$\begin{aligned} a_1a_2a_3a_4a_5a_6a_7 &= a_1a_2{a_1}^{-1}a_1a_3a_5{a_5}^{-1}a_4a_5a_6a_7 \\ &= [a_1a_2a_1^{-1}(a_1a_3a_5)(a_1a_2a_1^{-1})^{-1}][(a_1a_2a_1^{-1})(a_5^{-1}a_4a_5)a_7][a_7^{-1}a_6a_7] \end{aligned}$$

The resulting two singular fibers corresponding to $[a_1a_2a_1^{-1}(a_1a_3a_5)(a_1a_2a_1^{-1})^{-1}]$ and $[(a_1a_2a_1^{-1})(a_5^{-1}a_4a_5)a_7]$, are two disjoint spherical fibers with 3 nodes on each, and the third singular fiber corresponding to $a_7^{-1}a_6a_7$ is the Lefschetz type nodal fiber.

□

Lemma 3.3.6. *Let $f_4 : X \to \mathbb{D}^2$ and $f_5 : X \to \mathbb{D}^2$ denote the Lefschetz fibration given by the relations $(a_1a_2a_3a_4a_5\cdots a_{2g-1}a_{2g})$ and $(a_1a_2a_3a_4a_5\cdots a_{2g-1}a_{2g}a_{2g+1})$ in Γ_g. Then each of them can be deformed to contain a spherical g-nodal singular fiber given by the words below*

$$(a_1a_2a_3a_4a_5\cdots a_{2g-1}a_{2g}) = (a_1a_3a_5a\cdots a_{2g-1})W''. \tag{3.13}$$

$$(a_1a_2a_3a_4a_5\cdots a_{2g-1}a_{2g}a_{2g+1}) = (a_1a_3a_5a\cdots a_{2g-1})W'''. \tag{3.14}$$

Proof. This is relatively easy to verify by applying Hurwitz moves and the commutativity relation of disjoint curves.

□

More generally,

Lemma 3.3.7. *Let $f : X \to \mathbb{D}^2$ denote a Lefschetz fibration with k singular fibers and the monodromy $W = D_{\gamma_1}D_{\gamma_1}\cdots D_{\gamma_k}$ in Γ_g. Assume that $k \geq g$ and the word W contains a subword W', which consists of a product of g Dehn twists along the disjoint nonseparating vanishing cycles. Then the fibration can be deformed so that it contains a spherical g-nodal singular fiber (a sphere having g nodes, obtained from the genus 2 curve).*

Proof. Using the word $W = W'W''$ and deforming g homologically essential curves corresponding to the nonseparating vanishing cycles on genus g surface corresponding to the subword W', we obtain a spherical g-nodal singular fiber.

□

3.4 The Main Theorems

In this section, we will provide Lefschetz pencils in the Hirzebruch surfaces $\mathbb{F}_2$ and $\mathbb{F}_3$ and from these we will explicitly construct singular fibers of certain types in the Namikawa and Ueno's list. We would like to note that in [93, 94], Namikawa and Ueno constructed the singular fibers by using algebraic methods (by working with polynomials). Here we will give geometric constructions. In addition, we will apply our deformation technique to the complements of the fibers we constructed. From these we will obtain certain genus two Lefschetz fibration structures on $\mathbb{CP}^2\#13\,\overline{\mathbb{CP}}^2$ and $E(2)\#2\,\overline{\mathbb{CP}}^2$. Next, by using them we will construct the exotic copies of $\mathbb{CP}^2\#7\overline{\mathbb{CP}}^2$, $\mathbb{CP}^2\#6\overline{\mathbb{CP}}^2$, and $3\,\mathbb{CP}^2\#k\,\overline{\mathbb{CP}}^2$ for $k = 16, 17, 18, 19$.

Firstly we will work with the fibrations of types (VIII-1 - VIII-4), (IX-2 - IX-3), (VII - VII*), (V - V*), repsectively. These are studied in [49]. Moreover we have the following

Definition 3.4.1. *([49], p.85) Let $f : S \to C$ be a relatively minimal fibration of genus g over a smooth curve of genus b. Then the numerical invariants of the fibration are given as follows*

$$K_f^2 = c_1^2(S) - 8(g-1)(b-1) \tag{3.15}$$

$$\chi_f = \chi(\mathcal{O}_S) - (g-1)(b-1) \tag{3.16}$$

$$q_f = q(S) - g(C) \tag{3.17}$$

$$e_f = c_2(S) - 4(g-1)(b-1) = \sum_F (\chi_{top}(F) - (2-2g)) \tag{3.18}$$

where the summation is over the singular fibers and χ_{top} is the topological Euler characteristic.

The fibrations of types (VIII-1 - VIII-4), (IX-2 - IX-3), (VII - VII*), (V - V*) are genus 2 fibrations over S^2 and for each, the followings hold: $K_f^2 = 4, \chi_f = 2, q(S) = 0$ ([49], p.90). Hence

$$\begin{aligned} K_f^2 = 4 &= c_1^2(S) + 8 \Rightarrow \\ c_1^2(S) &= -4 \end{aligned} \tag{3.19}$$

and

$$\begin{aligned}\chi_f = 2 &= \chi(\mathcal{O}_S) + 1 \Rightarrow \\ \chi(\mathcal{O}_S) &= 1 \end{aligned} \tag{3.20}$$

Thereby $e(S) = 16$ and $\sigma(S) = -12$ showing that the total spaces of each of the 4 fibrations are $\mathbb{CP}^2 \# 13\overline{\mathbb{CP}}^2$.

3.4.1 1. Singular fibers of types VIII-1 and VIII-4

In this subsection, we will work with the singular fibers of Namikawa and Ueno, of types VIII-1 and VIII-4 as shown in Figure 3.5. First we will explicitly construct singular fiber of type VIII-4 starting from a Lefschetz pencil in the Hirzebruch surface $\mathbb{F}_2$. Then we will deform its complement, i.e., the fiber of type VIII-1. Then, by using them we will build the exotic copies of $\mathbb{CP}^2 \# 7\overline{\mathbb{CP}}^2$ and $\mathbb{CP}^2 \# 6\overline{\mathbb{CP}}^2$.

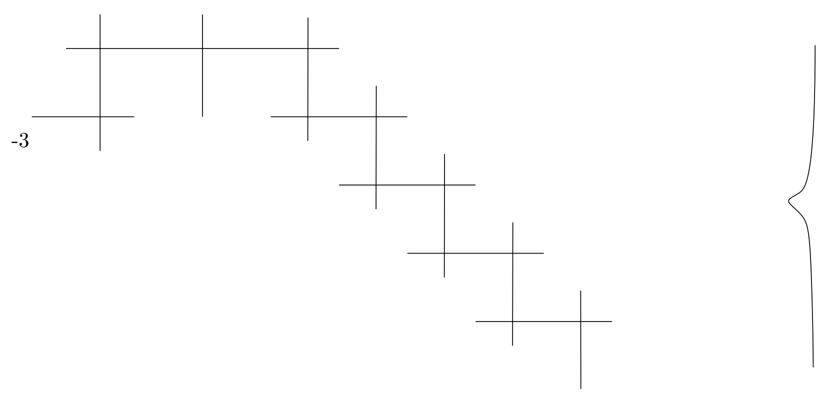


Figure 3.5: Fibers VIII-4 and VIII-1

More precisely we have the following result:

Theorem 3.4.2. *[20] Let M be one of the following 4-manifolds*

1. $\mathbb{CP}^2 \# 7\overline{\mathbb{CP}}^2$
2. $\mathbb{CP}^2 \# 6\overline{\mathbb{CP}}^2$

Then there exist an irreducible symplectic 4-manifold homeomorphic but non-diffeomorphic to M, obtained from a total space of genus two fibration with two singular fibers of types VIII-1 and VIII-4, using the combinations of 2-spherical deformations, the symplectic blow-ups, and the (generalized) rational blow-down surgery.

Proof. Our exotic symplectic 4-manifolds will be obtained from the blow-up of Hirzebruch surface $\mathbb{F}_2$ via a combination of the deformation, symplectic blow-up and rational blow-down. We will construct symplectic embeddings of the (generalized) rational blow-down plumbings C_{11} and $C_{23,11}$ in blow-ups of $\mathbb{F}_2$, and apply the (generalized) rational blow-down surgery to them.

It is known that $\mathbb{CP}^2\#13\overline{\mathbb{CP}}^2$ admits genus 2 fibrations over S^2, and the fiber of type VIII-4 is one of the singular fibers. Indeed the fiber VIII-4 can be constructed in $\mathbb{CP}^2\#13\overline{\mathbb{CP}}^2$ as follows. We first explicitly construct the genus two pencil in $\mathbb{F}_2$ which yields to the singularity type VIII-4 (See Figure 3.5). We take a reducible algebraic curve A consisting of the fiber $F = h - e_1$ with multiplicity 5 and the -2 section $C_0 = e_1 - e_2$ with multiplicity 2 in $\mathbb{F}_2$. Let the curve A be defined by a homogeneous polynomial p_1. Next, we take an irreducible algebraic curve $B = 2C_0 + 5F$ defined by a homogeneous polynomial p_2. For the existence of such curves A and B, we refer the reader to the Proposition 3.1.5 and 3.1.6, and also the references [113, 95, 73, 74], where these propositions are derived from.

Note that $B^2 = 12, F \cdot B = 2, F \cdot C_0 = 1$. Moreover, by the adjunction formula (Theorem 1.2.1 above), the genus of B is two:

$$
\begin{aligned}
(2C_0 + 5F)^2 + K_{\mathbb{F}_2} \cdot B &= 2g - 2 \\
(2C_0 + 5F)^2 - 2C_\infty \cdot B &= 2g - 2 \\
12 - 10 &= 2g - 2 \\
g &= 2.
\end{aligned}
$$

We represent A and B curves as at the beginning of Figure 3.6. (See also [74]). The curves A and B represent the same class in homology, and define a Lefschetz pencil $C_{[t_1:t_2]} = t_1p_1 + t_2p_2 = 0$ whose base locus is the point p.

We blow-up the given pencil at the base point p (see Figure 3.6, step 1), and the resulting exceptional divisor is $e = h - e_1 - e_2$. To obtain the desired singular fiber of type VIII-4, we need to redefine the curves A and B as follows: We take A' as the proper transform of curve A together with the divisor e with multiplicity six, and the curve B' as $B - e$, the proper transform of B. Notice that now both A' and B' curves represent the homology class $2C_0 + 5F - e = 4h - 2e_1 - e_2$, and they define a Lefschetz pencil of genus 2.

Now, as e is a part of A' curve and intersects $B - e$ curve at the point q, the base locus is nonempty. We blow-up at q and denote the resulting exceptional divisor by e_3 (see Figure 3.6, step 2). As above we will reset our curves, by adding e_3 with multiplicity ten to the proper transform of A' (which is colored black in Figure 3.6) and as the second curve we take $B - e - e_3$ (the blue curve). Both of the new curves represent the same homology class $2C_0 + 5F - e - e_3$, and thus define a Lefschetz pencil. The intersection point r of A' and B' curves becomes the base locus at which we blow-up and call the exceptional divisor e_4 (Figure 3.6).

We continue in the same fashion. To equate the homology classes of black and blue curves, we add e_4 with multiplicity nine to the black curve and blow-up its intersection point s with the blue curve. We call the exceptional divisor e_5. Then we add e_5 with multiplicity eight to the black curve and blow-up.

Notice that the multiplicities decrease by one at each step, so after the twelfth blow-up, we add e_{12} with multiplicity one to the black curve which intersects the blue curve at one point. Lastly we blow-up at that point and call the divisor e_{13} that separates the black and blue curves. Note that at this step, the total homology class of the black curve and the homology class of the blue curve $\tilde{B}$ are both $B - e - e_3 - \cdots - e_{13} = 2C_0 + 5F - e - e_3 - \cdots - e_{13}$. Since the homology classes are equal, we do not include the exceptional divisor e_{13} to any component of the Lefschetz pencil and we stop at this step (see the last part of Figure 3.6 where we ignored the multiplicities of the irreducible components of the black curve).

This configuration is symplectically embedded in $\mathbb{F}_2 \# 12\overline{\mathbb{CP}}^2 \cong \mathbb{CP}^2 \# 13\overline{\mathbb{CP}}^2$. Moreover, we note that the self intersection of the blue curve is zero; $(\tilde{B})^2 = (B - e - e_3 - \cdots - e_{13})^2 = (4h - 2e_1 - e_2 - e_3 - \cdots - e_{13})^2 = 0$ so $(\tilde{B})^2$ is the generic genus two fiber.

We remark that the black curve is the fiber VIII-4 in Namikawa-Ueno's list which is the complement of the $(2, 5)$ cusp singular fiber VIII-1 by Theorem 3.2.1 above ([49]).

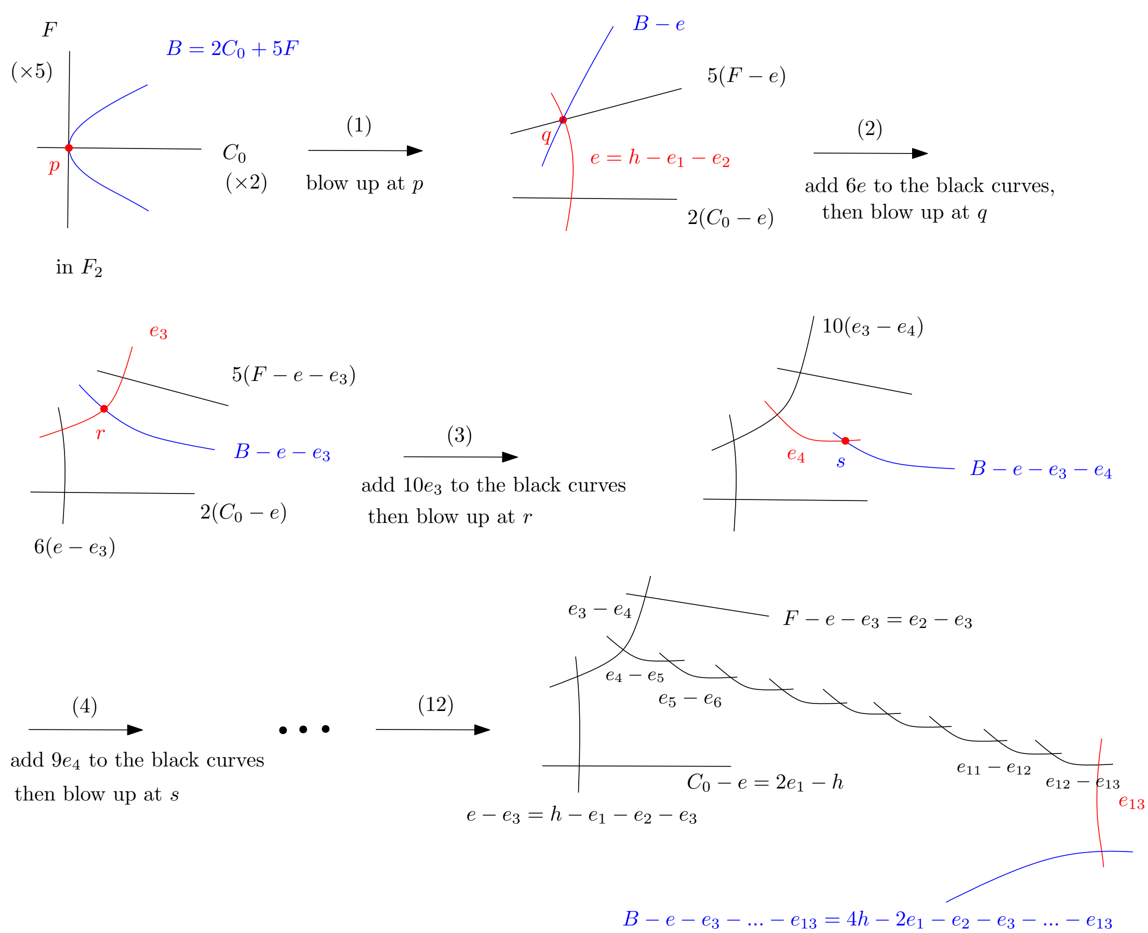


Figure 3.6: Fibers VIII-4, VIII-1

First Construction: Construction of an Exotic Copy of $\mathbb{CP}^2\#7\overline{\mathbb{CP}}^2$

After explicitly constructing a singular fiber of type VIII-4 in $\mathbb{F}_2\#12\overline{\mathbb{CP}}^2 \cong \mathbb{CP}^2\#13\overline{\mathbb{CP}}^2$ above, now we will apply our deformation technique and the rational blow-down surgery to construct an exotic copy of $\mathbb{CP}^2\#7\overline{\mathbb{CP}}^2$.

We have defined the generalized rational blow-down surgery in Chapter 1. Before continue with our constructions, let us retrieve the basics of this surgery here. First we will discuss the lens spaces and the rational homology balls.

Lens spaces

A lens space $L(p, q)$ is the 3-manifold obtained by gluing two solid tori $S^1 \times D^2$ along their boundaries $S^1 \times S^1$ by a homeomorphism such that the homology class $[m_1]$ of the meridian of the first one is sent to $q[m_2] + p[l_2]$, where $[m_2]$ and $[l_2]$ are the homology classes of the meridian and longitude of the second one, respectively. So the image of the meridian m_1 goes q times around m_2, and p times around l_2.

Example 3.4.3. *Note that $L(1, 0) = S^3$, $L(0, 1) = S^1 \times S^2$. Moreover, $L(2, 1) = \mathbb{RP}^3$.*

Rational homology balls

By a rational homology ball we mean a smooth 4-manifold with boundary, that has the same homology groups of a 4-ball with rational coefficients. Casson and Harer showed that for any pair of relatively prime integers p and q, the lens space $L(p^2, 1-pq)$ bounds a rational 4-ball $B_{p,q}$ ([27]). Moreover, $\pi_1(B_{p,q}) = \mathbb{Z}_p$ and $\pi_1(L(p^2, 1-pq)) \to \pi_1(B_{p,q})$ surjective ([101]).

When $q = 1$ the ball $B_{p,1} = B_p$ can be obtained as follows. We consider the Hirzebruch surface $\mathbb{F}_{p-1}$, $p \geq 2$. We resolve the fiber F with the $(p-1)$ section C_∞. The resulting sphere S has self intersection $(p+1)$ and intersects the $-(p-1)$ section C_0, once. Let us denote this configuration of intersecting 2-spheres S and C_0 by $\mathcal{C}$. We take out the regular neighborhood of the configuration $\mathcal{C}$ from $\mathbb{F}_{p-1}$. The neighborhood of $\mathcal{C}$ has boundary $L(p^2, p-1)$. Indeed, from the Hirzebruch-Jung continued fraction expansion we get

$$p^2/(1-p) = -(p+1) - \frac{1}{(p-1)} = [-(p+1), p-1] \tag{3.21}$$

and we see that $-(p+1)$ and $p-1$ are the negatives of the self intersections of the spheres S and C_0, respectively (cf. Definition 1.5.5). This shows that the boundary of the neighborhood of $\mathcal{C}$ is the lens space $L(p^2, p-1)$.

Moreover we note that from the Hirzebruch surface $\mathbb{F}_{p-1}$ we have taken out the neighborhood of two 2-spheres. Therefore, the remaining part is a rational homology ball whose boundary is the same as the boundary of $\mathcal{C}$ which is $L(p^2, p-1)$. Hence the rational homology ball is B_p. ([37])

Generalized rational blow-down surgery, recap

Now, let $C_{p,q}$ denote the configuration of transversally intersecting 2-spheres, where $p \geq q \geq 1$ and p, q are relatively prime integers. In fact, $C_{p,q}$ is the smooth 4-manifold with boundary, obtained by plumbing disk bundles over the 2-sphere according to the Hirzebruch-Jung continued linear fraction expansion $p^2/(pq-1) = [r_k, r_{k-1}, \cdots, r_1]$ of p^2 and $pq-1$. It is known that boundary of $C_{p,q}$ is the lens space $L(p^2, 1-pq)$ ([27]).

Next, assume that a smooth 4-manifold X contains the plumbing $C_{p,q}$. Then we can replace $C_{p,q}$ with rational homology ball $\mathbb{B}_{p,q}$, having the same boundary, to construct a new manifold $X_{p,q}$. This operation is called the generalized rational blow-down surgery ([101]).

Now let us continue our construction. Above we have built the singular fiber of type VIII-4 in $\mathbb{F}_2 \# 12\overline{\mathbb{CP}}^2 \cong \mathbb{CP}^2 \# 13\overline{\mathbb{CP}}^2$. Now we will apply our deformation technique and the rational blow-down surgery to construct an exotic copy of $\mathbb{CP}^2 \# 7\overline{\mathbb{CP}}^2$.

We first consider the complement of the fiber VIII-4 which is the $(2,5)$ cusp fiber VIII-1. Its monodromy is given by the word $(a_1 a_2 a_3 a_4)$ in the Mapping Class Group Γ_2 and by Lemma 3.3.3 this fibration can be deformed to contain two disjoint 2-nodal spherical singular fibers.

Geometrically we may sketch this as in Figure 3.7. The resulting two nodal spherical fibers, with 2 nodes on each, are in $\mathbb{CP}^2 \# 13\overline{\mathbb{CP}}^2$. We blow-up these four nodes of the two 2-nodal fibers, as in Figure 3.7 and resolve their proper transforms $\tilde{B} - 2e_{14} - 2e_{15}$ and $\tilde{B} - 2e_{16} - 2e_{17}$ with the sphere section e_{13}. The resulting sphere s has the following homology class

$$\begin{aligned} s &:= (\tilde{B} - 2e_{14} - 2e_{15}) + (\tilde{B} - 2e_{16} - 2e_{17}) + e_{13} \\ &= 8h - 4e_1 - 2e_2 - 2e_3 - \cdots - 2e_{12} - e_{13} - 2e_{14} - \cdots - 2e_{17}, \end{aligned}$$

where e_i's are the exceptional divisors and $\tilde{B}$ is as above. We note that $s^2 = -13$.

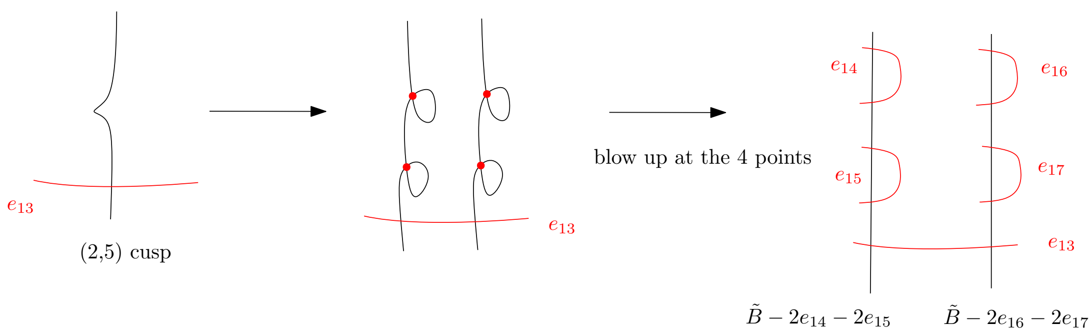


Figure 3.7: Perturbation of the $(2,5)$ cusp and the class s

Hence we have the singular fiber VIII-4 intersecting the sphere s once in $\mathbb{CP}^2\#17\overline{\mathbb{CP}}^2$. Consequently, we obtain a plumbing P of length ten as in Figure 3.8 which is symplectically embedded in $\mathbb{CP}^2\#17\overline{\mathbb{CP}}^2$. In the plumbing P, the homology class of leading -13 sphere is given by s above, and we denote the -2 spheres of the plumbing P by $u_2, \cdots, u_{10}$ respectively. We rationally blow-down this plumbing and call the resulting symplectic manifold $\mathcal{Y}$. In other words we have $\mathcal{Y} = (\mathbb{CP}^2\#17\overline{\mathbb{CP}}^2 - P) \cup B$, where B is the rational homology ball whose boundary is the lens space $L(121, 10)$ which also bounds P.

Next, we will show that the rationally blown down manifold $\mathcal{Y} = (\mathbb{CP}^2\#17\overline{\mathbb{CP}}^2 - P) \cup B$ is an exotic copy of $\mathbb{CP}^2\#7\overline{\mathbb{CP}}^2$.

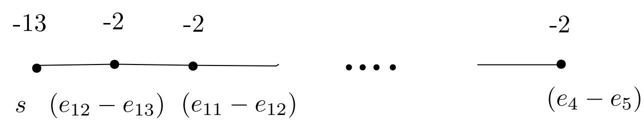


Figure 3.8: Plumbing of length ten

Let us first show that $\mathcal{Y}$ is homeomorphic to $\mathbb{CP}^2\#7\overline{\mathbb{CP}}^2$. This is an application of the lemma 1.5.6 and Freedman's classification theorem. For the reader's convenience, we spell out the details below.

We have $\mathcal{Y} = (\mathbb{CP}^2\#17\overline{\mathbb{CP}}^2 - P) \cup B$, where B is the rational homology ball whose boundary is the lens space $L(121, 10)$ which also bounds P. Note that (e_3-e_4) intersects P but it is not used in the rational blow-down surgery. Thus we first contract the generator of $\pi_1(\partial P)$ along the sphere (e_3-e_4). On the other hand we have the surjection $\pi_1(\partial B) \twoheadrightarrow \pi_1(B)$. Thus, $\mathcal{Y}$ is simply connected by Van Kampen's theorem. Applying well known formulas, we compute

$$\begin{aligned} e(\mathcal{Y}) &= e((\mathbb{CP}^2\#17\overline{\mathbb{CP}}^2) - e(P) + e(B) \\ &= 20 - 11 + 1 \\ &= 10, \\ \sigma(\mathcal{Y}) &= \sigma(\mathbb{CP}^2\#17\overline{\mathbb{CP}}^2) - \sigma(P) + \sigma(B) \\ &= -16 - (-10) \\ &= -6. \end{aligned}$$

Hence, by Freedman's classification the above follows.

Next, we will show that $\mathcal{Y}$ is not diffeomorphic to $\mathbb{CP}^2\#7\overline{\mathbb{CP}}^2$. First, we know that for every $k > 0$, the manifold $\mathbb{CP}^2\#k\overline{\mathbb{CP}}^2$ admits a symplectic structure whose cohomology class is given by $w = ah - b_1e_1 - \cdots - b_ke_k$ for some rational numbers $a, b_1, \cdots, b_k$ with $a > b_1 > \cdots > b_k$ and $a > b_1 + \cdots + b_k$ ([72], Lemma 5.4). Let

$$w = ah - b_1e_1 - \cdots - b_{17}e_{17} \tag{3.22}$$

be the cohomology class of a symplectic strucutre on $\mathbb{CP}^2\#17\overline{\mathbb{CP}}^2$ with $a > b_1 > \cdots >$

b_{17} and $a > b_1 + \cdots + b_{17}$. Let K be the canonical class of $\mathbb{CP}^2\#17\overline{\mathbb{CP}}^2$, so we have

$$K = -3h + e_1 + \cdots + e_{17}. \tag{3.23}$$

By direct computation, we see that K is disjoint from the all -2 spheres $u_2, \cdots, u_{10}$ of the plumbing P in Figure 3.8. Let us let $\gamma_1, \cdots, \gamma_{10}$ be the basis of $H^2(P, \mathbb{Q})$ which is dual to $s, u_2, \cdots, u_{10}$. Then from the adjunction formula

$$\begin{aligned} K|_P &= (K \cdot s)\gamma_1 + (K \cdot u_2)\gamma_2 + \cdots + (K \cdot u_{10})\gamma_{10} \\ &= (-3h + e_1 + \cdots + e_{17}) \cdot (8h - 4e_1 - 2e_2 - 2e_3 - \cdots - 2e_{12} - e_{13} \\ &- 2e_{14} - \cdots - 2e_{17})\gamma_1 \\ &= 11\gamma_1. \end{aligned}$$

We calculate the restriction of the symplectic class w on P

$$\begin{aligned} w|_P &= (w \cdot s)\gamma_1 + (w \cdot u_2)\gamma_2 + \cdots + (w \cdot u_{10})\gamma_{10} \\ &= (ah - b_1e_1 - \cdots - b_{17}e_{17}) \cdot (8h - 4e_1 - 2e_2 - 2e_3 - \cdots - 2e_{12} - e_{13} \\ &- 2e_{14} - \cdots - 2e_{17})\gamma_1 \\ &+ (ah - b_1e_1 - \cdots - b_{17}e_{17}) \cdot (e_{12} - e_{13})\gamma_2 + \cdots \\ &+ (ah - b_1e_1 - \cdots - b_{17}e_{17}) \cdot (e_4 - e_5)\gamma_{10} \\ &= (8a - 4b_1 - 2b_2 - 2b_3 - \cdots - 2b_{12} - b_{13} - 2b_{14} - \cdots - 2b_{17})\gamma_1 \\ &+ (b_{12} - b_{13})\gamma_2 + (b_{11} - b_{12})\gamma_3 + (b_{10} - b_{11})\gamma_4 + (b_9 - b_{10})\gamma_5 \\ &+ (b_8 - b_9)\gamma_6 + (b_7 - b_8)\gamma_7 + (b_6 - b_7)\gamma_8 + (b_5 - b_6)\gamma_9 + (b_4 - b_5)\gamma_{10}. \end{aligned}$$

Let M be the intersection matrix for the plumbing P:

$$M = \begin{bmatrix} -13 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

Then, the first column $[-1/121(10, 9, 8, 7, 6, 5, 4, 3, 2, 1)]$ of M^{-1} gives us $\gamma_1 \cdot \gamma_i$, $i = 1, \cdots, 10$. A direct, but lengthy computation shows that

$$K|_P \cdot w|_P = -11/121(80a - 40b_1 - 20(b_2 + b_3 + b_{14} + b_{15} + b_{16} + b_{17}) - 19(b_4 + b_5 + b_6 + \cdots + b_{13})).$$

Finally, we compute

$$\begin{aligned} K|_{\mathcal{Y}} \cdot w|_{\mathcal{Y}} &= K \cdot w - K|_P \cdot w|_P \\ &= (-3a + b_1 + b_2 + \cdots + b_{17}) + 11/121(80a - 40b_1 - 20(b_2 + b_3 + b_{14} + b_{15} \\ &+ b_{16} + b_{17}) - 19(b_4 + b_5 + b_6 + \cdots + b_{13})) \\ &= 1/121(517a - 319b_1 - 88(b_4 + \cdots + b_{13}) - 99(b_2 + b_3 + \cdots + b_{17})) \\ &> 0. \end{aligned}$$

This shows that $\mathcal{Y}$ is not diffeomorphic to $\mathbb{CP}^2\#7\overline{\mathbb{CP}}^2$. In fact, the standard symplectic form on $\mathbb{CP}^2\#k\overline{\mathbb{CP}}^2$ satisfies $K \cdot w < 0$. Moreover, there is a unique symplectic structure on $\mathbb{CP}^2\#k\overline{\mathbb{CP}}^2$ for $2 \leq k \leq 9$ up to diffeomorphism and deformation:

Theorem 3.4.4. *([83], Theorem D) There is a unique symplectic structure on* $\mathbb{CP}^2\#k\overline{\mathbb{CP}}^2$ *for* $2 \leq k \leq 9$ *up to diffeomorphisms and deformation. For* $k \leq 10$*, the symplectic structure is still unique for the standard canonical class.*

Hence $\mathbb{CP}^2\#7\overline{\mathbb{CP}}^2$ does not admit a symplectic structure with $K \cdot w > 0$.

Furthermore, using the adjunction formula and inequalities, and the methods of the article [98], we verify the minimality of $\mathcal{Y}$.

Second Construction: Construction of an Exotic Copy of $\mathbb{CP}^2\#6\overline{\mathbb{CP}}^2$

In this subsection, we will construct an exotic copy of $\mathbb{CP}^2\#6\overline{\mathbb{CP}}^2$. We will use the generalized rational blow-down plumbing of the form as shown in Figure 3.9 for $m = 3$ and $k = 9$.

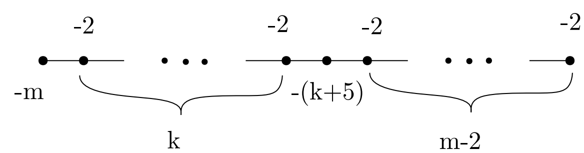


Figure 3.9: Plumbing for a generalized rational blow-down

First, the curve $\tilde{B} - 2e_{14} - 2e_{15}$ intersects the exceptional divisor e_{14} twice as in Figure 3.7 above. We blow-up one of their intersection points, and call the exceptional sphere e_{18}. We note that, after the blow-up, the -9 curve $\tilde{B} - 2e_{14} - 2e_{15} - e_{18}$ intersects the proper transform $e_{14} - e_{18}$ of e_{14} once. Let us take the symplectic resolution of the three curves:

$$\begin{aligned} s' &:= (\tilde{B} - 2e_{14} - 2e_{15} - e_{18}) + (\tilde{B} - 2e_{16} - 2e_{17}) + e_{13} \\ &= 8h - 4e_1 - 2e_2 - 2e_3 - \cdots - 2e_{12} - e_{13} - 2e_{14} - \cdots - 2e_{17} - e_{18}, \end{aligned}$$

where e_i's are the exceptional divisors and $\tilde{B}$ is as above. We have $s'^2 = -14$, it intersects $e_{14} - e_{18}$ and also $e_{12} - e_{13}$. Moreover, in Figure 3.6, we symplectically resolve three curves: $(2e_1 - h) + (h - e_1 - e_2 - e_3) + (e_3 - e_4) = e_1 - e_2 - e_4$ which is a -3 curve. Hence, we obtain a plumbing P' of length 12 for the rational blow-down (see Figure 3.10). (It can be obtained as the Hirzebruch-Jung continued fraction of 529/252, where $(529, 252) = 1$ and the boundary of P' is the lens space $L(529, 252)$. Let us label the spheres from left to right as $u'_1, \cdots, u'_{12}$.

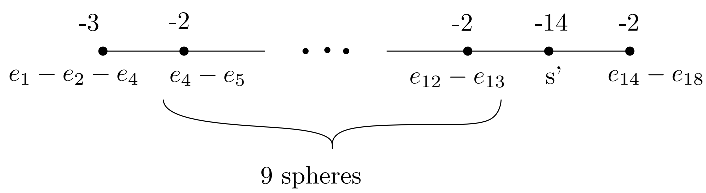


Figure 3.10: Generalized rational blow-down plumbing of length 12

Note that this plumbing is symplectically embedded in $(\mathbb{F}_2\#12\overline{\mathbb{CP}}^2)\#5\overline{\mathbb{CP}}^2$ $\cong \mathbb{CP}^2\#18\overline{\mathbb{CP}}^2$. Then we rationally blow-down this plumbing in $\mathbb{CP}^2\#18\overline{\mathbb{CP}}^2$, and call the resulting symplectic manifold $\mathcal{Z}$. Next, we will show that $\mathcal{Z}$ is an exotic copy of $\mathbb{CP}^2\#6\overline{\mathbb{CP}}^2$. As above, we first show that $\mathcal{Z}$ is homeomorphic to $\mathbb{CP}^2\#6\overline{\mathbb{CP}}^2$.

Let $\mathcal{Z} = (\mathbb{CP}^2\#18\overline{\mathbb{CP}}^2 - P') \cup B'$, where B' is the rational homology ball whose boundary is the lens space which also bounds P'. We contract the generator of $\pi_1(\partial P')$ along the sphere $F - e - e_3 = e_2 - e_3$. Note that $(e_2 - e_3)$ intersects P' but was not used in the rational blow-down plumbing P'. On the other, hand we have the surjection $\pi_1(\partial B') \twoheadrightarrow \pi_1(B')$. Thus, $\mathcal{Z}$ is simply connected by Van Kampen's theorem. As above, we compute the Euler characteristic e and signature σ of $\mathcal{Z}$, and by Freedman's classification theorem, we conclude that $\mathcal{Z}$ is homeomorphic to $\mathbb{CP}^2\#6\overline{\mathbb{CP}}^2$.

Next, we will show that $\mathcal{Z}$ is not diffeomorphic to $\mathbb{CP}^2\#6\overline{\mathbb{CP}}^2$. Let

$$w' = a'h - b_1'e_1 - \cdots - b_{18}'e_{18} \tag{3.24}$$

be the cohomology class of a symplectic strucutre on $\mathbb{CP}^2\#18\overline{\mathbb{CP}}^2$ with $a' > b_1' > \cdots > b_{18}'$ and $a' > b_1' + \cdots + b_{18}'$. Let K' be the canonical class of $\mathbb{CP}^2\#18\overline{\mathbb{CP}}^2$. We have

$$K' = -3h + e_1 + \cdots + e_{18}. \tag{3.25}$$

Next we compute

$$\begin{aligned} K' \cdot u_1' &= (-3h + e_1 + \cdots + e_{18}) \cdot (e_1 - e_2 - e_4) = 1 \\ K' \cdot u_i' &= 0, \quad i = 2, \cdots, 10, 12 \\ K' \cdot u_{11}' &= (-3h + e_1 + \cdots + e_{18}) \cdot (8h - 4e_1 - 2e_2 - \cdots - 2e_{12} - e_{13} \\ &- 2e_{14} - .. - 2e_{17} - e_{18}) \\ &= 12. \end{aligned}$$

Let $\gamma'_1, \cdots, \gamma'_{12}$ be the basis of $H^2(P', \mathbb{Q})$ which is dual to $u'_1, \cdots, u'_{12}$. Using the adjunction formula (Theorem 1.2.1 above),

$$\begin{aligned} K'|_{P'} &= \sum_{i=1}^{12} (K' \cdot u'_i)\gamma'_i \\ &= \gamma_1 + 12\gamma_{11}. \end{aligned}$$

Then we calculate the restriction of the symplectic class w' on P'

$$\begin{aligned} w'|_{P'} &= \sum_{i=1}^{12} (w' \cdot u'_i)\gamma'_i \\ &= (b'_1 - b'_2 - b'_4)\gamma'_1 + (b'_4 - b'_5)\gamma'_2 + (b'_5 - b'_6)\gamma'_3 + (b'_6 - b'_7)\gamma'_4 \\ &+ (b'_7 - b'_8)\gamma'_5 + (b'_8 - b'_9)\gamma'_6 + (b'_9 - b'_{10})\gamma'_7 + (b'_{10} - b'_{11})\gamma'_8 \\ &+ (b'_{11} - b'_{12})\gamma'_9 + (b'_{12} - b'_{13})\gamma'_{10} + (b'_{14} - b'_{18})\gamma'_{12} \\ &+ (8a' - 4b'_1 - 2b'_2 - \cdots - 2b'_{12} - b'_{13} - 2b'_{14} \cdots - 2b'_{17} - b'_{18})\gamma'_{11}. \end{aligned}$$

Let M' be the intersection matrix for the plumbing P':

$$M' = \begin{bmatrix} -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -14 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

To find $K'|'_P \cdot w'|'_P$, we need to read the first and the eleventh columns of M'^{-1} which are

$[-1/529(252, 227, 202, 177, 152, 127, 102, 77, 52, 27, 2, 1)]$ and

$[-1/529(2, 6, 10, 14, 18, 22, 26, 30, 34, 38, 42, 21)]$, respectively.

Hence, we find

$$\begin{aligned} K'|'_P \cdot w'|'_P &= -1/529[4048a' - 1748b'_1 - 1288b'_2 - 989(b'_4 + b'_5 + \cdots + b'_{13}) \\ &- 759(b'_{14} + b'_{18}) \\ &- 1012(b'_3 + b'_{15} + b'_{16} + b'_{17})]. \end{aligned}$$

We also have $K' \cdot w' = -3a' + b'_1 + \cdots + b'_{18}$, so we have

$$\begin{aligned} K'|_{\mathcal{Z}} \cdot w'|_{\mathcal{Z}} &= K' \cdot w' - K'|'_P \cdot w'|'_P \\ &= 1/529[2461a' - 1219b'_1 - 759b'_2 - 460(b'_4 + \cdots + b'_{13}) - 230(b'_{14} + b'_{18}) \\ &- 483(b'_3 + b'_{15} + b'_{16} + b'_{17})] \\ &> 0. \end{aligned}$$

This shows that $\mathcal{Z}$ is not diffeomorphic to $\mathbb{CP}^2\#6\overline{\mathbb{CP}}^2$, since $\mathbb{CP}^2\#6\overline{\mathbb{CP}}^2$ does not admit a symplectic structure with $K' \cdot w' > 0$ as explained above.

Furthermore, minimality of $\mathcal{Z}$ follows from [98].

□

3.4.2 2. Singular fibers of types IX-2 and IX-3

In this subsection, we will work with the singular fibers of types IX-2 and IX-3 as in the Namikawa-Ueno's list. First we will explicitly present a Lefschetz pencil in the Hirzebruch surface $\mathbb{F}_2$ and from this pencil we will geometrically construct the singular fiber of type IX-3. Then we will deform its complement, fiber of type IX-2. Then, by using them we will build an exotic copy of $\mathbb{CP}^2\#7\overline{\mathbb{CP}}^2$.

More precisely, our second theorem of this section is as follows.

Theorem 3.4.5. *[20] There exist an irreducible symplectic 4-manifold homeomorphic but non-diffeomorphic to $\mathbb{CP}^2\#7\overline{\mathbb{CP}}^2$, obtained from a total space of genus two fibration with two singular fibers of types IX-2 and IX-3, using the combinations of 2-spherical deformations, the symplectic blow-ups, and the (generalized) rational blow-down surgery.*

Proof. We take a reducible algebraic curve A consisting of the fiber $F = h - e_1$, and an irreducible curve $C_0 + 2F$ with multiplicity 2 in $\mathbb{F}_2$. Let us also take an irreducible algebraic curve B in the linear system $|2C_0 + 5F|$. Since $B^2 = 12$, $F \cdot B = 2$, $B \cdot (C_0 + 2F) = 5$, by adjunction formula (Theorem 1.2.1 above) the genus of B is two. We represent A and B curves as in Figure 3.11, where A consists of the black curves and B is the blue curve. The curves A and B represent the same class in homology, and define a Lefschetz pencil.

We blow-up the three intersection points of A and B curves, and denote the exceptional divisors $e = h - e_1 - e_2, e_3$ and e_4 as shown in the second step of Figure 3.11. We reset A and B curves as follows. A' consists of $F - e - e_3$, $C_0 + 2F - e_4$ with multiplicity two, and e_4; and B' is the proper transform $B - e - e_3 - e_4$ of B. Now A' and B' represent the same homology class $2C_0 + 5F - e - e_3 - e_4$, and they define a Lefschetz pencil. Moreover, these curves intersect at one point with multiplicity four as shown in the second step of Figure 3.11. We blow-up at that point and call the exceptional divisor e_5 (see the third step of Figure 3.11).

We continue in the same way; at each stage, to equate the homology classes of black and blue curves, we add the exceptional divisor with an appropriate multiplicity to the black curve, then we blow-up the intersection point of black and blue curves. We showed these steps together with the homology classes of each irreducible components of the curves with their multiplicities in Figure 3.11.

Note that at the sixth step, the blue curve $B - e - e_3 \cdots - e_8$ is separated from $C_0 + 2F - e_4 - \cdots - e_8$ by the exceptional divisor e_8. However, we need to add the divisor e_8 with multiplicity five to the black part to equate the homology classes of the two components of the pencil. Next, we blow-up the intersection point of e_8 with $B - e - e_3 \cdots - e_8$ (cf. sixth step of Figure 3.11). Then we continue as above. When we do the twelfth blow-up, where we denote the exceptional divisor by e_{13}, the total homology class of the black curve and that of the blue curve become equal; they are both $B - e - e_3 - \cdots - e_{13} = 2C_0 + 5F - e - e_3 - \cdots - e_{13}$. Therefore, we do not include e_{13} to any component of the pencil. As the black and blue curves are completely separated, we stop at this step (cf. the last part of Figure 3.11).

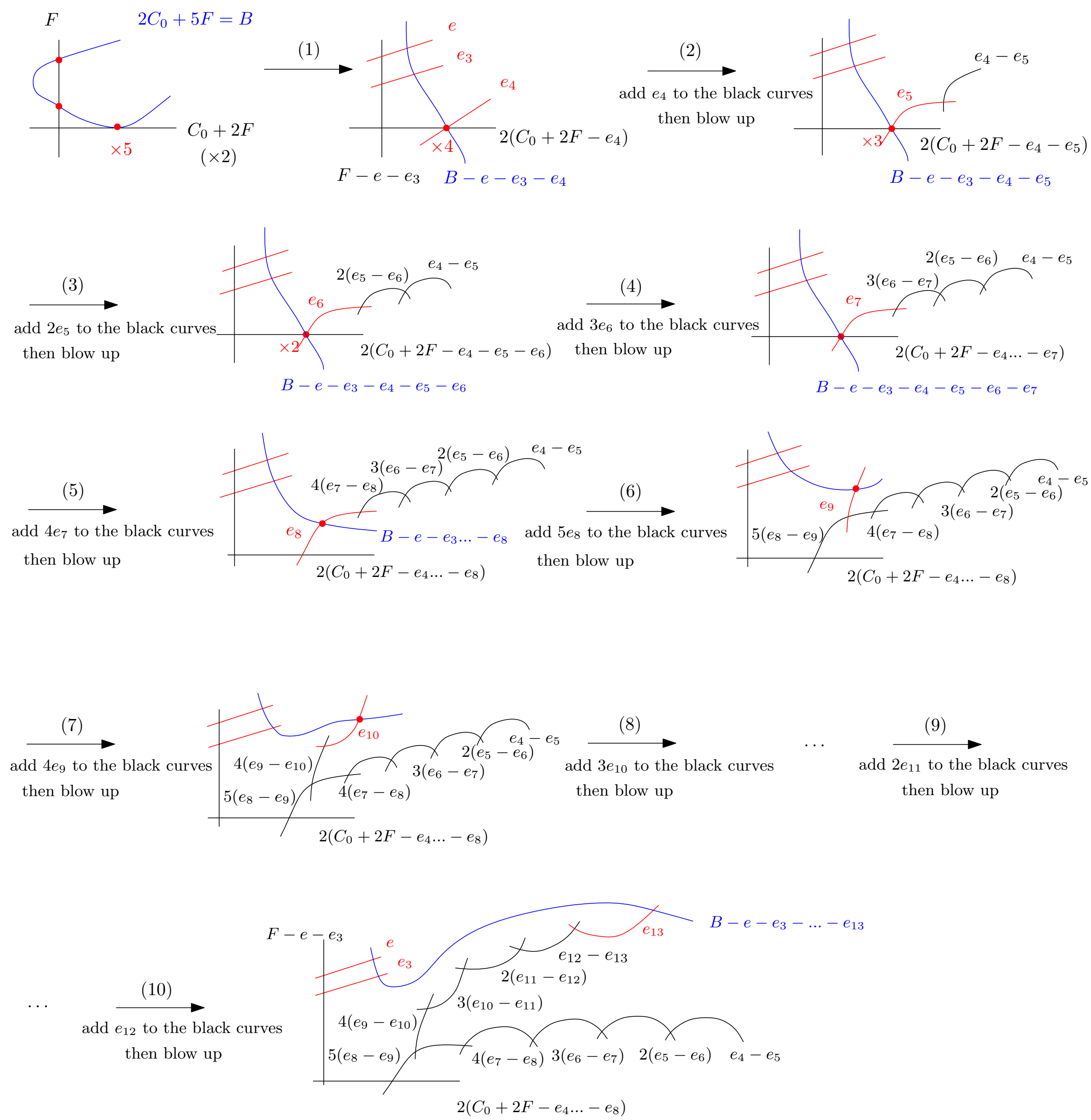


Figure 3.11: Fibers IX-2 and IX-3

This configuration is symplectically embedded in $\mathbb{F}_2\#12\overline{\mathbb{CP}}^2 \cong \mathbb{CP}^2\#13\overline{\mathbb{CP}}^2$ and for the blue curve, we have $(B - e - e_3 - \cdots - e_{13})^2 = (4h - 2e_1 - e_2 - e_3 - \cdots - e_{13})^2 = 0$. Lastly, we see that the resulting black curve is the fiber IX-3 in Namikawa-Ueno's list which is the dual of the fiber IX-2 (by Theorem 3.2.1 above).

The monodromy of type IX-2 fiber is $a_1a_2a_3a_4a_5^2$ in Γ_2 [68] and from above we have:

$$(a_1a_2a_3a_4)(a_5)^2 = (a_4^{-1}a_1a_3a_4)(a_4^{-1}a_3^{-1}a_2a_4a_3a_4)(a_5)^2 \tag{3.26}$$

which means we can consider the same deformation as in Figure 3.7. The resulting two nodal spherical fibers, with two nodes on each, are in $\mathbb{CP}^2\#13\overline{\mathbb{CP}}^2$. As in the previous case, we blow them up twice (cf. Figure 3.7) and resolve their proper transforms with the section e_{13}. Again, we obtain the sphere

$$s = 8h - 4e_1 - 2e_2 - 2e_3 - \cdots - 2e_{12} - e_{13} - 2e_{14} - \cdots - 2e_{17},$$

of square -13. We note that we used the section e_{13} in Figure 3.11 in our construction of the sphere s. Hence, we have the fiber IX-3 intersects the class s once in $\mathbb{CP}^2\#17\overline{\mathbb{CP}}^2$. We remark that also in this case, from the fiber IX-3 (Figure 3.11) and s, we obtain the same plumbing P of length ten as in Figure 3.8. P is symplectically embedded in $\mathbb{CP}^2\#17\overline{\mathbb{CP}}^2$, by rationally blowing it down we construct an exotic copy of $\mathbb{CP}^2\#7\overline{\mathbb{CP}}^2$ as above. Since the details are almost identical to the ones for the cases given above, we omit them.

□

3.4.3 3. Singular fibers of types VII and VII*

Now, we will work with the singular fibers of types VII and VII* from the Namikawa-Ueno's list. As above first we will build the singular fiber of type VII* starting from a Lefschetz pencil in the Hirzebruch surface $\mathbb{F}_2$. Then we will deform its complement, i.e., the fiber of type VII. Then, by using them we will construct the exotic copies of $\mathbb{CP}^2\#7\overline{\mathbb{CP}}^2$ and $\mathbb{CP}^2\#6\overline{\mathbb{CP}}^2$.

More specifically our third theorem of this chapter is as follows.

Theorem 3.4.6. *[20] Let M be one of the following 4-manifolds*

1. $\mathbb{CP}^2\#7\overline{\mathbb{CP}}^2$

2. $\mathbb{CP}^2\#6\overline{\mathbb{CP}}^2$

Then there exist an irreducible symplectic 4-manifold homeomorphic but non-diffeomorphic to M, obtained from a total space of genus two fibration with two singular fibers of types VII and VII, using the combinations of 2-spherical deformations, the symplectic blow-ups, and the (generalized) rational blow-down surgery.*

Proof. Let us take a reducible algebraic curve A consisting of the fiber $F = h - e_1$, the -2 section C_0 with multiplicity two, and another copy of F with multiplicity four in $\mathbb{F}_2$. On the other hand, we take an irreducible algebraic curve B in the linear system $|2C_0 + 5F|$. We have $B^2 = 12$, $F \cdot B = 2$, $B \cdot C_0 = 1$, and the genus of B is two. We represent A and B curves as in Figure 3.12, where A is in black and B is in blue. Once again, as A and B represent the same class in homology, and define a Lefschetz pencil.

We proceed as in the previous two cases. Namely, we blow-up the intersections of the curves A and B. Then to equate their homology classes we add the exceptional divisor with a multiplicity to A. Thus we obtain new intersection point of A and B then we blow-up again. We continue this process until A and B are completely separated. We show all the details of each step in Figure 3.12. At the end, yet again we acquire a configuration symplectically embedded in $\mathbb{F}_2\#12\overline{\mathbb{CP}}^2 \cong \mathbb{CP}^2\#13\overline{\mathbb{CP}}^2$ (the last step of Figure 3.12). The resulting black curve is the fiber VII* in Namikawa-Ueno's list, which is the dual of the fiber VII (by Theorem 3.2.1 above).

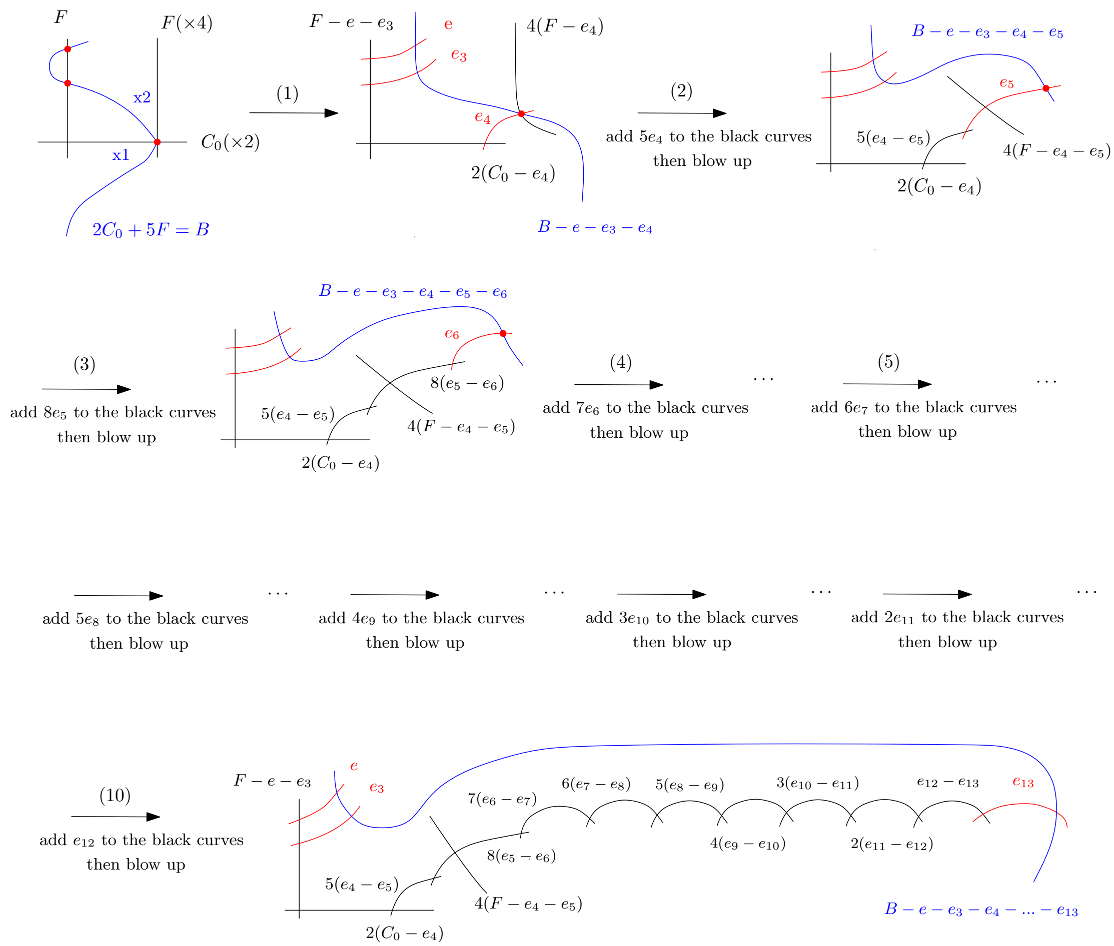


Figure 3.12: Fibers VII and VII*

The monodromy of the fiber VII is $a_1a_2a_3a_4^2$ in Γ_2 [68] and from above we have:

$$(a_1a_2a_3a_4)a_4 = (a_4^{-1}a_1a_3a_4)(a_4^{-1}a_3^{-1}a_2a_4a_3a_4)a_4 \tag{3.27}$$

so we use the same deformation as in Figure 3.7. The resulting two nodal spherical fibers, with 2 nodes on each, are in $\mathbb{CP}^2\#13\overline{\mathbb{CP}}^2$. As in the previous two cases, we blow them up twice (cf. Figure 3.7) and resolve their proper transforms $(\tilde{B}-2e_{14}-2e_{15}), (\tilde{B}-2e_{16}-2e_{17})$ with the section e_{13}, where $\tilde{B} = (B-e-e_3-\cdots-e_{13}) = (2C_0+5F-e-e_3-\cdots-e_{13})$. Again, we obtain a sphere

$$\begin{aligned} s &:= (\tilde{B}-2e_{14}-2e_{15})+(\tilde{B}-2e_{16}-2e_{17})+e_{13} \\ &= 8h-4e_1-2e_2-2e_3-\cdots-2e_{12}-e_{13}-2e_{14}-\cdots-2e_{17}, \end{aligned}$$

of square -13. Note that we used the section e_{13} in Figure 3.12 to construct s. Hence, we have that the fiber VII* intersects the class s once in $\mathbb{CP}^2\#17\overline{\mathbb{CP}}^2$. This gives us the same plumbing P of length ten as in Figure 3.8, symplectically embedded in $\mathbb{CP}^2\#17\overline{\mathbb{CP}}^2$. We rationally blow-down P to construct an exotic copy of $\mathbb{CP}^2\#7\overline{\mathbb{CP}}^2$.

Next, we will obtain a plumbing P'' of length 12 for the rational blow-down as in Figure 3.10. Note that we have not used all the spheres in the fiber VII* for the rational blow-down. One of the unused spheres C_0-e_4 intersects the sphere e_4-e_5 (cf. Figure 3.12) and $(C_0-e_4)^2=-3$. Thus, C_0-e_4 will be the leading sphere of the plumbing P'. Next, we take the spheres $(e_4-e_5),\cdots,(e_{12}-e_{13})$ of the fiber VII*. Now we will construct a (-14) sphere s' intersecting $(e_{12}-e_{13})$. We follow the same steps above. Namely, the curve $\tilde{B}-2e_{14}-2e_{15}$ intersects the exceptional divisor e_{14} twice as in Figure 3.7. We blow-up one of their intersection points, call the exceptional sphere e_{18}. We note that, after the blow-up, the -9 curve $\tilde{B}-2e_{14}-2e_{15}-e_{18}$ intersects the proper transform $e_{14}-e_{18}$ of e_{14} once. Now we take the symplectic resolution of the three curves:

$$\begin{aligned} s' &:= (\tilde{B}-2e_{14}-2e_{15}-e_{18})+(\tilde{B}-2e_{16}-2e_{17})+e_{13} \\ &= 8h-4e_1-2e_2-2e_3-\cdots-2e_{12}-e_{13}-2e_{14}-\cdots-2e_{17}-e_{18}, \end{aligned}$$

We have $s'^2=-14$, it intersects $e_{12}-e_{13}$ and also $e_{14}-e_{18}$. In addition to the spheres $(C_0-e_4),(e_4-e_5),\cdots,(e_{12}-e_{13})$ of the fiber VII*, we take s' and $e_{14}-e_{18}$. This gives the desired plumbing P'' of length 12, symplectically embedded in $(\mathbb{F}_2\#12\overline{\mathbb{CP}}^2)\#5\overline{\mathbb{CP}}^2 \cong \mathbb{CP}^2\#18\overline{\mathbb{CP}}^2$. We rationally blow it down. Following the same lines above we can show that the resulting manifold is an exotic copy of $\mathbb{CP}^2\#6\overline{\mathbb{CP}}^2$.

□

3.4.4 4. Singular fibers of types V and V*

Lastly, in this subsection we will work with the singular fibers of types V and V* which were constructed algebraically in [93, 94]. We begin with providing a Lefschetz pencil in the third degree Hirzebruch surface $\mathbb{F}_3$ and from this pencil we will geometrically construct the singular fiber of type V*. Then we will deform its complement, i.e., the fiber of type V. Then, by using them we will build an exotic copy of $\mathbb{CP}^2\#7\overline{\mathbb{CP}}^2$.

More specifically we will prove:

Theorem 3.4.7. *[20] There exist an irreducible symplectic 4-manifold homeomorphic but non-diffeomorphic to $\mathbb{CP}^2\#7\overline{\mathbb{CP}}^2$, obtained from a total space of genus two fibration with two singular fibers of types V and V*, using the combinations of 2-spherical deformations, the symplectic blow-ups, and the (generalized) rational blow-down surgery.*

Proof. In this construction, we will work in $\mathbb{F}_3 \cong \mathbb{CP}^2\#\overline{\mathbb{CP}}^2$. Let us take the (+3) section C_∞ with multiplicity two in $\mathbb{F}_3$, and denote it by A. As the second component B for our pencil, we take an irreducible algebraic curve in the linear system $|2C_\infty = 2(C_0 + 3F)|$, where C_0 is the (−3) section and F is the fiber. Since $(C_\infty)^2 = 3$, $(2C_\infty)^2 = 12$, by adjunction equality (Theorem 1.2.1 above), genus of $B = 2C_\infty$ is 2:

$$\begin{aligned}
(2C_\infty)^2 + K_{\mathbb{F}_3} \cdot 2C_\infty &= 2g - 2\\
12 + (-2C_\infty + F) \cdot 2C_\infty &= 2g - 2\\
12 - 12 + 2 &= 2g - 2\\
g &= 2.
\end{aligned}$$

Notice that $2C_\infty \cdot C_\infty = 6$ i.e., $2C_\infty$ intersects the (+3) section C_∞ of $\mathbb{F}_3$ at six points. Hence we represent A and B curves as at the beginning of Figure 3.13 where A is the black and B is the blue curve. Since A and B represent the same class in homology, they define a Lefschetz pencil with a nonempty base locus. In Figure 3.13 we denote it by the red point and its multiplicity by $\times 6$.

We blow-up that point and denote the exceptional divisor by e_2 in $\mathbb{F}_3\#\overline{\mathbb{CP}}^2$. We proceed as in the previous constructions. Namely, after each blow-up we reset the A and B curves so that we obtain a Lefschetz pencil. Then we blow-up the new intersection point of A and B. We pursue this process until A and B curves are separated and have the equal homology classes. We showed each step in Figure 3.13 with the homology classes and multiplicities of every irreducible components.

At the end of the blow-up process, we remark that the total homology class of the first curve (the black part) is $2C_\infty - e_2 - \cdots - e_{13}$ which is the same as the homology class of the blue curve. Let us call this blue curve at the end $\tilde{B}$ for which we have $\tilde{B} = 2C_\infty - e_2 - \cdots - e_{13} = 2(2h - e_1) - e_2 - \cdots - e_{13} = 4h - 2e_1 - e_2 - \cdots - e_{13}$, and so it is of self intersection zero.

Moreover, we also note that the resulting black curve is the fiber V* in Namikawa-Ueno's list. Hence we obtain a configuration as shown in the last step of Figure 3.13 that is symplectically embedded in $\mathbb{F}_3\#12\overline{\mathbb{CP}}^2 \cong \mathbb{CP}^2\#13\overline{\mathbb{CP}}^2$.

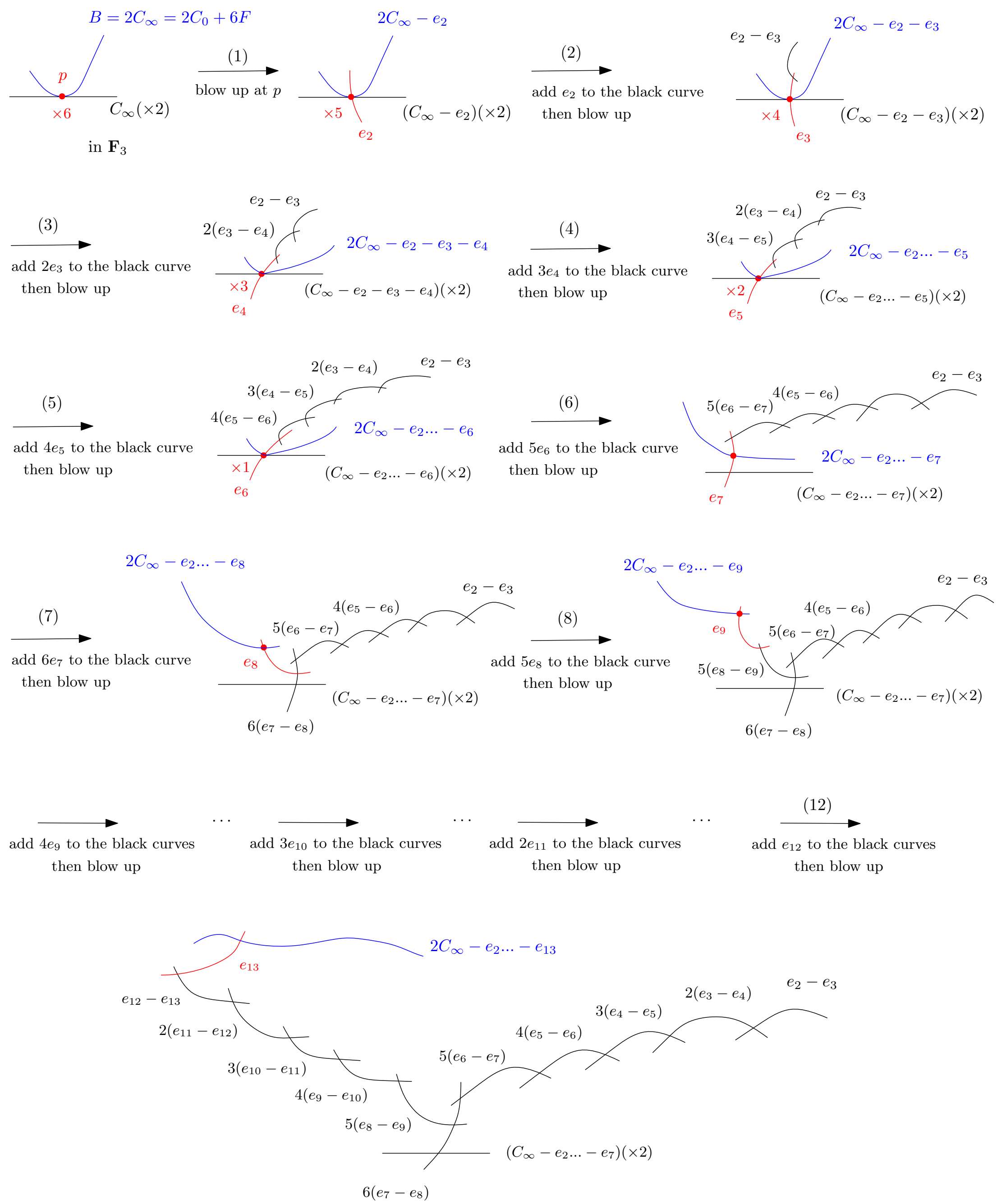


Figure 3.13: Fibers V and V*

Now we will work with the complement of the fiber of type V*. The fiber V is the dual of the fiber V* (by Theorem 3.2.1 above), and the monodromy of the former is $a_1a_2a_3a_4a_5$ in Γ_2 [68]. As we showed above we have:

$$(a_1a_2a_3a_4)a_5 = (a_4^{-1}a_1a_3a_4)(a_4^{-1}a_3^{-1}a_2a_4a_3a_4)a_5 \tag{3.28}$$

so as before we make the same perturbation shown in Figure 3.7. The resulting two fibers, with two nodes on each, are in $\mathbb{CP}^2\#13\overline{\mathbb{CP}}^2$. As in the previous cases, we blow them up twice (cf. Figure 3.7) and resolve their proper transforms $(\tilde{B}-2e_{14}-2e_{15}), (\tilde{B}-2e_{16}-2e_{17})$ with the section e_{13}. Again we obtain the class

$$\begin{aligned} s &:= (\tilde{B}-2e_{14}-2e_{15}) + (\tilde{B}-2e_{16}-2e_{17}) + e_{13} \\ &= 8h - 4e_1 - 2e_2 - 2e_3 - \cdots - 2e_{12} - e_{13} - 2e_{14} - \cdots - 2e_{17}, \end{aligned}$$

of square -13, where $\tilde{B} = 2C_\infty - e_2 - \cdots - e_{13} = 2(2h-e_1) - e_2 - \cdots - e_{13} = 4h - 2e_1 - e_2 - \cdots - e_{13}$ of self intersection zero. We note that we used the section e_{13} in Figure 3.13 to construct s. Hence we have that the fiber V* intersects the class s once in $\mathbb{CP}^2\#17\overline{\mathbb{CP}}^2$. This gives us the same plumbing P of length ten as in Figure 3.8, symplectically embedded in $\mathbb{CP}^2\#17\overline{\mathbb{CP}}^2$. We rationally blow it down. Moreover, the canonical class of $\mathbb{F}_3$ is $K_{\mathbb{F}_3} = -2C_\infty + F$ as we showed in the introduction of this chapter. So we have

$$K_{\mathbb{F}_3} = -2C_\infty + F = -2(2h-e_1) + (h-e_1) = -3h + e_1. \tag{3.29}$$

Therefore, in $(\mathbb{F}_3\#12\overline{\mathbb{CP}}^2)\#4\overline{\mathbb{CP}}^2 \cong (\mathbb{CP}^2\#13\overline{\mathbb{CP}}^2)\#4\overline{\mathbb{CP}}^2 \cong \mathbb{CP}^2\#17\overline{\mathbb{CP}}^2$ we have

$$K = -3h + e_1 + e_2 + \cdots + e_{17} \tag{3.30}$$

and the fiber class is

$$F = 4h - 2e_1 - e_2 - \cdots - e_{13}. \tag{3.31}$$

Hence we show that the resulting manifold after rational blow-down is an exotic copy of $\mathbb{CP}^2\#7\overline{\mathbb{CP}}^2$. The proof will follow the same lines of computations in the very first case above.

□

Remark 3.4.8. *In addition to the above constructions, we can also consider the dual fibers (III - III), (IV - IV), (VI - VI), (VIII-2 - VIII-3) and (IX-1 - IX-4) ([93, 94]). From each of these we obtain genus 2 fibration over* S^2 *with only two singular fibers. Then we construct exotic copies of* $\mathbb{CP}^2\#11\overline{\mathbb{CP}}^2$, $\mathbb{CP}^2\#9\overline{\mathbb{CP}}^2$, $\mathbb{CP}^2\#11\overline{\mathbb{CP}}^2$, $\mathbb{CP}^2\#10\overline{\mathbb{CP}}^2$, $\mathbb{CP}^2\#10\overline{\mathbb{CP}}^2$, *respectively.*

3.4.5 Singular fibers of types 5 (IX-2) and (IX-2) - 2 (IX-4)

In this last section, we will use the genus two fibrations on $K3\#2\,\overline{\mathbb{CP}}^2$ given in Section 3.2.2 (in [77]) to construct the exotic copies of $3\,\mathbb{CP}^2\#k\,\overline{\mathbb{CP}}^2$ for $k = 16, 17, 18, 19$. In our first construction we will work with such a fibration with five singular fibers of type IX-2 in the Namikawa and Ueno's classification list, and we will use the symplectic resolutions and rational blow-downs along -4 spheres. On the other hand, in our second construction we will work with the genus two fibration having one singular fiber of type IX-2 and two singular fibers of type IX-4 in the Namikawa and Ueno's list, and we will use the 2-nodal spherical deformations, symplectic blow-ups, symplectic resolutions, and (generalized) rational blow-down surgery. Let us begin with the following

Theorem 3.4.9. *[20] Let* M *be one of the following 4-manifolds*

1. $3\mathbb{CP}^2\#16\overline{\mathbb{CP}}^2$
2. $3\mathbb{CP}^2\#17\overline{\mathbb{CP}}^2$
3. $3\mathbb{CP}^2\#18\overline{\mathbb{CP}}^2$
4. $3\mathbb{CP}^2\#19\overline{\mathbb{CP}}^2$

Then there exist an irreducible symplectic 4-manifold homeomorphic but non-diffeomorphic to M*, obtained from a total space of genus two fibration with five singular fibers of type IX-2, using the combinations of symplectic resolution, and rational blow-down surgery along* -4 *sphere.*

Proof. We start with the genus two fibration structure on $K3\#2\,\overline{\mathbb{CP}}^2$ with five singular fibers of type IX-2 explained in Section 3.2.2 (see Lemma 3.2.2 and its proof). We recall that a singular fiber of type IX-2 is the union of three smooth rational curves F_i, G_i, E_i

(for $1 \leq i \leq 5$) passing through a single point, say x_i. Moreover, F_i is tangent to G_i at x_i, and the self-intersections of these rational curves are given as follows: $F_i^2 = G_i^2 = -3$, and $E_i^2 = -2$. For each of these five singular fibers, let us symplectically resolve the intersection point y_i of F_i and G_i (where $1 \leq i \leq 5$) to obtain a symplectic sphere $S_i = F_i + G_i$ of self-intersection -4 in $K3\#2\,\overline{\mathbb{CP}}^2$. Each of these -4 spheres S_i has a dual sphere F_i, which are all disjoint from each other. Let $M(i)$ denote the symplectic 4-manifold obtained from $K3\#2\,\overline{\mathbb{CP}}^2$ by performing the rational blow-down surgery along disjoint -4 spheres $S_1, \cdots, S_i$, where $1 \leq i \leq 5$.

Let us first verify that $M(i)$ is homeomorphic to $3\mathbb{CP}^2\#(21-i)\overline{\mathbb{CP}}^2$. This is a repated application of the Lemma 1.5.6 and Freedman's classification theorem (Theorem (1.5) of [42]). For the sake of completeness, let us give the details. Let P_2 be a tubular neighborhood of the sphere S_1 with self-intersection -4 in $K3\#2\,\overline{\mathbb{CP}}^2$. We have $M(1) = (K3\#2\overline{\mathbb{CP}}^2 - P_2) \cup B_2$, where B_2 is the rational homology ball whose boundary is the lens space $L(4,1)$ which also bounds P_2. We can contract the generator of $\pi_1(\partial P_2)$ using the sphere dual to S_1. Since we have the surjection $\pi_1(\partial B_2) \twoheadrightarrow \pi_1(B_2)$, $M(1)$ is simply connected by Van Kampen's Theorem. By applying the formulas of lemma 1.5.6, we compute

$$\begin{aligned}
e(M(1)) &= e((K3\#2\overline{\mathbb{CP}}^2) - e(P_2) + e(B_2) \\
&= 26 - 2 + 1 \\
&= 25, \\
\sigma(M(1)) &= \sigma(K3\#2\overline{\mathbb{CP}}^2) - \sigma(P_2) \\
&= -18 - (-1) \\
&= -17.
\end{aligned}$$

A repeated application of the Van Kampen's theorem shows that $M(i)$ is simply connected, and we have

$$\begin{aligned} e(M(i)) &= e((K3\#2\overline{\mathbb{CP}}^2) - ie(P_2) + ie(B_2) \\ &= -26 - 2i + i \\ &= 26 - i, \\ \sigma(M(i)) &= \sigma(K3\#2\overline{\mathbb{CP}}^2) - i\sigma(P_2) \\ &= -18 - (-i) \\ &= -18 + i. \end{aligned}$$

$M(i)$, for any $1 \leq i \leq 5$, contains a curve with odd self-intersection, so it is simply connected nonspin 4-manifold. Thus, we can conclude by Freedman's classification theorem that $M(i)$ is homeomorphic to $3\mathbb{CP}^2\#(21-i)\overline{\mathbb{CP}}^2$.

Next, we will verify that $M(i)$ is not diffeomorphic to $3\mathbb{CP}^2\#(21-i)\overline{\mathbb{CP}}^2$. By blow-up formula for the Seiberg-Witten function, we have $SW_{K3\#2\overline{\mathbb{CP}}^2} = SW_{K3}\prod_{k=1}^{2}(e^{e_k}+e^{-e_k}) = (e^{e_1} + e^{-e_1})(e^{e_2} + e^{-e_2})$, where e_k is an exceptional divisor class resulting from the k-th blow-up (of the base points of genus two pencil) in $K3\#2\overline{\mathbb{CP}}^2$. Consequently, the set of basic classes of $K3\#2\overline{\mathbb{CP}}^2$ are given by $\pm e_1 \pm e_2$, and the value of Seiberg-Witten invariants on $\pm e_1 \pm e_2$ are ± 1. It is routine to prove that after performing a single rational blow-down operation in $K3\#2\overline{\mathbb{CP}}^2$, along the sphere S_1, the resulting symplectic 4-manifold $M(1)$ is diffeomorphic to $K3\#\overline{\mathbb{CP}}^2$. $K3\#\overline{\mathbb{CP}}^2$ has a pair of basic classes $\pm K_{M(1)}$, which descends from the top classes $\pm(e_1 + e_2)$ of $K3\#2\overline{\mathbb{CP}}^2$. By applying Theorems from section 1.5.2, we completely determine the Seiberg-Witten invariants of $M(i)$ using the basic classes and invariants of $K3\#\overline{\mathbb{CP}}^2$: Up to sign the symplectic 4-manifold $M(i)$ has only one basic class which descends from the canonical class of $K3\#\overline{\mathbb{CP}}^2$. By Taubes theorem [120], the value of the Seiberg-Witten function on these classes $\pm K_{M_i}$ evaluates as ± 1 (see the Nonvanishing Theorems 1.4.4 above). By applying the connected sum theorem for the Seiberg-Witten invariant (see Theorem 1.4.3 above), Seiberg-Witten function is trivial for $3\mathbb{CP}^2\#(21-i)\overline{\mathbb{CP}}^2$. Thus, we have shown that $M(i)$ is not diffeomorphic to $3\mathbb{CP}^2\#(21-i)\overline{\mathbb{CP}}^2$.

Using Seiberg–Witten basic classes of $M(i)$, it is easy to verify that $M(i)$ is a minimal symplectic 4-manifold when $i \geq 2$. This follows from the the fact that for $i \geq 2$ $M(i)$ has no two basic classes K and K' such that $(K-K')^2 = -4$. Since symplectic minimality implies irreducibility for simply-connected 4-manifolds with $b_2^+ > 1$, it follows that $M(i)$ is also smoothly irreducible when $i \geq 2$.

□

In the following theorem we will use the 2-nodal spherical deformations introduced in Section 3.3, in addition to the symplectic blow-ups, symplectic resolutions, and (generalized) rational blow-down surgery.

Theorem 3.4.10. *[20] There exist an irreducible symplectic 4-manifold homeomorphic but non-diffeomorphic to* $3\mathbb{CP}^2\#17\overline{\mathbb{CP}}^2$*, obtained from a total space of genus two fibration with one singular fiber of type IX-2 and two singular fibers of type IX-4, using combinations of the* 2*-nodal spherical deformations, the symplectic blow-ups, symplectic resolution, and the rational blow-down surgery.*

Proof. We start with the genus two fibration structure on $K3\#2\,\overline{\mathbb{CP}}^2$ with one singular fiber of type IX-2 and two singular fibers of type IX-4 given in Section 3.2.2 (see Lemma 3.2.2). We recall that the singular fiber of type IX-4 consists of 9 rational curves E'_j, $(1 \leq j \leq 7)$, F_i, G_i. The intersection matrix corresponding to the first 7 smooth rational curves E'_j has the type D_7. This means that nonzero entries of this matrix are given by $E'_1 \cdot E'_2 = E'_2 \cdot E'_3 = E'_3 \cdot E'_4 = E'_4 \cdot E'_5 = E'_5 \cdot E'_6 = E'_5 \cdot E'_7 = 1$, $E'^2_j = -2$. If λ_i is a multiple root (see section 3.2.2), then F_i and G_i are disjoint and each of them meets one componet of D_7. We will assume that F_i meets E'_7 and G_i meets E'_6. Moreover it is easy to check that each of the two sphere sections e_1 and e_2, resulting from two blow-ups of the base points of genus two pencil in the $K3$ surface, hits only one of the components F_i and G_i. Let us assume that $e_1 \cdot F_i = e_2 \cdot G_i = 1$.

The monodromy relation corresponding to the above splitting of the singular fibers in $K3\#2\,\overline{\mathbb{CP}}^2$ is given by the following word

$$1 = (a_1a_2a_3a_4a_5^2)^5 \quad = \quad (a_1a_2a_3a_4a_5^2)(a_1a_2a_3a_4a_5^2)^2(a_1a_2a_3a_4a_5^2)^2,$$

where the monodromy $(a_1a_2a_3a_4a_5^2)$ corresponds to a type IX-2 singular fiber, and each $(a_1a_2a_3a_4a_5^2)^2$ corresponds to a type IX-4 singular fiber [69, 68, 94]. By Lemma 3.3.3 the word $(a_1a_2a_3a_4a_5^2)$ can be deformed to contain two disjoint 2-nodal spherical singular fibers and two nodal fibers in $K3\#2\,\overline{\mathbb{CP}}^2$. We will only use one of these 2-nodal spherical singular fibers to build a negative-definite plumbing tree for the rational blow-down. Let us blow-up two nodes of one of these 2-nodal spherical singular fibers, say B, and symplectically resolve the intersection point of its proper transform $\tilde{B} - 2e_3 - 2e_4$ with the sphere section e_1 and intersection point of the sphere section with -3 sphere F_i. The resulting symplectic sphere S has a self-intersection -8. Notice that S together with the spheres E'_7, E'_5, E'_4, E'_3 form a negative-definite plumbing tree P_6 symplectically embedded in $K3\#4\,\overline{\mathbb{CP}}^2$.

We remark that we can obtain the same plumbing by using the fibration with three singular members of type IX-2, and one singular member of type IX-4 (cf. Section 3.2.2).

Now let $M(3,17)$ denote the symplectic 4-manifold obtained from $K3\#4\,\overline{\mathbb{CP}}^2$ by performing the rational blow-down surgery along P_6. Similarly as before, we show $M(3,17)$ is simply connected; we contract the generator of $\pi_1(\partial P_6)$ using the sphere E'_6 that was not used in the rational blow-down plumbing P_6. Using the formulas, we compute the invariants Euler characteristic e, and signature σ of $M(3,17)$, and by Freedman's classification theorem we conclude that $M(3,17)$ is homeomorphic to $3\mathbb{CP}^2\#17\overline{\mathbb{CP}}^2$. Lastly, we verify the non-triviality of the Seiberg-Witten invariants of $M(3,17)$ and we conclude our proof as in the proof of the previous theorem.

□

■

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