QUANTUM THERMALIZATION DYNAMICS: FROM QUANTUM CHAOS TO EMERGENT HYDRO

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Part 1:
Eyal Leviatan, Frank Pollmann, Jens Bardarson, David Huse, EA
arXiv:1702.08894

Xiangyu Cao, Thomas Scaffidi, Ben Dickens (unpublished)

Part 2:
“Quantum-Classical correspondence for many-body chaos”
With Thomas Scaffidi arXiv:1711.04768
How to compute dynamical properties of strongly coupled quantum matter?

Explain unconventional transport: (Strange metals)

Describe relaxation following a quench in cold atomic systems:
Outline


2. A semiclassical perspective on maximal quantum chaos in the SYK model
   Thomas Scaffidi and E.A. (unpublished)
Characterizing thermalization dynamics

1. **Short times**: Scrambling / quantum chaos

\[ F(t) \equiv \langle [A(x, t), B(0, 0)]^2 \rangle \]

\[ F(x, t) \sim f_0(x, t)e^{\lambda t} \]

2. **Long times**: emergent hydrodynamic relaxation

See: Lux, Mueller, Mitra and Rosch, PRA 2014

\[ \partial_t e - D\nabla^2 e = \nabla f \quad \text{Thermal noise} \]
Can we compute dynamics of a thermalizing system?

Problem with DMRG (or MPS) calculation:
linear growth of entanglement entropy
(Flow of quantum info. From local operators to increasingly non-local ones)

Apparent paradox:
We expect to obtain emergent classical dynamics after thermalization time of order 1, long range entanglement should not matter
“Information paradox”

# of Bits needed to encode state

\[ e^{Jt} \]

Hydrodynamic regime

\[ \tau_{th} \]

Thermal state

Postulate: only entanglement at ranges of a “thermalization length” \( \xi_{th} \) encodes important structure.

How to truncate the entanglement safely? That is, without sacrificing information on local observables?
Time-dependent variational principle (TDVP) with matrix product states

Variational manifold: \[ |\psi\rangle = \sum_{\sigma_1 \cdots \sigma_N} A^{1}_{\sigma_1} \cdots A^{N}_{\sigma_N} |\sigma_1 \cdots \sigma_N\rangle \]

\[ \dim A^{i}_{\sigma_i} = \chi \]

The variational manifold defines a classical Lagrangian:

\[ \mathcal{L}[\alpha, \dot{\alpha}] = \langle \psi[\alpha] | i\partial_t | \psi[\alpha] \rangle - \langle \psi[\alpha] | H | \psi[\alpha] \rangle \]

Generates classical dynamics on the variational manifold.

Efficient algorithm to propagate the state:

Haegeman et. al. PRL 2011
Relaxation of observables in thermalizing systems

Model: \[ H = \sum_i J S_i^z S_{i+1}^z - h \perp S_i^x - h \parallel S_i^z \] 101 sites

Ensemble of initial states: \[ |\psi_{in}\rangle = S_0^+ |\psi_{MPS}\rangle \]

\[ |\psi_{MPS}\rangle = \text{random matrix product states with bond dimension } \chi \]

To characterize hydrodynamics we monitor the relaxation of the local spin and energy density over long times: \[ \langle S_0^z(t) \rangle, \langle H_0(t) \rangle \]

This scheme probes infinite T relaxation. Probe finite T by first evolving ensemble with \[ e^{-\beta H} \]
Results: evolution of the local spin

Entanglement growth (unentangled initial states)
Saturation value \( \log \chi \) depends on bond dim.

Relaxation of the perturbed spin (same initial conditions)
Obtain hydrodynamic tail.
Almost independent of \( \chi \)
Energy relaxation

\[ H = \sum_i J S_i^z S_{i+1}^z - h \perp S_i^x - h \parallel S_i^z \]
Extract energy diffusion constant

\[ H = \sum_i J S_i^z S_{i+1}^z - h_\perp S_i^x - h_\parallel S_i^z \]

Diffusion coefficient converges to \( \sim 5\% \) accuracy for bond dimension \( > 2 \).

The method captures the emergent hydrodynamic behavior.
Can it capture the characteristics of quantum chaos?
Computing a chaos diagnostic

\[ |\psi_2(t)\rangle \]

\[ |\psi_1(t)\rangle \]

The perturbation is a single-site unitary applied at the left edge.

We compute a normalized measure of the distance:

\[
\delta^2(x, t) = \frac{\text{tr}[\rho^R_1(x, t) - \rho^R_2(x, t)]^2}{\text{tr}[\rho^R_1(x, t)^2] + \text{tr}[\rho^R_2(x, t)^2]}
\]
Propagation of chaos from $\delta(x,t)$

$$\log \left[ \delta^2(x, t) \right]$$
Propagation of chaotic front

\[ \log [\delta^2(x, t)] \]

\[ \text{Jt} = 10 \]

\[ \text{Jt} = 130 \]

(a)
Propagation of chaotic front

\[ \log \left[ \delta^2(x, t) \right] \]

\[ \partial^2 \delta^2(x, t) = 10 \]

\[ v_B = 0.3J \]
Propagation of chaotic front

Front propagates ballistically, but at the same time broadens diffusively:

\[ \delta x = \sqrt{D_B t} \]

Temporal growth at fixed values of $x$

- Rapid convergence with bond $\chi$
- Growth rate $\lambda$ depends on the distance $x$
- No separation between $\lambda$ and (inverse) saturation time.
  
  ➔ No regime of exp. growth
Outlook

- Extend the calculations to lower temperature. T dependence of the transport and chaos on cooling toward a quantum critical point?

- Add disorder: Explore MBL transition by approaching from the thermal side.

- Systematic truncation of entanglement RG scheme?

- Extend method to 2D systems
Part 2: A semiclassical perspective on maximal quantum chaos in the SYK model

With Thomas Scaffidi (Berkeley)

The new results on quantum chaos (quantum bound, dynamics of SYK models, etc) depart from the long history of quantum chaos in the following ways:

1. They pertain to many-body systems
2. Not rooted in a semi-classical limit

In particular the quantum bound \( \lambda_{max} = \frac{2\pi T}{\hbar} \)
does not approach a finite value in the classical limit \( \hbar \to 0 \)

What is then the classical correspondence of a maximally chaotic system?
Classical limit of the SYK model

\[ \hat{H} = \frac{1}{N^{3/2}} \sum_{ijkl} J_{ijkl} \gamma_i \gamma_j \gamma_k \gamma_l \]

Write in terms of fermion bilinears = generators of so(N) algebra:

\[ \hat{L}_{ij} = -\frac{i\hbar}{2} \gamma_i \gamma_j \]
\[ [\hat{L}_a, \hat{L}_b] = i\hbar f_{abc} \hat{L}_c \quad a \equiv (i, j) \]

\[ a = 1, \ldots, M \quad \text{with} \quad M = N(N - 1)/2 \]

N-dimensional rotator with a random inverse moment of inertia tensor:

\[ H = \frac{1}{2M^{1/2}} \sum_{ab} L_a \mathcal{J}_{ab} L_b \quad \mathcal{J}_{ab} = J_{ab}/\hbar^2 \]
Classical limit of the SYK model

\[ H = \frac{1}{2} L_a \mathcal{J}_{ab} L_b \]

Replace commutation relations with Poisson brackets, \( \{ L_a, L_b \}_P = f_{abc} L_c \)

Classical equations of motion for the angular momenta:

\[ \partial_t L_a = -\{ H, L_a \}_P = -f_{abc} L_b \mathcal{J}_{cd} L_d \]

Conserved total (angular) momentum:

\[ P^2 = \frac{1}{M} \sum_a L_a^2 \]

Higher Casimirs:

\[ C_{2k} = \text{tr}(\hat{L}^{2k}) \]

Classically the integrals of motion are fixed by initial conditions. In the quantum system they are determined by the representation of SO(N) realized by the model.
Spinor representation: spin-S coherent states

\[
\hat{L} = \begin{pmatrix}
0 & \hbar S & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
-\hbar S & 0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & 0 & \hbar S & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & -\hbar S & 0 & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & 0 & \hbar S & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & -\hbar S & 0 & \ldots & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}
\]

\[
P^2 = \frac{1}{M} \sum_{i,j} L_{i,j}^2 = \frac{N}{2M} \hbar^2 S^2
\]

Classical limit: \( S/N \to \infty \)

SYK model: \( S = 1/2 \), \( N \to \infty \)
Diagnostic of Chaos

\[ C(t) = \frac{1}{M^2} \sum_{a,b} \left< \left( \frac{\partial L_a(t)}{\partial L_b(0)} \right)^2 \right> \]

Average over initial conditions with constant $E$ and $\mathcal{L}$

Direct numerical calculation of the Lyapunov exponent:

Exponential growth seen at all energy densities.

Extract Lyapunov.
The Lyapunov exponent also depends on N and S. We find:

$$\lambda_{cl} \sim \left( \frac{N}{S} \right) \frac{k_B T}{\hbar}$$
\[ \lambda_{cl} \sim \left( \frac{N}{S} \right) \frac{k_B T}{\hbar} \]

**S>>N:**
Semiclassical regime. Quantum bound satisfied

**N>>S:**
This is the case of interest. Semi-classics not valid apriori.

Quantum bound exceeded by the classical system! How is it re-established?
The chaotic mobility edge (S<<N)

The classical system has a spectrum of Lyapunov exponents due to the distribution of local curvatures on the SO(N) manifold.

The spectrum is separated into two distinct parts:
Lower = large curvature radii $R > l_{dB}$  $\rightarrow$ Semi-classical approx. valid
Upper = small curvature radii $R < l_{dB}$  $\rightarrow$ quantum interference kills chaos

The quantum bound emerges as a crossover from classical to quantum behavior. All chaotic modes are effectively classical!
The semiclassical criterion

Lyapunov exponent related to curvature radius

\[ \lambda_{\nu} = \frac{v}{R_{\nu}} \]

Condition for classical chaos:

\[ R_{\nu} \gg l_{dB} = \hbar / P \]

\[ \lambda_{\nu} < \frac{v}{l_{dB}} = \frac{vP}{\hbar} = \frac{2\epsilon}{\hbar} = \frac{k_B T}{\hbar} \]

Chaotic dynamics is essentially classical, bounded by the validity of the classical description.
Questions

• If chaos is classical, can we relate the growth of entanglement entropy to Kolmogorov-Sini entropy?

• Is there a phase transition analogous to Anderson localization upon crossing the mobility edge, e.g. by tuning S/N.

• How do these results translate to extended systems not in the large N limit?
Is there sub-diffusive Griffith-like phase in quasiperiodic disorder?

- No rare regions
- But indications of sub diffusive transport seen in experiment and numerics.

Need to calculate for longer times!
Energy relaxation on middle site

Random

Quasiperiodic

Weak disorder

Strong disorder
Dynamical exponent

Random

Quasiperiodic

Weak disorder

strong disorder
Implementation of TDVP

Haegeman et. al. PRL 2011

Construct an effective single site Hamiltonian:

\[ H \rightarrow \text{Construct an effective single site Hamiltonian:} \]

\[ H_{\text{eff}} = \chi \]

Evolve the matrices one by one using the single site evolution operator:

\[ e^{-iH_{\text{eff}} \delta t} \]
Possible experimental measurement?

\[ \text{Tr} \left[ (\rho_{B1} - \rho_{B2})^2 \right] = \text{Tr} \left[ \rho_{B1}^2 \right] + \text{Tr} \left[ \rho_{B2}^2 \right] - 2\text{Tr} [\rho_{B1}\rho_{B2}] \]

The first two terms (purities) are measured in Greiner’s experiments. Involves interference of two copies of the state 1 (or state 2). Islam et. al. Nature 2015, Kaufman et.al. Science 2016

The cross term can be measured by the same set up, but interfering state 1 with state 2 instead of two copies of the same state!