Equitable Loss Allocation in Distribution Systems

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Chapter 1

Introduction

Over the past decade, environmental and economic factors have led to an increased adoption of distributed generation (mainly rooftop and community-solar) in residential areas. This rise in renewable sources of electricity generation has led to an increase in the installation of inverters in the distribution network. As these inverters are present on the consumer side and are not centrally dispatchable, there is a need for regulation in the operation of distribution systems. Distribution loss allocation is one important topic that needs to be addressed. In the current scenario, some inverters might support larger fraction of active-power losses unfairly compared to others and the feeder head. This brings us a pressing need to allocate active-power losses to inverters fairly in the distribution network. To this end, a possible strategy in the form of an equitable-loss allocation method is presented in this thesis that allocates active-power losses to the inverters present in the distribution network based on their kVA ratings.

Previous work on loss allocation includes pro rata methods (PR), proportonal sharing methods, incremental transmission loss (ITL) methods and ciruit-based methods. Pro rata is not considered to be fair as it does not takes into account the topology of the network [1]. The proportional sharing method divides transmisson losses in terms of individual loads or generators [2]. In ITL method, the losses are assigned to generators and loads based on incremental transmission loss coefficients [3]. ITL method also depends on the location of the slack bus. A circuit-based method that emphasizes current rather than power for the loss allocation was also proposed as the currents are dominant factor in the determination of transmission losses [4]. Losses are also allocated by deriving a new quadratic loss expression in terms of nodal injections [5]. Many methods were developed to allocate losses in distribution systems. Optimal placement and sizing of DG units were discussed in [6]. Swarm intelligence methods are used in [7] to reconfigure distribution systems for loss minimization and allocation. In [8], initially the losses without DGs are allocated to consumers and then the difference in the losses with DGs are allocated to the inverters. A power summation algorithm where a direct relationship between losses in each branch and complex-power of the nodes in the network is proposed in [9]. Unlike previous approaches, we fairly allocate active-power losses to the inverters based on their kVA ratings by iteratively optimizing their active-power injections.

Now, we provide a short description of our method. Initially, a nonlinear power-flow method is used to determine the steady state operating point as well as the total losses of the distribution-network. This is followed by the application of equitable-loss allocation method to this network. As a first step of this iterative method, a power-flow solution is needed to be determined at every iteration. As the proposed method is iterative, using a traditional nonlinear power-flow method to determine the steady state operating point of the network in every iteration will increase the time-complexity of the method, thus making it ineffective when applied to large distribution networks. To execute this on a faster time-scale while preserving accuracy, a method to approximate the solution of nonlinear power flow is used from [10] and [11] . An approximation to the solution of nonlinear power-flow is obtained by solving for complex perturbations around the nominal voltage from a set of linear equations in active and reactive-power injections of the network. The voltage phase approximations of all the nodes that are obtained from this linear approximation method are used in subsequent steps.

Using the power-flow solution of the network obtained at every step, a circuit-theoretic method called power tracing is used to allocate losses in the network. Power tracing is the problem of disaggregating complex-power injections in a network. Two types of disaggregations can be considered in a network. In downstream tracing, the complex-power injections of a generator can be decomposed into terms that are attributed to both loads and losses.

Similarly in upstream tracing, the complex-power of a load can be decomposed into terms that are attributed to generators and losses in the network. For the purpose of allocating losses to inverters, only downstream tracing is considered in this thesis. Loss coefficients that describe the amount of losses that are allocable to these inverters are determined by downstream tracing. With these loss coefficients defined, the active-power of the inverters are optimized to reallocate losses based on their kVA ratings.

The remainder of this thesis is organized as follows. Mathematical notation utilized in this thesis and an overview of this method are presented in Chapter 2. Linear approximation of the nonlinear power-flow is explained in Chapter 3. In Chapter 4, we discuss how losses are allocated to generators by power tracing. Next, in Chapter 5, we outline the equitable-loss allocation method. In Chapter 6, we present numerical results to validate our approach. Finally, in Chapter 7, we provide concluding remarks and directions for future work.

Chapter 2

Preliminaries

2.1 Notation

The matrix transpose will be denoted by $(\cdot)^{\mathrm{T}}$, complex conjugate by $(\cdot)^*$, real and imaginary parts of a complex number by $\mathrm{Re}\{\cdot\}$ and $\mathrm{Im}\{\cdot\}$, respectively, magnitude and phase of a complex scalar by $|\cdot|$ and $\angle(\cdot)$, and $\mathrm{j} := \sqrt{-1}$. When applied to complex vectors (matrices), $\mathrm{Re}\{\cdot\}$, $\mathrm{Im}\{\cdot\}$, $|\cdot|$, and $\angle(\cdot)$ return real-valued vectors (matrices) of the same dimension with the respective operations performed element wise.

The spaces of $N \times 1$ real-valued and complex-valued vectors are denoted by $\mathbb{R}^N$ and $\mathbb{C}^N$, respectively; $\mathbb{T}^N$ denotes the N-dimensional torus. A diagonal matrix formed with entries of the vector X stacked along the main diagonal is denoted by $\mathrm{diag}(X)$; $\mathrm{diag}(X/Y)$ forms a diagonal matrix with the ℓth entry given by X_ℓ/Y_ℓ, where X_ℓ and Y_ℓ are the ℓth entries of vectors X and Y, respectively. Similarly, $\mathrm{diag}(X^{-1})$ denotes a diagonal matrix with the ℓth entry given by X_ℓ^{-1}. For a matrix X, $X_{\ell m}$ returns the entry in row ℓ and column m of X. We will routinely decompose the complex-valued vector $X \in \mathbb{C}^N$ (complex-valued matrix $Y \in \mathbb{C}^{N\times N}$) into its real and imaginary parts as follows: $X = X_{\mathrm{re}} + \mathrm{j}X_{\mathrm{im}}$ ($Y = Y_{\mathrm{re}} + \mathrm{j}Y_{\mathrm{im}}$, respectively).

$M \times N$ matrices with all zeros are denoted by $0_{M\times N}$. $N \times 1$ vectors with all zeros and ones are denoted by 0_N and 1_N, respectively. Let I_N denote the $N \times N$ identity matrix. For a vector

$\theta = [\theta_1, \ldots, \theta_N]^{\mathrm{T}} \in \mathbb{T}^N$, $\cos(\theta) := [\cos(\theta_1), \ldots, \cos(\theta_N)]^{\mathrm{T}}$, $\sin(\theta) := [\sin(\theta_1), \ldots, \sin(\theta_N)]^{\mathrm{T}}$, and $\mathrm{e}^{\mathrm{j}\theta} := [\mathrm{e}^{\mathrm{j}\theta_1}, \ldots, \mathrm{e}^{\mathrm{j}\theta_N}]^{\mathrm{T}}$. For vectors $X, Y \in \mathbb{R}^N$, the notation $X \preceq Y$ corresponds to the component-wise inequality, i.e., $X \preceq Y \iff X_\ell \leq Y_\ell, \forall \ell = 1, \ldots, N$ (the same applies for other inequalities as well).

2.2 Overview of Equitable Loss Allocation Method

The flowchart in Figure 2.1 describes the step-by-step process of equitable-loss allocation method. We first determine the power-flow solution of the distribution network using linear approximation to the solution of AC power-flow. Then the loss coefficients of the inverters are determined by the method of power tracing.

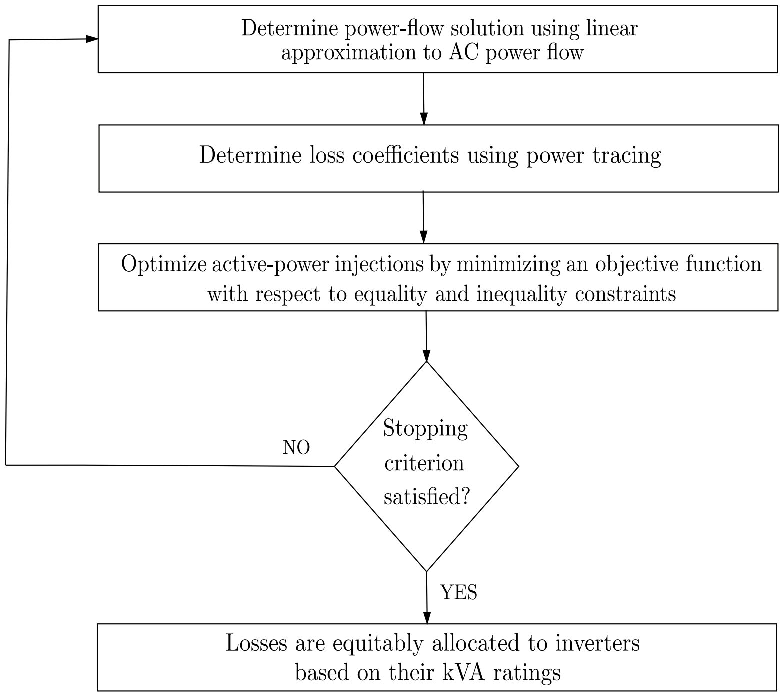

Figure 2.1: A flowchart describing equitable-loss allocation method.

With the calculated loss coefficients, we optimize for active-power injections by formulating an optimization problem with the specific objective of allocating losses fairly to inverters based on their kVA ratings. Finally, we check if a specific stopping criterion is satisfied or not. If it is satisfied, we stop the process and the optimized values ensure that the losses are allocated fairly to the inverters. If the stopping criterion is not satisfied, then we repeat the process until it is satisfied. This is the overview of an equitable-loss allocation method. In the next chapter, we shall describe the method of linear approximation to AC power-flow.

Chapter 3

Linear Approximations to AC Power Flow in Distribution Systems

3.1 Introduction

Linear approximation to AC power flow provides accurate estimates of the nodal voltages in the distribution system. Since our solution approach is iterative, linear approximation to nonlinear power flow aids in improving the time complexity of the algorithm.

In this method, complex-valued perturbations around the nominal voltage are solved by neglecting the quadratic terms of the nonlinear power-flow equations and accurate estimates of voltage magnitude and phase approximations are obtained. This linear approximation of AC power flow holds good for a distribution network where the secondary end of step-down transformer is treated as the slack bus along with power-injections (both positive and negative) from other nodes in the network. In the next section, distribution system model is discussed.

3.2 Distribution System Model

Let us consider a distribution system model comprising of N nodes. The first node is assumed to be the slack bus and it models the secondary of the step-down transformer. Let us capture the power injections at the other $N-1$ nodes in the vectors $P = [P_2, \ldots, P_N]^{\mathrm{T}} \in \mathbb{R}^{N-1}$ and $Q = [Q_2, \ldots, Q_N]^{\mathrm{T}} \in \mathbb{R}^{N-1}$.

Let $V = [V_2, \ldots, V_N]^{\mathrm{T}} \in \mathbb{C}^{N-1}$, where $V_a = |V_a|\angle\theta_a \in \mathbb{C}$ represents the voltage phasor at bus a. We shall also define the vectors $|V| = [|V_2|, \ldots, |V_N|]^{\mathrm{T}} \in \mathbb{R}^{N-1}_{>0}$ and $\theta = [\theta_2, \ldots, \theta_N]^{\mathrm{T}} \in \mathbb{T}^{N-1}$. Voltage vector V is expressed in rectangular coordinates as $V = V_{\mathrm{re}} + \mathrm{j}V_{\mathrm{im}}$, where $V_{\mathrm{re}}, V_{\mathrm{im}} \in \mathbb{R}^{N-1}$ denote the real and imaginary components of V.

Let $I = [I_2, \ldots, I_N]^{\mathrm{T}} \in \mathbb{C}^{N-1}$, where $I_a \in \mathbb{C}$ denotes the current injected into bus a. Kirchhoff's current law for the buses in the power system can be compactly represented in matrix-vector form as follows:

$$\begin{bmatrix} I_1 \\ I \end{bmatrix} = \begin{bmatrix} y & \overline{Y}^{\mathrm{T}} \\ \overline{Y} & Y \end{bmatrix} \begin{bmatrix} V_1 \\ V \end{bmatrix}, \tag{3.1}$$

where $V_1 = |V_1|\mathrm{e}^{\mathrm{j}\theta_1}$ is the slack-bus voltage, I_1 denotes the current injected into the slack bus, and the entries of the admittance matrix have the following dimensions: $Y \in \mathbb{C}^{N-1\times N-1}$, $\overline{Y} \in \mathbb{C}^{N-1}$, and $y \in \mathbb{C} \setminus \{0\}$.

3.3 Linearization in Rectangular Coordinates

Let us denote the vector of complex-power injections by $S = [S_2, \ldots, S_N]^{\mathrm{T}} \in \mathbb{C}^{N-1}$, where $S_a = P_a + jQ_a$. Then, using (3.1), complex-power injections can be expressed as:

$$S = \mathrm{diag}\,(V)\, I^* = \mathrm{diag}\,(V) \left(Y^* V^* + \overline{Y}^* V_1^* \right). \tag{3.2}$$

As we expand (3.2), we can see that it is nonlinear in nature. This hinders the possibility of obtaining a closed-form solution for the given problem. To linearize this equation, V is expressed as $V = V_{\mathrm{nom}} + \Delta V$, where V_{nom} is predefined nominal-voltage vector and entries of ΔV capture perturbations around V_{nom}. So, the perturbation vector ΔV can be solved from the equation:

$$S = \mathrm{diag}\,(V_{\mathrm{nom}} + \Delta V) \left(Y^* (V_{\mathrm{nom}} + \Delta V)^* + \overline{Y}^* V_1^* \right). \tag{3.3}$$

Expanding (3.3), we get the following:

$$S = \mathrm{diag}\left(V_{\mathrm{nom}}\right) Y^* V_{\mathrm{nom}}^* + \mathrm{diag}\left(V_{\mathrm{nom}}\right) Y^* \Delta V^* + \mathrm{diag}\left(\Delta V\right) Y^* V_{\mathrm{nom}}^* + \mathrm{diag}\left(\Delta V\right) Y^* \Delta V^* \\ + \mathrm{diag}\left(V_{\mathrm{nom}}\right) \overline{Y}^* V_1^* + \mathrm{diag}\left(\Delta V\right) \overline{Y}^* V_1^*. \tag{3.4}$$

Now, by neglecting the second-order term $\mathrm{diag}\left(\Delta V\right) Y^* \Delta V^*$ and recognizing that:

$$\begin{aligned} \mathrm{diag}\left(\Delta V\right) Y^* V_{\mathrm{nom}}^* &= \mathrm{diag}\left(Y^* V_{\mathrm{nom}}^*\right) \Delta V, \\ \mathrm{diag}\left(\Delta V\right) \overline{Y}^* V_1^* &= \mathrm{diag}\left(\overline{Y}^* V_1^*\right) \Delta V, \end{aligned} \tag{3.5}$$

we can arrange the terms in (3.4) as follows:

$$\alpha\left(V_{\mathrm{nom}}\right) \Delta V + \beta\left(V_{\mathrm{nom}}\right) \Delta V^* + S_{\mathrm{nom}}\left(V_{\mathrm{nom}}\right) = S, \tag{3.6}$$

where $\alpha : \mathbb{C}^{N-1} \to \mathbb{C}^{N-1\times N-1}$, $\beta : \mathbb{C}^{N-1} \to \mathbb{C}^{N-1\times N-1}$, and $S_{\mathrm{nom}} : \mathbb{C}^{N-1} \to \mathbb{C}^{N-1}$ are given by

$$\alpha\left(V_{\mathrm{nom}}\right) = \mathrm{diag}\left(Y^* V_{\mathrm{nom}}^* + \overline{Y}^* V_1^*\right), \tag{3.7}$$

$$\beta\left(V_{\mathrm{nom}}\right) = \mathrm{diag}\left(V_{\mathrm{nom}}\right) Y^*, \tag{3.8}$$

$$S_{\mathrm{nom}}\left(V_{\mathrm{nom}}\right) = \mathrm{diag}\left(V_{\mathrm{nom}}\right)\left(Y^* V_{\mathrm{nom}}^* + \overline{Y}^* V_1^*\right). \tag{3.9}$$

Now, by splitting (3.6) into real and imaginary parts, we can solve for ΔV_{re} and ΔV_{im} from:

$$\begin{bmatrix} \Delta V_{\mathrm{re}} \\ \Delta V_{\mathrm{im}} \end{bmatrix} = K^{-1} \begin{bmatrix} P \\ Q \end{bmatrix} - K^{-1} \begin{bmatrix} P_{\mathrm{nom}} \\ Q_{\mathrm{nom}} \end{bmatrix}, \tag{3.10}$$

where $P_{\mathrm{nom}} = \mathrm{Re}\{S_{\mathrm{nom}}\} \in \mathbb{R}^{N-1}$ and $Q_{\mathrm{nom}} = \mathrm{Im}\{S_{\mathrm{nom}}\} \in \mathbb{R}^{N-1}$ denote the active- and reactive-power injected into the network at the nominal voltage, and we define $K \in \mathbb{R}^{2(N-1)\times 2(N-1)}$ as follows:

$$K := \begin{bmatrix} \alpha_{\mathrm{re}} + \beta_{\mathrm{re}} & -\alpha_{\mathrm{im}} + \beta_{\mathrm{im}} \\ \alpha_{\mathrm{im}} + \beta_{\mathrm{im}} & \alpha_{\mathrm{re}} - \beta_{\mathrm{re}} \end{bmatrix}. \tag{3.11}$$

Next, we propose two choices for nominal voltage V_{nom}:

- We can determine the nominal voltage by assuming a flat start in the system, i.e., we can assume all the voltage magnitudes to have the same voltage profile as the slack bus V_1, so we have $V_{\mathrm{nom}} = V_1 1_{N-1}$.

- From expressions (3.6) and (3.7)–(3.9), with the following choice:

$$V_{\text{nom}} = -Y^{-1}\overline{Y}V_1, \tag{3.12}$$

we get $\alpha = 0_{N-1\times N-1}$, $S_{\text{nom}} = 0_{N-1}$, and subsequently recover the following linearized power-flow expressions

$$\operatorname{diag}(V_{\text{nom}}^{*})\, Y\Delta V = S^{*}. \tag{3.13}$$

Notice that V_{nom} in (3.12) is the non-zero voltage solution recovered when the current injections in the buses (i.e., $YV_{\text{nom}} + \overline{Y}V_1$) are zero. Since this corresponds to the non-zero solution to (3.2) when $S = 0_{N-1}$, it is referred as no-load voltage.

3.4 Voltage Magnitude and Phase Approximation

Leveraging the value of ΔV obtained in (3.10), it can be showed that voltage magnitude and phase vectors, $|V|$ and θ, can be written as linear functions of active and reactive power injection of the nodes in the system. Begin with the following equation:

$$V = V_{\text{nom}} + \Delta V = \operatorname{diag}(e^{j\theta_{\text{nom}}})|V|_{\text{nom}} + \Delta V. \tag{3.14}$$

3.4.1 Magnitude Approximation

Multiplying to the left of (3.14) by $\operatorname{diag}(e^{-j\theta_{\text{nom}}})$, we get:

$$\begin{aligned}\operatorname{diag}(e^{-j\theta_{\text{nom}}})V &= |V|_{\text{nom}} + \operatorname{diag}(e^{-j\theta_{\text{nom}}})\Delta V \\ &= \operatorname{diag}(|V|_{\text{nom}})\left(1_{N-1} + \operatorname{diag}\left(\frac{e^{-j\theta_{\text{nom}}}}{|V|_{\text{nom}}}\right)\Delta V\right).\end{aligned} \tag{3.15}$$

Taking the element-wise magnitude on both sides of the equation, we get:

$$|V| = \operatorname{diag}(|V|_{\text{nom}})\left|1_{N-1} + \operatorname{diag}\left(\frac{e^{-j\theta_{\text{nom}}}}{|V|_{\text{nom}}}\right)\Delta V\right|. \tag{3.16}$$

Now, since ΔV is a small perturbation around V_{nom}, we can assume that:

$$\left|\operatorname{diag}\left(\frac{e^{-j\theta_{\text{nom}}}}{|V|_{\text{nom}}}\right)\Delta V\right| \prec\prec 1_{N-1}. \tag{3.17}$$

Considering the element-wise approximation $|1_{N-1} + \nu| \approx 1_{N-1} + \mathrm{Re}(\nu)$ for $|\nu| \prec\prec 1_{N-1}$ where $\nu \in \mathbb{C}^{N-1}$, (3.16) becomes,

$$\begin{aligned} |V| &\approx \mathrm{diag}(|V|_{\mathrm{nom}}) \left(1_{N-1} + \mathrm{Re}\left\{\mathrm{diag}\left(\frac{\mathrm{e}^{-\mathrm{j}\theta_{\mathrm{nom}}}}{|V|_{\mathrm{nom}}}\right) \Delta V\right\}\right) \\ &= |V|_{\mathrm{nom}} + \mathrm{Re}\left\{\mathrm{diag}(\mathrm{e}^{-\mathrm{j}\theta_{\mathrm{nom}}})\Delta V\right\}. \end{aligned} \tag{3.18}$$

Now, by decomposing ΔV into real and imaginary parts, and substituting their values from (3.10) we get,

$$|V| = |V|_{\mathrm{nom}} - \Theta_1 K^{-1} \begin{bmatrix} P_{\mathrm{nom}} \\ Q_{\mathrm{nom}} \end{bmatrix} + \Theta_1 K^{-1} \begin{bmatrix} P \\ Q \end{bmatrix}, \tag{3.19}$$

where K is defined in (3.11) and we define $\Theta_1 \in \mathbb{R}^{N-1\times 2(N-1)}$ as follows:

$$\Theta_1 := [\mathrm{diag}(\cos(\theta_{\mathrm{nom}}))\ \mathrm{diag}(\sin(\theta_{\mathrm{nom}}))]\,. \tag{3.20}$$

From (3.19), it can be observed that the voltage magnitudes of the nodes in the network are expressed as linear functions of active and reactive-power injections.

3.4.2 Phase Approximation

Multiplying to the left of (3.14) with $\mathrm{diag}(\mathrm{e}^{-\mathrm{j}\theta_{\mathrm{nom}}}/|V|_{\mathrm{nom}})$, we get,

$$\mathrm{diag}\left(\frac{\mathrm{e}^{-\mathrm{j}\theta_{\mathrm{nom}}}}{|V|_{\mathrm{nom}}}\right) V = 1_{N-1} + \mathrm{diag}\left(\frac{\mathrm{e}^{-\mathrm{j}\theta_{\mathrm{nom}}}}{|V|_{\mathrm{nom}}}\right) \Delta V. \tag{3.21}$$

Using the property from (3.17) and applying the element-wise approximation $\angle(1_{N-1} + \nu) \approx \mathrm{Im}(\nu)$ for $|\nu| \prec\prec 1_{N-1}$ where $\nu \in \mathbb{C}$, we get

$$\begin{aligned} \theta - \theta_{\mathrm{nom}} &= \angle\left(1_{N-1} + \mathrm{diag}\left(\frac{\mathrm{e}^{-\mathrm{j}\theta_{\mathrm{nom}}}}{|V|_{\mathrm{nom}}}\right) \Delta V\right) \\ &\approx \mathrm{Im}\left\{\mathrm{diag}\left(\frac{\mathrm{e}^{-\mathrm{j}\theta_{\mathrm{nom}}}}{|V|_{\mathrm{nom}}}\right) \Delta V\right\}. \end{aligned} \tag{3.22}$$

Now, by decomposing ΔV into real and imaginary parts and by using (3.10), we get

$$\theta = \theta_{\mathrm{nom}} - \mathrm{diag}\left(|V|^{-1}_{\mathrm{nom}}\right) \Theta_2 K^{-1} \begin{bmatrix} P_{\mathrm{nom}} \\ Q_{\mathrm{nom}} \end{bmatrix} + \mathrm{diag}\left(|V|^{-1}_{\mathrm{nom}}\right) \Theta_2 K^{-1} \begin{bmatrix} P \\ Q \end{bmatrix}, \tag{3.23}$$

where K is defined in (3.11) and $\Theta_2 \in \mathbb{R}^{N-1\times 2(N-1)}$ is defined as follows

$$\Theta_2 := [-\mathrm{diag}(\sin(\theta_{\mathrm{nom}}))\ \mathrm{diag}(\cos(\theta_{\mathrm{nom}}))]\,. \tag{3.24}$$

From (3.23), it can be observed that voltage phase angle of nodes in the network can be expressed as a linear function of active and reactive-power injections. Now, as we have obtained a closed-form approximation of voltages in the network, we shall discuss how pertinent loss coefficients can be expressed as linear functions of these phase angles in the next chapter.

Chapter 4

Tracing Power in Distribution Systems

4.1 Introduction

Power Tracing refers to the task of decomposing the power injection of a generator (or a load) into terms that are the attributed to loads (or generators) and losses in the network. The method of power tracing can be classified into two types:

- In downstream tracing, the power injection of a generator is decomposed into terms that are attributed to loads and losses in the network.
- In upstream tracing, the power injection of a load is decomposed into terms that are attributed to generators and losses in the network.

As we allocate losses only to inverters in our method, we focus only on downstream tracing. Rather than directly tackling the tracing of power, we shall first follow the indirect path of tracing the currents and then relating it to trace the power in the networks. For the purpose of this thesis, we shall stick to the tracing of power in distribution networks with distributed generation. In the next section, we introduce the distribution system model.

4.2 Distribution System Model

Let us consider a distribution system model with N nodes collected in the set $\mathcal{N}$, operating in a sinusoidal steady state. At few of these nodes, let us suppose that inverters with distributed generation are installed and synchronized to the grid. Now, we partition the N nodes in the network into G generator nodes and L load nodes in the system. The G generator nodes are collected in the set $\mathcal{G} = \{1, \ldots, G\}$ which includes the feeder head that operates as the slack bus and the installed inverters present in the system. The L load nodes are captured in the set $\mathcal{L} = \mathcal{N} \setminus \mathcal{G} = \{G+1, \ldots, N\}$. The G generator nodes also have loads attached to it but are primarily treated as PV buses. To implement this method, we assume that a solved power flow is available and the complex-power injection information for all the nodes are available. The distribution lines are represented by edges $\mathcal{E} := \{(m,n)\} \subseteq \mathcal{N} \times \mathcal{N}$. In our distribution-system model, series admittance of the line is denoted by $y_{mn} = g_{mn} + \mathrm{j}b_{mn} \in \mathbb{C}$. We assume that there are no shunt elements connected to our system. The entries of the admittance matrix Y_{net} are as follows:

$$[Y_{\text{net}}]_{mn} := \begin{cases} \sum_{(m,k)\in\mathcal{E}} y_{mk}, & \text{if } m = n, \\ -y_{mn}, & \text{if } (m,n) \in \mathcal{E}, \\ 0, & \text{otherwise.} \end{cases} \tag{4.1}$$

Steady state nodal voltages of generators and loads are denoted by:

$$\begin{aligned} V_{\mathcal{G}} &= [V_1, \ldots, V_G]^{\mathrm{T}} =: [V_{\mathcal{G}}^1, \ldots, V_{\mathcal{G}}^G]^{\mathrm{T}} \in \mathbb{C}^G, \\ V_{\mathcal{L}} &= [V_{G+1}, \ldots, V_N]^{\mathrm{T}} =: [V_{\mathcal{L}}^{G+1}, \ldots, V_{\mathcal{L}}^N]^{\mathrm{T}} \in \mathbb{C}^L, \end{aligned} \tag{4.2}$$

where, $V_a = |V_a|\angle\theta_a \in \mathbb{C}$ represents the voltage phasor at bus a. Next, the current injections into generator nodes and load nodes are denoted as:

$$\begin{aligned} I_{\mathcal{G}} &= [I_1, \ldots, I_G]^{\mathrm{T}} =: [I_{\mathcal{G}}^1, \ldots, I_{\mathcal{G}}^G]^{\mathrm{T}} \in \mathbb{C}^G, \\ I_{\mathcal{L}} &= [I_{G+1}, \ldots, I_N]^{\mathrm{T}} =: [I_{\mathcal{L}}^{G+1}, \ldots, I_{\mathcal{L}}^N]^{\mathrm{T}} \in \mathbb{C}^L, \end{aligned} \tag{4.3}$$

By Kirchhoff's current law, we can express the system in matrix form as:

$$\begin{bmatrix} I_{\mathcal{G}} \\ I_{\mathcal{L}} \end{bmatrix} = \begin{bmatrix} Y_{\mathcal{G}\mathcal{G}} & Y_{\mathcal{G}\mathcal{L}} \\ Y_{\mathcal{G}\mathcal{L}}^{\mathrm{T}} & Y_{\mathcal{L}\mathcal{L}} \end{bmatrix} \begin{bmatrix} V_{\mathcal{G}} \\ V_{\mathcal{L}} \end{bmatrix}. \tag{4.4}$$

Notice here that the admittance matrix $Y_{\text{net}} \in \mathbb{C}^{N\times N}$ is decomposed into $Y_{\mathcal{G}\mathcal{G}} \in \mathbb{C}^{G\times G}$, $Y_{\mathcal{G}\mathcal{L}} \in \mathbb{C}^{G\times L}$, and $Y_{\mathcal{L}\mathcal{L}} \in \mathbb{C}^{L\times L}$. Then the complex-power injections into generator bus $g \in \mathcal{G}$ and load bus $\ell \in \mathcal{L}$ are represented as:

$$S_{\mathcal{G}}^{g} = V_{\mathcal{G}}^{g}(I_{\mathcal{G}}^{g})^{*}, \quad S_{\mathcal{L}}^{\ell} = V_{\mathcal{L}}^{\ell}(I_{\mathcal{L}}^{\ell})^{*}. \tag{4.5}$$

The complex-power injections above are decomposed into real and imaginary parts as follows: $S_{\mathcal{G}}^{g} = P_{\mathcal{G}}^{g} + \mathrm{j}Q_{\mathcal{G}}^{g}$ and $S_{\mathcal{L}}^{\ell} = P_{\mathcal{L}}^{\ell} + \mathrm{j}Q_{\mathcal{L}}^{\ell}$.

4.3 Tracing Currents

In this section, we address the problem of decomposing a particular generator current injection into constituent parts that serve loads as well as the dual problem of extracting generator contribution to serve a particular load. We shall discuss two types of current tracing:

4.3.1 Downstream Current Tracing

In downstream current tracing, the current injected by a generator $g \in \mathcal{G}$, $I_{\mathcal{G}}^{g}$ can be uniquely decomposed into the sum of linear combinations of load currents as follows:

$$I_{\mathcal{G}}^{g} = \sum_{\ell\in\mathcal{L}} \gamma_{g}^{\ell} I_{\mathcal{L}}^{\ell}, \quad \forall\, g \in \mathcal{G}. \tag{4.6}$$

Now, to achieve the disaggregation of generator currents, we need to determine the value of γ_{g}^{ℓ}. Let us assume that a vector $\Upsilon_{\mathcal{G}} \in \mathbb{C}^{G}$ satisfies the following relationship:

$$I_{\mathcal{G}} = \mathrm{diag}(\Upsilon_{\mathcal{G}})V_{\mathcal{G}}. \tag{4.7}$$

Now, by substituting (4.7) in (4.4), we get:

$$\begin{bmatrix} 0_{G} \\ I_{\mathcal{L}} \end{bmatrix} = \begin{bmatrix} Y_{\mathcal{G}\mathcal{G}} - \mathrm{diag}(\Upsilon_{\mathcal{G}}) & Y_{\mathcal{G}\mathcal{L}} \\ Y_{\mathcal{G}\mathcal{L}}^{\mathrm{T}} & Y_{\mathcal{L}\mathcal{L}} \end{bmatrix} \begin{bmatrix} V_{\mathcal{G}} \\ V_{\mathcal{L}} \end{bmatrix}, \tag{4.8}$$

By considering the first row of (4.8), and expressing $V_{\mathcal{G}}$ in terms of $V_{\mathcal{L}}$, we get:

$$V_{\mathcal{G}} = -(Y_{\mathcal{G}\mathcal{G}} - \mathrm{diag}(\Upsilon_{\mathcal{G}}))^{-1} Y_{\mathcal{G}\mathcal{L}} V_{\mathcal{L}}. \tag{4.9}$$

Substituting (4.9) into second row of (4.8) and expressing $I_{\mathcal{L}}$ in terms of $V_{\mathcal{L}}$,

$$\begin{aligned} I_{\mathcal{L}} &= \left(Y_{\mathcal{L}\mathcal{L}} - Y_{\mathcal{G}\mathcal{L}}^{\mathrm{T}}(Y_{\mathcal{G}\mathcal{G}} - \mathrm{diag}(\Upsilon_{\mathcal{G}}))^{-1} Y_{\mathcal{G}\mathcal{L}}\right) V_{\mathcal{L}} \\ &=: Y_{\mathcal{L}} V_{\mathcal{L}}. \end{aligned} \tag{4.10}$$

Using (4.9) and (4.10), $I_{\mathcal{G}}$ in (4.4) is expressed as,

$$I_{\mathcal{G}} = (Y_{\mathcal{G}\mathcal{L}} - Y_{\mathcal{G}\mathcal{G}}(Y_{\mathcal{G}\mathcal{G}} - \mathrm{diag}(\Upsilon_{\mathcal{G}}))^{-1} Y_{\mathcal{G}\mathcal{L}}) Y_{\mathcal{L}}^{-1} I_{\mathcal{L}} =: \Gamma I_{\mathcal{L}}. \tag{4.11}$$

By extracting the g-th row of Γ, we obtain the coefficients present in (4.6). Now, commenting about the vector $\Upsilon_{\mathcal{G}}$, notice that the the disaggregation in (4.6) would be consistent with any $G \times G$ matrix Υ, such that $I_{\mathcal{G}} = \Upsilon V_{\mathcal{G}}$. But only a diagonal matrix can be uniquely determined by preserving the topology of the network. This establishes the uniqueness of disaggregation and the entries of $\Upsilon_{\mathcal{G}}$ are equivalent-admittance representations of the generators. Also notice that, $Y_{\mathcal{L}}$ denotes the admittance matrix of the Kron reduced-network where all the generator nodes are eliminated.

4.3.2 Upstream Current Tracing

In upstream current tracing, the current injected by a load $\ell \in \mathcal{L}$, $I_{\mathcal{L}}^{\ell}$ can be uniquely decomposed into the sum of linear combinations of generator currents as follows:

$$I_{\mathcal{L}}^{\ell} = \sum_{g \in \mathcal{G}} \lambda_{\ell}^{g} I_{\mathcal{G}}^{g}. \tag{4.12}$$

Similar to downstream current tracing, we shall derive the current coefficients for upstream current tracing. Let us suppose that a vector Υ_L satisfies the power-flow solution at the L load buses given by:

$$I_{\mathcal{L}} = \mathrm{diag}(\Upsilon_{\mathcal{L}}) V_{\mathcal{L}}. \tag{4.13}$$

Now, by substituting (4.13) in (4.4), we get:

$$\begin{bmatrix} I_G \\ 0_{\mathcal{L}} \end{bmatrix} = \begin{bmatrix} Y_{\mathcal{G}\mathcal{G}} & Y_{\mathcal{G}\mathcal{L}} \\ Y_{\mathcal{G}\mathcal{L}}^{\mathrm{T}} & Y_{\mathcal{L}\mathcal{L}} - \mathrm{diag}(\Upsilon_{\mathcal{L}}) \end{bmatrix} \begin{bmatrix} V_{\mathcal{G}} \\ V_{\mathcal{L}} \end{bmatrix}, \tag{4.14}$$

By considering the second row of (4.14), expressing $V_{\mathcal{L}}$ in terms of $V_{\mathcal{G}}$, we get:

$$V_{\mathcal{L}} = -(Y_{\mathcal{L}\mathcal{L}} - \mathrm{diag}(\Upsilon_{\mathcal{L}}))^{-1} Y_{\mathcal{G}\mathcal{L}}^{\mathrm{T}} V_{\mathcal{G}}. \tag{4.15}$$

Substituting (4.15) into first row of (4.14) and expressing $I_{\mathcal{G}}$ in terms of $V_{\mathcal{G}}$,

$$\begin{aligned} I_{\mathcal{G}} &= \left(Y_{\mathcal{G}\mathcal{G}} - Y_{\mathcal{G}\mathcal{L}}(Y_{\mathcal{L}\mathcal{L}} - \mathrm{diag}(\Upsilon_{\mathcal{L}}))^{-1} Y_{\mathcal{G}\mathcal{L}}^{\mathrm{T}}\right) V_{\mathcal{G}} \\ &=: Y_{\mathcal{G}} V_{\mathcal{G}}. \end{aligned} \tag{4.16}$$

Here, $Y_{\mathcal{G}}$ refers to the admittance of the kron reduced network obtained by eliminating all the L load nodes. Next, by using (4.15) and (4.16), the value of $I_{\mathcal{L}}$ can be expressed in terms of $I_{\mathcal{G}}$ as follows:

$$I_{\mathcal{L}} = (Y_{\mathcal{G}\mathcal{L}}^{\mathrm{T}} - Y_{\mathcal{L}\mathcal{L}}(Y_{\mathcal{L}\mathcal{L}} - \mathrm{diag}(\Upsilon_{\mathcal{L}}))^{-1} Y_{\mathcal{G}\mathcal{L}}^{\mathrm{T}}) Y_{\mathcal{G}}^{-1} I_{\mathcal{G}} =: \Lambda I_{\mathcal{G}}. \tag{4.17}$$

By extracting the ℓ-th row of Λ, we obtain the coefficients present in (4.12). Since we have seen on how to trace currents in the network, we shall use these properties to trace complex-power in the next section.

4.4 Tracing Complex Power

In our optimization problem, as we are concerned on the effective allocation of losses to the generators, we shall focus only on downstream power tracing. Let us first determine the total complex-power loss in the system as:

$$L = \sum_{g\in\mathcal{G}} S_{\mathcal{G}}^{g} + \sum_{\ell\in\mathcal{L}} S_{\mathcal{L}}^{\ell}. \tag{4.18}$$

As we have stated before, downstream power tracing refers to the disaggregation of complex-power injections of a generator into constituent parts that are consumed by each load in the network and allocated to system losses. We can express the complex-power injection of a generator $g \in \mathcal{G}$ as follows:

$$S_{\mathcal{G}}^{g} = \omega_{\mathcal{G}}^{g} S_{\mathcal{G}}^{g} + \sum_{\ell\in\mathcal{L}} \mu_{\ell}^{g} S_{\mathcal{G}}^{g}. \tag{4.19}$$

where, $\omega_{\mathcal{G}}^{g} S_{\mathcal{G}}^{g}$ refers to the amount of loss allocated to generator g and $\mu_{\ell}^{g} S_{\mathcal{G}}^{g}$ denotes the contribution of generator g to load ℓ. Now, to determine the value of the coefficient μ_{ℓ}^{g}, let us consider the complex-power injection of a load ℓ from (4.5),

$$S_{\mathcal{L}}^{\ell} = V_{\mathcal{L}}^{\ell} (I_{\mathcal{L}}^{\ell})^{*}. \tag{4.20}$$

Substituting the value of $I_{\mathcal{L}}^{\ell}$ from (4.12), we get:

$$
\begin{aligned}
S_{\mathcal{L}}^{\ell} &= V_{\mathcal{L}}^{\ell}\Big(\sum_{g\in\mathcal{G}} \lambda_{\ell}^{g} I_{\mathcal{G}}^{g}\Big)^{*} = \sum_{g\in\mathcal{G}} V_{\mathcal{L}}^{\ell}(\lambda_{\ell}^{g} I_{\mathcal{G}}^{g})^{*} \\
&= \sum_{g\in\mathcal{G}} \frac{V_{\mathcal{L}}^{\ell}}{V_{\mathcal{G}}^{g}}(\lambda_{\ell}^{g})^{*} S_{\mathcal{G}}^{g}. \qquad (4.21)
\end{aligned}
$$

Observe that the term $\frac{V_{\mathcal{L}}^{\ell}}{V_{\mathcal{G}}^{g}}(\lambda_{\ell}^{g})^{*}$ refers to the fractional part of the load supplied by generator g. This is essentially the value of the coefficient μ_{ℓ}^{g} in (4.19). Therefore,

$$
\mu_{\ell}^{g} = -\frac{V_{\mathcal{L}}^{\ell}}{V_{\mathcal{G}}^{g}}(\lambda_{\ell}^{g})^{*}, \qquad (4.22)
$$

and,

$$
S_{\mathcal{L}}^{\ell} = -\sum_{g\in\mathcal{G}} \mu_{\ell}^{g} S_{\mathcal{G}}^{g} \quad \forall\, \ell \in \mathcal{L}. \qquad (4.23)
$$

Both the generator buses in the set $\mathcal{G}$ and the load buses in the set $\mathcal{L}$ are injected into their respective nodes. The negative sign in the above equation takes the direction of complex-power flow into account. Looking at (4.19) closely, we can determine the value of $\omega_{\mathcal{G}}^{g}$ as:

$$
\omega_{\mathcal{G}}^{g} = 1 - \sum_{\ell\in\mathcal{L}} \mu_{\ell}^{g}. \qquad (4.24)
$$

By substituting the value of μ_{ℓ}^{g} from (4.22) in the above equation and representing the voltage phasor in terms of their magnitude and phase, we can express the loss coefficient $\omega_{\mathcal{G}}^{g}$ as follows:

$$
\begin{aligned}
\omega_{\mathcal{G}}^{g} &= 1 + \sum_{\ell\in\mathcal{L}} \frac{V_{\mathcal{L}}^{\ell}}{V_{\mathcal{G}}^{g}}(\lambda_{\ell}^{g})^{*} \\
&= 1 + \sum_{\ell\in\mathcal{L}} \frac{|V_{\mathcal{L}}^{\ell}|}{|V_{\mathcal{G}}^{g}|}\big[\cos(\theta_{\ell} - \theta_{g}) + \mathrm{j}\sin(\theta_{\ell} - \theta_{g})\big](\lambda_{\ell}^{g})^{*}. \qquad (4.25)
\end{aligned}
$$

Hence from (4.25), it can be seen that the loss coefficient $\omega_{\mathcal{G}}^{g}$ is expressed as a function of voltage phase angles of all the nodes in the distribution-network. The voltage magnitudes of the nodes in the above equation are also approximated from the power-flow solution. Consider (4.18) and substituting the value of $S_{\mathcal{L}}^{\ell}$ from (4.23), we get:

$$
\begin{aligned}
L &= \sum_{g\in\mathcal{G}} S_{\mathcal{G}}^{g} - \sum_{\ell\in\mathcal{L}}\sum_{g\in\mathcal{G}} \mu_{\ell}^{g} S_{\mathcal{G}}^{g} = \sum_{g\in\mathcal{G}} S_{\mathcal{G}}^{g} - \sum_{g\in\mathcal{G}}\sum_{\ell\in\mathcal{L}} \mu_{\ell}^{g} S_{\mathcal{G}}^{g} \\
&= \sum_{g\in\mathcal{G}} \Big(1 - \sum_{\ell\in\mathcal{L}} \mu_{\ell}^{g}\Big) S_{\mathcal{G}}^{g} =: \sum_{g\in\mathcal{G}} \omega_{\mathcal{G}}^{g} S_{\mathcal{G}}^{g}. \qquad (4.26)
\end{aligned}
$$

Thus, this proves that the total loss of the system is split as the sum of the losses allocated to every $g \in \mathcal{G}$. By decomposing $\omega_{\mathcal{G}}^{g}$ and $S_{\mathcal{G}}^{g}$ into real and imaginary parts as $\omega_{\mathcal{G}}^{g} = \mathrm{Re}(\omega_{\mathcal{G}}^{g}) + \mathrm{jIm}(\omega_{\mathcal{G}}^{g})$ and $S_{\mathcal{G}}^{g} = P_{\mathcal{G}}^{g} + \mathrm{j}Q_{\mathcal{G}}^{g}$ respectively, and substituting them back into (4.26), we get:

$$\begin{aligned} L &= \sum_{g\in\mathcal{G}} \omega_{\mathcal{G}}^{g} S_{\mathcal{G}}^{g} = \sum_{g\in\mathcal{G}} \left[\mathrm{Re}(\omega_{\mathcal{G}}^{g}) + \mathrm{jIm}(\omega_{\mathcal{G}}^{g})\right]\left[P_{\mathcal{G}}^{g} + \mathrm{j}Q_{\mathcal{G}}^{g}\right] \\ &= \sum_{g\in\mathcal{G}} \Big[\mathrm{Re}(\omega_{\mathcal{G}}^{g})P_{\mathcal{G}}^{g} - \mathrm{Im}(\omega_{\mathcal{G}}^{g})Q_{\mathcal{G}}^{g}\Big] + \mathrm{j}\sum_{g\in\mathcal{G}} \Big[\mathrm{Im}(\omega_{\mathcal{G}}^{g})P_{\mathcal{G}}^{g} + \mathrm{Re}(\omega_{\mathcal{G}}^{g})Q_{\mathcal{G}}^{g}\Big]. \end{aligned} \tag{4.27}$$

From (4.27), it can be seen that both active and reactive-power losses of every generator $g \in \mathcal{G}$ depends on its active and reactive-power injections as well as its loss coefficients. With the loss coefficients derived, we are finally left with framing the optimization problem which will be the main focus of next chapter.

Chapter 5

Equitable Loss Allocation Method

In the last two chapters, we have formulated an approach to linearly approximate the power flow solution for a distribution network and to allocate losses independently to every inverter of the system. The objective is to fairly allocate losses based on the kVA ratings of the inverters. In the upcoming section, we determine the fraction of total loss that needs to be allocated to the slack bus and every inverter of the system.

5.1 Fractional Allocation by kVA Ratings

Let $M_1, M_2, \ldots, M_G$ be the kVA ratings of the slack bus and all the inverters present in the network. Let us express the ratio of the kVA rating of slack bus and each inverter separately as follows:

$$\frac{M_1}{M_2} = a_2, \quad \frac{M_1}{M_3} = a_3, \quad \ldots\ldots \quad \frac{M_1}{M_G} = a_G. \tag{5.1}$$

Let $x \in \mathbb{R}^G$ represent the fractions of total losses that needs to be allocated to the slack bus and every inverter in the system. It is obtained by solving the following linear system:

$$Ax = b, \tag{5.2}$$

with,

$$A = \begin{bmatrix} 1 & 1_{G-1}^{\mathrm{T}} \\ 1_{G-1} & -\mathrm{diag}(a) \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0_{\mathrm{G}-1} \end{bmatrix}, \tag{5.3}$$

where $a = [a_2, a_3, \ldots, a_G]^{\mathrm{T}} \in \mathbb{R}^{G-1}$. Let us consider the partition $x = [x_1 \ \overline{x}]^{\mathrm{T}}$ with $\overline{x} = [x_2, x_3, \ldots, x_G]^{\mathrm{T}} \in \mathbb{R}^{G-1}$. We can obtain the closed form solution of x as follows:

$$x = \begin{bmatrix} x_1 \\ \overline{x} \end{bmatrix} = \begin{bmatrix} \left(1 + \sum_{\mathrm{i}=2}^{G} a_{\mathrm{i}}^{-1}\right)^{-1} \\ a^{-1}\left(1 + \sum_{\mathrm{i}=2}^{G} a_{\mathrm{i}}^{-1}\right)^{-1} \end{bmatrix}. \tag{5.4}$$

where $a^{-1} \in \mathbb{R}^{G-1}$ denotes a vector with inverse operation performed element-wise.

5.2 Optimizing Real Power Injections of Inverters

In this section, we shall focus on the optimization of active-power injections of the inverters, a salient part of the equitable-loss allocation method. The optimization problem is framed as follows:

$$h = \underset{P_{\mathcal{G},\mathrm{gen}}}{\mathrm{minimize}} \quad \sqrt{\sum_{i=1}^{G} \left[\mathrm{Re}(\omega_{\mathcal{G}}^{i} S_{\mathcal{G}}^{i}) - x_i \sum_{j=1}^{G} \mathrm{Re}(\omega_{\mathcal{G}}^{j} S_{\mathcal{G}}^{j})\right]^2} \tag{5.5}$$

subject to

$$\sum_{i=1}^{G} P_{\mathcal{G},\mathrm{gen}}^{i} = P_{\mathcal{L}} + \mathrm{Re}(L), \forall \ \mathcal{G} \tag{5.6}$$

$$\sum_{i=1}^{G} \mathrm{Re}(\omega_{\mathcal{G}}^{i} S_{\mathcal{G}}^{i}) = \mathrm{Re}(L), \forall \ \mathcal{G} \tag{5.7}$$

$$P_{\mathcal{G},\mathrm{gen}}^{i,\min} \leq P_{\mathcal{G},\mathrm{gen}}^{i} \leq P_{\mathcal{G},\mathrm{gen}}^{i,\max}, \quad \forall i \in \mathcal{G}. \tag{5.8}$$

In (5.5), we are minimizing the sum of the square of the difference between active-power loss $\mathrm{Re}(\omega_{\mathcal{G}}^{g} S_{\mathcal{G}}^{g})$ of every generator g and the fraction x_g of the total active-power losses in the system that must be allocated to generator g. This is aligned with our objective of equitably allocating active-power losses based on the kVA ratings of the inverters. Also, considering the generator buses in the set $\mathcal{G}$ to have loads connected to them, we disintegrate real-power injections of every generator $g \in \mathcal{G}$ as $P_{\mathcal{G}}^{g} = P_{\mathcal{G},\mathrm{gen}}^{g} - P_{\mathcal{G},\mathrm{load}}^{g}$ and we optimize the problem over the vector of active-power generator injections $P_{\mathcal{G},\mathrm{gen}}$. The reactive-power injections of the generators which are used as a part in the expansion of real-power loss allocation terms are assumed to be fixed in this setup. In (5.6), we ensure the generation-demand balance of the system, with the total loss of the system pre-calculated by a nonlinear power-flow method denoted by L. In (5.7), we ensure that the total sum of active-power losses allocated to all the generators in the set $\mathcal{G}$ is equal to the pre-calculated value of $\mathrm{Re}(L)$. In (5.8), generator active-power injections are constrained within prespecified upper and lower bounds.

Define a matrix $\Pi \in \mathbb{R}^{G \times G}$ to have the following entries:

$$[\Pi]_{mn} := \begin{cases} 1 - x_m, & \text{if } m = n, \\ -x_m, & \text{if } m \neq n. \end{cases} \tag{5.9}$$

This contains information regarding the fractional allocation by kVA ratings of the generators. Considering active-power loss of a generator $g \in \mathcal{G}$ from (4.27), we get:

$$\begin{aligned} \mathrm{Re}(\omega_{\mathcal{G}}^{g} S_{\mathcal{G}}^{g}) &= \mathrm{Re}(\omega_{\mathcal{G}}^{g}) P_{\mathcal{G}}^{g} - \mathrm{Im}(\omega_{\mathcal{G}}^{g}) Q_{\mathcal{G}}^{g} \\ &= \mathrm{Re}(\omega_{\mathcal{G}}^{g}) P_{\mathcal{G},\mathrm{gen}}^{g} - \mathrm{Re}(\omega_{\mathcal{G}}^{g}) P_{\mathcal{G},\mathrm{load}}^{g} - \mathrm{Im}(\omega_{\mathcal{G}}^{g}) Q_{\mathcal{G}}^{g}. \end{aligned} \tag{5.10}$$

Using (5.9) and (5.10), and by representing the objective funtion as the square of a norm, we compactly represent the optimization problem as follows:

$$h = \underset{P_{\mathcal{G},\mathrm{gen}}}{\text{minimize}} \quad \left\| \Pi_P P_{\mathcal{G},\mathrm{gen}} - d \right\|_2^2 \tag{5.11}$$

subject to

$$1_{\mathcal{G}}^{\mathrm{T}} P_{\mathcal{G},\mathrm{gen}} = P_{\mathcal{L}} + \mathrm{Re}(L), \ \forall \ \mathcal{G} \tag{5.12}$$

$$\mathrm{Re}(\omega_{\mathcal{G}})^{\mathrm{T}} P_{\mathcal{G},\mathrm{gen}} = \mathrm{Re}(\omega_{\mathcal{G}})^{\mathrm{T}} P_{\mathcal{G},\mathrm{load}} + \mathrm{Im}(\omega_{\mathcal{G}})^{\mathrm{T}} Q_{\mathcal{G}} + \mathrm{Re}(L), \ \forall \ \mathcal{G} \tag{5.13}$$

$$P_{\mathcal{G},\mathrm{gen}}^{\min} \leq P_{\mathcal{G},\mathrm{gen}} \leq P_{\mathcal{G},\mathrm{gen}}^{\max}, \ \forall \ \mathcal{G}, \tag{5.14}$$

where,

$$\Pi_P = \Pi \mathrm{diag}(\mathrm{Re}(\omega_{\mathcal{G}})), \tag{5.15}$$

$$\Pi_Q = \Pi \mathrm{diag}(\mathrm{Im}(\omega_{\mathcal{G}})), \tag{5.16}$$

$$d = \Pi_P P_{\mathcal{G},\mathrm{load}} + \Pi_Q Q_{\mathcal{G}}, \tag{5.17}$$

with $d \in \mathbb{R}^G$, $\Pi_P \in \mathbb{R}^{G\times G}$ and $\Pi_Q \in \mathbb{R}^{G\times G}$. The vectors $Q_{\mathcal{G}}$, $\omega_{\mathcal{G}}$ denote the reactive-power injections and loss coefficients of all the generators respectively in the distribution-network. With $P_{\mathcal{G},\text{load}} \in \mathbb{R}^G$ being a constant vector, the objective (5.11) is only a function of active-power injections $P_{\mathcal{G},\text{gen}} \in \mathbb{R}^G$. The vectors $P^{\text{min}}_{\mathcal{G},\text{gen}} \in \mathbb{R}^G$ and $P^{\text{max}}_{\mathcal{G},\text{gen}} \in \mathbb{R}^G$ denote the minimum and maximum power limits of the generators in the set $\mathcal{G}$. We now write (5.11)-(5.14) in standard quadratic form:

$$h = \underset{P_{\mathcal{G}}}{\text{minimize}} \quad \frac{1}{2} P_{\mathcal{G}}^{\text{T}} N P_{\mathcal{G}} + c^{\text{T}} P_{\mathcal{G}} \tag{5.18}$$

subject to

$$R P_{\mathcal{G}} = s, \ \ \forall \ \mathcal{G} \tag{5.19}$$

$$U P_{\mathcal{G}} \leq v, \ \ \forall \ \mathcal{G}, \tag{5.20}$$

where,

$$N = 2\Pi_P^{\text{T}}\Pi_P, \tag{5.21}$$

$$c = -2\Pi_P^{\text{T}} d, \tag{5.22}$$

$$R = \begin{bmatrix} 1_{\mathcal{G}} & \text{Re}(\omega_{\mathcal{G}}) \end{bmatrix}^{\text{T}}, \tag{5.23}$$

$$s = \begin{bmatrix} P_{\mathcal{L}} + \text{Re}(L) & \text{Im}(\omega_{\mathcal{G}})^{\text{T}} Q_{\mathcal{G}} + \text{Re}(L) \end{bmatrix}^{\text{T}}, \tag{5.24}$$

$$U = \Big[I_{\mathcal{G}} \quad - I_{\mathcal{G}}]^{\text{T}}, \tag{5.25}$$

$$v = \begin{bmatrix} (P_{\mathcal{G}}^{max})^{\text{T}} & (P_{\mathcal{G}}^{min})^{\text{T}} \end{bmatrix}^{\text{T}}, \tag{5.26}$$

with $N \in \mathbb{R}^{G\times G} \succ 0$, $c \in \mathbb{R}^G$, $R \in \mathbb{R}^{2\times G}$, $s \in \mathbb{R}^2$, $U \in \mathbb{R}^{2G\times G}$ and $v \in \mathbb{R}^{2G}$. Here $P_{\mathcal{G}} \in \mathbb{R}^G$ denotes the vector of active-power injections of the generators. Thus, (5.18)-(5.20) denote the final form of the optimization problem.

5.3 Problem Formulation

The following steps describe the proposed equitable-loss allocation method:

Step 1: Express θ as a linear function of real and reactive-power injections of the network. With the obtained θ, determine the loss coefficient $\omega_{\mathcal{G}}^g$ for every generator g present in the network.

$$\theta = \theta_{\text{nom}} - \text{diag}\left(|V|^{-1}_{\text{nom}}\right)\Theta_2 K^{-1}\begin{bmatrix} P_{\text{nom}} \\ Q_{\text{nom}} \end{bmatrix} + \text{diag}\left(|V|^{-1}_{\text{nom}}\right)\Theta_2 K^{-1}\begin{bmatrix} P \\ Q \end{bmatrix}. \quad (5.27)$$

$$\omega^g_{\mathcal{G}} = 1 + \sum_{\ell\in\mathcal{L}} \frac{|V^\ell_{\mathcal{L}}|}{|V^g_{\mathcal{G}}|}\big[\cos(\theta_\ell - \theta_g) + \mathrm{j}\sin(\theta_\ell - \theta_g)\big](\lambda^g_\ell)^* \quad \forall g \in \mathcal{G}. \quad (5.28)$$

Step 2: With the loss coefficients calculated from the previous step, active-power injections of the generators are optimized using quadratic programming.

$$h = \underset{P_{\mathcal{G}}}{\text{minimize}} \quad \frac{1}{2}P^{\mathrm{T}}_{\mathcal{G}} N P_{\mathcal{G}} + c^{\mathrm{T}} P_{\mathcal{G}} \quad (5.29)$$

subject to

$$RP_{\mathcal{G}} = s, \ \forall \ \mathcal{G} \quad (5.30)$$

$$UP_{\mathcal{G}} \leq v, \ \forall \ \mathcal{G}. \quad (5.31)$$

Step 3: Once the vector of active-power of the generators $P_{\mathcal{G}}$ is optimized, go back to Step 1 and repeat the process. Now, if $|h_k - h_{k-1}| \leq \epsilon, k \geq 2$, then stop the process. If not, then repeat the process until the specific stopping criterion is satisfied (In the simulations, we use $\epsilon = 0.0001$.)

In this process, the ratio of voltage magnitude of a load ℓ and generator g and the upstream current tracing coefficient λ^g_ℓ are obtained from a nonlinear power flow solution. This is assumed to be constant through the iterative process. In the next chapter, we shall apply this algorithm to a distribution-network model and present some representative results.

Chapter 6

Numerical Simulation Results

In this chapter, we implement the proposed equitable-loss allocation method on a modified IEEE 37-node distribution feeder. In this distribution-network, the secondary of the step-down transformer is treated as the slack bus and there are ten photovoltaic inverters installed in the distribution system as shown in Figure 6.1. The inverters in Figure 6.1 are represented as green-colored boxes around the node to which they are connected. The resistance of the lines vary between the values 0.0135 Ω/km and 0.0110 Ω/km. The reactance of the lines vary between the values 0.0017 mH/km and 0.0045 mH/km. The base voltage is 4.8kV. The slack bus that models the secondary of the step-down transformer has a base rating of 14.4kVA. The kVA ratings of the inverters are chosen to demonstrate the advantages of the equitable-loss allocation method. The kVA ratings of these inverters are listed in Table 6.1.

Initially, a power-flow was solved for a given load in the distribution-network. The total active-power loss of the system was calculated to be 0.1704p.u. The linear-approximation to the AC power-flow solution is then used in the equitable-loss allocation method and Figure 6.2 shows the comparison of its estimated value of the voltage profile with the solution of the AC power-flow method. Now, by using the method of power tracing, the losses allocated to every inverter of the system are calculated. Next, by using the equitable-loss allocation method, the losses are allocated to every generator in the system based on their kVA ratings.

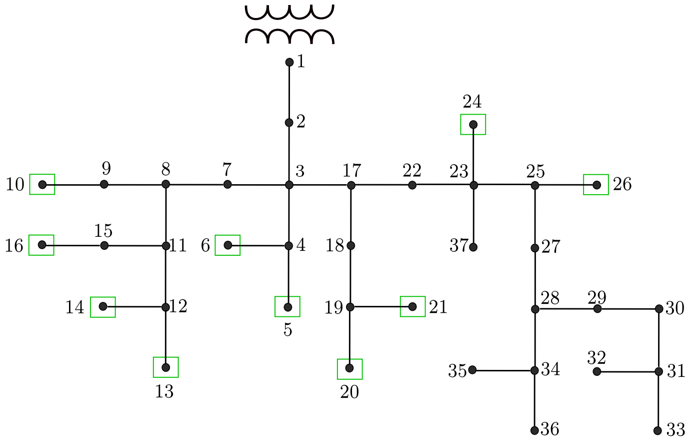

Figure 6.1: Modified IEEE 37-node distribution test feeder model with inverters.

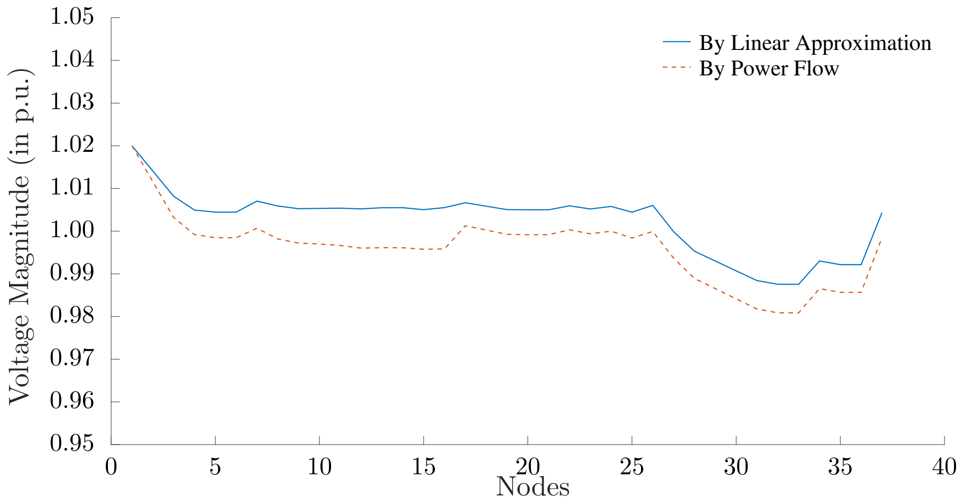

Figure 6.2: Voltage-profile comparison of AC power flow and Linear Approximation Method.

To determine the value of loss coefficients $\omega_{\mathcal{G}}^{g}$ $\forall g \in \mathcal{G}$, the upstream current coefficients λ_{ℓ}^{g} to be used in these calculations are assumed to be constant. Figure 6.3 shows the comparison of

Table 6.1: kVA ratings of Inverters

Inverter number	Node number	kVA rating
1	5	6.5
2	6	6.6
3	10	6.2
4	13	3.6
5	14	4.0
6	16	10.5
7	20	1.7
8	21	1.4
9	24	1.2
10	26	2.0

the losses allocated by a nonlinear power-flow method , an equitable-loss allocation method and the ideal value of losses that needs to be allocated by kVA ratings. With the application of the proposed equitable-loss allocation method, it can be noticed that the losses allocated to all the inverters are moving closer to the ideal values that are allocated based on their kVA ratings, thus achieving fairer operation in the distribution network.

From Figure 6.3, it is noticed that the objective function value in (5.29) is not exactly zero. This indicates that the actual optimized loss allocation values are not exactly equal to the expected values. Table 6.2 compares the expected and actual values of loss allocation obtained from the algorithm.

The optimized power of the generators that are obtained from the algorithm are different from those that were initially solved by the nonlinear power-flow method. So, with these optimized power of the generators as power-injections, a nonlinear power-flow method is solved again to perceive how losses vary due to these injections. Let us term this as the post-optimization power-flow. In this post-optimization power flow, the nodes with inverters are treated as load buses with fixed optimized real-power injections. Feeder head acts as the only generator (slack bus) in the system. Figure 6.4 compares the loss allocation by equitable-loss allocation method and post-optimization power-flow with optimized injections.

Table 6.2: Comparison between expected and actual values of loss allocation

Inverter number	Expected value (in p.u.)	Actual value (in p.u.)
1	0.0190	0.0222
2	0.0194	0.0223
3	0.0181	0.0203
4	0.0105	0.0118
5	0.0118	0.0140
6	0.0308	0.0310
7	0.0051	0.0061
8	0.0042	0.0043
9	0.0034	0.0013
10	0.0059	0.0086

From Figure 6.4, it can be noted that the losses allocated to every inverter is similar in the post-optimization power-flow case when compared with the equitable-loss allocation method. As the injections have changed with the equitable-loss allocation method, so the losses in the system. The slack bus compensates for the remaining losses by increasing or decreasing its real and reactive-power injections. The total losses in the post-optimization power-flow was calculated to be 0.1811p.u. For this particular case, the losses in the system has increased when compared to the value of 0.1704p.u. that was initially calculated by a traditional power-flow method without optimized active-power injections of the generators. This portrays the unpredictable nature of losses with the optimized injections. This highlights a limitation to our method.

Secondly, as a linear-approximation of a nonlinear power-flow method is being used in every iteration of this iterative-optimization, this affects the accuracy of the phase angles used in the optimization problem, thus affecting the objective function value of the algorithm. Along with the equality and inequality constraints, it prevents the algorithm to exactly allocate losses according to the expected value of the algorithm.

Inclusion of tight upper and lower bounds on voltage-magnitude constraints which is written as a linear function of active and reactive-power injections of the system as in (3.19)

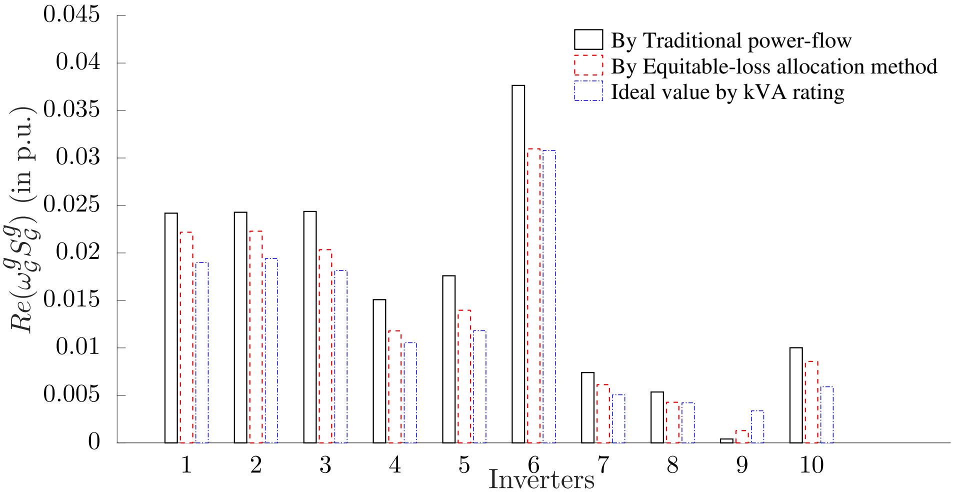

Figure 6.3: Comparison of active-power loss allocation by traditional power-flow method, equitable-loss allocation method and precalculated ideal values based on kVA ratings.

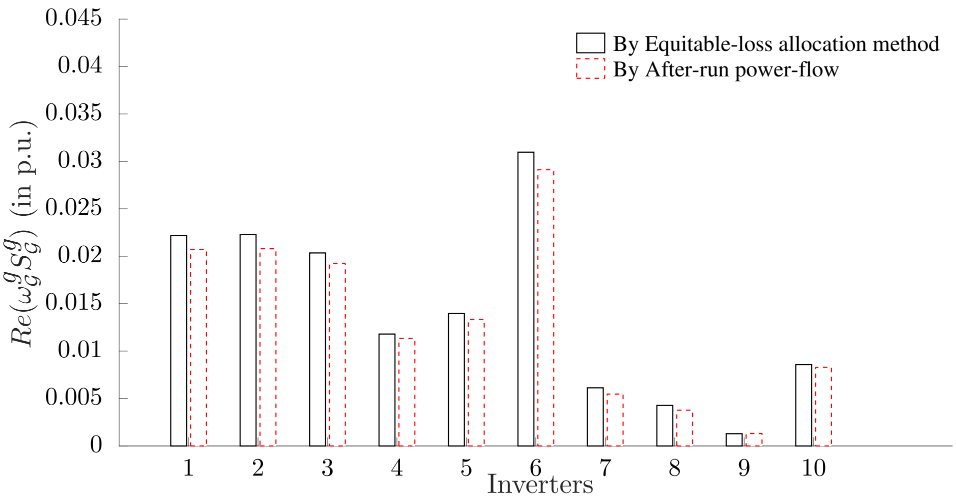

Figure 6.4: Comparison of active-power loss allocation by equitable-loss allocation method and post-optimization power-flow method.

does not provide effectively optimized values in a distribution-network. As voltage drop in a distribution-network is more pronounced, inclusion of tight upper and lower bounds on the voltage magnitudes of all the nodes distorts the objective-function value of the optimization problem. So, the inclusion of voltage-magnitude constraints is left as a future work.

However, as this algorithm uses only a linear approximation of a traditional nonlinear power-flow method, this algorithm can also be implemented on larger distribution-networks and can be solved with limited computational burden. The equitable-loss allocation method satisfies the stopping criterion in at most ten iterations for different networks with the objective function reaching a small steady-state value (greater than zero) ensuring the equitable allocation of losses. Thus, by this method, we have proposed a way to allocate losses fairly to the inverters present in the distribution system based on their kVA ratings.

Chapter 7

Concluding Remarks

This thesis developed a strategy in the form of an equitable-loss allocation method to fairly allocate active-power losses to the inverters present in the distribution networks. The proposed method leveraged linear approximations to power-flow solutions and a circuit-theoretic method called power tracing to iteratively optimize the active-power injections of the inverters to allocate losses equitably based on their kVA ratings. With simulation results, we demonstrated the fairness in the allocation of losses by our proposed method over that by an unoptimized system. Inclusion of voltage-magnitude constraints as a part of the proposed equitable-loss allocation method, providing acceptable bounds on the total active-power losses of the optimized network and improving the accuracy of the algorithm without compromising its time-complexity to better the results are few key directions for future work.

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