Continued Fractions with Irrational Numerators

A Thesis SUBMITTED TO THE FACULTY OF THE
UNIVERSITY OF MINNESOTA
BY

Kalani Thalagoda

IN PARTIAL FULFILLMENT OF THE REQUIERMENTS FOR
THE DEGREE OF MASTER OF SCIENCE

Adviser: Professor John Greene

June 2018

Abstract

A continued fraction is an expression of the form

$$x = a_0 + \cfrac{z}{a_1 + \cfrac{z}{a_2 + \cfrac{z}{a_3 + \cdots}}},$$

denoted by $[a_0, a_1, a_2, ...]_z$. A continued fraction is called a simple continued fraction when $z = 1$ and non-simple otherwise.
Simple continued fractions of $\sqrt{n}$ have particularly "nice" periodic patterns. For example,

$$\sqrt{19} = [4, \overline{2, 1, 3, 1, 2, 8}]_1,$$

where the sequence $2, 1, 3, 1, 2, 8$ is the periodic part which repeats. The pattern is "nice" because the first term is always half the last terms and the sequence $2, 1, 3, 1, 2$ is identical when reversed.

In this report, we study the periodic behavior of non-simple continued fractions of the form $\sqrt{a + b\sqrt{n}}$ with $z = \sqrt{n}$. We show that these have many similarities to simple continued fraction expansions of $\sqrt{n}$. For example,

$$\sqrt{100 + 35\sqrt{6}} = [13, \overline{3, 2, 3, 6, 3, 2, 3, 26}]_{\sqrt{6}}.$$

Here the sequence $3, 2, 3, 6, 3, 2, 3$ is also identical when reversed and last term is twice the first.

We show that when $x = \sqrt{a + b\sqrt{n}}$ with $z = \sqrt{n}$, the above pattern is correct for every period length less than 4. For period length up to 7, with some extra assumptions, we prove the pattern remains.

Contents

List of Tables iii

1 Introduction 1
- 1.1 Simple continued fractions . . . 3
- 1.2 Motivation . . . 4

2 Background 5
- 2.1 Generalized continued fraction expansions of $\sqrt{n}$ . . . 5
- 2.2 Generalized continued fraction expansions with rational numerators 6
- 2.3 Non-Simple continued fraction expansions of rationals . . . 8
- 2.4 Continued fraction expansions of $a + b\sqrt{n}$ . . . 9
  - 2.4.1 Period Length 1 . . . 12
  - 2.4.2 Period length 2 . . . 16

3 Continued fraction expansions of the form $\sqrt{a + b\sqrt{n}}$ 18
- 3.1 Introduction . . . 18
- 3.2 Period Length 1 . . . 19
- 3.3 Period Length 2 . . . 20
- 3.4 Period Length 3 . . . 23
- 3.5 Period Length 4 . . . 23
- 3.6 Longer period Lengths . . . 24
  - 3.6.1 Period length 5 . . . 24
  - 3.6.2 Period length 6 . . . 25
  - 3.6.3 Period length 7 . . . 26

4 Infinite Families 27
- 4.1 Period length 3 . . . 28
- 4.2 Period length 4 . . . 29
- 4.3 Longer period lengths . . . 31
  - 4.3.1 Period length 5 . . . 31
  - 4.3.2 Period length 6 . . . 32
  - 4.3.3 Period length 7 . . . 33

5 Future Work 36

References 37

6 Appendix 38
- 6.1 Computer search results . . . 38
  - 6.1.1 Results for $a + b\sqrt{n}$ . . . 38
  - 6.1.2 Results for $\sqrt{a + b\sqrt{n}}$ . . . 40
- 6.2 Mathematica code . . . 41

List of Tables

2.1 Summary of observed tail lengths for $a + b\sqrt{n}$ 9
2.3 Values of n arranged by different tail lengths for $a + b\sqrt{n}$ 10
2.4 Tail lengths observed arranged according to n for $a + b\sqrt{n}$ 10
2.5 Frequency of period lengths observed for $a + b\sqrt{n}$ 11
2.6 Value of n arranged by possible period lengths for $a + b\sqrt{n}$ 11
2.7 Possible period lengths arranged according to n for $a + b\sqrt{n}$ 11
3.1 Observed period lengths and percentages for $\sqrt{a + b\sqrt{n}}$ 18
3.2 Period lengths observed for different n values for $\sqrt{a + b\sqrt{n}}$ 18
4.1 Summary of Diophantine equations 35
6.1 Complete list of observed tail lengths for expansion of $a + b\sqrt{n}$ . . . 38
6.2 Complete list of n-values arranged by different tail lengths for expansion of $a + b\sqrt{n}$ 39
6.3 Complete list of Tail lengths for expansions of $\sqrt{a + b\sqrt{n}}$ 40
6.4 Complete list of observed period lengths for expansion of $\sqrt{a + b\sqrt{n}}$ 40
6.5 Period lengths observed for different n values 41

1 Introduction

A continued fraction is as expression of the form,

$$x = a_0 + \cfrac{b_0}{a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \cdots}}}$$

where a_i and b_i could be real or complex numbers for all integers i [9]. In this report, we restrict ourselves to the case where a_i is an integer for each i and b_i is equal to a constant, z, for all i. We call each a_i, the terms of the expansion and we call the constant z, the numerator of the expansion. With these extra conditions, we introduce the following more concise notation. A continued fraction of the format

$$x = a_0 + \cfrac{z}{a_1 + \cfrac{z}{a_2 + \cfrac{z}{a_3 + \cdots}}},$$

is represented by $x = [a_0, a_1, a_2, \ldots, a_j, a_{j+1}, \ldots]_z$.

Continued fractions can be categorized according to the nature of their numerators. A continued fraction is called a simple continued fraction if for all i we have $b_i = 1$. Otherwise, it is called a generalized continued fraction. Simple continued fraction expansions can also be finite or infinite. For example, any rational number can be expressed as a finite simple continued fraction expansion, while an irrational number has an infinite simple continued fraction expansion [4, 9]. For more concrete examples, lets consider a rational number

$$\frac{42}{19} = [2, 4, 1, 3]_1$$

and an irrational number,

$$\pi = [3, 7, 15, 1, ...]_1.$$

Another important behavior of continued fractions is periodicity. That is, after some point in the expansion the terms of the expansion repeat. For example,

$$\sqrt{30} = [5, 2, 10, 2, 10, 2, 10, ...]_1,$$

where the pattern $2, 10$ repeats.

The following notation is used to denote a continued fraction that eventually gives a periodic behavior:

$$x = [a_0, a_1, a_2, \ldots, a_{j-1}, \overline{a_j, a_{j+1}, \ldots, a_{j+k-1}}]_z.$$

We call the part $a_0, a_1, a_2, \ldots, a_{j-1}$ the tail of the expansion, while $a_j, a_{j+1}, \ldots, a_{j+k-1}$ is referred to as the periodic part of the expansion. For the above example the expansion is $[5, \overline{2, 10}]_1$, with a tail length of 1 and period length of 2.

In order to study continued fractions, we require an algorithm to obtain the continued fraction expansion of any given value, x. In the most general context, the continued fraction algorithm goes as follows.

Theorem 1.1. *Given $x, z \in \mathbb{R}$ and $x, z \geqslant 1$,*

1. $x_0 = x$
2. *Pick $a_i \in \mathbb{Z}^+$ such that $\lceil x_i - z \rceil \leqslant a_i \leqslant \lfloor x_i \rfloor$*
3. *If $x_i - a_i \neq 0$ then $x_{i+1} = \frac{z}{x_i - a_i}$*
4. *Go back to step 2.*

Then x has the expansion $[a_0, a_1, a_2, \ldots]_z$.

Proof. This is shown in [1]. □

For a unique expansion, we can select each a_i in a specific manner. If each $a_i = \max(1, \lceil x_i - z \rceil)$, we call that the minimal expansion. Another approach is to always chose $a_i = \lfloor x_i \rfloor$. We call this the maximal expansion. In this report, we only consider the maximal continued fraction expansion. It can be shown that the above algorithm can be carried out and it converges to x whenever $z \geqslant 1$ [5].

In the following chapters we see that it is helpful to do a computational search to recognize patterns prior to doing a more general analysis. This requires efficient algorithms to check if a number is periodic. We also require a method to extract the tail and the period of a periodic continued fraction expansion. We can see that a continued fraction expansion has periodic behavior if and only if there exist some distinct indexes i and j for which $x_i = x_j$. Here x_i and x_j are obtained using the Theorem 1.1 above [5]. Using the smallest pair of such indexes $i < j$, we can recover the lengths of the tail and periodic part. The value $j - i$ gives us the length of the period and $i - 1$ gives us the length of the tail. This can be used to extract the terms in the tail and the period of the expansion.

Computationally, we can do the above using a simple loop to compare each x_i and x_j values for all distinct indexes i and j. This method is efficient for shorter tail and period lengths, but less efficient for longer lengths. For example, if a number is purely periodic with a period length of 50, we need to make more than 4000 comparisons to recognize this. For numbers of the form $\sqrt{a + b\sqrt{n}}$ where periodicity is rare, the above method is time consuming. Therefore, Floyd's cycle finding algorithm was used to recognize the periodicity and later values with periodicity were further analyzed to extract tails and periods of the expansions. Floyd's cycle finding algorithm states that, for distinct i, j indexes, $x_i = x_j$ if and only if $x_k = x_{2k}$ for some index k [2]. With the above algorithm, if we were to check for periodicity within the first 100 terms, we only have to do 50 comparisons against more than 4000 required by the previous method. Although this algorithm helps speed up the search, the range needs to be sufficient to capture this behavior, especially for longer period lengths. Therefore, the code collaborating both the methods is used in the computer search to make the search efficient and accurate.

1.1 Simple continued fractions

Simple continued fractions have been extensively studied and there is a large body of research related to them [1, 5, 10]. They are now included as classical results in most undergraduate textbooks like [3, 6, 7] and they continue to be studied a great deal.

Although our main concentration is in generalized continued fractions, it is important to look for similarities. Previous studies have shown that generalized continued fractions follow similar patterns to those of simple continued fractions under certain circumstances [1]. Therefore it is important to make this comparison. Hence, we summarize some important results for simple continued fractions.

Simple continued fraction expansions are known for their nice behavior. It is shown in [9] that any quadratic surd has an eventually periodic continued fraction expansion. And when a quadratic surd, x, is reduced, that is $x > 1$ and the conjugate satisfies $0 \geqslant \overline{x} \geqslant -1$, the periodic part has a nicer pattern as given below.

Theorem 1.2. *If a quadratic surd, x, is reduced then for some integers $a_1, a_2, .., a_k$, $x = [\overline{a_1, a_2, ..., a_k}]_1$ and $-\frac{1}{\overline{x}} = [\overline{a_k, a_{k-1}, ..., a_1}]$.*

For example, $x = \frac{5+\sqrt{37}}{3} > 1$ and $0 \geqslant \frac{5-\sqrt{37}}{3} \geqslant -1$. The continued fraction expansion of $\frac{5+\sqrt{37}}{3}$ is $[\overline{3, 1, 2}]_1$ and continued fraction expansion of $-\frac{1}{\overline{x}} = \frac{-3}{5-\sqrt{37}} = \frac{5+\sqrt{37}}{4}$ is $[\overline{2, 1, 3}]_1$.

Another important fact about simple continued fractions is that the expansion of $\sqrt{n}$ is particularly nice. The proof of the following theorem uses the fact that $a + \sqrt{n}$ is a reduced quadratic surd, where $a = \lfloor \sqrt{n} \rfloor$ and n is not a perfect square [9].

Theorem 1.3. *The continued fraction expansions of $\sqrt{n}$ where n is not a perfect square has the form $[a_0, \overline{a_1, a_2, ..., a_{k-1}, a_k}]_1$ where $a_k = 2a_0$ and the sequence $a_1, a_2, .., a_{k-1}$ is palindromic, that is the sequence $a_1, a_2, .., a_{k-1}$ is the same as $a_{k-1}, a_{k-2}, ..., a_2, a_1$.*

For example,

$$\sqrt{71} = [8, \overline{2, 2, 1, 7, 1, 2, 2, 16}]_1.$$

There are many applications of continued fractions, specially of simple continued fractions. For example, simple continued fractions are used to obtain the best rational approximations for any real number, more importantly irrational numbers like $\sqrt{n}$ where n is square free. Simple continued fractions are also useful in finding solutions to the Pell's equation, $x^2 - ny^2 = \pm 1$.

A valuable tool for analysis on continued fractions are the convergents. Here we summarize general results about convergents given in [1], [5] and [10] that are used in different sections of this report. In an infinite continued fraction expansion, $x = [a_0, a_1, a_2, \ldots, a_k, a_{k+1}, \ldots]_z$, we define the k^{th} convergents to be the first k terms of the expansion, that is $C_k = [a_0, a_1, a_2, \ldots, a_{k-1}]_z$ [1, 2]. Given a sequence $a_0, a_1, a_2, ...$, we can define two polynomial sequences (p_k) and (q_k) as follows:

$$\begin{array}{llll} p_{-1} = 1, & p_0 = a_0, & p_k = a_k p_{k-1} + z p_{k-2} & \text{for } k \geqslant 1, \\ q_{-1} = 0, & q_0 = 1, & q_k = a_k q_{k-1} + z q_{k-2} & \text{for } k \geqslant 1. \end{array}$$

In [5], Theorem 2 it was shown that the k^{th} convergent of the expansion $[a_0, a_1, ...a_k, ...]_z$ satisfies

$$C_k = \frac{p_k}{q_k} = [a_0, a_1, ...a_{k-1}]_z.$$

It was also shown in [5] that

$$[a_0, a_1, ...a_k, x]_z = \frac{p_k x + z p_{k-1}}{q_k x + z q_{k-1}}.$$

1.2 Motivation

In this project, we study the continued fraction expansions of $a + b\sqrt{n}$ and $\sqrt{a + b\sqrt{n}}$ with $z = \sqrt{n}$, where a, b and n are positive integers with n not being a perfect square of an integer. The first case is an extension to the study done in [8]. The second case is motivated by the observation that the expansion of $\sqrt{a + b\sqrt{n}}$ with $z = \sqrt{n}$ mimics properties of simple continued fractions of $\sqrt{n}$.

To further motivate the two forms, we analyze the expansion with $z = \sqrt{n}$ using the theorem on convergents introduced in the previous section. By induction we can see that if $z = \sqrt{n}$, then for each n, the polynomials p_k and q_k have the following form,

$$p_k = f_k(n, a_0, a_1, ..., a_k) + \sqrt{n} g_k(n, a_0, a_1, ..., a_k)$$

and

$$q_k = h_k(n, a_1, ..., a_k) + \sqrt{n} l_k(n, a_1, ..., a_k)$$

for polynomial functions f_k, g_k, h_k and l_k with integer coefficients. Here f_k and h_k represent the rational portion of the polynomial while g_k and l_k represents the irrational portion. Also note that for constant a_i for all i, which are integers, the functions f_k, g_k, h_k and l_k also evaluate to be integers. Suppose for some periodic continued fraction expansion, $x = [a_0, a_1, ...a_{k-1}, \overline{a_k, .., a_{j+k}}]_z$. Then x_k has the expansion $[\overline{a_k, .., a_{j+k}}]_z$ which is purely periodic. Thus,

$$x_k = [a_k, .., a_{j+k}, x_k]_z = \frac{p_{j+k} x_k + \sqrt{n} p_{j+k-1}}{q_{j+k} x_k + \sqrt{n} q_{j+k-1}}.$$

Solving for x_k we get that x_k satisfy a quadratic equation

$$q_{j+k} x_k^2 + (\sqrt{n} q_{j+k-1} - p_{j+k}) x_k - \sqrt{n} p_{j+k-1} = 0,$$

which gives that

$$x_k = \frac{p_{j+k} - \sqrt{n} q_{j+k-1} \pm \sqrt{(p_{j+k} - \sqrt{n} q_{j+k-1})^2 + 4\sqrt{n} q_{j+k} p_{j+k-1}}}{2 q_{j+k}}.$$

Also, $x = [a_0, a_1, ...a_{k-1}, x_k]_z = \dfrac{p'_{k-1} x_k + z p'_{k-2}}{q'_{k-1} x_k + z q'_{k-2}}$ where p' and q' are polynomials defined based on x_k. Now each p, p', q, and q' have the form $c + d\sqrt{n}$ for some integers c and d. This force x to have the form,

$$x = P + Q\sqrt{n} + (R + S\sqrt{n})\sqrt{U + V\sqrt{n}},$$

where P, Q, R, S, U and V are rationals which depend on the sequence $a_0, a_1, ...,$ $a_{k-1}, a_k, .., a_{j+k}$ and n. So for $z = \sqrt{n}$, the simplest forms for a real number x with a periodic expansion are $\sqrt{a + b\sqrt{n}}$ and $a + b\sqrt{n}$, where a and b are integers.

2 Background

Continued fractions with $z > 1$, have not been extensively studied. However, they are starting to get a bit more attention. In this chapter we summarize studies done on generalized continued fractions.

We start with the relevant results from a paper by Anslem and Weintraub [1] where the case when z is a positive integer other than 1 was studied. Also we discuss results from [5] and [10] by J. Greene and J. Schmieg on continued fractions with z being a rational number. Finally, the more closely related case of continued fraction expansions of rational numbers with $z = \sqrt{n}$ will be discussed with reference to a research project by Pablo Mello [8].

Although each of the above cases are different from the main forms $a+b\sqrt{n}$ and $\sqrt{a+b\sqrt{n}}$, with $z = \sqrt{n}$ studied in this research, they have similarities. Thus, we summarize the findings of the above studies in this chapter and later the results obtained are compared. In the following chapters, only maximal expansions of continued fractions will be considered.

2.1 Generalized continued fraction expansions of $\sqrt{n}$

In [1] it was found that $\sqrt{n}$ can have particularly nice periodic patterns with integer z-values other than 1. In classical continued fraction theory, it is known that quadratic surds, that is a real numbers of the form $\frac{P+Q\sqrt{n}}{R}$, where P, Q, R and n are integers with n not being a perfect square, have expansions which are eventually periodic [9, Lagrange's Theorem]. It was shown that this theorem can also be extended to expansions of quadratic surds with positive integers $z \neq 1$.

Theorem 2.1. *Let x be a quadratic surd. Then for any positive integer N, x has a periodic continued fraction expansion when $z = N$.*

However, the maximal expansion of a quadratic surd when $z = N$ might not be periodic.

Conjecture 2.2. *For $N \geqslant 2$, the maximal expansion of a quadratic surd is not always periodic.*

Computational evidence was provided in [1] to justify the above conjecture. For example, the maximal expansion of $\sqrt{124}$ with $z = 2$ is not periodic within the first 6000 terms [1]. However for small z-values, the expansion of $\sqrt{n}$ can be periodic [1]. A frequently occurring requirement for a continued fraction expansion of a quadratic surd to have purely periodic behavior is for it to be reduced. A quadratic surd, $x = \frac{P+Q\sqrt{n}}{R}$ is $z-$reduced if $x > z$ and the conjugate root or the Galois conjugate of x, denoted by $\overline{x}$, satisfies $-1 < \overline{x} < 0$. For simple continued fractions it is known that a quadratic surd is purely periodic if and only if it is 1-reduced [9]. In [1] the following theorem goes parallel to this.

Theorem 2.3. *Suppose x is N-reduced and has a maximal periodic expansion when $z = N$. Then it is purely periodic. Moreover, if the maximal expansion of x is $[\overline{a_0, a_1, a_2, ..., a_{k-1}, a_k}]_N$ then $\frac{-N}{\overline{x}} = [\overline{a_k, a_{k-1}, ..., a_2, a_1, a_0}]_N$.*

It is also known that simple continued fraction expansions of $\sqrt{n}$ have periodic parts that follow a palindromic pattern [9]. The corresponding theorem in the generalized continued fraction case goes as follows.

Theorem 2.4. *Let $N < 2\sqrt{n}$ be a positive integer.If $\sqrt{n}$ has a periodic continued fraction expansion, then the expansion has the form*

$$\sqrt{n} = [a_0, \overline{a_1, ..., a_{k-1}, 2a_0}]_N,$$

with $a_i = a_{k-i}$ for each integer i, $1 \leqslant i \leqslant k-1$.

For example,

$$\sqrt{103} = [10, \overline{47, 497, 47, 20}]_7,$$
$$\sqrt{118} = [10, \overline{2, 6, 6, 2, 20}]_2.$$

It was also shown in [3] that when $N > 2\sqrt{n}$ the palindromic patterns tend to disappear. For example,

$$\sqrt{65} = [8, 2312, \overline{149, 702, 184, 341, 184, 702, 149, 2320}]_{144}$$

is palindromic but

$$\sqrt{31} = [5, 22, \overline{14, 26, 56, 23}]_{13}$$

is not.

So in summary, even though there may be cases of non-palindromic behavior, when N is small the behavior of periodic generalized continued fractions is similar to that of simple continued fractions.

2.2 Generalized continued fraction expansions with rational numerators

In [5], the case where z is a rational number was studied. A special focus was given to expansions of $\sqrt{n}$ and patterns were compared with simple continued fractions of $\sqrt{n}$ to find similarities and differences. Here we summarize the major results for the above case.

In this case it was shown that it is not necessary for x be a quadratic irrational to have periodic behavior. For example, $7/4 = [\,\overline{1}\,]_{21/16}$. However, it was also shown that if x has a periodic expansion it has to satisfy a quadratic equation.

Theorem 2.5. *If $x \in \mathbb{R}$ has a continued fraction expansion,*

$$[a_0, a_1, ..., a_{i-1}, \overline{a_i, a_{i+1}, ..., a_{i+k-1}}]_z$$

then x satisfies the quadratic equation

$$Qx^2 + Px + R = 0$$

where, P, Q and R are functions of $n, a_0, a_1, ..., a_{i-1}, ..., a_{i+k-1}$.

For example, if $x = [\,\overline{1}\,]_{21/16}$ then $x = 1 + \frac{21/16}{x}$, which implies that x satisfies the quadratic equation $x^2 - x - \frac{21}{16} = 0$ and $x = 7/4$ is a root for the above quadratic equation.

Unlike in continued fraction expansions where z is an integer, when z is a rational, a quadratic irrational x being z-reduced is not enough to guarantee a purely periodic expansion. In this case we require x to be strongly reduced. In [5] the concept of being strongly reduced is defined as follows.

Definition 2.6. *A quadratic irrational is said to be strongly reduced if it is reduced and the terms of the expansion satisfies* $a_i \geqslant z$ *for each* i.

The corresponding theorem is as follows.

Lemma 2.7. *If* x *has a strongly reduced periodic continued fraction expansion, then* x *is purely periodic. Moreover,* $-\frac{z}{x}$ *is also strongly reduced.*

Now looking at patterns of continued fractions expansions of $\sqrt{n}$ when z is a rational number, the following were found.

Theorem 2.8. *Suppose for some* n*, where* n *is not a perfect square, the continued fraction expansion of* $\sqrt{n}$ *is a strongly reduced periodic maximal expansion with* z*, a rational number. Let* $a = \lfloor \sqrt{n} \rfloor$*. Then,* $\sqrt{n} = [a, \overline{a_1, ..., a_{k-1}, 2a}]_z$ *when* $z < a + \sqrt{n}$ *and* $\sqrt{n} = [a, a_1, \overline{a_2, ..., a_k, a_1 + h}]_z$*, where* $h = \lfloor \frac{z}{a+\sqrt{n}} \rfloor$ *when* $z > a + \sqrt{n}$.

For example,

$$\sqrt{5} = [2, \overline{4, 1, 6, 11180, 6, 1, 4, 4}]_{20/17},$$

where $z = 20/17 < \sqrt{5} + 2$ and,

$$\sqrt{14} = [3, 18, \overline{66, 20, 5454, 20}]_{27/2},$$

where $z = 27/2 > \sqrt{14} + 3$.

Furthermore, the case when $z < a + \sqrt{n}$ has more structure.

Theorem 2.9. *Let* z *be a rational number with* $z < a + \sqrt{n}$ *where* $a = \lfloor \sqrt{n} \rfloor$*. If* $\sqrt{n}$ *has a strongly reduced periodic maximal expansion, then* $\sqrt{n} = [a, \overline{a_1, ..., a_{k-1}, 2a}]_z$ *where the sequence* $a_1, a_2, ..., a_{k-1}$ *is palindromic.*

Here are few such examples:

$$\sqrt{2} = [1, \overline{4, 70, 4, 2}]_{5/3},$$
$$\sqrt{11} = [3, \overline{8, 6}]_{8/3}.$$

Suppose for $x = \sqrt{2}$, we pick $z = 4/3$. Then the expansion is

$$\sqrt{2} = [1, \overline{3, 6, 14, 1, 2, 2}]_{4/3}.$$

The palindromic pattern is not preserved in the this case because the expansion is not strongly reduced as $1 < 4/3$.

Many infinite families of continued fraction expansions of $\sqrt{n}$ were found when $n = a^2 + b$ with $1 \leqslant b \leqslant 2a$ and $1 \leqslant z \leqslant 2a$. This case is also related to Pell's equation when the period length of the expansion is 4.

Theorem 2.10. *If* $n = a^2 + b$ *with* $1 \leqslant b \leqslant 2a$ *then* $\sqrt{n}$ *has the maximal expansion* $[a, \overline{c, d, c, 2a}]_z$ *if and only if* (x, d) *is a positive solution to the Pell's equation* $x^2 - nd^2 = b^2$ *for some integer* x*. When* d *is such a solution, an expansion will exist for* $z = \frac{c(x+b-ad)}{2a+d}$ *where* c *is chosen so that* $0 < z \leqslant \min(2a, d)$.

For example, if $a = b = 1$, then $n = 2$. Then the corresponding Pell's equation is $x^2 - 2d^2 = 1$. A solution for this is $(3, 2)$ which leads to $z = \frac{c}{2}$ where $0 < c \leqslant 3$. So the expansion becomes $\sqrt{2} = [1, \overline{c, 2, c, 2}]_z = [1, \overline{c, 2}]_z$. When $c = 2$ then $z = 1$, so $\sqrt{2} = [1, \overline{2}]$. If $c = 3$ then $z = \frac{3}{2}$, so $\sqrt{2} = [1, \overline{3, 2}]_{\frac{3}{2}}$. Also, $(99, 70)$ is a solution to $x^2 - 2d^2 = 1$. Consequently, $\sqrt{2}$ has expansions $[1, \overline{c, 70, c, 2}]_z$ with $z = \frac{5c}{12}$ for any $0 < c \leqslant 4$. For different values of c in the above range, following are the corresponding expansions. When $c = 1$, $\sqrt{2} = [1, \overline{1, 70, 1, 2}]_{\frac{5}{12}}$. When $c = 2$, $\sqrt{2} = [1, \overline{2, 70, 2, 2}]_{\frac{5}{6}}$. When $c = 3$, $\sqrt{2} = [1, \overline{3, 70, 3, 2}]_{\frac{5}{4}}$. Finally, when $c = 4$, $\sqrt{2} = [1, \overline{4, 70, 4, 2}]_{\frac{5}{3}}$.

2.3 Non-Simple continued fraction expansions of rationals

In a research project by Pablo Mello[8], non-simple continued fractions of rationals with numerators $z = \sqrt{n}$, where n is not a perfect square were studied. This case is related to the main discussion of this paper because of the following. If x has a purely period expansion of the form, $x = [\overline{d}]_{\sqrt{n}}$, then $x = \frac{d + \sqrt{d^2 + 4\sqrt{n}}}{2}$ which is very close to the form $\sqrt{a + b\sqrt{n}}$.

When considering expansions of rational numbers the following was said about each x_k obtained from the continued fraction algorithm.

Theorem 2.11. *If x is a rational number, then each x_k is either a rational number or a quadratic irrational.*

The case when x has period length of 1 was studied in the research project to give the following findings.

Theorem 2.12. *There are infinitely many quadratic surds that can be expanded in a purely periodic expansion with period length* 1*.*

An example of such an infinite family found in [8] is

$$k + 1 + \sqrt{k^2 + k} = [\overline{2k + 1}]_{\sqrt{k^2+k}}$$

with $k \geqslant 1$.

For rationals, the following infinite family was found.

Theorem 2.13. *Every rational number of the form $\frac{k+1}{k}$ with $k > 1$ has a periodic expansion with $z = \sqrt{4k^2 - 4}$ of the form $\frac{k+1}{k} = [1, 2k^2 - 2, 2k - 1, \overline{2k}]_z$.*

The case where the period length was 2 was also studied in the report to give the following theorems.

Theorem 2.14. *For any given n, there are at most finitely many purely periodic quadratic surds of period length* 2*.*

An example of an infinite family with period length 2 was obtained when $n = k^2 - 1$ for any $k > 1$.

Theorem 2.15. *If $n = k^2 - 1$ for some $k > 1$, there are infinitely many rational numbers of the form $\frac{k+1}{k}$ which have a periodic expansion with period length* 2*.*

2.4 Continued fraction expansions of $a + b\sqrt{n}$

In line with and extending the work done in [8], we analyze the case of periodic expansions of the format $a + b\sqrt{n}$ with $z = \sqrt{n}$. The relation to the case studied in [8] is as follows.

If we consider $x = \frac{p}{q}$ with $z = \sqrt{n}$ then each subsequent x_i of the continued fraction expansion has the format $s_i + t_i\sqrt{n}$ for some rational numbers s_i and t_i. Now if we consider the continued fraction expansion of $x = a + b\sqrt{n}$, the same will be true. Therefore, the above case has similarities to the case discussed in [8].

Prior to studying the theoretical aspect of the question, a Mathematica based computer search was done. This provided us a very general idea about possible patterns of the expansions. Also, this laid a basis to construct the theoretical results and helped to discover infinite families discussed in this chapter.

Numbers of the form $a + b\sqrt{n}$, where $0 \leqslant a \leqslant 100$ and $0 < b \leqslant 100$ were expanded for $z = \sqrt{n}$ with $1 < n < 100$ where n is not a perfect square. The search looked for periodic behavior within the first 100 terms of the continued fraction expansion. This limited the longest tail length recoverable to be 100 and the longest period length recoverable to be a period length 50. Results of the search can be summarized as follows. It was noted that not all values of n had periodic continued fraction expansions. Out of the 90 values of n only 18 numbers had periodic continued fraction expansions for the search range of a and b. Only about 2% of all the numbers searched displayed periodic behavior which is a total of 17958 numbers out of 909000 numbers. Tail length of the expansions were immensely varied in a range from 0 to 93 . The distribution of the tail lengths obtained are as follows, ranked according to the range of frequency of occurrence as given in Table 2.1.

Table 2.1: Summary of observed tail lengths for $a + b\sqrt{n}$

Tail length	Frequency Range
1	More than 2000
2, 5, 8	1000 – 2000
11	900 – 999
15, 19, 23	800 – 899
6, 22	700 – 799
25	600 – 699
9, 7	500 – 599
3, 4, 13, 16, 24, 27	400 – 499
10, 12, 40, 41, 42, 43, 44, 47, 62	300 – 399
14, 21, 26, 29, 30, 31, 32, 33, 36, 37, 38, 48	200 – 299
17, 20, 28, 34, 35, 39, 50, 55, 59, 60, 61, 64, 66, 71, 76, 80, 82, 85, 93	100 – 199
0	Less than 100

Table 2.3: Values of n arranged by different tail lengths for $a + b\sqrt{n}$

Tail length	n
0	$2, 8, 10, 12, 15, 20, 24, 30, 35, 42, 48, 56, 63, 72, 80, 90$
1	$2, 3, 6, 8, 10, 12, 15, 20, 24, 30, 35, 42, 48, 56, 63, 72, 80, 90$
2	$3, 8, 12, 15, 24, 32, 35, 48, 60, 63, 80, 96$
3	$8, 12, 32, 60, 96$
4	$2, 3, 6$
5	$2, 3, 6, 8, 10$
6	$2, 3, 8$
7	$3, 8, 12$
8	$2, 3, 6$
Longer tail lengths	$2, 3, 6, 8, 12$

Also, as n increased, longer period lengths tended to disappear. One reason for this could be that as n increases, the range of a and b becomes insufficient to capture such patters. In summary, following are the values of n and corresponding tail lengths which were found.

Table 2.4: Tail lengths observed arranged according to n for $a + b\sqrt{n}$

n	Tail lengths
2	$1, 4, 5, 6, 8, 9, 10, 11, 12, 13, 14, 15, 16, 19$
3	$1, 2, 4, 5, 6, 7, 8, 10, 11, 15, 19, 20, 21, 22, 23, 25, 26, 27, 31, 32, 33, 35, 41, 62, 64$
6	$1, 4, 5, 8, 9, 12, 13, 15, 16, 17, 19, 20, 22, 23, 24, 25, 26, 29, 30, 31, 36, 39, 40, 42, 43, 44, 47, 48, 50, 61, 62$
8	$1, 2, 5, 6, 7, 16, 22, 24, 28, 30, 32, 33, 34, 48, 55, 59, 60, 66, 71, 76, 80, 82, 85, 93$
10	$0, 1, 5$
12	$0, 1, 2, 3, 7, 10, 11, 13$
$15, 24, 35, 48, 63, 80$	$0, 1, 2$
$32, 60, 96$	$2, 3$
$20, 30, 42, 56, 72, 90$	$0, 1$

The only period lengths found were lengths $1, 2, 3, 12, 22$ and 72.

Table 2.5: Frequency of period lengths observed for $a + b\sqrt{n}$

Period length	Frequency
1	11746
2	6310
3	399
12	399
22	303
72	1399

It was also noted that almost all values of n appeared to be related to a unique period length as well as a periodic term. For example, when $n = 2$, the period length was always 1 and the term in the periodic part was always 3.
The following table summarizes the unique period lengths that were obtained for different n values.

Table 2.6: Value of n arranged by possible period lengths for $a + b\sqrt{n}$

Period length	n
1	$2, 6, 8, 12, 20, 30, 32, 42, 56, 60, 72, 90, 96$
2	$3, 8, 15, 24, 35, 48, 63, 80$
3	8
12	3
22	10
72	3

Table 2.7: Possible period lengths arranged according to n for $a + b\sqrt{n}$

n	Possible period lengths
$2, 6, 12, 20, 30, 32, 42, 56, 60, 72, 90, 96$	1
3	$2, 12, 72$
8	$2, 3$
$15, 24, 35, 48, 63, 80$	2
$15, 24, 35, 48, 63, 72$	3
10	22

Various infinite families were also observed in the search as well.

Theorem 2.16. *Each x_i of an expansion of $x = a + b\sqrt{n}$ has the format $p + q\sqrt{n}$ where a, b, p and q are rationals.*

Proof. We proceed by induction, with the base case

$$x_1 = \frac{\sqrt{n}}{a + b\sqrt{n} - a_0} = \frac{-\,bn}{(a-a_0)^2 - b^2 n} + \frac{a - a_0}{(a-a_0)^2 - b^2 n}\sqrt{n}.$$

Suppose for some x_k, where $k > 0$, the statement holds. Then

$$x_{k+1} = \frac{\sqrt{n}}{c + d\sqrt{n} - a_{k-1}} = \frac{-\,dn}{(c - a_{k-1})^2 - d^2 n} + \frac{c - a_{k-1}}{(c - a_{k-1})^2 - d^2 n}\sqrt{n}.$$

Since d, n, c, a_{k-1} are all integers, $\frac{-\,dn}{(c - a_{k-1})^2 - d^2 n}$ and $\frac{c - a_{k-1}}{(c - a_{k-1})^2 - d^2 n}$ are rational numbers. Therefore, by the principle of mathematical induction, the theorem holds. □

Shorter periods are much easier to analyses and longer period lengths get complicated. Let us first consider the case when the period length is 1.

2.4.1 Period Length 1

For tail length 1 we have the following.

Theorem 2.17. *If $a + b\sqrt{n} = [c, \overline{d}]_{\sqrt{n}}$ then $a = \frac{2c-d+u}{2}$ and $b = \frac{1}{u}$ for some u satisfying the biquadratic equation, $u^4 - u^2 d^2 + 4n = 0$. There are infinitely many numbers of the form $a+b\sqrt{n}$ with continued fraction expansion of the form $[c, \overline{d}]_{\sqrt{n}}$.*

Proof. If $x = [c, \overline{d}]_{\sqrt{n}}$ then $x_1 = [\overline{y}]_{\sqrt{n}}$. From the continued fraction algorithm, we have $x_1 = \frac{\sqrt{n}}{x-c}$ and $x_1 = d + \frac{\sqrt{n}}{x_1}$. So,

$$\frac{\sqrt{n}}{x - c} = d + x - c.$$

Simplifying we can obtain the following quadratic equation

$$(x - c)^2 + d(x - c) - \sqrt{n} = 0.$$

Solving for x we get,

$$x = \frac{2c - d + \sqrt{d^2 + 4\sqrt{n}}}{2}.$$

Since $x = a + b\sqrt{n}$ then $d^2 + 4\sqrt{n} = (u + v\sqrt{n})^2 = u^2 + nv^2 + 2uv\sqrt{n}$ for some rationals u and v. Then $d^2 = u^2 + nv^2$ and $uv = 2$. We have

$$x = \frac{2c - d + u}{2} + \frac{\sqrt{n}}{u}.$$

Solving for d^2 in terms of u,

$$d^2 = u^2 + \frac{4n}{u^2}.$$

□

To obtain infinite families, we can assume that u and v are integers. For $u = 1$ and $v = 2$, we have that $d^2 = 4n + 1$. Then $n = \frac{d^2-1}{4}$. Since n is an integer we need d to be odd. Then $d = 2k + 1$ for some k, therefore $n = k^2 + k$. For this case, $x = c - k + \sqrt{k^2 + k}$. Now we formally prove that x values of the above form do have the require continued fraction expansion.

Theorem 2.18. *For any $k > 0$, $x = c - k + \sqrt{k^2 + k}$ has the maximal continued fraction expansion $[c, \overline{2k + 1}]_{\sqrt{k^2+k}}$.*

Proof. If $x = c - k + \sqrt{k^2+k}$ then we have the following. Clearly,

$$0 \leqslant k < 2k+1.$$

Therefore,

$$k^2 \leqslant k^2 + k < k^2 + 2k + 1,$$
$$k \leqslant \sqrt{k^2+k} < k+1,$$

and

$$0 \leqslant -k + \sqrt{k^2+k} < 1.$$

This implies

$$c \leqslant c - k + \sqrt{k^2+k} < c+1.$$

Therefore, $\lfloor x \rfloor = c$. Then,

$$x_1 = \frac{\sqrt{k^2+k}}{\sqrt{k^2+k}-k} = k+1+\sqrt{k^2+k}.$$

From above,

$$k \leqslant \sqrt{k^2+k} < k+1,$$

and

$$2k+1 \leqslant k+1+\sqrt{k^2+k} < 2k+2.$$

So, $\lfloor x_1 \rfloor = 2k+1$ and

$$x_2 = \frac{\sqrt{k^2+k}}{k+1+\sqrt{k^2+k}-(2k+1)} = \frac{\sqrt{k^2+k}}{\sqrt{k^2+k}-k} = x_1.$$

Therefore,

$$c - k + \sqrt{k^2+k} = [c, \overline{2k+1}]_{\sqrt{k^2+k}}.$$

□

Another way to obtain an infinite family is to set $u = 2$ and $v = 1$, then $n = d^2 - 4$. With this, $x = c + 1 - \frac{d}{2} + \sqrt{\left(\frac{d}{2}\right)^2 - 1}$. By letting $d = 2k$ for some k we can obtain that $x = c + 1 - k + \sqrt{k^2-1}$, and this will have the expansion $[c, \overline{2k}]_{2\sqrt{k^2-1}}$.

Theorem 2.19. *For any $k > 1$, $x = c+1-k+\sqrt{k^2-1}$ has the maximal continued fraction expansion $[c, \overline{2k}]_{2\sqrt{k^2-1}}$.*

Proof. To see this, let $x = c + 1 - k + \sqrt{k^2-1}$ and $z = 2\sqrt{k^2-1}$. Now we have

$$x_0 = c + 1 - k + \sqrt{k^2-1} \text{ and } a_0 = \left\lfloor c + 1 - k + \sqrt{k^2-1} \right\rfloor.$$

When $k \geqslant 1$,

$$-2k \leqslant -2 < 0,$$

and

$$k^2 - 2k + 1 \leqslant k^2 - 1 < k^2.$$

So

$$k - 1 \leqslant \sqrt{k^2 - 1} < k,$$

which implies

$$c \leqslant c + 1 - k + \sqrt{k^2 - 1} < c + 1,$$

and $a_0 = c$. Now $x_1 = \frac{2\sqrt{k^2-1}}{1-k+\sqrt{k^2-1}}$. Simplifying this we get

$$x_1 = 1 + k + \sqrt{k^2 - 1} \text{ and } a_1 = \left\lfloor 1 + k + \sqrt{k^2 - 1} \right\rfloor.$$

For a_1, from the calculation of a_0 we have that

$$k - 1 \leqslant \sqrt{k^2 - 1} < k.$$

Thus,

$$2k \leqslant 1 + k + \sqrt{k^2 - 1} < 2k + 1.$$

So $a_1 = 2k$. Now $x_2 = \frac{2\sqrt{k^2-1}}{1+k+\sqrt{k^2-1}-2k} = x_1$.

Therefore, the expansion of $x = c+1-k+\sqrt{k^2-1}$ is $[c, \overline{2k}]_{2\sqrt{k^2-1}}$ for all $k > 1$ and $c \geqslant k - 1$ for $a \geqslant 0$. □

In our computer search, every periodic continued fraction expansion with tail length 1 and period length 1 fell into the two infinite families given in Theorems 2.18 and 2.19, which are

$$c + 1 - k + \sqrt{k^2 - 1} = [c, \overline{2k}]_{2\sqrt{k^2-1}},$$

and

$$c - k + \sqrt{k^2 + k} = [c, \overline{2k + 1}]_{\sqrt{k^2+k}}.$$

Now for tail length 2 we have the following.

Theorem 2.20. *If the maximal continued fraction expansion of $x = a + b\sqrt{n}$ is $[c, d, \overline{e}]_{\sqrt{n}}$, then*

$$a = c + \frac{(d - e)(4dn + 2nv) + 2nv(d + v)}{4d^2v(d - e)^2 - 4nv}$$

and

$$b = \frac{(d - e)\left(2dv(2d - e) + 2dv^2\right) + 4n}{4d^2v(d - e)^2 - 4nv}$$

for some v, a rational number, satisfying the biquadratic equation $v^4 - e^2v^2 + 4n = 0$.

Proof. If the maximal continued fraction expansion of $x = a + b\sqrt{n}$ is $[c, d, \overline{e}]_{\sqrt{n}}$, then

$$\frac{\sqrt{n}}{\frac{\sqrt{n}}{x-c} - d} = e + \frac{\sqrt{n}}{\frac{\sqrt{n}}{\frac{\sqrt{n}}{x-c}-d}}.$$

Then $x - c$ satisfies the quadratic equation

$$(x - c)^2 \left(-d^2 + de + \sqrt{n}\right) + (x - c)\left(d\sqrt{n}d\sqrt{n} - e\sqrt{n}\right) - n = 0.$$

Therefore,

$$x = c + \frac{\sqrt{n}(2d-e) + \sqrt{e^2 n + 4n\sqrt{n}}}{2d(d-e) - 2\sqrt{n}}.$$

Now we need $e^2 n + 4n\sqrt{n} = (u + v\sqrt{n})^2$ for some u and v. Then $e^2 n = u^2 + nv^2$ and $2n = uv$. Then $4n = v^2(e^2 - v^2)$, therefore v satisfies the biquadratic equation $v^4 - e^2v^2 + 4n = 0$. Using these relations and the equation for x above, we can get that when $x = a + b\sqrt{n}$, we need

$$a = c + \frac{(d-e)(4dn + 2nv) + 2nv(d+v)}{4d^2v(d-e)^2 - 4nv}$$

and

$$b = \frac{(d-e)\left(2dv(2d-e) + 2dv^2\right) + 4n}{4d^2v(d-e)^2 - 4nv}.$$

□

Using Theorem 2.20, we can specify v and u so that an infinite family can be obtained. If we let $v = -2$ and $u = -n$ then $n = e^2 - 4$. For a and b to be integers, we can let $d = 2k - 1$ and $e = 2k$ to recover the format

$$x = c - 2k^2 + 2 + k\sqrt{4k^2 - 4}$$

with the expansion $[c, 2k-1, \overline{2k}]_{\sqrt{4k^2-4}}$, where $k > 1$. This raises the question whether the above case works for all possible k. We can prove that it does as follows.

Let us consider the expansion of $x = c - 2k^2 + 2 + k\sqrt{4k^2 - 4}$ for any arbitrary $k > 1$. Using similar computation as Theorem 2.19 we can show that $\lfloor x \rfloor = c$. Now,

$$x_1 = \frac{\sqrt{4k^2-4}}{x-c} = \frac{\sqrt{4k^2-4}}{2 - 2k^2 + \sqrt{4k^4 - 4k^2}}.$$

Simplifying,

$$x_1 = k + \sqrt{k^2 - 1}.$$

Again using the same argument we can show $\lfloor x_1 \rfloor = 2k - 1$. With this we have

$$x_2 = \frac{\sqrt{4k^2-4}}{k + \sqrt{k^2-1} - (2k-1)} = k + 1 + \sqrt{k^2 - 1}.$$

Since $x_2 = x_1 + 1$, $\lfloor x_2 \rfloor = 2k$. Now,

$$x_3 = \frac{2\sqrt{k^2-1}}{k + 1 + \sqrt{k^2-1} - 2k} = x_2.$$

So the expansion is $[c, 2k-1, \overline{2k}]_{\sqrt{4k^2-4}}$.

For tail length 3 we can say the following.

Similar to the above two cases, when x has the expansion $[c, d, e, \overline{f}]_{\sqrt{n}}$, then $x - c$ satisfies the quadratic equation,

$$(x-c)^2\left(\sqrt{n}(-d^2+2de-df\right)+de(e-f)+n)+(x-c)(2dn-2en \\ +fn+2de(f-e)\sqrt{n})+n\left(e(e-f)-\sqrt{n}\right)=0.$$

Solving for x from the above quadratic equation we get,

$$x = c + \frac{(2e-2d-f)n + 2de(e-f)\sqrt{n} + n\sqrt{f^2+4\sqrt{n}}}{2\left(n + d^2e(e-f) + d(2e-d-f)\sqrt{n}\right)}.$$

Since x has the form $a + b\sqrt{n}$, we need $\sqrt{f^2+4\sqrt{n}}$ to reduce to a quadratic surd, thus $f^2 + 4\sqrt{n} = (u + v\sqrt{n})^2$ for some u and v. Then, $f^2 = u^2 + nv^2$ and $2 = uv$. Therefore,

$$x = c + \frac{(2e-2d-f)n + 2de(e-f)\sqrt{n} + n(2/v + v\sqrt{n})}{2\left(n + d^2e(e-f) + d(2e-d-f)\sqrt{n}\right)}$$

To obtain infinite families we can assume that $v = 1$ and $u = 2$. Then, $n = f^2 - 4$ and

$$x = c + \frac{(2e-2d-f)n + 2de(e-f)\sqrt{n} + n(2+\sqrt{n})}{2\left(n + d^2e(e-f) + d(2e-d-f)\sqrt{n}\right)}.$$

Now if we let $f = 2k$ and $e = 2k - 1$ for some k then $n = 4(k^2 - 1)$ and

$$x = c + \frac{4(k-d)(k^2-1) + 2(2k^2-2-2kd+d)\sqrt{k^2-1}}{4(k^2-1) - d^2(2k-1) + 2d(2k-2-d)\sqrt{k^2-1}}.$$

If $d = 4k^2 - 3$,

$$x = c + \frac{4(k-4k^2+3)(k^2-1) + 2(2k^2-2-2k(4k^2-3)+(4k^2-3))\sqrt{k^2-1}}{4(k^2-1) - (4k^2-3)^2(2k-1) + 2d(2k-2-(4k^2-3))\sqrt{k^2-1}}.$$

Now if we set $c = L + 4k(k^2 - 1)$ for any k and L, then the above equation simplifies to give us the infinite family

$$L - k^3 + 4k + \left(2k^2-1\right)\sqrt{4k^2-4} = [L + 4k^3 - 4k, 4k^2 - 3, 2k - 1, \overline{2k}]_{\sqrt{4k^2-4}}.$$

2.4.2 Period length 2

Analysis of continued fraction expansions with period length 2 is similar to that of period length 1. If the tail length is 1 we can show the following.

Theorem 2.21. *If the maximal continued fraction expansion of $x = a + b\sqrt{n}$ is $[c, \overline{d, e}]_{\sqrt{n}}$, then $a = \frac{v(2c-e)+2e}{2v}$ and $b = \frac{v}{2d}\sqrt{n}$ for some v satisfying the biquadratic equation $nv^4 - v^2d^2e^2 + 4 = 0$.*

Proof. If $x = [c, \overline{d, e}]_{\sqrt{n}}$ then,

$$\frac{\sqrt{n}}{x-c} = d + \frac{\sqrt{n}}{e+x-c}.$$

Therefore, x satisfies the quadratic equation

$$dx^2 + d(e-2c)x + cd(c-e) - e\sqrt{n} = 0.$$

Solving for x,

$$x = \frac{d(2c-e) + \sqrt{d^2e^2 + 4de\sqrt{n}}}{2d}.$$

Because x has the form $a + b\sqrt{n}$, we need $d^2e^2 + 4ed\sqrt{n} = (u + v\sqrt{n})^2$ for some u and v. This implies that $d^2e^2 = u^2 + nv^2$ and $2de = uv$. Using these we can obtain that the following form for x.

$$x = \frac{v(2c-e) + 2e}{2v} + \frac{v}{2d}\sqrt{n}.$$

□

For infinite families we can let $v = 2d = e$. Then $n = d^2 - 1$ and $x = c - d + 1 + \sqrt{d^2-1}$. Now, for any $d > \sqrt{n}$, the continued fraction expansion of $x = c - d + 1 + \sqrt{d^2-1}$ has the form $[c, \overline{d, 2d}]_{\sqrt{d^2-1}}$. We prove this as follows. First,

$$x_0 = c - (d-1) + \sqrt{d^2-1} \text{ and } a_0 = \left\lfloor c - d + 1 + \sqrt{d^2-1} \right\rfloor.$$

Since $d > 1$, we can easily show that

$$c \leqslant c - d + 1 + \sqrt{d^2-1} < c + 1.$$

Therefore, $a_0 = c$. Now, $x_1 = \frac{\sqrt{d^2-1}}{c-(d-1)+\sqrt{d^2-1}-c}$. Rationalizing the denominator, we get

$$x_1 = \frac{d+1+\sqrt{d^2-1}}{2} \text{ and } a_1 = \left\lfloor \frac{d+1+\sqrt{d^2-1}}{2} \right\rfloor.$$

Since $d > 1$, we have $d \leqslant \frac{d+1+\sqrt{d^2-1}}{2} < d + 1$. Therefore $a_1 = d$. Now similar to above we can obtain x_2 and x_3 to be

$$x_2 = d + 1 + \sqrt{d^2-1} = 2x_1 \text{ and } a_1 = 2d,$$

$$x_3 = \frac{\sqrt{d^2-1}}{-d+1+\sqrt{d^2-1}} = x_1 \text{ and } a_3 = d.$$

Therefore, the expansion is $[c, \overline{d, 2d}]_{\sqrt{d^2-1}}$.

The equations become increasingly complicated as the tail length and the period length increase. However, by looking at the data obtained from the computer search, many infinite families can be obtained. For example,

$$L + k\sqrt{k^2-1} = \left[L + k^2 - 1, 2k - 1, \overline{k, 2k}\right]_{\sqrt{k^2-1}}.$$

3 Continued fraction expansions of the form $\sqrt{a+b\sqrt{n}}$

3.1 Introduction

In this section, our focus is on expansions of numbers of the format $\sqrt{a+b\sqrt{n}}$ with $z=\sqrt{n}$, where a, b and n are positive integers and n is not a perfect square.

First, a computer search was done to recognize periodic behavior and possible patterns of expansions of $\sqrt{a+b\sqrt{n}}$. The range of parameters was set to be $0 < a, b, n \leqslant 100$, where n was square free and b is non-zero. Out of the continued fraction expansion of these 909000 numbers, only 1455 numbers showed periodicity within the first 100 terms of the expansion. Tail lengths ranged from 0 to 5, while period lengths varied from 1 to 18. Numbers with tail length other than 1 always reduced to be quadratic surds leaving 1386 numbers of tail length 1 which fitted the form. These numbers were further analyzed to look for patterns.

Table 3.1 below gives the percentage of period lengths which were found for tail length 1.

Table 3.1: Observed period lengths and percentages for $\sqrt{a+b\sqrt{n}}$

Period Length	Percentage (%)
1	53
2	42
3	1.2
4	1.8
Longer lengths	2

It was also observed that when n was smaller there was more variation of period lengths present. The longest period length found was 18 and the shortest was 1. The following is the distribution of period lengths among different n.

Table 3.2: Period lengths observed for different n values for $\sqrt{a+b\sqrt{n}}$

n	Period lengths observed
2	$1, 2, 3, 4, 5, 6, 8, 9, 10, 18$
3	$1, 2, 4, 6, 8, 9, 12$
5	$1, 2, 4$
6	$1, 2, 4, 8, 10$
7	$1, 2, 4$
8	$1, 2, 3, 4, 8$
10	$1, 2$
11	$1, 2$
12	$1, 2, 6$
Longer than 12	$1, 2$

In the following chapters we see that when n increases so do a and b required for $\sqrt{a+b\sqrt{n}}$ to be periodic. Therefore, the range of a and b needs to be increased to capture longer period lengths. We also prove the nonexistence of longer tail lengths for shorter period lengths.

In the following sections of this chapter we characterize the cases of period length 1 and 2. It is shown that for each n there are infinitely many a and b values for which $\sqrt{a+b\sqrt{n}}$ has an expansion of period length 1 and 2 when $z=\sqrt{n}$. The discussion extends to expansions with period length 3 up to 7. However, extra assumptions and conditions are required for these cases. We also show that for shorter period lengths, tail lengths of 2 and 3 are not possible when x has the form $\sqrt{a+b\sqrt{n}}$.

We start by proving a tool that was used frequently, the independence of the terms $1, \sqrt{n}, \sqrt{a+b\sqrt{n}}$ and $\sqrt{n}\sqrt{a+b\sqrt{n}}$ over the rationals.

Theorem 3.1. *Let a, b and n be positive integers where n is not a perfect square and suppose $x=\sqrt{a+b\sqrt{n}}$, where x does not simplify to be a quadratic surd. Then the terms $1, \sqrt{n}, \sqrt{a+b\sqrt{n}}$ and $\sqrt{n}\sqrt{a+b\sqrt{n}}$ are independent over rationals. That is, when $A+B\sqrt{n}+C\sqrt{a+b\sqrt{n}}+D\sqrt{n}\sqrt{a+b\sqrt{n}}=0$ for some rational numbers A, B, C and D, then A, B, C and D must be equal to zero.*

Proof. Suppose $A+B\sqrt{n}+Cx+D\sqrt{n}x=0$. Assume for contradiction that $C+D\sqrt{n}$ is non-zero. Then x can be rewritten as, $x=\frac{-A-B\sqrt{n}}{C+D\sqrt{n}}$. Rationalizing the denominator, we have $x=\frac{(-A-B\sqrt{n})(C-D\sqrt{n})}{C^2-nD^2}=\frac{(-AC+BDn)+(AD-BC)\sqrt{n}}{C^2-nD^2}$. This forces x to be a quadratic surd which is a contradiction. Therefore, we should not be able to solve for x, meaning that $C+D\sqrt{n}=0$. Therefore, $C=0$ and $D=0$. Now this condition forces $A+B\sqrt{n}=0$, therefore $A=0$ and $B=0$. □

Another tool that we use is the fact that if x is periodic, then every x_i obtained from the continued fraction algorithm is also periodic. This follows from the fact that if $x=[a_0, a_1, ..., a_{k-1}, a_k, a_{k+1}, a_{k+2}, a_{k+3}, ...]_{\sqrt{n}}$, then $x_k=[a_k, a_{k+1}, a_{k+2}, a_{k+3}...]_{\sqrt{n}}$.

3.2 Period Length 1

In searching for periodic expansions of numbers of the form $\sqrt{a+b\sqrt{n}}$, the most common period length was period length 1. This case is also the easiest to analyze. In this section we give sufficient and necessary conditions on a, b and n in $\sqrt{a+b\sqrt{n}}$ to guarantee a period length 1.

Theorem 3.2. *For positive integers a, b, let $x=\sqrt{a+b\sqrt{n}}$, where x does not simplify to be a quadratic surd. Then x has a the maximal periodic expansion $[c, \overline{d}]_{\sqrt{n}}$ with $\sqrt{n}\notin\mathbb{N}$ if and only if $a=c^2$, $b=1$, $d=2c$ and $2c>\sqrt{n}$.*

Proof. If $x=\sqrt{a+b\sqrt{n}}=[c, \overline{d}]_{\sqrt{n}}$ then $x_1=\frac{\sqrt{n}}{x-c}=[\overline{d}]_{\sqrt{n}}$. Thus $x=[c, d, x_1]_{\sqrt{n}}=[c, d, \frac{\sqrt{n}}{x-c}]_{\sqrt{n}}$.

Then

$$x=\frac{\sqrt{n}}{x-c+d}+c.$$

Therefore, x satisfies the quadratic equation

$$x^2+(d-2c)x+c(c-d)-\sqrt{n}=0.$$

Now substituting $x=\sqrt{a+b\sqrt{n}}$ we have

$$a+c(c-d)+(b-1)\sqrt{n}+(d-2c)\sqrt{a+b\sqrt{n}}=0.$$

From Theorem 3.1 we have $a+c(c-d)=0$, $(b-1)=0$ and $d-2c=0$. Therefore, $d=2c$, $b=1$ and $a=c^2$. Therefore, $x=\sqrt{c^2+\sqrt{n}}$ and the expansion is $[c,\overline{2c}]_{\sqrt{n}}$.

Now suppose we consider the expansion of $\sqrt{c^2+\sqrt{n}}$ where $2c>\sqrt{n}$. We have $x_0=\sqrt{c^2+\sqrt{n}}$ and $a_0=\lfloor\sqrt{c^2+\sqrt{n}}\rfloor$. Clearly $\sqrt{c^2+\sqrt{n}}>c$. However, $2c>\sqrt{n}>\sqrt{n}-1$, which implies that $c^2+2c+1=(c+1)^2>c^2+\sqrt{n}$, therefore $c+1>\sqrt{c^2+\sqrt{n}}>c$. Thus, $a_0=c$.

Next, $x_1=\frac{\sqrt{n}}{\sqrt{c^2+\sqrt{n}}-c}=c+\sqrt{c^2+\sqrt{n}}$, so $a_1=\lfloor c+\sqrt{c^2+\sqrt{n}}\rfloor=2c$. And finally, $x_2=\frac{\sqrt{n}}{c+\sqrt{c^2+\sqrt{n}}-2c}=x_1$. Therefore, we get an expansion $\sqrt{c^2+\sqrt{n}}=[c,\overline{2c}]_{\sqrt{n}}$. □

Note in the above case as well as in the other cases that are discussed in this chapter, a increases quadratically with respect to c. Therefore, the range that is required to capture such patterns in a computer search also increases.

Theorem 3.3. *For positive integers a and b, let $x=\sqrt{a+b\sqrt{n}}$, where x does not simplify to be a quadratic surd. Then x cannot have a maximal expansion of the form $x=[c,d,\overline{e}]_{\sqrt{n}}$ with $d\neq e$.*

Proof. If $x=\sqrt{a+b\sqrt{n}}=[c,d,\overline{e}]_{\sqrt{n}}$ then $x=[c,d,y]_{\sqrt{n}}$ with $y=[\overline{e}]$. Setting $x_0=x$ and $a_0=\lfloor x\rfloor=c$, then $x_1=\frac{\sqrt{n}}{x-c}$, $a_1=d$ and $x_2=\frac{\sqrt{n}}{\frac{\sqrt{n}}{x-c}-d}=y$.

Therefore, $x_2=e+\frac{\sqrt{n}}{x_2}$, which implies

$$\frac{\sqrt{n}}{\frac{\sqrt{n}}{x-c}-d}=e+\frac{\sqrt{n}}{\frac{\sqrt{n}}{\frac{\sqrt{n}}{x-c}-d}}.$$

From this we get the quadratic equation,

$$(x-c)^2\left(d(e-d)+\sqrt{n}\right)+\sqrt{n}(x-c)(2d-e)-n=0.$$

When $x=\sqrt{a+b\sqrt{n}}$, x^2 becomes a quadratic irrationality. Therefore, the coefficient of x is $2cd(d-e)$. From Theorem 3.1, this implies $2cd(d-e)=0$. Since c and d are non-zero integers, we require $d=e$. This is a contradiction, thus proving the theorem.

□

3.3 Period Length 2

In the search for periodic expansions, it was observed that numbers of the form $\sqrt{c^2+b\sqrt{n}}$ tend to have periodic expansions of length 2 when $b\geqslant 2$. It was also observed that $c>\sqrt{n}$, therefore when n increased, $\sqrt{c^2+b\sqrt{n}}$ with smaller c values became aperiodic. After analyzing the periodic behavior, the above observation was concluded to be true.

Theorem 3.4. *For positive integers a and b, let $x=\sqrt{a+b\sqrt{n}}$, where x does not simplify to be a quadratic surd. Then x has a maximal periodic expansion of the form $[c,\overline{d,e}]_{\sqrt{n}}$ with $\sqrt{n}\notin\mathbb{N}$ if and only if $a=c^2$, $b=2c/d$, $e=2c$, $2c/b>\sqrt{n}$ and $b\geqslant 2$.*

Proof. Let the maximal expansion of x be as follows,

$$x = \sqrt{a + b\sqrt{n}} = [c, \overline{d, e}]_{\sqrt{n}}.$$

If $y = \frac{\sqrt{n}}{x-c}$, then

$$y = [\overline{d, e}]_{\sqrt{n}} = d + \frac{\sqrt{n}}{e + \frac{\sqrt{n}}{y}}.$$

Substituting for y,

$$\frac{\sqrt{n}}{x - c} = d + \frac{\sqrt{n}}{e + x - c}.$$

Solving this we get the quadratic equation

$$dx^2 + d(e - 2c)x + cd(c - e) - e\sqrt{n} = 0.$$

Since $x = \sqrt{a + b\sqrt{n}}$,

$$d(e - 2c)\sqrt{a + b\sqrt{n}} + (db - e)\sqrt{n} + d(a + c^2 - ce) = 0.$$

From Theorem 3.1, $d(e - 2c) = 0$, $(db - e) = 0$ and $d(a + c^2 - ce) = 0$.
Solving these equations, we can recover the format

$$\sqrt{c^2 + b\sqrt{n}} = [c, \overline{\frac{2c}{b}, 2c}]_{\sqrt{n}}.$$

Finally, for the above expansion to be maximal, we have that $2c/b > \sqrt{n}$ and $2c > \sqrt{n}$. However, since $b \geqslant 2$ we have that $2c \geqslant 2c/b$ which implies that $2c/b > \sqrt{n}$ is sufficient for the expansion to be maximal.

Now suppose, $x = \sqrt{c^2 + b\sqrt{n}}$ with $2c/b > \sqrt{n}$ and $b \geqslant 2$.
First set,

$$x_0 = \sqrt{c^2 + b\sqrt{n}} \text{ and } a_0 = \lfloor\sqrt{c^2 + b\sqrt{n}}\rfloor.$$

Since b is a positive integer we have that $\sqrt{c^2 + b\sqrt{n}} > c$. Since $\frac{2c}{b} > \sqrt{n}$ we have that $b\sqrt{n} < 2c + 1$. Rearranging terms we get $c^2 + b\sqrt{n} < (c + 1)^2$. Thus $c < \sqrt{c^2 + b\sqrt{n}} < c + 1$ which implies that $a_0 = c$.

Now

$$x_1 = \frac{\sqrt{n}}{\sqrt{c^2 + b\sqrt{n}} - c} = \frac{\sqrt{c^2 + b\sqrt{n}} + c}{b} > \frac{2c}{b}.$$

From above we have $c < \sqrt{c^2 + b\sqrt{n}} < c+1$. Therefore, $\frac{2c}{b} < \frac{\sqrt{c^2+b\sqrt{n}}+c}{b} < \frac{2c}{b} + \frac{1}{b}$. Here $b \geqslant 1$ which implies that $\frac{2c}{b} + \frac{1}{b} \leqslant \frac{2c}{b} + 1$. So, $\frac{2c}{b} < \frac{\sqrt{c^2+b\sqrt{n}}+c}{b} < \frac{2c}{b} + 1$.

Therefore,

$$a_1 = \left\lfloor \frac{\sqrt{c^2 + b\sqrt{n}} + c}{b} \right\rfloor = \frac{2c}{b}.$$

Next,

$$x_2 = \frac{\sqrt{n}}{\frac{\sqrt{c^2+b\sqrt{n}}+c}{b} - \frac{2c}{b}} = \frac{b\sqrt{n}}{\sqrt{c^2 + b\sqrt{n}} - c} = \sqrt{c^2 + b\sqrt{n}} + c = x_0 + c$$

and $a_2 = \lfloor x_2 \rfloor = 2c$.

Finally,

$$x_3 = \frac{\sqrt{n}}{\sqrt{c^2 + b\sqrt{n}} - c} = x_1.$$

Therefore, our expansion follows. □

Theorem 3.5. *There are infinitely many numbers of the form* $\sqrt{a + b\sqrt{n}}$ *with expansion of the form* $[c, \overline{\frac{2c}{b}, 2c}]_{\sqrt{n}}$.

Proof. For any integers b and c, where $b \geqslant 2$ and $\frac{2c}{b}$ is an integer satisfying $\frac{2c}{b} > \sqrt{n}$, the above expansion holds. There are infinitely many such integers, thus we have infinitely many numbers of the form $\sqrt{a + b\sqrt{n}}$ with expansion $[c, \overline{\frac{2c}{b}, 2c}]_{\sqrt{n}}$. □

From the above equation we can see that not all combinations of a and b result in a valid expansion. On top of the requirement to be a maximal expansion, we require the quantity $\frac{2c}{b}$ be an integer. Even when $\frac{2c}{b}$ is an integer, not all values of n are valid. For example, if $c = 10$ and $b = 5$, then $\sqrt{100 + 5\sqrt{n}}$ has the maximal expansion $[10, \overline{4, 20}]_{\sqrt{n}}$ for all $n \leqslant 15$. For $n = 17$ we can still recover the pattern, but the expansion is not a maximal expansion. However, when $n = 18$, the expansion becomes $[11, 437, 4, 13, 23, ..]_{\sqrt{n}}$, which deviates from the pattern from the first term because $\sqrt{n} \geqslant 2c/b$.

Now we go on to prove the non-existence of tail length 2 for the period length 2 case.

Theorem 3.6. *For positive integers* a, b*, let* $x = \sqrt{a + b\sqrt{n}}$ *, where* x *does not simplify to be a quadratic surd. Then* x *cannot have an expansion of the form* $x = [c, d, \overline{e, f}]_{\sqrt{n}}$ *with* $f \neq d$

Proof. Similar to previous proofs, if x has an expansion of the form $x = [c, d, \overline{e, f}]_{\sqrt{n}}$ with $f \neq d$, x satisfies the following quadratic equation

$$(x - c)^2 \left((d - f)de - f\sqrt{n}\right) + e\sqrt{n}(x - c)(f - 2d) + en = 0.$$

Now since x does not simplify to be a quadratic surd, the coefficient of x is $-2c(d^2e - def)$. From Theorem 3.1, we have that $2cde(d - f) = 0$ which forces $d = f$. Therefore, above statement holds. □

In conjunction with the Theorems 3.3, 3.6 and the results from the computer search, we conjecture the following.

Conjecture 3.7. *If* $\sqrt{a + b\sqrt{n}}$ *does not reduced to a quadratic surd and the continued fraction expansion with* $z = \sqrt{n}$ *is periodic, then the expansion has the format*

$$x = [a_0, \overline{a_1, a_2, ..., a_{k-1}, 2a_0}]_{\sqrt{n}}$$

with $a_i = a_{k-i}$ *for all* $i \leqslant k$.

3.4 Period Length 3

As the period length increases, the corresponding quadratic equations obtained become harder to analyze. However, we are still able to prove the palindromic pattern of some expansions when the discussion is limited to the case of tail length 1.

Theorem 3.8. *If the maximal expansion of* $\sqrt{a+b\sqrt{n}}$ *is* $[c,\overline{d,e,f}]_{\sqrt{n}}$*, then* $d=e$*,* $f=2c$*,* $a=c^2+\dfrac{dn(d-2c)}{d^4-n}$ *and* $b=\dfrac{2cd^3-n}{d^4-n}$*.*

Proof. If $\sqrt{a+b\sqrt{n}}=[c,\overline{d,e,f}]_{\sqrt{n}}$ then,

$$x_1=\frac{\sqrt{n}}{x-c}=[\overline{d,e,f}]_{\sqrt{n}}.$$

Also,

$$x_1=d+\frac{\sqrt{n}}{e+\frac{\sqrt{n}}{f+\frac{\sqrt{n}}{x_1}}}.$$

Using the above two equations we obtain the quadratic equation

$$(x-c)^2\left(de+\sqrt{n}\right)+(x-c)\left(def+d\sqrt{n}-e\sqrt{n}+f\sqrt{n}\right)-ef\sqrt{n}-n=0.$$

Now if x does not reduce to be a quadratic surd, the coefficient of x is $-2cde+def$ and the coefficient of $x\sqrt{n}$ is $-2c+d-e+f$. From Theorem 3.1, these must be zero. Therefore, we can recover that $d=e$ and $f=2c$. With this the quadratic equation becomes

$$(x-c)^2\left(d^2+\sqrt{n}\right)+(x-c)\left(2d^2c+2c\sqrt{n}\right)-2cd\sqrt{n}-n=0.$$

Now solving for x and rationalizing the denominator we get,

$$x=\sqrt{c^2+\frac{nd(d-2c)}{d^4-n}+\frac{2cd^3-n}{d^4-n}\sqrt{n}}.$$

□

3.5 Period Length 4

This case is similar to the previous case. Therefore, using the same techniques we can prove the following.

Theorem 3.9. *If the maximal expansion of* $\sqrt{a+b\sqrt{n}}$ *is* $[c,\overline{d,e,f,g}]_{\sqrt{n}}$*, then* $f=d$*,* $g=2c$*,* $a=c^2+\dfrac{en(e-2c)}{e^2d^2-4n}$ *and* $b=\dfrac{2c}{d}+\dfrac{2n(2c-e)}{d(e^2d^2-4n)}$*.*

Proof. If $x=[c,\overline{d,e,f,g}]_{\sqrt{n}}$ then $y=\dfrac{\sqrt{n}}{x-c}=[\overline{d,e,f,g}]_{\sqrt{n}}$. Therefore, $y=[c,d,e,f,y]_{\sqrt{n}}$. From the above equation we get the quadratic equation

$$(def+d\sqrt{n}+f\sqrt{n})(x-c)^2+(defg+(de+dg-ef+fg)\sqrt{n})(x-c)$$
$$-efg\sqrt{n}-en-gn=0.$$

From the above equation the coefficient of x is $def(-2c+g)$. From Theorem 3.1, this should be zero, therefore we get that $g = 2c$. With this the quadratic equation becomes

$$(def + d\sqrt{n} + f\sqrt{n})(x-c)^2 + (2cdef + (de + 2cd - ef + 2cf)\sqrt{n})(x-c) \\ - 2cef\sqrt{n} - en - 2cn = 0.$$

Now the coefficients of $x\sqrt{n}$ should also be equal to zero, therefore $e(d-f) = 0$ and we must have $d = f$. With this the quadratic equation simplifies to

$$(d^2e + 2d\sqrt{n})(x-c)^2 + (2cd^2e + 4cd\sqrt{n})(x-c) - 2ced\sqrt{n} - en - 2cn = 0.$$

Solving for x we get,

$$x = \sqrt{c^2 + \frac{en(e-2c)}{e^2d^2 - 4n} + \left(\frac{2c}{d} + \frac{2n(2c-e)}{d(e^2d^2-4n)}\right)\sqrt{n}}.$$

□

3.6 Longer period Lengths

As the length of the periodic part increases, it becomes increasingly harder to prove that the periodic part of the expansion has the familiar palindromic pattern followed by the last term which is equal to twice the first term. However, if we assume that the last term of the periodic part is twice the first, then it can be proven that the palindromic pattern follows for some period lengths. Thus, for the period lengths 5 through 7, we use this assumption.

In all the cases below we assume that variables in the expansion and a, b in $x = \sqrt{a + b\sqrt{n}}$ are positive integers. And we only consider the maximal expansion as well.

3.6.1 Period length 5

For period length 5, we can obtain the following.

Theorem 3.10. *If the maximal expansion of* $\sqrt{a + b\sqrt{n}}$ *is* $[c, \overline{d, e, f, g, 2c}]_{\sqrt{n}}$*, then* $d = g$*,* $e = f$*, with*

$$a = c^2 + \frac{d^2e^3n(e-2c) + (2c-d)(dn^2 + en^2) + en^2(e-d)}{d^4e^4 - d^4n - 4d^3en - 2d^2e^2n + n^2}$$

and

$$b = \frac{(2cd(d^2e^4 - d^2n - 3den - e^2n) - 2de^3n + n^2)}{d^4e^4 - d^4n - 4d^3en - 2d^2e^2n + n^2}.$$

Proof. If $x = [c, \overline{d, e, f, g, h}]_{\sqrt{n}}$, then $x_1 = \frac{\sqrt{n}}{x-c} = [\overline{d, e, f, g, h}]_{\sqrt{n}} = [d, e, f, g, h, x_1]_{\sqrt{n}}$. Therefore,

$$\frac{\sqrt{n}}{x-c} = [d, e, f, g, h, \frac{\sqrt{n}}{x-c}]_{\sqrt{n}}.$$

Using the continued fraction expansion, we can see that $x-c$ satisfies the quadratic equation

$$\begin{aligned}&(defg+n+\sqrt{n}(de+dg+fg))(x-c)^2\\&+(defgh+dn-en+fn-gn+hn+\sqrt{n}(def+deh+dgh-efn+fgh))(x-c)\\&\qquad\qquad-n(ef+eh+gh)-\sqrt{n}(efgh+n)=0.\end{aligned}$$

From Theorem 3.1, the coefficients of x and $x\sqrt{n}$ must be zero because x has the form $\sqrt{a+b\sqrt{n}}$. These conditions give

$$(defg+n)(h-2c)+n(d-g)+(f-e)n=0$$

for the coefficient of x and

$$(de+dg+fg)(h-2c)+ef(d-g)=0$$

for the coefficients of $x\sqrt{n}$. When $h=2c$, this forces $d=g$ and $f=e$. With this the quadratic equation becomes

$$\begin{aligned}&\left(d^2e^2+n+\sqrt{n}(2de+d^2)\right)(x-c)^2+(2cd^2e^2+2cn+\sqrt{n}(de^2+2cde\\&+2cd^2-e^2n+2ced))(x-c)-n(e^2+2ce+2cd)-\sqrt{n}(2ce^2d+n)=0.\end{aligned}$$

Solving, we obtain the equations for a and b. □

3.6.2 Period length 6

For period length 6, we obtain the following.

Theorem 3.11. *If the expansion of $\sqrt{a+b\sqrt{n}}$ is $[c,\overline{d,e,f,g,h,2c}]_{\sqrt{n}}$ then $g=e$, $h=d$,*

$$a=c^2+\frac{(2c-e)en(-d^2e^2f^2+2dfn+2d^2n+f^2n)-2e^2(d-f)(d+f)n^2}{d^4e^4f^2-4d^4e^2n-4d^3e^2fn-2d^2e^2f^2n+4d^2n^2+4dfn^2+f^2n^2}$$

and

$$b=\frac{2c(d^3e^4f^2-4d^3e^2n-3d^2e^2fn-de^2f^2n+2dn^2+fn^2)-2ne(de^2f^2+2dn+fn)}{d^4e^4f^2-4d^4e^2n-4d^3e^2fn-2d^2e^2f^2n+4d^2n^2+4dfn^2+f^2n^2}.$$

Proof. Similar to Theorem 3.10, if $x=\sqrt{a+b\sqrt{n}}$ has the expansion the $[c,\overline{d,e,f,g,h,i}]$, by setting the coefficients of x and the coefficients of $x\sqrt{n}$ to be zero we get equations

$$(i-2c)(defgh+dn+fn+hn)+(e+g)(d-h)n-fn(e-g)=0$$

and

$$(i-2c)(def+deh+dgh+fgh)+efg(d-h)=0.$$

Since we assumed $i=2c$, we get that $d=h$ and $e=g$. Now the expansion of x is $[c,\overline{d,e,f,e,d,2c}]$ therefore we get the above two equations for a and b. □

3.6.3 Period length 7

For the general case of period length 7, if a piece of the expansion is palindromic, the rest follows. Specifically, we have the following.

Theorem 3.12. *If the maximal expansion of* $\sqrt{a+b\sqrt{n}}$ *is* $[c, \overline{d, e, f, g, h, d, 2c}]_{\sqrt{n}}$ *then* $e = h$ *and* $f = g$. *Moreover,*

$$a = c^2 + \frac{2c(-d^2f^3g^2n + 2d^2fn^2 + 2dfgn^2 + fg^2n^2) + d^2f^4g^2n - 4d^2f^2n^2 - 2df^2gn^2 + f^2g^2n^2}{d^4f^4g^2 - 4d^4f^2n - 4d^3f^2gn - 2d^2f^2g^2n + 4d^2n^2 + 4dgn^2 + g^2n^2}$$

$$b = \frac{2cd^3f^4g^2 - 8cd^3f^2n - 6cd^2f^2gn - 2cdf^2g^2n + 4cdn^2 + 2cgn^2 - 2df^3g^2n + 4dfn^2 + 2fgn^2}{d^4f^4g^2 - 4d^4f^2n - 4d^3f^2gn - 2d^2f^2g^2n + 4d^2n^2 + 4dgn^2 + g^2n^2}.$$

Proof. Similar to the proof of period length 6, if x has the expansion $[c, \overline{d, e, f, g, h, i, j}]$ then we can obtain the following equations:

$$(defg+defi+dehi+dghi+fghi+n)(2c-j)+(efgh+n)(i-d)+n(e-h)+(g-f)n = 0$$

and

$$(defghi+den+dgn+fgn+din+fin+hin)(2c-j)+(ef+eh+gh)n(i-d)+fgn(e-h) = 0.$$

Since $j = 2c$ and $i = d$ we have

$$n(e-h) = n(f-g)$$

and

$$fg(e-h) = 0.$$

Thus, $e = h$ and $f = g$. With this follows the equations for a and b. □

4 Infinite Families

In chapter 3, we discussed possible forms of maximal expansions of numbers of the format $\sqrt{a+b\sqrt{n}}$. In this chapter the existence of infinite families of numbers with a certain form of expansion is discussed.

First let us consider the shorter period lengths. For period length 1, it was concluded that the expansion of $\sqrt{a+b\sqrt{n}}$ is $[c,\overline{d}]_{\sqrt{n}}$ if and only if $a=c^2, b=1$ and $d=2c$, where $c>\sqrt{n}$. For period length 2, it was concluded that the expansion of $\sqrt{a+b\sqrt{n}}$ is $[c,\overline{d,e}]_{\sqrt{n}}$ if and only if $a=c^2, b=2c/d$, where $2c/b>\sqrt{n}$ and $b\geqslant 2$. Therefore, these two cases were completely characterized. Hence, we have infinite family of numbers with such expansion. Here are several examples of such infinite families. For period length 1, when $c=n$,

$$\sqrt{n^2+\sqrt{n}}=[n,\overline{2n}]_{\sqrt{n}}$$

and when $c=2n$,

$$\sqrt{4n^2+\sqrt{n}}=[2n,\overline{4n}]_{\sqrt{n}}.$$

For period length 2, we can obtain examples for infinite families by setting values for b and c as follows. If $c=kn$ and $b=k$ for some integer k, the equation and the expansion become

$$\sqrt{k^2n^2+k\sqrt{n}}=[kn,\overline{2n,2kn}]_{\sqrt{n}}.$$

This infinite family works when $k>1$ because when $k=1$, the expansion simplifies to the period length 1 case. Similarly, if we let $c=kn$ and $b=2k$ for some integer k, the equation and the expansion become

$$\sqrt{k^2n^2+2k\sqrt{n}}=[kn,\overline{n,2kn}]_{\sqrt{n}},$$

for all $k>0$.

Unlike for shorter periods, when the period length is longer than 2, it is not easy to derive conditions on the terms of the expansion to guarantee the form $\sqrt{a+b\sqrt{n}}$, where a and b are positive integers. Let us consider period length 3. When $\sqrt{a+b\sqrt{n}}$ has the expansion $[c,\overline{d,d,2c}]_{\sqrt{n}}$, we have $a=c^2+\frac{dn(d-2c)}{d^4-n}$ and $b=\frac{2cd^3-n}{d^4-n}$. Now conditions on c,d and n for a and b to be positive integers are much more complicated. Clearly, not all cases of c,d and n works. For example, consider the case when $n=2$ and $d=2$. If $c=2$, then $x=\sqrt{\frac{24}{7}+\frac{15}{7}\sqrt{2}}$ and the continued fraction expansion is $[2,\overline{2,2,4}]_{\sqrt{2}}$, but neither a nor b are integers. When $n=2$ and $d=2$, the first value for c that gives the desired form is $c=8$, for which $x=\sqrt{60+9\sqrt{2}}$ and the next is $c=15$, for which $x=\sqrt{217+17\sqrt{2}}$. Therefore, many combinations of n,d and c do not work.

Also note, when c,n increase a grows much faster, thus expansions of the above form become rarer. This was also observed in the computer search as the longer period length cases disappeared for large n-values.

However, using different techniques we can still prove the existence of infinite families for some cases. As we shall see, each period length case discussed has a corresponding linear Diophantine equation on b and c, where c denotes the first terms of the expansion of $\sqrt{a+b\sqrt{n}}$. By finding solutions to these equations, we can find integer solutions for a and b which give the required expansion.

4.1 Period length 3

We show that there are infinitely many $\sqrt{a+b\sqrt{n}}$ with period 3, by obtaining solutions to a Diophantine equation. The corresponding Diophantine equation is obtained using the equation for b. By deriving conditions on coefficients of the Diophantine equation, we show the existence of infinitely many solutions which also correspond to infinitely many expansions with the required pattern.

Theorem 4.1. *For every $n > 1$, where n is not a perfect square, there are infinitely many numbers of the form $\sqrt{a+b\sqrt{n}}$, where a and b are positive integers, with expansion of the form $[c, \overline{d, d, 2c}]_{\sqrt{n}}$.*

Proof. The conditions from Theorem 3.8 give that $b = \dfrac{2cd^3 - n}{d^4 - n}$. This can be rewritten to get the equation $2d^3c - b(d^4 - n) = n$. This is a linear Diophantine equation in variables b and c. Therefore, this has infinitely many integer solutions if and only if $\gcd(2d^3, d^4 - n)$ is a divisor of n.

In addition, we also have that $a = c^2 + \dfrac{dn(d-2c)}{d^4 - n}$. But a has to be a positive integer, so $d^4 - n$ has to be a divisor of $dn(d - 2c)$.

Suppose m is a common factor of $2d^3$ and $d^4 - n$. Now suppose the parity of d and n are different then $d^4 - n$ is odd. This forces m to be an odd number. Using these equation we can also show that $m|2n$. With this and since m is odd, it must divide n. Thus, the Diophantine equation has infinitely many solutions.

This is sufficient to prove that the statement holds. However, we will analyze a few more cases for completion.

The Diophantine equation does not have any solutions when both n and d are odd because $\gcd(2d^3, d^4 - n)$ is even. Therefore, we are left with the case when d and n are even. Using unique factorization, we can see that when the largest powers of 2 that exactly divide d^4 and n are different, we can still recover that $\gcd(2d^3, d^4 - n)|n$. However, when they are the same, $d^4 - n$ might be divisible by a larger power of 2, depending on the factorization, which might lead to a case where $\gcd(2d^3, d^4 - n)$ is divisible by more powers of 2 than n. □

Using guidelines stated in the proof above, we recover the following infinite families. Considering the Diophantine equation $2d^3c - b(d^4 - n) = n$, we see that $(d/2, 1)$ is a positive integer solution when d is even. Thus, we can obtain infinitely many solutions by

$$c = (d^4 - n)t + \frac{d}{2}$$

and

$$b = 2d^3t + 1,$$

where t is an arbitrary parameter. Setting $d = 2k$ for some k, we have

$$x = \sqrt{(16k^4t - nt + k)^2 - 4knt + (16k^3t + 1)\sqrt{n}}.$$

Using the above format, we can recover more concrete examples of infinite families as given below. Here we have to keep in mind the assumption that the expansion is maximal. For example, if we set $k = 1$ and $n = 2$, we can still recover the family

$$\sqrt{(1+8k)^2 - 4k + (1+7k)\sqrt{2}} = [1+8k, \overline{2, 2, 2(1+8k)}]_{\sqrt{2}}.$$

But this does not carry over for $n > 2$. However, if we let $d = 2n$, a maximal expansion is guaranteed. Now, by setting $t = 1$ we recover the following infinite family that works for any integer n,

$$\sqrt{256n^8 - 4n^2 + (16n^3 + 1)\sqrt{n}} = [16n^4, \overline{2n, 2n, 32n^4}]_{\sqrt{n}}.$$

4.2 Period length 4

Similar to the previous case, we can obtain a linear Diophantine equation. However, as the coefficients of the Diophantine equation depends on three variables, we use unique factorization to prove that there are infinite families.

Theorem 4.2. *For every $n > 1$ which is not a perfect square, there exist infinitely many numbers of the form $\sqrt{a + b\sqrt{n}}$, where a and b are positive integers, with an expansion of the form $[c, \overline{d, e, d, 2c}]_{\sqrt{n}}$.*

Proof. For period length 4, from Theorem 3.9, $b = \dfrac{2c}{d} + \dfrac{2n(2c-e)}{d(e^2d^2 - 4n)}$. Therefore, we get a Diophantine equation of the form

$$2ne = c(2e^2d^2 - 4n) - b(e^2d^3 - 4nd).$$

The equation has infinitely many integer solutions if and only if $\gcd(2e^2d^2 - 4n, e^2d^3 - 4nd)$ divides $2ne$.

Suppose d and e are odd. Then, $\gcd(2e^2d^2 - 4n, e^2d^3 - 4nd)$ is also odd. Now if d and e are relatively prime to n, we have the following. If $\gcd(2e^2d^2 - 4n, e^2d^3 - 4nd) = k$, then $k | 2e^2d^2 - 4n$ and $k | e^2d^3 - 4nd$. This implies that $k | 4nd$. Since k is odd, we have $k | nd$. Now because $k | e^2d^3 - 4nd$, $k | n$. This means that k must be a common factor of d and n. Since d and n are relatively prime, $k = 1$. Therefore, the above Diophantine equation has infinitely many solutions.

Using unique factorization, we can show that for many cases of f, d and n, we are able obtain infinitely many solutions for the above Diophantine equation which provides us infinite families.

Let $n = 2^{x_1} \cdot p_1^{x_2} \cdot p_2^{x_3} \ldots$, $d = 2^{y_1} \cdot p_1^{y_2} \cdot p_2^{y_3} \ldots$ and $e = 2^{z_1} \cdot p_1^{z_2} \cdot p_2^{z_3} \ldots$, where each p_i is an odd prime.

Let us consider the case when n, d and e are odd. Now consider an odd prime p_i and suppose $p_i^{x_i}$, $p_i^{y_i}$ and $p_i^{z_i}$ exactly divide n, d and e respectively. Then p_i^k exactly divides $\gcd(2d^2e^2 - 4n, d^3e^2 - 4nd)$, where $k = \min(2y_i + 2z_i, x_i, 3y_i + 2z_i, x_i + y_i) = \min(2y_i + 2z_i, x_i)$. Here $k \leqslant x_i \leqslant x_i + z_i$, therefore p_i^k divides $\gcd(2d^2e^2 - 4n, d^3e^2 - 4nd)$. Any common divisor of $2d^2e^2 - 4n$ and $d^3e^2 - 4nd$ divides $2ne$. So if $\gcd(2d^2e^2 - 4n, d^3e^2 - 4nd)$ is odd, it will guarantee infinitely many solutions to the Diophantine equation.

If one or more of n, d or e are even, then we must consider divisibility by 2. However, for many cases of x_i, y_i and z_i we can obtain infinite families. Therefore, we can conclude that infinitely many families with continued fraction expansion of the form $[c, \overline{d, e, d, 2c}]_{\sqrt{n}}$ exists. □

To obtain examples of infinite families we can let $d = e$. In this case the Diophantine equation is $c(2d^4 - 4n)t - b(d^5 - 4nd) = 2nd$. For this, $(d/2, 1)$ is a solution, therefore all general solutions will have the form

$$c = (d^5 - 4nd) + d/2$$

and

$$b = (2d^4 - 4n)t + 1.$$

For this case we need d to be even. So $d = 2k$ for some integer k. From this we get the following doubly infinite family,

$$\sqrt{k^2(1 - 24nt + 1024k^8t^2 + 64n^2t^2 - 64k^4t(-1 + 8nt)) + (1 + 32k^4t - 4nt)\sqrt{n}}$$
$$= [t\left(32k^5 - 8kn\right) + k, \overline{2k, 2k, 2k, 2(t(32k^5 - 8kn) + k)}]_{\sqrt{n}}.$$

For the above case to work, we require $k > \sqrt{n}/2$ to guarantee a maximal expansion. For concrete examples of the above family, we let $k = n$ and $t = n$. In this case,

$$\sqrt{1024n^{12} - 512n^9 + 64n^7 + 64n^6 - 24n^4 + n^2 + (32n^5 - 4n^2 + 1)\sqrt{n}}$$
$$= [32n^6 - 8n^3 + n, \overline{2n, 2n, 2n, 2(32n^6 - 8n^3 + n)}]_{\sqrt{n}}.$$

If $t = 1$ for the same case above, we recover the following family,

$$\sqrt{(1024n^8 - 512n^5 + 64n^4 + 64n^2 - 24n + 1)\, n^2 + (32n^4 - 4n + 1)\sqrt{n}}$$
$$= [32n^5 - 8n^2 + n, \overline{2n, 2n, 2n, 2(32n^5 - 8n^2 + n)}]_{\sqrt{n}}.$$

Another way to obtain a different collection of infinite families is to let $e = 2d$. Then the Diophantine equation becomes,

$$c(2d^4 - n) - b(d^5 - nd) = nd.$$

Therefore, $(d, 2)$ is a solution, and all solutions to the Diophantine equation have the form

$$b = (2d^4 - n)t + 2,$$
$$c = (d^5 - nd)t + d$$

for some parameter t. With this we get a doubly infinite family

$$\sqrt{d^2(d^8t^2 - 2d^4t(nt - 1) + n^2t^2 - 3nt + 1) + (2d^4t - nt + 2)\sqrt{n}}$$
$$= [d^5t - dnt + d, \overline{d, 2d, d, 2d(d^4t - nt + 1)}]_{\sqrt{n}}.$$

Following are more concrete examples of the above format. If $d = n$ and $t = 1$ then,

$$\sqrt{n^2\left(n^8 - 2n^5 + 2n^4 + n^2 - 3n + 1\right) + (2n^4 - n + 2)\sqrt{n}}$$
$$= [n - n^2 + n^5, \overline{n, 2n, n, 2(n - n^2 + n^5)}]_{\sqrt{n}}.$$

If $d = n$ and $t = n$ then,

$$\sqrt{n^{12} - 2n^9 + 2n^7 + n^6 - 3n^4 + n^2 + (2n^5 - n^2 + 2)\sqrt{n}}$$
$$= [n^6 - n^3 + n, \overline{n, 2n, n, 2\left(n^6 - n^3 + n\right)}]_{\sqrt{n}}.$$

4.3 Longer period lengths

Since the Diophantine equation gets complicated for longer periods, the existence of infinite families is shown by constructing such families.

4.3.1 Period length 5

For period length 5, the following proves the existence of infinite families.

Theorem 4.3. *For any integer $n > 1$, where n is not a perfect square, there exist infinitely many numbers of the format $\sqrt{a+b\sqrt{n}}$ with expansion $[c, \overline{d, e, e, d, 2c}]_{\sqrt{n}}$.*

Proof. From Theorem 3.10 we have,

$$a = c^2 + \frac{-2cd^2e^3n + 2cdn^2 + 2cen^2 + d^2e^4n - d^2n^2 - 2den^2 + e^2n^2}{d^4e^4 - d^4n - 4d^3en - 2d^2e^2n + n^2}$$

and

$$b = \frac{c(2d^3e^4 - 2d^3n - 6d^2en - 2de^2n) + n^2 - 2de^3n}{d^4e^4 - d^4n - 4d^3en - 2d^2e^2n + n^2}.$$

From the equation for b we get the linear Diophantine equation

$$c(2d^3e^4 - 2d^3n - 6d^2en - 2de^2n) - b(d^4e^4 - d^4n - 4d^3en - 2d^2e^2n + n^2) = 2de^3n - n^2.$$

If we let $d = e$ the Diophantine equation becomes

$$c(2d^7 - 10d^3n) - b(d^8 - 7d^4n + n^2) = 2d^4n - n^2.$$

For this $(d/2, 1)$ is an integer solution if d is even. Therefore, all such solutions have the form

$$c = t\left(d^8 - 7d^4n + n^2\right) + \frac{d}{2}$$

and

$$b = 2dt\left(d^6 - 5d^2n\right) + 1.$$

For the above,

$$x = \sqrt{\left((d^8 - 7d^4n + n^2)\,t + \frac{d}{2}\right)^2 - 2dnt\,(d^4 - 2n) + \sqrt{n}\,(2d^3\,(d^3 - 5n)\,t + 1)}.$$

For a more concrete example we can let $d = 2n$ and $t = 1$ to obtain the infinite family

$$\begin{aligned}&\sqrt{(256n^8 - 112n^5 + n^2 + n)^2 - (32n^6 - 4n^3) + \sqrt{n}(128n^6 - 80n^4 + 1)}\\ &= [256n^8 - 112n^5 + n^2 + n, \overline{2n, 2n, 2n, 2n, 2(256n^8 - 112n^5 + n^2 + n)}]_{\sqrt{n}}.\end{aligned}$$

The above family works for every n since the expansion obtained is a maximal expansion.

□

4.3.2 Period length 6

For period length 6, the following proves the existence of infinite families.

Theorem 4.4. *For any integer $n > 1$, where n is not a perfect square, there exist infinitely many numbers of the format $\sqrt{a + b\sqrt{n}}$ with expansion $[c, \overline{d, e, f, e, d, 2c}]_{\sqrt{n}}$.*

Proof. If $\sqrt{a + b\sqrt{n}}$ has a periodic continued fraction expansion with a period length 6 of the form $[c, \overline{d, e, f, e, d, 2c}]_{\sqrt{n}}$, then a and b have the following formats:

$$a = c^2 + \frac{(2c - e)en(-d^2e^2f^2 + 2dfn + 2d^2n + f^2n) - 2e^2(d - f)(d + f)n^2}{d^4e^4f^2 - 4d^4e^2n - 4d^3e^2fn - 2d^2e^2f^2n + 4d^2n^2 + 4dfn^2 + f^2n^2}$$

and

$$b = \frac{2cd^3e^4f^2 + (-8cd^3e^2 - 6cd^2e^2f - 2cde^2f^2 - 2de^3f^2)n + (4cd + 4de + 2cf + 2ef)n^2}{d^4e^4f^2 - 4d^4e^2n - 4d^3e^2fn - 2d^2e^2f^2n + 4d^2n^2 + 4dfn^2 + f^2n^2}.$$

From the equation of b we can obtain the Diophantine equation

$$\begin{aligned}(d^4e^4f^2 - 4d^4e^2n - 4d^3e^2fn - 2d^2e^2f^2n + 4d^2n^2 + 4dfn^2 + f^2n^2)b \\ - 2c(d^3e^4f^2 + (-4d^3e^2 - 3d^2e^2f - de^2f^2)n + (2d + f)n^2) \\ = (-2de^3f^2n + 2efn^2 + 4den^2).\end{aligned}$$

If we specify n and the terms of the expansion except c, we obtain specific Diophantine equations. Using solutions to these equations, we still recover many infinite families. Some examples of these are

$$\sqrt{784t^2 + 762t + 186 + (37t + 18)\sqrt{2}} = [28t + 14, \overline{2, 2, 4, 2, 2, 2(28t + 14)}]_{\sqrt{2}}$$

and

$$\sqrt{121t^2 + 140t + 42 + (17t + 10)\sqrt{3}} = [11t + 7, \overline{2, 3, 2, 3, 2, 2(11t + 7)}]_{\sqrt{3}}.$$

However, to obtain infinite families for any general n, which is not a perfect square, we limit to the following case. If we let $e = d$ and $f = d$, then the expansion becomes $[c, \overline{d, d, d, d, d, 2c}]_{\sqrt{n}}$ and the Diophantine equation becomes

$$b(d^{10} - 10d^6n + 9d^2n^2) - 2c(d^9 - 8d^5n + 3dn^2) = -2d^6n + 6d^2n^2.$$

Now if $c = d/2$, the expansion reduces to an expansion with a period length of 1. From Theorem 3.1 this force $b = 1$. Therefore, $(d/2, 1)$ is a solution for the equation. We can also check this by plugging the above solution to the equation. Hence, any solutions to the Diophantine equation have the form

$$c = (d^{10} - 10d^6n + 9d^2n^2)t + d/2$$

and

$$b = 2(d^9 - 8d^5n + 3dn^2)t + 1$$

for some integer parameter t.

By letting $d = 2k$ for some k, we get the doubly infinite family

$$\sqrt{\begin{array}{r}(1024tk^{10} - 640tk^6n + 36tk^2n^2 + k)^2 - 16k^3(16k^4 - 5n)nt \\ +(1 + 1024k^9t - 512k^5nt + 12kn^2t)\sqrt{n},\end{array}}$$

with the expansion,

$$[(1024k^{10} - 640k^6n + 36k^2n^2)t + k, \overline{2k, 2k, 2k, 2k, 2k, 2(1024k^{10} - 640k^6n + 36k^2n^2)t + k)}]_{\sqrt{n}}.$$

By specifying n, we can get examples for many infinite families. For example, letting $n = 2$ and $k > 1$, we get

$$\begin{aligned}&\sqrt{(k + 9k^2 - 80k^6 + 64k^{10})^2 - 4k^3(-5 + 8k^4) + (1 + 3k - 64k^5 + 64k^9)\sqrt{2}} \\ &= [k + 9k^2 - 80k^6 + 64k^{10}, \overline{2k, 2k, 2k, 2k, 2k, 2(k + 9k^2 - 80k^6 + 64k^{10})}]_{\sqrt{2}}.\end{aligned}$$

Now to obtain an infinite family that works for any integer n, which is not a perfect square, we can let $k = n$. In this case we have the family

$$\begin{aligned}&\sqrt{(n + 36n^4 - 640n^7 + 1024n^{10})^2 + 80n^5 - 256n^8 + (1 + 12n^3 - 512n^6 + 1024n^9)\sqrt{n}} \\ &= [n + 36n^4 - 640n^7 + 1024n^{10}, \overline{2n, 2n, 2n, 2n, 2n, 2(n + 36n^4 - 640n^7 + 1024n^{10})}]_{\sqrt{n}}\end{aligned}$$

The existence of the above family proves that the theorem holds.

□

4.3.3 Period length 7

Similar to the above cases, we can recover infinite families with period length 7.

Theorem 4.5. *For any integer $n > 1$, where n is not a perfect square, there exist infinitely many numbers of the format $\sqrt{a + b\sqrt{n}}$ with expansion* $[c, \overline{d, e, f, f, e, d, 2c}]_{\sqrt{n}}$.

Proof. The corresponding Diophantine equation can be obtained using Theorem 3.12. For an expansion of the form $x = [c, \overline{d, e, f, f, e, d, 2c}]_{\sqrt{n}}$, we can use the equation of b to get the following Diophantine equation

$$\begin{aligned}b(&d^4e^4f^4 - d^4e^4n - 4d^4e^3fn - 2d^4e^2f^2n - 4d^3e^2f^3n - 2d^2e^2f^4n + d^4n^2 + 4d^3en^2 \\ &+2d^2e^2n^2 + 4d^3fn^2 + 4d^2efn^2 + 6d^2f^2n^2 + 4df^3n^2 + f^4n^2 - n^3) \\ -c(&2d^3e^4f^4 - 2d^3e^4n - 8d^3e^3fn - 4d^3e^2f^2n - 6d^2e^2f^3n - 2de^2f^4n + 2d^3n^2 \\ &+6d^2en^2 + 2de^2n^2 + 6d^2fn^2 + 4defn^2 + 6df^2n^2 + 2f^3n^2) \\ &= -2de^3f^4n + (2def^2 + 2ef^3)n^2 - n^3.\end{aligned}$$

For specific n, we can easily recover many infinite families by specifying the terms of the expansion. For example, when $n = 2$, we have the infinite families

$$\sqrt{(21t + 15)^2 + 10t + 7 + \sqrt{2}(8t + 6)} = [21t + 15, \overline{2, 1, 1, 1, 1, 2, 2(21t + 15)}]_{\sqrt{2}}$$

and

$$\sqrt{261121t^2 + 80376t + 6186 + (682t + 105)\sqrt{2}} = [51t{+}27, \overline{2, 3, 2, 2, 3, 2, 2(51t + 27)}]_{\sqrt{2}}.$$

However, we can obtain infinite families for any general n by restricting our selves to the format $[c, \overline{d, d, d, d, d, d, 2c}]_{\sqrt{n}}$. In this case,

$$a = c^2 + \frac{-2cd^9n + 16cd^5n^2 - 6cdn^3 + d^{10}n - 8d^6n^2 + 3d^2n^3}{d^{12} - 13d^8n + 26d^4n^2 - n^3}$$

and

$$b = \frac{(c\,(2d^{11} - 22d^7n + 28d^3n^2) - (2d^8n - 12d^4n^2 + n^3))}{d^{12} - 13d^8n + 26d^4n^2 - n^3}.$$

Also the Diophantine equation simplifies to be

$$2c\left(d^{11} - 11d^7n + 14d^3n^2\right) - b\left(d^{12} - 13d^8n + 26d^4n^2 - n^3\right) = 2d^8n - 12d^4n^2 + n^3.$$

Similar to the arguments of previous cases $(d/2, 1)$ is a solution. This is evident by studying the equation as well. Therefore, the general solution for b and c are,

$$b = 2t\left(d^{11} - 11d^7n + 14d^3n^2\right) + 1$$

and

$$c = t\left(d^{12} - 13d^8n + 26d^4n^2 - n^3\right) + \frac{d}{2}.$$

Here we require d to be even, therefore we can let $d = 2k$ for some k. Now we have a doubly infinite family where a and b in $\sqrt{a + b\sqrt{n}}$ have the format

$$\begin{aligned} a = {} & 16777216k^{24}t^2 - 27262976k^{20}nt^2 + 14483456k^{16}n^2t^2 + 8192k^{13}t \\ & - 2777088k^{12}n^3t^2 - 7680k^9nt + 179712k^8n^4t^2 + 1344k^5n^2t - 832k^4n^5t^2 + k^2 \\ & - 14kn^3t + n^6t^2, \\ b = {} & 4096k^{11}t - 2816k^7nt + 224k^3n^2t + 1. \end{aligned}$$

The expansion of the above infinite family has the form

$$\begin{aligned} x = [& 4096k^{12}t - 3328k^8nt + 416k^4n^2t + k - n^3t, \\ & \overline{2k, 2k, 2k, 2k, 2k, 2k, 2\,(4096k^{12}t - 3328k^8nt + 416k^4n^2t + k - n^3t)}]_{\sqrt{n}} \end{aligned}$$

for any n which is not a perfect square.

□

Below is a summary of all the Diophantine equations up to period length 6.

Table 4.1: Summary of Diophantine equations

Period length	Form	Diophantine equation
3	$[c, \overline{d, d, 2c}]_{\sqrt{n}}$	$2d^3c - (d^4 - n)b = n$
4	$[c, \overline{d, e, d, 2c}]_{\sqrt{n}}$	$(2e^2d^2 - 4n)c - (e^2d^3 - 4nd)b = 2ne$
5	$[c, \overline{d, e, e, d, 2c}]_{\sqrt{n}}$	$(d^4e^4 - d^4n - 4d^3en - 2d^2e^2n + n^2)b$ $-(2d^3e^4 - 2d^3n - 6d^2en - 2de^2n)c$ $= n^2 - 2de^3n$
6	$[c, \overline{d, e, f, e, d, 2c}]_{\sqrt{n}}$	$(d^4e^4f^2 - 4d^4e^2n - 4d^3e^2fn-$ $2d^2e^2f^2n + 4d^2n^2 + 4dfn^2 + f^2n^2)b$ $-2c(d^3e^4f^2 - 4d^3e^2n - 3d^2e^2fn$ $-de^2f^2n + 2dn^2 + fn^2)$ $= 2en^2(2d + f) - 2de^3f^2n$

For even longer period length, it is possible to obtain infinite families with the format $[c, \overline{d, d, ..., d, 2c}]_{\sqrt{n}}$. For example, for period length 8,

$$\sqrt{(28t+1)^2 - 10t + \sqrt{2}(29t+1)} = [28t+1, \overline{2, 2, 2, 2, 2, 2, 2, 2(28t+1)}]_{\sqrt{2}},$$

$$\sqrt{(284t+1)^2 - 438t + \sqrt{3}(467t+1)} = [284t+1, \overline{2, 2, 2, 2, 2, 2, 2, 2(284t+1)}]_{\sqrt{3}},$$

and for period length 9,

$$\sqrt{(25089847t + 12544925)^2 - 2152302(2t+1) + \sqrt{2}(17741106t + 8870554)}$$
$$= [25089847t + 12544925, \overline{3, 3, 3, 3, 3, 3, 3, 3, 2(25089847t + 12544925)}]_{\sqrt{2}}.$$

Therefore, we make the following conjecture.

Conjecture 4.6. *There are infinite families for every period length when* $x = \sqrt{a + b\sqrt{n}}$ *with* $z = \sqrt{n}$.

If we can show the existence of a linear Diophantine equation of variables b and c for every period length, then we can show that infinite families are possible for any period length. This can be done by setting every term of the period to be a constant d, except for the last term, which is $2c$. Now if $d = 2c$, expansion reduces to a period length 1 expansion. This case was completely characterized in section 3. Therefore, $b = 1$. This implies that $(d/2, 1)$ is a solution for the Diophantine equation. Now we can use this to show the Diophantine equation will have infinitely many solutions.

5 Future Work

From the cases observed in this paper, an irrational number of the form $\sqrt{a+b\sqrt{n}}$ has similarities to simple continued fraction expansion of $\sqrt{n}$, where n is not a perfect square. For example, when the expansion of the form $\sqrt{a+b\sqrt{n}}$ is periodic, the expansion appears to have a tail length of 1 followed by a periodic part, which has a palindromic pattern with the last term being twice the first term. This is similar to the simple continued fraction expansions of $\sqrt{n}$. An interesting expansion to this research would be to prove these similarities in a general context. For several shorter period lengths, we showed that longer tail lengths are not possible using algebraic manipulation. But generalization to any period length would be more complicated and might require the properties of convergents as well. Also, as observed in the computer search as well as from shorter periods, the periodic part of the expansion always has a palindromic pattern. Generalizing this finding to every period length would also be an expansion to the scope of this project.

Another possibility might be proving Conjecture 4.6, that is to prove that continued fraction expansions with every period length exists. This would use the fact that every periodic expansion has a corresponding Diophantine equation.

When $z=\sqrt{n}$, the polynomial sequences, p_n and q_n, used to define the convergent will become

$$\begin{aligned} &p_{-1}=1, && p_0=a_0, && p_k=a_kp_{k-1}+\sqrt{n}p_{k-2} && \text{for } k\geqslant 1,\\ &q_{-1}=0, && q_0=1, && q_k=a_kq_{k-1}+\sqrt{n}q_{k-2} && \text{for } k\geqslant 1. \end{aligned}$$

For each $k>0$, the polynomials p_k and q_k have a rational part and an irrational part. Therefore, it might be possible to find 4 recursive polynomial functions, corresponding to rational and irrational parts of p_k and q_k. Such a characterization might be a very useful tool to prove many results for the expansion of $\sqrt{a+b\sqrt{n}}$.

Another possible expansion for this research would be to find infinite families for any general n, that do not have an expansion of the form $[c,\overline{d,d,...,d,d,2c}]_{\sqrt{n}}$. For example, when $n=3$ we found the family

$$\sqrt{121t^2+140t+42+(17t+10)\sqrt{3}}=[11t+7,\overline{2,3,2,3,2,2(11t+7)}]_{\sqrt{3}}$$

for period length 6. This family does not have the form $[c,\overline{d,d,...,d,d,2c}]_{\sqrt{n}}$. The extension would be to check if such families are possible for any n, which is not a perfect square.

References

[1] M. Anslem and S. Weintraub, *A generalization continued fractions*, Journal of Number Theory, 2011.

[2] J. Aumasson, W. Meier, R. Phan, L. Henzen,*The Hash Function BLAKE*, Springer Link inc., 2014.

[3] Alan Baker, *A concise introduction to theory of numbers*, Cambridge University press, 1986.

[4] David M.Burton, *Elementary Number theory*, 3rd edition, Wm.C Brown Publishers, 1994.

[5] J. Greene and J. Schmieg, *Continued fractions with non-integer numerators*, Journal of integer sequences, 2017.

[6] G.H. Hardy and E.M. Wright, *An introduction to the theory of Numbers*, 6th edition, Oxford University press, 1979.

[7] William J. LeVeque, *Fundamentals of Number Theory*, Dover Publications, 1996.

[8] Pablo Mello, *Patterns of non-simple continued fractions*, an undergraduate research project, 2017.

[9] C.D. Olds, *Continued Fractions*, 5th Edition, New York: Random House, Inc., 1963.

[10] J. Schmieg, Patterns of non-Simple continued Fractions, an Undergraduate research project, 2015.

6 Appendix

6.1 Computer search results

The starting point of this research was to conduct a Mathematica based computer search to find periodic behavior and possible patterns of the forms $a + b\sqrt{n}$ and $\sqrt{a + b\sqrt{n}}$ with $x = \sqrt{n}$. In this section, we would provide a complete description of the results.

6.1.1 Results for $a + b\sqrt{n}$

For the form $a+b\sqrt{n}$, the search range of the parameters a, b and n was from 0 to 100. But n was forced not to be a perfect square and b was forced to be non-zero to preserve the pattern. A total of 909000 numbers were searched to obtain 22664 numbers, which were periodic.
Table 6.1 provides the frequency of occurrence of different tail lengths observed in the search.

Table 6.1: Complete list of observed tail lengths for expansion of $a + b\sqrt{n}$

Tail length	Frequency
0	16
1	2294
2	1310
3	404
4	401
11	808
5	1105
6	701
7	501
8	1303
9	502
10	302
11	908
12	301
13	402
14	202
15	802
16	401
$17, 20, 26, 29, 30, 31, 32, 33, 36, 37, 38, 48$	200
19	801
22	700
23	800
$24, 27$	400
25	600
$21, 28, 34, 35, 39, 50, 55, 59, 60, 61, 64, 66, 71, 76, 80, 82, 85, 93$	100
$40, 41, 42, 43, 44, 47, 62$	300

Table 6.2 provide values of n which displayed each of the above tail lengths.

Table 6.2: Complete list of n-values arranged by different tail lengths for expansion of $a + b\sqrt{n}$

Tail length	n
0	2, 8, 10, 12, 15, 20, 24, 30, 35, 42, 48, 56, 63, 72, 80, 90
1	2, 3, 6, 8, 10, 12, 15, 20, 24, 30, 35, 42, 48, 56, 63, 72, 80, 90
2	3, 8, 12, 15, 24, 32, 35, 48, 60, 63, 80, 96
3	8, 12, 32, 60, 96
4	2, 3, 6
5	2, 3, 6, 8, 10
6	2, 3, 8
7	3, 8, 12
8	2, 3, 6
9	2, 6
10, 11	2, 3, 12
12	2, 6
13	2, 6, 12
14	2
15	2, 3, 6
16	2, 6, 8
17	6
19	2, 3, 6
20	3, 6
21	3
22	3, 6, 8
23	3, 6
24	6, 8
25, 26	3, 6
27	3
28	8
29	6
30	6, 8
31	6, 3
32, 33	3, 8
34	8
35	3
36, 37, 38, 39, 40	6
41	3
42, 43, 44, 47	6
48	6, 8
50	6
55, 59, 60	8
61	6
62	3, 6
66	3
71, 76, 80, 82, 85, 93	8

6.1.2 Results for $\sqrt{a+b\sqrt{n}}$

Following tables, Table 6.3, 6.4 and 6.5 provide results for the computer search done for the form $\sqrt{a+b\sqrt{n}}$, where a, b and n are ranged from 0 to 100. Similar to the above case, we forced n to not be a perfect square and b to be non-zero to preserve the pattern. A total of 909000 numbers were searched and only 1455 were found to be periodic.

The frequency of occurrence of tail lengths are shown in Table 6.3 below.

Table 6.3: Complete list of Tail lengths for expansions of $\sqrt{a+b\sqrt{n}}$

Tail lengths	Frequency
0	21
1	1386
2	28
3	2
4	8
5	10

All except for values with tail length 1 simplified to be a quadratic surd. Therefore, only the values with tail length 1 were further analyzed. Table 6.4 shows the frequency of occurrence of different period lengths for the values with a tail length 1.

Table 6.4: Complete list of observed period lengths for expansion of $\sqrt{a+b\sqrt{n}}$

Period lengths	Frequency
1	736
2	582
3	16
4	25
5	3
6	5
9	3
10	4
12	2
18	1

Table 6.5 shows possible period lengths for different n values.

Table 6.5: Period lengths observed for different n values

n	Period lengths observed
2	$1, 2, 3, 4, 5, 6, 8, 9, 10, 18$
3	$1, 2, 4, 6, 8, 9, 12$
5	$1, 2, 4$
6	$1, 2, 4, 8, 10$
7	$1, 2, 4$
8	$1, 2, 3, 4, 8$
10	$1, 2$
11	$1, 2$
12	$1, 2, 6$
Longer than 12	$1, 2$

6.2 Mathematica code

The function to check if a number has a periodic expansion.

```
(*
Variables,
x0: Number that we want to check
x : List of all values of each x_i in the continued fraction algorithm
a : List of the corresponding a_i for each x_i,
    represent the integers in the expansion
z : Numerator of the expansion
iter: Number of maximum iteration done
*)

Ifperiod[x0_,z_,iter_]:=
Block[{x,a},

x=Table[0,iter];
a=Table[0,iter];

(*Initial values*)
x[[1]]=x0;
a[[1]]=Floor[x[[1]]];

(*
    To build the corresponding iterations x_i and a_i of the continued
fraction expansion. Only the maximal expansion was considered.
    *)
For[i=2,i<=iter ,i++,
{
x[[i]]=FullSimplify[z/(x[[i-1]]-a[[i-1]])],
a[[i]]=Floor[x[[i]]],
For[k=1,k<=i,k++,
```

```
        If[k==iter,
        {
        For[j=1,j<=i,j ++,
        For[l=1,l<=i,l++,
          If[(l!=j)&&(x[[l]]==x[[j]]),
           {Return[0],Goto[end1]},
              If[l==iterMax&&j==iterMax,Goto[end]]
             ]]];
     }]];
     Label[end1];
}];
SessionTime[]
]


 (*
The code using Floyd's cycle finding algorithm to check if a number
is periodic.
 *)

IfperiodF[x0_,z_,it_]:=
Block[{x,a},
iter=it;
x={};
a={};
tail={};
period={};

AppendTo[x,x0];
AppendTo[a,Floor[x0]];

For[i=2,i <= iter ,i++,
Block[{},
If[x[[i-1]]!= a[[i-1]],
Block[{},
AppendTo[x,FullSimplify[z/(x[[i-1]]-a[[i-1]])]];
AppendTo[a,Floor[x[[i]]]];
If[EvenQ[i]&&x[[i]]==x[[i/2]],
{
Return[1]
Goto[return];
},
If[i==iter,{Return[0];Goto[return]}];
]],{Goto[return];}
]
]];
Label[return];
]
```

Following is the code to extract the tail and the period of a periodic continued fraction expansion.

```
Cfrac[x0_,z_,iter_]:=
Block[{x,a,w,y,checkperiod,tail,period},

x=Table[0,iter];
a=Table[0,iter];

(* Initial values *)
    x[[1]]=x0;
a[[1]]=Floor[x[[1]]];

    For[i=2,i<=iter ,i++,
{
    x[[i]]=FullSimplify[z/(x[[i-1]]-a[[i-1]])],
    a[[i]]=Floor[x[[i]]]}];

    tail={};
period={};

    (*
    i,j are the minimum indexes with x_i=x_j
    Min(i, j)-1 gives the length of the tail.
    |i-j| gives the length of the period.
    *)

    For[i=1,i<=iter,i++,
    For[j=1,j<=iter,j ++,
        If[(i!=j)&&(x[[i]]==x[[j]]),
{
tail=Part[a,1;;Min[i,j]-1];
period=Part[a,Min[i,j];;Abs[i-j]+Min[i,j]-1];
Goto[ending]
}]]];
Label[ending];
Return[{x[[1]],a,tail,period}]
Print["Session time is ",SessionTime[]];
]
```