A BOTTOM-UP INDUCTIVE PROOF OF THE SINGULAR VALUE DECOMPOSITION

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A BOTTOM-UP INDUCTIVE PROOF OF THE SINGULAR VALUE DECOMPOSITION*

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Abstract. The singular value decomposition (SVD) has a long history. The first proofs of the SVD for real square matrices came out of the study of bilinear forms, first by Beltrami in 1873 and, independently, by Jordan in 1874. Beltrami recognized and used the relationship of the SVD to the eigenvalue decomposition of the matrices $A^T A$ and AA^T while Jordan used an inductive argument which constructs the SVD from the largest singular value and its associated singular vectors. Many proofs of the SVD in modern references are still based on one of these methods. The purpose of this note is to give a new simple "bottom-up" inductive proof of the SVD, starting from the smallest singular value, which is essentially different from either of these methods.

Key Words. Singular value decomposition.

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The singular value decomposition (SVD) has a long history, a detailed survey of which is given in [6, p.134-144]. The first proofs of the SVD for real square matrices came out of the study of bilinear forms, first by Beltrami in 1873 [2] and, independently, by Jordan in 1874 [7]. Beltrami recognized and used the relationship of the SVD to the matrices $A^T A$ and AA^T. Jordan used an inductive argument which constructs the SVD from the largest singular value and its associated singular vectors. The first proof of the SVD for square complex matrices seems to be by Autonne in 1915 [1], and later in 1939 Eckart and Young [3] dealt with the rectangular complex case. Many proofs of the SVD in modern references either rely on the eigenvalue decomposition of the positive semidefinite Hermitian matrices A^*A and AA^* [5, 8, 9, 10] or use a "top-down" inductive argument similar to Jordan's [4], [5, p.427].

The purpose of this note is to give a new simple "bottom-up" inductive proof of the SVD, starting from the smallest singular value. One should note that this proof is essentially different from the "top-down" one; there does not appear to be a direct dual to the "top-down" proof. The QR-decomposition is needed in our proof. The proof is motivated by ideas from a paper of Stewart [11]. Our proof also shows that, if the estimates of the smallest singular value and its associated right singular vector were exact in each step, Stewart's URV decomposition [11] renders the exact SVD.

In this paper, bold lower-case letters denote the column vectors, and $\|\cdot\|_2$ is either the Euclidean norm of a vector or the spectral norm of a matrix. The i-th column of an identity matrix is denoted by $\mathbf{e}_i$.

LEMMA 1. *For a square nonsingular complex matrix A one has*

$$\min_{\|\mathbf{x}\|_2=1} \|A\mathbf{x}\|_2 = \frac{1}{\|A^{-1}\|_2}.$$

Proof. Since A is nonsingular, it is easy to see that

$$\|A^{-1}\|_2 = \max_{\mathbf{x}\neq 0} \frac{\|A^{-1}\mathbf{x}\|_2}{\|\mathbf{x}\|_2} = \max_{\mathbf{y}\neq 0} \frac{\|\mathbf{y}\|_2}{\|A\mathbf{y}\|_2} = \frac{1}{\min\limits_{\mathbf{y}\neq 0} \dfrac{\|A\mathbf{y}\|_2}{\|\mathbf{y}\|_2}} = \frac{1}{\min\limits_{\|\mathbf{y}\|_2=1} \|A\mathbf{y}\|_2}.$$

□

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LEMMA 2. *If R is an $m \times n$ upper triangular matrix with $m \geq n$, then*

$$|r_{ii}| \;\geq\; \min_{\|\mathbf{x}\|_2=1} \|R\mathbf{x}\|_2$$

for each diagonal entry r_{ii} of R.

Proof. If R is rank deficient, the stated bound is immediate since $R\mathbf{x} = 0$ for some $\|\mathbf{x}\|_2 = 1$, so we may assume that R has full column rank.

First note that for any entry a_{ij} of a complex matrix A,

$$|a_{ij}| = |\mathbf{e}_i^* A \mathbf{e}_j| \leq \|\mathbf{e}_i\|_2 \|A\mathbf{e}_j\|_2 \leq \|\mathbf{e}_i\|_2 \|A\|_2 \|\mathbf{e}_j\|_2 = \|A\|_2.$$

The matrix R has the form $\begin{bmatrix} R_1 \\ 0 \end{bmatrix}$ with R_1 invertible and $(R_1^{-1})_{ii} = \frac{1}{r_{ii}}$. Thus $\|R_1^{-1}\|_2 \geq |(R_1^{-1})_{ii}| = |\frac{1}{r_{ii}}|$, from which, by Lemma 1, the result follows since

$$\min_{\|\mathbf{x}\|_2=1} \|R\mathbf{x}\|_2 = \min_{\|\mathbf{x}\|_2=1} \|R_1\mathbf{x}\|_2 = \frac{1}{\|R_1^{-1}\|_2} \leq |r_{ii}|.$$

□

THEOREM 1. *Each matrix $A \in \mathbf{C}^{m\times n}$ has a singular value decomposition. That is, there exist unitary matrices*

$$U \in \mathbf{C}^{m\times m} \quad \text{and} \quad V \in \mathbf{C}^{n\times n}$$

such that

$$U^* A V = \operatorname{diag}(\sigma_1, \cdots, \sigma_p) \in \mathbf{C}^{m\times n}, \quad p = \min\{m, n\}$$

where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p \geq 0$.

If $A \in \mathbf{R}^{m\times n}$, then U and V may be taken to be real orthogonal matrices.

Proof. Assume $m \geq n$ (otherwise, consider A^*). Let $\mathbf{x}_0$ be a unit vector such that $\|A\mathbf{x}_0\|_2 = \min_{\|\mathbf{x}\|_2=1} \|A\mathbf{x}\|_2$, and set $\sigma := \|A\mathbf{x}_0\|_2$. Let $V_1 \in \mathbf{C}^{n\times n}$ be a unitary matrix whose last column is $\mathbf{x}_0$ and let $U_1 \in \mathbf{C}^{m\times m}$ be a unitary matrix such that $U_1^*(AV_1) = R =: [r_{ij}]$ is upper triangular (here we use the QR-decomposition).

Since

$$\|R\mathbf{e}_n\|_2 = \|U_1^* A V_1 \mathbf{e}_n\|_2 = \|U_1^* A\mathbf{x}_0\|_2 = \|A\mathbf{x}_0\|_2 = \sigma,$$

from Lemma 2 we have

$$|r_{1n}|^2 + \cdots + |r_{nn}|^2 = \sigma^2 \leq |r_{nn}|^2$$

and hence $|r_{1n}|^2 + \cdots + |r_{n-1,n}|^2 = 0$. It follows that $|r_{nn}| = \sigma$ so that $r_{nn} = \sigma e^{it}$ for some t. Now define the diagonal unitary matrix $W_1 := \operatorname{diag}(1, \ldots, 1, e^{it}) \in \mathbf{C}^{n\times n}$ and observe that $A = U_1 R (V_1 W_1)^*$ with $R = \begin{bmatrix} R_1 & 0 \\ 0 & \sigma \\ 0 & 0 \end{bmatrix} \in \mathbf{C}^{m\times n}$.

Since $R_1 \in \mathbf{C}^{(n-1)\times(n-1)}$ and $V_1 W_1$ is unitary, a straightforward inductive argument proves that there exist unitary matrices $U \in \mathbf{C}^{m\times m}$ and $V \in \mathbf{C}^{n\times n}$ such that

$$A = U \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \\ \hline & 0 & \end{bmatrix} V^*$$

□

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#	Author/s	Title
909	**Richard A. Brualdi and Bryan L. Shader**, Strong hall matrices	
910	**Håkan Wennerström and David M. Anderson**, Difference versus Gaussian curvature energies; monolayer versus bilayer curvature energies applications to vesicle stability	
911	**Shmuel Friedland**, Eigenvalues of almost skew symmetric matrices and tournament matrices	
912	**Avner Friedman, Bei Hu and J.L. Velazquez**, A Free Boundary Problem Modeling Loop Dislocations in Crystals	
913	**Ezio Venturino**, The Influence of Diseases on Lotka-Volterra Systems	
914	**Steve Kirkland and Bryan L. Shader**, On Multipartite Tournament Matrices with Constant Team Size	
915	**Richard A. Brualdi and Jennifer J.Q. Massey**, More on Structure-Ranks of Matrices	
916	**Douglas B. Meade**, Qualitative Analysis of an Epidemic Model with Directed Dispersion	
917	**Kazuo Murota**, Mixed Matrices Irreducibility and Decomposition	
918	**Richard A. Brualdi and Jennifer J.Q. Massey**, Some Applications of Elementary Linear Algebra in Combinations	
919	**Carl D. Meyer**, Sensitivity of Markov Chains	
920	**Hong-Ming Yin**, Weak and Classical Solutions of Some Nonlinear Volterra Integrodifferential Equations	
921	**B. Leinkuhler and A. Ruehli**, Exploiting Symmetry and Regularity in Waveform Relaxation Convergence Estimation	
922	**Xinfu Chen and Charles M. Elliott**, Asymptotics for a Parabolic Double Obstacle Problem	
923	**Yongzhi Xu and Yi Yan**, An Approximate Boundary Integral Method for Acoustic Scattering in Shallow Oceans	
924	**Yongzhi Xu and Yi Yan**, Source Localization Processing in Perturbed Waveguides	
925	**Kenneth L. Cooke and Janos Turi**, Stability, Instability in Delay Equations Modeling Human Respiration	
926	**F. Bethuel, H. Brezis, B.D. Coleman and F. Hélein**, Bifurcation Analysis of Minimizing Harmonic Maps Describing the Equilibrium of Nematic Phases Between Cylinders	
927	**Frank W. Elliott, Jr.**, Signed Random Measures: Stochastic Order and Kolmogorov Consistency Conditions	
928	**D.A. Gregory, S.J. Kirkland and B.L. Shader**, Pick's Inequality and Tournaments	
929	**J.W. Demmel, N.J. Higham and R.S. Schreiber**, Block *LU* Factorization	
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931	**Bryan L. Shader**, Convertible, Nearly Decomposable and Nearly Reducible Matrices	
932	**Dianne P. O'Leary**, Iterative Methods for Finding the Stationary Vector for Markov Chains	
933	**Nicholas J. Higham**, Perturbation theory and backward error for $AX - XB = C$	
934	**Z. Strakos and A. Greenbaum**, Open questions in the convergence analysis of the lanczos process for the real symmetric eigenvalue problem	
935	**Zhaojun Bai**, Error analysis of the lanczos algorithm for the nonsymmetric eigenvalue problem	
936	**Pierre-Alain Gremaud**, On an elliptic-parabolic problem related to phase transitions in shape memory alloys	
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948	**E.G. Kalnins, Willard Miller, Jr. and G.C. Williams**, Recent advances in the use of separation of variables methods in general relativity	
949	**G.W. Stewart**, On the perturbation of LU, Cholesky and QR factorizations	
950	**G.W. Stewart**, Gaussian elimination, perturbation theory and Markov chains	
951	**G.W. Stewart**, On a new way of solving the linear equations that arise in the method of least squares	
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