Complex Monge-Ampère Equations and Chern-Ricci Flow on Hermitian Manifolds

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Abstract

The regularity of weak solutions of an elliptic complex Monge-Ampère equation is studied on compact Hermitian manifolds. Using the smoothing property for the corresponding parabolic flow, a weak solution is proved to be smooth if the background Hermitian metric satisfies a compatibility condition.

The Chern-Ricci flow is an evolution equation of Hermitian metrics on a complex manifold by their Chern-Ricci form. The existence and uniqueness for the Chern-Ricci flow with rough initial data is obtained on compact Hermitian manifolds satisfying a mild assumption. Then we prove the existence of weak solutions of the Chern-Ricci flow through blow downs of exceptional curves, as well as smooth convergence on compact subsets away from image points of the exceptional curves.

Contents

Chapter 1

Introduction

Complex Monge-Ampère equation is a powerful tool in the study of complex manifolds. Given a Hermitian metric ω on an n-dimensional compact complex manifold M, consider the equation:

$$(\omega + \sqrt{-1}\partial\bar{\partial}\phi)^n = e^F \omega^n, \quad \omega + \sqrt{-1}\partial\bar{\partial}\phi > 0 \tag{1.1}$$

where F is a smooth function on M and ϕ is the unknown function. Write $\omega = \sqrt{-1}g_{i\bar{j}}dz^i \wedge d\bar{z}^j$ in local holomorphic coordinates, then the above equation can be written as $\det\left(g_{i\bar{j}} + \frac{\partial^2\phi}{\partial z^i \partial \bar{z}^j}\right) = e^F \det(g_{i\bar{j}})$ with $\left(g_{i\bar{j}} + \frac{\partial^2\phi}{\partial z^i \partial \bar{z}^j}\right)$ a positive definite Hermitian matrix.

When ω is Kähler and F satisfies the necessary condition $\int_M e^F \omega^n = \int_M \omega^n$, Yau [49] proved in 1978 that there exists a smooth solution to equation (1.1), unique up to a constant. Thus he proved the Calabi conjecture which asserts that any closed (1, 1) form representing the first Chern class of a compact Kähler manifold can be realized as the Ricci form of a Kähler metric cohomologous to ω.

When ω is a general Hermitian metric (non-Kähler), the equation was first studied by Cherrier [8, 9, 10] and Hanani [21, 22] in eighties and nineties. Particularly, Cherrier solved the equation in the case when $n = 2$ or $d\omega^{n-1} = 0$ (balanced metric). Cherrier's result was rediscovered and generalized in the recent works of Guan-Li [18, 19] and Tosatti-Weinkove [44, 45]. The equation was solved in full generality by Tosatti-Weinkove [45].

Geometric problems require the study of more general complex Monge-Ampère equations where the right hand side F also depends on the unknown function ϕ. In [42], Székelyhidi and Tosatti studied weak solutions of the equation

$$(\omega + \sqrt{-1}\partial\bar{\partial}\phi)^n = e^{-F(\phi,z)}\omega^n \tag{1.2}$$

on Kähler manifolds, where $F : \mathbb{R} \times M \to \mathbb{R}$ is a smooth function. Denote

$$PSH(M,\omega) = \{u : M \to [-\infty, +\infty) | u \text{ is upper semicontinuous, } \omega + \sqrt{-1}\partial\bar{\partial}u \geq 0\}$$

the set of ω-plurisubharmonic functions on M. They proved that if $\phi \in PSH(M,\omega) \cap L^\infty(M)$ solves (1.2) in the sense of currents, then ϕ is smooth. In particular, if M is Fano, $\omega \in c_1(M)$ and $F(\phi, z) = \phi - h$, where h satisfies $\sqrt{-1}\partial\bar{\partial}h = \mathrm{Ric}(\omega) - \omega$, their result implies that Kähler-Einstein currents with bounded potentials are smooth.

It is natural to ask if the result of Székelyhidi and Tosatti can be extended to general Hermitian manifolds. As ω is not closed in general, we impose the following compatibility condition:

$$\forall\ u \in PSH(M,\omega) \cap L^\infty(M), \quad \int_M (\omega + \sqrt{-1}\partial\bar{\partial}u)^n = \int_M \omega^n \tag{1.3}$$

Then we obtain the same regularity result in chapter 2.

Theorem 1.1. *Let (M,ω) be an n-dimensional compact Hermitian manifold with ω satisfying condition (1.3) and $F : \mathbb{R} \times M \to \mathbb{R}$ be a smooth function. Suppose that $\phi \in PSH(M,\omega) \cap L^\infty(M)$ solves*

$$(\omega + \sqrt{-1}\partial\bar{\partial}\phi)^n = e^{-F(\phi,z)}\omega^n \tag{1.4}$$

in the sense of currents. Then ϕ is smooth.

The condition (1.3) is automatically true on Kähler manifolds. In [42], the proof of the above theorem on Kähler manifolds needs Kołodziej's stability result [26]. We use the assumption (1.3) from [12] to make sure the stability result holds on such Hermitian manifolds [28]. Particularly, if ω satisfies Guan-Li's [19] condition $\partial\bar{\partial}\omega^k = 0$, $k = 1, 2$, then (1.3) is satisfied. When M is a complex surface, there always exist such Hermitian metrics ($\partial\bar{\partial}$-closed) due to a result of Gauduchon [14].

In chapter 3, we study a parabolic version of the complex Monge-Ampère equation on Hermitian manifolds, the Chern-Ricci flow. It is an evolution equation of Hermitian metrics on a complex manifold by their Chern-Ricci form (see [31, 41] for some related parabolic flows of Hermitian metrics). Let (M,ω) be a compact Hermitian manifold. The Chern-Ricci flow starting at ω_0 is given by

$$\frac{\partial}{\partial t}\omega = -\operatorname{Ric}(\omega), \qquad \omega|_{t=0} = \omega_0, \tag{1.5}$$

where $\operatorname{Ric}(\omega) = -\sqrt{-1}\partial\bar{\partial}\log\det g$ is the Chern-Ricci form of ω. It was introduced by Gill [15] on manifolds with vanishing first Bott-Chern class and investigated by Tosatti and Weinkove on general Hermitian manifolds [46]. If the initial metric is Kähler, then it coincides with the Kähler-Ricci flow (see [4] and the references therein). Many nice properties of the flow have been found [15, 16, 46, 47, 48], some of which are analogous to those of the Kähler-Ricci flow. For example, it was shown in [46] there exists a unique maximal solution to (1.1) on [0,T), where T can be explicitly determined by the initial metric; the (normalized) Chern-Ricci flow on manifolds with negative first Chern class starting at any Hermitian metric will converge smoothly to the unique Kähler-Einstein metric on M.

In the case of $n = 2$ (complex surfaces), the behavior of the flow is particularly interesting. It is expected that the Chern-Ricci flow is closely related to the geometry of the underlying manifold. Given a compact complex surface M and ω_0 a Gauduchon metric (i.e. $\partial\bar{\partial}\omega_0 = 0$), it was proved in [46] that the Chern-Ricci flow starting at ω_0 exists until either the volume of M goes to zero, or the volume of a curve of negative self-intersection goes to zero. We say that the Chern-Ricci flow is *collapsing* (*non-collapsing*) at T if the volume of M with respect to $\omega(t)$ goes to zero (stays positive) as $t \to T^-$.

Suppose that the Chern-Ricci flow starting at ω_0 is non-collapsing at $T < \infty$. Then it was shown in [46] that M contains finitely many disjoint (-1)-curves $E_1, ..., E_k$ and thus there exists a map $\pi : M \to N$ onto a complex surface contracting each E_i to a point $y_i \in N$. Denote $M' = M \setminus \cup_{i=1}^k E_i$ and $N' = N \setminus \{y_1, ..., y_k\}$. The following theorems were proved in [47] .

Theorem 1.2. (Tosatti-Weinkove) *Let M be a compact complex surface and ω_0 a Gauduchon metric . Suppose that the Chern-Ricci flow (1.5) is non-collapsing at time $T < \infty$. Then the metrics $\omega(t)$ converge to a smooth Gauduchon metric ω_T on M' in $C^\infty_{\mathrm{loc}}(M')$ as $t \to T^-$.*

Theorem 1.3. (Tosatti-Weinkove) *Assume that*

$$\omega_0 - T\,\mathrm{Ric}(\omega_0) + \sqrt{-1}\partial\bar{\partial} f = \pi^*\beta \tag{1.6}$$

for some $f \in C^\infty(M, \mathbb{R})$ and a smooth (1,1) form β on N. Then with the notation above, there exists a distance function d_T on N such that (N, d_T) is a compact metric space and $(M, g(t)) \to (N, d_T)$ as $t \to T^-$ in the Gromov-Hausdorff sense.

A natural conjecture by Tosatti and Weinkove is that in the setup of Theorem 1.2, the Chern-Ricci flow will blow down finitely many (-1)-curves and continue in a unique way on N. They call this behavior a *canonical surgical contraction* (see section 3.1.2 for more details). The conjecture requires smooth convergence of the metrics away from the (-1)-curves and global Gromov-Hausdorff convergence as $t \to T^-$ and $t \to T^+$. The above two theorems give these (under condition (1.6)) as $t \to T^-$.

To continue the flow on N, we need to show that: (i) there exists a unique smooth maximal solution $\omega(t)$ of the Chern-Ricci flow on N for $t \in (T, T_N)$ with $T < T_N \leq \infty$ such that $\omega(t)$ converges to $(\pi^{-1})^*\omega_T$ in $C^\infty_{\mathrm{loc}}(N')$ as $t \to T^+$; (ii) $(N, g(t))$ converges to (N, d_T) in the Gromov-Hausdorff sense as $t \to T^+$. We establish step (i) under the condition (1.6). As $(\pi^{-1})^*\omega_T$ may not be smooth on N, first we need to show that the Chern-Ricci flow on N with rough initial data has a unique smooth solution on $(T, T']$ for some $T' > T$. To prove the smooth convergence on compact subsets of N', we need more precise estimates near the exceptional curves as $t \to T^-$ and then estimates on $[T, T'] \times N'$. Denote $\tilde{\omega}_T = (\pi^{-1})^*\omega_T$ the push forward of the limiting current to N. Following the construction of Song-Weinkove [39, 40] for the Kähler-Ricci flow, we prove the following theorem in chapter 3.

Theorem 1.4. *Assume that condition (1.6) is satisfied, then with the notation above,* *there exists a unique maximal solution* $\omega(t)$ *to the equation:*

$$\omega|_{t=T} = \tilde{\omega}_T, \quad \frac{\partial}{\partial t}\omega = -\operatorname{Ric}(\omega), \quad \textit{for } t \in (T, T_N),$$

which is smooth on (T, T_N)*. Moreover,* $\omega(t)$ *converges to* $\tilde{\omega}_T$ *in* $C^\infty_{\mathrm{loc}}(N')$ *as* $t \to T^+$*.*

We actually obtain a more general existence theorem.

Now let (M, ω_0) be an n-dimensional compact Hermitian manifold. Let Ω be a smooth volume form on M and $\hat{\omega}_t = \omega_0 + t\chi$, where χ is a closed (1,1) form locally defined by $\chi = \sqrt{-1}\partial\bar{\partial}\log\Omega$. Denote

$$PSH_p(\omega_0, \Omega) = \{\varphi \in PSH(M, \omega_0) \cap L^\infty(M) | \frac{(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi)^n}{\Omega} \in L^p(M)\}.$$

We prove the following smoothing property for the Chern-Ricci flow.

Theorem 1.5. *Suppose that* $\omega_0' = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_0$ *for some* $\varphi_0 \in PSH_p(\omega_0, \Omega)$, $p > 1$*. Assume that* ω_0 *satisfies the condition (1.3), then there exists a unique family of smooth metrics* $\omega(t)$ *on* $(0, T)$ *such that*

(i) $\frac{\partial}{\partial t}\omega = -\operatorname{Ric}(\omega)$*, for* $t \in (0, T)$*.*

(ii) There exists $\varphi \in C^0([0, T) \times M) \cap C^\infty((0, T) \times M)$ *such that* $\omega = \hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi$ *and* $\varphi(t) \to \varphi_0$ *in* $L^\infty(M)$ *as* $t \to 0^+$*.*

In particular, $\omega(t) \to \omega_0'$ *in the sense of currents as* $t \to 0^+$*.*

When (M, ω_0) is a Kähler manifold, the result is contained in the work of Song-Tian [38] (see also [7]). The conditions in the above theorem naturally arise in geometric setting, which can be seen in section 3.3.

Chapter 2

Regularity of A Complex Monge-Ampère Equation

In this chapter we study the regularity of weak solutions of a class of complex Monge-Ampère equations:

$$(\omega + \sqrt{-1}\partial\bar{\partial}\phi)^n = e^{-F(\phi,z)}\omega^n \tag{2.1}$$

on a compact Hermitian manifold (M, ω), where $F : \mathbb{R} \times M \to \mathbb{R}$ is a smooth function and $\omega + \sqrt{-1}\partial\bar{\partial}\phi \geq 0$ in the sense of currents. We use a parabolic flow method to study the problem. In section 2.1, we give some background on Hermitian geometry, plurisubharmonic functions and complex Monge-Ampère Equation. In section 2.2, we state our main result, Theorem 2.5, and sketch the idea of the proof. In section 2.3, we prove higher order estimates for the corresponding parabolic flow, which is the key part to prove the theorem. In section 2.4, we give the proof of Theorem 2.5.

The results of this chapter can be found in [32].

2.1 Background

In this section we give some background materials on Hermitian manifolds, ω-plurisubharmonic functions and complex Monge-Ampère Equation.

2.1.1 Basic Hermitian geometry

Let (M, J) be an n-dimensional compact complex manifold. A Hermitian metric on (M, J) is a Riemannian metric g such that $g(JX, JY) = g(X, Y)$ for all tangent vectors X, Y. The fundamental 2-form of g is a (1, 1) form defined by

$$\omega(X, Y) = g(JX, Y).$$

If ω is closed, then g is called a Kähler metric. Let $z_1, ..., z_n$ be holomorphic coordinates on M and write $\partial_i, \partial_{\bar{j}}$ for $\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_{\bar{j}}}$. Denote $g_{i\bar{j}} = g(\partial_i, \partial_{\bar{j}})$ the coefficients of the metric tensor in local coordinates. Then locally

$$\omega = \sqrt{-1} g_{i\bar{j}} dz^i \wedge d\bar{z}^j.$$

For convenience, we will often refer to ω as a Hermitian metric. Denote ∇ the Chern connection of g with Christoffel symbols Γ_{ij}^k and torsion T given by:

$$\Gamma_{ij}^k = g^{k\bar{l}} \partial_i g_{j\bar{l}}, \qquad T_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k.$$

The covariant derivatives of $X = X^j \frac{\partial}{\partial z^j}$ and $a = a_j dz^j$ are defined in components as

$$\nabla_i X^j = \partial_i X^j + \Gamma_{ik}^j X^k, \quad \nabla_i a_j = \partial_i a_j - \Gamma_{ij}^k a_k.$$

Then ∇ can be extended naturally to any tensors. Define the Chern curvature tensor of g in components to be

$$R_{i\bar{j}k}{}^{l} = -\partial_{\bar{j}}\Gamma_{ik}^{l}.$$

We lower and raise indices using g. Then

$$R_{i\bar{j}k\bar{l}} = -\partial_i\partial_{\bar{j}}g_{k\bar{l}} + g^{p\bar{q}}\partial_i g_{k\bar{q}}\partial_{\bar{j}}g_{p\bar{l}}.$$

and the Chern-Ricci tensor is given by

$$R_{i\bar{j}} = g^{k\bar{l}}R_{i\bar{j}k\bar{l}} = -\partial_i\partial_{\bar{j}}\log\det g.$$

We have the following commutation formulas:

$$\begin{aligned} &[\nabla_i, \nabla_{\bar{j}}]X^l = R_{i\bar{j}k}{}^{l}X^k, \quad [\nabla_i, \nabla_{\bar{j}}]a_k = -R_{i\bar{j}k}{}^{l}a_l \\ &[\nabla_i, \nabla_{\bar{j}}]\overline{X^l} = -R_{i\bar{j}}{}^{\bar{l}}{}_{\bar{k}}\overline{X^k}, \quad [\nabla_i, \nabla_{\bar{j}}]\overline{a_k} = R_{i\bar{j}}{}^{\bar{l}}{}_{\bar{k}}\overline{a_l} \end{aligned} \tag{2.2}$$

The Bianchi identities will not hold necessarily for general Hermitian manifolds. There are extra torsion terms in the following identities.

$$\begin{aligned} R_{i\bar{j}k\bar{l}} - R_{k\bar{j}i\bar{l}} &= -\nabla_{\bar{j}}T_{ik\bar{l}} \\ R_{i\bar{j}k\bar{l}} - R_{i\bar{l}k\bar{j}} &= -\nabla_i T_{\bar{j}\bar{l}k} \\ R_{i\bar{j}k\bar{l}} - R_{k\bar{l}i\bar{j}} &= -\nabla_{\bar{j}}T_{ik\bar{l}} - \nabla_k T_{\bar{j}\bar{l}i} \\ \nabla_p R_{i\bar{j}k\bar{l}} - \nabla_i R_{p\bar{j}k\bar{l}} &= -T_{pi}{}^{r}R_{r\bar{j}k\bar{l}} \\ \nabla_{\bar{q}} R_{i\bar{j}k\bar{l}} - \nabla_{\bar{j}} R_{i\bar{q}k\bar{l}} &= -T_{\bar{q}\bar{j}}{}^{\bar{s}}R_{i\bar{s}k\bar{l}}. \end{aligned} \tag{2.3}$$

2.1.2 ω-psh functions

A function $u : \Omega \to [-\infty, +\infty)$ defined on an open set $\Omega \subset \mathbb{C}^n$ is said to be plurisubharmonic if it is upper semicontinuous in Ω and subharmonic when restricted to any intersection of Ω with a complex line. If u is plurisubharmonic, then $dd^c u$ is a positive current. Recall that a current can be considered as a differential form with distribution coefficients. A current T of bidegree $(n-p, n-p)$ is positive if for any (1,0) forms $\alpha_1, ..., \alpha_p$,

$$i^p T \wedge \alpha_1 \wedge \bar{\alpha}_1 \wedge \alpha_2 \wedge \bar{\alpha}_2 ... \wedge \alpha_p \wedge \bar{\alpha}_p$$

is a positive measure. According to Bedford-Taylor [1], for a locally bounded plurisubharmonic function u on $\mathbb{C}^n$, the wedge products $(dd^c u)^k,\ 1 \le k \le n$ are well defined positive currents, where $d^c = \frac{\sqrt{-1}}{2}(\bar{\partial} - \partial)$ and $dd^c = \sqrt{-1}\partial\bar{\partial}$.

Let (M, ω) be a compact Hermitian manifold and $\Omega \subset M$ a small open set .

Definition 2.1. An upper semicontinuous function $u : M \to [-\infty, +\infty)$ is called ω-plurisubharmonic if $u \in L^1(M, \omega^n)$ and $\omega + dd^c u$ is a positive current.

We denote $PSH(M, \omega)$ the set of ω-plurisubharmonic (short for ω-psh) functions on M. Following Bedford-Taylor, one can define the wedge product $(\omega + dd^c u)^n$ in the Hermitian setting for $u \in PSH(M, \omega) \cap L^\infty(M)$ in the following way (see [29] for more details). First for $u \in PSH(\Omega, \omega) \cap L^\infty(\Omega)$, take a plurisubharmonic function ρ such that $T = dd^c\rho - \omega > 0$. Let $v = u + \rho$, then $\omega + dd^c u = dd^c v - T$. As the wedge products $(dd^c v)^k \wedge T^{n-k},\ 0 \le k \le n$ are well defined by Bedford-Taylor [1], we can define

$$(dd^c v - T)^n := (dd^c v)^n - n(dd^c v)^{n-1} \wedge T + ... + (-1)^n T^n$$

using Newton expansion. By Demailly's regularization theorem [11] for quasi-psh functions, there exists a sequence of smooth functions $\{u_j\}$ in $PSH(\omega, \Omega) \cap L^\infty(\Omega)$ which decreases to u. Let $v_j = u_j + \rho$, then $\{v_j\} \in PSH(\omega, \Omega) \cap L^\infty(\Omega)$ is a sequence of smooth functions decreasing to v. It follows from the convergence theorem in [2] that $(dd^c v_j)^k \wedge T^{n-k} \to (dd^c v)^k \wedge T^{n-k}$ in the sense of currents for $0 \le k \le n$. Thus

$$(dd^c v - T)^n = \lim_{j\to\infty} (dd^c v_j - T)^n = \lim_{j\to\infty} (\omega + dd^c u_j)^n. \tag{2.4}$$

This shows that if we take another $\tilde{\rho}$ such that $\tilde{T} = dd^c\tilde{\rho} - \omega > 0$. Let $\tilde{v} = u + \tilde{\rho}$, then $(dd^c\tilde{v} - \tilde{T})^n = (dd^c v - T)^n$. Therefore

$$(\omega + dd^c u)^n := (dd^c v - T)^n$$

is independent of the choice of v and T and is a well-defined positive current by (2.4). By partition of unity, $(\omega + dd^c u)^n$ is well-defined for $u \in PSH(M, \omega) \cap L^\infty(M)$.

2.1.3 Complex Monge-Ampère equation

In this section, we state two theorems on complex Monge-Ampère equation that will be used later.

The Complex Monge-Ampère equation on compact Hermitian manifolds was solved in full generality by Tosatti and Weinkove in 2009. (See also [8, 18, 19, 44]).

Theorem 2.2. (Tosatti-Weinkove [45])*Let (M, ω) be a compact Hermitian manifold. Then for any smooth function F on M, there exists a unique constant b and a unique smooth function φ on M solving*

$$(\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^{F+b}\omega^n, \quad \textit{with}$$
$$\omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0, \quad \sup_M \varphi = 0$$

The equation on Hermitian manifolds was also studied by Dinew and Kolodziej from the pluripotential aspects. They studied the case when F is not smooth and they made the assumption:

$$\text{for all u} \in PSH(M,\omega) \cap L^\infty(M), \quad \int_M (\omega + \sqrt{-1}\partial\bar{\partial}u)^n = \int_M \omega^n. \tag{2.5}$$

Theorem 2.3. (Dinew-Kołodziej [12]) *Let (M,ω) be a compact Hermitian manifold. Let $f \in L^p(M,\omega^n)$ for some $p > 1$ be a nonnegative function satisfying $\int_M f\omega^n = \int_M \omega^n$. Assume that ω satisfies the condition (2.5). Then there exists a continuous function $u \in PSH(M,\omega)$ solving the equation*

$$(\omega + \sqrt{-1}\partial\bar{\partial}u)^n = f\omega^n.$$

2.2 Smoothness of weak solutions

In this section, we state our main theorem, Theorem 2.5, which generalizes a result of Székelyhidi and Tosatti on Kähler manifolds. We also sketch the idea and techniques we will use to prove the theorem.

In [42], Székelyhidi and Tosatti studied regularity of weak solutions of the equation

$$(\omega + \sqrt{-1}\partial\bar{\partial}\phi)^n = e^{-F(\phi,z)}\omega^n \tag{2.6}$$

on an n-dimensional compact Kähler manifold (M,ω), where $F : \mathbb{R}\times M \to \mathbb{R}$ is smooth. If $\phi \in PSH(M,\omega) \cap L^\infty(M)$ solves equation (2.6) in the sense of currents, we say ϕ is a weak solution of the equation. The Hölder continuity of such weak solutions follows from Kołodziej [27]. Using a different approach, Székelyhidi and Tosatti proved the following theorem.

Theorem 2.4. (Székelyhidi-Tosatti [42])*Let (M,ω) be an n-dimensional compact Kähler* *manifold. Suppose that* $\phi \in PSH(M,\omega) \cap L^\infty(M)$ *solves equation*

$$(\omega + \sqrt{-1}\partial\bar{\partial}\phi)^n = e^{-F(\phi,z)}\omega^n$$

in the sense of Bedford-Taylor, where $F : \mathbb{R} \times M \to \mathbb{R}$ *is a smooth function. Then* ϕ *is smooth.*

Particularly, if M is Fano, $\omega \in c_1(M)$ and $F(\phi, z) = \phi - h$, where h satisfies $\sqrt{-1}\partial\bar{\partial}h = \mathrm{Ric}(\omega) - \omega$, the above theorem implies that Kähler-Einstein currents (see e.g., [19]) with bounded potentials are smooth.

As the wedge product $(\omega + \sqrt{-1}\partial\bar{\partial}\phi)^n$ is still well defined for a general Hermitian metric ω and $\phi \in PSH(M,\omega) \cap L^\infty(M)$, it is natural to consider the regularity of weak solutions of equation (2.6) in the setting of Hermitian manifolds. We obtain the same regularity result under the following compatibility condition:

$$\forall\ u \in PSH(M,\hat{\omega}) \cap L^\infty(M), \quad \int_M (\hat{\omega} + \sqrt{-1}\partial\bar{\partial}u)^n = \int_M \hat{\omega}^n \tag{2.7}$$

Theorem 2.5. *Let* $(M, \hat{\omega})$ *be an* n*-dimensional compact Hermitian manifold with* $\hat{\omega}$ *satisfying condition (2.7) and* $F : \mathbb{R} \times M \to \mathbb{R}$ *be a smooth function. Suppose that* $\phi \in PSH(M,\hat{\omega}) \cap L^\infty(M)$ *solves*

$$(\hat{\omega} + \sqrt{-1}\partial\bar{\partial}\phi)^n = e^{-F(\phi,z)}\hat{\omega}^n \tag{2.8}$$

in the sense of currents. Then ϕ *is smooth.*

The proof of Theorem 2.5 is given in section 2.3 and section 2.4. We follow the parabolic flow method used in [42] (see also [38]). Consider the corresponding parabolic flow

$$\frac{\partial \varphi}{\partial t} = \log \frac{(\hat{\omega} + \sqrt{-1}\partial\bar{\partial}\varphi)^n}{\hat{\omega}^n} + F(\varphi, z). \tag{2.9}$$

We show that the higher order estimates in [42] can be obtained on a general compact Hermitian manifold. Particularly, the flow (2.9) with smooth initial data φ_0 has a smooth solution for a time T depending only on $\sup|\varphi_0|$ and $\sup|\dot{\varphi}_0|$. Assuming the condition (2.7), we can construct a function $\varphi \in C^0([0,T] \times M) \cap C^\infty((0,T] \times M)$ with $\varphi(0) = \phi$ which solves equation (2.9) on $(0,T]$, where T depends only on $\sup|\phi|$, F and ω. Then we show that $\dot{\varphi}(t) = 0$ for $0 < t \leq T$ since the initial ϕ is a solution of (2.6). Therefore $\phi = \varphi(0) = \varphi(t)$ is smooth.

The main difference between the proof in Hermitian case and Kählar case lies in the C^2, C^3 estimates and bound for $|\operatorname{Ric}|$. The computation on Hermitian manifolds is more complicated due to the existence of torsion terms. The proof of the second order estimate follows closely the arguments of Gill [15] and Tosatti-Weinkove [44]. For the third order estimate we make use of the arguments in Phong-Sesum-Sturm [33] and Sherman-Weinkove [37]. Such estimate for the first derivative of the evolving Hermitian metrics was also established in [48], where the authors took a local reference Kähler metric to obtain a good bound. To bound $|\operatorname{Ric}|$, we need to deal with the new terms involving $|\nabla \operatorname{Ric}|$ very carefully.

2.3 Estimates for the parabolic flow

In this section, we will prove higher order estimates for the corresponding parabolic flow, which depend only on $\sup|\varphi_0|$ and $\sup|\dot{\varphi}_0|$. In particular, we show that the flow (with smooth initial data) exists for a short time depending only on $\sup|\varphi_0|$ and $\sup|\dot{\varphi}_0|$.

Consider the following parabolic equation on a compact Hermitian manifold $(M, \hat{\omega})$,

$$\frac{\partial \varphi}{\partial t} = \log \frac{(\hat{\omega} + \sqrt{-1}\partial\bar{\partial}\varphi)^n}{\hat{\omega}^n} + F(\varphi, z) \tag{2.10}$$

where $F : \mathbb{R} \times M \to \mathbb{R}$ is a smooth function and $\varphi|_{t=0} = \varphi_0$ is smooth. By the theory of parabolic equations, there exists a unique smooth solution $\varphi(t)$ with $\hat{\omega} + \sqrt{-1}\partial\bar{\partial}\varphi > 0$ for a short time. Denote $\dot{\varphi}$ for $\frac{\partial \varphi}{\partial t}$. We have the following proposition which generalizes the estimates in [42] to compact Hermitian manifolds.

Proposition 2.6. *Given a compact Hermitian manifold $(M, \hat{\omega})$, there exists $T > 0$ depending only on $\sup|\varphi_0|$ and F such that the above equation has a smooth solution $\varphi(t, z)$on $[0, T]$. Moreover, there exist smooth functions $C_k(t)$ for all k on $(0, T]$ depending only on $\sup|\varphi_0|$, $\sup|\dot{\varphi}_0|$, $\hat{\omega}$ and F which blow up as $t \to 0$ such that*

$$\|\varphi(t)\|_{C^k(M)} < C_k(t)$$

for $t \leq T$.

We write g for the metric associated to $\omega = \hat{\omega} + \sqrt{-1}\partial\bar{\partial}\varphi$, where $\hat{\omega}$ is the background Hermitian metric on a compact complex manifold M. Denote $|\cdot|$ the norm of tensors with respect to g, ∇ the Chern connection of g and $\Delta = g^{p\bar{q}}\nabla_p\nabla_{\bar{q}}$ the Laplacian of ∇. We use $\hat{\nabla}$, $\hat{R}_{i\bar{j}}$, $|\cdot|_{\hat{g}}$, $\Delta_{\hat{g}}$, etc. to denote the quantities associated to $\hat{\omega}$. Throughout the section, $C, C', c, c_i, ...$ will be some constants which depend only on $\sup|\varphi_0|$, $\sup|\dot{\varphi}_0|$ (and $\hat{\omega}, F$), and may vary from line to line. Also we may denote H to be different quantities.

First we have the following lemma from [42].

Lemma 2.7. *There exist $T, C > 0$ depending only on $\sup|\varphi_0|$ and F such that*

$$|\varphi(t)| < C, \quad |\dot{\varphi}(t)| \leq \sup |\dot{\varphi}(0)| e^{Ct}, \tag{2.11}$$

when the solution exists and $t \leq T$. *In particular,*

$$|\log \frac{(\hat{\omega} + \sqrt{-1}\partial\bar{\partial}\varphi)^n}{\hat{\omega}^n}| < C' \tag{2.12}$$

for some C' *depending only on* $\sup |\varphi_0|$, $\sup |\dot{\varphi}_0|$ *and* F.

The proof follows from [42, Lemma 3] as it does not need the Kähler condition. Now we can fix a $T' \leq T$ such that there exists a smooth solution to (2.10) on $[0, T']$.

The C^1 estimate in [42] was obtained by modifying Błocki's estimate [3] (see also [21], [34]). In Hermitian case, we need the following special local coordinate system from Guan-Li [19], which is also crucial for our second order estimate.

Lemma 2.8. *Around a point* $p \in M$, *there exist local coordinates such that at* p,

$$\hat{g}_{i\bar{j}} = \delta_{ij}, \qquad \frac{\partial \hat{g}_{i\bar{i}}}{\partial z_j} = 0. \tag{2.13}$$

for all i, j.

With the above lemma, we have the following gradient estimate.

Lemma 2.9. *There exists* $\alpha > 0$ *depending on* $\sup |\varphi_0|$ *and* $\sup |\dot{\varphi}_0|$ *such that*

$$|\nabla \varphi(t)|^2_{\hat{g}} < e^{\alpha/t}, \tag{2.14}$$

for $t \leq T'$.

Proof. Define

$$H = t \log |\nabla \varphi(t)|^2_{\hat{g}} - \gamma(\varphi),$$

where γ is a smooth function which will be determined later. If H achieves maximum on $[0,T']\times M$ at $t=0$, then H is bounded by a constant depending on F and $\sup|\varphi_0|$ by Lemma 2.7. Now assume that H achieves its maximum at a point (t_0,z_0), $t_0>0$. We use $\varphi_i,\varphi_{i\bar{j}},\varphi_{i\bar{j}k}$, etc. to denote partial derivatives. Choose a coordinate system around z_0 in Lemma 2.8 such that $\varphi_{i\bar{j}}$ is diagonal at z_0. Write $\rho=|\nabla\varphi(t)|^2_{\hat{g}}=\hat{g}^{i\bar{j}}\varphi_i\varphi_{\bar{j}}$ and $\dot{\rho}=\frac{\partial\rho}{\partial t}$. As $(\frac{\partial}{\partial t}-\Delta)H=\frac{\partial}{\partial t}H-t\frac{\Delta\rho}{\rho}+t\frac{|\nabla\rho|^2}{\rho^2}+\Delta\gamma$, we do the following calculations at z_0. First we have

$$\begin{aligned}\frac{\partial}{\partial t}H&=\log\rho+\frac{t\dot{\rho}}{\rho}-\gamma'\dot{\varphi}\\&=\log\rho-\gamma'\dot{\varphi}+2\frac{t}{\rho}(\sum_{i,k}\mathrm{Re}(\frac{\varphi_{ki\bar{i}}\varphi_{\bar{k}}}{1+\varphi_{i\bar{i}}})+F'\sum_i|\varphi_i|^2+\sum_i\mathrm{Re}(F_i\varphi_{\bar{i}})),\end{aligned}$$

where the second equality follows from

$$\begin{aligned}\dot{\rho}&=\sum_i\dot{\varphi}_i\varphi_{\bar{i}}+\varphi_i\dot{\varphi}_{\bar{i}}\\\dot{\varphi}_i&=g^{k\bar{l}}(\partial_i\hat{g}_{k\bar{l}}+\varphi_{i\bar{l}k})-\hat{g}^{k\bar{l}}\partial_i\hat{g}_{k\bar{l}}+F'\varphi_i+F_i\\&=\sum_k\frac{\varphi_{ik\bar{k}}}{1+\varphi_{k\bar{k}}}+F'\varphi_i+F_i.\end{aligned}$$

Here F' is the derivative in the φ direction. Also, using $\partial_i\partial_{\bar{i}}\hat{g}^{k\bar{l}}=-\partial_i\partial_{\bar{i}}\hat{g}_{l\bar{k}}+\sum_q\partial_i\hat{g}_{q\bar{k}}\partial_{\bar{i}}\hat{g}_{l\bar{q}}+\sum_p\partial_i\hat{g}_{l\bar{p}}\partial_{\bar{i}}\hat{g}_{p\bar{k}}$, we get

$$\begin{aligned}\Delta\rho&=g^{i\bar{i}}\partial_i\partial_{\bar{i}}(\hat{g}^{k\bar{l}}\varphi_k\varphi_{\bar{l}})\\&=\sum_{i,k}\frac{1}{1+\varphi_{i\bar{i}}}(-\sum_l\partial_i\partial_{\bar{i}}\hat{g}_{l\bar{k}}\varphi_k\varphi_{\bar{l}}+2\,\mathrm{Re}(\varphi_{k\bar{i}i}\varphi_{\bar{k}})\\&\quad+|\varphi_{k\bar{i}}-\sum_l\partial_{\bar{i}}\hat{g}_{\bar{l}k}\varphi_l|^2+|\varphi_{ki}-\sum_l\partial_i\hat{g}_{k\bar{l}}\varphi_l|^2).\end{aligned}$$

At (t_0,z_0), $\nabla H=0$ gives

$$H_i = \frac{t}{\rho}\rho_i - \gamma'\varphi_i = 0. \tag{2.15}$$

Then

$$\frac{|\nabla\rho|^2}{\rho^2} = \sum_i \frac{1}{1+\varphi_{i\bar{i}}}(\frac{\gamma'}{t})^2|\varphi_i|^2.$$

Also $\Delta\gamma(\varphi) = \sum_i \frac{1}{1+\varphi_{i\bar{i}}}(\gamma''|\varphi_i|^2 + \gamma'\varphi_{\bar{i}i})$. Therefore we get

$$\begin{aligned}
0 &\le (\frac{\partial}{\partial t} - \Delta)H \\
&= \frac{\partial}{\partial t}H - t\frac{\Delta\rho}{\rho} + t\frac{|\nabla\rho|_g^2}{\rho^2} + \Delta\gamma \\
&\le \log\rho - \gamma'\dot{\varphi} + ct + \frac{c_1 t}{\rho} - \sum_{i,k}\frac{t}{\rho}\frac{1}{1+\varphi_{i\bar{i}}}(|\varphi_{k\bar{i}} - \sum_l \partial_{\bar{i}}\hat{g}_{\bar{l}k}\varphi_l|^2 + |\varphi_{ki} - \sum_l \partial_i\hat{g}_{k\bar{l}}\varphi_l|^2) \\
&\quad + \sum_i \frac{|\varphi_i|^2}{1+\varphi_{i\bar{i}}}(\frac{(\gamma')^2}{t} + \gamma'') + \sum_i \frac{c_2 t - \gamma'}{1+\varphi_{i\bar{i}}} + n\gamma'
\end{aligned}$$

for some constants c, c_1 depending on F and c_2 depending on the curvature of $\hat{\omega}$. Now we use the same trick in [3, 42] to control the term containing γ'^2. From (2.15) we get

$$\gamma'\rho\varphi_i = t\rho_i = t(\varphi_i\varphi_{i\bar{i}} + \sum_k \varphi_{ki}\varphi_{\bar{k}} - \sum_{k,l}\partial_i\hat{g}_{l\bar{k}}\varphi_k\varphi_{\bar{l}})$$

which gives

$$\sum_k(\varphi_{ki} - \sum_l \partial_i\hat{g}_{k\bar{l}}\varphi_l)\varphi_{\bar{k}} = t^{-1}\gamma'\rho\varphi_i - \varphi_i\varphi_{i\bar{i}}.$$

So

$$\frac{t}{\rho}\sum_{i,k}\frac{|\varphi_{ki}-\sum_l \partial_i\hat{g}_{k\bar{l}}\varphi_l|^2}{1+\varphi_{i\bar{i}}} \geq \frac{t}{\rho^2}\sum_i \frac{|\sum_k(\varphi_{ki}-\sum_l \partial_i\hat{g}_{k\bar{l}}\varphi_l)\varphi_{\bar{k}}|^2}{1+\varphi_{i\bar{i}}}$$
$$= \frac{t}{\rho^2}\sum_i \frac{|t^{-1}\gamma'\rho\varphi_i - \varphi_i\varphi_{i\bar{i}}|^2}{1+\varphi_{i\bar{i}}}$$
$$\geq \frac{(\gamma')^2}{t}\sum_i \frac{|\varphi_i|^2}{1+\varphi_{i\bar{i}}} - 2\gamma'$$

where we assume $\gamma' > 0$. As $\dot{\varphi}$ is bounded from Lemma 2.7 for $t \leq T'$, the above estimates gives

$$0 \leq \log\rho + ct + \sum_i \frac{\gamma''|\varphi_i|^2}{1+\varphi_{i\bar{i}}} + \sum_i \frac{c_1 t - \gamma'}{1+\varphi_{i\bar{i}}} + (n+2+c)\gamma' + \frac{c_2 t}{\rho}.$$

Take $\gamma(x) = Ax - \frac{1}{A}x^2$. Assume that $\log\rho \geq 1$ at (t_0, z_0) and choose A to be sufficiently large, then we get

$$\sum_i \frac{|\varphi_i|^2}{1+\varphi_{i\bar{i}}} + \sum_i \frac{1}{1+\varphi_{i\bar{i}}} \leq c'\log\rho$$

for some constant c'. The above inequality together with (2.12) imply that

$$1+\varphi_{i\bar{i}} \leq c(c'\log\rho)^{n-1}.$$

Then we have

$$\rho = \sum_i |\varphi_i|^2 \leq nc(c'\log\rho)^n,$$

which shows that ρ is bounded at (t_0, z_0). Therefore H has a bound depending only on $\sup|\varphi_0|$, $\sup|\dot{\varphi}_0|$ and the estimate (2.14) follows. $\square$

Now we will give the second order estimate. We use the idea of [18, 44] and follow the argument in [15] closely. For local computations in the proof of the following proposition, we always use a coordinate system in Lemma 2.8 around a point p, such that $\hat{g}_{i\bar{j}} = \delta_{ij}$, $\frac{\partial \hat{g}_{i\bar{i}}}{\partial z_j} = 0$ and $\varphi_{i\bar{j}}$ is diagonal at p.

Proposition 2.10. *There exists $C > 0$ depending on $\sup|\varphi_0|$ and $\sup|\dot{\varphi}_0|$ such that*

$$\operatorname{tr}_{\hat{g}} g = n + \Delta\varphi(t) < e^{Ce^{\alpha/t}} \tag{2.16}$$

for $t \leq T'$, where α is the same as in Lemma 2.9.

Proof. Let

$$H = e^{-\frac{\alpha}{t}} \log \operatorname{tr}_{\hat{g}} g + e^{\Psi},$$

where $\Psi = A(\sup_{[0,T']\times M} \varphi - \varphi)$ and A is a constant to be chosen later. First we have

$$\begin{aligned}(\frac{\partial}{\partial t} - \Delta)H = & \frac{\alpha}{t^2} e^{-\frac{\alpha}{t}} \log \operatorname{tr}_{\hat{g}} g + \frac{e^{-\frac{\alpha}{t}}}{\operatorname{tr}_{\hat{g}} g} \Delta_{\hat{g}}\dot{\varphi} - Ae^{\Psi}\dot{\varphi} \\ & - e^{-\frac{\alpha}{t}} \Delta \log \operatorname{tr}_{\hat{g}} g - A^2|\nabla\varphi|^2 e^{\Psi} - A(\operatorname{tr}_g \hat{g} - n)e^{\Psi}.\end{aligned} \tag{2.17}$$

It follows from (2.10) that

$$\Delta_{\hat{g}}\dot{\varphi} = -\operatorname{tr}_{\hat{g}} \operatorname{Ric}(g) + \operatorname{tr}_{\hat{g}} \operatorname{Ric}(\hat{g}) + \Delta_{\hat{g}} F(\varphi, z) \tag{2.18}$$

where

$$\Delta_{\hat{g}} F(\varphi, z) = F''|\nabla\varphi|^2_{\hat{g}} + F'\Delta_{\hat{g}}\varphi + 2\operatorname{Re}(g^{i\bar{j}} F'_i \varphi_{\bar{j}}) + \Delta_{\hat{g}} F.$$

Here F' is the derivative in the φ direction, $\Delta_{\hat{g}}$ is the complex Laplacian of F in the z

variable. Use that

$$\begin{aligned}\operatorname{tr}_{\hat g}\operatorname{Ric}(g) &= \sum_{i,k} g^{i\bar i}(-\partial_k\partial_{\bar k}g_{i\bar i} + g^{j\bar j}\partial_k g_{i\bar j}\partial_{\bar k}g_{j\bar i}) \\ &= \sum_{i,k} g^{i\bar i}(-\varphi_{i\bar i k\bar k} - \partial_k\partial_{\bar k}\hat g_{i\bar i} + g^{j\bar j}\partial_k g_{i\bar j}\partial_{\bar k}g_{j\bar i})\end{aligned}$$

to rewrite (2.18) as

$$\begin{aligned}\sum_{i,k} g^{i\bar i}\varphi_{i\bar i k\bar k} &= -\sum_{i,k} g^{i\bar i}\partial_k\partial_{\bar k}\hat g_{i\bar i} + \sum_{i,j,k} g^{i\bar i}g^{j\bar j}\partial_k g_{i\bar j}\partial_{\bar k}g_{j\bar i} + \Delta_{\hat g}\dot\varphi - \operatorname{tr}_{\hat g}\operatorname{Ric}(\hat g) - \Delta_{\hat g}F(\varphi,z) \\ &\geq \sum_{i,j,k} g^{i\bar i}g^{j\bar j}\partial_k g_{i\bar j}\partial_{\bar k}g_{j\bar i} + \Delta_{\hat g}\dot\varphi - C_1|\nabla\varphi|^2_{\hat g} - C_2\operatorname{tr}_{\hat g} g\operatorname{tr}_g\hat g. \qquad (2.19)\end{aligned}$$

From the bound in (2.12), we have $\operatorname{tr}_{\hat g} g,\ \operatorname{tr}_g\hat g \geq C^{-1}$ for some constant C and then $\operatorname{tr}_{\hat g} g,\ \operatorname{tr}_g\hat g \leq C\operatorname{tr}_g\hat g\operatorname{tr}_{\hat g} g$. These are used in the above inequality. We will also use them frequently in the following. As the estimates in [44, (2.6)], we have

$$\Delta\operatorname{tr}_{\hat g} g \geq \sum_{i,k} g^{i\bar i}\varphi_{i\bar i k\bar k} - 2\operatorname{Re}\Big(\sum_{i,j,k} g^{i\bar i}\partial_{\bar i}\hat g_{j\bar k}\varphi_{k\bar j i}\Big) - C\operatorname{tr}_{\hat g} g\operatorname{tr}_g\hat g. \qquad (2.20)$$

To control $\sum_{i,j,k} g^{i\bar i}\partial_{\bar i}\hat g_{j\bar k}\varphi_{k\bar j i}$, we use a trick from [6].

$$\sum_{i,j,k} g^{i\bar i}\partial_{\bar i}\hat g_{j\bar k}\varphi_{k\bar j i} = \sum_i\sum_{j\neq k}(g^{i\bar i}\partial_{\bar i}\hat g_{j\bar k}\partial_k g_{i\bar j} - g^{i\bar i}\partial_{\bar i}\hat g_{j\bar k}\partial_k\hat g_{i\bar j}).$$

so

$$\begin{aligned}\Big|2\operatorname{Re}\Big(\sum_{i,j,k} g^{i\bar i}\partial_i\hat g_{j\bar k}\varphi_{k\bar j i}\Big| &\leq \sum_i\sum_{j\neq k}(g^{i\bar i}g^{j\bar j}\partial_k g_{i\bar j}\partial_{\bar k}g_{j\bar i} + g^{i\bar i}g_{j\bar j}\partial_{\bar i}\hat g_{j\bar k}\partial_i\hat g_{k\bar j}) + C\operatorname{tr}_g\hat g \\ &\leq \sum_i\sum_{j\neq k} g^{i\bar i}g^{j\bar j}\partial_k g_{i\bar j}\partial_{\bar k}g_{j\bar i} + C\operatorname{tr}_g\hat g\operatorname{tr}_{\hat g} g. \qquad (2.21)\end{aligned}$$

Combining (2.19), (2.20) and (2.21) we can get

$$\Delta \operatorname{tr}_{\hat{g}} g \geq \sum_{i,j} g^{i\bar{i}} g^{j\bar{j}} \partial_j g_{i\bar{j}} \partial_{\bar{j}} g_{j\bar{i}} + \Delta_{\hat{g}} \dot{\varphi} - C_1 |\nabla \varphi|^2_{\hat{g}} - C \operatorname{tr}_g \hat{g} \operatorname{tr}_{\hat{g}} g.$$

Now we will control $\frac{|\partial \operatorname{tr}_{\hat{g}} g|^2}{(\operatorname{tr}_{\hat{g}} g)^2}$. As

$$\partial_i \operatorname{tr}_{\hat{g}} g = \partial_i \sum_j \varphi_{j\bar{j}} = \sum_j \partial_j \varphi_{i\bar{j}} = \sum_j (\partial_j g_{i\bar{j}} - \partial_j \hat{g}_{i\bar{j}}),$$

then

$$\frac{|\partial \operatorname{tr}_{\hat{g}} g|^2}{(\operatorname{tr}_{\hat{g}} g)^2} \leq \frac{1}{(\operatorname{tr}_{\hat{g}} g)^2} \sum_{i,j,k} g^{i\bar{i}} \partial_j g_{i\bar{j}} \partial_{\bar{k}} g_{k\bar{i}} - \frac{2}{(\operatorname{tr}_{\hat{g}} g)^2} \operatorname{Re}(\sum_{i,j,k} g^{i\bar{i}} \partial_j \hat{g}_{i\bar{j}} \partial_{\bar{k}} g_{k\bar{i}}) + C \operatorname{tr}_g \hat{g}.$$

Assume that H achieves maximum at $(t_0, z_0),\ t_0 > 0$, then $\nabla H(t_0, z_0) = 0$ gives

$$\frac{e^{-\frac{\alpha}{t}} \partial_{\bar{i}} \operatorname{tr}_{\hat{g}} g}{\operatorname{tr}_{\hat{g}} g} - A e^{\Psi} \varphi_{\bar{i}} = 0.$$

That is,

$$\sum_k \partial_{\bar{i}} g_{k\bar{k}} = A e^{\frac{\alpha}{t}} \operatorname{tr}_{\hat{g}} g \varphi_{\bar{i}} e^{\Psi}.$$

Together with $\partial_{\bar{k}} g_{k\bar{i}} = \partial_{\bar{k}} \hat{g}_{k\bar{i}} + \partial_{\bar{i}} g_{k\bar{k}}$, we get

$$\begin{aligned}
|\frac{2}{(\operatorname{tr}_{\hat{g}} g)^2} \operatorname{Re}(\sum_{i,j,k} g^{i\bar{i}} \partial_j \hat{g}_{i\bar{j}} \partial_{\bar{k}} g_{k\bar{i}})| &\leq |\frac{2A e^{\frac{\alpha}{t}} e^{\Psi}}{\operatorname{tr}_{\hat{g}} g} \operatorname{Re} \sum_{i,j} g^{i\bar{i}} \partial_j \hat{g}_{i\bar{j}} \varphi_{\bar{i}}| + C \operatorname{tr}_g \hat{g} \\
&\leq e^{\frac{\alpha}{t}} e^{\Psi} (A^2 |\nabla \varphi|^2 + \frac{C \operatorname{tr}_g \hat{g}}{(\operatorname{tr}_{\hat{g}} g)^2}) + C \operatorname{tr}_g \hat{g} \\
&\leq e^{\frac{\alpha}{t}} e^{\Psi} (A^2 |\nabla \varphi|^2 + C' \operatorname{tr}_g \hat{g}) + C \operatorname{tr}_g \hat{g} \qquad (2.22)
\end{aligned}$$

Using the Cauchy-Schwarz inequality as in Yau's second order estimate [49] (see equation

(2.21) in [44]), we have

$$\frac{1}{\operatorname{tr}_{\hat{g}} g}\sum_{i,j,k} g^{i\bar{i}}\partial_j g_{i\bar{j}}\partial_{\bar{k}} g_{k\bar{i}} \le \sum_{i,j} g^{i\bar{i}} g^{j\bar{j}}\partial_j g_{i\bar{j}}\partial_{\bar{j}} g_{j\bar{i}} \tag{2.23}$$

Combining (2.22) and (2.23), we get

$$\frac{|\partial \operatorname{tr}_{\hat{g}} g|^2}{(\operatorname{tr}_{\hat{g}} g)^2} \le \frac{1}{\operatorname{tr}_{\hat{g}} g}\sum_{i,j} g^{i\bar{i}} g^{j\bar{j}}\partial_j g_{i\bar{j}}\partial_{\bar{j}} g_{j\bar{i}} + e^{\frac{\alpha}{t}} e^{\Psi}(A^2|\nabla\varphi|^2 + C' \operatorname{tr}_g \hat{g}) + C \operatorname{tr}_g \hat{g}.$$

So

$$\begin{aligned} e^{-\frac{\alpha}{t}}\Delta \log \operatorname{tr}_{\hat{g}} g =& e^{-\frac{\alpha}{t}}(\frac{\Delta \operatorname{tr}_{\hat{g}} g}{\operatorname{tr}_{\hat{g}} g} - \frac{|\partial \operatorname{tr}_{\hat{g}} g|^2}{(\operatorname{tr}_{\hat{g}} g)^2}) \\ &\ge \frac{e^{-\frac{\alpha}{t}}}{\operatorname{tr}_{\hat{g}} g}\Delta_{\hat{g}}\dot{\varphi} - C_1\frac{e^{-\frac{\alpha}{t}}}{\operatorname{tr}_{\hat{g}} g}|\nabla\varphi|^2_{\hat{g}} - Ce^{-\frac{\alpha}{t}} \operatorname{tr}_g \hat{g} \\ &\quad - A^2|\nabla\varphi|^2 e^{\Psi} - C' \operatorname{tr}_g \hat{g} e^{\Psi}. \end{aligned} \tag{2.24}$$

From (2.12) we have $\operatorname{tr}_{\hat{g}} g \le C(\operatorname{tr}_g \hat{g})^{n-1}$ for some constant C. Now putting (2.24) into (2.17) and using (2.14) and that φ, $\dot{\varphi}$ are bounded, we have

$$\begin{aligned} (\frac{\partial}{\partial t} - \Delta)H \le& C \log \operatorname{tr}_{\hat{g}} g - A\dot{\varphi}e^{\Psi} + C_1 + Ce^{-\frac{\alpha}{t}} \operatorname{tr}_g \hat{g} \\ &+ C \operatorname{tr}_g \hat{g}e^{\Psi} + Ane^{\Psi} - A \operatorname{tr}_g \hat{g}e^{\Psi} \\ \le& C' \log \operatorname{tr}_{\hat{g}} g + AC'e^{\Psi} - (A - C)e^{\Psi} \operatorname{tr}_g \hat{g} + Ce^{-\frac{\alpha}{t}} \operatorname{tr}_g \hat{g} \\ \le& -(A - C - C_1)e^{\Psi} \operatorname{tr}_g \hat{g} + AC'e^{\Psi}. \end{aligned}$$

Choosing A large enough such that $A - C - C_1 \ge 0$, then at (t_0, z_0),

$$0 \le -(A - C - C_1) \operatorname{tr}_g \hat{g} + AC'.$$

for $t \leq T'$ gives $\operatorname{tr}_g \hat{g} \leq C'$ at (t_0, z_0), which implies that $H \leq C$ for some constant C depending on $\sup|\varphi_0|$ and $\sup|\dot{\varphi}_0|$. Then we obtain the desired estimate (2.16). $\square$

Now we give the third order estimate. Our proof is based on the arguments in [33, 37](see also [48]). As in [49], consider $S = g^{i\bar{p}} g^{q\bar{j}} g^{k\bar{r}} \hat{\nabla}_k \varphi_{i\bar{j}} \hat{\nabla}_{\bar{r}} \varphi_{\bar{p}q}$. We introduce the tensor ${\Phi_{ij}}^k = \Gamma_{ij}^k - \hat{\Gamma}_{ij}^k$ and then

$$S = |\Phi|^2 = g^{i\bar{p}} g^{j\bar{q}} g_{k\bar{r}} {\Phi_{ij}}^k \Phi_{\bar{p}\bar{q}}^{\ \ \bar{r}}.$$

From now on, we will write $k(t), k_1(t), k_2(t), \ldots$ for functions of the form $Ke^{\lambda Ce^{\alpha/t}}$ where $e^{Ce^{\alpha/t}}$ is the bound in Proposition 2.10, and K, λ are constants depending only on $\hat{\omega}$ and F. In the proof of the following proposition, we will use the estimates $|\nabla\varphi(t)|^2_{\hat{g}} \leq k(t)$, $\operatorname{tr}_{\hat{g}} g \leq k(t)$ repeatedly.

Proposition 2.11. *There exists a smooth function $C(t) > 0$ on $(0, T']$ depending only on $\sup|\varphi_0|$, $\sup|\dot{\varphi}_0|$ and blowing up as $t \to 0$ such that $S < C(t)$ for $t \leq T'$.*

Proof. We write $\Delta = g^{p\bar{q}} \nabla_p \nabla_{\bar{q}}$. As the calculations in [33, 37], first we have

$$\begin{aligned}
\Delta S =& |\overline{\nabla}\Phi|^2 + |\nabla\Phi|^2 - {\Phi_{ij}}^k \left({R_{p\ k}^{\ p\ q}} {\Phi^{ij}}_q - {R_{p\ q}^{\ p\ i}} {\Phi^{qj}}_k - {R_{p\ q}^{\ p\ j}} {\Phi^{iq}}_k \right) \\
& + 2\operatorname{Re}\left(\Delta {\Phi_{ij}}^k {\Phi^{ij}}_k \right), \\
\frac{\partial}{\partial t} S =& {\Phi_{ij}}^k \left(\frac{\partial}{\partial t} g^{i\bar{q}} {\Phi_{\bar{q}\ k}^{\ j}} + \frac{\partial}{\partial t} g_{k\bar{q}} \Phi^{ij\bar{q}} + \frac{\partial}{\partial t} g^{j\bar{q}} {\Phi^i}_{\bar{q}k} \right) + 2\operatorname{Re}\left(\frac{\partial}{\partial t} {\Phi_{ij}}^k {\Phi^{ij}}_k \right) \\
=& {\Phi_{ij}}^k \left(g^{q\bar{r}} \frac{\partial}{\partial t} g_{k\bar{r}} {\Phi^{ij}}_q - g^{i\bar{r}} \frac{\partial}{\partial t} g_{q\bar{r}} {\Phi^{qj}}_k - g^{j\bar{r}} \frac{\partial}{\partial t} g_{q\bar{r}} {\Phi^{iq}}_k \right) \\
& + 2\operatorname{Re}\left(\frac{\partial}{\partial t} {\Phi_{ij}}^k {\Phi^{ij}}_k \right).
\end{aligned}$$

Thus

$$(\frac{\partial}{\partial t}-\Delta)S = -|\overline{\nabla}\Phi|^2 - |\nabla\Phi|^2 + {\Phi_{ij}}^k\left({B_k}^q{\Phi^{ij}}_q - {B_q}^i{\Phi^{qj}}_k - {B_q}^j{\Phi^{iq}}_k\right)$$
$$+ 2\,\mathrm{Re}\left((\frac{\partial}{\partial t}-\Delta){\Phi_{ij}}^k{\Phi^{ij}}_k\right) \tag{2.25}$$

where $B_i^j = g^{j\bar{r}}\frac{\partial}{\partial t}g_{i\bar{r}} + {R_p}^{p}{}_i{}^j$. From equation (2.10) and formula (2.3), we get

$$\frac{\partial}{\partial t}g_{i\bar{j}} = -R_{i\bar{j}} + \hat{R}_{i\bar{j}} + F_{i\bar{j}}(\varphi, z),$$
$${R_p}^{p}{}_k{}^q = {R_k}^q - \nabla^p {T_{pk}}^q - \nabla_k {T^{pq}}_p.$$

Hence

$${B_i}^j = g^{j\bar{r}}(\hat{R}_{i\bar{r}} + F_{i\bar{r}}(\varphi, z)) - \nabla^p {T_{pi}}^j - \nabla_i {T^{pj}}_p. \tag{2.26}$$

Here $F_{i\bar{r}}(\varphi, z) = F_{i\bar{r}} + F''\varphi_i\varphi_{\bar{r}} + F'\varphi_{i\bar{r}} + F'_i\varphi_{\bar{r}} + F'_{\bar{r}}\varphi_i$.

Now we compute the evolution of ${\Phi_{ij}}^k$. First

$$\begin{aligned}\frac{\partial}{\partial t}{\Phi_{ij}}^k &= g^{k\bar{l}}\nabla_i\frac{\partial}{\partial t}g_{j\bar{l}}\\ &= -\nabla_i {R_j}^k + g^{k\bar{l}}(\nabla_i\hat{R}_{j\bar{l}} + \nabla_i F_{j\bar{l}}(\varphi, z)).\end{aligned}$$

Note that

$$\nabla_{\bar{q}}{\Phi_{ij}}^k = -{R_{i\bar{q}j}}^k + {\hat{R}_{i\bar{q}j}}{}^k. \tag{2.27}$$

Then

$$\begin{aligned}\Delta \Phi_{ij}{}^{k} &= -\nabla^{\bar p} R_{i\bar p j}{}^{k} + \nabla^{\bar p} \hat R_{i\bar p j}{}^{k} \\ &= \nabla_i(-R_j{}^k + \nabla^q T_{qj}{}^k + \nabla_j T^{pk}{}_p) - T_{iq}{}^r R_r{}^q{}_j{}^k + \nabla^{\bar p} \hat R_{i\bar p j}{}^{k}.\end{aligned}$$

So we have

$$\begin{aligned}(\frac{\partial}{\partial t} - \Delta)\Phi_{ij}{}^k &= \nabla_i (g^{k\bar l}(\hat R_{j\bar l} + F_{j\bar l}(\varphi, z)) - \nabla^q T_{qj}{}^k - \nabla_j T^{pk}{}_p) + T_{iq}{}^r R_r{}^q{}_j{}^k - \nabla^{\bar p} \hat R_{i\bar p j}{}^{k} \\ &= \nabla_i B_j{}^k + T_{iq}{}^r R_r{}^q{}_j{}^k - \nabla^{\bar p} \hat R_{i\bar p j}{}^{k}.\end{aligned}$$

Combining with (2.25), we get

$$\begin{aligned}(\frac{\partial}{\partial t} - \Delta) S = &- |\overline\nabla \Phi|^2 - |\nabla \Phi|^2 + \Phi_{ij}{}^k \left(B_k{}^q \Phi^{ij}{}_q - B_q{}^i \Phi^{qj}{}_k - B_q{}^j \Phi^{iq}{}_k \right) \\ &+ 2\,\mathrm{Re}\left(\nabla_i B_j{}^k + T_{iq}{}^r R_r{}^q{}_j{}^k - \nabla^{\bar p} \hat R_{i\bar p j}{}^k \right) \Phi^{ij}{}_k.\end{aligned}$$

As $T_{ij\bar k} = \hat T_{ij\bar k}$,

$$\nabla^p T_{pk}{}^q = g^{p\bar l} g^{q\bar r} (\hat\nabla_{\bar l} \hat T_{pk\bar r} - \Phi_{\bar l \bar r}{}^{\bar s} \hat T_{pk\bar s}). \tag{2.28}$$

By (2.26)

$$|B_{i\bar j}| \le k(t)(S^{1/2} + 1 + |\nabla\varphi|_g^2 + |\varphi_{i\bar j}|_g^2) \le k(t)(S^{1/2} + 1).$$

Now we want to control $\nabla_i B_j{}^k$. From (2.26) we need the following estimates from [37] obtained by similar calculations as (2.28),

$$|\nabla_i \nabla^q T_{qj}^{\ k}| \le k(t)(S + |\overline\nabla \Phi| + 1),$$

$$|\nabla_i \nabla_j T_{\bar p}^{\ k\bar p}| \le k(t)(S + |\nabla \Phi| + 1).$$

Also

$$|T_{iq}^{\ \ r} R_{r\ \ j}^{q\ \ k}| \le k(t)(|\overline{\nabla}\Phi| + 1),$$

$$|\nabla^{\bar{p}} R_{\bar{p}ij}^{\ \ \ \ k}| \le k(t)(S^{1/2} + 1).$$

We bound the terms with φ_{ij} and $\Phi^{ij}_{\ \ k}$ in $\mathrm{Re}(\nabla_i B_j^{\ k} \Phi^{ij}_{\ \ k})$ by $|\varphi_{ij}|^2 + k(t)S$. Together with the above estimates we get

$$(\frac{\partial}{\partial t} - \Delta)S \le k(t)(S^{3/2} + S + 1) + \sum_{i,j} |\varphi_{ij}|^2 - \frac{1}{2}(|\nabla\Phi|^2 + |\overline{\nabla}\Phi|^2). \tag{2.29}$$

We will use a similar way as in [35, 37] to control the term $S^{3/2}$. The evolution equations below can be obtained by following the computations in [35, 42],

$$\begin{aligned}(\frac{\partial}{\partial t} - \Delta)\,\mathrm{tr}_{\hat{g}}\, g &\le -\frac{S}{k_2(t)} + k_2(t),\\ (\frac{\partial}{\partial t} - \Delta)|\nabla\varphi|^2_{\hat{g}} &\le -\sum_{i,j} \frac{|\varphi_{ij}|^2}{k_3(t)} + k_3(t).\end{aligned} \tag{2.30}$$

Now we will apply a maximum principle argument to the quantity

$$H = \frac{S}{(C_1(t) - \mathrm{tr}_{\hat{g}}\, g)^2} + \frac{\mathrm{tr}_{\hat{g}}\, g}{C_2(t)} + \frac{|\nabla\varphi|^2_{\hat{g}}}{C_3(t)}.$$

Here we can take $C_i(t)$ to be the form of $Le^{\lambda C e^{\alpha/t}}$ where $C,\ \alpha$ are the same as in (2.16) and L, λ will be determined later. Let $L,\ \lambda > 2$ such that

$$\frac{C_1(t)}{2} \le C_1(t) - \mathrm{tr}_{\hat{g}}\, g \le C_1(t), \quad 0 < -\frac{C_i'(t)}{C_i^2(t)} \le \frac{1}{\sqrt{C_i(t)}}, \quad i = 1, 2, 3. \tag{2.31}$$

We calculate the evolution of H.

$$
\begin{aligned}
(\frac{\partial}{\partial t}-\Delta)H =& \frac{1}{(C_1(t)-\operatorname{tr}_{\hat{g}} g)^2}(\frac{\partial}{\partial t}-\Delta)S+\frac{2S}{(C_1(t)-\operatorname{tr}_{\hat{g}} g)^3}(\frac{\partial}{\partial t}-\Delta)\operatorname{tr}_{\hat{g}} g \\
&-\frac{4\operatorname{Re}\nabla\operatorname{tr}_{\hat{g}} g\cdot\overline{\nabla}S}{(C_1(t)-\operatorname{tr}_{\hat{g}} g)^3}-\frac{6S|\nabla\operatorname{tr}_{\hat{g}} g|^2}{(C_1(t)-\operatorname{tr}_{\hat{g}} g)^4}-\frac{2C_1'(t)S}{(C_1(t)-\operatorname{tr}_{\hat{g}} g)^3} \\
&+\frac{1}{C_2(t)}(\frac{\partial}{\partial t}-\Delta)\operatorname{tr}_{\hat{g}} g+\frac{1}{C_3(t)}(\frac{\partial}{\partial t}-\Delta)|\nabla\varphi|^2_{\hat{g}}-\frac{C_2'(t)\operatorname{tr}_{\hat{g}} g}{C_2(t)^2}-\frac{C_3'(t)|\nabla\varphi|^2_{\hat{g}}}{C_3(t)^2}.
\end{aligned}
$$

Taking $C_2(t), C_3(t)$ large enough and using (2.14), (2.16) and (2.31), the last two terms can be bounded by a constant C. Assuming $S>1$ at the maximum point of H, from (2.29) we have

$$
(\frac{\partial}{\partial t}-\Delta)S\leq k_1(t)(S^{3/2}+1)+\sum_{i,j}|\varphi_{ij}|^2-\frac{1}{2}|\overline{\nabla}\Phi|^2.
$$

Together with (2.30) and (2.31), we get

$$
\begin{aligned}
0\leq\ & (\frac{\partial}{\partial t}-\Delta)H \\
\leq\ & \left(\frac{4k_1(t)}{C_1^2(t)}S^{3/2}+\frac{4k_1(t)}{C_1^2(t)}+\frac{4}{C_1^2(t)}\sum_{i,j}|\varphi_{ij}|^2-\frac{|\overline{\nabla}\Phi|^2}{2C_1^2(t)}\right)+\left(-\frac{2S^2}{k_2(t)C_1^3(t)}+\frac{16k_2(t)S}{C_1^3(t)}\right) \\
&+\frac{4|\operatorname{Re}\nabla\operatorname{tr}_{\hat{g}} g\cdot\overline{\nabla}S|}{(C_1(t)-\operatorname{tr}_{\hat{g}} g)^3}+\frac{2S}{\sqrt{C_1^3(t)}}+\left(-\frac{1}{k_2(t)C_2(t)}S+\frac{k_2(t)}{C_2(t)}\right) \\
&+\left(-\frac{1}{k_3(t)C_3(t)}\sum_{i,j}|\varphi_{ij}|^2+\frac{k_3(t)}{C_3(t)}\right)+C.
\end{aligned}
$$

As $|\nabla\operatorname{tr}_{\hat{g}} g|\leq\frac{1}{64}k_5(t)S^{1/2}$ and $|\overline{\nabla}S|\leq 2S^{1/2}|\overline{\nabla}\Phi|$,

$$
\frac{4|\operatorname{Re}\nabla\operatorname{tr}_{\hat{g}} g\cdot\overline{\nabla}S|}{(C_1(t)-\operatorname{tr}_{\hat{g}} g)^3}\leq\frac{k_5(t)S|\overline{\nabla}\Phi|}{C_1^3(t)}\leq\frac{|\overline{\nabla}\Phi|^2}{2C_1^2(t)}+\frac{k_5^2(t)S^2}{2C_1^4(t)}.
$$

We will also use

$$\frac{4k_1(t)S^{3/2}}{C_1^2(t)} \leq \frac{S^2}{k_2(t)C_1^3(t)} + \frac{4k_1^2(t)k_2(t)S}{C_1(t)}.$$

Recall that all $k_i(t), C_i(t)$ are functions of the form $Le^{\lambda Ce^{\alpha/t}}$. First choose $C_i(t) > k_i(t)$, then fix $C_2(t), C_3(t)$. Now take the constant L, λ in $C_1(t)$ to be large enough such that $\frac{k_5^2(t)}{2C_1^4(t)} \leq \frac{1}{k_2(t)C_1^3(t)}$, $\frac{4}{C_1^2(t)} \leq \frac{1}{k_3(t)C_3(t)}$ and $\frac{16k_2(t)}{C_1^3(t)} + \frac{2}{\sqrt{C_1^3(t)}} + \frac{4k_1^2(t)k_2(t)}{C_1(t)} \leq \frac{1}{2k_2(t)C_2(t)}$. The above estimates then give that at (t_0, z_0),

$$0 \leq \frac{-1}{2k_2(t)C_2(t)}S + C',$$

for some constant C'. Therefore $S \leq 4C'k_2(t)C_2(t) \leq C'C_1(t)$ at (t_0, z_0). It follows that H is bounded by some constant C depending only on $\sup|\varphi_0|$ and $\sup|\dot{\varphi}_0|$, which gives the desired estimate of S. $\square$

Using (2.16), the above estimate $S \leq C(t)$ implies that $\|\varphi(t)\|_{C^{2+\alpha}(M,g)}$ can be bounded by a smooth function $C(t)$ on $(0, T']$, which depends only on $\sup|\varphi_0|$ and $\sup|\dot{\varphi}_0|$. Differentiating the equation (2.10) in t, we get

$$\frac{\partial\dot{\varphi}}{\partial t} = \triangle\dot{\varphi} + F'(\varphi, z)\dot{\varphi} \tag{2.32}$$

To apply parabolic Schauder estimates to obtain higher order estimates, we still need to bound the derivatives of $g_{i\bar{j}}$ in the t-direction (cf. [6]). Then it is sufficient to bound $|\operatorname{Ric}(g)|$.

Lemma 2.12. *There exists a smooth function $C(t) > 0$ on $(0, T']$ depending only on $\sup|\varphi_0|$ and $\sup|\dot{\varphi}_0|$ and blowing up as $t \to 0$ such that $|\operatorname{Ric}| < C(t)$ for $t \leq T'$.*

Proof. To compute the evolution of $|\operatorname{Ric}|$, first

$$\frac{\partial}{\partial t}R_{j\bar{k}} = -g^{l\bar{q}}\nabla_{\bar{k}}\nabla_j \frac{\partial}{\partial t}g_{l\bar{q}} = -g^{l\bar{q}}\nabla_{\bar{k}}\nabla_j(-R_{l\bar{q}} + \hat{R}_{l\bar{q}} + F_{l\bar{q}}(\varphi, z)).$$

Using (2.2) and (2.3), we have

$$\begin{aligned}\nabla_{\bar{k}}\nabla_j R_{l\bar{q}} = \nabla_l\nabla_{\bar{q}}R_{j\bar{k}} - \nabla_l T_{\bar{k}\bar{q}}{}^{\bar{s}}R_{j\bar{s}} + T_{\bar{k}\bar{q}}{}^{\bar{s}}\nabla_l R_{j\bar{s}} + R_{l\bar{k}j}{}^{r}R_{r\bar{q}}\\ - R_{l\bar{k}}{}^{\bar{s}}{}_{\bar{q}}R_{j\bar{s}} + \nabla_{\bar{k}}T_{lj}^{r}R_{r\bar{q}} + T_{lj}^{r}\nabla_{\bar{k}}R_{r\bar{q}}.\end{aligned}$$

So

$$\begin{aligned}(\frac{\partial}{\partial t} - \Delta)R_{j\bar{k}} =& \nabla_{\bar{k}}T_{lj}^{r}R_r{}^{l} + T_{lj}^{r}\nabla_{\bar{k}}R_r{}^{l} + R_{l\bar{k}j}{}^{r}R_r{}^{l}\\ &- R_{l\bar{k}}{}^{\bar{s}l}R_{j\bar{s}} + \nabla^{\bar{q}}T_{\bar{k}\bar{q}}{}^{\bar{s}}R_{j\bar{s}} + T_{\bar{k}\bar{q}}{}^{\bar{s}}\nabla^{\bar{q}}R_{j\bar{s}}\\ &- g^{l\bar{q}}\nabla_{\bar{k}}\nabla_j(\hat{R}_{l\bar{q}} + F_{l\bar{q}}(\varphi, z)).\end{aligned}$$

From (2.28) we get

$$|\nabla T| \le k(t)(1 + S^{1/2}), \quad |\overline{\nabla}T| \le k(t)(1 + S^{1/2}).$$

where S is bounded by some $k(t)$ by Proposition 2.11. Note that (2.27) gives

$$R_{i\bar{j}} = \hat{R}_{i\bar{j}} + \nabla_{\bar{j}}\Phi_{ik}{}^{k}, \quad |\overline{\nabla}\Phi| \le |\operatorname{Rm}| + k(t). \tag{2.33}$$

Use this and similar calculation as (2.28) to get

$$|g^{l\bar{q}}\nabla_{\bar{k}}\nabla_j\hat{R}_{l\bar{q}}| \le k(t)(|\operatorname{Rm}| + 1).$$

Also we have

$$|g^{l\bar{q}}\nabla_{\bar{k}}\nabla_j F_{l\bar{q}}(\varphi, z)| \leq k(t)(|\operatorname{Rm}| + 1).$$

Therefore

$$\begin{aligned}|(\frac{\partial}{\partial t} - \Delta)R_{j\bar{k}}| &\leq k(t)(|\nabla \operatorname{Ric}| + |\operatorname{Rm}|^2 + |\operatorname{Rm}| + 1) \\ &\leq k(t)(|\nabla \operatorname{Ric}| + |\operatorname{Rm}|^2 + 1).\end{aligned}$$

As $|\frac{\partial}{\partial t}g_{i\bar{j}}| = | - R_{i\bar{j}} + \hat{R}_{i\bar{j}} + F_{i\bar{j}}(\varphi, z)| \leq |\operatorname{Ric}| + k(t)$, direct computation gives

$$(\frac{\partial}{\partial t} - \Delta)|\operatorname{Ric}|^2 \leq k(t)(|\operatorname{Ric}|^3 + |\operatorname{Ric}|^2) + 2|(\frac{\partial}{\partial t} - \Delta)\operatorname{Ric}||\operatorname{Ric}| - 2|\nabla \operatorname{Ric}|^2.$$

We then obtain the following

$$\begin{aligned}(\frac{\partial}{\partial t} - \Delta)|\operatorname{Ric}| &= \frac{1}{2|\operatorname{Ric}|}(\frac{\partial}{\partial t} - \Delta)|\operatorname{Ric}|^2 + 2|\nabla|\operatorname{Ric}||^2) \\ &\leq k_1(t)(|\nabla \operatorname{Ric}| + |\operatorname{Rm}|^2 + 1) - \frac{|\nabla \operatorname{Ric}|^2}{|\operatorname{Ric}|} + \frac{|\nabla|\operatorname{Ric}||^2}{|\operatorname{Ric}|}.\end{aligned}$$

Let us consider

$$H = \frac{|\operatorname{Ric}|}{C_1(t)} + \frac{S}{C_2(t)},$$

as in [41] where $C_1(t), C_2(t)$ are the functions of the form $Le^{\lambda C e^{\alpha/t}}$ as in the proof of Propostion 2.11 such that $-\frac{C_i'(t)}{C_i^2(t)} \leq \frac{1}{\sqrt{C_i(t)}}, i = 1, 2$. Assume that H achieves maximum at a point (t_0, z_0), $t_0 > 0$, and assume $|\operatorname{Ric}| \geq 1$ at (t_0, z_0). From (2.29) and Proposition 2.10, 2.11 we have

$$(\frac{\partial}{\partial t} - \Delta)S \leq -\frac{1}{2}Q + k_2(t)$$

where $Q = |\nabla\Phi|^2 + |\overline{\nabla}\Phi|^2$. Take $C_1(t) > k_1(t), C_2(t) \geq \max\{S, S^2, k_2(t)\}$. Direct computation gives

$$
\begin{aligned}
(\frac{\partial}{\partial t} - \Delta)H &\leq \frac{k_1(t)(|\nabla \operatorname{Ric}| + |\operatorname{Rm}|^2)}{C_1(t)} - \frac{|\nabla \operatorname{Ric}|^2}{C_1(t)|\operatorname{Ric}|} + \frac{|\nabla|\operatorname{Ric}||^2}{C_1(t)|\operatorname{Ric}|} \\
&\quad + \frac{|\operatorname{Ric}|}{\sqrt{C_1(t)}} + \left(-\frac{Q}{2C_2(t)} + \frac{k_2(t)}{C_2(t)}\right) + \frac{S}{\sqrt{C_2(t)}} \\
&\leq \frac{k_3(t)|\operatorname{Rm}|^2}{C_1(t)} - \frac{|\nabla \operatorname{Ric}|^2}{2C_1(t)|\operatorname{Ric}|} + \frac{|\nabla|\operatorname{Ric}||^2}{C_1(t)|\operatorname{Ric}|} - \frac{Q}{2C_2(t)} + C \qquad (2.34)
\end{aligned}
$$

where the last inequality we use

$$
\frac{k_1(t)|\nabla \operatorname{Ric}|}{C_1(t)} \leq \frac{|\nabla \operatorname{Ric}|^2}{2C_1(t)|\operatorname{Ric}|} + \frac{k_1^2(t)|\operatorname{Ric}|}{2C_1(t)}.
$$

Using $\nabla H = 0$ at (t_0, z_0) and $|\nabla|\operatorname{Ric}|| \leq |\nabla \operatorname{Ric}|$, we get

$$
\begin{aligned}
\frac{|\nabla|\operatorname{Ric}||^2}{C_1(t)|\operatorname{Ric}|} &= \frac{|\nabla S \cdot \overline{\nabla}|\operatorname{Ric}||}{C_2(t)|\operatorname{Ric}|} \\
&\leq \frac{|\nabla \operatorname{Ric}|^2}{2C_1(t)|\operatorname{Ric}|} + \frac{C_1(t)|\nabla S|^2}{2C_2^2(t)|\operatorname{Ric}|} \\
&\leq \frac{|\nabla \operatorname{Ric}|^2}{2C_1(t)|\operatorname{Ric}|} + \frac{C_1(t)k_4(t)Q}{C_2^2(t)|\operatorname{Ric}|}
\end{aligned}
$$

where the last inequality we use $|\nabla S|^2 \leq 2S(|\overline{\nabla}\Phi|^2 + |\nabla\Phi|^2)$. From (2.27),

$$
|\operatorname{Rm}|^2 \leq \frac{3}{2}Q + k_5(t), \quad |\operatorname{Ric}| \leq \sqrt{Q} + k_6(t).
$$

Choose $C_2(t) \geq 8k_4(t)$. Fix $C_2(t)$ and choose $C_1(t) \geq \max\{k_3(t)k_5(t), k_6(t)\}$ large enough such that $\frac{3k_3(t)}{2C_1(t)} \leq \frac{1}{4C_2(t)}$ and then fix $C_1(t)$. Combining the above estimates, we obtain that at (x_0, t_0),

$$
0 \leq (\frac{\partial}{\partial t} - \Delta)H \leq -\frac{Q}{4C_2(t)} + \frac{C_1(t)Q}{8C_2(t)|\operatorname{Ric}|} + C'
$$

for some constant C'. If $\frac{|\,\mathrm{Ric}\,|}{C_1(t)} \leq 1$, then $H \leq 2$ at (t_0, z_0) and we obtain the estimate for $|\,\mathrm{Ric}\,|$. Otherwise at (t_0, z_0)

$$0 \leq -\frac{Q}{8C_2(t)} + C'.$$

Therefore $Q \leq 8C'C_2(t) \leq 8C'C_1(t)$ at (t_0, z_0). By our choice of $C_1(t), C_2(t)$, H is bounded by some constant C depending only on $\sup|\varphi_0|$ and $\sup|\dot{\varphi}_0|$, which gives the bound for $|\,\mathrm{Ric}\,|$. $\square$

The estimates we have obtained imply that the parabolic $C^{\alpha,\alpha/2}$ norm of the coefficients in equation (2.32) can be bounded. The parabolic Schauder estimates then give a $C^{2+\alpha,1+\alpha/2}$ bound for $\dot{\varphi}$ in $[\epsilon, T']\times$ M for any $\epsilon > 0$, with the bounds only depending on ϵ, $\sup|\varphi_0|$ and $\sup|\dot{\varphi}_0|$. Similarly we can obtain a $C^{2+\alpha,1+\alpha/2}$ bound for $\varphi_k, \varphi_{\bar{k}}$ in $[\epsilon, T'] \times M$. Differentiating the flow again and repeatedly using Schauder estimates, we obtain all higher order estimates for φ. Let $\epsilon \to 0$, we obtain the bounds in Proposition 2.6 which blow up as $t \to 0$. Particularly, there exists a smooth solution on $[0, T]$ where T is the same as in Lemma 2.7 and depends only on $\sup|\varphi_0|$ and F.

2.4 Proof of Theorem 2.5

Assume that $\hat{\omega}$ satisfies the condition (2.7), then it follows from [28] that Kołodziej's stability result (Corollary 4.4 in [26]) is also true. In particular if

$$(\hat{\omega} + \sqrt{-1}\partial\bar{\partial}\phi_1)^n = (\hat{\omega} + \sqrt{-1}\partial\bar{\partial}\phi_2)^n = f\hat{\omega}^n,$$

with $f \geq 0 \in L^p(M, \hat{\omega})$, $p > 1$ and $\int_M f\hat{\omega}^n = \int_M \hat{\omega}^n$, then $\phi_1 - \phi_2 = const$. Now suppose that $\phi \in PSH(M, \hat{\omega}) \cap L^\infty(M)$ is a weak solution of the equation

$$(\hat{\omega}+\sqrt{-1}\partial\bar{\partial}\phi)^n = e^{-F(\phi,z)}\hat{\omega}^n, \tag{2.35}$$

then $f(z) = e^{-F(\phi(z),z)} \in L^p(M,\hat{\omega}),\ p > 1$ as ϕ is bounded for $t \leq T$. Also the condition (2.7) gives that $\int_M f\hat{\omega}^n = \int_M \hat{\omega}^n$. Therefore Theorem 5.2 in [12] shows that ϕ is continuous. Approximate ϕ with a sequence of smooth functions ϕ_j such that

$$\sup_M |\phi_j - \phi| \to 0, \quad as \quad j \to \infty. \tag{2.36}$$

It follows from [45] there exist smooth functions ψ_j such that

$$(\hat{\omega}+\sqrt{-1}\partial\bar{\partial}\psi_j)^n = c_j e^{-F(\phi_j,z)}\hat{\omega}^n \tag{2.37}$$

where $c_j > 0$ are constants chosen to satisfy the integration equality of the above equation. From assumption (2.7) we have $c_j \to 1$ as $j \to \infty$. Normalize ψ_j as in [26]

$$\sup(\psi_j - \phi) = \sup(\phi - \psi_j). \tag{2.38}$$

The stability result [28] gives

$$\lim_{j\to\infty} \|\psi_j - \phi\|_{L^\infty} = 0. \tag{2.39}$$

Consider the equations

$$\frac{\partial \varphi_j}{\partial t} = \log \frac{(\hat{\omega}+\sqrt{-1}\partial\bar{\partial}\varphi_j)^n}{\hat{\omega}^n} + F(\varphi_j, z) - \log c_j. \tag{2.40}$$

Applying Proposition 2.6, there exist a sequence of smooth functions φ_j with $\varphi_j(0) = \psi_j$ such that φ_j solves the equations on $[0, T_j]$ where T_j depends only on $\sup|\psi_j|$ and

$\sup|\dot{\varphi}_j(0)|$. Using (2.37) and (2.40),

$$\dot{\varphi}_j(0) = F(\psi_j, z) - F(\phi_j, z). \tag{2.41}$$

It follows from (2.36) and (2.39) that $\sup|\psi_j|$ and $\sup|\dot{\varphi}_j(0)|$ can be bounded by a constant depending only on $\sup|\phi|$. Therefore there exists a $T > 0$ independent of j such that φ_j solve the equation (2.40) on $[0, T]$. By Lemma 6 in [42], $\{\varphi_j\}$ is a Cauchy sequence in $C^0([0, T] \times M)$. Let

$$\beta(t, z) = \lim_{j\to\infty} \varphi_j,$$

which is continuous on $[0, T] \times M$. For any $\epsilon > 0$, from the proof of Proposition 2.6, we have bounds on all derivatives of φ_j for $t \in [\epsilon, T]$. Then $\beta \in C^\infty([\epsilon, T] \times M)$ and

$$\lim_{j\to\infty} \|\beta - \varphi_j\|_{C^k([\epsilon,T]\times M)} = 0.$$

Lemma 2.7 gives that $|\dot{\varphi}_j(t)| \leq \sup|\dot{\varphi}_j(0)|e^{Ct}$, for $t \in [0, T]$. From (2.41) we get

$$\dot{\varphi}_j(0) \to 0 \;\; as \;\; j \to \infty.$$

Therefore for any $t > 0$,

$$\dot{\beta}(t) = \lim_{j\to\infty} \dot{\varphi}_j(t) = 0.$$

As it is continuous on $[0, T]$, we have $\beta(0) = \beta(t)$ for $t \in (0, T]$ is smooth. But $\beta(0) = \lim_{j\to\infty} \varphi_j(0) = \lim_{j\to\infty} \psi_j = \phi$, thus we get the smoothness of ϕ.

Chapter 3

Chern-Ricci flow with rough initial data

In this chapter, we study weak solutions of the Chern-Ricci flow with rough initial data and investigate the behavior of the Chern-Ricci flow on a compact complex surface after contracting exceptional curves. In section 3.1, we provide some background on the Chern-Ricci flow and describe the canonical surgical contraction conjecture by Tosatti and Weinkove. In section 3.2, we prove the smoothing property for the Chern-Ricci flow, Theorem 3.3. In section 3.3.1, we obtain the existence of weak solutions of the Chern-Ricci flow through blow downs of exceptional curves. Then we prove global higher order estimates for the evolving metrics as $t \to T^-$. In section 3.3.3, we obtain sharper estimates after the singular time to establish the smooth convergence and then complete the proof of Theorem 3.11 .

3.1 Background

In this section we collect some results on the Chern-Ricci flow and describe the canonical surgical contraction conjecture precisely.

3.1.1 Chern-Ricci flow

Let M be an n-dimensional compact complex manifold. Let g_0 be a Hermitian metric on M and ω_0 the associated (1, 1) form. The Chern-Ricci flow starting at ω_0 is a flow of Hermitian metrics given by

$$\frac{\partial}{\partial t}\omega = -\operatorname{Ric}(\omega), \qquad \omega|_{t=0} = \omega_0, \tag{3.1}$$

where $\operatorname{Ric}(\omega) = -\sqrt{-1}\partial\bar{\partial}\log\det g$ is the Chern-Ricci form of ω which represents the first Chern class of M (up to a factor of 2π). The flow was introduced by Gill [15] on manifolds with vanishing first Bott-Chern class and investigated by Tosatti and Weinkove [46, 47] (see also [16, 48]) on general Hermitian manifolds.

A fundamental result on the Chern-Ricci flow is the explicit characterization of the maximal existence time using the initial metric ω_0 and its Chern-Ricci form. Note that equation (3.1) can be written as

$$\frac{\partial}{\partial t}\omega = -\operatorname{Ric}(\omega_0) + \sqrt{-1}\partial\bar{\partial}\beta(t),$$

where $\beta(t) = \log\frac{\det g(t)}{\det g_0}$. Thus a solution of the flow (3.1) must be of the form

$$\omega(t) = \omega_0 - t\operatorname{Ric}(\omega_0) + \sqrt{-1}\partial\bar{\partial}\rho(t)$$

for some function $\rho(t)$. Let

$$T = \sup\{t \geq 0 | \exists \psi \in C^\infty(M),\ \ \omega_0 - t\,\mathrm{Ric}(\omega_0) + \sqrt{-1}\partial\bar{\partial}\psi > 0\}.$$

Then it was shown in [46] the following theorem.

Theorem 3.1. (Tosatti-Weinkove) *There exists a unique maximal solution to the Chern-Ricci flow (3.1) on [0, T).*

In the case when ω_0 is Kähler, this is obtained by the result of Tian-Zhang in [43], where $T = \sup\{t \geq 0 | [\omega_0] - tc_1(M) > 0\}$ is determined by the cohomology class of ω_0 and the first Chern class.

In the case of a complex surface, we can always start the Chern-Ricci flow at a $\partial\bar{\partial}$-closed (Gauduchon) metric. Tosatti and Weinkove give the following geometric characterization of the maximal existence time T.

Theorem 3.2. (Tosatti-Weinkove [46]) *Let M be a compact complex surface and ω_0 a Gauduchon metric. Let T be the maximal existence time of the Chern-Ricci flow (3.1) starting at ω_0. Then*

(i) If $T = \infty$, then M is minimal.

(ii) If $T < \infty$ and the volume of M goes to zero as $t \to T^-$, then M is either birational to a ruled surface or it is a surface of class VII.

(iii) If $T < \infty$ and the volume of M stays positive, then M contains (-1)-curves.

Furthermore, if M is minimal then $T = \infty$ unless M is $\mathbb{CP}^2$, a ruled surface, a Hopf surface or a surface of class VII with $b_2 > 0$, in which cases (ii) holds.

We say that the Chern-Ricci flow is *collapsing* (*non-collapsing*) at T if the volume of M with respect to $\omega(t)$ goes to zero (stays positive) as $t \to T^-$.

3.1.2 Canonical surgical contraction conjecture

Consider the Chern-Ricci flow

$$\frac{\partial}{\partial t}\omega = -\operatorname{Ric}(\omega), \qquad \omega|_{t=0} = \omega_0, \tag{3.2}$$

starting at a Gauduchon metric ω_0 on a compact complex surface M. Suppose that the flow (3.2) is non-collapsing in finite time. It was shown in [46] that M contains finitely many disjoint (-1)-curves $E_1, ...E_k$ giving a map $\pi : M \to N$ onto a complex surface contracting each E_i to a point $y_i \in N$. Denote $M' = M \backslash \cup_{i=1}^k E_i$ and $N' = N \backslash \{y_1, ..., y_k\}$. A natural conjecture by Tosatti and Weinkove is that in our setup, the Chern-Ricci flow will blow down finitely many (-1)-curves and continue in a unique way on N. More precisely, they conjecture that $g(t)$ performs a *canonical surgical contraction*. That is, the following occurs:

(i) As $t \to T^-$, the metrics $g(t)$ converge to a smooth Gauduchon metric g_T on M' smoothly on compact subsets of M'.

(ii) Let d_{g_T} be the distance function on N' given by $(\pi^{-1})^* g_T$. Then there is a unique metic d_T on N extending d_{g_T} such that (N, d_T) is a compact metric space homeomorphic to the manifold N and (N, d_T) is the metric completion of (N', d_{g_T}).

(iii) $(M, g(t))$ converges to (N, d_T) in the Gromov-Hausdorff sense as $t \to T^-$.

(iv) There exists a unique smooth maximal solution $g(t)$ of the Chern-Ricci flow on N for $t \in (T, T_N)$ with $T < T_N \leq \infty$ such that $g(t)$ converges to $(\pi^{-1})^* g_T$ on N' smoothly on compact subsets of N' as $t \to T^+$.

(v) $(N, g(t))$ converges to (N, d_T) in the Gromov-Hausdorff sense as $t \to T^+$.

In the case of the Käler-Ricci flow, the above results were proved by Song and Weinkove in [39, 40], where they use the terminology *canonical surgical contraction.* For the Chern-Ricci flow, Tosatti and Weinkove proved (i) in ([47], Theorem 1.2). In addition, assuming that

$$\omega_0 - T\,\mathrm{Ric}(\omega_0) + \sqrt{-1}\partial\bar{\partial} f = \pi^*\beta \tag{3.3}$$

for some $f \in C^\infty(M, \mathbb{R})$ and a smooth (1,1) form β on N, they also obtain (ii) and (iii) ([46], Theorem 1.3) except identifying (N, d_T) with the metric completion of (N', d_{g_T}).

As the current $(\pi^{-1})^* g_T$ may not be smooth, to continue the flow on N, first one need to prove that the Chern-Ricci flow with rough initial data has a unique smooth solution on $(0, T']$ for some $T' > 0$. We will show the existence and uniqueness of such weak solutions on an n-dimensional compact Hermitian manifold (under a condition) in the next section and then return to the setup of the above conjecture to establish (iv) in section 3.3.

3.2 Chern-Ricci flow with rough initial data

In this section, we prove the smoothing property for the Chern-Ricci flow with rough initial data. First we give some notations.

Let (M, ω_0) be an n-dimensional compact Hermitian manifold. Let Ω be a smooth volume form on M. Suppose that $\varphi_0 \in PSH_p(M, \omega_0, \Omega)$ for some $p > 1$, where

$$PSH_p(\omega_0, \Omega) = \{\varphi \in PSH(M, \omega_0) \cap L^\infty(M) | \frac{(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi)^n}{\Omega} \in L^p(M)\}.$$

Assume that ω_0 satisfies the condition:

$$\forall\ u \in PSH(M,\omega_0) \cap L^\infty(M), \quad \int_M (\omega_0 + \sqrt{-1}\partial\bar{\partial}u)^n = \int_M \omega_0^n, \tag{3.4}$$

then using the same arguments as in section 2.4, there exist functions $\psi_j \in PSH(M,\omega_0) \cap C^\infty(M)$ such that

$$\lim_{j\to\infty} \|\psi_j - \varphi_0\|_{L^\infty(M)} = 0. \tag{3.5}$$

Write $\omega_{0,j} = \omega_0 + \sqrt{-1}\partial\bar{\partial}\psi_j$. The Chern-Ricci flow starting at $\omega_{0,j}$ can be reduced to a parabolic complex Monge-Ampère equation. First denote

$$T = \sup\{t \geq 0 | \exists \psi \in C^\infty(M),\ \ \omega_0 - t\,\mathrm{Ric}(\omega_0) + \sqrt{-1}\partial\bar{\partial}\psi > 0\}.$$

Then for any $T' < T$, there exists $\psi_{T'} \in C^\infty(M))$ such that

$$\beta = \omega_0 - T'\,\mathrm{Ric}(\omega_0) + \sqrt{-1}\partial\bar{\partial}\psi_{T'} > 0.$$

Fix $T' < T$ and define smooth Hermitian metrics

$$\hat{\omega}_t = (1 - \frac{t}{T'})\omega_0 + \frac{t}{T'}\beta = \omega_0 + t\chi$$

on $[0,T']$, where $\chi = \frac{1}{T'}\sqrt{-1}\partial\bar{\partial}\psi_{T'} - \mathrm{Ric}(\omega_0)$. Let Ω be a volume form satisfying $\sqrt{-1}\partial\bar{\partial}\log\Omega = \frac{\partial}{\partial t}\hat{\omega}_t = \chi$. It follows that if φ_j solves the parabolic complex Monge-Ampère equation

$$\frac{\partial \varphi_j}{\partial t} = \log \frac{(\hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi_j)^n}{\Omega}, \qquad \varphi_j(0) = \psi_j \tag{3.6}$$

for $t \in [0, T']$, then $\omega_j = \hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi_j$ solves the Chern-Ricci flow starting at $\omega_{0,j} = \omega_0 + \sqrt{-1}\partial\bar{\partial}\psi_j$. We will show uniform C^∞ bounds for φ_j and prove the following theorem.

Theorem 3.3. *Suppose that $\omega_0' = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_0$ for some $\varphi_0 \in PSH_p(\omega_0, \Omega)$, $p > 1$. Assume that ω_0 satisfies the condition (3.4), then there exists a unique family of smooth metrics $\omega(t)$ on $(0, T)$ such that*

(i) $\frac{\partial}{\partial t}\omega = -\operatorname{Ric}(\omega)$, *for* $t \in (0, T)$.

(ii) There exists $\varphi \in C^0([0, T) \times M) \cap C^\infty((0, T) \times M)$ such that $\omega = \hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi$ and $\varphi(t) \to \varphi_0$ in $L^\infty(M)$ as $t \to 0^+$.

In particular, $\omega(t) \to \omega_0'$ in the sense of currents as $t \to 0^+$.

If M is a compact complex surface with a Gauduchon metric ω_0, then condition (3.4) holds and the above result follows. When (M, ω_0) is a Kähler manifold, the result is contained in the work of Song and Tian [38] (see also [7]).

Let

$$F = \frac{(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_0)^n}{\Omega} \in L^p(M).$$

We use $C, C', C_i, ...$ to denote uniform constants depending only on ω_0, $||\varphi_0||_{L^\infty(M)}$ and $||F||_{L^p(M)}$ and varying from line to line.

First we have the following two lemmas from [38, Lemma 3.1 and 3.2]. The proof is exactly the same as in [38].

Lemma 3.4. *There exists $C > 0$ such that for any $t \in [0, T']$,*

$$||\varphi_j||_{L^\infty(M)} \leq C.$$

Moreover, $\{\varphi_j\}$ is a Cauchy sequence in $C^0([0, T'] \times M)$, i.e.,

$$\lim_{j,k\to\infty} ||\varphi_j - \varphi_k||_{L^\infty([0,T']\times M)} = 0.$$

Lemma 3.5. *There exists $C > 0$ such that*

$$\frac{t^n}{C} \leq \frac{(\hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi_j)^n}{\Omega} \leq e^{\frac{C}{t}}.$$

for any $t \in [0, T']$.

For convenience, we write $\omega' = \omega_j$ the solution of the Chern-Ricci flow starting at $\omega_{0,j}$ and g', Δ', $|\cdot|_{g'}, ...$ the notations corresponding to ω_j for a fix j. Similarly, we use ∇^0 to denote the covariant derivative with respect to g_0. All the bounds obtained in the following lemmas are independent of j.

To prove the second order estimate, we will need the following proposition. It follows from Proposition 3.1 in [46] as $(T_0)_{ki\bar{l}} = (T_{0,j})_{ki\bar{l}}$, where T_0 and $T_{0,j}$ are the torsions corresponding to ω_0 and $\omega_{0,j}$.

Proposition 3.6. (Tosatt-Weinkove) *Assume that at a point $\mathrm{tr}_{\omega_0}\omega' \geq 1$, then*

$$(\frac{\partial}{\partial t} - \Delta')\log \mathrm{tr}_{\omega_0}\omega' \leq \frac{2}{(\mathrm{tr}_{\omega_0}\omega')^2}\,\mathrm{Re}(g'^{p\bar{q}}(T_0)^i_{pi}\nabla^0_{\bar{q}}\,\mathrm{tr}_{\omega_0}\omega') + C\,\mathrm{tr}_{\omega'}\omega_0$$

at this point for some constant C depending only on g_0.

Lemma 3.7. *There exists $C > 0$ such that for $t \in (0, T]$,*

$$\mathrm{tr}_{\omega_0}\omega' \leq e^{\frac{C}{t}}.$$

Proof. Let

$$H = t\log \mathrm{tr}_{\omega_0}\omega' + e^{\Psi},$$

where $\Psi = A(\sup\limits_{[0,T']\times M}\phi_j - \phi_j)$ and A is a constant to be chosen later. Assume that H achieves its maximum at (t_0, z_0) and $\mathrm{tr}_{\omega_0}\omega' > 1$ (otherwise we obtain the upper bound for $\mathrm{tr}_{\omega_0}\omega'$ directly). Choose coordinates around (t_0, z_0) such that at this point,

$(g_0)_{i\bar{j}} = \delta_{ij}$ and $(g'_{i\bar{j}})$ is diagonal. First we have

$$(\frac{\partial}{\partial t} - \Delta')H = t(\frac{\partial}{\partial t} - \Delta') \log \operatorname{tr}_{\omega_0} \omega' + \log \operatorname{tr}_{\omega_0} \omega' - Ae^{\Psi} \dot{\varphi}_j \\ + Ae^{\Psi} \Delta' \varphi_j - A^2 e^{\Psi} |\nabla \varphi_j|^2_{g'}.$$

It follows from Proposition 3.6 that

$$(\frac{\partial}{\partial t} - \Delta') \log \operatorname{tr}_{\omega_0} \omega' \leq \frac{2}{(\operatorname{tr}_{\omega_0} \omega')^2} \operatorname{Re}(g'^{k\bar{k}} (T_0)^i_{ki} \partial_{\bar{k}} \operatorname{tr}_{\omega_0} \omega') + C \operatorname{tr}_{\omega'} \omega_0.$$

At (t_0, z_0), $\nabla_{\bar{k}} H = 0$ gives

$$t \frac{\partial_{\bar{k}} \operatorname{tr}_{\omega_0} \omega'}{\operatorname{tr}_{\omega_0} \omega'} - Ae^{\Psi} \partial_{\bar{k}} \varphi_j = 0.$$

Then

$$\begin{aligned} \frac{2t}{(\operatorname{tr}_{\omega_0} \omega')^2} \operatorname{Re}(g'^{k\bar{k}} (T_0)^i_{ki} \partial_{\bar{k}} \operatorname{tr}_{\omega_0} \omega') &\leq \frac{2}{\operatorname{tr}_{\omega_0} \omega'} e^{\Psi} |\operatorname{Re}(g'^{k\bar{k}} (T_0)^i_{ki} \partial_{\bar{k}} \varphi_j)| \\ &\leq e^{\Psi} (A^2 |\nabla \varphi_j|^2_{g'} + C_1 \operatorname{tr}_{\omega'} \omega_0). \end{aligned} \tag{3.7}$$

Also

$$\operatorname{tr}_{\omega_0} \omega' \leq \frac{1}{(n-1)!} (\operatorname{tr}_{\omega'} \omega_0)^{n-1} \frac{(\omega')^n}{\omega_0^n}.$$

Combining all the above inequalities, we get

$$
\begin{aligned}
&(\frac{\partial}{\partial t} - \Delta')H \\
&\leq C_1 e^{\Psi} \operatorname{tr}_{\omega'} \omega_0 + Ct \operatorname{tr}_{\omega'} \omega_0 + \log \operatorname{tr}_{\omega_0} \omega' + Ane^{\Psi} - Ae^{\Psi} \operatorname{tr}_{\omega'} \hat{\omega}_t - Ae^{\Psi} \dot{\varphi}_j \\
&\leq - e^{\Psi} \operatorname{tr}_{\omega'}(A\hat{\omega}_t - C_1\omega_0 - Ct\omega_0) + (1 - Ae^{\Psi}) \log \frac{(\omega')^n}{\omega_0^n} + (n-1) \log \operatorname{tr}_{\omega'} \omega_0 + C_2 \\
&\leq - C \operatorname{tr}_{\omega'} \omega_0 - C_3 \log t + C_4
\end{aligned}
\tag{3.8}
$$

if we choose A to be large enough. Then at (t_0, z_0),

$$C_5 \left(\frac{\omega_0^n}{(\omega')^n} \right)^{\frac{1}{n-1}} (\operatorname{tr}_{\omega_0} \omega')^{\frac{1}{n-1}} \leq C \operatorname{tr}_{\omega'} \omega_0 \leq -C_3 \log t + C_4.$$

So

$$\log \operatorname{tr}_{\omega_0} \omega' \leq \log \left((\log \frac{1}{t})^{n-1} \left(\frac{(\omega')^n}{\omega_0^n} \right) \right) + C_6 \leq \frac{C_7}{t} + C_8.$$

Thus H is uniformly bounded for $t \in (0, T]$ and we obtain the required estimate. $\square$

Consider $S = |\nabla_{g_0} g'|^2_{g'}$. Denote $(\Gamma_0)^k_{ij}$ the Christoffel symbols associated to g_0. It is convenient to compute using the tensor ${\Phi_{ij}}^k = \Gamma'^k_{ij} - (\Gamma_0)^k_{ij}$ as in [37] (see also [35]) , then

$$S = |\Phi|^2_{g'} = g'^{i\bar{p}} g'^{j\bar{q}} g'_{k\bar{r}} {\Phi_{ij}}^k \Phi_{\bar{p}\bar{q}}{}^{\bar{r}}.$$

Lemma 3.8. *There exists $C > 0$ and $\lambda > 0$ such that for $t \in (0, T']$,*

$$S \leq C e^{\frac{\lambda}{t}}.$$

Proof. By the evolution equation for $\operatorname{tr}_{\omega_0} \omega'$ in [46] and the second order estimate

(Lemma 3.7), we have

$$(\frac{\partial}{\partial t} - \Delta') \operatorname{tr}_{\omega_0} \omega' \le -C_1 e^{-\frac{\alpha}{t}} S + C_2 e^{\frac{\alpha}{t}}$$

and

$$(\frac{\partial}{\partial t} - \Delta') S \le e^{\frac{\beta}{t}} (S^{3/2} + 1) - \frac{1}{2} (|\overline{\nabla}' \Phi|^2_{g'} + |\nabla' \Phi|^2_{g'}).$$

for some positive constants α and β. Take $\lambda_1 > 0$ such that $\frac{1}{2} e^{\frac{\lambda_1}{t}} < e^{\frac{\lambda_1}{t}} - \operatorname{tr}_{\omega_0} \omega' < e^{\frac{\lambda_1}{t}}$. We will apply a maximal principle argument to

$$H = \frac{S}{(e^{\frac{\lambda_1}{t}} - \operatorname{tr}_{\omega_0} \omega')^2} + e^{-\frac{\lambda_2}{t}} \operatorname{tr}_{\omega_0} \omega'$$

to show the bound for S. Compute

$$\begin{aligned}
(\frac{\partial}{\partial t} - \Delta') H =& \frac{1}{(e^{\frac{\lambda_1}{t}} - \operatorname{tr}_{\omega_0} \omega')^2} (\frac{\partial}{\partial t} - \Delta') S + \frac{2S}{(e^{\frac{\lambda_1}{t}} - \operatorname{tr}_{\omega_0} \omega')^3} (\frac{\partial}{\partial t} - \Delta') \operatorname{tr}_{\omega_0} \omega' \\
& - \frac{4 \operatorname{Re} \nabla' \operatorname{tr}_{\omega_0} \omega' \cdot \overline{\nabla}' S}{(e^{\frac{\lambda_1}{t}} - \operatorname{tr}_{\omega_0} \omega')^3} - \frac{6S |\nabla' \operatorname{tr}_{\omega_0} \omega'|^2}{(e^{\frac{\lambda_1}{t}} - \operatorname{tr}_{\omega_0} \omega')^4} + \frac{\frac{2\lambda_1}{t^2} e^{\frac{\lambda_1}{t}} S}{(e^{\frac{\lambda_1}{t}} - \operatorname{tr}_{\omega_0} \omega')^3} \\
& + e^{-\frac{\lambda_2}{t}} (\frac{\partial}{\partial t} - \Delta') \operatorname{tr}_{\omega_0} \omega' + \frac{\lambda_2}{t^2} e^{-\frac{\lambda_2}{t}} \operatorname{tr}_{\omega_0} \omega' \\
\le & \, 4 e^{-\frac{2\lambda_1}{t}} e^{\frac{\beta}{t}} (S^{3/2} + 1) - \frac{1}{2} e^{-\frac{2\lambda_1}{t}} (|\overline{\nabla}' \Phi|^2_{g'} + |\nabla' \Phi|^2_{g'}) \\
& + \left(-2C_1 e^{-\frac{\alpha}{t}} e^{-\frac{3\lambda_1}{t}} S^2 + 16 C_2 e^{\frac{\alpha}{t}} e^{-\frac{3\lambda_1}{t}} S \right) + 32 e^{-\frac{3\lambda_1}{t}} |\operatorname{Re} \nabla' \operatorname{tr}_{\omega_0} \omega' \cdot \overline{\nabla}' S| \\
& + C_3 e^{-\frac{\lambda_1}{t}} S + \left(-C_1 e^{-\frac{\alpha}{t}} e^{-\frac{\lambda_2}{t}} S + C_2 e^{\frac{\alpha}{t}} e^{-\frac{\lambda_2}{t}} \right) + C_4
\end{aligned}$$

where we choose λ_2 large enough such that $2 \frac{\lambda_2}{t^2} e^{-\frac{\lambda_2}{t}} \operatorname{tr}_{\omega_0} \omega' \le C_4$ for some constant C_4.

Note that $|\nabla' \operatorname{tr}_{\omega_0} \omega'|_{g'} \leq \frac{1}{64} e^{\frac{\gamma}{t}} S^{1/2}$ for some $\gamma > 0$ and $|\overline{\nabla}' S|_{g'} \leq 2S^{1/2} |\overline{\nabla}' \Phi|_{g'}$, we have

$$
\begin{aligned}
32e^{-\frac{3\lambda_1}{t}} |\operatorname{Re} \nabla' \operatorname{tr}_{\omega_0} \omega' \cdot \overline{\nabla}' S| &\leq e^{-\frac{3\lambda_1}{t}} e^{\frac{\gamma}{t}} S |\overline{\nabla}' \Phi|_{g'} \\
&\leq \frac{1}{2} e^{-\frac{2\lambda_1}{t}} |\overline{\nabla}' \Phi|^2_{g'} + \frac{1}{2} e^{-\frac{4\lambda_1}{t}} e^{\frac{2\gamma}{t}} S^2.
\end{aligned}
$$

Also we have

$$
4e^{-\frac{2\lambda_1}{t}} e^{\frac{\beta}{t}} S^{3/2} \leq C_1 e^{-\frac{\alpha}{t}} e^{-\frac{3\lambda_1}{t}} S^2 + \frac{4}{C_1} e^{\frac{\alpha}{t}} e^{\frac{2\beta}{t}} e^{-\frac{\lambda_1}{t}} S.
$$

Take λ_2 sufficiently large such that $e^{\frac{\alpha}{t}} e^{-\frac{\lambda_2}{t}} < 1$, then fix λ_2. Let $\lambda_1 \geq \alpha + \lambda_2$ be large enough such that

$$
(\frac{\partial}{\partial t} - \Delta') H \leq -\frac{1}{2} C_1 e^{-\frac{\alpha}{t}} e^{-\frac{\lambda_2}{t}} S + C
$$

for some constant C. Assume that H achieves its maximal at $(t_0, z_0), t_0 > 0$, then at this point

$$
0 \leq -\frac{1}{2} C_1 e^{-\frac{\alpha}{t}} e^{-\frac{\lambda_2}{t}} S + C.
$$

It follows that H is bounded by some constant. Therefore $S \leq C e^{\frac{\lambda}{t}}$ for some constants $C > 0$ and $\lambda > 0$. $\square$

In addition, the bounds on derivatives of ω_j in the t-direction follow from Lemma 2.12. Then by the standard parabolic estimates [30], we obtain all the higher order estimates.

Proposition 3.9. *For any $0 < \epsilon < T'$ and $k \geq 0$, there exists $C_{\epsilon, T', k} > 0$, such that*

$$
||\varphi_j||_{C^k([\epsilon, T'] \times M)} \leq C_{\epsilon, T', k}.
$$

Proposition 3.10. *There exists a function $\varphi \in C^0([0, T) \times M) \cap C^\infty((0, T) \times M)$ such* *that φ is the unique solution of the equation*

$$\frac{\partial \varphi}{\partial t} = \log \frac{(\hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi)^n}{\Omega}, \quad for\ t \in (0,T), \quad \varphi|_{t=0} = \varphi_0. \tag{3.9}$$

Proof. By Lemma 3.4, φ_j is a Cauchy sequence in $C^0([0,T'] \times M)$ and so we can define $\varphi = \lim_{j\to\infty} \varphi_j$ which is in $C^0([0,T'] \times M)$. Then it follows from Proposition 3.9 that for any $0 < \epsilon < T' < T$, φ_j converges to φ in $C^\infty([\epsilon, T'] \times M)$. Therefore $\varphi \in C^\infty((0,T) \times M)$ satisfying the above equation on $(0,T)$. Note that

$$\lim_{t\to 0^+} ||\varphi(t,\cdot) - \varphi_0(\cdot)||_{L^\infty(M)} = 0$$

as $\varphi_j(0) = \psi_j$ and $\psi_j \to \varphi_0$ in $L^\infty(M)$ as $j \to \infty$. Then $\varphi|_{t=0} = \varphi_0$ and we have the existence of a solution $\varphi \in C^0([0,T) \times M) \cap C^\infty((0,T) \times M)$ for equation (3.9). To prove the uniqueness, we assume that there is another solution $\tilde{\varphi} \in C^0([0,T) \times M) \cap C^\infty((0,T) \times M)$ of equation (3.9). Let $\psi = \tilde{\varphi} - \varphi$. Then ψ solves the equation

$$\frac{\partial \psi}{\partial t} = \log \frac{(\hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi + \sqrt{-1}\partial\bar{\partial}\psi)^n}{(\hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi)^n}, \quad for\ t \in (0,T), \quad \psi|_{t=0} = 0.$$

At any given time t, the maximum of ψ is achieved at some point $z \in M$, then $\frac{d\psi_{max}(t)}{dt} \leq 0$ a.e. in $[0,T)$. Similarly we have $\frac{d\psi_{min}(t)}{dt} \geq 0$ a.e. in $[0,T)$. As both $\psi_{max}(t)$ and $\psi_{min}(t)$ are absolutely continuous on $([0,T)$ with $\psi_{max}(0) = \psi_{min}(0) = 0$, we have $\psi_{max}(t) \leq 0 \leq \psi_{min}(t)$ for $t \in [0,T)$. Hence $\psi(t) = 0$ for $t \in [0,T)$. $\square$

Now we are ready to prove the smoothing property for the Chern-Ricci flow with rough initial data.

Proof of Theorem 3.3. If φ is a solution of (3.9), then taking $\sqrt{-1}\partial\bar{\partial}$ of (3.9) shows that $\omega = \hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi$ solves the Chern-Ricci flow on $(0,T)$ with $\lim_{t\to 0^+} \|\varphi(t,\cdot) -$

$\varphi_0(\cdot)\|_{L^\infty(M)} = 0$ by Lemma 3.5 in [38]. Conversely, if ω solves (3.1), then

$$\frac{\partial}{\partial t}(\omega - \hat{\omega}_t) = \sqrt{-1}\partial\bar{\partial}\log\frac{\omega^n}{\Omega}.$$

Thus $\omega(t)$ must be of the form $\omega(t) = \hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi$ for some φ solving the equation

$$\sqrt{-1}\partial\bar{\partial}(\frac{\partial\varphi}{\partial t} - \log\frac{(\hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi)^n}{\Omega}) = 0. \tag{3.10}$$

Proposition 3.10 gives a solution of the above equation. Suppose that there exists another solution $\tilde{\varphi} \in C^\infty((0,T)\times M) \cap C^0([0,T)\times M)$ of equation (3.10). Then

$$\frac{\partial\tilde{\varphi}}{\partial t} = \log\frac{(\hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\tilde{\varphi})^n}{\Omega} + f(t)$$

with $\lim_{t\to 0^+}\tilde{\varphi}(t) = \varphi_0$ for some smooth function $f(t)$. So we get $\varphi = \tilde{\varphi} - \int_0^t f(s)ds$ is a solution of the equation (3.9) which is unique. Therefore $\tilde{\varphi} = \varphi + \int_0^t f(s)ds$ and we prove the uniqueness. □

3.3 Continuing the flow after blowing down exceptional curves

In this section, we will establish step (iv) of the conjecture in section 3.1.2. More precisely, given a compact complex surface M, suppose that the Chern-Ricci flow (3.2) starting at a Gauduchon metric ω_0 is non-collapsing at $T < \infty$ and denote $\tilde{\omega}_T = (\pi^{-1})^*\omega_T$ the push forward of the limiting current to N, then with the notation in section 3.1.2, we prove the following theorem.

Theorem 3.11. *Assume that condition (3.3) is satisfied, then with the notation above,* *there exists a unique maximal solution* $\omega(t)$ *to the equation:*

$$\omega|_{t=T} = \tilde{\omega}_T, \quad \frac{\partial}{\partial t}\omega = -\operatorname{Ric}(\omega), \quad for\ t \in (T, T_N),$$

which is smooth on (T, T_N)*. Moreover,* $\omega(t)$ *converges to* $\tilde{\omega}_T$ *in* $C^\infty_{\mathrm{loc}}(N')$ *as* $t \to T^+$.

We will use some of the techniques for the Kähler-Ricci flow in [39].

3.3.1 Construction of the solution

Without loss of generality, assume that M contains only one exceptional curve E for simplicity. Recall that

$$T = \sup\{t \geq 0 | \exists \psi \in C^\infty(M),\ \ \omega_0 - t\operatorname{Ric}(\omega_0) + \sqrt{-1}\partial\bar{\partial}\psi > 0\}.$$

Assuming that the condition (3.3) holds, it follows from [47, Lemma 3.2] that there exists a smooth function ψ and a Gauduchon metric on N such that

$$\omega_0 - T\operatorname{Ric}(\omega_0) + \sqrt{-1}\partial\bar{\partial}\psi = \pi^*\omega_N. \tag{3.11}$$

Define

$$\hat{\omega}_t = (1 - \frac{t}{T})\omega_0 + \frac{t}{T}\pi^*\omega_N,$$

which are smooth nonnegative forms on $[0, T]$. Let $\Omega = e^{\psi/T}\omega_0^n$. (From now on, we will write n instead of 2 whenever our calculations hold for $n \geq 2$.) If φ solves the parabolic complex Monge-Ampère equation

$$\frac{\partial \varphi}{\partial t} = \log \frac{(\hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi)^n}{\Omega}, \qquad \varphi|_{t=0} = 0 \tag{3.12}$$

for $t < T$, then $\omega = \hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi$ solves the Chern-Ricci flow (3.2) on $[0, T)$.

By Lemma 3.3 of [47], there exists a uniform constant C such that $|\varphi| \leq C$ and $\dot{\varphi} \leq C$, where we write $\dot{\varphi}$ for $\frac{\partial \varphi}{\partial t}$. Then it follows that as $t \to T^-$, $\varphi(t)$ converges pointwise on M to a bounded function φ_T with

$$\omega_T = \hat{\omega}_T + \sqrt{-1}\partial\bar{\partial}\varphi_T \geq 0.$$

In particular, $\omega(t) \to \omega_T$ in the sense of currents as $t \to T^-$. From Lemma 5.1 of [38], φ_T must be constant on E since $\sqrt{-1}\partial\bar{\partial}\varphi_T|_E = \omega_T|_E \geq 0$. Thus $\psi_T = (\pi^{-1})^*\varphi_T$ is a bounded function on N which is smooth on $N \setminus \{y_0\}$ as π is the blow down map contracting E to y_0. Define

$$\tilde{\omega}_T = \omega_N + \sqrt{-1}\partial\bar{\partial}\psi_T \geq 0, \tag{3.13}$$

which is the push down of the limiting current ω_T to N.

Lemma 3.12. *There exists $p > 1$ such that $\tilde{\omega}_T^n/\omega_N^n \in L^p(N)$. Moreover, ψ_T is continuous on N.*

Proof. The argument of Lemma 5.2 in [39] shows that $\tilde{\omega}_T^n/\omega_N^n \in L^p(N)$. The continuity of ψ_T then follows from section 2.4 (Kołodziej's stability result and Dinew-Kołodziej [12]). $\square$

Using the construction of Song-Weinkove for the Kähler-Ricci flow [39], we can construct the solution in Theorem 3.11 explicitly in the following way. Given a smooth volume form Ω_N, let $\chi = \sqrt{-1}\partial\bar{\partial}\log\Omega_N$. Then there exists $T' > T$ such that

$$\hat{\omega}_{t,N} = \omega_N + (t-T)\chi$$

are smooth Gauduchon metrics for $t \in [T, T']$. We will now construct a family of functions $\psi_{T,\epsilon}$ on N which converge to ψ_T in $L^\infty(N)$. For sufficiently small $\epsilon > 0$ and K large enough, define

$$\Omega_\epsilon = (\pi|^{-1}_{M\setminus E})^*(\frac{|s|_h^{2K}\omega^n(T-\epsilon)}{\epsilon + |s|_h^{2K}}) + \epsilon\Omega_N \quad \text{on } N \setminus \{y_0\}.$$

and $\Omega_\epsilon|_{y_0} = \epsilon\Omega_N|_{y_0}$. Here s is a holomorphic section of the line bundle $[E]$ vanishing along the exceptional curve E to order 1. Choose h to be a smooth Hermitian metric on $[E]$ as in [46] with curvature $R_h = -\sqrt{-1}\partial\bar{\partial}\log|s|_h^2$ such that for sufficiently small $\epsilon > 0$, $\pi^*\omega_N - \epsilon R_h > 0$.(see [17, P. 187] for a argument of this.) Then the volume form $\Omega_\epsilon \in C^k(N)$ for a fixed constant k as π is the blow down map and K can be chosen to be sufficiently large. Moreover, Ω_ϵ converges to $\tilde{\omega}_T^n$ in C^∞ on compact subsets of $N \setminus \{y_0\}$ as ϵ goes to zero. By the result of Tosatti and Weinkove [45], there exist functions $\psi_{T,\epsilon} \in C^k(N) \cap C^\infty(N \setminus \{y_0\})$ such that

$$(\omega_N + \sqrt{-1}\partial\bar{\partial}\psi_{T,\epsilon})^n = C_\epsilon\Omega_\epsilon$$

where the constants C_ϵ are chosen so that the integrals of both sides of the above equation match. Write $F_\epsilon = C_\epsilon\Omega_\epsilon/\Omega_N$ and $F = \tilde{\omega}_T^n/\Omega_N$. Note that $C_\epsilon \to 1$ as $\epsilon \to 0$, then by the definition of Ω_ϵ we have

$$\lim_{\epsilon\to 0} ||F_\epsilon - F||_{L^1(N)} = 0.$$

By Lemma 3.12 and Kolodziej's stability result, we have

$$\lim_{\epsilon\to 0} ||\psi_{T,\epsilon} - \psi_T||_{L^\infty(N)} = 0. \tag{3.14}$$

Consider the parabolic complex Monge-Ampère equations

$$\frac{\partial \varphi_\epsilon}{\partial t} = \log \frac{(\hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi_\epsilon)^n}{\Omega_N}, \quad \varphi_\epsilon|_{t=T} = \psi_{T,\epsilon}$$

on $[T, T']$. Then Theorem 3.3 and Lemma 3.12 give the following proposition.

Proposition 3.13. *There exists* $\varphi \in C^0(([T,T'] \times N) \cap C^\infty((T,T'] \times N)))$ *such that*

(i) $\varphi_\epsilon \to \varphi$ *in* $L^\infty([T,T'] \times N)$.

(ii) $\omega(t) = \hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi$ *is the unique solution to the equation,*

$$\omega|_{t=T} = \tilde{\omega}_T, \quad \frac{\partial}{\partial t}\omega = -\operatorname{Ric}(\omega), \quad \textit{for } t \in (T, T'].$$

To prove smooth convergence of $\omega(t)$ on compact subsets of $N \setminus \{y_0\}$ as $T \to T^+$, we need uniform estimates for $\omega_\epsilon(t) = \hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi_\epsilon$ on $[T,T'] \times N \setminus \{y_0\}$, independent of ϵ. To obtain these estimates, we need the bounds at time T, i.e. the bounds for $\tilde{\omega}_{T,\epsilon} = \omega_N + \sqrt{-1}\partial\bar{\partial}\psi_{T,\epsilon}$. We will prove these in section 3.3.3.

3.3.2 Higher order estimates as $t \to T^-$

In this subsection, we will prove a third order estimate and a bound for $|\operatorname{Ric}|$, which will be used to obtain the bounds for $\tilde{\omega}_{T,\epsilon}$ in the next subsection.

As $t \to T^-$, the smooth convergence of $\omega(t)$ on compact subsets of $M \setminus \{E\}$ follows from Theorem 1.1 of [47] . In particular, C^∞ a priori estimates for $\omega(t)$ on compact subsets away from the exceptional curves have already been obtained. However, we need more precise higher order global estimates on M as $t \to T^-$ to obtain the bounds for $\tilde{\omega}_{T,\epsilon}$.

From [47, Lemma 2.3 and 2.4], there exist positive constants C and K such that

$$\frac{|s|_h^{2K}}{C}\omega_0 \leq \omega(t) \leq \frac{C}{|s|_h^{2K}}\omega_0. \tag{3.15}$$

We may assume that $|s|_h^2 \leq 1$ on M for convenience. By Lemma 2.8, we can choose local coordinates around a point such that at this point

$$(g_0)_{i\bar{j}} = \delta_{ij}, \quad \partial_i (g_0)_{j\bar{j}} = 0 \tag{3.16}$$

for all i, j and $(g_{i\bar{j}})$ is diagonal. Now choose such a coordinate system around a point. Following an argument in [44] (see also [49]), we can get the inequality,

$$\frac{|\nabla \operatorname{tr}_{\omega_0} \omega|^2}{\operatorname{tr}_{\omega_0} \omega} \leq \sum_{i,j,k} g^{i\bar{i}} g^{j\bar{j}} \partial_k g_{i\bar{j}} \partial_{\bar{k}} g_{j\bar{i}} + \sum_{i,j} g^{i\bar{i}} g^{i\bar{i}} |\partial_j (g_0)_{i\bar{j}}|^2. \tag{3.17}$$

To see this, first applying the Cauchy-Schwarz inequality

$$\begin{aligned}
|\nabla \operatorname{tr}_{\omega_0} \omega|^2 &= \sum_{i,j,k} g^{i\bar{i}} \partial_i g_{j\bar{j}} \partial_{\bar{i}} g_{k\bar{k}} \\
&= \sum_{j,k} \sum_i g^{i\bar{i}} \partial_i g_{j\bar{j}} \partial_{\bar{i}} g_{k\bar{k}} \\
&\leq \sum_{j,k} \left(\sum_i g^{i\bar{i}} |\partial_i g_{j\bar{j}}|^2 \right)^{1/2} \left(\sum_i g^{i\bar{i}} |\partial_i g_{k\bar{k}}|^2 \right)^{1/2} \\
&= \left(\sum_j \left(\sum_i g^{i\bar{i}} |\partial_i g_{j\bar{j}}|^2 \right)^{1/2} \right)^2 \\
&= \left(\sum_j \sqrt{g_{j\bar{j}}} \left(\sum_i g^{i\bar{i}} g^{j\bar{j}} |\partial_i g_{j\bar{j}}|^2 \right)^{1/2} \right) \\
&\leq (\operatorname{tr}_{\omega_0} \omega) \left(\sum_i g^{i\bar{i}} g^{j\bar{j}} \partial_i g_{j\bar{j}} \partial_{\bar{i}} g_{j\bar{j}} \right).
\end{aligned} \tag{3.18}$$

Note that $\omega = \omega_0 + \theta(t)$ for a closed (1,1) form $\theta(t)$. Hence

$$\partial_i g_{j\bar{j}} - \partial_j g_{i\bar{j}} = \partial_i (g_0)_{j\bar{j}} - \partial_j (g_0)_{i\bar{j}}.$$

Using (3.16), we get

$$\partial_i g_{j\bar{j}} = \partial_j g_{i\bar{j}} - \partial_j (g_0)_{i\bar{j}}.$$

Similarly $\partial_{\bar{i}} g_{j\bar{j}} = \partial_{\bar{j}} g_{i\bar{j}} - \partial_{\bar{j}} (g_0)_{i\bar{j}}$. Then (3.18) gives

$$\begin{aligned}
\frac{|\nabla \operatorname{tr}_{\omega_0} \omega|^2}{\operatorname{tr}_{\omega_0} \omega} \leq\ & \sum_{i,j} g^{i\bar{i}} g^{j\bar{j}} \partial_j g_{i\bar{j}} \partial_{\bar{j}} g_{j\bar{i}} - 2\operatorname{Re}\left(\sum_{i,j} g^{i\bar{i}} g^{j\bar{j}} \partial_j g_{i\bar{j}} \partial_{\bar{j}} (g_0)_{j\bar{i}} \right) \\
& + \sum_{i,j} g^{i\bar{i}} g^{j\bar{j}} \partial_j (g_0)_{i\bar{j}} \partial_{\bar{j}} (g_0)_{j\bar{i}} \\
=\ & \sum_{i,j} g^{i\bar{i}} g^{j\bar{j}} \partial_j g_{i\bar{j}} \partial_{\bar{j}} g_{j\bar{i}} - 2\operatorname{Re}\left(\sum_{i,j} g^{i\bar{i}} g^{j\bar{j}} \partial_i g_{j\bar{j}} \partial_{\bar{j}} (g_0)_{j\bar{i}} \right) \\
& - \sum_{i,j} g^{i\bar{i}} g^{j\bar{j}} \partial_j (g_0)_{i\bar{j}} \partial_{\bar{j}} (g_0)_{j\bar{i}} \\
\leq\ & \sum_{i,j} g^{i\bar{i}} g^{j\bar{j}} \partial_j g_{i\bar{j}} \partial_{\bar{j}} g_{j\bar{i}} - 2\operatorname{Re}\left(\sum_{i\neq j} g^{i\bar{i}} g^{j\bar{j}} \partial_i g_{j\bar{j}} \partial_{\bar{j}} (g_0)_{j\bar{i}} \right) \\
\leq\ & \sum_{i,j} g^{i\bar{i}} g^{j\bar{j}} \partial_j g_{i\bar{j}} \partial_{\bar{j}} g_{j\bar{i}} + \sum_{i\neq j} g^{j\bar{j}} g^{j\bar{j}} |\partial_i g_{j\bar{j}}|^2 + \sum_{i\neq j} g^{i\bar{i}} g^{i\bar{i}} |\partial_j (g_0)_{i\bar{j}}|^2 \\
\leq\ & \sum_{i,j,k} g^{i\bar{i}} g^{j\bar{j}} \partial_k g_{i\bar{j}} \partial_{\bar{k}} g_{j\bar{i}} + \sum_{i,j} g^{i\bar{i}} g^{i\bar{i}} |\partial_j (g_0)_{i\bar{j}}|^2. \qquad (3.19)
\end{aligned}$$

Thus we get the inequality (3.17). We will use it to prove our third order estimate.

Recall that $(\Gamma_0)^k_{ij}$ denotes the Christoffel symbols associated to g_0 and ${\Phi_{ij}}^k = \Gamma^k_{ij} - (\Gamma_0)^k_{ij}$. Consider

$$S = |\nabla_{g_0} g|^2_g = g^{i\bar{p}} g^{j\bar{q}} g_{k\bar{r}} {\Phi_{ij}}^k {\Phi_{\bar{p}\bar{q}}}^{\bar{r}}.$$

Proposition 3.14. *There exist positive constants λ and C such that for $t \in [0, T)$,*

$$S \leq \frac{C}{|s|_h^{2\lambda}}.$$

Proof. Let

$$H = \frac{S}{(|s|_h^{-\alpha} - \operatorname{tr}_{\omega_0} \omega)^2} + |s|_h^{\beta} \operatorname{tr}_{\omega_0} \omega - At,$$

where α and β are constants to be determined and at least large enough such that

$$\frac{1}{2}|s|_h^{-\alpha} < |s|_h^{-\alpha} - \operatorname{tr}_{\omega_0} \omega < |s|_h^{-\alpha}, \quad \operatorname{tr}_{\omega_0} \omega \leq C|s|_h^{-\beta}.$$

Computing the evolution of H,

$$\begin{aligned}
(\frac{\partial}{\partial t} - \Delta)H =& \frac{1}{(|s|_h^{-\alpha} - \operatorname{tr}_{\omega_0} \omega)^2}(\frac{\partial}{\partial t} - \Delta)S \\
& - \frac{2S}{(|s|_h^{-\alpha} - \operatorname{tr}_{\omega_0} \omega)^3}(\frac{\partial}{\partial t} - \Delta)(|s|_h^{-\alpha} - \operatorname{tr}_{\omega_0} \omega) \\
& + \frac{4 \operatorname{Re} \nabla S \cdot \overline{\nabla}(|s|_h^{-\alpha} - \operatorname{tr}_{\omega_0} \omega)}{(|s|_h^{-\alpha} - \operatorname{tr}_{\omega_0} \omega)^3} - \frac{6S|\nabla(|s|_h^{-\alpha} - \operatorname{tr}_{\omega_0} \omega)|^2}{(|s|_h^{-\alpha} - \operatorname{tr}_{\omega_0} \omega)^4} - A \\
& + (\frac{\partial}{\partial t} - \Delta)(|s|_h^{\alpha} \operatorname{tr}_{\omega_0} \omega).
\end{aligned} \tag{3.20}$$

From [37] and (3.15), we have the estimates

$$(\frac{\partial}{\partial t} - \Delta)S \leq C|s|_h^{-\alpha_1}(1 + S^{3/2}) - \frac{1}{2}|\overline{\nabla}\Phi|^2, \tag{3.21}$$

and

$$(\frac{\partial}{\partial t} - \Delta) \operatorname{tr}_{\omega_0} \omega \leq -\frac{|s|_h^{\alpha_2} S}{C} + C|s|_h^{-\alpha_2} - \frac{1}{2} g^{j\bar{j}} g^{i\bar{i}} \partial_k g_{i\bar{j}} \partial_{\bar{k}} g_{j\bar{i}}. \tag{3.22}$$

Also, by (3.16)

$$\frac{|\nabla \operatorname{tr}_{\omega_0}\omega|^2}{\operatorname{tr}_{\omega_0}\omega} \leq g^{j\bar{j}} g^{i\bar{i}} \partial_k g_{i\bar{j}} \partial_{\bar{k}} g_{j\bar{i}} + C|s|_h^{-\alpha_3}.$$

Compute

$$\begin{aligned}
&(\frac{\partial}{\partial t} - \Delta)(|s|_h^\beta \operatorname{tr}_{\omega_0}\omega) \\
&= |s|_h^\beta (\frac{\partial}{\partial t} - \Delta) \operatorname{tr}_{\omega_0}\omega (\operatorname{tr}_{\omega_0}\omega) \Delta |s|_h^{2\beta} - 2\operatorname{Re}(\nabla |s|_h^{2\beta} \cdot \overline{\nabla} \operatorname{tr}_{\omega_0}\omega) \\
&\leq -\frac{1}{C'}|s|_h^{2\beta} S + C'.
\end{aligned}$$

In the last inequality we use

$$\begin{aligned}
2\operatorname{Re}(\nabla |s|_h^{2\beta} \cdot \overline{\nabla} \operatorname{tr}_{\omega_0}\omega) &\leq C + \frac{1}{C}|\nabla |s|_h^\beta|^2 |\nabla \operatorname{tr}_{\omega_0}\omega|^2 \\
&\leq C + \frac{1}{2}|\nabla |s|_h^\beta|^2 (\operatorname{tr}_{\omega_0}\omega)(g^{i\bar{i}} g^{j\bar{j}} \partial_k g_{i\bar{j}} \partial_{\bar{k}} g_{j\bar{i}} + C|s|_h^{-\alpha_0}) \\
&\leq C + \frac{1}{2}|s|_h^\beta g^{i\bar{i}} g^{j\bar{j}} \partial_k g_{i\bar{j}} \partial_{\bar{k}} g_{j\bar{i}}.
\end{aligned}$$

for β large enough. Then fix β. Together with (3.21) and (3.22) we get

$$\begin{aligned}
(\frac{\partial}{\partial t} - \Delta)H \leq\ & C|s|_h^{2\alpha-\alpha_1}(1 + S^{3/2}) - \frac{1}{2}|s|_h^{2\alpha}|\overline{\nabla}\Phi|^2 \\
&+ C_0|s|_h^{3\alpha}|\Delta |s|_h^{-\alpha}| S - |s|_h^{3\alpha+\alpha_2}\frac{S^2}{C_0} + C_0|s|_h^{3\alpha-\alpha_2} S \\
&+ 4\operatorname{Re}\frac{\nabla S \cdot \overline{\nabla}(|s|_h^{-\alpha} - \operatorname{tr}_{\omega_0}\omega)}{(|s|_h^{-\alpha} - \operatorname{tr}_{\omega_0}\omega)^3} + (-\frac{1}{C'}|s|_h^{2\beta} S + C') - A.
\end{aligned}$$

As

$$|\overline{\nabla} S| \leq 2S^{1/2}|\overline{\nabla}\Phi|, \quad |\nabla \operatorname{tr}_{\omega_0}\omega| \leq C|s|_h^{-\alpha_4} S^{1/2}$$

and $|\nabla |s|_h^{-\alpha}| \leq C|s|_h^{-\alpha-\alpha_5}$, we have

$$\left|4\,\mathrm{Re}\,\frac{\nabla S \cdot \overline{\nabla}(|s|_h^{-\alpha} - \mathrm{tr}_{\omega_0}\,\omega)}{(|s|_h^{-\alpha} - \mathrm{tr}_{\omega_0}\,\omega)^3}\right| \leq CS^{1/2}|\overline{\nabla}\Phi|(|s|_h^{2\alpha-\alpha_5} + |s|_h^{3\alpha-\alpha_4}S^{1/2})$$
$$\leq \frac{1}{2}|s|_h^{2\alpha}|\overline{\nabla}\Phi|^2 + C_1(|s|_h^{2\alpha-2\alpha_5}S + |s|_h^{4\alpha-2\alpha_4}S^2).$$

Also

$$C|s|_h^{2\alpha-\alpha_1}S^{3/2} \leq |s|_h^{3\alpha+\alpha_2}\frac{S^2}{2C_0} + C_1|s|_h^{\alpha-2\alpha_1-\alpha_2}S.$$

By choosing α and then A sufficiently large, we have

$$(\frac{\partial}{\partial t} - \Delta)H \leq 0.$$

Thus H has a uniform upper bound and we obtain the desired estimate for S. □

Proposition 3.15. *There exist positive constants λ and C such that for $t \in [0, T)$,*

$$|\,\mathrm{Ric}\,| \leq \frac{C}{|s|_h^{2\lambda}}.$$

Proof. First, we have the evolution equation

$$(\frac{\partial}{\partial t} - \Delta)R_{j\bar{k}} = \nabla_{\bar{k}}T_{lj}^{\ r}R_r^{\ l} + T_{lj}^{\ r}\nabla_{\bar{k}}R_r^{\ l} + R_{l\bar{k}j}^{\ \ \ r}R_r^{\ l}$$
$$- R_{l\bar{k}}^{\ \ \bar{s}l}R_{j\bar{s}} + \nabla^{\bar{q}}T_{\bar{k}\bar{q}}^{\ \ \bar{s}}R_{j\bar{s}} + T_{\bar{k}\bar{q}}^{\ \ \bar{s}}\nabla^{\bar{q}}R_{j\bar{s}}$$

Then it follows from Lemma 2.12 and Proposition 3.14 that

$$(\frac{\partial}{\partial t} - \Delta)|\,\mathrm{Ric}\,| = \frac{1}{2|\,\mathrm{Ric}\,|}((\frac{\partial}{\partial t} - \Delta)|\,\mathrm{Ric}\,|^2 + 2|\nabla|\,\mathrm{Ric}\,||^2)$$
$$\leq |s|_h^{-\alpha}(|\nabla\,\mathrm{Ric}\,| + |\,\mathrm{Rm}\,|^2 + 1) - \frac{|\nabla\,\mathrm{Ric}\,|^2}{|\,\mathrm{Ric}\,|} + \frac{|\nabla|\,\mathrm{Ric}\,||^2}{|\,\mathrm{Ric}\,|}. \tag{3.23}$$

for some constant $\alpha > 0$. Consider

$$H = |s|_h^{3\alpha} |\operatorname{Ric}| + |s|_h^{4\beta} S - At,$$

where α and β are constants to be determined and are at least large enough such that

$$|\overline{\nabla} |s|_h^{3\alpha}| \le C|s|_h^{2\alpha}, \qquad |\Delta |s|_h^{3\alpha}| \le C|s|_h^{2\alpha} \tag{3.24}$$

and

$$|\overline{\nabla} |s|_h^{4\beta}| \le C|s|_h^{3\beta}, \qquad |\Delta |s|_h^{4\beta}| \le C|s|_h^{3\beta}. \tag{3.25}$$

Assume that H achieves maximum at a point (t_0, z_0), $t_0 > 0$ and $|\operatorname{Ric}| > 1$ at $(t_0,\ z_0)$. By (3.21) and the estimate for S,

$$(\frac{\partial}{\partial t} - \Delta) S \le C|s|_h^{-\beta} - \frac{1}{2}|\overline{\nabla}\Phi|^2, \tag{3.26}$$

for sufficiently large α. Together with (3.23), (3.24) and (3.25), we obtain

$$\begin{aligned}(\frac{\partial}{\partial t} - \Delta) H \le & |s|_h^{2\alpha}(|\nabla \operatorname{Ric}| + |\operatorname{Rm}|^2) + |s|_h^{3\alpha}\left(\frac{|\nabla |\operatorname{Ric}||^2}{|\operatorname{Ric}|} - \frac{|\nabla \operatorname{Ric}|^2}{|\operatorname{Ric}|}\right) \\ & - 2\operatorname{Re}\left(\nabla |\operatorname{Ric}| \cdot \overline{\nabla} |s|_h^{3\alpha}\right) - \Delta |s|_h^{3\alpha} |\operatorname{Ric}| - 2\operatorname{Re}\left(\nabla S \cdot \overline{\nabla} |s|_h^{4\beta}\right) \\ & + C|s|_h^{3\beta} - \frac{1}{2}|s|_h^{4\beta}|\overline{\nabla}\Phi|^2 - \Delta |s|_h^{4\beta} S - A. \end{aligned} \tag{3.27}$$

As $\nabla H = 0$ at (t_0, z_0), we have

$$|s|_h^{3\alpha}\overline{\nabla}|\operatorname{Ric}| = -\overline{\nabla}|s|_h^{3\alpha}|\operatorname{Ric}| - |s|_h^{4\beta}\overline{\nabla} S - \nabla |s|_h^{4\beta} S$$

Combining with (3.24) and (3.25) and $|\nabla|\operatorname{Ric}|| \le |\nabla \operatorname{Ric}|$, we get

$$\begin{aligned}\frac{|\nabla|\operatorname{Ric}||^2}{|s|_h^{-3\alpha}|\operatorname{Ric}|} &\le C|s|_h^{2\alpha}|\nabla \mathrm{Ric}| + \frac{|\overline{\nabla}S||\nabla\mathrm{Ric}|}{|s|_h^{-4\beta}|\operatorname{Ric}|} + C\frac{S|\nabla\mathrm{Ric}|}{|s|_h^{-3\beta}|\operatorname{Ric}|} \\ &\le \frac{|\nabla\operatorname{Ric}|^2}{4|s|_h^{-3\alpha}|\operatorname{Ric}|} + C_1|s|_h^{\alpha}|\operatorname{Ric}| + C_2\frac{|s|_h^{7\beta}|\overline{\nabla}\Phi|^2}{|s|_h^{3\alpha}|\operatorname{Ric}|} \\ &\quad + C_3\frac{|s|_h^{4\beta}}{|s|_h^{3\alpha}|\operatorname{Ric}|} \end{aligned} \tag{3.28}$$

where we use $|\overline{\nabla}S| \le 2S|\overline{\nabla}\Phi|^2$ and $S \le C|s|_h^{\beta}$ in the second inequality.

$$\begin{aligned}(\frac{\partial}{\partial t} - \Delta)H \le & C|s|_h^{\alpha}|\operatorname{Ric}| + C\frac{|s|_h^{7\beta}|\overline{\nabla}\Phi|^2}{|s|_h^{3\alpha}|\operatorname{Ric}|} + C\frac{|s|_h^{4\beta}}{|s|_h^{3\alpha}|\operatorname{Ric}|} \\ & + C|s|_h^{2\beta}|\overline{\nabla}\Phi| - \frac{1}{2}|s|_h^{4\beta}|\overline{\nabla}\Phi|^2 + C - A.\end{aligned}$$

Assume at (t_o, z_0), $|s|_h^{3\alpha}|\operatorname{Ric}| \le 1$ (otherwise, $H \le 2$ and the bound for $|\operatorname{Ric}|$ follows). By (2.27) and (3.15),

$$|\operatorname{Ric}| \le |\overline{\nabla}\Phi| + |s|_h^{-\alpha_1}.$$

For α large enough, the term $|s|_h^{\alpha}|\operatorname{Ric}|$ can be controlled by $\frac{1}{4}|s|_h^{4\beta}|\overline{\nabla}\Phi|^2$. Choosing α and then A sufficiently large, we get

$$(\frac{\partial}{\partial t} - \Delta)H \le 0,$$

therefore we have the uniform upper bound for H and the proposition follows. □

3.3.3 Smooth convergence as $t \to T^+$

In this section, we will prove sharper bounds on φ_ϵ and φ and then obtain the smooth convergence of the metrics $\omega(t)$ to $\tilde{\omega}_T$ on compact subsets of $N \setminus \{y_0\}$ as $t \to T^+$.

Recall that $\tilde{\omega}_{T,\epsilon} = \omega_N + \sqrt{-1}\partial\bar{\partial}\psi_{T,\epsilon}$. For simplicity, write $\tilde{\omega} = \tilde{\omega}_{T,\epsilon}$, $\hat{\omega} = \hat{\omega}_T = \omega_N$ and $|s|_h^2 = (\pi|_{M\setminus E}^{-1})^*(|s|_h^2)$ in the proof of the following lemma. Then

$$\tilde{\omega}^n = (\hat{\omega} + \sqrt{-1}\partial\bar{\partial}\psi_{T,\epsilon})^n = e^{F_\epsilon}\hat{\omega}^n, \quad \text{where} \quad F_\epsilon = \log\left(\frac{C_\epsilon \Omega_\epsilon}{\hat{\omega}^n}\right).$$

Lemma 3.16. *There exist constants $\lambda > 0$ and $C > 0$, independent of ϵ, such that*

$$\frac{|s|_h^{2\lambda}}{C}\omega_N \leq \tilde{\omega}_{T,\epsilon} \leq \frac{C}{|s|_h^{2\lambda}}\omega_N$$

on $N \setminus \{y_0\}$.

Proof. As $\psi_{T,\epsilon}$ is uniformly bounded by (3.14), there exists a constant C_0 such that $\psi_{T,\epsilon} + C_0 \geq 1$. Take ϵ_0 small enough such that $\hat{\omega} - \epsilon_0 R_h \geq c\hat{\omega}$ for some positive constant c, where $R_h = -\sqrt{-1}\partial\bar{\partial}\log|s|_h^2$ is the curvature of the Hermitian metric h. Let $\tilde{\psi}_{T,\epsilon} = \psi_{T,\epsilon} - \epsilon_0 \log|s|_h^2$ and define

$$H = \log \operatorname{tr}_{\hat{\omega}} \tilde{\omega} - A\tilde{\psi}_{T,\epsilon} + \frac{1}{\tilde{\psi}_{T,\epsilon} + C_0}.$$

Note that $H(t,y)$ goes to negative infinity as y tends to y_0. Compute at a point in $N \setminus \{y_0\}$. Assume $\operatorname{tr}_{\hat{\omega}} \tilde{\omega} \geq 1$ at this point. From [46, Section 9], we have

$$\Delta_{\tilde{\omega}} \log \operatorname{tr}_{\hat{\omega}} \tilde{\omega} \geq \frac{2}{(\operatorname{tr}_{\hat{\omega}} \tilde{\omega})^2} \operatorname{Re}\left(\tilde{g}^{k\bar{q}}\hat{T}^p_{pk}\partial_{\bar{q}} \operatorname{tr}_{\hat{\omega}} \tilde{\omega}\right) - C \operatorname{tr}_{\tilde{\omega}} \hat{\omega} - |\Delta_{\hat{\omega}} F_\epsilon| - C. \tag{3.29}$$

Assume that H achieves a maximum at (t_0, z_0). As $\partial_q H = 0$ at (t_0, z_0), that is

$$\frac{\partial_q \operatorname{tr}_{\hat{\omega}} \tilde{\omega}}{\operatorname{tr}_{\hat{\omega}} \tilde{\omega}} - A\partial_q \psi_{T,\epsilon} - \frac{\partial_q \psi_{T,\epsilon}}{(\psi_{T,\epsilon} + C_0)^2} = 0,$$

we get

$$\begin{aligned}
&\left|\frac{2}{(\operatorname{tr}_{\hat{\omega}}\tilde{\omega})^2}\operatorname{Re}(\tilde{g}^{k\bar{q}}\hat{T}^j_{jk}\hat{\nabla}_{\bar{q}}\operatorname{tr}_{\hat{\omega}}\tilde{\omega})\right| \\
= &\left|\frac{2}{(\operatorname{tr}_{\hat{\omega}}\tilde{\omega})^2}\operatorname{Re}(\tilde{g}^{k\bar{q}}\hat{T}^j_{jk}(A+\frac{1}{\psi_{T,\epsilon}+C_0)^2})\partial_{\bar{q}}\psi_{T,\epsilon}\right| \\
\leq &\frac{|\partial\psi_{T,\epsilon}|^2_{\tilde{\omega}}}{(\psi_{T,\epsilon}+C_0)^3}+\frac{(\psi_{T,\epsilon}+C_0)^3 CA^2\operatorname{tr}_{\tilde{\omega}}\hat{\omega}}{(\operatorname{tr}_{\hat{\omega}}\tilde{\omega})^2}.
\end{aligned} \tag{3.30}$$

If at (t_0, z_0), $(\operatorname{tr}_{\hat{\omega}}\tilde{\omega})^2 \leq A^2(\tilde{\varphi}_{T,\epsilon}+C_0)^3$, then

$$H \leq \log A + \frac{3}{2}\log(\tilde{\varphi}_{T,\epsilon}+C_0) - A\tilde{\varphi}_{T,\epsilon} + \frac{1}{\tilde{\varphi}_{T,\epsilon}+C_0}.$$

As $\tilde{\varphi}_{T,\epsilon}+C \geq 1$, we have an upper bound for H and thus $\tilde{\omega}$ is bounded from above. Otherwise, $A^2(\tilde{\varphi}_{T,\epsilon}+C_0)^3 \leq (\operatorname{tr}_{\hat{\omega}}\tilde{\omega})^2$ at the maximum point. Moreover, by the definition of Ω_ϵ, it follows from (3.15) and Proposition 3.15 that

$$|\Delta_{\hat{\omega}}F_\epsilon| \leq \frac{C}{|s|^{2\beta}}$$

for uniform constants C and β. Together with (3.29) and (3.30) we have

$$\Delta_{\tilde{\omega}} H = \Delta_{\tilde{\omega}} \log \operatorname{tr}_{\hat{\omega}} \tilde{\omega} + \frac{2|\partial \psi_{T,\epsilon}|^2_{\tilde{\omega}}}{(\psi_{T,\epsilon} + C_0)^3}$$

$$- \left(A + \frac{1}{(\tilde{\psi}_{T,\epsilon} + C_0)^2} \right) \operatorname{tr}_{\tilde{\omega}} (\tilde{\omega} - \hat{\omega} + \epsilon_0 R_h)$$

$$\geq \Delta_{\tilde{\omega}} \log \operatorname{tr}_{\hat{\omega}} \tilde{\omega} + A \operatorname{tr}_{\tilde{\omega}} \hat{\omega} + \frac{2|\partial \psi_{T,\epsilon}|^2_{\tilde{\omega}}}{(\psi_{T,\epsilon} + C_0)^3} - (A+1)n$$

$$\geq \frac{2}{(\operatorname{tr}_{\hat{\omega}} \tilde{\omega})^2} \operatorname{Re}(\tilde{g}^{k\bar{q}} \hat{T}^j_{jk} \hat{\nabla}_{\bar{q}} \operatorname{tr}_{\hat{\omega}} \tilde{\omega}) - C \operatorname{tr}_{\tilde{\omega}} \hat{\omega} - C - |\Delta_{\hat{\omega}} F_\epsilon|. \tag{3.31}$$

Then

$$\Delta_{\tilde{\omega}} H \geq A \operatorname{tr}_{\tilde{\omega}} \hat{\omega} - \frac{(\psi_{T,\epsilon} + C_0)^3 C A^2 \operatorname{tr}_{\tilde{\omega}} \hat{\omega}}{(\operatorname{tr}_{\hat{\omega}} \tilde{\omega})^2} - \frac{C}{|s|_h^{2\lambda}}$$
$$\geq (A - C) \operatorname{tr}_{\tilde{\omega}} \hat{\omega} - \frac{C}{|s|_h^{2\lambda}}.$$

At the maximum point, $\Delta_{\tilde{\omega}} H \leq 0$, therefore

$$\operatorname{tr}_{\tilde{\omega}} \hat{\omega} \leq \frac{C}{|s|_h^{2\lambda}}$$

for sufficiently large A. Then at (t_0, z_0),

$$\operatorname{tr}_{\hat{\omega}} \tilde{\omega} \leq \frac{1}{(n-1)!} (\operatorname{tr}_{\tilde{\omega}} \hat{\omega})^{n-1} \frac{\tilde{\omega}^n}{\hat{\omega}^n} \leq \frac{C}{|s|_h^{2\beta}}$$

as $\tilde{\omega}^n = C_\epsilon \Omega_\epsilon$ has an upper bound by definition of Ω_ϵ and (3.15). Thus there exists $C > 0$, independent of ϵ, such that $H \leq C$ for sufficiently large α. Since $\tilde{\varphi}_{T,\epsilon} + C_0 \geq 1$, we see that $\tilde{\omega}$ is uniformly bounded from above. $\square$

Define

$$\omega_\epsilon(t) = \hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi_\epsilon \tag{3.32}$$

for $t \in [T, T']$. From Lemma 5.4 of [39], we have the following volume bound.

Lemma 3.17. *There exist constants $\lambda > 0$ and $C > 0$, independent of ϵ, such that*

$$\frac{\omega_\epsilon^n}{\Omega_N} \leq \frac{C}{|s|_h^{2\lambda}}$$

on $[T, T'] \times (N \setminus \{y_0\})$.

With Lemma 3.16 and Lemma 3.17, the upper bound for $\omega(t)$ can be obtained by using the argument of Tosatti-Weinkove [47] (see also [34]). For simplicity, we write $\hat{\omega} = \omega_N, \quad \tilde{\omega} = \omega_\epsilon$ in the proof of the following lemma.

Lemma 3.18. *There exist constants $\lambda > 0$ and $C > 0$, independent of ϵ, such that on $[T, T'] \times (N \setminus \{y_0\})$,*

$$\frac{|s|_h^{2\lambda}}{C}\omega_N \leq \omega_\epsilon \leq \frac{C}{|s|_h^{2\lambda}}\omega_N.$$

Proof. Take ϵ_0 small enough such that $\hat{\omega}_t - \epsilon_0 R_h \geq c\hat{\omega}$ for any $t \in [T, T']$ for some constant $c > 0$. Let $\tilde{\varphi}_\epsilon = \varphi_\epsilon - \epsilon_0 \log |s|_h^2$. Then there exists a positive constant C, such that $\tilde{\varphi}_\epsilon + C_0 \geq 1$. Define

$$H = \log \mathrm{tr}_{\hat{\omega}}\, \tilde{\omega} - A\tilde{\varphi}_\epsilon + \frac{1}{\tilde{\varphi}_\epsilon + C_0},$$

where A and α are positive constants to be determined. We have

$$H|_{t=T} \leq \log \mathrm{tr}_{\omega_N}\, \tilde{\omega}_{T,\epsilon}|s|_h^{A\epsilon_0} - A\psi_{T,\epsilon} + 1.$$

By Lemma 3.16, H is uniformly bounded from above at time T. Moreover, $H(t,y)$ tends to negative infinity as y tends to y_0, for any $t \in [T, T']$. Compute at a point in $N \setminus \{y_0\}$ with $\operatorname{tr}_{\hat{\omega}} \tilde{\omega} \geq 1$. From Proposition 3.6,

$$\left(\frac{\partial}{\partial t} - \Delta_{\tilde{\omega}}\right) \log \operatorname{tr}_{\hat{\omega}} \tilde{\omega} \leq \frac{2}{(\operatorname{tr}_{\hat{\omega}} \tilde{\omega})^2} \operatorname{Re}\left(\tilde{g}^{k\bar{q}} \hat{T}^p_{kp} \partial_{\bar{q}} \operatorname{tr}_{\hat{\omega}} \tilde{\omega}\right) + C \operatorname{tr}_{\tilde{\omega}} \hat{\omega}. \tag{3.33}$$

Assume H achieves its maximum at (t_0, z_0), we have $\partial_{\bar{q}} \operatorname{tr}_{\hat{\omega}} \tilde{\omega} = 0$ at this point, thus

$$\begin{aligned} & \left| \frac{2}{(\operatorname{tr}_{\hat{\omega}} \tilde{\omega})^2} \operatorname{Re}\left(\tilde{g}^{k\bar{q}} \hat{T}^p_{kp} \partial_{\bar{q}} \operatorname{tr}_{\hat{\omega}} \tilde{\omega}\right) \right| \\ \leq & \left| \frac{2}{\operatorname{tr}_{\hat{\omega}} \tilde{\omega}} \operatorname{Re}\left(\tilde{g}^{k\bar{q}} \hat{T}^p_{kp} \left(A + \frac{1}{(\tilde{\varphi}_\epsilon + C_0)^2}\right) \partial_{\bar{q}} \tilde{\varphi}_\epsilon \right) \right| \\ \leq & \frac{|\partial \tilde{\varphi}_\epsilon|^2}{(\tilde{\varphi}_\epsilon + C_0)^3} + C A^2 (\tilde{\varphi}_\epsilon + C_0)^3 \frac{\operatorname{tr}_{\tilde{\omega}} \hat{\omega}}{(\operatorname{tr}_{\hat{\omega}} \tilde{\omega})^2}. \end{aligned} \tag{3.34}$$

If at (t_0, z_0), $(\operatorname{tr}_{\hat{\omega}} \tilde{\omega})^2 \leq A^2 (\tilde{\varphi}_\epsilon + C_0)^3$, then

$$H \leq \log A + \frac{3}{2} \log(\tilde{\varphi}_\epsilon + C_0) - A \tilde{\varphi}_\epsilon + \frac{1}{\tilde{\varphi}_\epsilon + C_0}.$$

As $\tilde{\varphi}_\epsilon + C \geq 1$, we have an upper bound for H and thus $\tilde{\omega}$ is bounded from above. Otherwise, $A^2(\tilde{\varphi}_\epsilon + C_0)^3 \leq (\operatorname{tr}_{\hat{\omega}} \tilde{\omega})^2$ at the maximum point. Computing the evolution of H, it follows from (3.33) and (3.34) at (t_0, z_0)

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta_{\tilde{\omega}}\right) H \leq\, & C \operatorname{tr}_{\tilde{\omega}} \hat{\omega} - \left(A + \frac{1}{\tilde{\varphi}_\epsilon + C_0)^2}\right) \dot{\varphi}_\epsilon \\ & + \left(A + \frac{1}{\tilde{\varphi}_\epsilon + C_0)^2}\right) \operatorname{tr}_{\tilde{\omega}}(\tilde{\omega} - \hat{\omega}_t + \epsilon_0 R_h) \\ \leq\, & C \operatorname{tr}_{\tilde{\omega}} \hat{\omega} + (A+1) \log \frac{\Omega_N}{\tilde{\omega}}^n + C + (A+1)n \\ & - A \operatorname{tr}_{\tilde{\omega}}(\hat{\omega}_t - \epsilon_0 R_h).\end{aligned}$$

As $\hat{\omega}_t - \epsilon_0 R_h \geq c\hat{\omega}$ and $\left(\frac{\partial}{\partial t} - \Delta_{\tilde{\omega}}\right) H \leq 0$ at this point, we have

$$\operatorname{tr}_{\tilde{\omega}} \hat{\omega} \leq C \log \frac{\Omega}{\omega^n} + C_1.$$

for A large enough. Then at (t_0, z_0),

$$\begin{aligned}\operatorname{tr}_{\hat{\omega}} \tilde{\omega} &\leq \frac{1}{(n-1)!} (\operatorname{tr}_{\tilde{\omega}} \hat{\omega})^{n-1} \frac{\det \tilde{\omega}}{\det \hat{\omega}} \\ &\leq C \frac{\tilde{\omega}^n}{\Omega_N} \left(\log \frac{\Omega_N}{\tilde{\omega}^n}\right)^{n-1} + C' \\ &\leq \frac{C}{|s|_h^{2\beta}}\end{aligned}$$

as $\hat{\omega}^n/\Omega_N \leq \frac{C}{|s|_h^{2\lambda}}$ by Lemma 3.16. Hence there exists $C > 0$, independent of ϵ, such that $H \leq C$ for sufficiently large α. Since $\tilde{\varphi}_\epsilon + C_0 \geq 1$, we see that $\tilde{\omega}$ is uniformly bounded from above. The lower bound follows from an argument similar to the proof of Lemma 2.3 in [47]. □

Proof of Theorem 3.11. The existence and uniqueness follows from Proposition 3.13. By Lemma 3.18, for any compact subset $K \subset N \setminus \{y_0\}$, there exists a positive constant

C_K such that

$$\frac{\omega_N}{C_K} \leq \omega(t) \leq C_K \omega_N \qquad \text{on } [T, T'] \times N.$$

The local estimates of Gill [15] then gives uniform C^∞ estimates for $\omega(t)$ on compact subsets of $N \setminus \{y_0\}$. The smooth convergence follows from this and we finish the proof.

$\square$

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