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ON THE CONNECTION BETWEEN MULTIPLICITY THEORY AND O. HANNER'S TIME DOMAIN ANALYSIS OF WEAKLY STATIONARY STOCHASTIC PROCESSES

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1. Introduction:

In an early paper full of original ideas, O. Hanner [1] obtained a decomposition for a mean-continuous, purely non deterministic, weakly stationary stochastic process depending on a continuous time parameter. Such a representation had already been derived by K. Karhunen in his 1947 paper [2] and is now generally known by his name. The interest of Hanner's work arises from the fact that his method is based on a time-domain analysis of the process itself and is entirely free of spectral considerations. In recent years, in the light of the extensive development of multivariate stationary processes, it has appeared desirable to separate the time-domain analysis from spectral studies, and interest in the former has revived. As an example, we mention the paper of P. Masani and J. Robertson [3] whose approach considers the discretized process corresponding to the given continuous one and makes essential use of the Cayley transform associated with the unitary group of the process. The extension of this method to (finite dimensional) multivariate stationary processes has been carried out by J. Robertson in his thesis [4]. Hanner's paper, nevertheless, has remained an isolated piece of work. We propose to show in this paper that a proper modification of Hanner's arguments reveals its intimate connection with the notion of multiplicity of a stochastic process. In fact, it will appear that the essence of the matter is to show that the self-adjoint operator A whose resolution of identity (r.o.i.) $\{E(t)\}$ is determined by the process is cyclic, i.e. has multiplicity one. The study of the multiplicity theory of second order, in general non-stationary, stochastic processes has been introduced by T. Hida [5] (who applied them to Gaussian processes) and by H. Cramer ([6], [7]), but its application to weakly stationary processes has not been adequately explored.

The interest of this approach to Hanner's work is not only methodological since it indicates that the true extension of his method to the multidimensional case lies in the study of multiplicity theory. We have undertaken such a study in another paper in which using these ideas we are able to study the corresponding time domain problems for a large class of infinite dimensional stationary processes.

2. Relation between the Karhunen representation and multiplicity.

Let $x_t(-\infty < t < \infty)$ be a one dimensional (complex valued) weakly stationary (w.s.) process with mean assumed zero and satisfying the following assumptions (A):

A -(i). x_t is purely non deterministic, and

A -(ii). x_t is continuous in quadratic mean.

Following [1] let $L_2(x)$ be the Hilbert space of the process $\{x_t\}$, and let $L_2(x;a) = \mathfrak{S}\{x_t, t \leq a\}$ where $\mathfrak{S}\{\ldots\}$ denote the closed linear subspace of $L_2(x)$ spanned by the set of random variables in $\{\ldots\}$. We shall also write $L_2(x;a,b) = L_2(x;b) \ominus L_2(x;a)$ $(a \leq b)$. As we proceed further, whenever necessary, we shall draw on Hanner's notation and also on some of the notation and terminology of [5].

The following representation was obtained in [2] for a w.s. process $\{x_t\}$ satisfying (A).

$$(2.1) \qquad x_t = \int_{-\infty}^{t} G(u-t)d\xi(u) ,$$

where ξ is a homogeneous random set function, i.e., a random set function with the property

$$\mathcal{E}\left[\xi(\Delta_1)\,\overline{\xi(\Delta_2)}\right] = \mu(\Delta_1 \cap \Delta_2) ,$$

μ being Lebesgue measure and Δ_1, Δ_2 any two sets of finite μ measure;

$L_2(x;t) = L_2(\xi;t) = \mathfrak{S}\left\{\xi(\Delta), \Delta\ (-\infty,t]\right\}$ for every t and

$\int_{-\infty}^{0} |G(u)|^2 d\mu(u)$ is finite.

It has been shown by Hida in [5] that every second order process $\{x_t\}$ satisfying (A) and not necessarily stationary has the representation

$$(2.2) \qquad x_t = \sum_{n=1}^{N} \int_{-\infty}^{t} F_n(t,u)\, dE(u) f^{(n)}$$

where (i) $\{E(u)\}$ $(-\infty < u < \infty)$ is the resolution of the identity determined by the projection operator with range $L_2(x;u)$, (ii) N is the multiplicity of x_t (N may be finite or ∞), (iii) $f^{(n)}$ $(n = 1,\ldots,N)$ are elements of $L_2(x)$ with the following properties: (a) the processes $Z_n(\Delta) = E(\Delta)f^{(n)}$ have orthogonal increments and are mutually orthogonal, (b) the variance function of $Z_n(\Delta)$ is given by $\rho_{f^{(n)}}(\Delta)$ where $\rho_{f^{(n)}}(\Delta) = ||E(\Delta)f^{(n)}||^2$,

(c) $\rho_{f^{(1)}} \gg \rho_{f^{(2)}} \gg \ldots \gg \rho_{f^{(N)}}$, (d) $F_n(t, \cdot)$ is square integrable with respect to $\rho_{f^{(n)}}$, and (e) $\sum_{h=1}^{N} \int_{-\infty}^{+\infty} |F_n^2(t,u)|\, d\rho_{f^{(n)}}(u) < \infty$. The representation (2.2) can be chosen so as to satisfy

$$(2.3) \qquad L_2(x;t) = \sum_{n=1}^{N} \oplus\, L_2(z_n;t) \qquad \text{for each } t.$$

It is convenient at this point to recall some of the terminology of multiplicity theory in a separable Hilbert space H. We shall introduce just those ideas that will be used in this paper. Let A be any self-adjoint operator with spectral measure function $E(\cdot)$. For any element f of H let ρ_f be the finite measure on the Borel sets of the line given by

$\rho_f(\Delta) = ||E(\Delta)f||^2$. The family of all finite measures on the line is divided into equivalence classes by the relation of equivalence between measures (equivalence here means mutual absolute continuity). If ρ is used to denote the equivalence class to which the measure ρ_f belongs, ρ will be called the spectral type of f with respect to A. ρ is also referred to as the spectral type belonging to A. If elements f and g are such that $\rho_f = \rho_g$, they obviously have the same spectral type ρ. We shall say that the spectral type ρ dominates the spectral type σ ($\rho > \sigma$, or $\sigma < \rho$) if any (and thus every) measure belonging to σ is absolutely continuous with respect to any measure belonging to ρ. ρ and σ are said to be independent spectral types if for any spectral type ν such that $\nu < \rho$ and $\nu < \sigma$ we have $\nu = 0$. An element f is said to be of maximal spectral type ρ (with respect to A) if for every g in H, $\rho_g \ll \rho_f$. The subspace $\mathfrak{S}\{E(\Delta)f, \Delta$ over all finite intervals$\}$ is called the cyclic subspace with respect to A generated by f. If this space coincides with H, A is said to be cyclic and f is called a cyclic or generating element of A. A has multiplicity one in this case. Also, if f is a generating element of A then it is of maximal spectral type with respect to A. The spectral type of a generating element of A is referred to as the spectral type of A. The reader is referred to the article by A. I. Plessner and V. A. Rohlin [8] for further details.

The following result gives the relationship between the Karhunen representation (2.1) and the multiplicity of $\{x_t\}$, i.e. the multiplicity N of the self adjoint operator A. Comparing (2.1) with (2.2) it might seem obvious that N = 1. Although this is true and we shall give a proof of it below, it should be noted that in the representation (2.1) the function ξ has for its variance function the Lebesgue measure μ, so that it is not possible to write $\xi(u) = E(u)f$ (for all u) for some f in $L_2(x)$.

Proposition. The x_t- process has the Karhunen representation if and only if in the representation (2.2) $N = 1$ and $\rho_{f^{(1)}} \equiv \mu$.

The statement $N = 1$ and $\rho_{f^{(1)}} \equiv \mu$ is, of course, equivalent to saying that the operator A is cyclic with spectral type equivalent to Lebesgue measure. The "only if" part of the above proposition makes possible an alternative purely "time-domain" proof of the Karhunen representation. This, in fact, is the idea that is implicit in Hanner's method as we propose to show in the next section. We need the following useful fact.

Lemma 1: If x_t is a w.s. process, satisfying assumption A -(ii), then for any element $f \in L_2(x)$ $\rho_f \ll \mu$.

Proof: Let $\{T_h\}$ $(-\infty < h < +\infty)$, be the strongly continuous unitary group of the x_t- process ([2] (p.55)). We recall from [1] that for every h, and any t real, we have

$$T_h E(t-h) = E(t) T_h \quad (2.4)$$

Now $\rho_f(\Delta - h) = ||E(\Delta-h)f||^2$ where $\Delta - h = \{u-h | u \in \Delta\}$ and Δ is a Borel measurable set. Therefore, by (2.4) $\rho_f(\Delta-h) = ||E(\Delta)T_h f||^2$. By strong continuity of the group,

$$\rho_f(\Delta-h) \rightarrow \rho_f(\Delta) \quad \text{as} \quad h \rightarrow 0 .$$

The assertation of the lemma is now an immediate consequence of a theorem due to N. Wiener and R. C. Young (see [9], p.91).

Proof of the Proposition.

Necessity: By a property of the Karhunen representation and property (2.3) of representation (2.2), we have $L_2(x:t) = L_2(\xi:t) = \sum_{n=1}^{n} \oplus L_2(Z_n:t)$ for all t. Hence, (see Doob [10], p. 425-428) $Z_n(t) = \int_{-\infty}^{t} h_n(t,u) d\xi(u)$. But $\{Z_n(t), -\infty < t < +\infty\}$

being a process with orthogonal increments we have $h_n(t,u)$ independent of t; i.e., $Z_n(t) = \int_{-\infty}^{t} h_n(u)d\xi(u)$.

We thus have by mutual orthogonality of Z_n's, for $n \neq 1$

$$EZ_n(t)Z_1(t) = \int_{-\infty}^{t} h_1(v)\,\overline{h_n(v)}\,d\mu(v) = 0 \quad \text{for all } t.$$

This implies

(2.5) $$\mu\{v|h_1(v)\,\overline{h_n(v)} \neq 0\} = 0 \quad .$$

Define,

$$S = \{v|h_n(v) \neq 0\} \quad \text{for } n = 1,2,\ldots,N.$$

By (2.5) $\mu\{S_1 \cap S_n\} = 0$ and hence by lemma 2.1 $\rho_{f^{(n)}}(S_1 \cap S_n) = 0$ for all n. Clearly, $\rho_{f^{(1)}}(S_1^c) = 0$ and $\rho_{f^{(n)}}(S_n^c) = 0$, and therefore, by maximality of $\rho_{f^{(1)}}$, $\rho_{f^{(n)}}(S_1^c \cup S_n^c) = 0$ for all $n \neq 1$. Thus we obtain $\rho_{f^{(n)}} = 0$ for $n \neq 1$, giving $N = 1$ since $L_2(x) \neq \{0\}$. Now by (2.2) $L_2(\xi;t) = L_2(Z_1;t)$; i.e., $\xi(t) = \int_{-\infty}^{t} \nu(u)dZ,(v)$. Therefore, for every measurable set Δ $\mu(\Delta) = \mathcal{E}|\xi(\Delta)|^2 = \int_{\Delta} |\nu(u)|^2 d\rho_{f^{(1)}}(u)$. This along with with lemma 1.1 implies that $\rho_{f^{(1)}} \equiv \mu$.

<u>Sufficiency</u>: Suppose $N = 1$ and that f is a generating element of A. Let us denote by $\rho_f^{(h)}(\Delta) = ||E(\Delta)T_h f||^2$ for Δ measurable in $(-\infty,t]$. Clearly by generating property of f

$$T_h f = \int_{-\infty}^{+\infty} r(h,u)dE(u)f, \text{ giving } E(\Delta)T_h f = \int_{\Delta} r(h,u)dE(u)f. \text{ Also,}$$

(2.6) $$dE(u)T_h f = r(h,u)dE(u)f.$$

This implies that $\rho_f^{(h)} \ll \rho_f$ for all h and $r(h,u) = \left[\frac{d\rho_f^{(h)}}{d\rho_f}(u)\right]^{\frac{1}{2}}$. Where

$\frac{d\rho_f^{(h)}}{d\rho_f}$ denotes the Radon-Nikodym derivative of $\rho_f^{(h)}$ ρ_f. Again, from

the fact that f is a generating element and $x_0 \epsilon L_2(x)$, one has

$$x_0 = \int_{-\infty}^{0} F(0,u)dE(u)f \quad \text{with} \quad \int_{-\infty}^{0} F^2(0,u)d\rho_f(u) < \infty$$

and therefore,

$$x_t = T_t\, x_0 = \int_{-\infty}^{t} F(0,u-t)dE(u)T_t f \quad . \tag{2.7}$$

Now, since $\rho_f \equiv \mu$, we can define an orthogonal random set function ξ, by,

$$\xi(\Delta) = \int_{\Delta} \left[\frac{d\rho_f}{d\mu}\right]^{-\frac{1}{2}} dE(u)f, \tag{2.8}$$

having Lebesgue measure as its variance function. Inverting (2.7) we obtain

$$dE(u)f = \left[\frac{d\rho_f}{d\mu}(u)\right]^{\frac{1}{2}} d\xi(u). \tag{2.9}$$

From (2.6) and (2.9), we can write (2.7) as

$$x_t = \int_{-\infty}^{t} F(0,u-t) \frac{d\rho_f^{(t)}}{d\mu}^{\frac{1}{2}} d\xi(u). \tag{2.10}$$

However, in view of (2.4)

$$\rho_f^{(t)}(\Delta) = \rho_f(\Delta-t)$$

and therefore $\frac{d\rho_f^{(t)}}{d\mu}(u) = \frac{d\rho_f}{d\mu}(u-t)$ for all u and t. If we define

$G(u-t) = F(0,u-t)\left[\frac{d\rho_f}{d\mu}(u-t)\right]^{\frac{1}{2}}$, (2.10) has the form (2.1), since by

(2.3), (2.8) and (2.9) it follows that $L_2(x:t) = L_2(Z;t) = L_2(\xi:t)$.

3. The operator A and its spectral type.

In this section we shall closely examine Hanner's arguments and modify them to show directly that the operator A has multiplicity one and spectral type $\rho \equiv \mu$. It is convenient to first study the operator A on $L_2(x;a,b)$. Since $\{E(u)\ -\infty < u < +\infty\}$ is the resolution of the identity of A, every subspace of the type $E(I)L_2(x)$, where I is any interval, reduces the operation A. Let A_I denote the reduced operator. It will be shown (Theorem 1) that A_I has multiplicity one with $Z(I_a^b)$ defined by Hanner ([1], p. 166) for its generating element. We shall write $g_a^b = Z(I_a^b)$.

Let us recall the definition of $Z(I_a^b)$:

$$Z(I_a^b) = \mathrm{Proj}_{L_2(x;a,b)} \int_A^B T_h z dh \tag{3.1}$$

where $z \in L_2(x:0,u)$ for $u > 0$, $A < a-u$, and $B > b$. Proposition C of [1] shows that $Z(I_a^b)$ can be chosen so that $||Z(I_a^b)||^2 = (b-a)$. Now, if $a < \alpha < \beta \leqq b$, we have

$$Z(I_\alpha^\beta) = \{E(\beta) - E(\alpha)\} g_a^b \ .$$

If we consider further, the resolution of identity $\{E_a^b(u)\}$ (where $E_a^b(u) = E(u)(E(b)-E(a))\ a < u \leqq b)$ corresponding to $A_I (I = (a,b])$, then

$$L_2(Z;a,b) = \{Z(I_\alpha^\beta) | a < \alpha < \beta \leqq b\} = \{(E_a^b(\beta) - E_a^b(\alpha)) g_a^b ,\ a < \alpha < \beta \leqq b\}.$$

Also, by the properties of the homogeneous orthogonal random set function Z, proved in [1], it is easy to deduce that

$$\rho_{g_a^b}(\Delta) = \mu^I(\Delta) \quad \text{where } \mu^I(\Delta) = \mu(\Delta \quad (a,b]) \ .$$

<u>Theorem 1.</u> If the w. s. process x_t satisfies (A) then for every interval $(a,b]$ and for t such that $a < t \leqq b$,

$$\mathrm{Proj}_{L_2(x;a,b)} x_t = \int_a^t K(u-t) dE_a^b(u) g_a^b \ , \text{ where } \int_a^b |K(u-t)|^2 d\mu(u) < \infty \ . \tag{3.2}$$

Remark:

(a) Since $L_2(x;a,b) = \mathfrak{S}\{\text{Proj}_{L_2(x:a,b)}\, x_t,\ a < t \leq b\}$ one deduces from (3.2),

$$L_2(x;a,b) = \mathfrak{S}\{E_a^b(\Delta)g_a^b,\ \Delta\ \ (a,b]\};\ \text{i.e., } A_I \text{ is cyclic.}$$

(b) The spectral-type of A_I is equivalent to μ^I.

We shall present the proof in a number of lemmas. The first of these, stated below, reduces the problem from an arbitrary interval as stated in (3.2) to an interval of the type (c,0].

Lemma 1. If for any $t(c < t \leq 0,\ c < 0)$

$$\text{(3.3)} \qquad \text{Proj}_{L_2(x;c,0)}\, x_t = \int_c^t K(u-t)dE_c^0(u)g_c^0\ , \text{ where } \int_c^0 |K(u-t)|^2 d\mu < \infty,$$

then (3.2) holds for any interval (a,b].

Proof: Let us assume that (3.3) holds, then putting $c = a-b$, we have

$$\text{Proj}_{L_2(x;c,0)}\, x_t = \int_c^t K(u-t)dE_c^0(u)g_c^0 \qquad a - b < t \leq 0.$$

This can be rewritten as,

$$\text{(3.4)} \qquad T_{-b}\ \text{Proj}_{L_2(x:a,b)}\ T_b\ x_t = \int_{a-b}^t K(u-t)dE_c^0(u)g_c^0\ .$$

Since $a - b < t \leq 0$, $\tau = t + b \in (a,b]$; further, from [1], (p. 163 and 167) we have $T_b\ E_c^0(u)g_c^0 = E_a^b(u+b)g_a^b$. Therefore (3.4) yields

$$\text{Proj}_{L_2(x:a,b)}\, x_\tau = \int_a^\tau K(u-\tau)dE_a^{\ b}(u)g_a^b \quad , \text{ where}$$

$$||\text{Proj}_{L_2(x;a,b)}\, x_b||^2 = ||T_b \int_c^0 K(u-t)dE_c^0(u)g_c^0||^2 = \int_c^0 |K(u-t)|^2 \mu(u) < \infty,$$

we obtain for $\tau \in (a,b]$

$$\int_a^b |K(u-\tau)|^2 d\mu(u) < \infty\ .$$

Let us now set $\hat{x}_t = \text{Proj}_{L_2(x;c,0)}\, x_t$ and

$\hat{x}_t^{(1)} = \text{Proj}_{\{(E_c^0(\beta) - E_c^0(\alpha))g_c^0,\ c < \alpha \leq \beta \leq 0\}}\, \hat{x}_t$. Since

$|Z(I_c^0)|^2 \neq 0$, it easily follows that $\hat{x}_0^{(1)} \not\equiv 0$. Further, from the definition of $\hat{x}_0^{(1)}$, we have

$$(3.5) \qquad \hat{x}_0^{(1)} = \int_c^0 K(u)dE_c^0(u)g_c^0 \quad \text{where} \int_c^0 |K(u)|^2 d\mu(u) < \infty \ .$$

Now, we first obtain a representation of type (3.3) for $\hat{x}_t^{(1)}$ $c < t \leq 0$.

Lemma 2: For $c < t \leq 0$,

$$\hat{x}_t^{(1)} = \int_c^t K(u-t)dE_c^0(u)g_c^0, \text{ where} \int_c^0 |K(u-t)|^2 d\mu(u) < \infty \ .$$

Proof: Form (3.5), it suffices to prove that

$T_t\, \hat{x}_0^{(1)} = \hat{x}_t^{(1)}$, for in that case,

$$\hat{x}_t^{(1)} = T_t\, \hat{x}_0^{(1)} = \int_c^0 K(u)dE_c^0(u+t)g_{c+t}^t = \int_{c+t}^t K(u-t)dE_c^0(u)g_{c+t}^t$$

$$= \int_c^t K(u-t)dE_c^0(u)g_{c+t}^t \ .$$

Also, it is clear that $E_c^0(u)g_{c+t}^t = E_c^0(u)\left\{g_{c+t}^c + g_c^t\right\} = E_c^0(u)g_c^t$.

Hence, $\hat{x}_t^{(1)} = \int_c^t K(u-t)dE_c^0(u)g_c^t$. However, for $c < u \leq t$,

$$E_c^0(u)g_c^0 = E_c^0(u)\left\{g_c^t + g_t^0\right\} = E_c^0(u)g_c^t + \left\{E(u) - E(c)\right\}g_t^0 = E_c^0(u)g_c^t \ .$$

Thus, we have

$$\hat{x}_t^{(1)} = \int_c^t K(u-t)dE_c^0(u)g_c^0 \ .$$

Now for t, such that $c < t \leq 0$,

$$T_t \hat{x}_0^{(1)} = T_t \operatorname{Proj}_{\mathfrak{S}\{(E_c^0(\beta) - E_c^0(\alpha))g_c^0 \; c < \alpha \leq \beta \leq 0\}} \hat{x}_0$$

$$= \operatorname{Proj}_{\mathfrak{S}\{g_{\alpha+t}^{\beta+t}, \; c < \alpha \leq \beta \leq 0\}} \hat{x}_t \quad ; \quad \text{i.e.,}$$

$$T_t \hat{x}_0^{(1)} = \operatorname{Proj}_{\mathfrak{S}\{g_\alpha^\beta, \; c + t < \alpha \leq \beta \leq t\}} \hat{x}_t \, . \tag{3.6}$$

However, $\mathfrak{S}\{g_\alpha^\beta, \; c + t < \alpha \leq \beta \leq c\}$, being a subspace of $L_2(x;c)$, is orthogonal to $\hat{x}_t$ for $c < t \leq 0$. Therefore

$$T_t \hat{x}_0^{(1)} = \operatorname{Proj}_{\mathfrak{S}\{g_\alpha^\beta, \; c < \alpha \leq \beta \leq t\}} \hat{x}_t \, . \tag{3.7}$$

Also, since $t \leq 0$ and from the fact proved in [1] (p. 170) viz., $x_t \perp \mathfrak{S}\{g_\alpha^\beta, \; t < \alpha \leq \beta \leq 0\}$, we have

$$\operatorname{Proj}_{\mathfrak{S}\{g_\alpha^\beta, \; c < \alpha \leq \beta \leq t\}} \hat{x}_t$$

$$= \operatorname{Proj}_{\mathfrak{S}\{g_\alpha^\beta, \; c < \alpha \leq \beta \leq t\}} \left[\operatorname{Proj}_{L_2(x:0)} x_t - \operatorname{Proj}_{L_2(x:c)} x_t\right]$$

$$= \operatorname{Proj}_{\mathfrak{S}\{g_\alpha^\beta, \; c < \alpha \leq \beta \leq t\}} x_t = \operatorname{Proj}_{\mathfrak{S}\{g_\alpha^\beta, \; c < \alpha \leq \beta \leq 0\}} x_t$$

$$= \operatorname{Proj}_{\mathfrak{S}\{g_\alpha^\beta, \; c < \alpha \leq \beta \leq 0\}} \hat{x}_t \, .$$

Thus

$$\operatorname{Proj}_{\mathfrak{S}\{g_\alpha^\beta, \; c < \alpha \leq \beta \leq t\}} \hat{x}_t = \hat{x}_t^{(1)} \qquad \text{for } c < t \leq 0 \, . \tag{3.8}$$

From (3.7) and (3.8), it follows that

$$T_t \hat{x}_0^{(1)} = \hat{x}_t^{(1)} \qquad \text{for } c < t \leq 0.$$

We now prove $\hat{y}_t = \hat{x}_t - \hat{x}_t^{(1)} \equiv 0$ for $c < t \leq 0$. We recall here the definition $x_t^{(1)} = P_{L_2(Z)} x_t$ from [1] (p. 170). It can be easily seen that

$\hat{y}_t = \text{Proj}_{L_2(x^{(1)};c,0)}\, x_t^{(1)}$ for $c < t \leq 0$. Also as proved in [1], $\{x_t^{(1)}, -\infty < t < +\infty\}$ is a w. s. process satisfying (A). Hence, proceeding with $\hat{y}_t$ exactly as we did with $\hat{x}_t$, we obtain for $c < t \leq 0$,

$$\hat{y}_t = \hat{y}_t^{(1)} + \hat{Z}_t \quad \text{where} \quad \hat{Z}_t \; -\infty < t < +\infty \quad \text{is orthogonal to}$$

$L_2(\hat{x}^{(1)}) + L_2(\hat{y}^{(1)})$ and

$$\hat{y}_t^{(1)} = \int_c^t K'(u-t)dE_c^{'0}(u)g_c^{0'} \quad \text{with} \int_c^t K'^2(u-t)d\mu(u) < \infty$$

and $E_c^{'0}(u) = E'(u)[E'(0) - E'(c)]$, where $\{E'(u) \; -\infty < u < +\infty\}$ denotes the r.o.i. in $L_2(x^{(1)})$ given by projections onto $L_2(x^{(1)}:u)$. Furthermore, $g_a^{'b} = Z'(I_a^{\,b})$ as in [1] (p.172).

Now, $\hat{x}_t = \hat{x}_t^{(1)} + \hat{y}_t^{(1)} + \hat{Z}_t$ and by definitions of random variables $Z(I_a^{\,b})$ and $Z'(I_a^{\,b})$, we have

(3.9) $$\mathfrak{S}\{g_\alpha^\beta,\; c < \alpha \leq \beta \leq 0\},\; \mathfrak{S}\{g_\alpha^{'\beta},\; c < \alpha \leq \beta \leq 0\} \subset L_2(x:c,0)\{ = L_2(\hat{x}:c,0)\}.$$

The rest of the reasoning follows very closely the concluding arguments of Proposition D of [1]. Consider the element,

$$w_s = \int_s^0 K'(u-s)dE_c^0(u)g_c^0 - \int_s^0 K(u-s)dE_c^{0'}(u)g_c^{0'} \quad \text{for} \quad s \geq c.$$

Then w_s is well-defined because $\int_s^0 |K'(u-s)|^2 d\mu(u) < \infty$ and $\int_s^0 |K'(u-s)|^2 d\mu(u) < \infty$ and $\int_s^0 K'(u-s)dE_c^0 \neq \int_s^0 K(u-s)dE_c^{0'}(u)g_c^{0'}$ as they are in mutually orthogonal subspace. Also $T_t\, \hat{x}_0^{(1)} \neq 0$. By definitions of g_c^0 and $g_c^{0'}$

$$||w_s||^2 = \int_s^0 |K'(u-s)|^2 d\mu(u) + \int_s^0 |K(u-s)|^2 d\mu(u)$$

$$s \rightrightarrows c \quad \int_c^0 |K'(u-c)|^2 d\mu(u) + \int_c^0 |K(u-\;)|^2 d\mu(u) \geq ||x_0^{(1)}||^2 > 0.$$

Hence we can chose an s such that $w_s \neq 0$. (3.9) implies $w_s \in L_2(x;c,0)$. If we now show that $w_s \perp \hat{x}_t$ for $c < t \leq 0$, we arrive at a contradiction. But, in fact, for all t such that $c < t \leq 0$

$$E w_s \overline{\hat{x}_t} = \int_s^t K'(u-s)\, K(u-t) d\mu - \int_s^t K'(u-t)\, K(u-s) d\mu(u) = 0,$$ since for

$s > t$ the equation is obvious. Thus $\hat{y}_t = 0$. We summerize these assertions in the following lemma:

Lemma 3: For all t, $(c < t \leq 0)$.

$$\hat{x}_t^{(1)} = \hat{x}_t$$

The proof of theorem 1 now follows immediately from lemmas 3.1, 3.2 and 3.3.

Now let $I_j = (a_j, b_j]$ $(j = 1,2,\ldots)$ be disjoint finite intervals whose union is the real line. Then the subspaces $L_2(x{:}a_j, b_j)$ reduce A and we know, from the remarks after theorem 1, that the reduced operators A_{I_j} are all cyclic and that the spectral type ρ_j of A_{I_j} is equivalent to μ_{I_j}. It is further easy to verify that the ρ_j's are independent spectral types. For, let j and m be arbitrary $(j \neq m)$ and suppose that σ is a measure whose spectral type is dominated by both ρ_j and ρ_m. For all $k \neq j$ since $\mu^{I_j}(I_k) = 0$ we have $\sigma(I_k) = 0$. But $\sigma(I_j)$ is also equal to zero since $\mu^{Im}(I_j) = 0$. Hence $\sigma = 0$.

Assembling all the above facts together we find that we have a representation of A as the orthogonal sum of cyclic operators A_{I_j}, whose corresponding spectral types ρ_j are independent. It then follows immediately, that ([8] p. 152) A itself is cyclic, i.e. has multiplicity one.

We need one more fact before we are ready to obtain our final result. Since A is cyclic it has a generating element say, f. From Lemma 2.1, $\rho_f \ll \mu$. We have also shown, in the discussion preceding Theorem 1 that

$\rho_{g_a^b} = \mu^I$. Since the spectral type of ρ_f is maximal it follows that $\mu^I \ll \rho_f^I$, where $\rho_f^I(\Delta) = \rho_f(I \cap \Delta)$. This being true for every interval I = (a,b], we have $\mu << \rho_f$. Thus $\rho_f \equiv \mu$.

Theorem 2: The self-adjoint operator A of a weakly stationary, continuous in quadratic mean, purely non-deterministic process x_t is cyclic and has a spectral type that is equivalent to Lebesgue measure. Furthermore the x_t-process has a Karhunen representation.

The last statement of the theorem follows from the proposition proved in Section 2.

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ERRATA

Page	Line	Reads	Should Read
1	12	discretized	discrete
4	11	$\sum_{n=1}^{N} \int_{-\infty}^{+\infty} \lvert F_n^2(t,u) \rvert dp_n(u)$	$\sum_{n=1}^{N} \int_{-\infty}^{+\infty} \lvert F_n(t,u) \rvert^2 dp_n(u)$
4	13	$\sum_{n=1}^{N} \oplus L_2(z_n;t)$	$\sum_{n=1}^{N} \oplus L_2(Z_n;t)$
6	2 from bottom	all- :'s	all- ;'s
7	bottom	$\left[\frac{d\rho_f^{(h)}}{d\rho_f}(u)\right] \frac{1}{2}$	$\left[\frac{d\rho_f^{(h)}}{d\rho_f}(u)\right]^{\frac{1}{2}}$
8	1	$\rho_f^{(h)} \quad \rho_f$	$\rho_f^{(h)}$ w.r.t. ρ_f
8	3	$\int_{-\infty}^{0} F^2(0,u)d\rho_f(u)$	$\int_{-\infty}^{0} \lvert F(0,u) \rvert^2 d\rho_f(u)$
8	12	$\frac{d\rho_f^{(t)}}{d\rho_f} \ \frac{1}{2}$	$\left[\frac{d\rho_f^{(t)}}{d\rho_f}(u)\right]^{\frac{1}{2}}$
8	bottom	all - :'s	all - ;'s
9	6	operation	operator
9	17	$a < u \leq b)$	$(a < u \leq b)$
9	21	$\mu(\triangle \ (a,b])$	$\mu(\triangle_\mu(a,b])$
10	3	$\triangle \ (a;b]$	$\triangle c(a,b]$
10	17	$\mathrm{Proj}_{L_2}(x:a,b)^x_\gamma$	$\mathrm{Proj}_{L_2}(x;a,b)^x_\gamma$
11	9	Form	From
13	5	$L_2(\hat{x}^{(1)}) + L_2(\hat{y}^{(1)})$	$L_2(\hat{x}^{(1)}) \oplus L_2(\hat{y}^{(1)})$
13	Last	$\int_c^0 \lvert K(u-) \rvert^2 d\mu(u)$	$\int_c^0 \lvert K(u-c) \rvert^2 d\mu(u)$
13	Last	$\lVert x_0^{(1)} \rVert^2$	$\lVert \hat{x}_0^{(1)} \rVert^2$
14	1	$W_s = 0$	$W_s \neq 0$
14	4	$Lw_s \bar{x}_t$	$\sum w_s \hat{x}_t$
14	9	theorem 1	Theorem 1
14	12	theorem 1	Theorem 1