December, 1965

ON MULTIVARIATE WIDE-SENSE MARKOV PROCESSES*

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Technical Report No. 67

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Minneapolis, Minnesota

*This research was supported by U.S. Army Research Office, Grant No. DA-ARO-D-31-124-G562 and U.S. Air Force Office of Scientific Research, Grant No. AFOSR-381-65.

1. Introduction. In this note we obtain a characterization of a multivariate wide-sense Markov process and study its applications. As an immediate consequence of it, we derive in Section 3 a partial converse to a recent theorem of F. J. Beutler [1]. Section 4 is devoted to the study of the representations of purely non-deterministic (not necessarily stationary) processes which generalize the main theorem of J. L. Doob [5] on stationary Gaussian Markov processes. It is proved that the multiplicity of a q-variate Wide-sense Markov process does not exceed q. The comparison with the above-mentioned work of Doob is carried out in Section 5. In studying these representations we make extensive use of the multiplicity theory of stochastic processes as developed by H. Cramér ([2], [3]. See also G. Kallianpur and the author [7]). Since it is basic to the understanding of the results of this note we devote the next Section to recalling some known results and notation of the theory.

2. Multivariate second order processes and associated subspaces. Let $\{\underline{x}_t\}$ $(-\infty < t < +\infty)$ denote a second order q-variate continuous parameter process; i.e., for each t, $\underline{x}_t$ is a column vector $(x_1(t),\ldots,x_q(t))^*$ such that $E|x_i(t)|^2$ is finite $(i = 1,2,\ldots,q)$. If $L_2(\Omega,P)$ is the Hilbert-space of the complex-valued square integrable functions on the basic probability space (Ω,P), we can regard for each (i,t), $x_i(t)$ as an element of $L_2(\Omega,P)$. We associate with $\{\underline{x}_t\}$ the following subspaces of $L_2(\Omega,P)$.

The "past" and "present" of the process up to t, is the subspace $L_2(\underline{x};t)$ generated by the random variables $\{x_i(\tau)\ \ i = 1,2,\ldots,q,\ \ \tau \leq t\}$. The "remote past" of the process $L_2(\underline{x};-\infty)$ is the intersection of $L_2(\underline{x};t)$ for all real t and the space of the process $L_2(\underline{x})$ is the smallest subspace of $L_2(\Omega,P)$ containing all $L_2(\underline{x};t)$ for each t.

Definition 2.1. A q-variate second order process $\{\underline{x}_t\}$ is deterministic if $L_2(\underline{x};-\infty) = L_2(\underline{x})$ and is purely non-deterministic $L_2(\underline{x};-\infty) = \{0\}$.

For a q-variate purely non-deterministic, continuous in quadratic mean $\underline{x}_t$-process,

H. Cramér [3] has obtained the following representation,

$$x_i(t) = \sum_{k=1}^{M} \int_{-\infty}^{t} g_{ik}(t,u)\, dz_i(u) \tag{2.1}$$

where $\{z_i(u), -\infty < u < +\infty\}$ $(i = 1,2,\ldots,M)$ are mutually orthogonal processes of orthogonal increments with covariance measure $\rho_i(\Delta) = \mathcal{E}|z_i(\Delta)|^2$, $\sum_1^M \int_{-\infty}^{t} |g_{ik}(t,u)|^2 d\rho_i(u)$ is finite, and M which is at most countable is called the multiplicity of the representation. In case $\underline{x}_t$ is stationary process it is proved in [7] that M is equal to the rank of the process.

Before we conclude this section we want to introduce the idea of direct-product Hilbert spaces. If H is a Hilbert-space we shall mean by $H^{(q)}$ the space of all vectors $\underline{h} = (h_1, h_2, \ldots, h_q)$ where for each i, $h_i \in H$. In $H^{(q)}$ is introduced a norm $|||\underline{h}||| = \sqrt{\sum_1^q ||h_i||_H^2}$ and an inner product given by the Gramian matrix $[h,h^*] = ((\langle h_i, h_j\rangle_H))$. A linear manifold in $H^{(q)}$ is a non-void subset $\mathfrak{M}$ of $H^{(q)}$ such that if $\underline{h},\underline{h}' \in \mathfrak{M}$ then $A\underline{h} + B\underline{h}' \in \mathfrak{M}$ for all $q \times q$ matrices A,B. A subspace of $H^{(q)}$ is a linear manifold closed under the topology $|||\ \ |||$. We recall here a lemma due to N. Wiener and P. Masani [10] which proves the existence of the projection of an element $\underline{h}$ and gives its structure.

<u>Lemma</u> WM (Lemma 5.8 [10]) (a). If $\mathfrak{M}$ is a subspace of $H^{(q)}$ then there exists a subspace $\mathcal{M}$ of H such that $\mathfrak{M} = \mathcal{M}^{(q)}$, where $\mathcal{M}^{(q)}$ denotes the Cartisian product $\mathcal{M} \otimes \ldots \otimes \mathcal{M}$ with q-factors. $\mathcal{M}$ is a set of all components of all elements in $\mathfrak{M}$.

(b). If $\mathfrak{M}$ is a subspace of $H^{(q)}$ and $\underline{h} \in H^{(q)}$, then there is a unique $\underline{h}' \in \mathfrak{M}$ such that $|||\underline{h}-\underline{h}'||| \leqq |||\underline{h}-\underline{g}|||$ for all $\underline{g} \in \mathfrak{M}$. For this $\underline{h}'$, $h_i' = P_{\mathcal{M}} h_i$ $\mathcal{M}$ being as in (a). An element $\underline{h}'$ satisfies preceding condition iff $[\underline{h}-\underline{h}', \underline{g}] = 0$ for all $\underline{g} \in \mathfrak{M}$. The part (c), (d), and (e) of the original lemma are omitted since they won't be referred here.

<u>Definition 2.2</u> The unique element $\underline{h}'$ of Lemma WM (b) is called the orthogonal projection of $\underline{h}$ onto $\mathfrak{M}$ and is denoted by $(\underline{h}|\mathfrak{M})$.

3. Wide-sense Markov process and its characterization. Let $\underline{x}_t$ be a q-variate second order stochastic process. Then $\underline{x}_t \in L_2^{(q)}(\underline{x})$. We now introduce multivariate wide-sense martingale and Markov process as an extension of the similar concepts due to J. Doob in univariate case (see Doob [4], pp. 164, 90).

Definition 2.1. a) A q-variate process $\{\underline{x}_t\}$ $(-\infty < t < +\infty)$ is a wide-sense martingale if for each t, $(\underline{x}_t | L_2^{(q)}(\underline{x};s)) = \underline{x}_s$ for $s < t$.

b) A process $\{\underline{x}_t\}$ is called side-sense Markov if for each t, $(\underline{x}_t | L_2^{(q)} | (\underline{x};s)) = (\underline{x}_t | \underline{x}_s)$.

The assumption (D) given below will be made throughout this paper.

(D.1) $\underline{x}_t$-process is continuous in q. m., i.e.,

$$\lim_{s \to t} |||\underline{x}_t - \underline{x}_s||| = 0.$$

(D.2) For all t, s real the covariance matrix $\Gamma(t,s)$ is non-singular.

The assumption (D.2) and the definition of wide-sense Markov process imply $(\underline{x}_t | L_2^{(q)}(\underline{x};s) = A(t,s)\underline{x}_s$ where the matrix $A(t,s)$ is given by $A(t,s) = \Gamma(t,s)\Gamma^{-1}(s,s)$ for $s \leq t$. The function $A(t,s)$ is called a transition matrix function and is defined only for $s \leq t$. Observe that if $\underline{x}_t$ is wide-sense Markov then for $s \leq t \leq u$ $A(u,s) = A(u,t)A(t,s)$.

Theorem 3.1. A q-variate second order continuous parameter process satisfying (D) is wide-sense Markov if and only if $\underline{x}_t = \Psi(t)\underline{u}_t$ with probability one, where for each t, $\Psi(t)$ is a non-singular $q \times q$ matrix and $\underline{u}_t$ is a q-dimensional wide-sense martingale such that $L_2(\underline{u};t) = L_2(\underline{x};t)$.

Proof: Sufficiency. Let $\underline{x}_t = \Psi(t)\underline{u}_t$ where $\Psi(t)$ and $\underline{u}_t$ are as described above. Then for $s \leq t$ $(\underline{x}_t | L_2^{(q)}(\underline{x};s)) = (\Psi(t)\underline{u}_t | L_2^{(q)}(\underline{x};s)) = (\Psi(t)\underline{u}_t | L_2^{(q)}(u;s)) = \Psi(t)\underline{u}_s$. Since $\underline{u}_s = \Psi^{-1}(s)\underline{x}_s$ with probability one, we obtain that the transition matrix function $A(t,s) = \Psi(t)\Psi^{-1}(s)$.

Necessity: Let $\underline{x}_t$-process be wide-sense Markov. Then denoting by $A(t,s)$ the transition matrix function we have for $s \leq t$,

$$(\underline{x}_t | L_2^{(q)}(\underline{x};s)) = A(t,s)\underline{x}_s \quad \text{with probability one} \tag{3.1}$$

(3.2) $$A(u,s) = A(u,t)A(t,s) \quad \text{for} \quad s \leq t \leq u.$$

Let us now define, following Hida [6], for every real t, the function

$$\underline{\Psi}(t) = A(t,s_0) \quad \text{if} \quad s_0 \leq t$$
$$= A^{-1}(s_0,t) \quad \text{if} \quad t < s_0$$

where s_0 is a fixed real number. We now show that for all s,t $(s \leq t)$ real

(3.3) $$A(t,s) = \underline{\Psi}(t)\underline{\Psi}^{-1}(s).$$

First of all, if $s < s_0 \leq t$, (3.3) is a restatement of (3.2): i.e., $A(t,s) = A(t,s_0)A(s_0,s)$. If $s_0 \leq s < t$ from (3.2) we have $A(t,s)A(s,s_0) = A(t,s_0)$, i.e., $A(t,s) = A(t,s_0)A^{-1}(s,s_0)$ giving (3.3) again. Finally if $s < t \leq s_0$ we again get from (3.3), $A(s_0,s) = A(s_0,t)A(t,s)$ and hence $A(t,s) = \underline{\Psi}(t)\underline{\Psi}^{-1}(s)$. $\underline{\Psi}(t)$ is non-singular since $A(t,s)$ is. Therefore from (3.1) and (3.3) we have for $s < t$

(3.4) $$(\underline{x}_t | L_2^{(q)}(\underline{x};s)) = \underline{\Psi}(t)\underline{\Psi}^{-1}(s)\underline{x}_s \quad \text{with probability one.}$$

Hence $\underline{u}_t = \underline{\Psi}^{-1}(t)\underline{x}_t$ is a martingale and $L_2(\underline{u};t) = L_2(\underline{x};t)$. The proof of the Theorem is now complete.

The characterization of Theorem 3.1 will be used later to study purely non-deterministic wide-sense Markov processes and their multiplicity.

However as a first application we show that if $\underline{x}_0 = 0$ and $\underline{\Psi}(t)$ is differentiable, then it satisfies the following differential equation with probability one.

(3.5) $$\underline{x}_t = A(t)\underline{x}_t + M(t)\underline{\eta}(t) \qquad t \geq 0$$

where $\underline{\eta}(\cdot)$ is a multivariate "white noise" random process and $A(t) = \mathring{\underline{\Psi}}(t)\underline{\Psi}^{-1}(t)$, $M(t) = \underline{\Psi}(t)$. The equation (3.5) is to be interpreted as $\underline{x}_t = \int A(t)\underline{x}(t)dt + \int M(t)d\underline{u}(t)$ $\underline{\eta}_t$ being the "fictitious derivative" of $\underline{u}_t$.

<u>Theorem 3.2</u>. Let $\{\underline{x}_t, 0 \leq t < \infty\}$ be a wide-sense Markov process satisfying (D). If further $\underline{x}_0 = 0$ and $\underline{\Psi}(t)$ of Theorem 3.1 is continuously differentiable

then $\underline{x}_t$ satisfies equation (3.5) for $t \geq 0$ where η_t is a q-variate white noise process and the matrix function $A(t) = \mathring{\Psi}(t)\Psi^{-1}(t)$, $M(t) = \Psi(t)$. The proof of the Theorem follows by substituting in (3.5) $\dot{x}_t = \Psi(t)\underline{u}_t$.

We now take up the study of covariance function of a stationary wide-sense Markov process.

Definition 3.2. We say that a q-variate second order process $\{\underline{x}_t, -\infty < t < +\infty\}$ is stationary if $\Gamma(t,s) = [\underline{x}(t),\underline{x}(s)] = R(t-s)$. for $s < t$.

By Theorem 3.1 and the definition of wide-sense martingale we get for $h \geq 0$

$$(3.6) \qquad R(h) = [\underline{x}(t+h),\underline{x}(t)] = \Psi(t+h)J(t,t)\Psi^*(t).$$

Let $h = 0$, we get

$$(3.7) \qquad R(0) = \Psi(t)J(t,t)\Psi^*(t)$$

With $t = 0$ in (3.6), one has

$$(3.8) \qquad R(h) = \Psi(h)J(0,0)\Psi^*(0).$$

Relations (3.6) and (3.8) imply $h \geq 0$, $t \geq 0$

$$(3.9) \qquad R(h) = R(t+h)R^{-1}(t)R(0)$$

With $R_1(t) = R(t)R^{-1}(0)$, (3.9) reduces to

$$(3.10) \qquad R_1(t+h) = R_1(t)R_1(h).$$

We prove now the following theorem

Theorem 3.3. Let $\{\underline{x}_t\}$ $(-\infty < t < \infty)$ be a q-dimensional stationary process satisfying assumption (D) then it is wide-sense Markov if and only if the transition matrix function $B(t) = e^{tQ}$ for $t \geq 0$ where $B(t) = A(t,0)$ and Q is uniquely determined constant $q \times q$ matrix none of whose eigenvalues has positive real part.

Proof. Necessity. We have already shown that for $R(t) = R(t)R^{-1}(0)$ the equation (3.10) holds. Further, from (D.1) it follows that $R_1(t)$ is a continuous function and therefore $R_1(t) = e^{tQ}$ $(t \geq 0)$ is the solution of (3.10) where Q is a $q \times q$ constant matrix. The assumption (D.2) in addition implies that $R_1(t)$ is non-singular and hence Q is uniquely determined by $R_1(t)$. Since $B(t) = R(t)R^{-1}(0)$ for $t \geq 0$ we have $B(t) = e^{tQ}$. Due to the fact that $\lambda(t) = \max_{j \leq q} \lambda_j(t)$ (where $\lambda_j(t)$ is jth eigenvalue of $B_1(t)$) satisfies for all t,

$|\lambda(t)| \leq tr[R^{-1}(0)(R^{-1}(0))^*] \; (\sum_{i=1}^{q} E|x_i(0)|^2)^2$ it follows that the eigenvalues of $Q = \lim_{t \to 0} \frac{B(t)-I}{t}$ has non-negative real parts.

The above result was first proved by J. L. Doob [5] for Stationary Gaussian Markov processes. It was reproved by Beutler [1] for wide-sense Markov processes. We have proved it because our proof is based directly on the characterization of Theorem 3.1. Furthermore it brings out the form of $\Psi(t)$ in stationary case which will be utilized in Theorem 5.1. It is also interesting to note that the fact that $R(t-s) = \psi(t)J(s,s)\psi^*(t)$ could enable one to obtain a general form for the covariance function of stationary wide-sense Markov processes (See Kalmykov [8]).

4. Representations of purely non-deterministic wide-sense Markov processes.

In general representation(2.1) of Cramér ([4]) the multiplicity M is in general countable for non-stationary processes. We show that for a q-variate wide-sense Markov process satisfying (D) the multiplicity M is finite and is actually less than q. The following general lemma will be found useful in the study of multiplicity theory.

Lemma 4.1. Let H be a separable Hilbert space. H_1, H_2 be two subspaces of such that $H_1 \perp H_2$ and $H = H_1 \oplus H_2$. Suppose that $\{E_1(t)\}$ is a resolution of the identity in H_1 and $\{E_2(t)\}$ is the resolution of the identity in H_2 then the multiplicity N of $\{E(t)\}$ is equal to the sum $N_1 + N_2$ of the multiplicities of N_i of $E_i(t)$ $(i = 1,2)$, where $E(t) = E_1(t) + E_2(t)$.

Proof:

$$H_1 = \sum_{i=1}^{N_1} \oplus \mathfrak{S}\{E_1(\Delta)f_1^{(i)}, \Delta \subset (-\infty, +\infty)\}$$

$$H_2 = \sum_{i=1}^{N_2} \oplus \mathfrak{S}\{E_2(\Delta)f_2^{(i)}, \Delta \subset (-\infty, +\infty)\} .$$

It is easy to check that $E(t)$ is a resolution of the identity in H. Under the assumptions, $E(\Delta)f_i^{(k)} = E_i(\Delta)f_i^{(k)}$ for $i = 1,2,$ $k = 1,2,\ldots \max(N_1,N_2)$.

Hence $H = \sum_{i=1}^{N_1+N_2} \bigoplus \mathfrak{S}\{E(\Delta)g^{(i)}, \ \Delta \subset (-\infty,\infty)\}$ where $g^{(i)} = f_1^{(i)}$, $i = 1,2,\ldots,N_1$; $g^{(k)} = f_2^{(k)}$, $k = N_1+1,\ldots, N_1+N_2$. The multiplicity of $\{E(t)\}$ is therefore N_1+N_2. The above theorem will be used to determine the multiplicity of a (in general) non-stationary q-variate martingale.

Lemma 4.2. A q-variate purely nondeterministic continuous in q.m. martingale has multiplicity $M \leq q$.

Proof. We shall prove this result for $q = 2$. The general proof differs only in induction. Let us define $v_1(t) = u_1(t)$, $v_2(t) = (I - P_{L_2(u_1)})u_2(t)$. Now $P_{L_2(u_1)} = P_{L_2(u_1;t)} + P_{L(t;\infty)}$ where $L(t;\infty) = \mathfrak{S}\{u_1(\tau)-u_1(t), \ \tau \geq t\}$. Since $P_{L_2(\underline{u};t)}\, u_1(\tau) = u_1(t)$ $\tau > t$ we get $L(t,\infty) \perp L_2(\underline{u};t)$ and hence $u_2(t) \perp L(t,\infty)$. Thus $P_{L_2(u_1;\infty)}\, u_2(t) = P_{L_2(u_1;t)}\, u_2(t)$. Now we have $L_2(\underline{u};t) = L_2(v_1;t) \bigoplus L_2(v_2;t)$; furthermore by definition, $L_2(v_1;\infty) \perp L_2(v_2;\infty)$. We now prove that $\{v_2(t), -\infty < t < \infty\}$ is purely non-deterministic process of orthogonal increments. The only thing one has to prove is that $\{v_2(t), -\infty < t < \infty\}$ is a process of orthogonal increments.

Consider
$$
\begin{aligned}
P_{L_2(v_2;s)}v_2(t) &= P_{L_2(v_2;s)}P_{L_2(u;t)\ominus L_2(u_1;t)}u_2(t) = P_{L_2(v_2;s)}u_2(t) \\
&= P_{L_2(\underline{u};s)}u_2(t) - P_{L_2(u_1;s)}u_2(t) \\
&= P_{L_2(\underline{u};s)\ominus L_2(u_1;s)}u_2(s) \\
&= v_2(s).
\end{aligned}
$$

The following sublemma will now complete the proof in view of Lemma 4.1.

Sublemma A: Every purely non-deterministic $\{u(t), -\infty < t < \infty\}$ univariate process of orthogonal increments has unit multiplicity.

Proof: Let $f(\tau) > 0$ a.e. ρ_u where $\rho_u(t)-\rho_u(s) = \mathcal{E}\,|u(t)-u(s)|^2$. Then $f_1 = \int_{-\infty}^{\infty} f(\tau)du(\tau)$ is a generating element of $L_2(u)$.

The following representation theorem is an extension of Doob's result ([5], Theorem 4.3).

Theorem 4.1. If $\underline{x}_t$ is a continuous parameter purely non-deterministic process satisfying assumption (D) then it is wide-sense Markov iff

$$x_i(t) = \sum_{k=1}^{q} \sum_{j=1}^{M} \int_{-\infty}^{t} \psi_{ik}(t) h_{kj}(u) dz_j(u)$$

where (1) $\{\psi_{ik}(t)\}$ $(i,k = 1,2,\ldots,q)$ are elements of the non-singular $q \times q$ matrix $\Psi(t)$, (ii) $h_{k_j}(\cdot)$ for each j belong to $L_2(\rho_j)$ where $\rho_j(\Delta) = \mathcal{E}|z_j(\Delta)|^2$ for every measurable set Δ, (iii) M is the multiplicity of $\underline{x}(t)$ and $M \leq q$ (iv) $L_2(\underline{z};t) = L_2(\underline{x};t)$.

Proof: Sufficiency. Let $u_i(t) = \sum_{j=1}^{M} \int_{-\infty}^{t} h_{kj}(u) dz_j(u)$. Then $\underline{x}_t = \Psi(t)\underline{u}_t$ and $\Psi(t)$ is non-singular $L_2(\underline{u};t) = L_2(\underline{x};t) = L_2(\underline{z};t)$. Therefore

$$P_{L_2(\underline{u};s)} u_i(t) = P_{L_2(\underline{z};s)} \sum_{j=1}^{M} \int_{-\infty}^{t} h_{kj}(u) dz_j(u) = \sum_{j=1}^{M} \int_{-\infty}^{s} h_{kj}(u) dz_j(u) = u_i(s).$$

Thus from Theorem 3.1 it follows that $\underline{x}_t$ is a wide-sense Markov process.

Necessity. $\underline{x}_t$-process is wide-sense Markov implies from (3.1) that $\underline{x}_t = \Psi(t)\underline{u}_t$ with $\Psi(t)$ satisfying the conditions of the theorem and $\underline{u}_t$ is a q-variate wide-sense Martingale such that $L_2(\underline{u};t) = L_2(\underline{x};t)$. Hence $\{\underline{u}_t, -\infty < t < +\infty\}$ is a purely non-deterministic wide-sense martingale. Thus by Cramér's theorem and Lemma 4.2 we obtain that

$$u_k(t) = \sum_{j=1}^{M} \int_{-\infty}^{t} h_{kj}(t,u) dz_j(u)$$

where M is the multiplicity and $M \leq q$.

Since $\sum_{j=1}^{M} \int_{-\infty}^{+\infty} |h_k(t,u)|^2 \, d\rho_j(u)$ is finite for each t implies that for any t, s, $\mathcal{E}|u_k(t)-u_k(s)|^2 = \sum_{1}^{M} \int_{s}^{t} |h_{kj}(t,u)|^2 \, d\rho_j(u) \to 0$ as $s,t \to \infty$; i.e., $u_k(+\infty)$ exists in q.m. Therefore $u_k(+\infty) \in L_2(z)$ and this gives that

$u_k(+\infty) = \sum_{1}^{M} \int_{-\infty}^{+\infty} h_{kj}(u) dz_j(u)$ [See Doob [4]).

We have

$$u_k(t) = P_{L_2(\underline{u};t)} u_k(+\infty) = P_{L_2(\underline{z};t)} \sum_1^M \int_{-\infty}^{+\infty} h_{kj}(u) dz_j(u) = \sum_1^M \int_{-\infty}^{t} h_{kj}(u) dz_j(u).$$

Thus we get the required representation for $\underline{x}_t$ satisfying the given conditions.

Using the definition of a vector valued integral with respect to an orthogonally scattered measure (M. Rosenberg [9]) we can write the representation in the vector form

$$\underline{x}_t = \int_{-\infty}^{t} \Psi(t)H(u)d\underline{z}(u) \qquad \text{where} \quad \operatorname{tr}\left[\int_{-\infty}^{+\infty} H(u)d\rho(u)H^*(u)\right] \quad \text{is}$$

finite and $\Psi(t)$ is the non-singular matrix of Theorem 3.1 and $H(u)$ is the $q \times M$ matrix function $H(u) = (h_{kj}(u))$ $k = 1,2,\dots,q$ $j = 1,2,\dots,M$.

5. Stationary Wide-sense Markov processes: Results of Doob. It has been proved (See G. Kallianpur and V. Mandrekar [7]) that in case $\underline{x}_t$ is a q-variate purely non-deterministic stationary process continuous in q.m.

$$\text{(5.1)} \qquad \underline{x}_t = \int_{-\infty}^{t} F(t-u)d\underline{\xi}(u) \quad \text{with} \quad L_2(\underline{x};t) = L_2(\underline{\xi};t).$$

Further $[\underline{\xi}(\Delta),\underline{\xi}(\Delta')] = \mu(\Delta\cap\Delta')I$ where I is an $M\times M$ identity matrix.. From the representation of Theorem 4.2 and (5.1) it follows that a stationary wide-sense Markov process has the representaion

$$\text{(5.2)} \qquad \underline{x}_t = \int_{-\infty}^{t} \Psi(t)K(u)d\underline{\xi}(u)$$

where $[\underline{\xi}(\Delta),\underline{\xi}(\Delta')] = \mu(\Delta\cap\Delta')I$ and $\operatorname{tr}\left[\int_{-\infty}^{+\infty} K(u)d\mu(u)K^*(u)\right]$ is finite. Hence we get for $u\leq t$,

$$\text{(5.3)} \qquad F(t-u) = \Psi(t)K(u) \quad \text{a.e.}[\mu].$$

Without loss of generality we can assume that $F(t-u)$ and $K(u)$ are defined for $u = 0$. Equation (5.3) then yields

$$\text{(5.4)} \qquad F(t) = \Psi(t)K(0) \text{ for } t\geq 0$$

i.e., $F(t) = e^{tQ}S$ where $S = \Psi(0)K(0)$.

Hence

$$(5.5) \qquad \underline{x}_t = \int_{-\infty}^{t} e^{(t-u)Q}\, S d\underline{\xi}(u)$$

We therefore have the following result.

Theorem 5.1. Let $\underline{x}_t$ be continuous in q.m.. Then $\{\underline{x}_t\}$ is wide-sense Markov stationary purely non-deterministic process iff

$$(5.6) \qquad \underline{x}_t = \int_{-\infty}^{t} e^{(t-u)Q}\, S d\underline{\xi}(u)$$

where Q is as in Theorem 3.3 and $[\underline{\xi}(\Delta), \xi(\Delta')] = \mu(\Delta \cap \Delta')I$, I being $M \times M$ identity matrix where M equals the rank of the process.

In conclusion we compare the above representation to that of Doob ([5]). First of all observe that if $\{\underline{x}_t\}$ is Gaussian then it is wide-sense Markov if and only if it is Markov. Now let $\zeta(t) = \int_{-\infty}^{t} e^{-uQ}\, S d\underline{\xi}(u)$ then $\zeta(t)$ is the ζ-process of Doob ([5], (4.3.14)). Comparing our results with the further analysis of Doob, it can be shown that the number of ones occurring in his diagonal matrix U(see (4.3.18) in [5]) is equal to the multiplicity of $\{\underline{x}_t\}$. Finally one can write (5.6) in his form (4.3.2).

Acknowledgements

I would like to thank Professor G. Kallianpur for interesting discussions and suggestions.

References

1. Beutler, F. J. "Multivariate wide-sense Markov processes and Prediction theory." Annals of Math. Statist. 34 1963. p. 424-438.

2. Cramér, H. "On some classes of non-stationary processes," Proc. 4th Berkeley Sympos. Math. Stat. and Prob. II, 1961, 57-77.

3. Cramér, H. "On the structure of purely non-deterministic processes" Arkiv For Math. 4, 2-3 1961, 249-266.

4. Doob, J. L. "Stochastic Processes", New York 1953.

5. Doob, J. L. "The Elementary Gaussian processes" Annals of Math. Stat. 15 1944.

6. Hida, T. "Canonical representations of Gaussian processes and their applications" Mem. Coll. Sci., Kyoto, A33, 1960.

7. Kallianpur, G. and Mandrekar, V. "Multiplicity and representation theory of purely non-deterministic stochastic processes." To appear in Teoria Veryatnosteii i ee Primeneniya 1965.

8. Kalmykov, G. I. "Correlation Functions of a Gaussian Markov Process" Doklady Akad. Nauk. USSR 1965.

9. Rosenberg, M. "Square integrability of matrix valued functions with respect to a non-negative definite Hermitian measure." Duke Math. J. 31 1964.

10. Wiener, N. and Masani, P. "The prediction theory of multivariate stochastic processes I," Acta Mathematica 98 1957 111-149.