Construction of New Exotic Symplectic 4-Manifolds and Fillings of Contact 3-Manifolds

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Abstract

This dissertation studies the geography problem for symplectic/non-symplectic 4-manifolds. Examples of new families of 4-manifolds that is homeomorphic but not diffeomorphic to $(2n+2k-3)\mathbb{CP}^2\#(6n+2k-3)\overline{\mathbb{CP}}^2$, for any $n \geq 1$, $k \geq 1$, and $(n,k) \neq (1,1)$, are constructed using higher genus Lefschetz fibrations introduced by Yusuf Gürtaş, product 4-manifolds and certain symplectic operations such as Luttinger surgery and symplectic connected sum. As well as, examples of the families of Lefschetz fibrations over $\mathbb{S}^2$ are discussed which are obtained by using the cyclic group actions on product manifolds. The latter symplectic 4-manifolds has nice applications using rational blow-down and equivariant sum.

Contents

List of Tables

List of Figures

Chapter 1

Introduction

Discovery of exotic smooth structures on simply-connected 4-manifolds has played an important role in the study of topology of smooth 4-manifolds. The existence of an exotic smooth structure on a 4-manifold was first proved by Donaldson in [**Do3**]. He showed that a Dolgachev surface $E(1)_{2,3}$ is homeomorphic but not diffeomorphic to $\mathbb{CP}^2\#9\overline{\mathbb{CP}}^2$ using $SU(2)$ gauge theory. And consequently infinitely many irreducible smooth structures on $\mathbb{CP}^2\#9\overline{\mathbb{CP}}^2$ were constructed by Friedman and Morgan [**FM1**].

For the sake of convenience of the reader, let us recall some recent history on this problem. We refer the reader to [**A1**], [**A2**], [**A3**] and [**AP**] for the complete account of the history.

In early 2004, Jongil Park [**P2**] has constructed the first example of exotic smooth structure on $\mathbb{CP}^2\#7\overline{\mathbb{CP}}^2$, i.e. 4-manifold homeomorphic but not diffeomorphic to $\mathbb{CP}^2\#7\overline{\mathbb{CP}}^2$ using the Rational blow-down surgery. Shortly after, András Stipsicz and Zoltán Szabó [**SS1**] constructed an exotic smooth structure on $\mathbb{CP}^2\#6\overline{\mathbb{CP}}^2$ using a technique similar to Park's. Furthermore, Fintushel and Stern [**FS3**] introduced a new technique, surgery in double nodes, which verify that $\mathbb{CP}^2\#k\overline{\mathbb{CP}}^2$, $k = 6, 7, 8$, have infinitely many distinct smooth structures. Then using similar ideas, Park, Stipsicz and Szabó [**PSS**] constructed exotic smooth structures (not known if symplectic) when $k = 5$, Park [**P3**] for $k = 8$ and Stipsicz and Szabó [**SS2**] constructed exotic smooth structures on $3\mathbb{CP}^2\#9\overline{\mathbb{CP}}^2$. All these infinite family of manifolds were constructed by applying the sequence of knot surgery in double nodes, blow-ups and rational blowdown to elliptic surfaces $E(1)$ and $E(2)$. In [**A3**], Anar Akhmedov obtain similar result starting from $E(n)$ for $n \geq 3$ and constructed an infinite family of simply connected, non-symplectic and pairwise nondiffeomorphic manifolds with nontrivial SW-invariants. In [**A2**], he constructed an exotic smooth structure on $3\mathbb{CP}^2\#7\overline{\mathbb{CP}}^2$ and the first known such exotic symplectic $\mathbb{CP}^2\#5\overline{\mathbb{CP}}^2$. The difference from the other constructions is that he didn't use any rational blowdown surgery and he used non-simply connected building blocks to produce simply connected examples.

The main goal of Chapter 3 is to exhibit a new family of simply connected minimal symplectic and infinitely many non-symplectic 4-manifolds that is homeomorphic but not diffeomorphic to $(2n+2k-3)\mathbb{CP}^2\#(6n+2k-3)\overline{\mathbb{CP}}^2$ for any $n \geq 1$, $k \geq 1$, and $(n,k) \neq (1,1)$ (cf. [**AS**]). Our motivation for constructing such 4-manifolds comes from [**A2**], where the first known exotic minimal symplectic $\mathbb{CP}^2\#5\overline{\mathbb{CP}}^2$ (the case when $(n,k) = (1,1)$) was constructed by A. Akhmedov using Y. Matsumoto's genus two Lefschetz fibrations on $\mathbb{T}^2 \times \mathbb{S}^2\#4\overline{\mathbb{CP}}^2$ over $\mathbb{S}^2$ along with the fake symplectic $\mathbb{S}^2 \times \mathbb{S}^2$ construction in [**A1**] obtained via a combination of a knot surgery along the fibered knots in $\mathbb{S}^3$ and the twisted fiber sums. It was an interesting question if a similar construction can be carried out using M. Korkmaz's and Y. Gürtaş' higher genus Lefschetz fibrations on $\Sigma_k \times \mathbb{S}^2\#4n\overline{\mathbb{CP}}^2$ (i.e. the genus $g \geq 2$). These higher genus Lefschetz fibrations arise from the Dehn twist factorization of a certain involution of the genus $2k+n-1$ surface, which are the generalizations of Matsumoto's genus two fibration. If $k \geq 2$, the fundamental group of these Lefschetz fibrations are not abelian, and the relations in the fundamental group coming from the vanishing cycles are more complicated. Therefore, the fundamental group computations are subtle one than in [**A2**]. Our approach is also different than in [**A2**] in the sense that we use the Luttinger surgery instead of the knot surgery. The methods highlighted can also be used to get new symplectic 4-manifolds with various fundamental groups (and with a small size) by applying the Luttinger surgeries to a certain family of Lefschetz fibrations. (cf. [**AS**]).

In Chapter 4, we work on constructing symplectic 4-manifold using finite order cyclic group actions on product manifolds with cyclic quotient singularities. One of the most useful tools in the study of algebraic surfaces is the analysis of fibrations, that is morphisms with connected fibers from a surface X onto a curve C. When all smooth fibers of a fibration $\phi : X \to C$ are isotopic to each other, we call ϕ an *isotrivial fibration* which is studied in [**Po**]. A smooth, projective surface S is called a *standard isotrivial fibration* if there exists a finite group G, acting faithfully on two smooth projective curves C_1 and C_2, so that S is isomorphic to the minimal desingularization of $T := (C_1 \times C_2)$, where G acts diagonally on the product. A special case of such manifolds have been investigated in [**Ma**]. Another inspiring example was given by Akhmedov and Park [**AP**]. They first blow-up the union of the graphs of finite powers of genus $g{+}1$ and $2g{+}1$ actions on $\Sigma_g \times \Sigma_g$ at their intersection points. Then using branched covering techniques, they constructed symplectic, non-spin, irreducible, simply connected 4-manifolds with $0 \leq \sigma(X) \leq 4$. These examples have the smallest Euler characteristic among all known simply-connected examples with non-negative signature which admits more than one smooth structure.

Finally, in Chapter 5, by generalizing Ghiggini's examples in [**Gh2**], we aim to give more examples of contact 3-manifolds which are strongly fillable but not Stein fillable. We start with the 3-manifold M_g obtained by 0-framed surgery on $(2, 2g+1)$-torus knot in $\mathbb{S}^3$, instead of the trefoil knot, $(2, 3)$-torus knot, which admits a structure of Σ_g-bundle over $\mathbb{S}^1$.

Chapter 2

Background

2.1 Symplectic Manifolds

In this section, we will give some basic definitions and results on symplectic 4-manifolds. (cf. [**Si**])

Definition 1. *A symplectic form* on $2n$-dimensional smooth manifold X is a differential 2-form ω that is closed ($dw = 0$) and non-degenerate (for every nonzero vector $u \in TX$ there is a vector $v \in TX$ such that $\omega(u, v) \neq 0$).

The pair (X, ω) is called *symplectic manifold*.

Definition 2. Let (M, ω) be a $2n$-dimensional symplectic manifold, and $N \subset M$ be a submanifold of M. The tangent bundle TN is a subbundle of $TM_{|_N}$. Its symplectic orthogonal is defined as

$$TN^{\omega} = \bigcup_{q \in N} \{u \in TM_q : \omega(u, v) = 0 \text{ for all } m \in TN_q\},$$

which is also a subbundle of $TM_{|_N}$.

1. N is called *symplectic submanifold* (i.e. $w_{|N}$ is nondegenerate) if

$$TN \cap TN^{w} = \{0\}$$

2. N is called *isotropic submanifold* if $TN \subset TN^w$.

3. N is called *coisotropic submanifold* if $TN^w \subset TN$.

4. N is called *Lagrangian submanifold* if at each $p \in N$, T_pN is a Lagrangian subspace of T_pM, i.e. $\omega_p|_{T_pN} = 0$ and $dimT_pN = \frac{1}{2}dimT_pM$.

   Equivalently, if $i : N \to M$ is the *the inclusion map*, then N is *lagrangian* iff $i^*\omega = 0$ and $dimN = \frac{1}{2}dimM$.

Definition 3. Let (M_1, ω_1) and (M_2, ω_2) be $2n$-dimensional symplectic manifolds, and let $g : M_1 \to M_2$ be a diffeomorphism. Then g is a *symplectomorphism* if $g^*\omega_2 = \omega_1$.

Example 2.1.1. Let $M = \mathbb{R}^{2n}$ with linear coordinates $x_1, \cdots, x_n, y_1, \cdots, y_n$. The form $\omega_0 = \Sigma_{i=1}^n dx_i \wedge dy_i$ is symplectic as can be easily checked, which is called *the standard symplectic form* on $\mathbb{R}^{2n}$, and the set

$$\{(\frac{\partial}{\partial x_1})_p, \cdots, (\frac{\partial}{\partial x_n})_p, (\frac{\partial}{\partial y_1})_p, \cdots, (\frac{\partial}{\partial y_n})_p\}$$

is a symplectic basis of T_pM.

Example 2.1.2. The Kodaira-Thurston Manifold (Thurston, 1976 [Th]) Take $\mathbb{R}^4$ with the standard symplectic form $dx_1 \wedge dy_1 + dx_2 \wedge dy_2$, and Γ the discrete group generated by the following symplectomorphisms:

1. $\gamma_1 := (x_1, x_2, y_1, y_2) \longrightarrow (x_1, x_2 + 1, y_1, y_2)$

2. $\gamma_2 := (x_1, x_2, y_1, y_2) \longrightarrow (x_1, x_2, y_1, y_2 + 1)$

3. $\gamma_3 := (x_1, x_2, y_1, y_2) \longrightarrow (x_1 + 1, x_2, y_1, y_2)$

4. $\gamma_4 := (x_1, x_2, y_1, y_2) \longrightarrow (x_1, x_2 + y_2, y_1 + 1, y_2)$

Then $M = \mathbb{R}^4 \backslash \Gamma$ is a flat 2-torus bundle over a 2-torus. Kodaira had shown that M has a complex structure. Since, $\Pi_1(M) = \Gamma$, we have $H^1(\mathbb{R}^4 \backslash \Gamma; \mathbb{Z}) = \Gamma \backslash [\Gamma, \Gamma]$ has rank 3, i.e. $b_1 = 3$ is odd. It is well known that the odd-dimensional Betti numbers of a compact Kähler manifold are even. Therefore, M is *not* Kähler.

Example 2.1.3. Product Symplectic Manifolds $\Sigma_g \times \Sigma_h$

Let Σ_g denote the closed, connected, oriented surface of genus g. A volume form, i.e., any never vanishing 2-form, of a closed 2-manifold Σ_g is a symplectic form. Thus, any oriented hypersurface $\Sigma_g \subset \mathbb{R}^3$ carries a natural symplectic form and a natural compatible almost complex structure induced by the standard inner (or dot) and exterior (or vector) products. They are given by the formulas $\omega(u, v) := \langle v_p, u \times v \rangle$ and $J_p(v) = v_p \times v$ for $v \in T_p\Sigma$, where v_p is the outward-pointing unit normal vector at $p \in \Sigma$. The corresponding Riemannian metric is the restriction of the standard euclidean metric $\langle ., . \rangle$. Products of symplectic manifolds are naturally symplectic by taking the sum of the pullbacks of the symplectic forms from the factors. In other words, $\pi_1^*\omega_1 + \pi_2^*\omega_2$ gives a symplectic structure on $\Sigma_g \times \Sigma_h$. (cf. [**GS**], pg. 386)

2.2 Mapping Class Group

In this section, we will give necessary tools from the theory of mapping class group. We refer the reader to [**FM**] for details.

Let Σ_g denote a 2-dimensional, closed, oriented, and connected surface of genus $g > 0$ surface.

Definition 4. Let $Diff^+(\Sigma_g)$ denote the group of all orientation-preserving diffeomorphisms $\Sigma_g \to \Sigma_g$, and $Diff_0^+(\Sigma_g)$ be the subgroup of $Diff^+(\Sigma_g)$ consisting of all orientation-preserving diffeomorphisms $\Sigma_g \to \Sigma_g$ that are isotopic to the identity. *The mapping class group* M_g of Σ_g is defined to be the group of isotopy classes of orientation-preserving diffeomorphisms of Σ_g, i.e.

$$M_g = Diff^+(\Sigma_g) / Diff_0^+(\Sigma_g).$$

Definition 5. Let α be a simple closed curve on Σ_g. A *right handed Dehn twist* t_α about α is the isotopy class of a self-diffeomophism of Σ_g obtained by cutting the surface Σ_g along α and gluing the ends back after rotating one of the ends 2π to the right. (See Figure 2.1.)

Definition 6. Let $S(\Sigma_g)$ denote be the set of all isotopy classes of simple closed curves

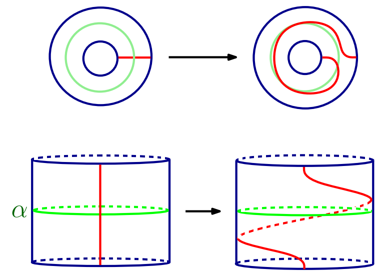


Figure 2.1: Two views of a Dehn twist

in Σ_g. The *intersection number* of two classes $[\alpha]$ and $[\beta]$ is

$$\mu([\alpha],[\beta]) = min\{|a \cap b| | a \in [\alpha], b \in [\beta]\}$$

Some Facts.

1. The conjugate of a Dehn twist is again a Dehn twist.

   Indeed, if $f : \Sigma_g \to \Sigma_g$ is an orientation-preserving diffeomorphism, then

$$f \circ t_\alpha \circ f^{-1} = t_{f(\alpha)}$$

2. If α and β are two simple closed curves with $\mu([\alpha],[\beta]) = k \geq 0$, then

$$t_\alpha(\beta) = \alpha^k \cdot \beta$$

Example 2.2.1. Let a and b be the simple closed curves on the torus $\mathbb{T}^2$ given in Figure 2.2, where t_a and t_b denote Dehn twists about a and b, respectively. It is well known that the mapping class group of $\mathbb{T}^2 = a \times b$ is $M_1 = SL(2,\mathbb{Z})$ and is generated by

$$t_a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$t_b = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

with relations $t_a t_b t_a = t_b t_a t_b$ and $(t_a t_b)^6 = 1$. (cf. [**Ro1**])

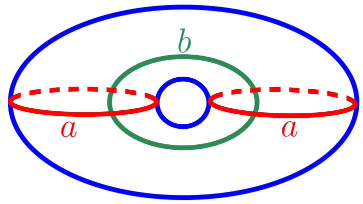


Figure 2.2: vanishing cycles on T^2

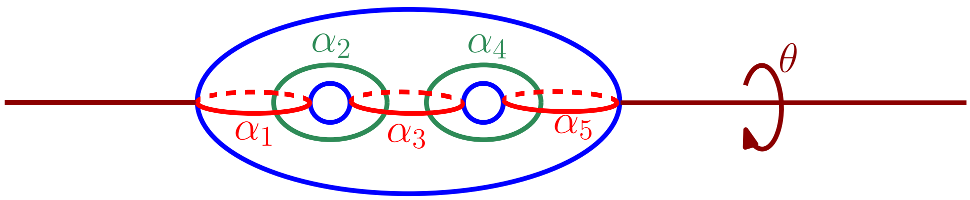


Figure 2.3: vanishing cycles on Σ_2

Example 2.2.2. Consider the hyperelliptic involution θ of Σ_2 which is a rotation by π. Let $\alpha_1, \ldots, \alpha_5$ denote the collection of simple closed curves given in Figure 2.3, and t_i denote the right handed Dehn twists t_{α_i} along the curve α_i. Then, θ can be expressed as

$$\theta = t_5 t_4 t_3 t_2 t_1 t_1 t_2 t_3 t_4 t_5$$

$M_2 = \langle t_1, t_2, t_3, t_4, t_5 | t_i t_j = t_j t_i \quad for \quad |i-j| \geq 2,$

$$t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}, \tag{2.1}$$

$$(t_1 t_2 t_3)^4 = t_5^2, \tag{2.2}$$

$$[\theta, t_1] = 1 \tag{2.3}$$

$$\theta^2 = 1 \rangle \tag{2.4}$$

The first relation is known as *commutativity relation*, the second is called *braid relation* and the third one is known as 3-*chain relation*.(cf. [**FM**])

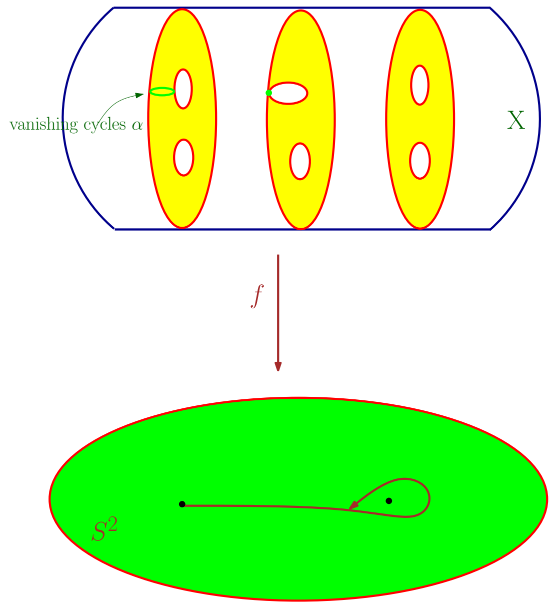


Figure 2.4: Lefschetz Fibration

2.3 Lefschetz Fibration

In this section, we review the basics on a fundamental object we deal with, i.e. the Lefschetz fibrations. (cf. [**Fu**])

Definition 7. Let X be a compact, connected, oriented, smooth 4-manifold. A *Lefschetz fibration* on X is a smooth map $f : X \to \Sigma_h$, (see Figure 2.4), where Σ_h is a compact, oriented, smooth 2-manifold of genus h, such that

1. f is surjective
2. each critical point of f has an orientation preserving chart on which
   $f : \mathbb{C}^2 \to \Sigma_h$ is given by $f(z_1, z_2) = z_1^2 + z_2^2$.

It follows from Sard's theorem that f is a smooth fiber bundle away from finitely many critical points. Let us denote the critical points of f by $p_1, \ldots, p_s$.

Definition 8. The genus of the regular fiber of f is defined to be *the genus of the Lefschetz fibration*.

Definition 9. A fiber of f passing through the critical point set $p_1, \ldots, p_s$ is called a *singular fiber* which is an immersed surface with a single transverse self-intersection.

A singular fiber of the genus g Lefschetz Fibration can be described by its monodromy, i.e., an element of the mapping class group M_g.

Definition 10. This element is a right-handed (or a positive) Dehn twist along a simple closed curve on Σ_g, called the *vanishing cycle*. If this curve is a nonseparating curve, then the singular fiber is called *nonseparating*, otherwise it is called *separating*.

Now, we will recall an important fact on Lefschetz fibrations. For a genus g Lefschetz fibration over $\mathbb{S}^2$, the product of right handed Dehn twists t_{α_i} along the vanishing cycles α_i, for $i = 1, \ldots, s$, determines the global monodromy of the Lefschetz fibration, the relation $t_{\alpha_1} t_{\alpha_2} \cdots t_{\alpha_s} = 1$ in M_g. Conversely, such a relation in M_g determines a genus g Lefschetz fibration over $\mathbb{S}^2$ with the vanishing cycles $\alpha_1, \ldots, \alpha_s$.

Lemma 2.3.1. *([**GS**]) Let $f : X \to \mathbb{S}^2$ be a genus g Lefschetz fibration with global monodromy given by the relation $t_{\alpha_1} t_{\alpha_2} \cdots t_{\alpha_s} = 1$. Suppose that f has a section. Then the fundamental group of X is isomorphic to the fundamental group of Σ_g divided out by the normal closure of the simple closed curves $\alpha_1, \alpha_2, \cdots, \alpha_s$, considered as elements in $\pi_1(\Sigma_g)$. In particular, there is an epimorphism $\pi_1(\Sigma_g) \to \pi_1(X)$.*

Example 2.3.2. (cf. [**Fu**]) The relation in example 2.2.1 defines an elliptic fibration over $\mathbb{D}^2$, which can be extended to an elliptic fibration $E(1) \to \mathbb{S}^2$. We can then form the n-fold fiber sum (using the identity homeomorphism on regular fibers) $E(n) = \#_F(nE(1))$. The monodromy relation corresponding to this genus 1 Lefschetz fibration over $\mathbb{S}^2$ is given by $(t_a t_b)^{6n} = 1$. It was proven by Moishezon that the global monodromy of any elliptic Lefschetz fibration is equivalent to this relation (cf. [**Moi**]). Hence, the family of $E(n)$'s are a complete classification of genus 1 Lefschetz fibration with at least one singular fiber. Each $E(n)$ is complex.

$E(1) = \mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2$

$E(2) = K3$ surface.

Example 2.3.3. Higher genus Lefschetz fibrations Let $\alpha_1, \alpha_2, \ldots, \alpha_{2g}, \alpha_{2g+1}$ denote the collection of simple closed curves given in Figure 2.5, and c_i denote the right handed

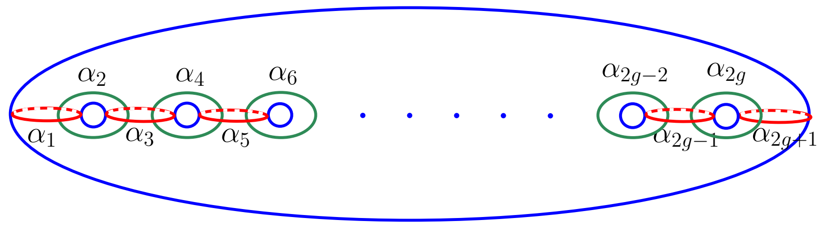


Figure 2.5: Vanishing Cycles of the Genus g Lefschetz Fibration on $X(g,1) = \mathbb{CP}^2\,\#(4g+5)\overline{\mathbb{CP}}^2$

Dehn twists t_{α_i} along the curve α_i. It is well-known that the following relations hold in M_g:

$$\Gamma_1(g) = (c_1 \cdot c_2 \cdots c_{2g-1} \cdot c_{2g} \cdot c_{2g+1}^2 \cdot c_{2g} \cdot c_{2g-1} \cdots c_2 \cdot c_1)^2 = 1.$$
$$\Gamma_2(g) = (c_1 \cdot c_2 \cdots c_{2g-1} \cdot c_{2g} \cdot c_{2g+1})^{2g+2} = 1.$$
$$\Gamma_3(g) = (c_1 \cdot c_2 \cdots c_{2g-1} \cdot c_{2g})^{2(2g+1)} = 1.$$

The monodromy relation $\Gamma_1 = 1$ given above, corresponding to the genus g Lefschetz fibration over $\mathbb{S}^2$, has total space $X(g,1) = \mathbb{CP}^2\,\#(4g+5)\overline{\mathbb{CP}}^2$, the complex projective plane blown up at $4g+5$ points. Furthermore, for $g \geq 2$, the above fibration on $X(g,1)$ admits $4g+4$ disjoint -1-sphere sections (see [**Tan**] for a proof of this fact using a mapping class group argument or [**AO1**] for a geometric argument).

Definition 11. A 4-manifold X is called *minimal* if there is no 4-manifold Y with $X = Y\,\#\overline{\mathbb{CP}}^2$, that is, if X is not the blow-up of another manifold.

Definition 12. Lefschetz fibration is called *relatively minimal* if there is no sphere with self-intersection -1 contained in a fiber.

Proposition 2.3.4. *([**St**]) If $f : X \to \Sigma_h$ is a Lefschetz fibration with $g(\Sigma_h) \geq 1$ then f is a relatively minimal Lefschetz fibration if and only if X is minimal.*

Proposition 2.3.5. *([**GS**]) For any fiber F of a Lefschetz fibration $f : X \to \Sigma_h$, the maps $F \hookrightarrow X \to \Sigma_h$ induces an exact sequence*

$$\pi_1(F) \to \pi_1(X) \to \pi_1(\Sigma_h) \to \pi_0(F) \to 0.$$

Theorem 2.3.6.

a) ***[Donaldson]****(cf.* **[Do1]***)For any symplectic* 4*-manifold* X*, there exists a non-negative integer* n *such that the* n*-fold blowup* $X\#n\overline{\mathbb{CP}}^2$ *of* X *admits a Lefschetz fibration* $f : X\#n\overline{\mathbb{CP}}^2 \to \mathbb{S}^2$*.*

b) ***[Gompf]****(cf.* **[GS]***) Assume that the closed* 4*-manifold* X *admits a Lefschetz fibration* $\pi : X \to \Sigma_h$*, and let* $[F]$ *denote the homology class of the fiber. Then* X *admits a symplectic structure with symplectic fibers iff* $[F] \neq 0$ *in* $H_2(X;\mathbb{R})$*. If* $e_1, e_2, ..., e_n$ *is a finite set of sections of the Lefschetz fibration, the symplectic form* ω *can be chosen in such a way that all these sections are symplectic.*

2.4 Symplectic Connected Sum

Definition 13. Let X_1 and X_2 be closed, oriented, smooth 4-manifolds and $F_i \subset X_i$ are 2-dimensional, smooth, closed, connected submanifolds in them. Suppose that $[F_1]^2 + [F_2]^2 = 0$ and the genera of F_1 and F_2 are equal. We choose an orientation-preserving diffeomorphism $\psi : F_1 \to F_2$ and lift it to an orientation-reversing diffemorphism $\Psi : \partial\nu F_1 \to \partial\nu F_2$ between the boundaries of the tubular neighborhoods of F_i. Using Ψ, we glue $X_1 \setminus \nu F_1$ and $X_2 \setminus \nu F_2$ along the boundary. This new oriented smooth 4-manifold $X\#_\psi Y$ is called the a *symplectic connected sum* of X_1 and X_2 along F_1 and F_2, determined by Ψ.

Lemma 2.4.1. *Let* X *and* Y *be closed, oriented, smooth* 4*-manifolds containing an embedded surface* Σ *of self-intersection* 0*. Then,*

1. $e(X\#_\psi Y) = e(X) + e(Y) - 2e(\Sigma)$
2. $\sigma(X\#_\psi Y) = \sigma(X) + \sigma(Y)$
3. $c_1^2(X\#_\psi Y) = c_1^2(X) + c_1^2(Y) + 8(g-1)$
4. $\chi_h(X\#_\psi Y) = \chi_h(X) + \chi_h(Y) + (g-1)$

where g *is the genus of the surface* Σ *and* $\chi_h := (\sigma + e)/4$ *and* $c_1^2 := 3\sigma + 2e$*.*

If X_1, X_2 are symplectic manifolds and F_1 and F_2 are symplectic submanifolds then according to theorem of Gompf (cf. [**Go2**]) $X_1 \#_{\psi} X_2$ admits a symplectic structure.

Next, we state a theorem on minimality of symplectic connected sums, which is due to M. Usher.

Theorem 2.4.2. *(cf.[**Us**])* **(Minimality of Symplectic Sums)**

Let $Z = X_1 \#_{F_1=F_2} X_2$ be symplectic fiber sum of manifolds X_1 and X_2. Then,

(i) If either $X_1 \backslash F_1$ or $X_2 \backslash F_2$ contains an embedded symplectic sphere of square -1, then Z is not minimal.

(ii) If one of the summands X_i (say X_1) admits the structure of an $\mathbb{S}^2$-bundle over a surface of genus g such that F_i is a section of this fiber bundle, then Z is minimal if and only if X_2 is minimal.

(iii) In all other cases, Z is minimal.

2.5 Seiberg-Witten Invariants

Next, we will provide background for Seiberg-Witten invariants introduced by Seiberg and Witten. Let X be a smooth closed oriented 4-manifold X with $b_2^+(X) > 1$. Assuming $H_1(X, \mathbb{Z})$ has no 2-torsion, there is a one-to-one correspondence between the set of *spin*^{c} structures over X and the set characteristic elements of $H^2(X, \mathbb{Z})$ as follows: there is a bundle of positive spinors W_s^+over X corresponding to each *spin*^{c} structure s over X. Let $c(s) \in H_2(X)$ denote the Poincaré dual of $c_1(W_s^+)$. Each $c(s)$ is a characteristic element of $H_2(X; \mathbb{Z})$ (i.e. its Poincaré dual $\hat{c}(s) = c_1(W_s^+)$ reduces mod 2 to $w_2(X)$). Then, the Seiberg-Witten invariant of X is an integer valued function which is defined on the set of *spin*^{c} structures over X (cf. [**Wi**]).

$$\mathcal{SW}_X : \{k \in H^2(X, \mathbb{Z}) | k = w_2(TX)(mod \ \ 2)\} \longrightarrow \mathbb{Z}.$$

When $b_2^+(X) > 1$, the Seiberg-Witten invariant $\mathcal{SW}_X$ is a diffeomorphism invariant. It does not depend on the choice of generic metric on X or a generic perturbation of

Seiberg-Witten equations and its sign depends on an orientation of

$$H^0(X;\mathbb{R}) \otimes det \;\; H^2_+(X;\mathbb{R}) \otimes det \;\; H^1(X;\mathbb{R}).$$

If $SW_X(\beta) \neq 0$, then we call β a *basic class of* X. It is a fundamental fact that the set of basic classes is finite. It can be shown that, if β is a basic class, then so is $-\beta$ with

$$SW_X(-\beta) = (-1)^{(e(X)+\sigma(X))/4} SW_X(\beta)$$

where $e(X)$ is the Euler number and $\sigma(X)$ is the signature of X. Now let $\{\pm\beta_1, \cdots, \pm\beta_n\}$ be the set of nonzero basic classes for X. Consider variables $t_\beta = exp(\beta)$ for each $\beta \in H^2(X;\mathbb{Z})$ which satisfy the relations $t_{\alpha+\beta} = t_\alpha t_\beta$. We may then view the Seiberg-Witten invariant of X as the symmetric Laurent polynomial

$$\mathcal{SW}_X = b_0 + \sum_{j=1}^{n} b_j \left(t_{b_j} + (-1)^{(e(X)+\sigma(X))/4} t_{\beta_j}^{-1} \right)$$

Example 2.5.1. Let $E(n)$ be a simply connected minimally elliptic surface with holomorphic Euler characteristic $\chi_h = n$ and with no multiple fibers. Then we have $\mathcal{SW}_{E(n)} = (t - t^{-1})^{n-2}$ where $t = t_F = exp(F)$ and F is the cohomology class Poincaré dual to the fiber class. Thus

$$\mathcal{SW}_{E(n)}((n-2i)F) = (-1)^{i-1} \binom{n-2}{i-1} \qquad for \quad i = 1, \cdots, n-1$$

and $\mathcal{SW}_{E(n)}(\alpha) = 0$ for any other class α.

Definition 14. The 4-manifold X is of *simple type* if each basic class β satisfies the equation $\beta^2 = c_1^2(X) = 3\sigma(X) + 2e(X)$. If X is symplectic manifold of $b_2^+(X) > 1$ then it has simple type.

For the proof of the following theorems, we refer the reader to [**GS**].

Theorem 2.5.2. *(**Vanishing theorems**) Suppose that X is a smooth, closed, oriented, simply connected 4-manifold with $b_2^+ > 1$ and odd.*

1. *If $X = X_1 \# X_2$ and $b_2^+(X_i > 0$ $(i = 1, 2)$, then $\mathcal{SW}_X \equiv 0$.*

2. *If X admits a metric with positive scalar curvature, then $\mathcal{SW}_X \equiv 0$.*

3. *If $\Sigma \subset X$ is an embedded sphere with $[\Sigma]^2 \geq 0$ and $0 \neq [\Sigma]$ in $H_2(X;\mathbb{Z})$, then $\mathcal{SW}_X \equiv 0$.*

Theorem 2.5.3. ***(Non-vanishing theorems)***

1. *If S is a simply connected complex surface (hence $b_2^+(S)$ is odd) and $b_2^+(S) > 1$, then $\mathcal{SW}_S(\pm c_1(S)) \neq 0$*

2. *([**Ta**]) If (X, ω) is a simply-connected symplectic manifold with $b_2^+(X) > 1$, then*

$$\mathcal{SW}_X(\pm c_1(X, w)) = \pm 1.$$

Theorem 2.5.4. *[**KM**, **OS**] (**Generalized adjunction formula for** $\mathbf{b_2^+ \geq 1}$) Assume that $\Sigma \subset X$ is an embedded, oriented, connected surface of genus $g(\Sigma)$ with self-intersection $|\Sigma|^2 \geq 0$ and represents nontrivial homology class. Then for every Seiberg-Witten basic class β,*

$$2g(\Sigma) - 2 \geq |\Sigma|^2 + |\beta(\Sigma)|.$$

If X is of simple type and $g(\Sigma) \geq 0$, then the same inequality holds for Σ with arbitrary square $|\Sigma|^2$.

Theorem 2.5.5. *([**LL**])(**Generalized adjunction formula for** $\mathbf{b_2^+ = 1}$) Assume that $\Sigma \subset X$ is an embedded, oriented, connected surface of genus $g(\Sigma)$ with self-intersection $|\Sigma|^2 \geq 0$ and represents nontrivial homology class. Then any characteristic class β with $\mathcal{SW}_X^0(\beta) \neq 0$ satisfy,*

$$2g(\Sigma) - 2 \geq |\Sigma|^2 + |\beta(\Sigma)|.$$

2.6 Rational Blow-Down

In this section, we will introduce a construction called *rational blow-down* (cf. [**FS**]) which replaces a neighborhood of a certain configuration of spheres by a rational-homology 4-ball.

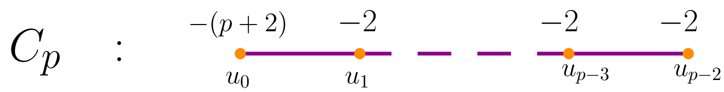


Figure 2.6: Plumbing diagram for $\mathcal{C}_p$

The rational blow-down construction is useful in many ways. First, any logarithmic transformation can be described as a sequence of blow-ups followed by a rational blow-down. Second, the change of the Seiberg-Witten invariants under a rational blow-down is not difficult to determine. Finally, rational blow-downs can be used to reduce the homology of a 4-manifold, and they have been used to build exotic 4-manifolds with small homology.

Take $p-1$ copies of $\mathbb{S}^2$ and build on them disk-bundles of Euler classes $-(p+2), -2, \cdots, -2, -2$, then plumb these according to the diagram in Figure 2.6. We obtain a simply connected 4-manifold $\mathcal{C}_p$, whose boundary is the lens space $L(p^2, p-1)$. In particular, $\pi_1(\partial\mathcal{C}_p) = \mathbb{Z}_{p^2}$.

Lemma 2.6.1. *The* 4*-manifold* $\mathcal{C}_p$ *can be embedded in* $\#(p-1)\overline{\mathbb{CP}^2}$

Lemma 2.6.2. *The* 4*-manifold* $\mathcal{C}_p$ *has the same boundary as the rational-homology* 4*-ball* $\mathcal{B}_p$*.*

Lemma 2.6.3. $L(p^2, p-1) = \partial\mathcal{C}_p$ *bounds a rational-homology* 4*-ball* $\mathcal{B}_p$ *with* $\pi_1(\mathcal{B}_p) = \mathbb{Z}_p$ *and the inclusion induced homomorphism* $\pi_1\left(L(p^2, p-1)\right) \to \pi_1\left(\mathcal{B}_p\right)$ *is surjective.*

If $\mathcal{C}_p$ is embedded in some 4-manifold M, then we could cut it out of M and replace it by a copy of $\mathcal{B}_p$. Specially, if M contains a configuration of embedded 2-spheres as described in the plumbing diagram of $\mathcal{C}_p$, then a neighborhood of this configuration in M must be a copy of $\mathcal{C}_p$. Then, M can be split as $M = M\backslash\mathcal{C}_p \bigcup_\partial \mathcal{C}_p$.

By replacing $\mathcal{C}_p$ by $\mathcal{B}_p$, we obtain the new manifold

$$M_{(p)} = (M\backslash\mathcal{C}_p)\bigcup_{\partial}\mathcal{B}_p$$

which is called **rational blow-down** of $\mathcal{C}_p$ from M.

Lemma 2.6.4.

$$
\begin{aligned}
&b_2^+\left(M_{(p)}\right) = b_2^+(M), && e\left(M_{(p)}\right) = e(M) - (p-1), && c_1^2(M_{(p)} = c_1^2(M) + (p-1)\\
&b_2^-\left(M_{(p)}\right) = b_2^-(M) - (p-1), && \sigma\left(M_{(p)}\right) = \sigma(M) + (p-1), && \chi_h(M_{(p)} = \chi_h(M)
\end{aligned}
$$

Theorem 2.6.5. *([**FS**, **Pa**]) Suppose that X is a smooth 4-manifold with $b_2^+(X) > 1$ which contains a configuration $\mathcal{C}_p$. If L is a Seiberg-Witten basic class of X satisfying $L \cdot u_i = 0$ for any i with $1 \leq i \leq p-2$ and $L \cdot u_{p-1} = \pm p$, then L induces an $\mathcal{SW}$ basic class $\overline{L}$ of $M_{(p)}$ such that $\mathcal{SW}_{M_{(p)}}(\overline{L}) = \mathcal{SW}_M(L)$.*

Remark. Note that, if both M and $M \backslash \mathcal{C}_p$ are simply-connected, then so is $M_{(p)}$.

Theorem 2.6.6. *([**FS**, **Pa**]) If a simply connected smooth 4-manifold M contains a configuration $\mathcal{C}_p$, then the Seiberg-Witten invariants of $M_{(p)}$ are completely determined by those of M. That is, for any characteristic line bundle $\overline{L}$ on $M_{(p)}$ with $\mathcal{SW}_{M_{(p)}}(\overline{L}) \neq 0$, there exists a characteristic line bundle L on M such that $\mathcal{SW}_M(L) = \mathcal{SW}_{M_{(p)}}(\overline{L})$.*

Remark. The homology $H_2(M_{(p)};\mathbb{Z})$ can be identified with the Q_M-orthogonal complement of $H_2(\mathcal{C}_p;\mathbb{Z})$ in $H_2(M;Z)$; in order words, with the complement of the classes represented by the spheres in M used to embed $\mathcal{C}_p$. Since moving from M to M_P eliminates *only* classes of negative self-intersection, it follows that $b_2^+(M) = b_2^+(M_{(p)})$, while the signature has increased.

2.7 Knot Surgery

Definition 15. Let X be a 4-manifold (with $b_2^+ > 1$) which contains a homologically essential torus T of self-intersection 0.

Let N_K be a tubular neighborhood of K in $\mathbb{S}^3$, and let $T \times D^2$ be a tubular neighborhood of T in X. Then the manifold X_K after we apply knot surgery is defined by

$$X_K = [X \setminus (T \times D^2)] \cup_\varphi [\mathbb{S}^1 \times (\mathbb{S}^3 \setminus N_K)]$$

where $\varphi : \partial(X \setminus (T \times D^2)) \longrightarrow \partial(\mathbb{S}^1 \times (\mathbb{S}^3 \setminus N_K)$ identifies $[\partial D^2]$ with a longitude λ of K. i.e. $\varphi([pt \times \partial D^2]) = [pt \times \lambda]$.

Definition 16. A Laurent polynomial $P(t) = a_0 + \Sigma_{j=1}^{n} a_j(t^j + t^{-j})$ of one variable with coefficient sum $a_0 + 2 \cdot \Sigma_{j=1}^{n} a_j = \pm 1$ is called an *A-polynomial.* If, in addition, $a_n = \pm 1$ we call $P(t)$ a *monic* A-polynomial.

Let X be any simply connected smooth 4-manifold with $b_2^+ > 1$.

Definition 17. A *cusp* in X is a PL embedded 2-sphere of self intersection 0 with a single non-locally flat point whose neighborhood is the cone on the right-handed trefoil knot.

The regular neighborhood N of a cusp in a 4-manifold is a *cusp neighborhood*; it is the manifold obtained by performing 0-framed surgery on a trefoil knot in the boundary of the 4-ball. Since the trefoil knot is a fibered knot with a genus 1 fiber, N is fibered by a smooth tori with one singular fiber, the cusp.

Definition 18. If T is a smoothly embedded torus representing a *nontrivial* homology class $[T]$, we say that T is *c-embedded* if T is a smooth fiber in a cusp neighborhood N_K; equivalently, T has two vanishing cycles. Note that a c-embedded torus has self-intersection 0.

Theorem 2.7.1. *[***FS2***] Let X be a simply connected smooth 4-manifold with $b_2^+ > 1$. Suppose that X contains a smoothly c-embedded torus T such that $\pi_1(X \setminus T)$. Then for any A-polynomial $P(t)$, there is a smooth 4-manifold X_P which is homeomorphic to X and has Seiberg-Witten invariant $SW_{X_P} = SW_X \cdot P(t)$, where $t = \exp 2[T]$.*

Corollary 2.7.2. *If $P(t)$ of is not monic then X_P does not admit a symplectic structure. Furthermore, if X contains a surface Σ_g of genus g disjoint from T with $0 \neq [\Sigma_g] \in H_2(X;\mathbb{Z})$ and with $\Sigma_g^2 < 2 - 2 \cdot g$ if $g > 0$, or $\Sigma_g^2 \leq 0$ if $g = 0$, then X_P with the opposite orientation does not admit a symplectic structure.*

Theorem 2.7.3. *[***FS2***] Assume that T c-embedded torus in X, then Seiberg-Witten invariants of X_K are $SW_{X_K} = SW_X \cdot \Delta_K(t)$, where the Alexander polynomial Δ_K is evaluated on $t = \exp 2[T]$.*

Proposition 2.7.4. *[***FS2***] If $X \setminus T$ is simply connected and $\pi_1(X) = 1$ then X_K is homeomorphic to X.*

2.8 Luttinger Surgery

In this section, we will review an important symplectic operation, the Luttinger surgery ([**Lu**]), which is one of the main tools used in our construction.

Definition 19. Let (M, w) be a symplectic 4-manifold and the torus Γ be a Lagrangian submanifold of M with self intersection 0. Assume that γ is a simple closed loop on Γ, and γ' is a simple loop on $\partial(\nu\Gamma)$ that is parallel to γ under the Lagrangian framing.

For any integer m, the $(\Gamma, \gamma, 1/m)$ *Luttinger surgery* on M is defined as

$$M_{\Gamma,\gamma}(1/m) = (M - \nu(\Gamma)) \cup_{\phi} (S^1 \times S^1 \times D^2),$$

where μ_Γ is a meridian of Γ, and the gluing map $\phi : S^1 \times S^1 \times \partial D^2 \to \partial(M - \nu(\Gamma)$ satisfies $\phi([\partial D^2]) = m[\gamma'] + [\mu_\Gamma]$ in $H_1(\partial(M - \nu(\Gamma))$.

It was shown in [**ADK**] that $M_{\Gamma,\gamma}(1/m)$ possesses a symplectic form that restricts to the original symplectic form ω on $M \setminus \nu\Gamma$.

In other words, given an embedded Lagrangian torus Γ in a symplectic 4-manifold (M, w), a homotopically non-trivial embedded loop $\gamma \subset \Gamma$ and an integer m, Luttinger surgery is an operation that consists in cutting out from M a tubular neighborhood of Γ, foliated by parallel Lagrangian tori, and gluing it back in such a way that the new meridian loop differs from the odd one by m twists along the loop γ (while longitudes are not affected), yielding in a new symplectic manifold $M_{\Gamma,\gamma}(1/m)$.

Proposition 2.8.1. *(cf. [**ADK**]) Luttinger surgery is well-defined symplectically.*

Proof. It is well-known that a neighborhood of Γ in M can be identified symplectically with a neighborhood of the zero section in the cotangent bundle $\Gamma^*\Gamma \simeq \Gamma \times \mathbb{R}^2$ with its standard symplectic structure. Moreover, Γ itself can be identified with $\mathbb{R}^2/\mathbb{Z}$ in such a way that γ is identified with the first coordinate axis and its co-orientation coincides with the standard orientation of the second coordinate axis.

Let (x_1, x_2) denote the corresponding coordinates on Γ and (y_1, y_2) denote the dual coordinates in the cotangent fibers. Then, the symplectic form is given by $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$. Define $U_r = (\mathbb{R}^2/\mathbb{Z}^2) \times [-r, r] \times [-r, r] \subset (\mathbb{R}^2/\mathbb{Z}^2)$, where $r > 0$, is contained in the neighborhood of Γ over which the identification holds. Now, choose a smooth step function $\chi : [-r, r] \to [0, 1]$ such that $\chi(t) = 0$ for $t \leq -\frac{r}{3}$, $\chi(t) = 1$ for $t \geq \frac{r}{3}$, and $\int_{-r}^{r} t\chi'(t)\, dx = 0$

Given an integer $k \in \mathbb{Z}$, define $\phi_k : U_r - U_{r/2} \to U_r - U_{r/2}$ by the formulas $\phi k(x_1, x_2, y_1, y_2) = (x_1 + k\chi(y_1), x_2, y_1, y_2)$ if $y_2 \geq r/2$ and $\phi_k = Id$ otherwise.

Since the support of $d\chi$ is contained in $[-r/3, r/3]$, the map ϕ_k is a diffeomorphism of $U_r - U_{r/2}$; moreover, ϕ_k preserves by the symplectic form. So we can make the following definition:

$M(\Gamma, \gamma, k)$ is the manifold obtained from M by removing a small neighborhood of Γ and gluing back the standard piece U_r, using the symplectomorphism ϕ_k to identify the two sides near their boundaries. i.e.

$$M(\Gamma, \gamma, k) = (M - U_{r/2}) \cup_{\phi_k} U_r.$$

This surgery operation is equivalent to that introduced by Luttinger in [**Lu**].

This construction is well-defined as a consequence of the Moser‘s stability theorem.

□

Theorem 2.8.2 (Moser's stability theorem)**.** *$M(\Gamma, \gamma, k)$ caries a natural symplectic form $\tilde{\omega}$, well-defined up to isotopy independently of the choices made in the construction. Moreover, deforming Γ among Lagrangian tori and $\gamma \subset \Gamma$ by smooth isotopies induces a deformation (pseudo-isotopy) of the symplectic structure $\tilde{\omega}$, and if the symplectic area swept by γ is equal to zero then this deformation preserves the cohomology class $[\tilde{\omega}]$ and is therefore an isotopy.*

Next, we state a lemma which we will use later in the proof of the main theorem.

Lemma 2.8.3. *Let $M_{\Gamma,\gamma}(1/m) = (M - \nu(\Gamma)) \cup_\phi (S^1 \times S^1 \times D^2)$ be the $(\Gamma, \gamma, 1/m)$* Luttinger surgery *on M. Then we have:*

1. $b_1 (M_{\Gamma,\gamma}(1/m)) = b_1(M) - 1$
2. $b_2 (M_{\Gamma,\gamma}(1/m)) = b_2(M) - 2$

3. $e\left(M_{\Gamma,\gamma}(1/m)\right) = e(M)$

4. $\sigma\left(M_{\Gamma,\gamma}(1/m)\right) = \sigma(M)$

5. $\pi_1\left(M_{\Gamma,\gamma}(1/m)\right) = \pi_1(M-\Gamma)/N(\mu_\Gamma \gamma'^m)$.

Proof. 3. Follows from part 1 and 2.

4. The signature formula comes from Novikov additivity.

   Let M and N be two 4-manifolds with non-empty boundaries. Assume that their boundary 3-manifolds ∂M and ∂N admit an orientation-reversing diffeomorphism $\partial M \cong \overline{\partial N}$. Then the closed manifold $M \cup_\partial N$, built by identifying the boundaries ∂M and ∂N, has signature

$$\sigma(M \cup_\partial N) = \sigma(M) + \sigma(N)$$

1. The normal bundle to Γ along γ comes equipped with a natural framing, so that the loop γ can be pushed away from Γ in a canonical way (up to homotopy). Therefore, we can define the homotopy class of γ in $\pi_1(M-\Gamma)$. Comparing the fundamental groups of M and $M_{\Gamma,\gamma}(1/m)$ with $\pi_1(M-\Gamma)$ and using Seifert-van Kampen Theorem we finish the proof. Note that the surgery operation preserves the fundamental group whenever γ^m is homologically trivial in $M-\Gamma$.

□

Proposition 2.8.4. *Luttinger surgery preserves the minimality (cf. [**HL**]) .*

Proof. Luttinger surgery can be reversed and the reverse operation is also a Luttinger surgery. Therefore it is enough to show that, if we start with a non-minimal symplectic 4-manifold, then after a Luttinger surgery, the resulting symplectic manifold is still non-minimal and this fact is a consequence of the following theorem Theorem 2.8.5. □

Theorem 2.8.5. *(cf. [**Wel**]) Given a Lagrangian torus Γ and a symplectic -1 class, there is an embedded symplectic -1-sphere in that class which is disjoint from Γ.*

Theorem 2.8.6. *([**HL**]) The Luttinger surgery preserves the symplectic Kodaira dimension.*

Chapter 3

New Exotic 4 Manifolds via Luttinger Surgery on Lefschetz Fibrations

Main Theorem. *([**AS**]) Let M be $(2n+2k-3)\mathbb{CP}^2\#(6n+2k-3)\overline{\mathbb{CP}}^2$ for any $n \geq 1$, $k \geq 1$, and $(n,k) \neq (1,1)$. There exists a new family of smooth closed simply-connected minimal symplectic 4-manifold and an infinite family of non-symplectic 4-manifolds that are homeomorphic but not diffeomorphic to M that can obtained by the sequence of Luttinger surgeries and a single generalized torus surgery on Lefschetz fibrations.*

3.1 Matsumoto and Korkmaz Fibrations

In this section, we will introduce the 4-manifold $Y(k) = \Sigma_k \times \mathbb{S}^2\#4\overline{\mathbb{CP}}^2$, which is the total space of the well known genus $g = 2k$ Lefschetz fibration over $\mathbb{S}^2$. This was shown by Y. Matsumoto for $k = 1$ (cf. [**Ma**]).

3.1.1 Matsumoto fibration

Let $w : \Sigma_2 \to \Sigma_2$ be the involution shown in Figure 3.1 and $\tau : \mathbb{S}^2 \to \mathbb{S}^2$ be the $180°$ rotation of a 2-sphere about the axis through the poles.

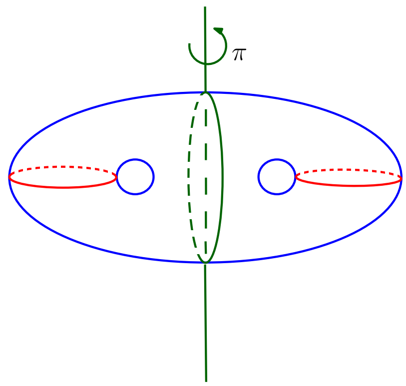


Figure 3.1: Involution w

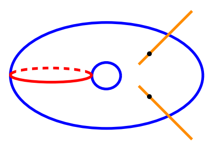


Figure 3.2: Singular fiber F_w

The quotient space $\mathbb{S}^2 \times \Sigma_2/\tau \times w$ has 4 singular points. By blowing up these singularities, we obtain a compact complex surface $Y(1)$ and a holomorphic map $f_w : Y(1) \to \mathbb{S}^2$ induced by the first projection $\mathbb{S}^2 \times \Sigma_2 \to \mathbb{S}^2$.

The fibration $f_w : Y(1) \to \mathbb{S}^2$ has two singular fibers over the north and the south poles of $\mathbb{S}^2$. They are topologically equivalent, and the monodromy around each of them is the involution w. We denote either singular fiber by F_w.

Structure of F_w: The quotient space Σ_2/w is homeomorphic to $\mathbb{T}^2$ with two cusps. Blowing them up, we obtain the configuration of F_w as shown in Figure 3.2. F_w splits into 4 Lefschetz singular fibers (3 of them have type I, and one has type II. See [**Ma**] for more detail.) We may assume that the new critical values are in small disk D centered at the critical value of F_w. Taking loops and changing them by elementary transformations and/or their inverses, we can arrange so that the monodromy representation

$$\rho : \pi_1(D - \{B_0, B_1, B_2, c\}, x_0) \to \mathcal{M}_2$$

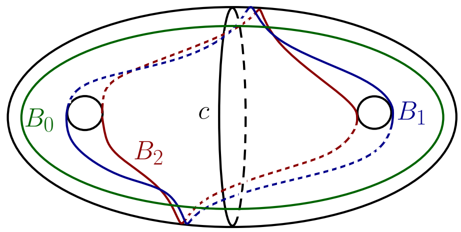


Figure 3.3: Vanishing Cycles in Matsumoto Fibration

is given by the quadruple of negative Dehn twists about the simple closed curves B_0, B_1, B_2 and c as shown in Figure 3.3.

The product of the negative Dehn twist about simple closed curves in Figure 3.3 is w and juxtaposing two copies of w gives the global monodromy of the fibration $f_w : Y(1) \to \mathbb{S}^2$. Namely, this Lefschetz fibration can be perturbed into Lefschetz one with the global monodromy given by the word $(t_{B_0}t_{B_1}t_{B_2}t_c)^2 = 1$ in the mapping class group M_2 of genus 2 surface, where t_{B_0}, t_{B_1}, t_{B_2} and t_c denotes the Dehn twists along the curves B_0, B_1, B_2 and c respectively.

Proposition 3.1.1. *The total space $Y(1)$ of the Lefschetz fibration f_w is diffeomorphic to $\mathbb{T}^2 \times \mathbb{S}^2 \# 4\overline{\mathbb{CP}}^2$.*

Proof. Let $\psi : Y(1) \to \Sigma_2/w$ (recall that $\Sigma_2/w \cong \mathbb{T}^2$) be the projection induced by the projection to the second factor $\mathbb{S}^2 \times \Sigma_2 \to \Sigma_2$. A general fiber of ψ is a 2-sphere and there are two singular fibers over $Fix(w)$. Blowing down the -1-sphere 4 times, we obtain a sphere bundle over $\mathbb{T}^2$, $\mathbb{T}^2 \tilde{\times} \mathbb{S}^2$. Therefore,

$$Y(1) \cong \mathbb{T}^2 \tilde{\times} \mathbb{S}^2 \# 4\overline{\mathbb{CP}}^2 \cong \mathbb{T}^2 \times \mathbb{S}^2 \# 4\overline{\mathbb{CP}}^2$$

□

3.1.2 Branched-cover description of Matsumoto fibration

Let us take a double branched cover of $\mathbb{T}^2 \times \mathbb{S}^2$ along the union of two disjoint copies of $\mathbb{T}^2 \times \{pt\}$ and two disjoint copies of $\{pt\} \times \mathbb{S}^2$. (See Figure 3.4). The resulting branched cover has 4 singular points, corresponding to the number of the intersection points of the

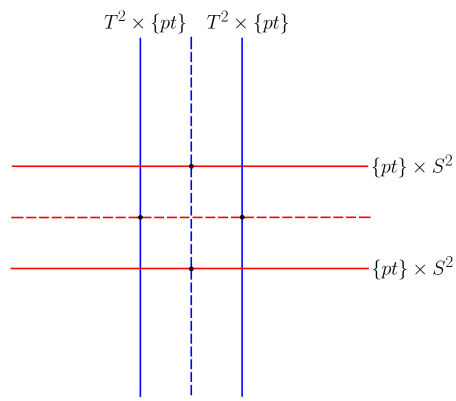


Figure 3.4: The branch locus for $\mathbb{T}^2 \times \mathbb{S}^2 \# 4\overline{\mathbb{CP}}^2$

vertical tori and the horizontal spheres in the branch set. By desingularizing the above singular manifold, we obtain the total space $Y(k) = \mathbb{T}^2 \times \mathbb{S}^2 \# 4\overline{\mathbb{CP}}^2$ of the Matsumoto fibration.

Note that vertical $\mathbb{T}^2$ fibration on $\mathbb{T}^2 \times \mathbb{S}^2$ pulls back to give a fibration of $Y(k) = \mathbb{T}^2 \times \mathbb{S}^2 \# 4\overline{\mathbb{CP}}^2$ over $\mathbb{S}^2$. A generic fiber of the vertical fibration is a genus 2 surface since it is the double cover of $\mathbb{T}^2$, branched over two points. According to [**Ma**], each of the two singular fibers of the vertical fibration can be perturbed into 4 Lefschetz type singular fibers.

A generic fiber of the horizontal fibration is the double cover of $\mathbb{S}^2$ branched over two points. This gives a sphere fibration on $Y(k) = \mathbb{T}^2 \times \mathbb{S}^2 \# 4\overline{\mathbb{CP}}^2$.

3.1.3 Fundamental Group Computation of $Y(1)$

Lemma 3.1.2. *Let a_1, b_1, a_2, and b_2 denote the standard generators of the regular fiber Σ in the fundamental group $\pi_1(Y(1))$.*

$$\pi_1(Y(1)) = \langle a_1, b_1, a_2, b_2 \mid b_1 b_2 = [a_1, b_1] = [a_2, b_2] = b_2 a_2 b_2^{-1} a_1 = 1 \rangle = \mathbb{Z} \oplus \mathbb{Z}$$

Proof. Using the homotopy long exact sequence for a Lefschetz fibration and existence of -1-sphere sections of Matsumoto‘s fibration, we have the following identification of

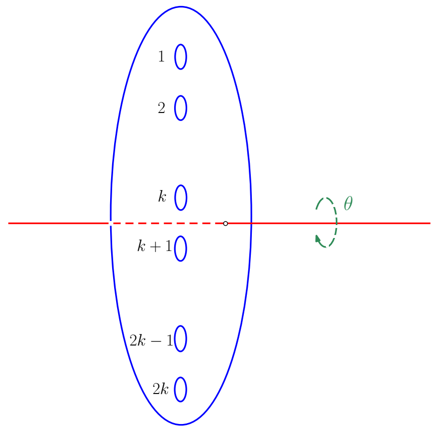


Figure 3.5: The vertical involution θ of the genus $2k$ surface

the fundamental group of $Y(1)$ [**OzSt2**]:

$$\pi_1(Y(1) = \pi_1(\Sigma)/\langle B_0, B_1, B_2, c\rangle$$

$$\begin{aligned} B_0 &= b_1 b_2 \\ c &= a_1 b_1 a_1^{-1} b_1^{-1} = a_2 b_2 a_2^{-1} b_2^{-1} \\ B_1 &= b_2 a_2 b_2^{-1} a_1 \\ B_2 &= b_2 a_2 a_1 b_1 \end{aligned}$$

Hence, $\pi_1(Y(1)) = \mathbb{Z} \oplus \mathbb{Z}$, which completes the proof. □

3.1.4 Korkmaz fibration

Then, M. Korkmaz (cf. [**Ko**]) generalized this construction for $k \geq 2$ by factorizing the *vertical* involution of θ of genus $2k$ surface as shown in Figure 3.5.

3.1.5 Branched-cover description of Korkmaz's fibration

Let us take a double branched cover of $\Sigma_k \times \mathbb{S}^2$ along the union of two disjoint copies of $\Sigma_k \times \{pt\}$ and two disjoint copies of $\{pt\} \times \mathbb{S}^2$. (See Figure 3.6). The resulting branched

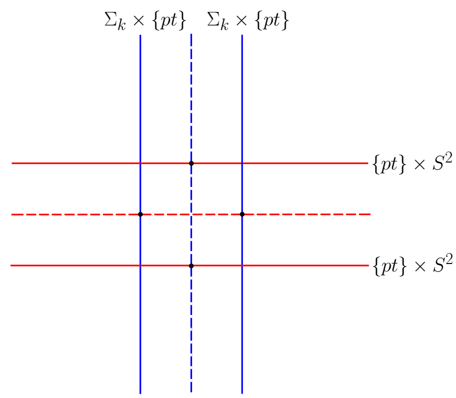


Figure 3.6: The branch locus for $\Sigma_k \times \mathbb{S}^2 \# 4\overline{\mathbb{CP}}^2$

cover has 4 singular points, corresponding to the number of the intersection points of the vertical genus k surfaces and the horizontal spheres in the branch set. By desingularizing the above singular manifold, we obtain the total space $Y(k) = \Sigma_k \times \mathbb{S}^2 \# 4\overline{\mathbb{CP}}^2$ of the Korkmaz's fibration.

A generic fiber of the horizontal fibration is the double cover of $\mathbb{S}^2$ branched over two points. This gives a sphere fibration on $Y(k) = \Sigma_k \times \mathbb{S}^2 \# 4\overline{\mathbb{CP}}^2$.

A generic fiber of the vertical fibration is the double cover of Σ_k branched over two points. Thus, a generic fiber is a genus $2k$ surface. According to [**Ko**, **Ma**], each of the two singular fibers of the vertical fibration can be perturbed into $2k+2$ Lefschetz type singular fibers.

3.1.6 Fundamental Group Computation of $Y(k)$

The following theorem was proved in [**Ko**], which computes the global monodromy of the given genus g Lefschetz fibration for both an even and an odd g.

Theorem 3.1.3. *Let θ denote the vertical involution of the genus g surface with two fixed points. In the mapping class group M_g, the following relations between right Dehn twists hold:*

a) $(t_{B_0} t_{B_1} t_{B_2} \cdots t_{B_g} t_c)^2 = \theta^2 = 1$ *if g is even,*

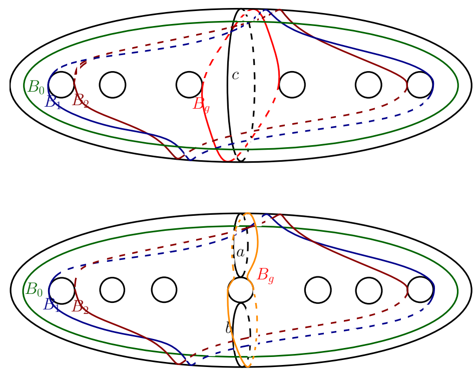


Figure 3.7: Vanishing Cycles in Korkmaz' s Fibration

b) $(t_{B_0}t_{B_1}t_{B_2}\cdots t_{B_g}(t_a)^2(t_b)^2)^2 = \theta^2 = 1$ *if g is odd,*

where B_k, a, b, c are the simple closed curves defined as in Figure 3.7.

Let Σ_{2k} denote a regular fiber of the given Lefschetz fibration and a_1, b_1,..., a_{2k} and b_{2k} denote the standard generators of fundamental group of Σ_{2k} under the inclusion.

Using the homotopy exact sequence for a Lefschetz fibration, we have

$$\pi_1(\Sigma_{2k}) \to \pi_1(Y(k)) \to \pi_1(\mathbb{S}^2)$$

Proposition 3.1.4. *(cf.[**Ko**]) The following identification hold in the fundamental group of $Y(k)$:*

$\pi_1(Y(k)) = \pi_1(\Sigma_{2k}) \backslash < B_0, B_1, \cdots, B_{g-1}, B_g, c >$

It follows that the fundamental group of $Y(k)$ has a presentation with the generators a_1, b_1, a_2, b_2, ... , a_g, b_g and the relations

$[a_1, b_1][a_2, b_2]\cdots[a_g, b_g] = 1,$

$B_0 = B_1 = B_2 = \cdots = B_g = c = 1.$

Lemma 3.1.5. *(cf. [**Ko**]) The following identities hold in $\pi_1(Y(k))$*

$B_0 = b_1 b_2 \cdots b_g,$

$B_{2i-1} = a_i b_i b_{i+1} \cdots b_{g+1-i} c_{g+1-i} a_{g+1-i}$, $1 \leq i \leq k$ *(Here, for $i = 1$ we ignore c_{g+1-i})*

$B_{2i} = a_i b_{i+1} b_{i+2} \cdots b_{g-i} c_{g-i} a_{g+1-i},\ 1 \le i \le k-1$

$B_g = B_{2k} = a_k c_k a_{k+1}$

$c = c_k = [a_1, b_1][a_2, b_2] \cdots [a_k, b_k]$

Lemma 3.1.6. *(cf. [**Ko**]) The following relations hold in* $\pi_1(Y(k))$
$a_1 a_{2k} = 1,\ \ a_2 a_{2k-1} = 1,\ \ \cdots,\ \ a_k a_{k+1} = 1,$
$b_1 b_2 \cdots b_{2k} = 1,$
$b_2 b_3 \cdots b_{2k-1} = [a_{2k}, b_{2k}],$
$\cdots$
$b_{i+1} b_{i+2} \cdots b_{g-i} = [a_{2k-i+1}, b_{2k-i+1}] \cdots [a_{2k-1}, b_{2k-1}][a_{2k}, b_{2k}].$

Proof. Using the relations $B_0 = b_1 b_2 \cdots b_g = 1$, $B_1 = a_1 b_1 b_2 \cdots b_g c_g a_g = 1$ and $c_g = 1$ in the fundamental group of $Y(k)$, we easily see that

$$a_1 a_g = a_1 a_{2k} = 1.$$

Next, using the relations

$$B_2 = a_1 b_2 b_3 \cdots b_{g-1} c_{g-1} a_g = 1,$$

$$B_3 = a_2 b_2 b_3 \cdots b_{g-1} c_{g-1} a_{g-1} = 1,$$

$$\text{and} \ \ a_1 a_g = 1,$$

we obtain $a_2 a_{g-1} = a_2 a_{2k-1} = 1$.

By continuing in this fashion, i.e. using the relations

$$B_{2i-2} = a_{i-1} b_i b_{i+1} \cdots b_{g-i+1} c_{g-i+1} a_{g-i+2},$$

$$B_{2i-1} = a_i b_i b_{i+1} \cdots b_{g-i+1} c_{g-i+1} a_{g+1-i} = 1,$$

$$a_{i-1} a_{g-i+2} = 1,$$

we have $a_i a_{g-i+1} = 1$ for any i between 1 and k.

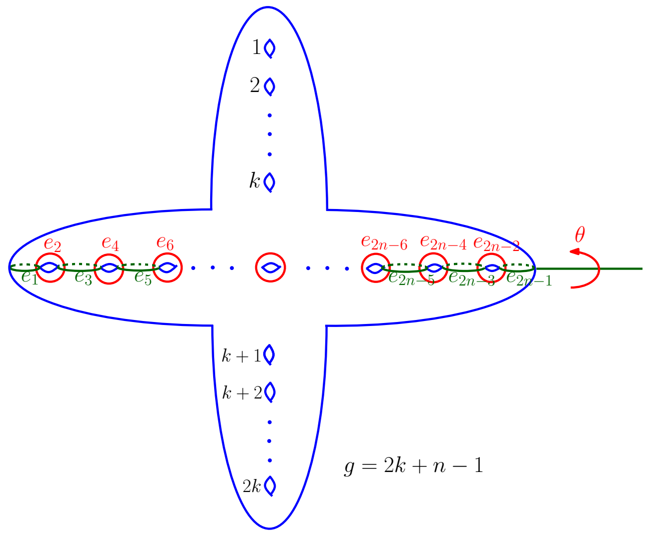


Figure 3.8: The involution θ of the surface Σ_{2k+n-1}

Furthermore, relations $B_0 = b_1 b_2 \cdots b_g = 1$, $c_{g-1} = [a_g, b_g]^{-1}$ and $a_1 a_g = 1$ imply

$$1 = B_2 = a_1 b_2 b_3 \cdots b_{g-1} c_{g-1} a_g = b_2 b_3 \cdot b_{g-1} c_{g-1} = b_2 b_3 \cdots b_{g-1}(b_g a_g b_g^{-1} a_g^{-1}) = 1.$$

Consequently, the relations $B_{2i} = a_i b_{i+1} b_{i+2} \cdots b_{g-i} c_{g-i} a_{g+1-i} = 1$ and $a_i a_{g+i-1} = 1$ results in $b_{i+1} b_{i+2} \cdots b_{g-i} = [a_{g-i+1}, b_{g-i+1}] \cdots [a_{g-1}, b_{g-1}][a_g, b_g]$ for any i between 1 and k. $\square$

We conclude that $\pi_1(Y(k)) = \prod_k$, where $\prod_k = \pi_1(\Sigma_k)$ is the surface group generated by the loops $a_1, b_1, \cdots, a_k$ and b_k. Moreover, the fundamental group $\pi_1(Y(k)) \setminus \nu(\Sigma_{2k}))$ of the complement is also $\prod_k$. The normal circle $\mu = \{pt\} \times \partial(\mathbb{D}^2)$ to Σ_{2k} is trivial in $\pi_1(Y(k) \setminus \nu\Sigma_{2k})$, since we can deform it using an exceptional sphere section.

3.2 Gürtaş' Fibration

Let M_{h+v} denote the mapping class group of a compact, closed, oriented 2-dimensional surface Σ_{h+v} of genus $h + v$ and θ denote the involution on the surface Σ_{h+v}, as shown in Figure 3.8. Y. Gürtaş (cf. [**Gu**]) generalized the constructions in [**Ko**, **Ma**] even further by presenting the positive Dehn twist expression for a new set of involutions in M_{h+v} which are obtained by gluing the horizontal involution on a surface Σ_h of genus h and the vertical involution on a surface Σ_v of genus v, where v is a positive even number.

According to Gürtaş (cf. [**Gu**]), θ can be expressed as a product of $8h+2v+4$ positive Dehn twists.

Theorem 3.2.1. *(cf. [**Gu**])The positive Dehn twist expression for θ is given by*

$$\theta = (e_{2i+2}\cdots e_{2n-2}e_{2n-1})(e_{2i}\cdots e_2e_1)B_0(e_{2n-1}e_{2n-2}\cdots e_{2i+2})(e_1e_2\cdots e_{2i})(B_1B_2\cdots B_{2k-1}B_{2k})e_{2i+1}.$$

Now, setting $n = h+1$ and $v = 2k$, the word $\theta^2 = 1$ in the mapping class group M_{2k+n-1} defines a genus $g = 2k+n-1$ Lefschetz fibration over $\mathbb{S}^2$ which has $s = 8h+2v+4 = 8(n-1)+2(2k)+4 = 8n+4k-4$ singular fibers and the vanishing cycles all are about nonseparating curves (cf. [**Gu**]). Let $Y(n,k)$ denote the total space of this Lefschetz fibration.

Lemma 3.2.2. *(cf. [**AS**]) $Y(n,k)$ has the following topological invariants.*

1. $e(Y(n,k)) = 4n-4k+4$
2. $\sigma(Y(n,k)) = -4n$ *(see [**Gu**] for the proof)*
3. $c_1^2(Y(n,k)) = -4(n+2k-2)$ *and* $\chi_h(Y(n,k)) = 1-k$

where $\chi_h := (\sigma+e)/4$, $c_1^2 := 3\sigma+2e$.

Proof. The Euler characteristic of the symplectic 4-manifold $Y(n,k)$ can be computed using the following formula:

$$e(Y(n,k)) = e(\mathbb{S}^2)e(F)+s = 2(2-2(2k+n-1))+8n+4k-4 = 4n-4k+4$$

The signature of the Lefschetz fibration described by the word $\theta^2 = 1$ was computed in [**Gu**] and also in the related work of K. Yun ([**Yu**]):

$$\sigma(Y(n,k)) = -4n$$

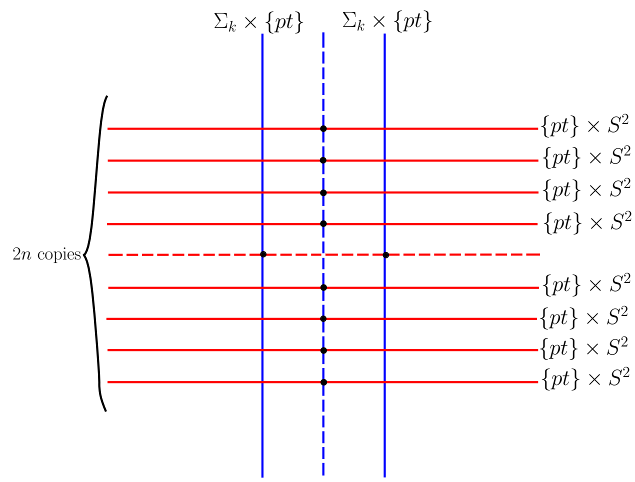


Figure 3.9: The branch locus for $\Sigma_k \times \mathbb{S}^2 \# 4n\overline{\mathbb{CP}}^2$

Now, using the formulas $\chi_h := (\sigma + e)/4$ and $c_1^2 := 3\sigma + 2e$, we get $c_1^2(Y(n,k)) = -4(n + 2k - 2)$ and $\chi_h(Y(n,k)) = 1 - k$. □

Lemma 3.2.3. *The genus $2k + n - 1$ Lefschetz fibration on $Y(n,k)$ admits at least $4n$ disjoint -1 sphere sections.*

Proof. We first observe that $Y(n,k)$ is the symplectic sum of $Y(k) = \Sigma_k \times \mathbb{S}^2 \# 4\overline{\mathbb{CP}}^2$ and $X(n-1,1) = \mathbb{CP}^2 \# (4n+1)\overline{\mathbb{CP}}^2$ (see ex. 2.3.3), along the spheres $\{pt\} \times \mathbb{S}^2$ and the sphere fiber of horizontal fibration on $X(n-1,1)$. Since a generic sphere fiber of $X(n-1,1)$ intersects a generic genus $n-1$ fiber at two points, after the symplectic sum we obtain a genus $2k+n-1$ fibration on $Y(n,k)$ over $\mathbb{S}^2$. Since for $n \geq 2$, the genus $n-1$ fibration on $X(n-1,1)$ admits $4n$ disjoint (-1)-sphere sections (see [**Lu**]), these -1-sphere sections extends to $Y(n,k)$. □

3.2.1 Branched cover description of Gürtaş's fibrations

In this section, we will provide the branched cover description of Gürtaş' s fibrations which can also be found in K.H. Yun's paper [**Yu**].

Take a double branched cover of $\Sigma_k \times \mathbb{S}^2$ along the union of $2n$ disjoint copies of $\{pt\} \times \mathbb{S}^2$ and two disjoint copies of $\Sigma_k \times \{pt\}$. (See Figure 3.9.) The resulting branched cover has $4n$ singular points, corresponding to the number of the intersection points of the horizontal spheres and the vertical genus k surfaces in the branch set. By desingularizing this manifold we obtain $Y(n,k) = \Sigma_k \times \mathbb{S}^2 \# 4n\overline{\mathbb{CP}}^2$. A generic horizontal fiber of $Y(n,k)$ is the double cover of $\mathbb{S}^2$ branched over two points. Thus, we have a sphere fibration on $Y(n,k) = \Sigma_k \times \mathbb{S}^2 \# 4\overline{\mathbb{CP}}^2$. A generic fiber of the vertical fibration is the double cover of Σ_k branched over $2n$ points. Thus, a generic fiber of the vertical fibration has genus $2k+n-1$. Furthermore, two complicated singular fibers of the vertical fibration can be perturbed into $4n+2k-2$ Lefschetz type singular fibers.

3.2.2 Fundamental Group of $Y(n,k)$

The following lemma will be used in the fundamental group computation of our example.

Lemma 3.2.4. *(cf. [**AS**]) The following relations hold in $\pi_1(Y(n,k))$.*

$e_1 = 1, \cdots, e_{2n-2} = 1, e_{2n-1} = 1,$

$a_1 a_{2k} = 1, a_2 a_{2k-1} = 1, \cdots, a_k a_{k+1} = 1$

$b_1 b_2 \cdots b_{2k} = 1, b_2 b_3 \cdots b_{2k-1} = [a_{2k} b_{2k}], \cdots,$

$b_{i+1} b_{i+2} \cdots b_{g-i} = [a_{2k-i+1} b_{2k-i+1}] \cdots [a_{2k-1} b_{2k-1}][a_{2k} b_{2k}].$

Proof. Notice that the first set of relations simply follows from the fact that the Dehn twists along the curves $e_1, e_2, \cdots, e_{2n-1}$ appear in the factorization of θ. By using the relations $e_1 = 1, \cdots, e_{2n-2} = 1, e_{2n-1} = 1$, we obtain the relations

$$\begin{aligned}
B_0 &= b_1 b_2 \cdots b_{2k} = 1 \\
B_1 &= a_1 b_1 b_2 \cdots b_{2k} c_{2k} a_{2k} = 1 \\
&\cdots \\
B_{2i-1} &= a_i b_i b_{i+1} \cdots b_{2k+1-i} c_{2k+1-i} a_{2k+1-i} = 1 \\
B_{2i} &= a_i b_{i+1} b_{i+2} \cdots b_{2k-i} c_{2k-i} a_{2k+1-i} = 1 \\
B_{2k} &= a_k c_k a_{k+1} = 1
\end{aligned}$$

To prove the remaining relations, we use the relations $B_0 = 1$, $B_1 = 1$, ..., B_{2i-1}, $B_{2i} = 1$, ..., $B_{2k} = 1$ and $c_k = 1$ in the fundamental group of $Y(n,k)$. $\square$

3.3 Luttinger Surgeries On Product Manifolds $\Sigma_n \times \Sigma_2$ and $\Sigma_n \times \mathbb{T}^2$

One of our symplectic building blocks will be a family of symplectic manifolds that are obtained from $\Sigma_n \times \mathbb{T}^2$ by performing a sequence of Luttinger surgeries along the Lagrangian tori (cf. [**APU**], [**FPS**]). The symplectic manifolds in this family have $b_1 = 2$. Our second family of symplectic manifolds is obtained from $\Sigma_n \times \Sigma_2$ by performing $2n + 4$ Luttinger surgeries along the Lagrangian tori (see Figure 3.10). The manifolds in this family have $b_1 = 0$.

Let us fix integers $n \geq 2$,$p_i \geq 0$ and $q_i \geq 0$, where $1 \leq i \leq n$. We denote by $Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)$ the symplectic 4-manifold obtained by performing the following $2n+4$ Luttinger surgeries on $\Sigma_n \times \Sigma_2$, which consist of the following 8 surgeries:

$$\begin{gathered}
(a_1' \times c_1', a_1', -1) \ , \ (b_1' \times c_1'', b_1', -1), \\
(a_2' \times c_2', a_2', -1) \ , \ (b_2' \times c_2'', b_2', -1), \\
(a_2' \times c_1', c_1', +1/p_1) \ , \ (a_2'' \times d_1', d_1', +1/q_1), \\
(a_1' \times c_2', c_2', +1/p_2) \ , \ (a_1'' \times d_2', d_2', +1/q_2)
\end{gathered}$$

followed by the following set of additional $2(n-2)$ Luttinger surgeries

$$\begin{gathered}
b_1' \times c_3', c_3', -1/p_3) \ , \ (b_2' \times d_3', d_3', -1/q_3), \\
\cdots \ , \ \cdots \\
(b_1' \times c_n', c_n', -1/p_n) \ , \ (b_2' \times d_n', d_n', -1/q_n)
\end{gathered}$$

where a_i, b_i $(i = 1, 2)$ and c_j, d_j $(j = 1, \cdots, n)$ denote the standard loops that generate $\pi_1(\Sigma_2)$ and $\pi_1(\Sigma_n)$, respectively.

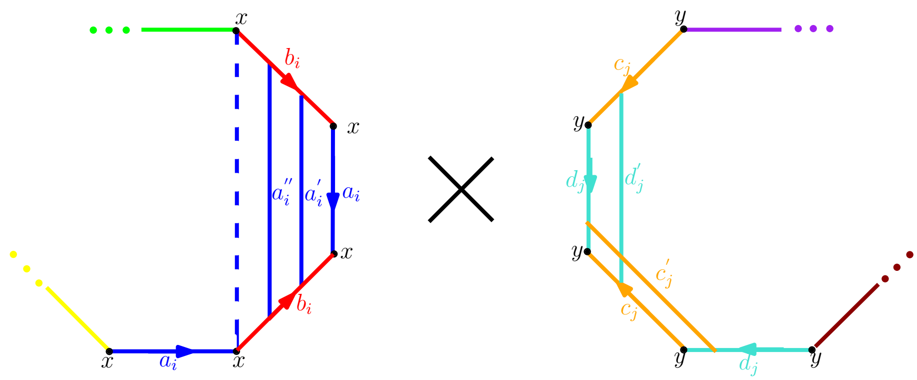


Figure 3.10: Lagrangian tori $a_i' \times c_j'$ and $a_i'' \times d_j''$

Lemma 3.3.1.

$$e(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n) = 4n - 4$$
$$\sigma(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n) = 0$$

Proof. By Lemma 2.8.3,

$$e(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n) = e(\Sigma_n \times \Sigma_2) = e(\Sigma_n)e(\Sigma_2) = (2 - 2n)(-2) = 4n - 4.$$
$$\sigma(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n) = \sigma(\Sigma_n \times \Sigma_2) = 0.$$

□

Lemma 3.3.2. *$\pi_1(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n))$ is generated by the loops a_i, b_i, c_j, d_j, ($i = 1, 2$) and $j = 1, ..., n$) and the following relations hold in $\pi_1(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n))$:*

$$[b_1^{-1}, d_1^{-1}] = a_1, [a_1^{-1}, d_1] = b_1, [b_2^{-1}, d_2^{-1}] = a_2, [a_2^{-1}, d_2] = b_2,$$
$$[d_1^{-1}, b_2^{-1}] = c_1^{p_1}, [c_1^{-1}, b_2] = d_1^{q_1}, [d_2^{-1}, b_1^{-1}] = c_2^{p_2}, [c_2^{-1}, b_1] = d_2^{q_2},$$
$$[a_1, c_1] = 1, [a_1, c_2] = 1, [a_1, d_2] = 1, [b_1, c_1] = 1,$$
$$[a_2, c_1] = 1, [a_2, c_2] = 1, [a_2, d_1] = 1, [b_2, c_2] = 1,$$

$$[a_1, b_1][a_2, b_2] = 1, \Pi_{j=1}^n [c_j, d_j] = 1,$$

$$[a_1^{-1}, d_3^{-1}] = c_3^{p_3}, [a_2^{-1}, c_3^{-1}] = d_3^{q_3}, \cdots, [a_1^{-1}, d_n^{-1}] = c_n^{p_n}, [a_2^{-1}, c_n^{-1}] = d_n^{q_n},$$

$$[b_1, c_3] = 1, [b_2, d_3] = 1, \cdots, [b_1, c_n] = 1, [b_2, d_n] = 1.$$

Proof. Follows from Lemma 2.8.3. □

Note that since the surfaces $\Sigma_2 \times \{pt\}$ and $\{pt\} \times \Sigma_n$ in $\Sigma_2 \times \Sigma_n$ are not affected by the above Luttinger surgeries, they descend to surfaces in $Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)$, say Σ_2 and Σ_n. Note also that $[\Sigma_2]^2 = [\Sigma_n]^2 = 0$ and $[\Sigma_2] \cdot [\Sigma_n] = 1$.

Now, let us fix a quadruple of integers $n \geq 2$, $m \geq 1$, $p \geq 1$ and $q \geq 1$. Let $Y_n(1/p, m/q)$ denote smooth 4-manifold obtained by performing the following $2n$ torus surgeries on $\Sigma_n \times \mathbb{T}^2$:

$$\begin{gathered}
(a_1' \times c', a_1', -1) \ , \ (b_1' \times c'', b_1', -1) \, , \\
(a_2' \times c', a_2', -1) \ , \ (b_2' \times c'', b_2', -1) \, , \\
\cdots \, , \, \cdots \\
(a_{n-1}' \times c', a_{n-1}', -1) \ , \ (b_{n-1}' \times c'', b_{n-1}', -1) \, , \\
(a_n' \times c', c', +1/p) \ , \ (a_n'' \times d', d', +m/q) \, .
\end{gathered}$$

where a_i, b_i $(i = 1, 2, \cdots, n)$ and c, d denote the standard generators $\pi_1(\Sigma_n)$ and $\pi_1(\mathbb{T}^2)$, respectively.

Lemma 3.3.3. *$\pi_1(Y_n(1/p, m/q))$ is generated by the loops a_i, b_i $(i = 1, 2, \cdots, m)$ and c, d and the following relations hold in $\pi_1(Y_n(1/p, m/q)$:*

$$\begin{gathered}
[a_1^{-1}, d] = b_1, [a_2^{-1}, d] = b_2 \ , \ \cdots, [a_{n-1}^{-1}, d] = b_{n-1} \\
[b_1^{-1}, d^{-1}] = a_1, [b_2^{-1}, d^{-1}] = a_2 \ , \ \cdots, [b_{n-1}^{-1}, d^{-1}] = a_{n-1},
\end{gathered}$$

$$[d^{-1}, b_n^{-1}] = c^p, [c^{-1}, b_n]^{-m} = d^q,$$

$$[a_1, c] = 1, [b_1, c] = 1,$$
$$\cdots \ , \ \cdots$$
$$[a_{n-1}, c] = 1 \ , \ [b_{n-1}, c] = 1,$$
$$[a_n, c] = 1, [a_n, d] = 1,$$

$$[a_1, b_1] \cdot [a_2, b_2] \cdots [a_n, b_n] = 1, [c, d] = 1.$$

Proof. Follows from Lemma 2.8.3. □

Let $\Sigma'_n \subset Y_n(1/p, l/q)$ denote a genus n surface that descend from the surface $\Sigma_n \times pt$ in $\Sigma_n \times \mathbb{T}^2$.

3.4 Construction of Exotic 4-Manifolds

3.4.1 Building Blocks

Our first building block will be the symplectic 4-manifold $Y(n, k) = \Sigma_k \times \mathbb{S}^2 \# 4n\overline{\mathbb{CP}}^2$ with a genus $2k + n - 1$ symplectic submanifold $\Sigma_{2k+n-1} \subset Y(n, k)$, which is a regular fiber of the Lefschetz fibration. Here, we endowed $Y(n, k) = \Sigma_k \times \mathbb{S}^2 \# 4n\overline{\mathbb{CP}}^2$ with the symplectic structure induced from the given Lefschetz fibration.

The other building block will be the symplectic 4-manifold $Y_g(1, 1)$ obtained via Luttinger surgeries along the symplectic submanifold Σ'_g, where we set $g = 2k + n - 1$ and $p = q = m = 1$.

3.4.2 Construction of the symplectic manifold $X(n, k)$

Let $X(n, k)$ denote the symplectic 4-manifold obtained by forming the symplectic fiber sum of $Y(n, k)$ and $Y_g(1, 1)$ along the surfaces Σ_{2k+n-1} and Σ'_{2k+n-1}.

Recall that loops $a_1, b_1, \cdots, a_{2k}$ and b_{2k} generate the inclusion-induced image of $\pi_1(\Sigma_{2k+n-1} \times S^1)$ inside $\pi_1 (Y(n, k) \setminus \nu\Sigma_{2k+n-1})$. This is because the loops $e_1, e_2, \cdots, e_{2n-2}, e_{2n-1}$ and the normal circle to $\mu = \{pt\} \times S^1$ to Σ_{2k+n-1} are all nullhomotopic in $\pi_1(Y(n, k) \setminus \nu\Sigma_{2k+n-1})$. Choose a base point x on $\partial\nu\Sigma_{2k+n-1}$

such that $\pi_1(Y(n,k) \setminus \nu\Sigma_{2k+n-1})$ is generated by the homotopy classes of the loops $a_1, b_1, \cdots a_{2k}, b_{2k}$ based at x.

Let $a'_1, b'_1, \cdots, a'_g, b'_g$ and $\mu' = [c,d]$ generate $\pi_1(\Sigma'_g \times S^1)$ in $\pi_1(Y_g(1,1) \setminus \nu\Sigma'_g)$. Choose the base point y to lie on the boundary $\partial(\nu\Sigma'_g)$.

We choose an orientation-reversing gluing diffeomorphism

$$\psi : \Sigma_{2k+n-1} \times S^1 \longrightarrow \Sigma'_g \times S^1$$

that maps the generators of the fundamental groups as follows:

$$\begin{aligned}
\psi_*(a_1) = a'_1, \psi_*(b_1) = b'_1 &\ , \ \psi_*(e_1) = a'_{k+1}, \psi_*(e_2) = b'_{k+1},\\
\psi_*(a_2) = a'_2, \psi_*(b_2) = b'_2 &\ , \ \psi_*(e_3) = a'_{k+2}, \psi_*(e_4) = b'_{k+2},\\
\cdots &\ , \ \cdots\\
\psi_*(a_k) = a'_k, \psi_*(b_k) = b'_k &\ , \ \psi_*(e_{2n-3}) = a'_{k+n-1}, \psi_*(e_{2n-2}) = b'_{k+n-1},\\
\psi_*(a_{k+1}) = a'_{k+n} &\ , \ \psi_*(b_{k+1}) = b'_{k+n},\\
\cdots &\ , \ \cdots\\
\psi_*(a_{2k}) = a'_{2k+n-1} &\ , \ \psi_*(b_{2k}) = b'_{2k+n-1},\\
\psi_*(\mu) &= (\mu')^{-1}.
\end{aligned}$$

Lemma 3.4.1. *$X(n,k)$ is symplectic.*

Proof. Follows from Gompf's theorem in [**Go2**]. □

Lemma 3.4.2. *$X(n,k)$ is simply-connected.*

Proof. By applying the Seifert-Van Kampen theorem, we see that

$$\pi_1(X(n,k)) = \frac{\pi_1\left(Y(n,k) \setminus \nu\Sigma_{2k+n-1}\right) * \pi_1\left(Y_g(1,1) \setminus \nu\Sigma'_g\right)}{\langle a_1 = a'_1, b_1 = b'_1, \cdots, a_{2k} = a'_{2k+n-1}, b_{2k} = b'_{2k+n-1}, \mu = \mu' = 1\rangle}$$

Since the loops $e_1, e_2, \cdots, e_{2n-3}, e_{2n-2}$, corresponding to the vanishing cycles, and the normal circle to $\mu = \{pt\} \times \mathbb{S}^1$ are all nullhomotopic $Y(n,k) \setminus \nu\Sigma_{2k+n-1}$, we get the following presentation for the fundamental group of $X(n,k)$.

$$\pi_1\left(X(n,k)\right) = \langle a_1, b_1, \cdots, a_{2k}, b_{2k}; c, d; |$$

$$[b_1^{-1}, d^{-1}] = a_1, [a_1^{-1}, d] = b_1,$$
$$\cdots, \cdots,$$
$$[b_{2k-1}^{-1}, d^{-1}] = a_{2k-1}, [a_{2k-1}^{-1}, d] = b_{2k-1},$$
$$[d^{-1}, b_{2k}^{-1}] = c, [c^{-1}, b_{2k}]^{-1} = d,$$
$$[a_1, c] = 1, [b_1, c] = 1, [a_2, c] = 1, [b_2, c] = 1, \cdots [b_{2k}, c] = 1, [a_{2k}, c] = 1, [a_{2k}, d] = 1,$$
$$[a_1, b_1] \cdot [a_2, b_2] \cdots [a_n, b_n] = 1, [c, d] = 1,$$
$$b_1 b_2 \cdots b_{2k} = 1, b_2 b_3 \cdots b_{2k-1} = [a_{2k}, b_{2k}],$$
$$\cdots$$
$$b_{i+1} b_{i+2} \cdots b_{2k-i} = [a_{2k-I+1}, b_{2k-I+1}] \cdots [a_{2k-1}, b_{2k-1}][a_{2k}, b_{2k}],$$
$$[a_1, b_1][a_2, b_2] \cdots [a_k, b_k] = 1\rangle$$

To prove $\pi_1(X(n,k)) = 1$, it is enough to prove that $b_1 = 1$, which in turn will imply that all other generators are trivial. Using the last set of identities, we have $a_1^{-1} = a_{2k}, \cdots a_k^{-1} = a_{k+1}$. Let us rewrite the relation $[a_1^{-1}, d] = b_1$ as $[a_{2k}, d]^{-1} = b_1$. Since $[a_{2k}, d] = 1$, we obtain $b_1 = 1$. This in turn imply that $a_1 = a_{2k} = b_{2k} = c = d = 1$ using the relations $[b_1^{-1}, d^{-1}] = a_1$, $a_{2k} = a_1^{-1}$, $b_1 b_2 \cdots b_{2k} = 1$, $b_2 b_3 \cdots b_{2k-1} = [a_{2k}, b_{2k}]$, $[d^{-1}, b_{2k}^{-1}] = c$, and $[c^{-1}, b_{2k}] = d$. Since $[b_i^{-1}, d^{-1}] = a_i$ and $[a_i^{-1}, d] = b_i$ for any $1 \leq i \leq 2k-1$, we have $a_2 = \cdots = a_{2k-1} = b_2 = \cdots = b_{2k-1} = 1$. Thus, we conclude that $\pi_1\left(X(n,k)\right)$ is trivial. $\square$

Lemma 3.4.3. *Topological Invariants of $X(n,k)$ are given as follows:*

1. $e(X(n,k)) = 8n + 4k - 4$,
2. $\sigma(X(n,k)) = -4n$,
3. $c_1^2(X(n,k)) = 4n + 8k - 8$,
4. $\chi_h(X(n,k)) = n + k - 1$.

Proof. Using Lemma 2.4.1, we have

$$\begin{aligned} e(X(n,k)) &= e(Y(n,k)) + e(Y_g(1,1)) + 4(2k+n-2), \\ \sigma(X(n,k)) &= \sigma(Y(n,k)) + (Y_g(1,1)), \\ c_1^2(X(n,k)) &= c_1^2(Y(n,k)) + c_1^2(Y_g(1,1)) + 8(2k+n-2), \quad \textit{and} \\ \chi_h(X(n,k)) &= \chi_h(Y(n,k)) + \chi_h(Y_g(1,1)) + (2k+n-2) \end{aligned}$$

Since $e\left(Y_g(1,1)\right) = 0$, $\sigma\left(Y_g(1,1)\right) = 0$, $c_1^2\left(Y_g(1,1)\right) = 0$, $\chi_h\left(Y_g(1,1)\right) = 0$,$e\left(Y(n,k)\right) = 4 - 4k + 4n$, $\sigma\left(Y(n,k)\right) = -4n$, $c_1^2\left(Y(n,k)\right) = 8 - 8k - 4n$, and $\chi_h\left(Y(n,k)\right) = 1 - k$, the proof of lemma follows. $\square$

Proof of the Main Theorem (1)**.** By Freedman's classification theorem for simply-connected 4-manifolds (cf. [**Fr**]) and by the lemma above, $X(n,k)$ is homeomorphic to $M = (2n+2k-3)\mathbb{CP}^2 \,\#(6n+2k-3)\overline{\mathbb{CP}}^2$.

Since $X(n,k)$ is symplectic, by Taubes's theorem (cf. [**Tau**]), $SW_{X(n,k)}(K_{X(n,k)}) = \pm 1$, where $K_{X(n,k)}$ denote the canonical class of $X(n,k)$. Next, using Donaldson's connected sum theorem for the SW-invariants (Theorem B in [**Do2**]), we deduce that the SW-invariant of M is trivial. Since the SW-invariant is a diffeomorphism invariant, we conclude that $X(n,k)$ is not diffeomorphic to M.

Proof of the Main Theorem (2)**.** $X(n,k)$ being an exotic copy of $M = (2n+2k-3)\mathbb{CP}^2 \,\#(6n+2k-3)\overline{\mathbb{CP}}^2$ also follows from the fact that it is a minimal symplectic 4-manifold, which we show next. Notice that all $4n$ exceptional spheres $E_1, E_2, \cdots, E_{4n-1}$, and E_{4n}, which are the sections of the genus $2k+n-1$ Lefschetz fibration on $Y\left(n,k\right) = \Sigma_k \times \mathbb{S}^2 \,\#4n\overline{\mathbb{CP}}^2$, meet with the fiber $\Sigma = 2\left(\Sigma_k \times \{pt\}\right) + n\left(\{pt\} \times \mathbb{S}^2\right) - \sum_{i=1}^{4n} E_i$ by Lemma 3.4.4. Furthermore, by Lemma 3.4.5, any embedded symplectic -1 sphere in $Y\left(n,k\right)$ has the form $r\left(\{pt\} \times \mathbb{S}^2\right) \mp E_{i_0}$, thus intersect non-trivially with the fiber $\Sigma = 2\left(\Sigma_k \times \{pt\}\right) + n\left(\{pt\} \times \mathbb{S}^2\right) - \sum_{i=1}^{4n} E_i$. Finally, using the Usher's Theorem, we see that $X\left(n,k\right)$ is a minimal symplectic 4-manifold. Since symplectic minimality implies smooth minimality (cf. [**Li**]), $X\left(n,k\right)$ is smoothly minimal as well.

Lemma 3.4.4. *All $4n$ exceptional spheres E_1, E_2,...,E_{4n-1} and E_{4n}, which are the sections of the genus $2k+n-1$ Lefschetz fibration on $Y\left(n,k\right) = \Sigma_k \times S^2 \,\#4n\overline{\mathbb{CP}}^2$, meet with the fiber $\Sigma = 2\left(\Sigma_k \times \{pt\}\right) + n\left(\{pt\} \times \mathbb{S}^2\right) - \sum_{i=1}^{4n} E_i$.*

Proof. For all $i = 1, 2, \cdots 4n$

$$E_i \cdot \Sigma = 2 \cdot (E_i \cdot (\Sigma_k \times \{pt\})) + n \left(E_i \cdot (\{pt\} \times S^2)\right) - \sum_{j=1}^{4n} E_i \cdot E_j = -E_i \cdot E_i = 1$$

This simply follows from

$$E_i \cdot E_i = -1,\ E_i \cdot E_j = 0,\ E_i \cdot (\Sigma_k \times \{pt\}) = 0,\ E_i \cdot (\{pt\} \times S^2) = 0.$$

□

Lemma 3.4.5. *Any embedded symplectic -1 sphere in $Y(n,k)$ has the form $r(\{pt\} \times S^2) \mp E_{i_0}$, thus intersect non-trivially with the fiber*

$$\Sigma = 2\,(\Sigma_k \times \{pt\}) + n\left(\{pt\} \times \mathbb{S}^2\right) - \sum_{i=1}^{4n} E_i.$$

Proof. Let Σ' be an embedded symplectic -1 sphere in $Y\,(n,k)$.
Then it should be in the following form

$$\Sigma' = r(\{pt\} \times S^2) + t(\Sigma_k \times \{pt\}) + \sum_{i=1}^{4n} a_i E_i$$

Actually, we don't have $\Sigma_k \times \{pt\}$ term because genus can not go up.
In other words, genus doesn't change.

Hence; we get,

$$\begin{aligned}
-1 &= (\Sigma') \cdot (\Sigma') \\
&= r^2 \left[\{pt\} \times S^2\right] \cdot \left[\{pt\} \times S^2\right] + 2r \sum_{i=1}^{4n} a_i \cdot (\{pt\} \times S^2) \cdot E_i + 2 \sum_{i \neq j} a_i a_j \cdot E_i \cdot E_j + \sum_{i=1}^{4n} a_i^2 \cdot E_i \cdot E_i \\
&= -\sum_{i=1}^{4n} a_i^2
\end{aligned}$$

since $E_i \cdot E_i = -1$, $E_i \cdot E_j = 0$, $\left[\{pt\} \times S^2\right]^2 = 0$ and $\left(\{pt\} \times S^2\right) \cdot E_i = 0$.

Therefore, $a_i = \mp 1$ for some i_0 and $a_i = 0 \quad \forall\ \ i \neq i_0$.

So, we conclude $\Sigma' = r(\{pt\} \times S^2) \mp E_{i_0}$.

Enough to show Σ' intersects non-trivially with the fiber Σ.

$$\begin{aligned}
\Sigma' \cdot \Sigma = & \left(r(\{pt\} \times S^2) \mp E_{i_0}\right) \cdot \left(2\left(\Sigma_k \times \{pt\}\right) + n\left(\{pt\} \times \mathbb{S}^2\right) - \sum_{i=1}^{4n} E_i\right) \\
= & \; 2r\left(\{pt\} \times S^2\right) \cdot \left(\Sigma_k \times \{pt\}\right) + rn\left(\{pt\} \times S^2\right) \cdot \left(\{pt\} \times S^2\right) - r\sum_{i=1}^{4n}\left(\{pt\} \times S^2\right) \cdot E_i \\
\mp & \; 2E_{i_0} \cdot \left(\Sigma_k \times \{pt\}\right) \mp nE_{i_0} \cdot \left(\{pt\} \times S^2\right) \pm \sum_{j=1}^{4n} E_{i_0} \cdot E_j \\
= & \; 2r\left(\{pt\} \times S^2\right) \cdot \left(\Sigma_k \times \{pt\}\right) \pm \sum_{j=1}^{4n} E_{i_0} \cdot E_j \\
= & \; 2r\left(\{pt\} \times S^2\right) \cdot \left(\Sigma_k \times \{pt\}\right) \pm E_{i_0}^2 \\
= & \; 2r \mp 1 \neq 0
\end{aligned}$$

In addition, using adjunction formula, we get $\Sigma' \cdot \Sigma' + K_{Y(n,k)} \cdot \Sigma' = 2g\left(\Sigma'\right) - 2$, where $K_{Y(n,k)}$ is the canonical class of $Y(n,k)$. Thus, $K_{Y(n,k)} \cdot \Sigma' = -1$.

Again, by applying adjunction formula to E_i and $\{pt\} \times S^2$, we

$$\begin{aligned}
-1 = & \; K_{Y(n,k)} \cdot \left(\mp E_{i_0} + r\left(\{pt\} \times S^2\right)\right) \cdot \\
= & \mp K_{Y(n,k)} \cdot E_{i_0} + rK_{Y(n,k)} \cdot \left(\{pt\} \times S^2\right) \\
= & \mp \left(2g\left(E_{i_0}\right) - 2 - E_{i_0} \cdot E_{i_0}\right) + r\left(2g\left(\{pt\} \times S^2\right) - 2 - \left(\{pt\} \times S^2\right)^2\right) \\
= & \mp \left(2 \cdot 0 - 2 + 1\right) + r\left(2 \cdot 0 - 2 - 0\right) \\
= & \mp(-1) - 2r
\end{aligned}$$

Thus, $r = 0$ or $r = 1$. $\square$

3.4.3 Infinitely many non-symplectic exotic $(2n+2k-3)\mathbb{CP}^2\#(6n+2k-3)\overline{\mathbb{CP}}^2$

Using the building block $Y_g(1,m)$, where $|m| \geq 1$, instead of $Y_g(1,1)$ used in our construction above, we obtain an infinite family of exotic smooth 4-manifolds which we will denote by $X(n,k,m)$. This corresponds to replacing a single relation $[c^{-1}, b_{2k}]^{-1} = d$ in

$\pi_1(X(n,k))$, representing the Luttinger surgery $(a''_{2k+n-1} \times d', d', 1)$, with $[c^{-1}, b_{2k}]^{-m} = d$. Our fundamental group computations follow the same steps as in $\pi_1(X(n,k))$ case, since the above relation does not affect the fundamental group computation. Therefore, we conclude that $X(n,k,m)$ is simply connected.

Furthermore, $X(n,k)$ has one basic class up to sign, the canonical class $\pm K_{X(n,k)}$, which can be seen by using an argument similar to that in [**ABP**] (Section 4, pages $12-18$). Now, consider the manifold obtained by performing the Luttinger surgery $(a''_{2k+n-1} \times d', d', 0/1)$ and denote the resulting symplectic 4-manifold by $X(n,k)_0$. Note that, $\pi_1(X(n,k)_0) = \mathbb{Z}$ and the canonical class can be written as $K_{X(n,k)_0} = 2[\Sigma_k] + \sum_{j=1}^{4n}[R_j]$, where R_j are rim tori of self-intersection -1. Moreover, the basic class $\beta_{n,k,m}$ of $X(n,k,m)$ corresponding to the canonical class $K_{X(n,k)_0}$ has SW-invariant equal to m.

$$SW_{X(n,k,m)}(\beta_{n,k,m}) = SW_{X_{n,k}}(K_{X(n,k)}) + (m-1)SW_{X(n,k)_0}(K_{X(n,k)_0}) = 1 + (m-1) = m.$$

Hence, $X(n,k,m)$ is non-symplectic for any $m \geq 2$.

3.4.4 New symplectic 4 manifolds with $\delta = -4n$ and various Fundamental Groups

In this section we modify the above construction to obtain symplectic 4-manifolds with the various finitely generated fundamental groups, such as the free groups of rank $s \geq 1$, and the finite free products of cyclic groups, using Luttinger surgery. Our construction can also be generalized further to obtain symplectic 4-manifolds with arbitrarily finitely presented fundamental groups and with *small size*.

Our first building block will be the symplectic 4-manifold $Y(n,k) = \Sigma_k \times \mathbb{S}^2 \# 4n\overline{\mathbb{CP}}^2$ along with a regular fiber of the genus $2k+n-1$ Lefschetz fibration on $Y(n,k)$. We equip $Y(n,k) = \Sigma_k \times \mathbb{S}^2 \# 4n\overline{\mathbb{CP}}^2$ with the symplectic structure induced from the given Lefschetz fibration. The second building block will be the symplectic 4-manifold $Y_l(1/p_1, 1/q_1, \cdots, 1/p_l, 1/q_l)$ along with the symplectic submanifold Σ_l, where we set $l = 2k+n-1$. To simplify the notation, we set $\overline{p} = (p_1, \cdots, p_l)$ and $\overline{q} = (q_1, \cdots, q_l)$ throughout this section. Let $X(n,k,\overline{p},\overline{q})$ denote the symplectic 4-manifold obtained by forming the symplectic fiber sum of $Y(n,k)$ and $Y_l(1/p_1, 1/q_1, \cdots, 1/p_l, 1/q_l)$ along the surfaces Σ_{2k+n-1} and Σ_l. Let $c_1, d_1, \cdots, c_l, d_l$, and $\mu' = [a_1, b_1][a_2, b_2]$ generate $\pi_1\left(\Sigma_l \times \mathbb{S}^1\right)$ in $\pi_1\left(Y_l(1/p_1, 1/q_1, \cdots, 1/p_l, 1/q_l) \setminus \nu\Sigma_l\right)$.

We choose the gluing diffeomorphism $\psi : \Sigma_{2k+n-1} \times \mathbb{S}^1 \longrightarrow \Sigma_l \times \mathbb{S}^1$ that maps the generators of the fundamental groups as follows:

$$\begin{aligned}
\psi_*(\alpha_1) &= c_1, \psi_*(\beta_1) = d_1, \\
\psi_*(\alpha_2) &= c_2, \psi_*(\beta_2) = d_2, \\
\cdots &, \cdots \\
\psi_*(\alpha_{2k}) &= c_{2k}, \psi_*(\beta_{2k}) = d_{2k}, \\
\psi_*(e_1) &= c_{2k+1}, \psi_*(e_2) = d_{2k+1}, \\
\cdots &, \\
\psi_*(e_{2n-3}) &= c_{2k+n-1}, \psi_*(e_{2n-2}) = d_{2k+n-1}, \\
\psi_*(\mu) &= \mu'
\end{aligned}$$

By Gompf's theorem in [**Go2**], $X(n,k,\overline{p},\overline{q})$ is symplectic.

By Van Kampen' s theorem, we have $\pi_1(X(n,k,\overline{p},\overline{q})) =$

$$\frac{\pi_1\left(Y(n,k) \setminus \nu\Sigma_{2k+n-1}\right) * \pi_1\left(Y_l(1/p_1, 1/q_1, \cdots, 1/p_l, 1/q_l) \setminus \nu\Sigma_l\right)}{\langle \alpha_1 = c_1, \beta_1 = d_1, \cdots, \alpha_{2k} = c_{2k}, \beta_{2k} = d_{2k}, e_1 = c_{2k+1}, \cdots, e_{2n-2} = d_{2k+n-1}, \mu = \mu' \rangle}$$

Since the loops $e_1, e_2, \cdots, e_{2n-3}, e_{2n-2}$ and the normal circle to $\mu = \{pt\} \times S^1$ are all nullhomotopic in $\pi_1(Y(n,k) \setminus \nu\Sigma_{2k+n-1})$, we get the following presentation for the fundamental group of $X(n,k,\overline{p},\overline{q})$.

$$\pi_1(X(n,k,\overline{p},\overline{q})) = \langle c_1, d_1 \cdots c_{2k}, d_{2k}; a_1, b_1, a_2, b_2; |$$

$$\begin{aligned}
[b_1^{-1}, d_1^{-1}] &= a_1, [a_1^{-1}, d_1] = b_1,\\
[b_2^{-1}, d_2^{-1}] &= a_2, [a_2^{-1}, d_2] = b_2,\\
[d_1^{-1}, b_2^{-1}] &= c_1^{p_1}, [c_1^{-1}, b_2] = d_1^{q_1},\\
[d_2^{-1}, b_1^{-1}] &= c_2^{p_2}, [c_1^{-1}, b_1] = d_2^{q_2},\\
[a_1, c_1] &= 1, [a_1, c_2] = 1, [a_1, d_2] = 1, [b_1, c_1] = 1\\
[a_2, c_1] &= 1, [a_2, c_2] = 1, [a_2, d_1] = 1, [b_2, c_2] = 1,
\end{aligned}$$

$$[a_1, b_1][a_2, b_2] = 1, \Pi_{j=1}^{2k}[c_j, d_j] = 1.$$

$$\begin{aligned}
[a_1^{-1}, d_3^{-1}] &= c_3^{p_3}, [a_2^{-1}, c_3^{-1}] = d_3^{p_3},\\
&\cdots,\\
[a_1^{-1}, d_{2k}^{-1}] &= c_{2k}^{p_{2k}}, [a_2^{-1}, c_{2k}^{-1}] = d_{2k}^{q_{2k}},
\end{aligned}$$

$$[b_1, c_3] = 1, [b_2, d_3] = 1, \cdots, [b_1, c_{2k}] = 1, [b_2, d_{2k}] = 1,$$

$$c_1 c_{2k} = 1, c_2 c_{2k-1} = 1, \cdots, c_k c_{k+1} = 1,$$

$$d_1 d_2 \cdots d_{2k} = 1, d_2 d_3 \cdots d_{2k-1} = [c_{2k}, d_{2k}], \cdots,$$

$$d_{i+1} d_{i+2} \cdots d_{2k-i} = [c_{2k-I+1}, d_{2k-i+1}] \cdots [c_{2k-1}, d_{2k-1}][c_{2k}, d_{2k}]\rangle.$$

- To realize the free group of rank $s \geq 1$ as the fundamental groups, we simply set $p_3 = \cdots = p_{2k} = 0, p_1 = p_2 = q_1 = \cdots = q_{2k} = 1$ in the above presentation. Using the identity $c_{2k}^{-1} = c_1$, we rewrite the relation $[a_2^{-1}, c_{2k}^{-1}] = d_{2k}$ as $[a_2^{-1}, c_{2k}^{-1}] = d_{2k}$. Since $[a_2, c_1] = 1$, we obtain $d_{2k} = 1$. $d_{2k} = 1$ in turn implies $d_1 = 1$ using the relations $d_2 d_3 \cdots d_{2k-1} = [c_{2k}, d_{2k}]$ and $d_1 d_2 \cdot d_{2k} = 1$. Next, using $[b_1^{-1}, d_1^{-1}] = a_1$ $[a_1^{-1}, d_1] = b_1$ and $[d_1^{-1}, b_2^{-1}] = c_2$, we obtain $a_1 = b_1 = c_1 = 1$. Since $[c_2^{-1}, b_1] = d_2$ and $[d_2^{-1}, b_1^{-1}] = c_2$, we have $d_2 = c_2 = 1$, which in turn lead $a_2 = b_2 = 1$. Next, using the relations $[a_1^{-1}, c_i^{-1}] = d_i$ for any $3 \leq i \leq 2k$, we have $d_3 = d_4 = \cdots = d_{2k} = 1$. Since $c_{2k-i+1}^{-1} = c_i$ for any $i \leq k$ and $c_1 = c_2 = 1$, we conclude that $\pi_1(X(n, k, \overline{p}, \overline{q}))$ is a free group of rank $s := k - 2$ generated by $c_3, \cdots, c_k$.

- To realize the finite free products of cyclic groups as the fundamental groups, we simply set $p_1 = p_2 = 1, p_3 = \cdots = p_l = 0$, and let $q_i \geq 1$ vary arbitrarily in the above presentation.

- By varying $p_j \geq 1$ and $q_i \geq 1$ arbitrarily, we can realize many other finitely presented groups as fundamental groups.

Chapter 4

Lefschetz Fibrations Using Cyclic Quotient Singularities

4.1 Finite Order Cyclic Group Actions

4.1.1 Order $2g + 1$ cyclic action

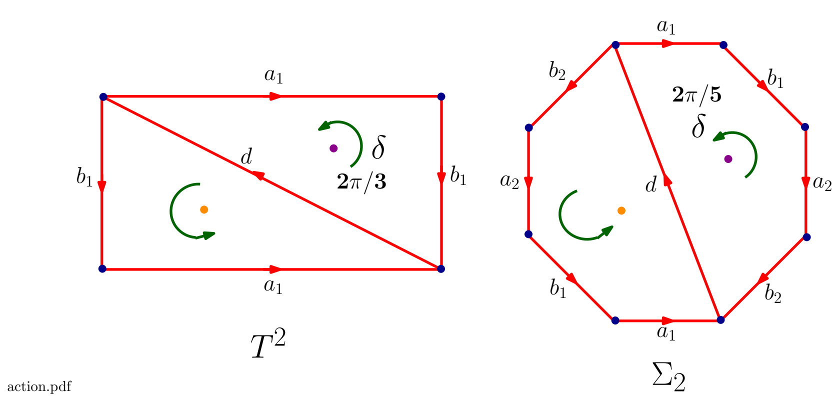


Figure 4.1: Order $2g + 1$ cyclic group action

Let g be a positive integer. We can think of the genus g surface Σ_g as a $4g$-gon with diametrically opposite edges identified so that the word given by the boundary of the

$4g$-gon is

$$a_1 a_2 \cdots a_{2g} a_1^{-1} a_2^{-1} \cdots a_{2g}^{-1}.$$

Divide this $4g$-gon into two $(2g+1)$-gons by cutting along a diagonal d such that the boundaries of the resulting two $(2g+1)$-gons give the words

$$a_1 a_2 \cdots a_{2g} d \qquad \textit{and} \qquad a_1^{-1} a_2^{-1} \cdots a_{2g}^{-1} d^{-1}.$$

Viewing each $(2g+1)$-gon as a regular polygon, we can rotate each $(2g+1)$-gon by angle $\dfrac{2\pi}{2g+1}$ and then reglue them to obtain an orientation-preserving self-diffeomorphism $\delta : \Sigma_g \to \Sigma_g$ of order $2g+1$ with 3 fixed points.

4.1.2 Order $g+1$ cyclic action

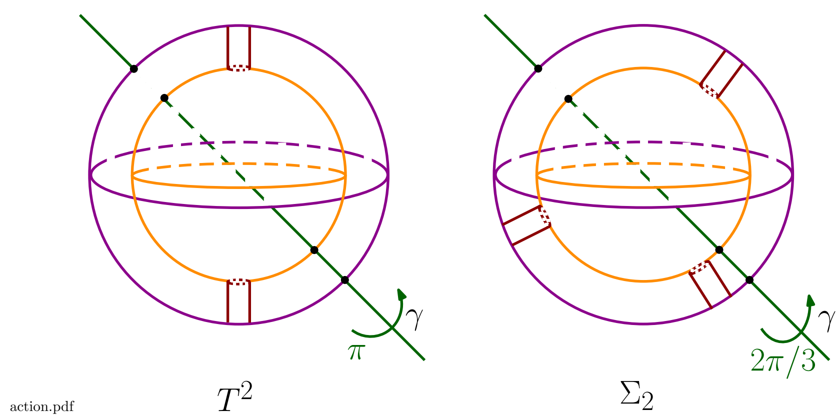


Figure 4.2: Order $g+1$ cyclic group action

Let g be a positive integer and Σ_g be a closed genus g Riemann surface. Consider Σ_g as two concentric spheres with $g+1$ tubes connecting them. Now, take an orientation-preserving self-diffeomorphism $\gamma : \Sigma_g \to \Sigma_g$ which is the rotation of this surface by angle $\dfrac{2\pi}{g+1}$. This action is of order $g+1$ and has 4 fixed points, the axis of rotation goes through two points on each sphere.

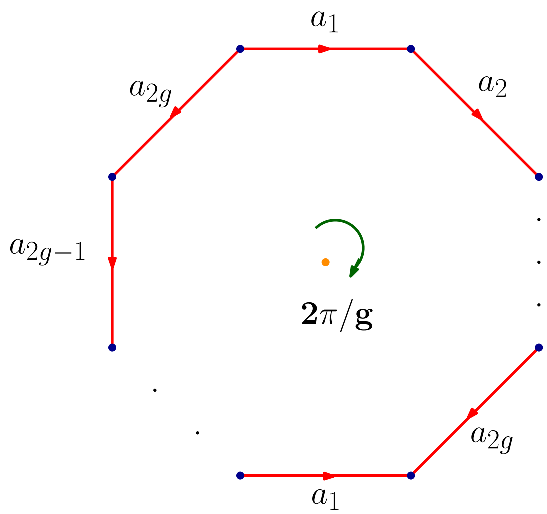


Figure 4.3: order g action

4.1.3 Order g action

Let g be a positive integer. Again, we can think of the genus g surface Σ_g as a $4g$-gon with diametrically opposite edges identified. Rotating this $4g$-gon by angle $\frac{2\pi}{g}$, we can obtain an orientation-preserving self-diffeomorphism $\alpha : \Sigma_g \to \Sigma_g$ of order g with 2 fixed points. (See Figure 4.3)

4.1.4 Order $2g$ action

Again, if we can think of the genus g surface Σ_g as a $4g$-gon with diametrically opposite edges identified and rotate this $4g$-gon by angle $\frac{2\pi}{2g}$, we obtain an orientation-preserving self-diffeomorphism $\beta : \Sigma_g \to \Sigma_g$ of order $2g$ with 2 fixed points.

4.1.5 Order $4g$ action

Similar to the previous actions, we can also rotate this $4g$-gon by angle $\frac{2\pi}{4g}$ and obtain an orientation-preserving self-diffeomorphism $\lambda : \Sigma_g \to \Sigma_g$ of order $4g$ with 2 fixed points.

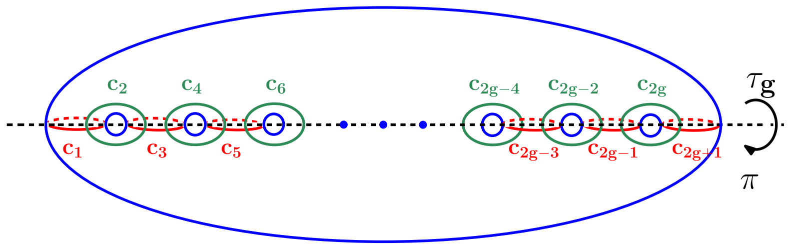


Figure 4.4: hyperelliptic involution

4.1.6 Composition with Hyperelliptic Action

Imagine $\Sigma_g \subset \mathbb{R}^3$ such that the y-axis intersect it in $2g+2$ points and Σ_g is invariant under the 180° rotation around the y-axis. See Figure 4.4. This rotation defines a $\mathbb{Z}_2$-action $\tau_g : \Sigma_g \to \Sigma_g$ with $2g+2$ fixed points, called *hyperelliptic involution.*

Now, by combining hyperelliptic involution with the involutions we have described above, we can obtain more actions as follows.

$\gamma \circ \tau_g$

The hyperelliptic involution τ_2 gives $\mathbb{Z}_2$ action on Σ_2. γ gives $\mathbb{Z}_3$ action on Σ_2. Combining them, we obtain $\mathbb{Z}_6$ action on Σ_2.

The hyperelliptic involution τ_3 gives $\mathbb{Z}_2$ action on Σ_3. γ gives $\mathbb{Z}_4$ action on Σ_3. Combining them, we obtain $\mathbb{Z}_8$ action on Σ_3.

More generally, the hyperelliptic involution τ_g gives $\mathbb{Z}_2$ action on Σ_g. γ gives $\mathbb{Z}_{g+1}$ action on Σ_g. Combining them, we obtain $\mathbb{Z}_{2(g+1)}$ action on Σ_g.

$\delta \circ \tau_g$

The hyperelliptic involution τ_2 gives $\mathbb{Z}_2$ action on Σ_2. δ gives $\mathbb{Z}_5$ action on Σ_2. Combining them, we obtain $\mathbb{Z}_{10}$ action on Σ_2.

The hyperelliptic involution τ_3 gives $\mathbb{Z}_2$ action on Σ_3. δ gives $\mathbb{Z}_7$ action on Σ_3. Combining them, we obtain $\mathbb{Z}_{14}$ action on Σ_3.

More generally, the hyperelliptic involution τ_g gives $\mathbb{Z}_2$ action on Σ_g. δ gives $\mathbb{Z}_{2g+1}$

action on Σ_g. Combining them, we obtain $\mathbb{Z}_{2(2g+1)}$ action on Σ_g.

$\alpha \circ \tau_g$

The hyperelliptic involution τ_2 gives $\mathbb{Z}_2$ action on Σ_2. α gives $\mathbb{Z}_2$ action on Σ_2. Combining them, we obtain $\mathbb{Z}_4$ action on Σ_2.

The hyperelliptic involution τ_3 gives $\mathbb{Z}_2$ action on Σ_3. α gives $\mathbb{Z}_3$ action on Σ_3. Combining them, we obtain $\mathbb{Z}_6$ action on Σ_3.

More generally, the hyperelliptic involution τ_g gives $\mathbb{Z}_2$ action on Σ_g. α gives $\mathbb{Z}_g$ action on Σ_g. Combining them, we obtain $\mathbb{Z}_{2g}$ action on Σ_g.

$\beta \circ \tau_g$

The hyperelliptic involution τ_2 gives $\mathbb{Z}_2$ action on Σ_2. β gives $\mathbb{Z}_4$ action on Σ_2. Combining them, we obtain $\mathbb{Z}_8$ action on Σ_2.

The hyperelliptic involution τ_3 gives $\mathbb{Z}_2$ action on Σ_3. β gives $\mathbb{Z}_6$ action on Σ_3. Combining them, we obtain $\mathbb{Z}_{12}$ action on Σ_3.

More generally, the hyperelliptic involution τ_g gives $\mathbb{Z}_2$ action on Σ_g. β gives $\mathbb{Z}_{2g}$ action on Σ_g. Combining them, we obtain $\mathbb{Z}_{4g}$ action on Σ_g.

$\lambda \circ \tau_g$

The hyperelliptic involution τ_2 gives $\mathbb{Z}_2$ action on Σ_2. λ gives $\mathbb{Z}_8$ action on Σ_2. Combining them, we obtain $\mathbb{Z}_{16}$ action on Σ_2.

The hyperelliptic involution τ_3 gives $\mathbb{Z}_2$ action on Σ_3. λ gives $\mathbb{Z}_{12}$ action on Σ_3. Combining them, we obtain $\mathbb{Z}_{24}$ action on Σ_3.

More generally, the hyperelliptic involution τ_g gives $\mathbb{Z}_2$ action on Σ_g. λ gives $\mathbb{Z}_{4g}$ action on Σ_g. Combining them, we obtain $\mathbb{Z}_{8g}$ action on Σ_g.

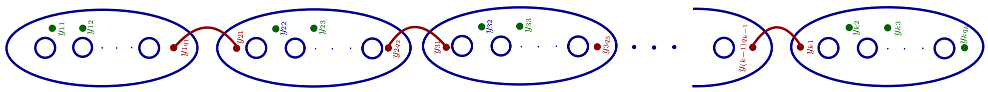


Figure 4.5: equivariant sum

4.2 Gluing self-diffeomorpfhisms of surfaces

Let $k \geq 2$ be an integer and $\alpha_i : \Sigma_{g_i} \to \Sigma_{g_i}$ be an orientation preserving diffeomorphism for $i = 1, 2, \cdots k$. Let $\{y_{i,1}, y_{i,2}, \cdots y_{i,q_i}\}$ be the set fixed points of α_i. Assume α_i generates a semi-free $\mathbb{Z}_p$-action on Σ_{g_i} which is a rotation by angle $\dfrac{2\pi\rho_{i,j}}{p}$ at $T_{y_{i,j}}\Sigma_{g_i}$, where $\rho_{i,j}$ is the rotational number of α_i at $y_{i,j}$ which are defined by

- $(\rho_{i,j}, p) = 1$
- $\sum_{j=1}^{q_i} \dfrac{1}{\rho_{i,j}} \equiv 0 \qquad mod(p)$

where $\dfrac{1}{\rho_{i,j}}$ is the multiplicative inverse of $\rho_{i,j}$ in $(\mathbb{Z}_p)^*$.

Choose some fixed points as in the following Figure 4.5.

- Remove small $\mathbb{Z}_p$ equivariant neighborhoods of these chosen fixed points.
- Then glue the boundary circles.
- If $\rho_{i,q_i} = -\rho_{i+1,1}$, for all $i = 1, 2, \cdots, k-1$, then $\alpha_i|_{\Sigma^0_{g_i}}$ can also be glued together to form an orientation preserving diffeomorphism $\zeta = \alpha_1 \# \alpha_2 \# \cdots \# \alpha_k$

$$\zeta : \Sigma_g \longrightarrow \Sigma_g$$

of genus $g = \sum_{i=1}^k g_i$ surface with $q = \left(\sum_{i=1}^k q_i\right) - 2(k-1)$ fixed points which is called *equivariant sum*.

4.3 Cyclic Quotient Singularities

This section is borrowed from Polizzi's paper, [**Po**].

Figure 4.6: dual grapf of E

Let n and q be coprime natural numbers with $1 \leq q \leq n-1$, and let ξ_n be a primitive nth root of unity. Let us consider the action of the cyclic group $\mathbb{Z}_n = \langle \xi_n \rangle$ on $\mathbb{C}^2$ defined by $\xi_n \cdot (x, y) = (\xi_n x, \xi_n^q y)$. Then the analytic space $X_{n,q} = \mathbb{C}^2/\mathbb{Z}_n$ contains a cyclic quotient singularity of type $\frac{1}{n}(1, q)$. Denoting by q' the unique integer $1 \leq q' \leq n-1$ such that $qq' \equiv 1\,(mod, \;\; n)$, we have $X_{n_1,q_1} \cong X_{n,q}$ if and only if $n_1 = n$ and either $q_1 = q$ or $q_1 = q'$. The exceptional divisor on the minimal resolution $\tilde{X}_{n,q}$ of $X_{n,q}$ is a HJ-string (Hirzebruch -Jung string), that is to say, a connected union of $\mathrm{E} = \bigcup_{i=1}^{k} Z_i$ of smooth rational curves $Z_1, \cdots Z_k$ with self intersection ≤ -2, and ordered linearly so that $Z_i Z_{i+1} = 1$ for all i, and $Z_i Z_j = 0$ if $|i-j| \geq 2$. More precisely, given the continued fraction

$$\frac{n}{q} = [b_1, \cdots, b_k] = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\cdots - \cfrac{1}{b_k}}}, \qquad b_i \geq 2,$$

the dual graph of E is as in Figure 4.6. Moreover,

$$\frac{n}{q} = [b_1, \cdots, b_k] \;\; if \;\; and \;\; only \;\; if \;\; \frac{n}{q'} = [b_k, \cdots, b_1].$$

Definition 20. Let x be a cyclic quotient singularity of type $\frac{1}{n}(1, q)$ and let E be the corresponding HJ-string. If $\frac{n}{q} = [b_1, \cdots, b_k]$, we write $\mathrm{E} : \frac{n}{q} = [b_1, \cdots, b_k]$ and we set

$$\begin{aligned}
l_x &= l(\mathrm{E}) = l\left(\frac{q}{n}\right) := k, \\
h_x &= h(\mathrm{E}) = h\left(\frac{q}{n}\right) := 2 - \frac{2+q+q'}{n} - \sum_{i=1}^{k}(b_i - 2), \\
e_x &= e(\mathrm{E}) = e\left(\frac{q}{n}\right) := k + 1 - \frac{1}{n}, \\
B_x &= B(\mathrm{E}) = B\left(\frac{q}{n}\right) := 2e_x - h_x = \frac{q+q'}{n} + \sum_{i=1}^{k} b_i.
\end{aligned}$$

Definition 21. We say that a projective surface S is *standard isotrivial fibration* if there exists finite group G, acting faithfully on two smooth projective curves C_1 and C_2 so that S is isomorphic to the minimal desingularization of $T := (C_1 \times C_2)/G$, where G acts diagonally on the product. The two maps $\alpha_1 : S \to C_1/G$, $\alpha_2 : S \to C_2/G$ will be referred as the *natural projections*. If T is smooth then $S = T$ is called *quasi-bundle*.

The stabilizer $H \subseteq G$ of a point $y \in C_2$ is a cyclic group ([**FK**] pg. 106). If H acts freely on C_1, then T is smooth along the scheme-theoretic fibre of $\sigma : T \to C_2/G$ over $\overline{y} \in C_2/G$, and this fibre consists of the curve C_1/H counted with multiplicity $|H|$. Thus, the smooth fibres of σ are all isomorphic to C_1. On the contrary, if $x \in C_1$ is fixed by some non-zero element of H, then one has cyclic quotient singularity over the point $\overline{(x,y)} \in T$.

Theorem 4.3.1. *Let $\lambda : S \to T = (C_1 \times C_2)/G$ be a standard isotrivial fibration and let us consider the natural projection $\alpha_2 : S \to C_2/G$. Take a point over $\overline{y} \in C_2/G$. Then*

(i) The reduced structure of F is the union of an irreducible curve Y, called the central component of F, and either none or at least two mutually disjoint HJ-strings, each meeting Y at one point, and each being contracted by λ to a singular point of T. These strings are in one-to one correspondence with the branch points of $C_1 \to C_1/H$, where $H \subset G$ is the stabilizer of y.

(ii) The intersection of a string with Y is transversal, and it takes place at only one of the end components of the string.

(iii) Y is isomorphic to C_1/H, and has multiplicity equal to $|H|$ in F.

An analogous statement holds if one consider the natural projection $\alpha_1 : S \to C_1/G$

Proposition 4.3.2. *Let $\lambda : S \to T = (C_1 \times C_2)/G$ be a standard isotrivial fibration.*

Then the invariant of S are given by

$$
\begin{aligned}
(i), \quad K_S^2 &= \frac{8\,(g(C_1)-1)\,(g(C_2-1))}{|G|} + \sum_{x \in Sing(T)} h_x; \\
(ii), \quad e(S) &= \frac{4\,(g(C_1)-1)\,(g(C_2)-1)}{|G|} + \sum_{\times in Sing(T)} e_x; \\
(iii), \quad q(S) &= g(C_1/G) + g(C_2/G).
\end{aligned}
$$

Let us consider now minimal resolution of a cyclic quotient singularity $x \in T$. If Y_1 and Y_2 are the strict transforms of C_1 and C_2, by Theorem 4.3.1 we obtain situation illustrated in Figure 4.7.

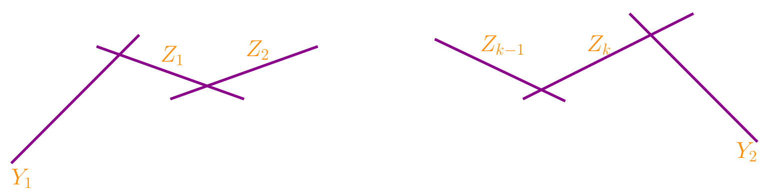


Figure 4.7: Resolution of a cyclic quotient singularity $x \in T$

The curves Y_1 and Y_2 are the central components of two reducible fibers F_1 and F_2 of $\alpha_2 : S \to C_2/G$ and $\alpha_1 : S \to C_1/G$, respectively. Then there exist $\lambda_1, \cdots, \lambda_k, \mu_1, \cdots, \mu k \in \mathbb{N}$ such that

$$
\begin{aligned}
F_1 &= \rho_1 Y_1 + \sum_{i=1}^{k} \lambda_i Z_i + \Gamma_1 \\
F_2 &= \rho_2 Y_2 + \sum_{i=1}^{k} \mu_i Z_i + \Gamma_2
\end{aligned}
$$

where the supports of both divisors Γ_1 and Γ_2 are union of HJ-strings disjoint from the Z_i; moreover if x is of type $\dfrac{1}{n}(1,q)$, then n divides both ρ_1 and ρ_2.

Definition 22. We say that a reducible fibre F_1 of $\alpha_2 : S \to C_2/G$ is of type $\left(\dfrac{q_1}{n_1}, \cdots, \dfrac{q_r}{n_r}\right)$ if it contains exactly r HJ-strings $E_1, \cdots, E_r$, where each E_i is of type $\dfrac{1}{n_i}(1, q_i)$. The same definition holds for a reducible fibre F_2 of $\alpha_1 : S \to C_1/G$.

Proposition 4.3.3. *([**Po**]) Let F_1 be of type $\left(\frac{q_1}{n_1}, \cdots, \frac{q_r}{n_r}\right)$ and let Y_1 be its central component. Then*

$$(Y_1)^2 = -\sum_{i=1}^{r} \frac{q_i}{n_i}$$

Analogously, if F_2 is of type $\left(\frac{q_1}{n_1}, \cdots, \frac{q_r}{n_r}\right)$ then

$$(Y_2)^2 = -\sum_{i=1}^{r} \frac{q_i'}{n_i}$$

4.4 Mapping Class Group

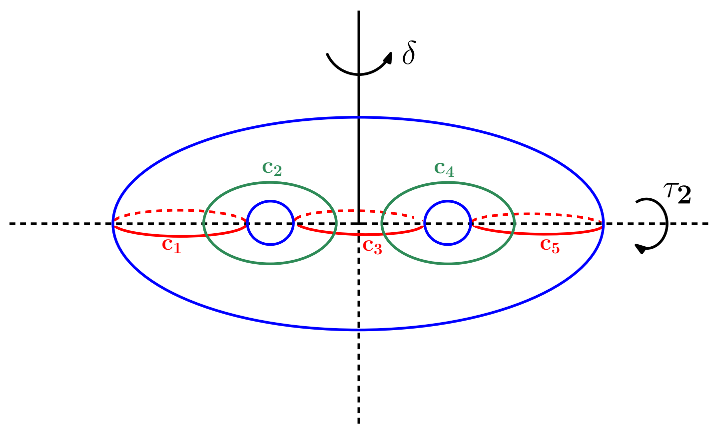


Figure 4.8: hyperelliptic involution on Σ_2

Lemma 4.4.1. *(cf. [**Luo**])*

*a) ([**De**]) Let a and b be two simple loops in the torus $\Sigma_{1,0}$ so that they intersect transversely at one point. Let A and B be the positive Dehn-twist on a and b respectively. Then the standard symmetries of the torus are the following:*

the hyperelliptic involution $\tau_2 = ABABAB$, the 4-fold symmetry $\tau_4 = ABA$ and the 6-fold symmetry $\tau_6 = AB$.

*b) ([**Bi**]) Let $a_1, \cdots, a_{r-1}$ be the pairwise disjoint arcs in the planar surface $\Sigma_{0,r}$ so that a_i joins the i-th boundary B_i to B_{i+1}. Let A_i be the half-twist about the arc a_i. Then $\tau_r = A_1 \cdots A_{r-1}$ and $\tau_{r-1} = A_1 \cdots A_{r-2}$ are $2\pi/r$ and $2\pi/(r-1)$-rotation of the surface sending a_i to a_{i+1} for $1 \leq i \leq r-3$.*

*c) ([**Bi**]) Let $C_1, \cdots C_5$ be the positive Dehn-twist on the five simple loops $c_1, \cdots, c_5$ in the genus-2 surface (see Figure 4.8). Then the hyperelliptic involution $\tau_2 = C_1C_2C_3C_4C_5^2C_4C_3C_2C_1$ and the 5-fold symmetry is $\tau_5 = \tau_2 C_1C_2C_3C_4$.*

4.5 Construction

We first start with the product manifold $\Sigma_g \times \Sigma_g$. Then, we take the cyclic group actions of finite order n with k fixed points. So, we obtain singular manifolds $S(g,n,t) = (\Sigma_g \times \Sigma_g)/\mathbb{Z}_n$ with cyclic quotient singularities, where t denotes the type of the singular fibers which we list in the following sections. The singular manifolds $S(g,n,t)$ has k^2 singular points, respectively. Desingularizing these manifolds, we obtain families of simply-connected Lefschetz fibrations $X(g,n,t)$ over $\mathbb{S}^2$.

We desingularize them as follows. By first removing the cone like neighborhood of the singular points of $S(g,n,t)$, we get a manifold $S'(g,n,t)$ with $\partial S'(g,n,t) = \bigcup_1^{k^2} \mathbb{RP}^3$. Then, we glue k^2 copies of W yo $S'(g,n,t)$, where W is the unit disk bundle of the cotangent bundle of the sphere $\mathbb{S}^2$, namely

$$W = \{v \in T^*\mathbb{S}^2 | \|v\| \leq 1\}$$

which is a smooth manifold with $\partial W = \mathbb{RP}^3$.

Lemma 4.5.1. *The total space $X(g,n,t)$ of these Lefschetz fibrations described above has euler characteristic*

$$e(X(g,n,t)) = k\ e\,(F(g,n,t)) + (2-k)(2-2g)$$

where k denotes the number of the fixed points of the cyclic group actions of order n and $F(g,k,t)$ denotes the singular fiber of the Lefschetz fibration.

Proof. By decomposing the total space as

$$X(g,n,t) = (X(g,n,t) \setminus Singular\ sets) \bigcup_k (Singular\ sets)$$

Since is $X(g,n,t)\setminus Singular\ sets$ is an Σ_g bundle over D^2 with k-points deleted (denote the base by D^2_k), we get

$$\begin{aligned} e(X(g,n,t)) &= e\left(X(g,n,t) \setminus Singular\ sets\right) + k\ e\left(Singular\ sets\right) \\ &= e(\Sigma_g)e(D^2_k) + k\ e(Singular\ sets) \\ &= (2-2g)(2-k) + k\ e(Singular\ sets) \end{aligned}$$

□

Next, we will provide the following two lemmas and conclusion (cf. [**Ur**]) which prove that $X(g,n,t)$ is simply connected.

Let $f : X \to B$ be a fibration. Let $F = f^{-1}(b)$ with $b \in B \setminus sing(f)$. Hence, the inclusion $f \hookrightarrow X$ induces a homomorphism $\pi_1(F) \to \pi_1(X)$. Let $\mathcal{V}_f$ be the image of this homomorphism, which is called the *vertical part* of $\pi_1(X)$.

Lemma 4.5.2. *The vertical part $\mathcal{V}_f$ is a normal subgroup of $\pi_1(X)$ and is independent of the choice of F.*

The *horizontal part* of π_1 is $\mathcal{H}_f := \pi_1(X) \setminus \mathcal{V}_f$, and so we have

$$1 \to \mathcal{V}_f \to \pi_1(X) \to \mathcal{H}_f \to 1$$

Let F be any fiber of f, we write $F = f^*(b) = \Sigma_{i=1}^n \mu_i \Gamma_i$ for some positive integers μ_i. The multiplicity of F is $m = gcd(\mu_1, \mu_2, \cdots, \mu_n)$. F is called *multiple fiber* of f if $m > 1$. Let $\{x_1, x_2, \cdots, x_s\}$ be the images of all the multiple fibers of f (it may be empty), and $\{m_1, m_2, \cdots, m_s\}$ the corresponding multiplicities. Let $B' = B \setminus \{x_1, x_2, \cdots, x_s\}$ and γ_i be a small loop around x_i. Then, there are generators $\alpha_1, \cdots, \alpha_b, \beta_1, \cdots, \beta_b$ such that

$$\pi_1(B') \simeq \langle \alpha_1, \cdots, \alpha_b, \beta_1, \cdots, \beta_b, \gamma_1, \cdots, \gamma_s : \alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1} \cdots \alpha_b\beta_b\alpha_b^{-1}\beta_b^{-1}\gamma_1 \cdots \gamma_s = 1 \rangle$$

Lemma 4.5.3. *The horizontal part $\mathcal{H}_f$ is the quotient of $\pi_1(B')$ by the normal subgroup generated by the conjugates of $\gamma_i^{m_i}$ for all i.*

Lemma 4.5.4. *Let F be any fiber of f with multiplicity m. Then the image of $\pi_1(F)$ in $\pi_1(X)$ contains $\mathcal{V}_f$ as a normal subgroup, whose quotient group is cyclic of order m, which maps isomorphically onto the subgroup of $\mathcal{H}_f$ generated by the class of a small loop around the image of F in B. In particular, $\mathcal{V}_f$ is trivial if f has a simply connected fiber.*

Corollary 4.5.5. *If f has a section, then*

$$1 \to \mathcal{V}_f \to \pi_1(X) \to \pi_1(B) \to (B) \to 1.$$

Moreover, if f has a simply connected fiber, then $\pi_1(X) \simeq \pi_1(B)$.

4.6 Examples of order $g+1$ action

As mentioned above, in this case we are using order $g+1$ cyclic action on $\Sigma_g \times \Sigma_g$ which has 4 fixed points. So, the singular manifold $S(g, g+1, t) = \Sigma_g \times \Sigma_g/\mathbb{Z}_{g+1}$ has 16 singular points $\{x_{ij} | 1 \le i, j \le 4\}$ corresponding to the fixed points of the action.

4.6.1 $\mathbf{h = 1}$

The singular manifold $\mathbb{T}^2/\mathbb{Z}^2$ is homeomorphic to $\mathbb{S}^2$ and has 4 corner points (corresponding to the fixed points of the $\mathbb{Z}^2$-action) (cf. [**GS**] pg 78). So, the singular manifold $S(1,2,t) = \mathbb{T}^2 \times \mathbb{T}^2/\mathbb{Z}_2 \times \mathbb{Z}_2$ has 16 singular points $\{x_{ij} | 1 \le i, j \le 4\}$ corresponding to the fixed points of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ action $\mathbb{T}^2 \times \mathbb{T}^2$. We can desingularize $S(1,2,t)$ and get a smooth manifold X as follows. By first removing the cone like neighborhood of the singular points of $S(1,2,t)$, we get a manifold $S'(1,2,t)$ with $\partial S'(1,2,t) = \bigcup_1^{16} \mathbb{RP}^3$. Then, we glue 16 copies of W yo $S'(1,2,t)$ where W is the unit disk bundle of the cotangent bundle of the sphere $\mathbb{S}^2$, namely $W = \{v \in T^*\mathbb{S}^2 | \|v\| \le 1\}$, which is a smooth manifold with $\partial W = \mathbb{RP}^3$.

$$n = 2, \qquad q = 1$$

$\mathbf{h = 1}$, Type 1: fiber of type $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$

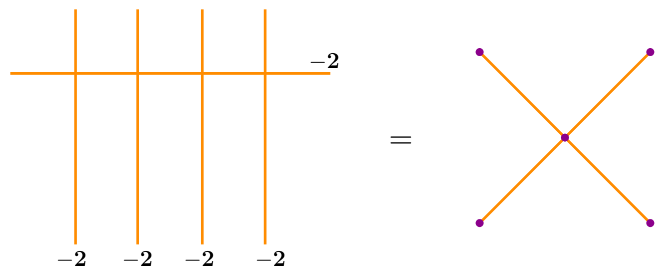


Figure 4.9: $h = 1$, Type 1

$$(Y)^2 = -\sum_{i=1}^{4} \frac{1}{2} = -2$$

$$\frac{n_i}{q_i} = \frac{2}{1} = [2], \qquad 1 \le i \le 4$$

In this case, see Figure 4.9, there is no -1-sphere and the singular fibers correspond to type I_0^* in Table 1^* in [**KM**]. (Also see [**Ko**])

The quotient $N = \mathbb{T}^2/\gamma$ is a 2-sphere, and the projection $\mathbb{T}^2 \to N$ is a 2-fold cover branched at four points corresponding to the center and vertex of rectangle. Set $M = \mathbb{T}^2 \times B^2/(\gamma \times \gamma)$ and $D = B^2/\gamma \cong B^2$. Then the natural projection $M \to D$ has a singular fiber over $0 \in D$, hence on N, and is a $\mathbb{T}^2$-bundle over $D/0$ with monodromy $-I$. M is a 4-manifold except that at the four branch points on N which are locally cones on $\mathbb{RP}^3$. These singular points can be resolved by removing the open cones on $\mathbb{RP}^3$ and gluing in cotangent disk bundles of $\mathbb{S}^2$. This gives a neighborhood $N(I_0^*)$ of the singular fiber I_0^*, as shown in Figure 4.9.

$$l_{x_{ij}} = l\left(\frac{n_i}{q_i}\right) = k = 1$$
$$q_i = 1, \qquad q_i' = 1$$

$$h_{x_{ij}} = h\left(\frac{q}{n}\right) := 2 - \frac{2 + q + q'}{n} - \sum_{i=1}^{k}(b_i - 2) = 2 - \frac{2+1+1}{2} - (2-2) = 0$$

$$B_{x_{ij}} = \frac{q+q'}{n} + \sum_{i=1}^{k} b_i = \frac{1+1}{2} + 2 = 3$$

$$e_{x_{ij}} = 1 + 1 - \frac{1}{2} = \frac{3}{2}, \qquad 1 \leq i, j \leq 4$$

Lemma 4.6.1.

$$\begin{aligned} e(X(1,2,1)) &= 24, & c_1^2(X(1,2,1)) = 0, \\ \sigma(X(1,2,1)) &= -16, & \chi_h(X(1,2,1)) = 2. \end{aligned}$$

Hence, $X(1,2,1)$ is homeomorphic to the elliptic surface $E(2)$.

Proof.

$$K^2_{X(1,2,1)} = \frac{8\,(g(C_1)-1)\,(g(C_2)-1)}{|G|} + \sum_{x \in Sing(T)} h_x = \frac{8(0)(0)}{2} + 16(0) = 0$$

$$e(X(1,2,1)) = \frac{4\,(g(C_1)-1)\,(g(C_2)-1)}{|G|} + \sum_{x_{ij} \in Sing(T)} e_{x_{ij}} = \frac{4(0)(0)}{2} + 16\left(\frac{3}{2}\right) = 24$$

Therefore, $\sigma(X(1,2,1)) = -16$ and $\chi_h(X(1,3,1)) = 2$, which follows from the formulas $c_1^2(X) = 2e(X) + 3\sigma(X)$ and $\chi_h(X) = \dfrac{e(X)+\sigma(X)}{4}$.

We can compute the euler characteristic also using Lemma 4.5.1. Each singular fiber has $e(F(1,2,1)) = 5 \cdot 2 - 4 = 6$. Hence,

$$e(X(1,2,1)) = k \cdot e(F(1,2,1)) + (2-k) \cdot (2-2g) = 4 \cdot 6 - 2(2-2) = 24.$$

$\square$

There are 4 singular fibers each has monodromy $(UV)^3$ of order 2.(cf. [**KM**], [**Ogg**, **Ko**]). So the total monodromy of $X(1,2,1)$ is $((UV)^3)^4 = (UV)^{12} = 1$.

4.6.2 $\mathbf{h = 2}$

In this case we are using order 3 cyclic action on Σ_2 and again recall from section 4.1.2 that it has 4 fixed points. So, the singular manifold $S(2,3,t) = \Sigma_2 \times \Sigma_2/\mathbb{Z}_3$ has 16 singular points $\{x_{ij} | 1 \leq i,j \leq 4\}$ corresponding to the fixed points of the action.

$$n = 3, \qquad 1 \leq q \leq 2$$

h = 2, Type 1: fiber of type $\left(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$

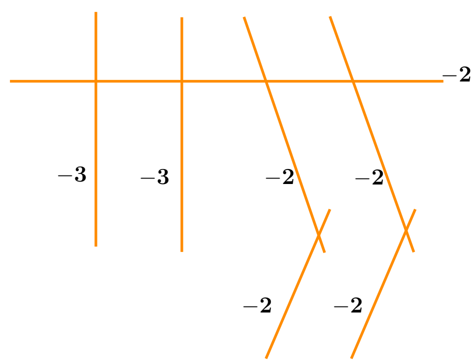


Figure 4.10: $h = 2$, Type 1

$$(Y)^2 = -\left(\frac{1}{3} + \frac{1}{3} + \frac{2}{3} + \frac{2}{3}\right) = -2$$

$$\frac{n_1}{q_1} = \frac{n_2}{q_2} = \frac{3}{1} = [3],$$
$$\frac{n_3}{q_3} = \frac{n_4}{q_4} = \frac{3}{2} = 2 - \frac{1}{2} = [2,2]$$

In this case, the singular fibers (See Figure 4.10) correspond to type 42 in [**NU**] (Also type 42 in the table on pg. 359 in [**Ogg**]).

$$\begin{aligned} l_{x_{i1}} &= l_{x_{i2}} = 1, \qquad l_{x_{i3}} = l_{x_{i4}} = 2 \\ q_1 &= q_2 = 1, \qquad q_3 = q_4 = 2 \\ q_1' &= q_2' = 1, \qquad q_3' = q_4' = 2 \end{aligned}$$

$$\begin{aligned} h_{x_{i1}} &= h_{xi2} = 2 - \frac{2+1+1}{3} - (3-2) = -\frac{1}{3} \\ h_{x_{i3}} &= h_{xi4} = 2 - \frac{2+2+2}{3} - 2(2-2) = 0 \end{aligned}$$

$$\begin{aligned} B_{x_{i1}} &= B_{x_{i2}} = \frac{q+q'}{n} + \sum_{i=1}^{k} b_i = \frac{1+1}{3} + 3 = \frac{11}{3} \\ B_{x_{i3}} &= B_{x_{i4}} = \frac{q+q'}{n} + \sum_{i=1}^{k} b_i = \frac{2+2}{3} + 2 + 2 = \frac{16}{3} \end{aligned}$$

$$e_{x_{i1}} = e_{x_{i2}} = 1 + 1 - \frac{1}{3} = \frac{5}{3}, \qquad e_{x_{i3}} = e_{x_{i4}} = 2 + 1 - \frac{1}{3} = \frac{8}{3}$$

Lemma 4.6.2.

$$e(X(2,3,1)) = 36, \qquad \sigma(X(2,3,1)) = -24, \qquad c_1^2(X(2,3,1)) = 0, \qquad \chi_h(X(2,3,1)) = 3.$$

Hence, $X(2,3,1)$ is homeomorphic to the elliptic surface $E(3)$.

Proof.

$$e(X(2,3,1)) = \frac{4\,(g(C_1)-1)\,(g(C_2)-1)}{|G|} + \sum_{x_{ij}\in Sing(T)} e_{x_{ij}} = \frac{4}{3} + 4\left(2\cdot\frac{5}{3} + 2\cdot\frac{8}{3}\right) = 36$$

$$K^2_{X(2,3,1)} = \frac{8\,(g(C_1)-1)\,(g(C_2)-1)}{|G|} + \sum_{x\in Sing(T)} h_x = \frac{8}{3} + 4\left(2\cdot\left(-\frac{1}{3}\right) + 2\cdot 0\right) = 0$$

Therefore, $\sigma(X(2,3,1)) = -24$ and $\chi_h(X(2,3,1)) = 3$, which follows from the formulas

$c_1^2(X) = 2e(X) + 3\sigma(X)$ and $\chi_h(X) = \dfrac{e(X)+\sigma(X)}{4}$.

Each singular fiber has $e(F(2,3,1)) = 7 \cdot 2 - 6 = 8$. Hence, again by Lemma 4.5.1, $e(X(2,3,1)) = k \cdot e(F(2,3,1)) + (2-k)\cdot(2-2g) = 4 \cdot 8 - 2(2-4) = 36$. □

4.6.3 $\mathbf{h = 3}$

In this case we are using order 4 cyclic action on Σ_3 and again recall from section 4.1.2 that in this case we have 4 fixed points. So, the singular manifold $S(3,4,t) = \Sigma_3 \times \Sigma_3/\mathbb{Z}_4$ has 16 singular points $\{x_{ij} | 1 \leq i,j \leq 4\}$ corresponding to the fixed points of the action.

$$n = 4, \qquad 1 \leq q \leq 3$$

$\mathbf{h = 3}$, Type 1: fiber of type $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$

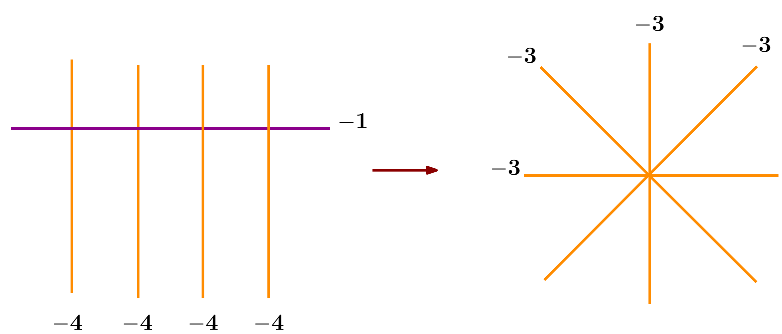


Figure 4.11: $h = 3$, Type 1

$$(Y)^2 = -\sum_{i=1}^{4} \frac{1}{4} = -1$$

$$\frac{n_i}{q_i} = \frac{4}{1} = [4], \qquad 1 \leq i \leq 4$$

In this case, see Figure 4.11, there is a central -1-sphere which can be blown down.

$$
\begin{aligned}
l_{x_{ij}} &= 1 \\
q_1 &= 1, \qquad q_1' = 1 \\
h_{x_{ij}} &= 2 - \frac{2+1+1}{4} - (4-2) = -1 \\
B_{x_{ij}} &= \frac{1+1}{4} + 4 = \frac{9}{2} \\
e_{x_{ij}} &= 1 + 1 - \frac{1}{4} = \frac{7}{4}, \qquad 1 \leq i, j \leq 4
\end{aligned}
$$

Once we blow down this manifold, we obtain a manifold $X(3,4,1)$ with singular fibers as in Figure 4.11.

Lemma 4.6.3.

$$
\begin{aligned}
e(X(3,4,1)) &= 28, \qquad c_1^2(X(3,4,1)) = 4, \\
\sigma(X(3,4,1)) &= -20, \qquad \chi_h(X(3,4,1)) = 2.
\end{aligned}
$$

Proof. Before blow-down we have;

$$
e(S) = \frac{4\,(g(C_1)-1)\,(g(C_2)-1)}{|G|} + \sum_{x_{ij} \in Sing(T)} e_{x_{ij}} = \frac{4(2)(2)}{4} + 16\left(\frac{7}{4}\right) = 32
$$

$$
K_S^2 = \frac{8\,(g(C_1)-1)\,(g(C_2)-1)}{|G|} + \sum_{x \in Sing(T)} h_x = \frac{8(2)(2)}{4} + 16(-1) = -8
$$

Once we blow-down we get;

$$
K^2_{X(3,4,1)} = K_S^2 + 4 = -8 + 4 = -4
$$

$$
e(X(3,4,1)) = e(S) - 4 = 28
$$

Therefore, $\sigma(X(3,4,1)) = -20$ and $\chi_h(X(3,4,1)) = 2$, which follows from the formulas

$$
c_1^2(X) = 2e(X) + 3\sigma(X) \qquad \chi_h(X) = \frac{e(X)+\sigma(X)}{4}.
$$

We can compute the euler characteristic also using Lemma 4.5.1. Each singular fiber has $e(F(3,4,1)) = 4\cdot 2 - 3 = 5$. Hence,

$$e(X(3,4,1)) = k\cdot e(F(3,4,1)) + (2-k)\cdot(2-2g) = 4\cdot 5 - 2(2-6) = 28.$$

$\square$

h = 3, Type 2: fiber of type $\left(\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4}\right)$

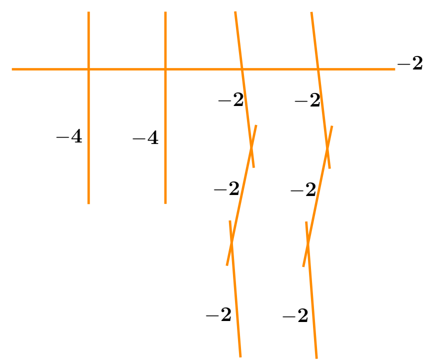


Figure 4.12: $h = 3$, Type 2

$$(Y)^2 = -\left(\frac{1}{4}+\frac{1}{4}+\frac{3}{4}+\frac{3}{4}\right) = -2$$

$$\begin{aligned}
\frac{n_1}{q_1} &= \frac{n_2}{q_2} = \frac{4}{1} = [4],\\
\frac{n_3}{q_3} &= \frac{n_4}{q_4} = \frac{4}{3} = 2-\frac{2}{3} = 2-\frac{1}{\frac{3}{2}} = 2-\frac{1}{2-\frac{1}{2}} = [2,2,2]
\end{aligned}$$

$$\begin{aligned}
l_{x_{i1}} &= l_{x_{i2}} = 1, & l_{x_{i3}} &= l_{x_{i4}} = 3\\
q_1 &= q_2 = 1, & q_3 &= q_4 = 3\\
q_1' &= q_2' = 1, & q_3' &= q_4' = 3
\end{aligned}$$

$$h_{x_{i1}} = h_{x_{i2}} = 2 - \frac{2+1+1}{4} - (4-2) = -1, \quad h_{x_{i3}} = h_{x_{i4}} = 2 - \frac{2+3+3}{4} - 3(2-2) = 0$$

$$\begin{aligned} B_{x_{i1}} &= B_{x_{i2}} = \frac{1+1}{4} + 4 = \frac{9}{2}, \qquad B_{x_{i3}} = B_{x_{i4}} = \frac{3+3}{4} + 2 + 2 + 2 = \frac{15}{2} \\ e_{x_{i1}} &= e_{x_{i2}} = 1 + 1 - \frac{1}{4} = \frac{7}{4}, \qquad e_{x_{i3}} = e_{x_{i4}} = 3 + 1 - \frac{1}{4} = \frac{15}{4} \end{aligned}$$

Lemma 4.6.4.

$$\begin{aligned} e(X(3,4,2)) &= 48, \qquad c_1^2(X(3,4,2)) = 0, \\ \sigma(X(3,4,2)) &= -32, \qquad \chi_h(X(3,4,2)) = 4. \end{aligned}$$

Proof.

$$K^2_{X(3,4,2)} = \frac{8\,(g(C_1)-1)\,(g(C_2)-1)}{|G|} + \sum_{x \in Sing(T)} h_x = \frac{8(2)(2)}{4} + 4\,(-1-1+0+0) = 0$$

$$e(X(3,4,2)) = \frac{4\,(g(C_1)-1)\,(g(C_2)-1)}{|G|} + \sum_{x_{ij} \in Sing(T)} e_{x_{ij}} = \frac{4(2)(2)}{4} + 4\left(2 \cdot \frac{7}{4} + 2 \cdot \frac{15}{4}\right) = 48$$

Therefore, $\sigma(X(3,4,2)) = -32$ and $\chi_h(X(3,4,2)) = 4$, which follows from the formulas $c_1^2(X) = 2e(X) + 3\sigma(X)$ and $\chi_h(X) = \dfrac{e(X)+\sigma(X)}{4}$.
In another way, since each singular fiber has euler characteristic $e(F(3,4,2)) = 9 \cdot 2 - 8 = 10$, by Lemma 4.5.1 we have

$$e(X(3,4,2)) = k \cdot e(F(3,4,2)) + (2-k) \cdot (2-2g) = 4 \cdot 10 - 2(2-6) = 48.$$

$\square$

$\mathbf{h=3}$, Type 3: fiber of type $\left(\frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4}\right)$

$$(Y)^2 = -\left(\frac{1}{4} + \frac{1}{2} + \frac{1}{2} + \frac{3}{4}\right) = -2$$

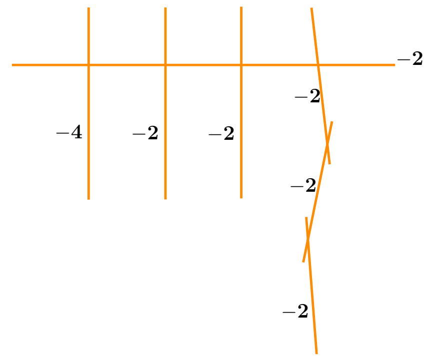


Figure 4.13: $h = 3$, Type 3

$$\frac{n_1}{q_1} = \frac{4}{1} = [4], \qquad \frac{n_2}{q_2} = \frac{n_3}{q_3} = \frac{2}{1} = [2], \qquad \frac{n_4}{q_4} = \frac{4}{3} = [2,2,2]$$

$$\begin{aligned} l_{x_{i1}} &= l_{x_{i2}} = l_{x_{i3}} = 1, \qquad l_{x_{i4}} = 3 \\ q_1 &= q_2 = q_3 = 1, \qquad q_4 = 3 \\ q_1' &= q_2' = q_3' = 1, \qquad q_4' = 3 \end{aligned}$$

$$\begin{aligned} h_{x_{i1}} &= 2 - \frac{2+1+1}{4} - (4-2) = -1 \\ h_{x_{i2}} &= h_{x_{i3}} = 2 - \frac{2+1+1}{2} - (2-2) = 0 \\ h_{x_{i4}} &= 2 - \frac{2+3+3}{4} - 3(2-2) = 0 \end{aligned}$$

$$\begin{aligned} B_{x_{i1}} &= \frac{1+1}{4} + 4 = \frac{9}{2}, \quad B_{x_{i2}} = B_{x_{i3}} = \frac{1+1}{2} + 2 = 3, \quad B_{x_{i4}} = \frac{3+3}{4} + 2 + 2 + 2 = \frac{15}{2} \\ e_{x_{i1}} &= e_{x_{i2}} = 1 + 1 - \frac{1}{4} = \frac{7}{4}, \qquad e_{x_{i3}} = e_{x_{i4}} = 3 + 1 - \frac{1}{4} = \frac{15}{4} \end{aligned}$$

Lemma 4.6.5. $e(X(3,4,3)) = 40$

Proof.

$$e(X(3,4,3)) = \frac{4\,(g(C_1)-1)\,(g(C_2)-1)}{|G|} + \sum_{x_{ij} \in Sing(T)} e_{x_{ij}} = \frac{4(2)(2)}{4} + 4\left(3 \cdot \binom{7}{4} + \frac{15}{4}\right) = 40$$

We can compute the euler characteristic also using Lemma 4.5.1. Each singular fiber has $e(F(3,4,3)) = 7 \cdot 2 - 6 = 8$. Hence,

$$e(X(3,4,3)) = k \cdot e(F(3,4,3)) + (2-k) \cdot (2-2g) = 4 \cdot 8 - 2(2-6) = 40.$$

$\square$

$\mathbf{h = 3}$, Type 4: fiber of type $\left(\frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}\right)$

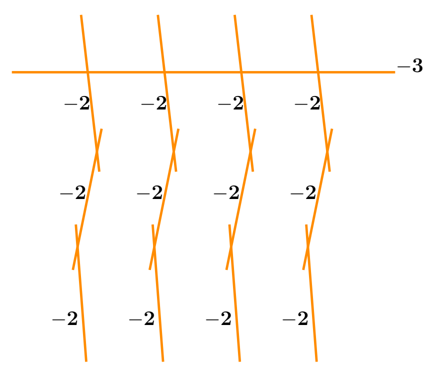


Figure 4.14: $h = 3$, Type 4

$$(Y)^2 = -\sum_{i=1}^{4} \frac{3}{4} = -3$$

$$\frac{n_i}{q_i} = \frac{4}{3} = [2,2,2], \qquad 1 \le i \le 4$$

$$l_{x_{ij}} = 3, \qquad q_i = 3, \quad q_i' = 3$$

$$\begin{aligned} h_{x_{ij}} &= 2 - \frac{2+3+3}{4} - 3(2-2) = 0 \\ e_{x_{ij}} &= 3 + 1 - \frac{1}{4} = \frac{15}{4} \\ B_{x_{ij}} &= \frac{3+3}{4} + 2 + 2 + 2 = \frac{19}{2} \end{aligned}$$

Lemma 4.6.6.

$$\begin{aligned} e(X(3,4,4)) &= 64, \quad c_1^2(X(3,4,4)) = 8, \\ \sigma(X(3,4,4)) &= -40, \quad \chi_h(X(3,4,4)) = 6. \end{aligned}$$

Proof.

$$K^2_{X(3,4,4)} = \frac{8\,(g(C_1)-1)\,(g(C_2)-1)}{|G|} + \sum_{x \in Sing(T)} h_x = \frac{8(2)(2)}{4} + 16(0) = 8$$

$$e(X(3,4,4)) = \frac{4\,(g(C_1)-1)\,(g(C_2)-1)}{|G|} + \sum_{x_{ij} \in Sing(T)} e_{x_{ij}} = \frac{4(2)(2)}{4} + 16\left(\frac{15}{4}\right) = 64$$

Therefore, $\sigma(X(3,4,4)) = -40$ and $\chi_h(X(3,4,4)) = 6$, which follows from the formulas $c_1^2(X) = 2e(X) + 3\sigma(X)$ and $\chi_h(X) = \dfrac{e(X)+\sigma(X)}{4}$. We can compute the euler characteristic also using Lemma 4.5.1. Each singular fiber has $e(F(3,4,4)) = 13 \cdot 2 - 12 = 14$. Hence,

$$e(X(3,4,4)) = k \cdot e(F(3,4,4)) + (2-k) \cdot (2-2g) = 4 \cdot 14 - 2(2-6) = 64.$$

□

4.6.4 h = 4

In this case we are using order 5 cyclic action on Σ_4 and again recall from section 4.1.2 that in this case we have 4 fixed points. So, the singular manifold $T = \Sigma_4 \times \Sigma_4/\mathbb{Z}_5$ has 16 singular points $\{x_{ij} | 1 \le i, j \le 4\}$ corresponding to the fixed points of the action.

$$n = 5, \qquad 1 \le q \le 4$$

h = 4, Type 1: fiber of type $\left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{2}{5}\right)$

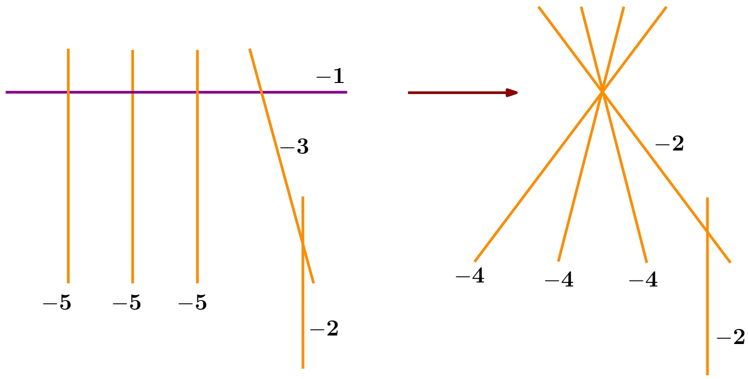


Figure 4.15: $h = 4$, Type 1

$$(Y)^2 = -1$$

$$\begin{aligned} \frac{n_1}{q_1} &= \frac{n_2}{q_2} = \frac{n_3}{q_3} = \frac{5}{1} = [5], \\ \frac{n_4}{q_4} &= \frac{5}{2} = 3 - \frac{1}{2} = [3, 2] \end{aligned}$$

$$\begin{aligned} l_{x_{i1}} &= l_{x_{i2}} = l_{x_{i3}} = 1, \qquad l_{x_{i4}} = 2 \\ q_1 &= q_2 = q_3 = 1, \qquad q_4 = 2 \\ q_1' &= q_2' = q_3' = 1, \qquad q_4' = 3 \end{aligned}$$

$$\begin{aligned} h_{x_{i1}} &= h_{x_{i2}} = h_{x_{i3}} = 2 - \frac{2+1+1}{5} - (5-2) = -\frac{9}{5} \\ h_{x_{i4}} &= 2 - \frac{2+2+3}{5} - ((3-2) + (2-2)) = -\frac{2}{5} \end{aligned}$$

$$\begin{aligned} B_{x_{i1}} &= B_{x_{i2}} = B_{x_{i3}} = \frac{1+1}{5} + 5 = \frac{27}{5}, \qquad B_{x_{i4}} = \frac{2+3}{5} + 3 + 2 = 6 \\ e_{x_{i1}} &= e_{x_{i2}} = e_{x_{i3}} = 1 + 1 - \frac{1}{5} = \frac{9}{5}, \quad e_{x_{i4}} = 2 + 1 - \frac{1}{5} = \frac{14}{5} \end{aligned}$$

Lemma 4.6.7. $e(X(4,5,1)) = 36$

Proof. Before blow-down we have;

$$e(S) = \frac{4\,(g(C_1)-1)\,(g(C_2)-1)}{|G|} + \sum_{x_{ij}\in Sing(T)} e_{x_{ij}} = \frac{4(3)(3)}{5} + 4\left(\frac{9}{5}+\frac{9}{5}+\frac{9}{5}+\frac{14}{5}\right) = 40$$

Once we blow-down we get; $e(X(4,5,1)) = e(S) - 4 = 36$.

We can compute the euler characteristic also using Lemma 4.5.1. Each singular fiber has $e(F(4,5,1)) = 5 \cdot 2 - 4 = 6$. Hence,

$$e(X(4,5,1)) = k \cdot e(F(4,5,1)) + (2-k)\cdot(2-2g) = 4\cdot 6 - 2(2-8) = 36.$$

$\square$

h = 4, Type 2**: fiber of type** $\left(\frac{1}{5},\frac{1}{5},\frac{4}{5},\frac{4}{5}\right)$

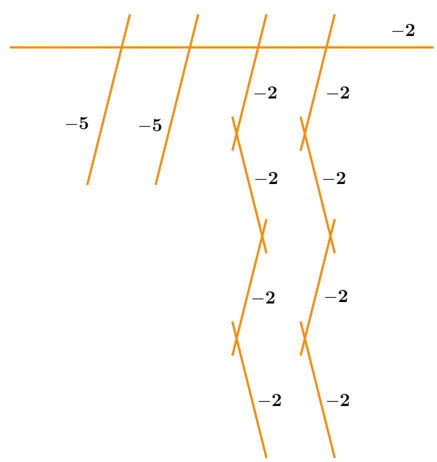


Figure 4.16: $h = 4$, Type 2

$$(Y)^2 = -2$$

$$\frac{n_1}{q_1} = \frac{n_2}{q_2} = \frac{5}{1} = [5],$$
$$\frac{n_3}{q_3} = \frac{n_4}{q_4} = \frac{5}{4} = 2 - \frac{3}{4} = 2 - \frac{1}{\frac{4}{3}} = 2 - \frac{1}{2 - \frac{2}{3}} = 2 - \frac{1}{2 - \frac{1}{\frac{3}{2}}} = 2 - \frac{1}{2 - \frac{1}{2 - \frac{1}{2}}} = [2,2,2,2]$$

$$\begin{aligned} l_{x_{i1}} &= l_{x_{i2}} = 1, \qquad l_{x_{i3}} = l_{x_{i4}} = 4 \\ q_1 &= q_2 = 1, \qquad q_3 = q_4 = 4 \\ q'_1 &= q'_2 = 1, \qquad q'_3 = q'_4 = 4 \end{aligned}$$

$$h_{x_{i1}} = h_{x_{i2}} = 2 - \frac{2+1+1}{5} - (5-2) = -\frac{9}{5}, \qquad h_{x_{i3}} = h_{x_{i4}} = 2 - \frac{2+4+4}{5} - 4(2-2) = 0$$

$$B_{x_{i1}} = B_{x_{i2}} = \frac{1+1}{5} + 5 = \frac{27}{5}, \quad B_{x_{i3}} = B_{x_{i4}} = \frac{4+4}{5} + 2+2+2+2 = \frac{48}{5}$$

$$e_{x_{i1}} = e_{x_{i2}} = 1 + 1 - \frac{1}{5} = \frac{9}{5}, \qquad e_{x_{i3}} = e_{x_{i4}} = 4 + 1 - \frac{1}{5} = \frac{24}{5}$$

Lemma 4.6.8.

$$\begin{aligned} e(X(4,5,2)) &= 60, \qquad c_1^2(X(4,5,2)) = 0, \\ \sigma(X(4,5,2)) &= -40, \qquad \chi_h(X(4,5,1)) = 5. \end{aligned}$$

Proof.

$$K^2_{X(4,5,2)} = \frac{8\,(g(C_1)-1)\,(g(C_2)-1)}{|G|} + \sum_{x \in Sing(T)} h_x = \frac{8(3)(3)}{5} + 4\left(-\frac{9}{5} - \frac{9}{5} + 0 + 0\right) = 0$$

$$e(X(4,5,2)) = \frac{4\,(g(C_1)-1)\,(g(C_2)-1)}{|G|} + \sum_{x_{ij} \in Sing(T)} e_{x_{ij}} = \frac{4(3)(3)}{5} + 4\left(2 \cdot \frac{9}{5} + 2 \cdot \frac{24}{5}\right) = 60$$

Therefore, $\sigma(X(4,5,2)) = -40$ and $\chi_h(X(4,5,2)) = 5$.

We can compute the euler characteristic also using Lemma 4.5.1. Each singular fiber has $e(F(4,5,2)) = 11 \cdot 2 - 10 = 12$. Hence,

$$e(X(4,5,2)) = k \cdot e(F(4,5,2)) + (2-k) \cdot (2-2g) = 4 \cdot 12 - 2(2-8) = 60.$$

$\square$

h = 4, Type 3: fiber of type $\left(\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\right)$

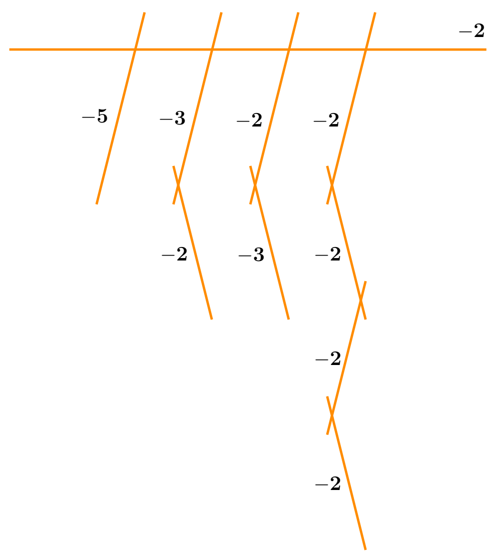


Figure 4.17: $h = 4$, Type 3

$$(Y)^2 = -2$$

$$\frac{n_1}{q_1} = \frac{5}{1} = [5], \qquad \frac{n_2}{q_2} = \frac{5}{2} = [3,2],$$
$$\frac{n_3}{q_3} = \frac{5}{3} = 2 - \frac{1}{3} = [2,3], \qquad \frac{n_4}{q_4} = \frac{5}{4} = [2,2,2,2]$$

$$\begin{aligned}
l_{x_{i1}} &= 1, \qquad l_{x_{i2}} = l_{x_{i3}} = 2, \qquad l_{x_{i4}} = 4 \\
q_1 &= 1, \qquad q_2 = 2, \qquad q_3 = 3, \qquad q_4 = 4 \\
q'_1 &= 1, \qquad q'_2 = 3, \qquad q'_3 = 2, \qquad q'_4 = 4
\end{aligned}$$

$$\begin{aligned}
h_{x_{i1}} &= 2 - \frac{2+1+1}{5} - (5-2) = -\frac{9}{5} \\
h_{x_{i2}} &= 2 - \frac{2+2+3}{5} - ((3-2)+(2-2)) = -\frac{2}{5} \\
h_{x_{i3}} &= 2 - \frac{2+3+2}{5} - ((2-2)+(3-2)) = -\frac{2}{5} \\
h_{x_{i4}} &= 2 - \frac{2+4+4}{5} - 3(2-2) = 0
\end{aligned}$$

$$\begin{aligned}
B_{x_{i1}} &= \frac{1+1}{5} + 5 = \frac{27}{5} \\
B_{x_{i2}} &= \frac{2+3}{5} + 3 + 2 = 6, \qquad B_{x_{i3}} = \frac{3+2}{5} + 2 + 3 = 6 \\
B_{x_{i4}} &= \frac{4+4}{5} + 2 + 2 + 2 + 2 = \frac{48}{5}
\end{aligned}$$

$$e_{x_{i1}} = 1 + 1 - \frac{1}{5} = \frac{9}{5}, \qquad e_{x_{i2}} = e_{x_{i3}} = 2 + 1 - \frac{1}{5} = \frac{14}{5}, \qquad e_{x_{i4}} = 4 + 1 - \frac{1}{5} = \frac{24}{5}$$

Lemma 4.6.9.

$$\begin{aligned}
e(X(4,5,3)) &= 56, \qquad c_1^2(X(4,5,3)) = 4, \\
\sigma(X(4,5,3)) &= -36, \qquad \chi_h(X(4,5,3)) = 5.
\end{aligned}$$

Proof.

$$K^2_{X(4,5,3)} = \frac{8\,(g(C_1)-1)\,(g(C_2)-1)}{|G|} + \sum_{x \in Sing(T)} h_x = \frac{8(3)(3)}{5} + 4\left(-\frac{9}{5} - \frac{2}{5} - \frac{2}{5} + 0\right) = 4$$

$$e(X(4,5,3)) = \frac{4\,(g(C_1)-1)\,(g(C_2)-1)}{|G|} + \sum_{x_{ij} \in Sing(T)} e_{x_{ij}} = \frac{4(3)(3)}{5} + 4\left(\frac{9}{5} + 2 \cdot \frac{14}{5} + \frac{24}{5}\right) = 56$$

Therefore, $\sigma(X(4,5,3)) = -36$ and $\chi_h(X(4,5,3)) = 5$, which follows from the formulas

$c_1^2(X) = 2e(X) + 3\sigma(X)$ and $\chi_h(X) = \dfrac{e(X)+\sigma(X)}{4}$. We can compute the euler characteristic also using Lemma 4.5.1. Each singular fiber has $e(F(4,5,3)) = 10\cdot 2 - 9 = 11$. Hence,

$$e(X(4,5,3)) = k \cdot e(F(4,5,3)) + (2-k)\cdot(2-2g) = 4\cdot 11 - 2(2-8) = 56.$$

□

h = 4, Type 4: fiber of type $\left(\frac{1}{5}, \frac{3}{5}, \frac{3}{5}, \frac{3}{5}\right)$

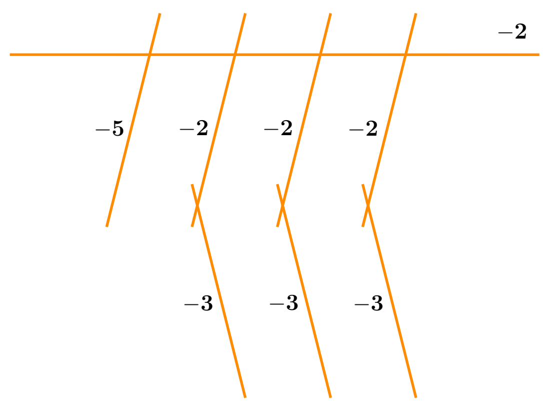


Figure 4.18: $h = 4$, Type 4

$$(Y)^2 = -2$$

$$\frac{n_1}{q_1} = \frac{5}{1} = [5],$$
$$\frac{n_2}{q_2} = \frac{n_3}{q_3} = \frac{n_4}{q_4} = \frac{5}{3} = [2,3]$$

$$l_{x_{i1}} = 1, \qquad l_{x_{i2}} = l_{x_{i3}} = l_{x_{i4}} = 2$$
$$q_1 = 1, \qquad q_2 = q_3 = q_4 = 3$$
$$q_1' = 1, \qquad q_2' = q_3' = q_4' = 2$$

$$\begin{aligned} h_{x_{i1}} &= 2 - \frac{2+1+1}{5} - (5-2) = -\frac{9}{5} \\ h_{x_{i2}} &= h_{x_{i3}} = h_{x_{i4}} = 2 - \frac{2+3+2}{5} - ((2-2)+(3-2)) = -\frac{2}{5} \end{aligned}$$

$$\begin{aligned} B_{x_{i1}} &= \frac{1+1}{5} + 5 = \frac{27}{5}, \qquad B_{x_{i2}} = B_{x_{i3}} = B_{x_{i4}} = \frac{3+2}{5} + 2 + 3 = 6 \\ e_{x_{i1}} &= 1 + 1 - \frac{1}{5} = \frac{9}{5}, \qquad e_{x_{i2}} = e_{x_{i3}} = e_{x_{i4}} = 2 + 1 - \frac{1}{5} = \frac{14}{5} \end{aligned}$$

Lemma 4.6.10. $e(X(4,5,4)) = 48$

Proof.

$$e(X) = \frac{4\,(g(C_1)-1)\,(g(C_2)-1)}{|G|} + \sum_{x_{ij}\in Sing(T)} e_{x_{ij}} = \frac{4(3)(3)}{5} + 4\left(\frac{9}{5} + \frac{14}{5} + \frac{14}{5} + \frac{14}{5}\right) = 48$$

We can compute the euler characteristic also using Lemma 4.5.1. Each singular fiber has $e(F(4,5,4)) = 8\cdot 2 - 7 = 9$. Hence,

$$e(X(4,5,4)) = k\cdot e(F(4,5,4)) + (2-k)\cdot(2-2g) = 4\cdot 9 - 2(2-8) = 48.$$

□

h = 4, Type 5**: fiber of type** $\left(\frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{4}{5}\right)$

$$(Y)^2 = -2$$

$$\begin{aligned} \frac{n_1}{q_1} &= \frac{n_2}{q_2} = \frac{n_3}{q_3} = \frac{5}{2} = [3,2], \\ \frac{n_4}{q_4} &= \frac{5}{4} = [2,2,2,2] \end{aligned}$$

$$\begin{aligned} l_{x_{i1}} &= l_{x_{i2}} = l_{x_{i3}} = 2, \qquad l_{x_{i4}} = 4 \\ q_1 &= q_2 = q_3 = 2, \qquad q_4 = 4 \\ q_1' &= q_2' = q_3' = 3, \qquad q_4' = 4 \end{aligned}$$

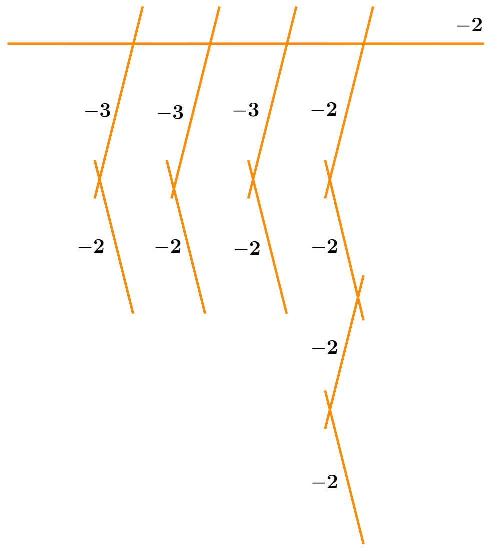


Figure 4.19: $h = 4$, Type 5

$$h_{x_{i1}} = h_{x_{i2}} = h_{x_{i3}} = 2 - \frac{2+2+3}{5} - ((3-2)+(2-2)) = -\frac{2}{5}$$
$$h_{x_{i4}} = 2 - \frac{2+4+4}{5} - 4(2-2) = 0$$

$$B_{x_{i1}} = B_{x_{i2}} = B_{x_{i3}} = \frac{2+3}{5} + 3 + 2 = 6, \qquad B_{x_{i4}} = \frac{4+4}{5} + 2 + 2 + 2 + 2 = \frac{48}{5}$$

$$e_{x_{i1}} = e_{x_{i2}} = e_{x_{i3}} = 2 + 1 - \frac{1}{5} = \frac{14}{5}, \qquad e_{x_{i4}} = 4 + 1 - \frac{1}{5} = \frac{24}{5}$$

Lemma 4.6.11. $e(X(4,5,5)) = 60$

Proof.

$$e(X) = \frac{4\,(g(C_1)-1)\,(g(C_2)-1)}{|G|} + \sum_{x_{ij} \in Sing(T)} e_{x_{ij}} = \frac{4(3)(3)}{5} + 4\left(3 \cdot \frac{14}{5} + \frac{24}{5}\right) = 60$$

Each singular fiber has $e(F(4,5,5)) = 11 \cdot 2 - 10 = 12$. Hence, by Lemma 4.5.1,

$$e(X(4,5,5)) = k \cdot e(F(4,5,5)) + (2-k) \cdot (2-2g) = 4 \cdot 12 - 2(2-8) = 60.$$

$\square$

$\mathbf{h = 4}$, Type 6: fiber of type $\left(\frac{2}{5}, \frac{2}{5}, \frac{3}{5}, \frac{3}{5}\right)$

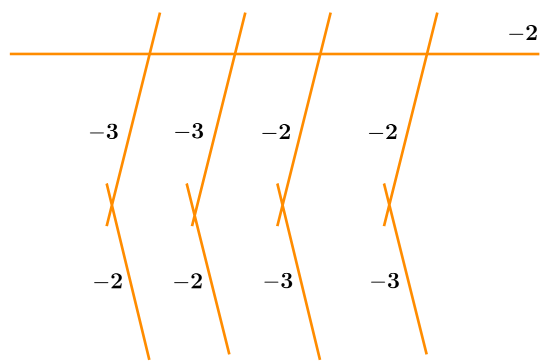


Figure 4.20: $h = 4$, Type 6

$$(Y)^2 = -2$$

$$\frac{n_1}{q_1} = \frac{n_2}{q_2} = \frac{5}{2} = [3,2], \qquad \frac{n_3}{q_3} = \frac{n_4}{q_4} = \frac{5}{3} = [2,3]$$

$$\begin{aligned} l_{x_{ij}} &= 2 \\ q_1 &= q_2 = 2, \qquad q_3 = q_4 = 3 \\ q_1' &= q_2' = 3, \qquad q_3' = q_4' = 2 \end{aligned}$$

$$\begin{aligned} h_{x_{i1}} &= h_{x_{i2}} = 2 - \frac{2+2+3}{5} - ((3-2)+(2-2)) = -\frac{2}{5} \\ h_{x_{i3}} &= h_{x_{i4}} = 2 - \frac{2+3+2}{5} - ((2-2)+(3-2)) = -\frac{2}{5} \end{aligned}$$

$$B_{x_{i1}} = B_{x_{i2}} = \frac{2+3}{5} + 3 + 2 = 6, \qquad B_{x_{i3}} = B_{x_{i4}} = \frac{3+2}{5} + 2 + 3 = 6$$

$$e_{x_{i1}} = e_{x_{i2}} = e_{x_{i3}} = e_{x_{i4}} = 2 + 1 - \frac{1}{5} = \frac{14}{5}$$

Lemma 4.6.12.

$$\begin{aligned} e(X(4,5,6)) &= 52, & c_1^2(X(4,5,6)) = 8, \\ \sigma(X(4,5,6)) &= -32, & \chi_h(X(4,5,6)) = 5. \end{aligned}$$

Proof.

$$K^2_{X(4,5,6)} = \frac{8\,(g(C_1)-1)\,(g(C_2)-1)}{|G|} + \sum_{x \in Sing(T)} h_x = \frac{8(3)(3)}{5} + 16\left(-\frac{2}{5}\right) = 8$$

$$e(X(4,5,6)) = \frac{4\,(g(C_1)-1)\,(g(C_2)-1)}{|G|} + \sum_{x_{ij} \in Sing(T)} e_{x_{ij}} = \frac{4(3)(3)}{5} + 16\left(\frac{14}{5}\right) = 52$$

Therefore, $\sigma(X(4,5,6)) = -32$ and $\chi_h(X(4,5,6)) = 8$.

We can compute the euler characteristic also using Lemma 4.5.1. Each singular fiber has $e(F(4,5,6)) = 9 \cdot 2 - 8 = 10$. Hence,

$$e(X(4,5,6)) = k \cdot e(F(4,5,6)) + (2-k) \cdot (2-2g) = 4 \cdot 10 - 2(2-8) = 52.$$

□

h = 4, Type 7: fiber of type $\left(\frac{3}{5}, \frac{4}{5}, \frac{4}{5}, \frac{4}{5}\right)$

$$(Y)^2 = -3$$

$$\begin{aligned} \frac{n_1}{q_1} &= \frac{5}{3} = [2,3], \\ \frac{n_2}{q_2} &= \frac{n_3}{q_3} = \frac{n_4}{q_4} = \frac{5}{4} = [2,2,2,2] \end{aligned}$$

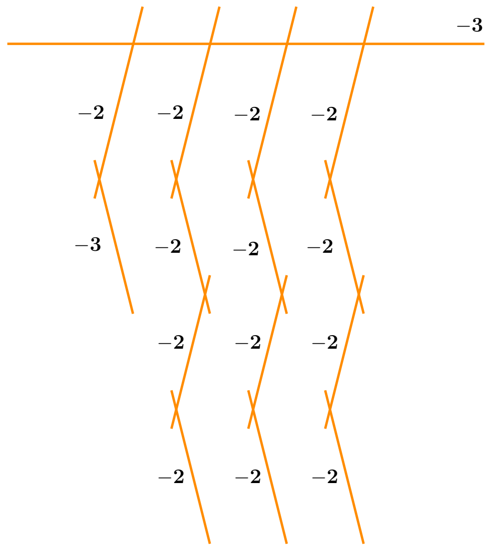


Figure 4.21: $h = 4$, Type 7

$$\begin{aligned} l_{x_{i1}} &= 2, \qquad l_{x_{i2}} = l_{x_{i3}} = l_{x_{i4}} = 4 \\ q_1 &= 3, \qquad q_2 = q_3 = q_4 = 4 \\ q'_1 &= 2, \qquad q'_2 = q'_3 = q'_4 = 4 \end{aligned}$$

$$\begin{aligned} h_{x_{i1}} &= 2 - \frac{2+3+2}{5} - ((2-2)+(3-2)) = -\frac{2}{5} \\ h_{x_{i2}} &= h_{x_{i3}} = h_{x_{i4}} = 2 - \frac{2+4+4}{5} - 4(2-2) = 0 \end{aligned}$$

$$\begin{aligned} B_{x_{i1}} &= \frac{3+2}{5} + 2 + 3 = 6, \qquad B_{x_{i2}} = B_{x_{i3}} = B_{x_{i4}} = \frac{4+4}{5} + 2 + 2 + 2 + 2 = \frac{48}{5} \\ e_{x_{i1}} &= 2 + 1 - \frac{1}{5} = \frac{14}{5}, \qquad e_{x_{i2}} = e_{x_{i3}} = e_{x_{i4}} = 4 + 1 - \frac{1}{5} = \frac{24}{5} \end{aligned}$$

Lemma 4.6.13. $e(X(4,5,7)) = 76$

Proof.

$$e(X(4,5,7)) = \frac{4\,(g(C_1)-1)\,(g(C_2)-1)}{|G|} + \sum_{x_{ij}\in Sing(T)} e_{x_{ij}} = \frac{4(3)(3)}{5} + 4\left(\frac{14}{5} + 3\cdot\frac{24}{5}\right) = 76$$

We can compute the euler characteristic also using Lemma 4.5.1. Each singular fiber has $e(F(4,5,7)) = 15\cdot 2 - 14 = 16$. Hence,

$$e(X(4,5,7)) = k\cdot e(F(4,5,7)) + (2-k)\cdot(2-2g) = 4\cdot 16 - 2(2-8) = 76.$$

□

4.7 Examples of order $2g+1$ action

In this case we are using order $2g+1$ cyclic action on $\Sigma_g\times\Sigma_g$ and recall from section 4.1.1 that in this case we have 3 fixed points. So, the singular manifold $S(g,2g+1,t) = \Sigma_g\times\Sigma_g/\mathbb{Z}_{2g+1}$ has 9 singular points $\{x_{ij}|1\le i,j\le 3\}$ corresponding to the fixed points of the action.

4.7.1 $\mathbf{g=1}$

$\mathbf{g=1}$, Type 1: fiber of type $\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$

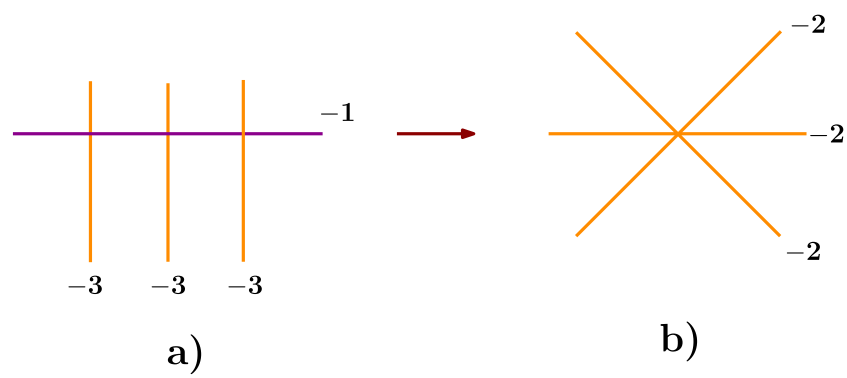


Figure 4.22: $g=1$, Type 1

$$(Y)^2 = -\sum_{i=1}^{3} \frac{1}{3} = -1 \qquad and$$

$$\frac{n_i}{q_i} = \frac{3}{1} = [3], \quad 1 \leq i \leq 3.$$

Now, the reducible fiber has a -1-sphere which is the central component and three -3-sphere intersecting it at three points as illustrated in Figure 4.22 a).

$$\begin{aligned} l_{x_{ij}} &= 1, \qquad where, \;\; 1 \leq i, j \leq 3 \\ q_i &= 1, \qquad q_i' = 1, \qquad where, \;\; \leq i, j \leq 3 \end{aligned}$$

$$h_{x_{ij}} = 2 - \frac{2+1+1}{3} - (3-2) = -\frac{1}{3}$$

$$B_{x_{ij}} = \frac{1+1}{3} + 3 = \frac{11}{3}$$

$$e_{x_{ij}} = 1 + 1 - \frac{1}{3} = \frac{5}{3}, \qquad where, \;\; 1 \leq i, j \leq 3$$

Once we blow down this -1-sphere, we obtain a manifold $X(1,3,1)$ which has a singular fiber consists of 3 -2-spheres intersecting at one point (See Figure 4.22 b)).

Lemma 4.7.1.

$$\begin{aligned} e(X(1,3,1)) &= 12, \qquad c_1^2(X(1,3,1)) = 0, \\ \sigma(X(1,3,1)) &= -8, \qquad \chi_h(X(1,3,1)) = 1. \end{aligned}$$

Hence, $X(1,3,1)$ is homeomorphic to the elliptic surface $E(1) = \mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2$.

Proof. Before blow-down we have;

$$K_S^2 = \frac{8\,(g(C_1)-1)\,(g(C_2)-1)}{|G|} + \sum_{x \in Sing(T)} h_x = \frac{8(1-1)(1-1)}{3} + 9\left(-\frac{1}{3}\right) = -3$$

$$e(S)=\frac{4\left(g(C_1)-1\right)\left(g(C_2)-1\right)}{|G|}+\sum_{x_{ij}\in Sing(T)}e_{x_{ij}}=\frac{4(1-1)(1-1)}{3}+9\left(\frac{5}{3}\right)=15$$

Once we blow-down we get;

$$K^2_{X(1,3,1)}=K^2_S+3=-3+3=0$$

$$e(X(1,3,1))=e(S)-3=15-3=12$$

Therefore, $\sigma(X(1,3,1))=-8$ and $\chi_h(X(1,3,1))=1$, which follows from the formulas $c_1^2(X)=2e(X)+3\sigma(X)$ and $\chi_h(X)=\dfrac{e(X)+\sigma(X)}{4}$. We can compute the euler characteristic also using Lemma 4.5.1. Each singular fiber has euler characteristic $e(F(1,3,1))=3\cdot 2-2=4$. Hence,

$$e(X(1,3,1))=k\cdot e(F(1,3,1))+(2-k)\cdot(2-2g)=3\cdot 4-(2-2)=12.$$

$\square$

There are 3 singular fibers each has monodromy $(UV)^2$ which corresponds to type IV in Table 1 in [**KM**]. (Also see [**Ogg**, **Ko**]). So the total monodromy of X is $((UV)^2)^3=(UV)^6=1$.

$\mathbf{g=1}$, Type 2: fiber of type $\left(\frac{2}{3},\frac{2}{3},\frac{2}{3}\right)$

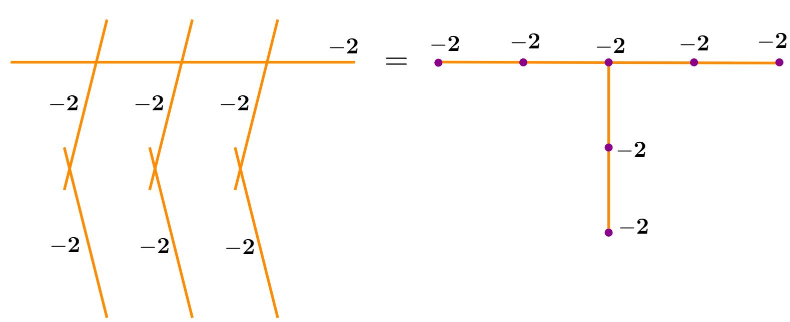


Figure 4.23: $g=1$, Type 2

$$(Y)^2 = -\sum_{i=1}^{3} \frac{2}{3} = -2$$

$$\frac{n_i}{q_i} = \frac{3}{2} = 2 - \frac{1}{2} = [2,2], \qquad 1 \leq i \leq 3$$

In this case the singular fibers looks like as in Figure 4.23.

$$\begin{aligned}
l_{x_{ij}} &= 2 \\
q_i &= 2, \qquad q_i' = 2, \qquad where, \ \ \leq i,j \leq 3 \\
h_{x_{ij}} &= 2 - \frac{2+2+2}{3} - ((2-2)+(2-2)) = 0 \\
B_{x_{ij}} &= \frac{2+2}{3} + 2 + 2 = \frac{16}{3} \\
e_{x_{ij}} &= 2 + 1 - \frac{1}{3} = \frac{8}{3}, \qquad where, \ \ 1 \leq i,j \leq 3
\end{aligned}$$

Lemma 4.7.2.

$$\begin{aligned}
e(X(1,3,2)) &= 24, \qquad c_1^2(X(1,3,2)) = 0, \\
\sigma(X(1,3,2)) &= -16, \qquad \chi_h(X(1,3,2)) = 2.
\end{aligned}$$

Hence, $X(1,3,2)$ is homeomorphic to the elliptic surface $E(2)$.

Proof.

$$K^2_{X(1,3,2)} = \frac{8\,(g(C_1)-1)\,(g(C_2)-1)}{|G|} + \sum_{x \in Sing(T)} h_x = \frac{8(1-1)(1-1)}{3} + 9(0) = 0$$

$$e(X(1,3,2)) = \frac{4\,(g(C_1)-1)\,(g(C_2)-1)}{|G|} + \sum_{x_{ij} \in Sing(T)} e_{x_{ij}} = \frac{4(1-1)(1-1)}{3} + 9\left(\frac{8}{3}\right) = 24$$

Therefore, $\sigma(X(1,3,2)) = -16$ and $\chi_h(X(1,3,2)) = 2$.

We can compute the euler characteristic also using Lemma 4.5.1. Each singular fiber has euler characteristic $e(F(1,3,2)) = 7 \cdot 2 - 6 = 8$. Hence,

$$e(X(1,3,2)) = k \cdot e(F(1,3,2)) + (2-k) \cdot (2-2g) = 3 \cdot 8 - (2-2) = 24.$$

□

There are 3 singular fibers each has monodromy $(UV)^4$ which corresponds to type IV^* in Table 1^* in [**KM**]. (Also see [**Ogg**, **Ko**]). So the total monodromy of X is $((UV)^4)^3 = (UV)^{12} = 1$.

4.7.2 g = 2

In this case we are using order 5 cyclic action on Σ_2 with 3 fixed points. So, the singular manifold $T = \Sigma_2 \times \Sigma_2/\mathbb{Z}_5$ has 9 singular points $\{x_{ij} | 1 \leq i, j \leq 3\}$ corresponding to the fixed points of the action.

g = 2, Type 1: fiber of type $\left(\frac{1}{5}, \frac{1}{5}, \frac{3}{5}\right)$

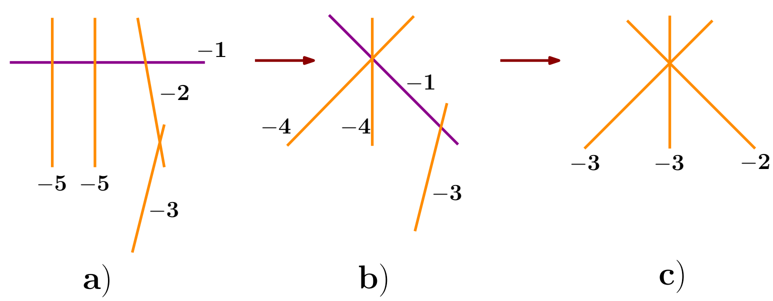


Figure 4.24: $g = 2$, Type 1

$$(Y)^2 = -\left(\frac{1}{5} + \frac{1}{5} + \frac{3}{5}\right) = -1$$

$$\begin{aligned} \frac{n_1}{q_1} &= \frac{n_2}{q_2} = \frac{5}{1} = [5], \\ \frac{n_3}{q_3} &= \frac{5}{3} = 2 - \frac{1}{3} = [2, 3] \end{aligned}$$

In this case the singular fibers has central -1-sphere as illustrated in Figure 4.24a.

$$\begin{aligned} l_{x_{i1}} &= l_{x_{i2}} = 1, \qquad l_{x_{i3}} = 2 \\ q_1 &= q_2 = 1, \qquad q_3 = 3 \\ q_1' &= q_2' = 1, \qquad q_3' = 2 \end{aligned}$$

$$\begin{aligned} h_{x_{i1}} &= h_{x_{i2}} = 2 - \frac{2+1+1}{5} - (5-2) = -\frac{9}{5}, \\ h_{x_{i3}} &= 2 - \frac{2+3+2}{5} - ((0)+(3-2)) = -\frac{2}{5} \\ B_{x_{i1}} &= B_{x_{i2}} = \frac{1+1}{5} + 5 = \frac{27}{5}, \qquad B_{x_{i3}} = \frac{3+2}{5} + 2 + 3 = 6 \\ e_{x_{i1}} &= e_{x_{i2}} = 1 + 1 - \frac{1}{5} = \frac{9}{5}, \qquad e_{x_{i3}} = 2 + 1 - \frac{1}{5} = \frac{14}{5} \end{aligned}$$

Now, we blow down the central -1-sphere and get a manifold which now has the singular fiber with configuration as in Figure 4.24b. The new fiber still has a central -1-sphere. Blowing down once more we get a singular fiber as in Figure 4.24c which corresponds to type 36 in the table on pg. 359 in [**Ogg**](see also [**NU**]).

Lemma 4.7.3.

$$\begin{aligned} e(X(2,5,1)) &= 14, \qquad c_1^2(X(2,5,1)) = -2, \\ \sigma(X(2,5,1)) &= -10, \qquad \chi_h(X(2,5,1)) = 1. \end{aligned}$$

Hence, $X(2,5,1)$ is homeomorphic to $\mathbb{CP}^2 \# 11\overline{\mathbb{CP}}^2$.

Proof. Before blow-down we have;

$$K_S^2 = \frac{8\,(g(C_1)-1)\,(g(C_2)-1)}{|G|} + \sum_{x \in Sing(T)} h_x = \frac{8(1)(1)}{5} + 3\left(-\frac{9}{5} - \frac{9}{5} - \frac{2}{5}\right) = -\frac{52}{5}$$

$$e(S) = \frac{4\,(g(C_1)-1)\,(g(C_2)-1)}{|G|} + \sum_{x_{ij} \in Sing(T)} e_{x_{ij}} = \frac{4(1)(1)}{5} + 3\left(\frac{9}{5} + \frac{9}{5} + \frac{14}{5}\right) = 20$$

Once we blow-down we get; $e(X(2,5,1)) = e(S) - 3 - 3 = 20 - 6 = 14$.

We can compute the euler characteristic also using Lemma 4.5.1. Each singular fiber has euler characteristic $e(F(2,5,1)) = 3 \cdot 2 - 2 = 4$. Hence,

$$e(X(2,5,1)) = k \cdot e(F(2,5,1)) + (2-k) \cdot (2-2g) = 3 \cdot 4 - (2-4) = 14.$$

We determine the signature by eliminating all possibilities using the inequality $c_1^2(X) \leq 6\chi_h(X) - 3$ for genus 2 hyperelliptic fibrations (See Corollary 10 in [**Oz**]. This inequality results in $\sigma(X(2,5,1) \leq -\frac{20}{3}$. On the other hand, we know that $b_2(X(2,5,1)) = 12$, since $X(2,5,1)$ is simply connected and $e\left(X(2,5,1)\right) = 14$. In addition, $b_2^+(X(2,5,1)$ can not be even. So, we only left with one possibility, which is $b_2^+(X(2,5,1) = 1$ and $b_2^-(X(2,5,1) = 11$. Hence,

$$\sigma(X(2,5,1)) = b_2^+(X(2,5,1) - b_2^-(X(2,5,1) = -10.$$

which gives $c_1^2(X(2,5,1)) = -2$ and $\chi_h(X(2,5,1)) = 1$. □

g = 2, Type 2**: fiber of type** $\left(\frac{1}{5}, \frac{2}{5}, \frac{2}{5}\right)$

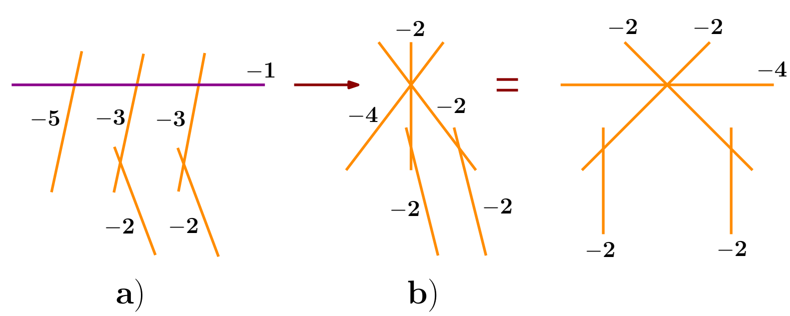


Figure 4.25: $g = 2$, Type 2

$$(Y)^2 = -\left(\frac{1}{5} + \frac{2}{5} + \frac{2}{5}\right) = -1$$

$$\frac{n_1}{q_1} = \frac{5}{1} = [5],$$
$$\frac{n_2}{q_2} = \frac{n_3}{q_3} = \frac{5}{2} = 3 - \frac{1}{2} = [3,2]$$

In this case the singular fibers central -1-sphere as illustrated in Figure 4.25a.

$$\begin{aligned} l_{x_{i1}} &= 1, \qquad l_{x_{i2}} = l_{x_{i3}} = 2 \\ q_1 &= 1, \qquad q_2 = q_3 = 2 \\ q_1' &= 1, \qquad q_2' = q_3' = 3 \end{aligned}$$

$$\begin{aligned} h_{x_{i1}} &= 2 - \frac{2+1+1}{5} - (5-2) = -\frac{9}{5} \\ h_{x_{i2}} &= h_{x_{i3}} = 2 - \frac{2+2+3}{5} - ((3-2)+(2-2)) = -\frac{2}{5} \end{aligned}$$

$$\begin{aligned} B_{x_{i1}} &= \frac{1+1}{5} + 5 = \frac{27}{5}, \qquad B_{x_{i2}} = B_{x_{i3}} = \frac{2+3}{5} + 3 + 2 = 6 \\ e_{x_{i1}} &= 1 + 1 - \frac{1}{5} = \frac{9}{5}, \qquad e_{x_{i2}} = e_{x_{i3}} = 2 + 1 - \frac{1}{5} = \frac{14}{5} \end{aligned}$$

Now, we blow down the central -1-sphere and get $X(2,5,2)$ which has a singular fiber with configuration as in Figure 4.25b. The new fiber now corresponds to type 8 in the table on pg. 357 in [**Ogg**] (see also [**NU**]).

Lemma 4.7.4. $e(X(2,5,2)) = 20$

Proof. Before blow-down we have;

$$e(S) = \frac{4\,(g(C_1)-1)\,(g(C_2)-1)}{|G|} + \sum_{x_{ij} \in Sing(T)} e_{x_{ij}} = \frac{4(1)(1)}{5} + 3\left(\frac{9}{5} + \frac{14}{5} + \frac{14}{5}\right) = 23$$

Once we blow-down we get; $e(X(2,5,2)) = e(S) - 3 = 20$.

Again using Lemma 4.5.1, since each singular fiber has euler characteristic $e(F(2,5,2)) = 5 \cdot 2 - 4 = 6$, we have

$$e(X(2,5,2)) = k \cdot e(F(2,5,2)) + (2-k) \cdot (2-2g) = 3 \cdot 6 - (2-4) = 20.$$

$\square$

$\mathbf{g=2}$, Type 3: fiber of type $\left(\frac{2}{5}, \frac{4}{5}, \frac{4}{5}\right)$

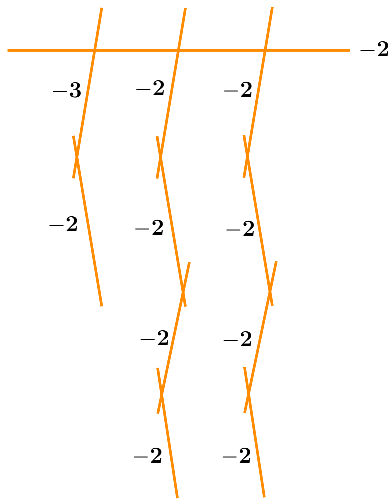


Figure 4.26: $g = 2$, Type 3

$$(Y)^2 = -\left(\frac{2}{5} + \frac{4}{5} + \frac{4}{5}\right) = -2$$

$$\frac{n_1}{q_1} = \frac{5}{2} = [3, 2],$$
$$\frac{n_2}{q_2} = \frac{n_3}{q_3} = \frac{5}{4} = [2, 2, 2, 2]$$

In this case, as you see in, each singular fiber looks as in Figure 4.26 and corresponds to type 21 in the table on pg. 358 in [**Ogg**] (see also [**NU**]).

$$l_{x_{i1}} = 2, , \qquad l_{x_{i2}} = l_{x_{i3}} = 4$$
$$q_1 = 2, \qquad q_2 = q_3 = 4$$
$$q_1' = 3, \qquad q_2' = q_3' = 4$$

$$h_{x_{i1}} = 2 - \frac{2+2+3}{5} - ((3-2)+(2-2)) = -\frac{2}{5}$$
$$h_{x_{i2}} = h_{x_{i3}} = 2 - \frac{2+4+4}{5} - 4(2-2) = 0$$

$$B_{x_{i1}} = \frac{2+3}{5} + 3 + 2 = 6, \quad B_{x_{i2}} = B_{x_{i3}} = \frac{4+4}{5} + 2 + 2 + 2 + 2 = \frac{48}{5}$$

$$e_{x_{i1}} = 2 + 1 - \frac{1}{5} = \frac{14}{5}, \qquad e_{x_{i2}} = e_{x_{i3}} = 4 + 1 - \frac{1}{5} = \frac{24}{5}$$

Lemma 4.7.5. *$e(X(2,5,3)) = 38$.*

Proof.

$$e(X(2,5,3)) = \frac{4\,(g(C_1)-1)\,(g(C_2)-1)}{|G|} + \sum_{x_{ij} \in Sing(T)} e_{x_{ij}} = \frac{4(1)(1)}{5} + 3\left(\frac{14}{5} + 2 \cdot \frac{24}{5}\right) = 38$$

Again using Lemma 4.5.1 we get

$$e(X(2,5,3)) = k \cdot e(F(2,5,3)) + (2-k) \cdot (2-2g) = 3 \cdot 12 - (2-4) = 38.$$

since each singular fiber has euler characteristic $e(F(2,5,3)) = 11 \cdot 2 - 10 = 12$. □

g = 2, Type 4: fiber of type $\left(\frac{3}{5}, \frac{3}{5}, \frac{4}{5}\right)$

$$(Y)^2 = -\left(\frac{3}{5} + \frac{3}{5} + \frac{4}{5}\right) = -2$$

$$\frac{n_1}{q_1} = \frac{n_2}{q_2} = \frac{5}{3} = [2,3],$$
$$\frac{n_3}{q_3} = \frac{5}{4} = [2,2,2,2]$$

Again, in this case as you see in Figure 4.27, there is no -1-sphere and the singular

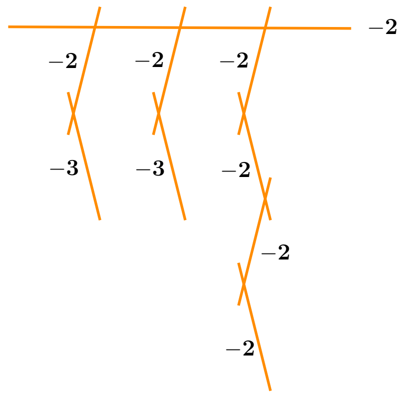


Figure 4.27: $g = 2$, Case 4

fibers correspond to type 44 in the table on pg. 359 in [**Ogg**] (see also [**NU**]).

$$\begin{aligned} l_{x_{i1}} &= l_{x_{i2}} = 2, \qquad l_{x_{i3}} = 4 \\ q_1 &= q_2 = 3, \qquad q_3 = 4 \\ q_1' &= q_2' = 2, \qquad q_3' = 4 \end{aligned}$$

$$h_{x_{i1}} = h_{x_{i2}} = 2 - \frac{2+3+2}{5} - ((2-2)+(3-2)) = -\frac{2}{5}, \quad h_{x_{i3}} = 2 - \frac{2+4+4}{5} - 4(2-2) = 0$$

$$B_{x_{i1}} = B_{x_{i2}} = \frac{3+2}{5} + 2 + 3 = 6, \qquad B_{x_{i3}} = \frac{4+4}{5} + 2 + 2 + 2 + 2 = \frac{48}{5}$$

$$e_{x_{i1}} = e_{x_{i2}} = 2 + 1 - \frac{1}{5} = \frac{14}{5}, \qquad e_{x_{i3}} = 4 + 1 - \frac{1}{5} = \frac{24}{5}$$

Lemma 4.7.6. $e(X(2,5,4)) = 32$

Proof.

$$e(X(2,5,4)) = \frac{4\,(g(C_1)-1)\,(g(C_2)-1)}{|G|} + \sum_{x_{ij} \in Sing(T)} e_{x_{ij}} = \frac{4(1)(1)}{5} + 3\left(2 \cdot \frac{14}{5} + \frac{24}{5}\right) = 32$$

We can compute the euler characteristic also using Lemma 4.5.1. Each singular fiber has euler characteristic $e(F(2,5,4)) = 9 \cdot 2 - 8 = 10$. Hence,

$$e(X(2,5,4)) = k \cdot e(F(2,5,4)) + (2-k) \cdot (2-2g) = 3 \cdot 10 - (2-4) = 32.$$

□

4.7.3 g = 3

In this case we are using order 7 cyclic action on Σ_3 with 3 fixed points. So, the singular manifold $T = \Sigma_3 \times \Sigma_3/\mathbb{Z}_7$ has 9 singular points $\{x_{ij} | 1 \leq i,j \leq 3\}$ corresponding to the fixed points of the action.

g = 3, Type 1: fiber of type $\left(\frac{1}{7}, \frac{1}{7}, \frac{5}{7}\right)$

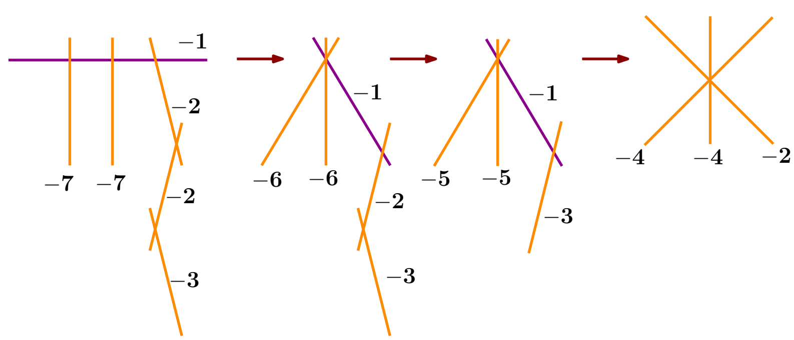


Figure 4.28: $g = 3$, Type 1

$$(Y)^2 = -\left(\frac{1}{7} + \frac{1}{7} + \frac{5}{7}\right) = -1$$

$$\frac{n_1}{q_1} = \frac{n_2}{q_2} = \frac{7}{1} = [7],$$
$$\frac{n_3}{q_3} = \frac{7}{5} = 2 - \frac{3}{5} = 2 - \frac{1}{\frac{5}{3}} = 2 - \frac{1}{2 - \frac{1}{3}} = [2,2,3]$$

In this case, the reducible fiber has a -1-sphere which is the central component as illustrated in Figure 4.28.

$$\begin{aligned} l_{x_{i1}} &= l_{x_{i2}} = 1, \qquad l_{x_{i3}} = 3 \\ q_1 &= q_2 = 1, \qquad q_3 = 5 \\ q_1' &= q_2' = 1, \qquad q_3' = 3 \end{aligned}$$

$$\begin{aligned} h_{x_{i1}} &= h_{x_{i2}} = 2 - \frac{2+1+1}{7} - (7-2) = -\frac{25}{7} \\ h_{x_{i3}} &= 2 - \frac{2+3+5}{7} - ((2-2)+(2-2)+(3-2)) = -\frac{3}{7} \end{aligned}$$

$$B_{x_{i1}} = B_{x_{i2}} = \frac{1+1}{7} + 7 = \frac{51}{7}, \quad B_{x_{i3}} = \frac{5+3}{7} + 2 + 2 + 3 = \frac{57}{7}$$

$$e_{x_{i1}} = e_{x_{i2}} = 1 + 1 - \frac{1}{7} = \frac{13}{7}, \qquad e_{x_{i3}} = 3 + 1 - \frac{1}{7} = \frac{27}{7}$$

Once we blow down three times, we end up with a singular fiber which doesn't include a -1-sphere. Finally, we obtain a manifold $X(3,7,1)$ which has a singular fiber consists of two -4-spheres and one -2 sphere intersecting at one point (See Figure 4.28).

Lemma 4.7.7. *$e(X(3,7,1)) = 16$.*

Proof. Before blow-down we have;

$$e(S) = \frac{4\,(g(C_1)-1)\,(g(C_2)-1)}{|G|} + \sum_{x_{ij}\in Sing(T)} e_{x_{ij}} = \frac{4(2)(2)}{7} + 3\left(\frac{13}{7} + \frac{13}{7} + \frac{27}{7}\right) = 25$$

Once we blow-down we get; $e(X(3,7,1)) = e(S) - 3 - 3 - 3 = 25 - 9 = 16$.

We can compute the euler characteristic also using Lemma 4.5.1. Each singular fiber has euler characteristic $e(F(3,7,1)) = 3 \cdot 2 - 2 = 4$. Hence,

$$e(X(3,7,1)) = k \cdot e(F(3,7,1)) + (2-k)\cdot(2-2g) = 3 \cdot 4 - (2-6) = 16.$$

$\square$

g = 3, Type 2: fiber of type $\left(\frac{1}{7}, \frac{2}{7}, \frac{4}{7}\right)$

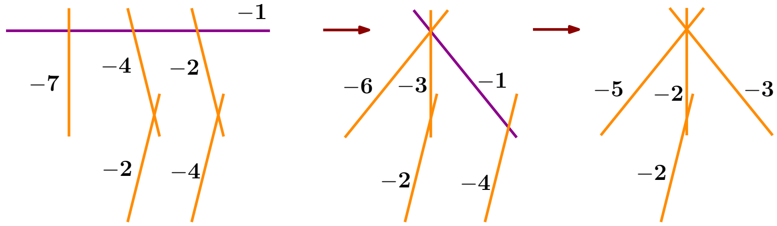


Figure 4.29: $g = 3$, Type 2

$$(Y)^2 = -\left(\frac{1}{7} + \frac{2}{7} + \frac{4}{7}\right) = -1$$

$$\begin{aligned}
\frac{n_1}{q_1} &= \frac{7}{1} = [7], \\
\frac{n_2}{q_2} &= \frac{7}{2} = 4 - \frac{1}{2} = [4, 2], \\
\frac{n_3}{q_3} &= \frac{7}{4} = 2 - \frac{1}{4} = [2, 4]
\end{aligned}$$

$$\begin{aligned}
l_{x_{i1}} &= 1, \qquad l_{x_{i2}} = l_{x_{i3}} = 2 \\
q_1 &= 1, \qquad q_2 = 2, \qquad q_3 = 4 \\
q'_1 &= 1, \qquad q'_2 = 4, \qquad q'_3 = 2
\end{aligned}$$

$$\begin{aligned}
h_{x_{i1}} &= 2 - \frac{2+1+1}{7} - (7-2) = -\frac{25}{7} \\
h_{x_{i2}} &= 2 - \frac{2+2+4}{7} - ((4-2) + (2-2)) = -\frac{8}{7} \\
h_{x_{i3}} &= 2 - \frac{2+4+2}{7} - ((2-2) + (4-2)) = -\frac{8}{7}
\end{aligned}$$

$$B_{x_{i1}} = \frac{1+1}{7} + 7 = \frac{51}{7}, \quad B_{x_{i2}} = \frac{2+4}{7} + 4 + 2 = \frac{48}{7}, \quad B_{x_{i3}} = \frac{4+2}{7} + 2 + 4 = \frac{48}{7}$$

$$e_{x_{i1}} = 1 + 1 - \frac{1}{7} = \frac{13}{7}, \qquad e_{x_{i2}} = e_{x_{i3}} = 2 + 1 - \frac{1}{7} = \frac{20}{7}$$

This time we apply blow down twice to end up with singular fibers which doesn't include a -1-sphere. Finally, we obtain a manifold $X(3,7,2)$ which has singular fibers as seen in Figure 4.29.

Lemma 4.7.8.

$$e(X(3,7,2)) = 19, \qquad c_1^2(X(3,7,2)) = -7,$$
$$\sigma(X(3,7,2)) = -15, \qquad \chi_h(X(3,7,2)) = 1.$$

Proof. Before blow-down we have;

$$K_S^2 = \frac{8\,(g(C_1)-1)\,(g(C_2)-1)}{|G|} + \sum_{x \in Sing(T)} h_x = \frac{8(2)(2)}{7} + 3\left(-\frac{25}{7} - \frac{8}{7} - \frac{8}{7}\right) = -13$$

$$e(S) = \frac{4\,(g(C_1)-1)\,(g(C_2)-1)}{|G|} + \sum_{x_{ij} \in Sing(T)} e_{x_{ij}} = \frac{4(2)(2)}{7} + 3\left(\frac{13}{7} + \frac{13}{7} + \frac{27}{7}\right) = 25$$

Once we blow-down we get;

$$K^2_{X(3,7,2)} = K_S^2 + 3 + 3 = -13 + 3 + 3 = -7$$

$$e(X(3,7,2)) = e(S) - 3 - 3 = 25 - 6 = 19$$

Therefore, $\sigma(X(3,7,2)) = -15$ and $\chi_h(X(3,7,2)) = 1$.

We can compute the euler characteristic also using Lemma 4.5.1. Each singular fiber has euler characteristic $e(F(3,7,2)) = 4 \cdot 2 - 3 = 5$. Hence,

$$e(X(3,7,2)) = k \cdot e(F(3,7,2)) + (2-k) \cdot (2-2g) = 3 \cdot 5 - (2-6) = 19.$$

$\square$

g = 3, Type 3: fiber of type $\left(\frac{1}{7}, \frac{3}{7}, \frac{3}{7}\right)$

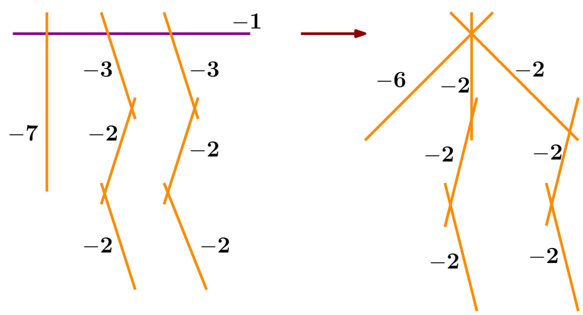


Figure 4.30: $g = 3$, Type 3

$$(Y)^2 = -\left(\frac{1}{7} + \frac{3}{7} + \frac{3}{7}\right) = -1$$

$$\frac{n_1}{q_1} = \frac{7}{1} = [7],$$
$$\frac{n_2}{q_2} = \frac{n_3}{q_3} = \frac{7}{3} = 3 - \frac{2}{3} = 3 - \cfrac{1}{\frac{3}{2} = 3 - \cfrac{1}{2 - \cfrac{1}{2}}} = [3, 2, 2]$$

In this case the singular fibers looks like as in Figure 4.30. Blowing-down the central -1-sphere, we end up with a singular fiber which doesn't include a -1-sphere and we obtain a manifold $X(3, 7, 3)$ which has singular fibers as in the right in Figure 4.30.

$$\begin{aligned} l_{x_{ij}} &= 1 \\ q_1 &= 1, \qquad q_2 = q_3 = 3 \\ q_1' &= 1, \qquad q_2' = q_3' = 5 \end{aligned}$$

$$\begin{aligned} h_{x_{i1}} &= 2 - \frac{2+1+1}{7} - (7-2) = -\frac{25}{7} \\ h_{x_{i2}} &= h_{x_{i3}} = 2 - \frac{2+3+5}{7} - ((3-2) + (2-2) + (2-2)) = -\frac{3}{7} \end{aligned}$$

$$B_{x_{i1}} = \frac{1+1}{7} + 7 = \frac{51}{7}, \qquad B_{x_{i2}} = B_{x_{i3}} = \frac{3+5}{7} + 3 + 2 + 2 = \frac{57}{7}$$

$$e_{x_{i1}} = 1 + 1 - \frac{1}{7} = \frac{13}{7}, \qquad e_{x_{i2}} = e_{x_{i3}} = 3 + 1 - \frac{1}{7} = \frac{27}{7}$$

Lemma 4.7.9. $e(X(3,7,3)) = 28$.

Proof. Before blow-down we have;

$$e(S) = \frac{4\,(g(C_1)-1)\,(g(C_2)-1)}{|G|} + \sum_{x_{ij} \in Sing(T)} e_{x_{ij}} = \frac{4(2)(2)}{7} + 3\left(\frac{13}{7} + \frac{27}{7} + \frac{27}{7}\right) = 31$$

Once we blow-down we get; $e(X(3,7,3)) = e(S) - 3 = 31 - 3 = 28$.

Again, we can compute the euler characteristic also using Lemma 4.5.1. Each singular fiber has euler characteristic $e(F(3,7,3)) = 7 \cdot 2 - 6 = 8$. Hence,

$$e(X(3,7,3)) = k \cdot e(F(3,7,3)) + (2-k) \cdot (2-2g) = 3 \cdot 8 - (2-6) = 28.$$

□

$\mathbf{g = 3}$, Type 4: fiber of type $\left(\frac{2}{7}, \frac{2}{7}, \frac{3}{7}\right)$

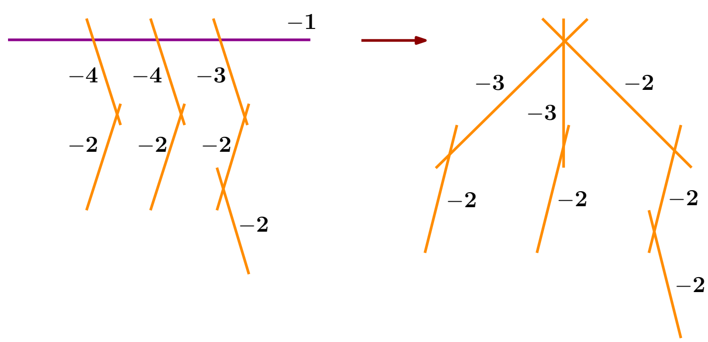


Figure 4.31: $g = 3$, Type 4

$$(Y)^2 = -\left(\frac{2}{7} + \frac{2}{7} + \frac{3}{7}\right) = -1$$

$$\begin{aligned}\frac{n_1}{q_1} &= \frac{n_2}{q_2} = \frac{7}{2} = [4,2],\\ \frac{n_3}{q_3} &= \frac{7}{3} = [3,2,2]\end{aligned}$$

In this case the singular fibers looks like as in 4.31.

$$\begin{aligned} l_{x_{i1}} &= l_{x_{i2}} = 2, \qquad l_{x_{i3}} = 3\\ q_1 &= q_2 = 2, \qquad q_3 = 3\\ q'_1 &= q'_2 = 4, \qquad q'_3 = 5\end{aligned}$$

$$\begin{aligned} h_{x_{i1}} &= h_{x_{i2}} = 2 - \frac{2+2+4}{7} - ((4-2)+(2-2)) = -\frac{8}{7}\\ h_{x_{i3}} &= 2 - \frac{2+3+5}{7} - ((3-2)+(2-2)+(2-2)) = -\frac{3}{7}\end{aligned}$$

$$B_{x_{i1}} = B_{x_{i2}} = \frac{2+4}{7} + 4 + 2 = \frac{48}{7}, \qquad B_{x_{i3}} = \frac{3+5}{7} + 3 + 2 + 2 = \frac{57}{7}$$

$$e_{x_{i1}} = e_{x_{i2}} = 2 + 1 - \frac{1}{7} = \frac{20}{7}, \qquad e_{x_{i3}} = 3 + 1 - \frac{1}{7} = \frac{27}{7}$$

We need to blow down the central -1-sphere to obtain a singular fiber which doesn't include a -1-sphere. (See Figure 4.31). We obtain a manifold $X(3,7,4)$ with singular fibers as in Figure 4.31.

Lemma 4.7.10. $e(X(3,7,4)=28$.

Proof. Before blow-down we have;

$$e(S) = \frac{4\,(g(C_1)-1)\,(g(C_2)-1)}{|G|} + \sum_{x_{ij}\in Sing(T)} e_{x_{ij}} = \frac{4(2)(2)}{7} + 3\left(\frac{20}{7} + \frac{20}{7} + \frac{27}{7}\right) = 31$$

Once we blow-down we get; $e(X(3,7,4)) = e(S) - 3 = 31 - 3 = 28$.

Each singular fiber has euler characteristic $e(F(3,7,4)) = 7 \cdot 2 - 6 = 8$. Hence, Lemma 4.5.1

$$e(X(3,7,4)) = k \cdot e(F(3,7,4)) + (2-k) \cdot (2-2g) = 3 \cdot 8 - (2-6) = 28.$$

$\square$

g = 3, Type 5: fiber of type $\left(\frac{2}{7}, \frac{6}{7}, \frac{6}{7}\right)$

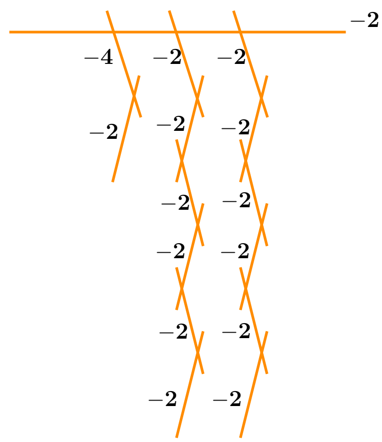


Figure 4.32: $g = 3$, Type 5

$$(Y)^2 = -\left(\frac{2}{7} + \frac{6}{7} + \frac{6}{7}\right) = -2$$

$$\frac{n_1}{q_1} = \frac{7}{2} = [4,2],$$

$$\frac{n_2}{q_2} = \frac{n_3}{q_3} = \frac{7}{6} = [2,2,2,2,2,2]$$

$$\begin{aligned} l_{x_{i1}} &= 2, \qquad l_{x_{i2}} = l_{x_{i3}} = 6 \\ q_1 &= 2, \qquad q_2 = q_3 = 6 \\ q_1' &= 4, \qquad q_2' = q_3' = 6 \end{aligned}$$

$$\begin{aligned} h_{x_{i1}} &= 2 - \frac{2+2+4}{7} - ((4-2)+(2-2)) = -\frac{8}{7} \\ h_{x_{i2}} &= h_{x_{i3}} = 2 - \frac{2+6+6}{7} - 6(2-2) = 0 \end{aligned}$$

$$B_{x_{i1}} = \frac{2+4}{7} + 4 + 2 = \frac{48}{7}, \qquad B_{x_{i2}} = B_{x_{i3}} = \frac{6+6}{7} + 6(2) = \frac{96}{7}$$

$$e_{x_{i1}} = 2 + 1 - \frac{1}{7} = \frac{20}{7}, \qquad e_{x_{i2}} = e_{x_{i3}} = 6 + 1 - \frac{1}{7} = \frac{48}{7}$$

Lemma 4.7.11. $e(X(3,7,5)) = 52$

Proof.

$$e(X(3,7,5)) = \frac{4\,(g(C_1)-1)\,(g(C_2)-1)}{|G|} + \sum_{x_{ij} \in Sing(T)} e_{x_{ij}} = \frac{4(2)(2)}{7} + 3\left(\frac{20}{7} + 2 \cdot \frac{48}{7}\right) = 52$$

Each singular fiber has euler characteristic $e(F(3,7,5)) = 15 \cdot 2 - 14 = 16$. Hence, by Lemma 4.5.1

$$e(X(3,7,5)) = k \cdot e(F(3,7,5)) + (2-k) \cdot (2-2g) = 3 \cdot 16 - (2-6) = 52.$$

□

g = 3, Type 6: fiber of type $\left(\frac{3}{7}, \frac{5}{7}, \frac{6}{7}\right)$

$$(Y)^2 = -\left(\frac{3}{7} + \frac{5}{7} + \frac{6}{7}\right) = -2$$

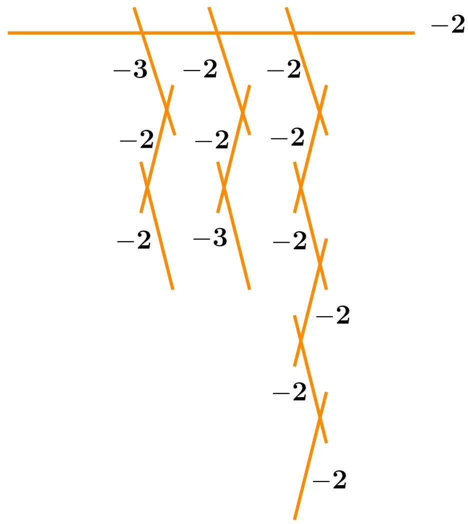


Figure 4.33: $g = 3$, Type 6

$$\begin{aligned}\frac{n_1}{q_1} &= \frac{7}{3} = [3,2,2],\\ \frac{n_2}{q_2} &= \frac{7}{5} = [2,2,3],\\ \frac{n_3}{q_3} &= \frac{7}{6} = [2,2,2,2,2,2]\end{aligned}$$

$$\begin{aligned}l_{x_{i1}} &= l_{x_{i2}} = 3, \qquad l_{x_{i3}} = 6\\ q_1 &= 3, \qquad q_2 = 5, \qquad q_3 = 6\\ q_1' &= 5, \qquad q_2' = 3, \qquad q_3' = 6\end{aligned}$$

$$\begin{aligned}h_{x_{i1}} &= 2 - \frac{2+3+5}{7} - ((3-2)+(2-2)+(2-2)) = -\frac{3}{7}\\ h_{x_{i2}} &= 2 - \frac{2+5+3}{7} - ((2-2)+(2-2)+(3-2)) = -\frac{3}{7}\\ h_{x_{i3}} &= 2 - \frac{2+6+6}{7} - 6(2-2) = 0\end{aligned}$$

$$B_{x_{i1}} = \frac{3+5}{7} + 3 + 2 + 2 = \frac{57}{7}, \quad B_{x_{i2}} = \frac{5+3}{7} + 2 + 2 + 3 = \frac{57}{7}, \quad B_{x_{i3}} = \frac{6+6}{7} + 6(2) = \frac{96}{7}$$
$$e_{x_{i1}} = e_{x_{i2}} = 3 + 1 - \frac{1}{7} = \frac{27}{7}, \qquad e_{x_{i3}} = 6 + 1 - \frac{1}{7} = \frac{48}{7}$$

Lemma 4.7.12.

$$\begin{aligned} e(X(3,7,6)) &= 46, & c_1^2(X(3,7,6)) = 2, \\ \sigma(X(3,7,6)) &= -30, & \chi_h(X(3,7,6)) = .4 \end{aligned}$$

Proof.

$$K^2_{X(3,7,6)} = \frac{8\,(g(C_1)-1)\,(g(C_2)-1)}{|G|} + \sum_{x \in Sing(T)} h_x = \frac{8(2)(2)}{7} + 3\left(-\frac{3}{7} - \frac{3}{7} + 0\right) = 2$$

$$e(X(3,7,6)) = \frac{4\,(g(C_1)-1)\,(g(C_2)-1)}{|G|} + \sum_{x_{ij} \in Sing(T)} e_{x_{ij}} = \frac{4(2)(2)}{7} + 3\left(2 \cdot \frac{27}{7} + \frac{48}{7}\right) = 46$$

Therefore, $\sigma(X(3,7,6)) = -30$ and $\chi_h(X(3,7,6)) = 2$.

Each singular fiber has euler characteristic $e(F(3,7,6)) = 13 \cdot 2 - 12 = 14$, which by Lemma 4.5.1, gives that

$$e(X(3,7,6)) = k \cdot e(F(3,7,6)) + (2-k) \cdot (2-2g) = 3 \cdot 14 - (2-6) = 46.$$

□

g = 3, Type 7**: fiber of type** $\left(\frac{4}{7}, \frac{4}{7}, \frac{6}{7}\right)$

$$(Y)^2 = -\left(\frac{4}{7} + \frac{4}{7} + \frac{6}{7}\right) = -2$$

$$\begin{aligned} \frac{n_1}{q_1} &= \frac{n_2}{q_2} = \frac{7}{4} = [2,4], \\ \frac{n_3}{q_3} &= \frac{7}{6} = [2,2,2,2,2,2] \end{aligned}$$

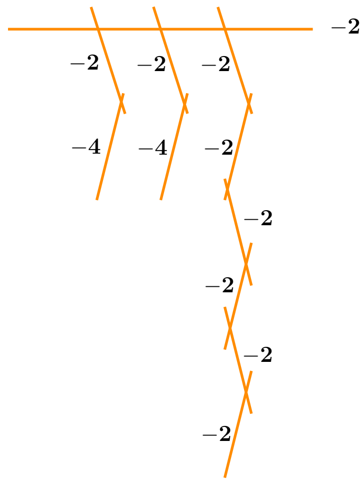


Figure 4.34: $g = 3$, Type 7

$$\begin{aligned} l_{x_{i1}} &= l_{x_{i2}} = 2, \qquad l_{x_{i3}} = 6 \\ q_1 &= q_2 = 4, \qquad q_3 = 6 \\ q'_1 &= q'_2 = 2, \qquad q'_3 = 6 \end{aligned}$$

$$\begin{aligned} h_{x_{i1}} &= h_{x_{i2}} = 2 - \frac{2+4+2}{7} - ((2-2)+(4-2)) = -\frac{8}{7} \\ h_{x_{i3}} &= 2 - \frac{2+6+6}{7} - 6(2-2) = 0 \end{aligned}$$

$$\begin{aligned} B_{x_{i1}} &= B_{x_{i2}} = \frac{4+2}{7} + 2 + 4 = \frac{48}{7}, \qquad B_{x_{i3}} = \frac{6+6}{7} + 6(2) = \frac{96}{7} \\ e_{x_{i1}} &= e_{x_{i2}} = 2 + 1 - \frac{1}{7} = \frac{20}{7}, \qquad e_{x_{i3}} = 6 + 1 - \frac{1}{7} = \frac{48}{7} \end{aligned}$$

Lemma 4.7.13. $e(X(3,7,7)) = 40$

Proof.

$$e(X(3,7,7)) = \frac{4\,(g(C_1)-1)\,(g(C_2)-1)}{|G|} + \sum_{x_{ij} \in Sing(T)} e_{x_{ij}} = \frac{4(21)(2)}{7} + 3\left(2 \cdot \frac{20}{7} + \frac{48}{7}\right) = 40$$

which can also be shown using Lemma 4.5.1. Each singular fiber has euler characteristic $e(F(3,7,7)) = 11 \cdot 2 - 10 = 12$. Hence,

$$e(X(3,7,7)) = k \cdot e(F(3,7,7)) + (2-k) \cdot (2-2g) = 3 \cdot 12 - (2-6) = 40.$$

$\square$

g = 3, Type 8**: fiber of type** $\left(\frac{4}{7}, \frac{5}{7}, \frac{5}{7}\right)$

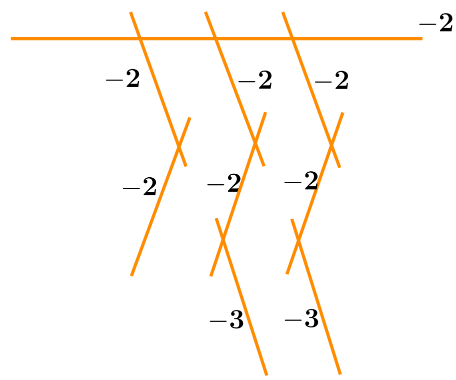


Figure 4.35: $g = 3$, Type 8

$$(Y)^2 = -\left(\frac{4}{7} + \frac{5}{7} + \frac{5}{7}\right) = -2$$

$$\frac{n_1}{q_1} = \frac{7}{4} = [2,4],$$
$$\frac{n_2}{q_2} = \frac{n_3}{q_3} = \frac{7}{5} = [2,2,3]$$

$$l_{x_{i1}} = 2, \qquad l_{x_{i2}} = l_{x_{i3}} = 3$$
$$q_1 = 4, \qquad q_2 = q_3 = 5$$
$$q_1' = 2, \qquad q_2' = q_3' = 3$$

$$h_{x_{i1}} = 2 - \frac{2+4+2}{7} - ((2-2)+(4-2)) = -\frac{8}{7}$$
$$h_{x_{i2}} = h_{x_{i3}} = 2 - \frac{2+5+3}{7} - ((2-2)+(2-2)+(3-2)) = -\frac{3}{7}$$

$$B_{x_{i1}} = \frac{4+2}{7} + 2 + 4 = \frac{48}{7}, \qquad B_{x_{i2}} = B_{x_{i3}} = \frac{5+3}{7} + 2 + 2 + 3 = \frac{57}{7}$$

$$e_{x_{i1}} = 2 + 1 - \frac{1}{7} = \frac{20}{7}, \qquad e_{x_{i2}} = e_{x_{i3}} = 3 + 1 - \frac{1}{7} = \frac{27}{7}$$

Lemma 4.7.14. $e(X(3,7,8)) = 34$

Proof.

$$e(X(3,7,8)) = \frac{4\,(g(C_1)-1)\,(g(C_2)-1)}{|G|} + \sum_{x_{ij} \in Sing(T)} e_{x_{ij}} = \frac{4(2)(2)}{7} + 3\left(\frac{20}{7} + 2 \cdot \frac{27}{7}\right) = 34$$

Again, using Lemma 4.5.1, since each singular fiber has euler characteristic $e(F(3,7,8)) = 9 \cdot 2 - 8 = 10$, we get

$$e(X(3,7,8)) = k \cdot e(F(3,7,8)) + (2-k) \cdot (2-2g) = 3 \cdot 10 - (2-6) = 34.$$

□

4.7.4 g = 4

In this case we are using order 9 cyclic action on Σ_4 with 3 fixed points. So, the singular manifold $T = \Sigma_4 \times \Sigma_4 / \mathbb{Z}_9$ has 9 singular points $\{x_{ij} | 1 \leq i, j \leq 3\}$ corresponding to the fixed points of the action.

g = 4, Type 1: fiber of type $\left(\frac{1}{9}, \frac{1}{9}, \frac{7}{9}\right)$

$$(Y)^2 = -1$$

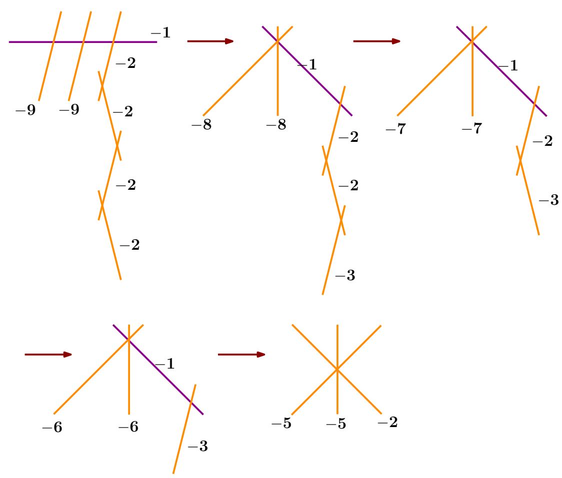


Figure 4.36: $g = 4$, Type 1

$$\frac{n_1}{q_1} = \frac{n_2}{q_2} = \frac{9}{1} = [9],$$
$$\frac{n_3}{q_3} = \frac{9}{7} = 2 - \frac{5}{7} = 2 - \frac{1}{\frac{7}{5}} = 2 - \frac{1}{2 - \frac{3}{5}} = 2 - \frac{1}{2 - \frac{1}{\frac{5}{3}}} = 2 - \frac{1}{2 - \frac{1}{2 - \frac{1}{3}}} = [2, 2, 2, 3]$$

$$\begin{aligned}
l_{x_{i1}} &= l_{x_{i2}} = 1, \qquad l_{x_{i3}} = 4 \\
q_1 &= q_2 = 1, \qquad q_3 = 7 \\
q_1' &= q_2' = 1, \qquad q_3' = 4 \\
h_{x_{i1}} &= h_{x_{i2}} = 2 - \frac{2+1+1}{9} - (9-2) = -\frac{49}{9} \\
h_{x_{i3}} &= 2 - \frac{2+7+4}{9} - (3(2-2) + (3-2)) = -\frac{4}{9}
\end{aligned}$$

$$B_{x_{i1}} = B_{x_{i2}} = \frac{1+1}{9} + 9 = \frac{83}{9}, \qquad B_{x_{i3}} = \frac{7+4}{7} + 2 + 2 + 2 + 3 = \frac{92}{9}$$

$$e_{x_{i1}} = e_{x_{i2}} = 1 + 1 - \frac{1}{9} = \frac{17}{9}, \qquad e_{x_{i3}} = 4 + 1 - \frac{1}{9} = \frac{44}{9}$$

Lemma 4.7.15. $e(X(4,9,1)) = 18$

Proof. Before blow-down we have;

$$e(S) = \frac{4\,(g(C_1)-1)\,(g(C_2)-1)}{|G|} + \sum_{x_{ij}\in Sing(T)} e_{x_{ij}} = \frac{4(3)(3)}{9} + 3\left(\frac{17}{9} + \frac{17}{9} + \frac{44}{9}\right) = 30$$

Once we blow-down we get; $e(X(4,9,1)) = e(S) - 4(3) = 30 - 12 = 18.$

Each singular fiber has euler characteristic $e(F(4,9,1)) = 3 \cdot 2 - 2 = 4$. Hence, by Lemma 4.5.1

$$e(X(4,9,1)) = k \cdot e(F(4,9,1)) + (2-k)\cdot(2-2g) = 3\cdot 4 - (2-8) = 18.$$

□

$\mathbf{g = 4}$, Type 2: fiber of type $\left(\frac{1}{9}, \frac{2}{9}, \frac{2}{3}\right)$

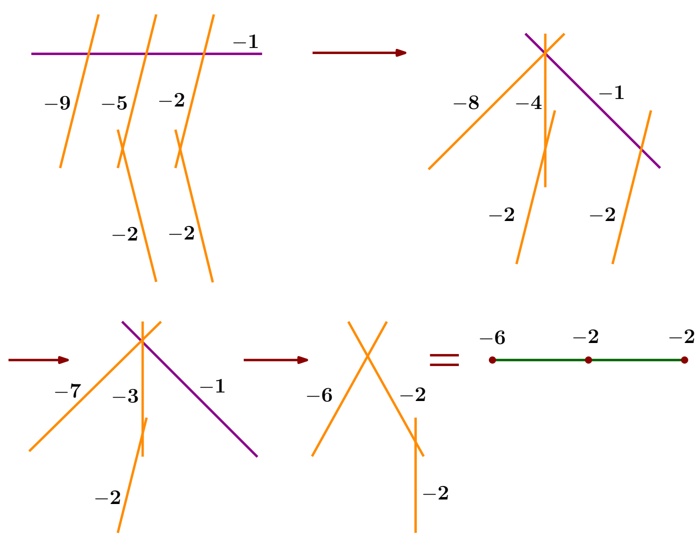


Figure 4.37: $g = 4$, Type 2

$$(Y)^2 = -1$$

$$\begin{aligned}
\frac{n_1}{q_1} &= \frac{9}{1} = [9], \\
\frac{n_2}{q_2} &= \frac{9}{2} = 5 - \frac{1}{2} = [5,2], \\
\frac{n_3}{q_3} &= \frac{3}{2} = 2 - \frac{3}{2} = 2 - \frac{1}{2} = [2,2]
\end{aligned}$$

$$\begin{aligned}
l_{x_{i1}} &= 1, \qquad l_{x_{i2}} = l_{x_{i3}} = 2 \\
q_1 &= 1, \qquad q_2 = 2, \qquad q_3 = 2 \\
q_1' &= 1, \qquad q_2' = 5, \qquad q_3' = 2
\end{aligned}$$

$$\begin{aligned}
h_{x_{i1}} &= 2 - \frac{2+1+1}{9} - (9-2) = -\frac{49}{9} \\
h_{x_{i2}} &= 2 - \frac{2+2+5}{9} - ((5-2)+(2-2)) = -2 \\
h_{x_{i3}} &= 2 - \frac{2+2+2}{3} - ((2-2)+(2-2)) = 0
\end{aligned}$$

$$B_{x_{i1}} = \frac{1+1}{9} + 9 = \frac{83}{9}, \qquad B_{x_{i2}} = \frac{2+5}{9} + 5 + 2 = \frac{70}{9}, \qquad B_{x_{i3}} = \frac{2+2}{3} + 2 + 2 = \frac{16}{3}$$

$$e_{x_{i1}} = 1 + 1 - \frac{1}{9} = \frac{17}{9}, \qquad e_{x_{i2}} = e_{x_{i3}} = 2 + 1 - \frac{1}{9} = \frac{26}{9}$$

Lemma 4.7.16. $e(X(4,9,2)) = 18$

Proof. Before blow-down we have;

$$e(S) = \frac{4\,(g(C_1)-1)\,(g(C_2)-1)}{|G|} + \sum_{x_{ij} \in Sing(T)} e_{x_{ij}} = \frac{4(3)(3)}{9} + 3\left(\frac{17}{9} + \frac{26}{9} + \frac{26}{9}\right) = 27$$

Once we blow-down we get; $e(X(4.9.2)) = e(S) - 3(3) = 27 - 9 = 18$.

We can compute the euler characteristic also using Lemma 4.5.1. Each singular fiber has euler characteristic $e(F(4,9,2)) = 3 \cdot 2 - 2 = 4$. Hence,

$$e(X(4,9,2)) = k \cdot e(F(4,9,2)) + (2-k) \cdot (2-2g) = 3 \cdot 4 - (2-8) = 18. \qquad \square$$

g = 4, Type 3: fiber of type $\left(\frac{1}{9}, \frac{1}{3}, \frac{5}{9}\right)$

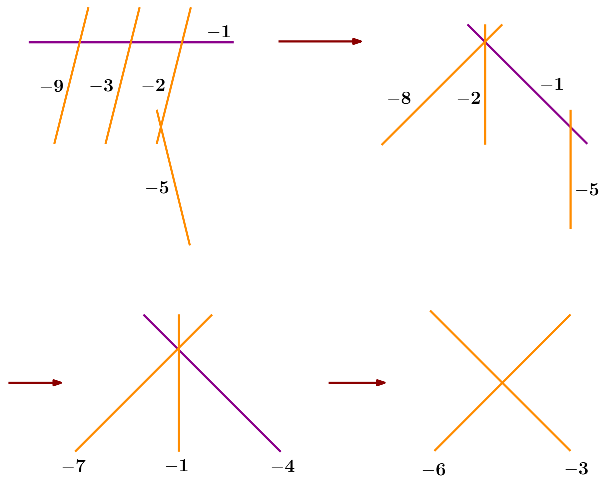


Figure 4.38: $g = 4$, Type 3

$$(Y)^2 = -1$$

$$\begin{aligned}
\frac{n_1}{q_1} &= \frac{9}{1} = [9], \\
\frac{n_2}{q_2} &= \frac{3}{1} = [3], \\
\frac{n_3}{q_3} &= \frac{9}{5} = 2 - \frac{1}{5} = [2, 5]
\end{aligned}$$

$$\begin{aligned}
l_{x_{i1}} &= l_{x_{i2}} = 1, \qquad l_{x_{i3}} = 2 \\
q_1 &= q_2 = 1, \qquad q_3 = 5 \\
q_1' &= q_2' = 1, \qquad q_3' = 2
\end{aligned}$$

$$\begin{aligned} h_{x_{i1}} &= 2 - \frac{2+1+1}{9} - (9-2) = -\frac{49}{9} \\ h_{x_{i2}} &= 2 - \frac{2+1+1}{3} - (3-2) = -\frac{1}{3} \\ h_{x_{i3}} &= 2 - \frac{2+5+2}{9} - ((2-2)+(5-2)) = -2 \end{aligned}$$

$$B_{x_{i1}} = \frac{1+1}{9} + 9 = \frac{83}{9}, \qquad B_{x_{i2}} = \frac{1+1}{3} + 3 = \frac{11}{3}, \qquad B_{x_{i3}} = \frac{5+2}{9} + 2 + 5 = \frac{70}{9}$$

$$e_{x_{i1}} = e_{x_{i2}} = 1 + 1 - \frac{1}{9} = \frac{17}{9}, \qquad e_{x_{i3}} = 2 + 1 - \frac{1}{9} = \frac{26}{9}$$

Lemma 4.7.17. $e(X(4,9,3)) = 15$

Proof. Before blow-down we have;

$$e(S) = \frac{4\,(g(C_1)-1)\,(g(C_2)-1)}{|G|} + \sum_{x_{ij} \in Sing(T)} e_{x_{ij}} = \frac{4(3)(3)}{9} + 3\left(\frac{17}{9} + \frac{17}{9} + \frac{26}{9}\right) = 24$$

Once we blow-down we get; $e(X(4,9,3)) = e(S) - 3(3) = 24 - 9 = 15$.

We can again double check this using Lemma 4.5.1. Each singular fiber has euler characteristic $e(F(4,9,3)) = 2 \cdot 2 - 1 = 3$. Hence,

$$e(X(4,9,3)) = k \cdot e(F(4,9,3)) + (2-k) \cdot (2-2g) = 3 \cdot 3 - (2-8) = 15.$$

$\square$

g = 4, Type 4**: fiber of type** $\left(\frac{1}{9}, \frac{4}{9}, \frac{4}{9}\right)$

$$(Y)^2 = -1$$

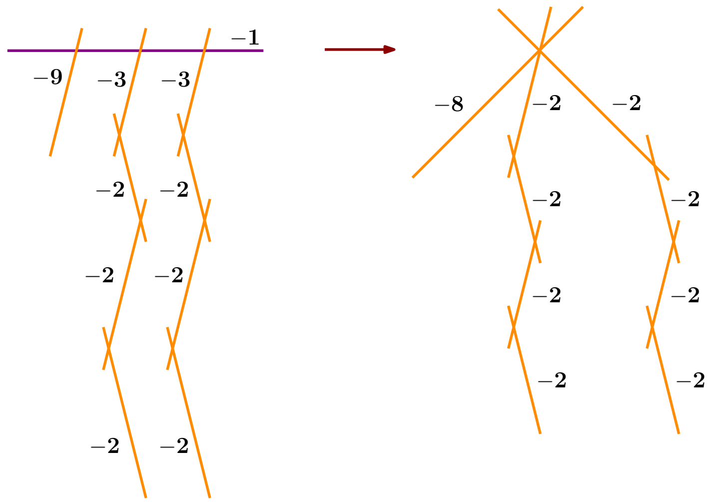


Figure 4.39: $g = 4$, Type 4

$$\frac{n_1}{q_1} = \frac{9}{1} = [9],$$
$$\frac{n_2}{q_2} = \frac{n_3}{q_3} = \frac{9}{4} = 3 - \frac{3}{4} = 3 - \frac{1}{4/3} = 3 - \cfrac{1}{2 - \cfrac{2}{3}} = 3 - \cfrac{1}{2 - \cfrac{1}{3/2}} = 3 - \cfrac{1}{2 - \cfrac{1}{2 - \cfrac{1}{2}}} = [3,2,2,2]$$

$$\begin{aligned} l_{x_{i1}} &= 1, \qquad l_{x_{i2}} = l_{x_{i3}} = 4 \\ q_1 &= 1, \qquad q_2 = q_3 = 4 \\ q'_1 &= 1, \qquad q'_2 = q'_3 = 7 \end{aligned}$$

$$\begin{aligned} h_{x_{i1}} &= 2 - \frac{2+1+1}{9} - (9-2) = -\frac{49}{9} \\ h_{x_{i2}} &= h_{x_{i3}} = 2 - \frac{2+4+7}{9} - ((3-2) + 3(2-2)) = -\frac{4}{9} \end{aligned}$$

$$\begin{aligned} B_{x_{i1}} &= \frac{1+1}{9} + 9 = \frac{83}{9}, \qquad B_{x_{i2}} = B_{x_{i3}} = \frac{4+7}{9} + 3 + 2 + 2 + 2 = \frac{92}{9} \\ e_{x_{i1}} &= 1 + 1 - \frac{1}{9} = \frac{17}{9}, \qquad e_{x_{i2}} = e_{x_{i3}} = 4 + 1 - \frac{1}{9} = \frac{44}{9} \end{aligned}$$

Lemma 4.7.18. $e(X(4,9,4)) = 36$

Proof. Before blow-down we have;

$$e(S) = \frac{4\left(g(C_1)-1\right)\left(g(C_2)-1\right)}{|G|} + \sum_{x_{ij}\in Sing(T)} e_{x_{ij}} = \frac{4(3)(3)}{9} + 3\left(\frac{17}{9}+\frac{44}{9}+\frac{44}{9}\right) = 39$$

Once we blow-down we get; $e(X(4,9,4)) = e(S) - 3 = 39 - 3 = 36$.

Again, euler characteristic can also be computed easily using Lemma 4.5.1. Each singular fiber has euler characteristic $e(F(4,9,4)) = 9\cdot 2 - 8 = 10$. Hence,

$$e(X(4,9,4)) = k\cdot e(F(4,9,4)) + (2-k)\cdot(2-2g) = 3\cdot 10 - (2-8) = 36.$$

□

g = 4, Type 5: fiber of type $\left(\frac{2}{9},\frac{2}{9},\frac{5}{9}\right)$

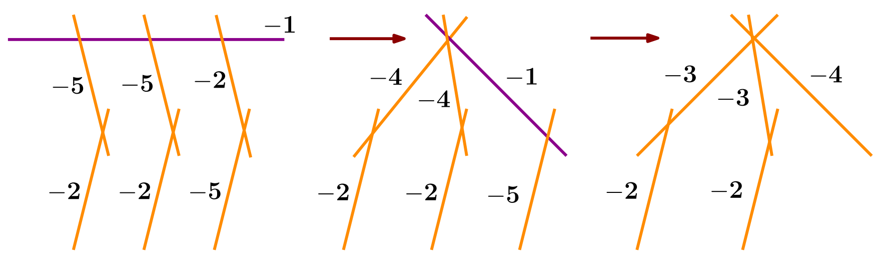


Figure 4.40: $g = 4$, Type 5

$$(Y)^2 = -1$$

$$\frac{n_1}{q_1} = \frac{n_2}{q_2} = \frac{9}{2} = [5,2],$$
$$\frac{n_3}{q_3} = \frac{9}{5} = [2,5]$$

$$\begin{aligned} l_{x_{ij}} &= 2 \\ q_1 &= q_2 = 2, \qquad q_3 = 5 \\ q_1' &= q_2' = 5, \qquad q_3' = 2 \end{aligned}$$

$$\begin{aligned} h_{x_{i1}} &= h_{x_{i2}} = 2 - \frac{2+2+5}{9} - ((5-2)+(2-2)) = -2 \\ h_{x_{i3}} &= 2 - \frac{2+5+2}{9} - ((2-2)+(5-2)) = -2 \end{aligned}$$

$$B_{x_{i1}} = B_{x_{i2}} = \frac{2+5}{9} + 5 + 2 = \frac{70}{9}, \qquad B_{x_{i3}} = \frac{5+2}{9} + 2 + 5 = \frac{70}{9}$$

$$e_{x_{i1}} = e_{x_{i2}} = e_{x_{i3}} = 2 + 1 - \frac{1}{9} = \frac{26}{9}$$

Lemma 4.7.19. $e(X(4,9,5)) = 24$

Proof. Before blow-down we have;

$$e(S) = \frac{4\,(g(C_1)-1)\,(g(C_2)-1)}{|G|} + \sum_{x_{ij} \in Sing(T)} e_{x_{ij}} = \frac{4(3)(3)}{9} + 9\left(\frac{26}{9}\right) = 30$$

Once we blow-down we get; $e(X(4,9,5)) = e(S) - 2(3) = 24$.

In this type, each singular fiber has euler characteristic $e(F(4,9,5)) = 5 \cdot 2 - 4 = 6$. Thus,

$$e(X(4,9,5)) = k \cdot e(F(4,9,5)) + (2-k) \cdot (2-2g) = 3 \cdot 6 - (2-8) = 24.$$

$\square$

g = 4, Type 6: fiber of type $\left(\frac{2}{9}, \frac{1}{3}, \frac{4}{9}\right)$

$$(Y)^2 = -1$$

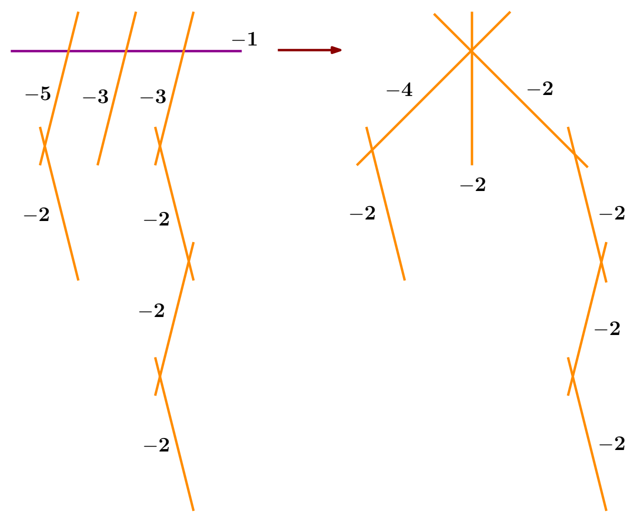


Figure 4.41: $g = 4$, Type 6

$$\begin{aligned}\frac{n_1}{q_1} &= \frac{9}{2} = [5, 2],\\ \frac{n_2}{q_2} &= \frac{3}{1} = [3],\\ \frac{n_3}{q_3} &= \frac{9}{4} = [3, 2, 2, 2]\end{aligned}$$

$$\begin{aligned} l_{x_{i1}} &= 2, & l_{x_{i2}} &= 1, & l_{x_{i3}} &= 4\\ q_1 &= 2, & q_2 &= 1, & q_3 &= 4\\ q_1' &= 5, & q_2' &= 1, & q_3' &= 7\end{aligned}$$

$$\begin{aligned} h_{x_{i1}} &= 2 - \frac{2+2+5}{9} - ((5-2) + (2-2)) = -2\\ h_{x_{i2}} &= 2 - \frac{2+1+1}{3} - (3-2) = -\frac{1}{3}\\ h_{x_{i3}} &= 2 - \frac{2+4+7}{9} - ((3-2) + 3(2-2)) = -\frac{4}{9}\end{aligned}$$

$$B_{x_{i1}} = \frac{2+5}{9} + 5 + 2 = \frac{70}{9}, \quad B_{x_{i2}} = \frac{1+1}{3} + 3 = \frac{11}{3}, \quad B_{x_{i3}} = \frac{4+7}{9} + 3 + 2 + 2 + 2 = \frac{92}{9}$$

$$e_{x_{i1}} = 2+1-\frac{1}{9} = \frac{26}{9}, \qquad e_{x_{i2}} = 1+1-\frac{1}{9} = \frac{17}{9}, \qquad e_{x_{i3}} = 4+1-\frac{1}{9} = \frac{44}{9}$$

Lemma 4.7.20. $e(X(4,9,6)) = 30$

Proof. Before blow-down we have;

$$e(S) = \frac{4\,(g(C_1)-1)\,(g(C_2)-1)}{|G|} + \sum_{x_{ij}\in Sing(T)} e_{x_{ij}} = \frac{4(3)(3)}{9} + 3\left(\frac{26}{9}+\frac{17}{9}+\frac{44}{9}\right) = 33$$

Once we do blow-down we get;

$$e(X(4,9,6)) = e(S) - 3 = 30$$

After blow down, the singular fiber we get has euler characteristic $e(F(4,9,6)) = 7\cdot 2 - 6 = 8$. So, by Lemma 4.5.1

$$e(X(4,9,6)) = k\cdot e(F(4,9,6)) + (2-k)\cdot(2-2g) = 3\cdot 8 - (2-8) = 30.$$

$\square$

$\mathbf{g = 4}$, Type 7: fiber of type $\left(\frac{2}{9}, \frac{8}{9}, \frac{8}{9}\right)$

$$(Y)^2 = -2$$

$$\frac{n_1}{q_1} = \frac{9}{2} = [5,2],$$
$$\frac{n_2}{q_2} = \frac{n_3}{q_3} = \frac{9}{8} = [2,2,2,2,2,2,2,2]$$

$$l_{x_{i1}} = 2, \qquad l_{x_{i2}} = l_{x_{i3}} = 8$$
$$q_1 = 2, \qquad q_2 = q_3 = 8$$
$$q_1' = 5, \qquad q_2' = q_3' = 8$$

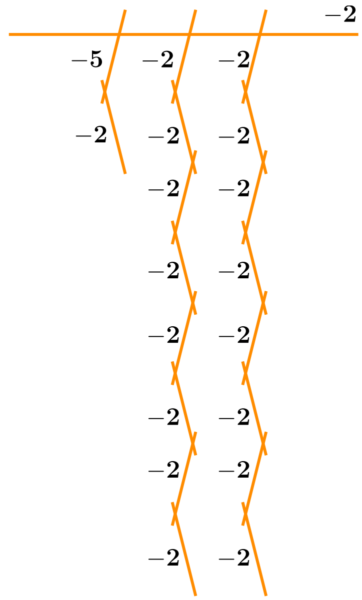


Figure 4.42: $g = 4$, Type 7

$$h_{x_{i1}} = 2 - \frac{2+2+5}{9} - ((5-2)+(2-2)) = -2$$
$$h_{x_{i2}} = h_{x_{i3}} = 2 - \frac{2+8+8}{9} - 8(2-2) = 0$$

$$B_{x_{i1}} = \frac{2+5}{9} + 5 + 2 = \frac{70}{9}, \qquad B_{x_{i2}} = B_{x_{i3}} = \frac{8+8}{9} + 8(2) = \frac{160}{9}$$

$$e_{x_{i1}} = 2 + 1 - \frac{1}{9} = \frac{26}{9}, \qquad e_{x_{i2}} = e_{x_{i3}} = 8 + 1 - \frac{1}{9} = \frac{80}{9}$$

Lemma 4.7.21. $e(X(4,9,7)) = 66$

Proof.

$$e(X(4,9,7)) = \frac{4\,(g(C_1)-1)\,(g(C_2)-1)}{|G|} + \sum_{x_{ij} \in Sing(T)} e_{x_{ij}} = \frac{4(3)(3)}{9} + 3\left(\frac{26}{9} + 2 \cdot \frac{80}{9}\right) = 66$$

This time, each singular fiber has euler characteristic $e(F(4,9,7)) = 19 \cdot 2 - 18 = 20$.

So, by Lemma 4.5.1,

$$e(X(4,9,7)) = k \cdot e(F(4,9,7)) + (2-k)\cdot(2-2g) = 3\cdot 20 - (2-8) = 66.$$

$\square$

g = 4, Type 8: fiber of type $\left(\frac{1}{3}, \frac{7}{9}, \frac{8}{9}\right)$

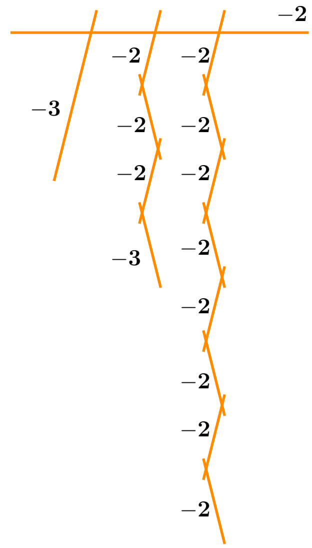


Figure 4.43: $g = 4$, Type 8

$$(Y)^2 = -2$$

$$\frac{n_1}{q_1} = \frac{3}{1} = [3],$$
$$\frac{n_2}{q_2} = \frac{9}{7} = [2,2,2,3],$$
$$\frac{n_3}{q_3} = \frac{9}{8} = [2,2,2,2,2,2,2,2]$$

$$\begin{aligned} l_{x_{i1}} &= 1, \qquad l_{x_{i2}} = 4, \qquad l_{x_{i3}} = 8 \\ q_1 &= 1, \qquad q_2 = 7, \qquad q_3 = 8 \\ q_1' &= 1, \qquad q_2' = 4, \qquad q_3' = 8 \end{aligned}$$

$$\begin{aligned} h_{x_{i1}} &= 2 - \frac{2+1+1}{3} - (3-2) = -\frac{1}{3} \\ h_{x_{i2}} &= 2 - \frac{2+7+4}{9} - (3(2-2) + (3-2)) = -\frac{4}{9} \\ h_{x_{i3}} &= 2 - \frac{2+8+8}{9} - 8(2-2) = 0 \end{aligned}$$

$$B_{x_{i1}} = \frac{1+1}{3} + 3 = \frac{11}{3}, \quad B_{x_{i2}} = \frac{7+4}{3} + 2 + 2 + 2 + 3 = \frac{92}{9}, \quad B_{x_{i3}} = \frac{8+8}{9} + 8(2) = \frac{160}{9}$$

$$e_{x_{i1}} = 1 + 1 - \frac{1}{9} = \frac{17}{9}, \qquad e_{x_{i2}} = 4 + 1 - \frac{1}{9} = \frac{44}{9}, \qquad e_{x_{i3}} = 8 + 1 - \frac{80}{9}$$

Lemma 4.7.22. $e(X(4,9,8)) = 51$

Proof.

$$e(X(4,9,8)) = \frac{4\,(g(C_1)-1)\,(g(C_2)-1)}{|G|} + \sum_{x_{ij} \in Sing(T)} e_{x_{ij}} = \frac{4(3)(3)}{9} + 3\left(\frac{17}{9} + \frac{44}{9} + \frac{80}{9}\right) = 51$$

Again, we can compute the euler characteristic also using Lemma 4.5.1. Each singular fiber has euler characteristic $e(F(4,9,8)) = 14 \cdot 2 - 13 = 15$. Hence,

$$e(X(4,9,8)) = k \cdot e(F(4,9,8)) + (2-k) \cdot (2-2g) = 3 \cdot 15 - (2-8) = 51.$$

□

g = 4, Type 9**: fiber of type** $\left(\frac{4}{9}, \frac{2}{3}, \frac{8}{9}\right)$

$$(Y)^2 = -2$$

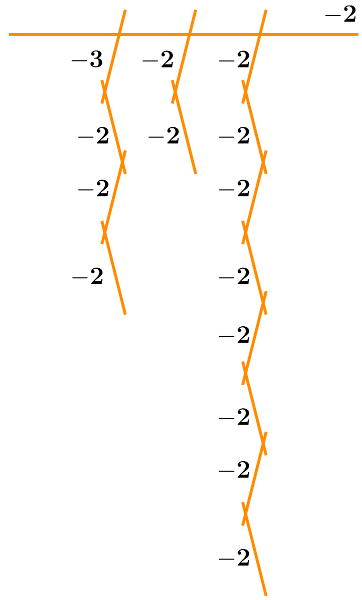


Figure 4.44: $g = 4$, Type 9

$$\begin{aligned}
\frac{n_1}{q_1} &= \frac{9}{4} = [3, 2, 2, 2], \\
\frac{n_2}{q_2} &= \frac{3}{2} = [2, 2], \\
\frac{n_3}{q_3} &= \frac{9}{8} = [2, 2, 2, 2, 2, 2, 2, 2]
\end{aligned}$$

$$\begin{aligned}
l_{x_{i1}} &= 4, & l_{x_{i2}} &= 72, & l_{x_{i3}} &= 8 \\
q_1 &= 4, & q_2 &= 2, & q_3 &= 8 \\
q_1' &= 7, & q_2' &= 2, & q_3' &= 8
\end{aligned}$$

$$\begin{aligned}
h_{x_{i1}} &= 2 - \frac{2+4+7}{9} - ((3-2) + 3(2-2)) = -\frac{4}{9} \\
h_{x_{i2}} &= 2 - \frac{2+2+2}{3} - 2(2-2) = 0 \\
h_{x_{i3}} &= 2 - \frac{2+8+8}{9} - 8(2-2) = 0
\end{aligned}$$

$$
\begin{aligned}
B_{x_{i1}} &= \frac{4+7}{9}+3+2+2+2=\frac{92}{9}\\
B_{x_{i2}} &= \frac{2+2}{3}+2+2=\frac{16}{3}\\
B_{x_{i3}} &= \frac{8+8}{9}+8(2)=\frac{160}{9}
\end{aligned}
$$

$$
e_{x_{i1}} = 4+1-\frac{1}{9}=\frac{44}{9}, \qquad e_{x_{i2}} = 2+1-\frac{1}{9}=\frac{26}{9}, \qquad e_{x_{i3}} = 8+1-\frac{80}{9}
$$

Lemma 4.7.23. $e(X(4,9,9)) = 54$

Proof.

$$
e(X(4,9,9)) = \frac{4\,(g(C_1)-1)\,(g(C_2)-1)}{|G|}+\sum_{x_{ij}\in Sing(T)} e_{x_{ij}} = \frac{4(3)(3)}{9}+3\left(\frac{44}{9}+\frac{26}{9}+\frac{80}{9}\right) = 54
$$

Each singular fiber has euler characteristic $e(F(4,9,9)) = 15\cdot 2-14 = 16$, which also shows that

$$
e(X(4,9,9)) = k\cdot e(F(4,9,9)) + (2-k)\cdot(2-2g) = 3\cdot 16-(2-8) = 54.
$$

□

g = 4, Type 10**: fiber of type** $\left(\frac{4}{9},\frac{7}{9},\frac{7}{9}\right)$

$$
(Y)^2 = -2
$$

$$
\begin{aligned}
\frac{n_1}{q_1} &= \frac{9}{4} = [3,2,2,2],\\
\frac{n_2}{q_2} &= \frac{n_3}{q_3} = \frac{9}{7} = [2,2,2,3]
\end{aligned}
$$

$$
\begin{aligned}
l_{x_{ij}} &= 4\\
q_1 &= 4, \qquad q_2 = q_3 = 7\\
q_1' &= 7, \qquad q_2' = q_3' = 4
\end{aligned}
$$

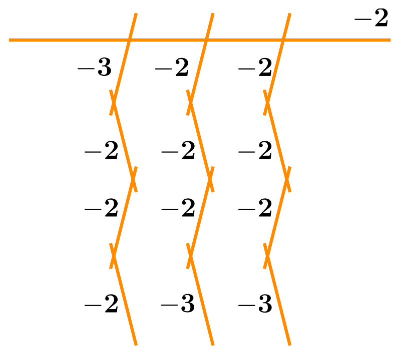


Figure 4.45: $g = 4$, Type 10

$$\begin{aligned} h_{x_{i1}} &= 2 - \frac{2+4+7}{9} - ((3-2) + 3(2-2)) = -\frac{4}{9} \\ h_{x_{i2}} &= h_{x_{i3}} = 2 - \frac{2+7+4}{9} - (3(2-2) + (3-2)) = -\frac{4}{9} \end{aligned}$$

$$B_{x_{i1}} = \frac{4+7}{9} + 3 + 2 + 2 + 2 = \frac{92}{9}, \qquad B_{x_{i2}} = B_{x_{i3}} = \frac{7+4}{9} + 2 + 2 + 2 + 3 = \frac{92}{9}$$

$$e_{x_{i1}} = e_{x_{i2}} = e_{x_{i3}} = 4 + 1 - \frac{1}{9} = \frac{44}{9}$$

Lemma 4.7.24. $e(X(4,9,10)) = 48$

Proof.

$$e(X(4,9,10)) = \frac{4\,(g(C_1)-1)\,(g(C_2)-1)}{|G|} + \sum_{x_{ij} \in Sing(T)} e_{x_{ij}} = \frac{4(3)(3)}{9} + 9\left(\frac{44}{9}\right) = 48$$

Again, since each singular fiber has euler characteristic $e(F(4,9,10)) = 13 \cdot 2 - 12 = 14$, byLemma 4.5.1 we get

$$e(X(4,9,10)) = k \cdot e(F(4,9,10)) + (2-k) \cdot (2-2g) = 3 \cdot 14 - (2-8) = 48.$$

□

g = 4, Type 11: fiber of type $\left(\frac{5}{9}, \frac{5}{9}, \frac{8}{9}\right)$

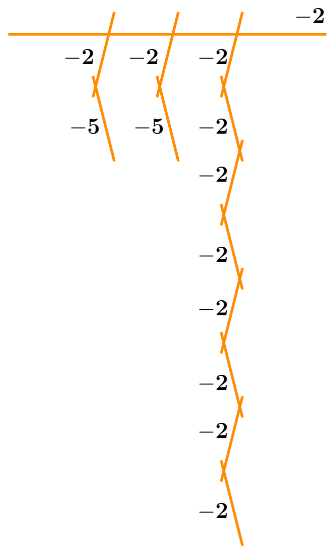


Figure 4.46: $g = 4$, Type 11

$$(Y)^2 = -2$$

$$\frac{n_1}{q_1} = \frac{n_2}{q_2} = \frac{9}{5} = [2, 5],$$
$$\frac{n_3}{q_3} = \frac{9}{8} = [2, 2, 2, 2, 2, 2, 2, 2]$$

$$l_{x_{i1}} = l_{x_{i2}} = 2, \qquad l_{x_{i3}} = 8$$
$$q_1 = q_2 = 5, \qquad q_3 = 8$$
$$q_1' = q_2' = 2, \qquad q_3' = 8$$

$$h_{x_{i1}} = h_{x_{i2}} = 2 - \frac{2+5+2}{9} - ((2-2) + (5-2)) = -2, \quad h_{x_{i3}} = 2 - \frac{2+8+8}{9} - 8(2-2) = 0$$

$$B_{x_{i1}} = B_{x_{i2}} = \frac{5+2}{9} + 2 + 4 = \frac{160}{9}, \qquad B_{x_{i3}} = \frac{8+8}{9} + 8(2) = \frac{70}{9}$$

$$e_{x_{i1}} = e_{x_{i2}} = 2 + 1 - \frac{1}{9} = \frac{26}{9}, \qquad e_{x_{i3}} = 8 + 1 - \frac{80}{9}$$

Lemma 4.7.25. $e(X(4,9,11)) = 48$

Proof.

$$e(X(4,9,11)) = \frac{4\,(g(C_1)-1)\,(g(C_2)-1)}{|G|} + \sum_{x_{ij} \in Sing(T)} e_{x_{ij}} = \frac{4(3)(3)}{9} + 3\left(2 \cdot \frac{26}{9} + \frac{80}{9}\right) = 48$$

Again, we can confirm this using Lemma 4.5.1. Each singular fiber has euler characteristic $e(F(4,9,11)) = 13 \cdot 2 - 12 = 14$. Hence,

$$e(X(4,9,11)) = k \cdot e(F(4,9,11)) + (2-k) \cdot (2-2g) = 3 \cdot 14 - (2-8) = 48.$$

$\square$

$\mathbf{g = 4}$, Type 12: fiber of type $\left(\frac{5}{9}, \frac{2}{3}, \frac{7}{9}\right)$

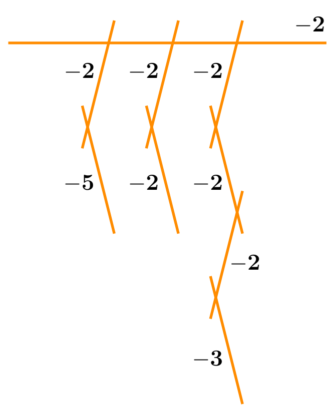


Figure 4.47: $g = 4$, Type 12

$$(Y)^2 = -2$$

$$\begin{aligned}
\frac{n_1}{q_1} &= \frac{9}{5} = [2,5], \\
\frac{n_2}{q_2} &= \frac{3}{2} = [2,2], \\
\frac{n_3}{q_3} &= \frac{9}{7} = [2,2,2,3]
\end{aligned}$$

$$\begin{aligned}
l_{x_{i1}} &= l_{x_{i2}} = 2, \qquad l_{x_{i3}} = 4 \\
q_1 &= 5, \qquad q_2 = 2, \qquad q_3 = 7 \\
q_1' &= 2, \qquad q_2' = 2, \qquad q_3' = 4
\end{aligned}$$

$$\begin{aligned}
h_{x_{i1}} &= 2 - \frac{2+5+2}{9} - ((2-2)+(5-2)) = -2 \\
h_{x_{i2}} &= 2 - \frac{2+2+2}{3} - 2(2-2) = 0 \\
h_{x_{i3}} &= 2 - \frac{2+7+4}{9} - (3(2-2)+(3-2)) = -\frac{4}{9}
\end{aligned}$$

$$B_{x_{i1}} = \frac{5+2}{9} + 2 + 5 = \frac{70}{9}, \quad B_{x_{i2}} = \frac{2+2}{3} + 2 + 2 = \frac{16}{3}, \quad B_{x_{i3}} = \frac{7+4}{9} + 2 + 2 + 2 + 3 = \frac{92}{9}$$

$$e_{x_{i1}} = e_{x_{i2}} = 2 + 1 - \frac{1}{9} = \frac{26}{9}, \qquad e_{x_{i3}} = 4 + 1 - \frac{1}{9} = \frac{44}{9}$$

Lemma 4.7.26. $e(X(4,9,12)) = 36$

Proof.

$$e(X(4,9,12)) = \frac{4\,(g(C_1)-1)\,(g(C_2)-1)}{|G|} + \sum_{x_{ij} \in Sing(T)} e_{x_{ij}} = \frac{4(3)(3)}{9} + 3\left(2 \cdot \frac{26}{9} + \frac{44}{9}\right) = 36$$

Each singular fiber has euler characteristic $e(F(4,9,12)) = 9 \cdot 2 - 8 = 10$, which verify that $e(X(4,9,12)) = 36$,since by Lemma 4.5.1, $e(X(4,9,12)) = k \cdot e(F(4,9,12)) + (2-k) \cdot (2-2g) = 3 \cdot 10 - (2-8) = 36$. □

4.8 More Examples

4.8.1 Rational Blow-Down

In some cases, by perturbing these Lefschetz fibrations we can obtain singular fibers for which we apply *rational blow-down* surgeries (See section 2.6). In this section, we will provide examples of these kind of configurations.

Table 4.1: Rational Blow-Down-$g+1$-action

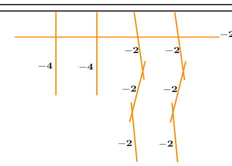
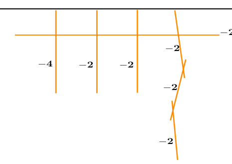
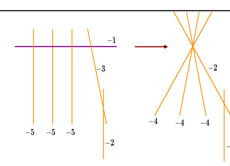
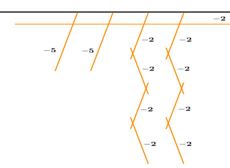
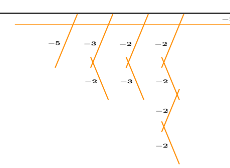
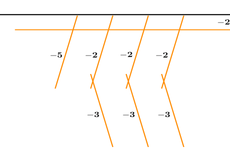
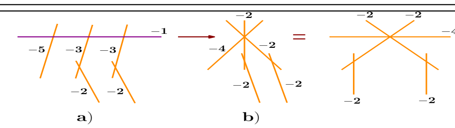
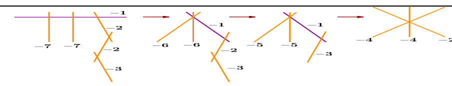
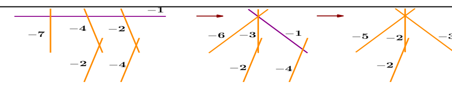
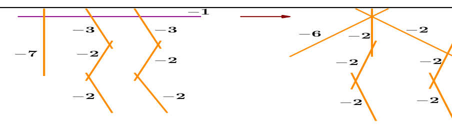
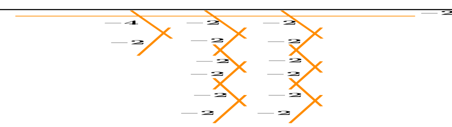
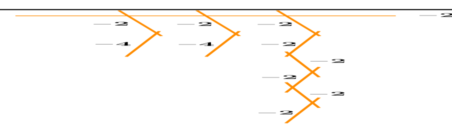
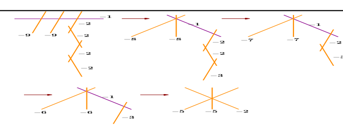
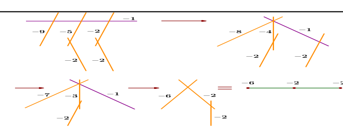
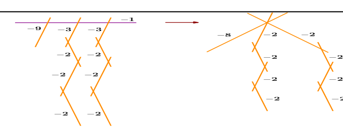
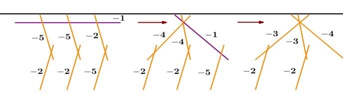
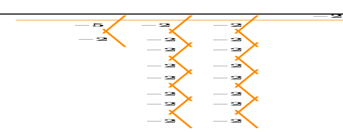
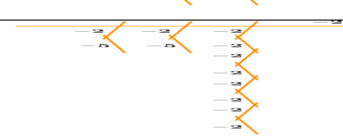
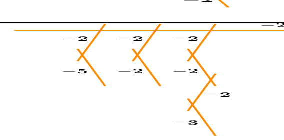
g	action	e	σ	c_1^2	χ_h	singular fiber
3	$\mathbf{Z}_4$	48	-32	0	4	
3	$\mathbf{Z}_4$	40				
4	$\mathbf{Z}_5$	36				
4	$\mathbf{Z}_5$	60	-40	0	5	
4	$\mathbf{Z}_5$	56	-36	4	5	
4	$\mathbf{Z}_5$	48				

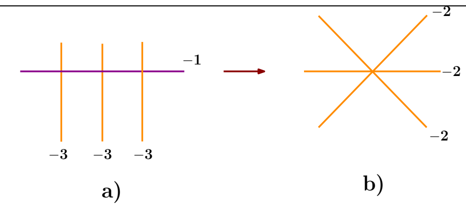
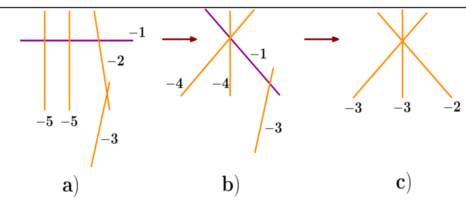
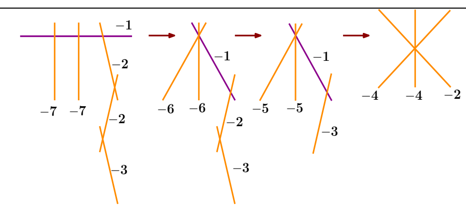
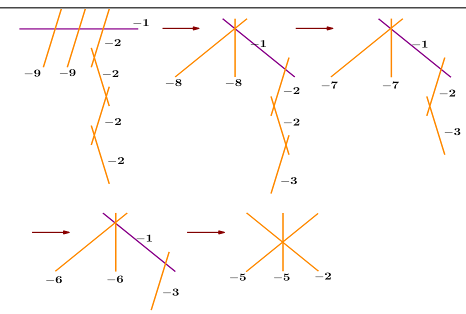
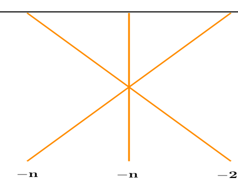
Table 4.2: Rational Blow-Down-$2g+1$-action

g	action	e	σ	c_1^2	χ_h	singular fiber
2	$\mathbf{Z}_5$	20				
3	$\mathbf{Z}_7$	16				
3	$\mathbf{Z}_7$	19	-15	-7	1	
3	$\mathbf{Z}_7$	28				
3	$\mathbf{Z}_7$	52				
3	$\mathbf{Z}_7$	40				
4	$\mathbf{Z}_9$	18				
4	$\mathbf{Z}_9$	18				
4	$\mathbf{Z}_9$	36				
4	$\mathbf{Z}_9$	24				
4	$\mathbf{Z}_9$	66				
4	$\mathbf{Z}_9$	48				
4	$\mathbf{Z}_9$	36				

4.8.2 The manifolds $X(g)$

By looking at the list of types of all possible singular fibers, we obtain some pattern as shown in Table 4.3.

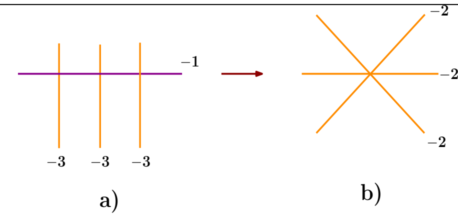
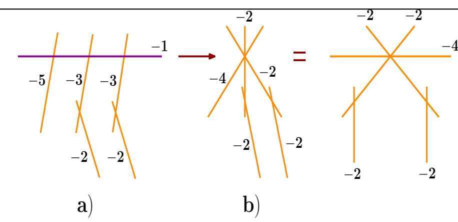
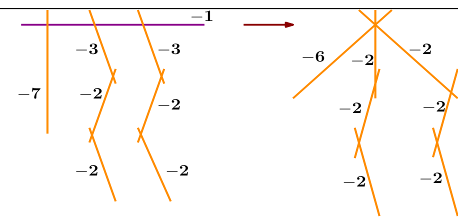
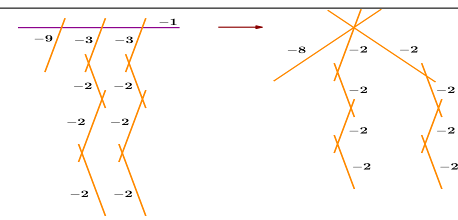
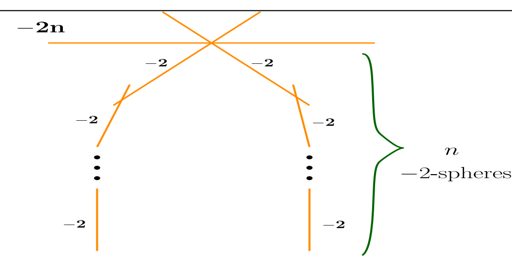
Table 4.3: The manifolds $X(g)$

g	action	e	σ	c_1^2	χ_h	singular fiber
1	$\mathbf{Z}_3$	12	-8	0	1	
2	$\mathbf{Z}_5$	14				
3	$\mathbf{Z}_7$	16				
4	$\mathbf{Z}_9$	18				
g-1	$\mathbf{Z}_{2g-1}$	$12 + 2(g-2)$				

4.8.3 The manifolds $Y(g)$

Table 4.4 shows another pattern we observed.

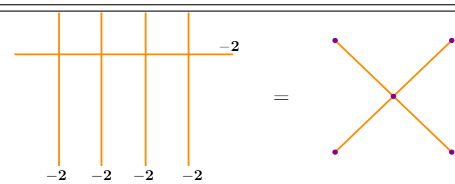
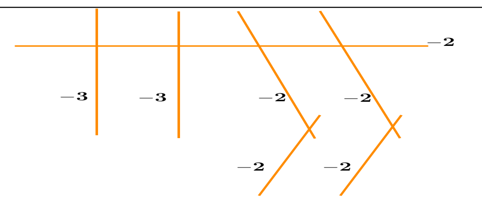
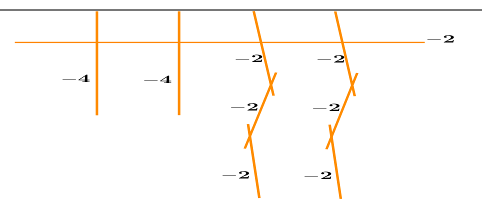
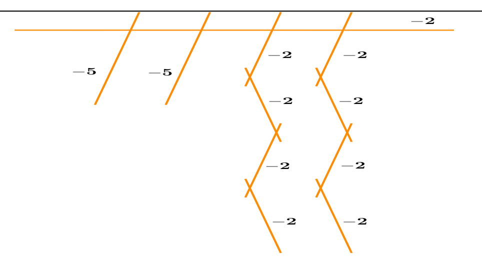
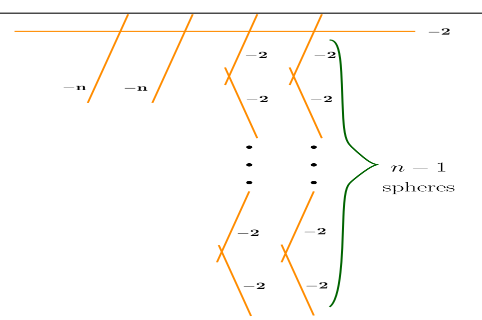
Table 4.4: The manifolds $Y(g)$

g	action	e	σ	c_1^2	χ_h	singular fiber
1	$\mathbf{Z}_3$	12	-8	0	1	
2	$\mathbf{Z}_5$	20				
3	$\mathbf{Z}_7$	28				
4	$\mathbf{Z}_9$	36				
g	$\mathbf{Z}_{2g+1}$	4(2g+1)				

4.8.4 $E(n)$

It is also interesting that we observe all elliptic surfaces $E(n)$.

Table 4.5: The manifolds $E(n)$

n	g	action	e	σ	c_1^2	χ_h	singular fiber
2	1	$\mathbf{Z}_2$	24	-16	0	2	
3	2	$\mathbf{Z}_3$	36	-24	0	3	
4	3	$\mathbf{Z}_4$	48	-32	0	4	
5	4	$\mathbf{Z}_5$	60	-40	0	5	
n	n-1	$\mathbf{Z}_n$	12n	-8n	0	n	

Chapter 5

Strongly Fillable Contact 3-Manifolds Without Stein Fillings

In this chapter we will progress to prove the following theorem:

Theorem 5.0.1. *For any $n > 1$ odd and $g > 1$ the 3-manifold $-\Sigma\,(2, 2g+1, 2(2g+1)n-1)$ admits a strongly symplectically fillable contact structure which is not Stein fillable.*

5.1 Contact 3-Manifolds

Having discussed symplectic 4-manifolds, we will now give a quick review of contact topology (cf. [**Ge**]), which is the odd-dimensional analogue of symplectic topology.

Definition 23. Let Y be a $(2n+1)$-dimensional manifold, and let $\xi \subset TY$ be a hyperplane field, which is defined by a 1-form α, i.e., $\xi = ker(\alpha)$. ξ is called a *contact structure* if $d\alpha$ is non-degenerate when restricted to ξ, or equivalently, $\alpha \wedge (d\alpha)^n \neq 0$. The 1-form α is called a *contact form* associated to ξ, and the pair $(Y;\xi)$ is called a *contact manifold*.

Definition 24. Two contact manifolds (Y_1, ξ_1) and (Y_2, ξ_2) are said to be *contactomorphic* if there exists a diffeomorphism $\phi : Y_1 \to Y_2$ such that $\phi_*(\xi_1) = \xi_2$.

Definition 25. A submanifold L of a contact manifold $\left(Y^{2n+1}, \xi\right)$ is called an *isotropic submanifold* if $T_pL \subset \xi_p$ for all $p \in L$.

Definition 26. Let (Y, ξ) be a contact manifold of dimension $2n+1$, and let L be a n-dimensional submanifold of Y. L is called *Legendrian* if $TL \subset \xi_L$. i.e. It is an isotropic submanifold of Y of maximal dimension n.

Example 5.1.1. *The standard contact structure on* $\mathbb{S}^3$.

Consider $\mathbb{S}^3$ as the subset of $\mathbb{R}^4$ with coordinates (x_1, y_1, x_2, y_2) such that

$$x_1^2 + y_2^2 + x_2^2 + y_2^2 = 1$$

Let $\omega_{st} = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$ denote the standard symplectic 2-form on $\mathbb{R}^4$ and $\alpha = (x_1dy_1 - y_1dx_1 + x_2dy_2 - y_2dx_2)_{|_{\mathbb{S}^3}}$. Then, the standard contact structure on $\mathbb{S}^3 \subset \mathbb{R}^4$ is defied as $\xi_{st} = ker\alpha$.

Definition 27. A vector field V on a symplectic manifold (X, ω) is called a *Liouville vector field* if $\mathcal{L}_V\omega = \omega$, where $\mathcal{L}$ is the Lie derivative.

Definition 28. An embedded disk D in a contact manifold (Y, ξ) is called *overtwisted disk* if at each point $p \in \partial D$ we have $T_pD = \xi_p$. A contact 3-manifold which contains such an overtwisted disk is called *overtwisted*, otherwise it is called *tight*.

Note that ∂D of an overtwisted disk is a Legendrian unknot with $tb\,(\partial D) = 0$.

If (Y, ξ) admits a topologically unknotted Legendrian knot with $tb(K) = 0$, then (Y, ξ) is overtwisted. This can be taken as the definition of an overtwisted manifold.

5.2 Symplectic and Stein Fillings of Contact 3-Manifolds

In this section, we recall the definitions of different type of symplectic fillings of contact 3-manifolds whose inclusion relations are summarized as in the diagram of inclusions in Figure 5.1. This table is borrowed from [**Et3**, **EH**] with a small modification.

There are several different notions of symplectic fillability each of which are defined under some compatibility conditions between a given contact 3-manifold (Y, ξ) and a symplectic 4-manifold (X, ω) with $\partial X = Y$. Example of a weakly fillable but not

						Tight
						$\bigcup$ $\not\!\bigcup$
				Strongly symplectically semi-fillable	$\subset$	Weakly symplectically semi-fillable
				$\bigcup$		$\bigcup$
Stein fillable	$\subset_1$	Exactly fillable	$\subset_2$	Strongly symplectically fillable	$\subset_3$	Weakly symplectically fillable
				$\bigcup$		
				Holomorphically fillable		

Figure 5.1: Table of Inclusions

strongly fillable contact structure has been constructed first by Eliashberg [**El5**] and then more examples have been constructed by Ding and Geiges [**DG**] which shows that inclusion 3 in Figure 5.1 is strict. Examples of tight contact structures which are not weakly fillable were first constructed by Etnyre and Honda [**EH**] and then more examples are given by Lisca and Stipsicz [**LS1**, **LS2**]. Later, Ghiggini proved in [**Gh2**] that for any $n \geq 2$ and even, the 3-manifold $-\Sigma\,(2, 3, 6n + 5)$ admits a strongly fillable contact structure which is not Stein fillable. These examples are not even exactly fillable, hence proves that inclusion 2 in Figure 5.1 is also strict. Recently, examples of exactly fillable but not Stein fillable contact structures have been found by Bowden [**Bw**], which proves that inclusion 1 in Figure 5.1 is also strict.

Definition 29. A contact 3-manifold (Y, ξ) is said to be *weakly symplectically fillable* if there is a compact symplectic 4-maifold (W, ω) such that $\partial X = Y$ as oriented manifolds and $\omega|_\xi \geq 0$. In this case (X, ω) is called *a weak symplectic filling of* (Y, ξ).

Definition 30. A closed contact manifold (Y, ξ) is said to be *strongly symplectically fillable* if there is a compact symplectic manifold (X, ω) such that

- $\partial X = Y$ as oriented manifolds,
- ω is exact near the boundary,
- α can be chosen in such a way that $ker\left(\alpha_{|_Y}\right) = \xi$.

In this case, (X, ω) is called *a strongly symplectic filling of* (Y, ξ).

Example 5.2.1. The vector field $V = x_1 \frac{\partial}{\partial x_1} + y_1 \frac{\partial}{\partial y_1} + x_2 \frac{\partial}{\partial x_2} + y_2 \frac{\partial}{\partial y_2}$ is a Liouville vector field for ω_{st} which is transverse to $\mathbb{S}^3$ (pointing outward). Then, by Cartan formula, this implies ω_{st} is exact near the boundary. Therefore, $\left(D^4, \omega_{st}\right)$ is a strong symplectic filling of 3-sphere $\left(\mathbb{S}^3, \xi_{st}\right)$.

Definition 31. A *Stein manifold* is an affine complex manifold, i.e. a complex manifold that admits a proper holomorphic embedding into some $\mathbb{C}^N$.

Definition 32. A compact complex manifold (X, J) with boundary $\partial X = Y$ is a *Stein domain* if it admits an exhausting J-convex function $\phi : X \to \mathbb{R}$ such that Y is a regular level set (i.e. $Y = \phi^{-1}(t)$).

Then, we say that the contact manifold $\left(Y, \xi = ker(\alpha_{|_Y})\right)$ is *Stein fillable* and (X, J) is called a *Stein filling* of $\left(Y, \xi = ker(\alpha_{|_Y})\right)$.

Moreover, $(X, \omega) = \left(\phi^{-1}\left((0, \infty]\right), \omega_\phi)\right)$ where $d^{\mathbb{C}}\phi := d\phi \circ J$ (which is a 1-form), and $\omega := -dd^{\mathbb{C}}\phi$ is a 2-form which is skew-symmetric.

Definition 33. A compact 4-manifold X with $\partial X = Y$ is said to be *Stein filling* of the closed contact manifold (Y, ξ) if it has a complex structure J and a plurisubharmonic function such that $\xi = TY \cap J(TY)$.

Note that the contact structure induced on ∂X by complex tangencies is contactomorphic to (Y, ξ).

A Stein filling is a strong symplectic filling, where the symplectic form is exact, because $\nabla\phi$ is a Liouville vector field for ω_ϕ.

Strongly symplectically fillable contact structures are weakly symplectically fillable.

Theorem 5.2.2. *([**El1**])* $\left(D^4, \omega_{st}\right)$ *is the unique Stein filling of* $\left(\mathbb{S}^3, \xi_{st}\right)$ *up to diffeomorphism.*

Theorem 5.2.3. *([**El3**, **EG**, **Gr**]) (Eliashberg-Gromov)*
A weakly symplectically fillable contact 3*-manifold* (Y, ξ) *is tight.*

Definition 34. Given a contact 3-manifold (Y, ξ) and a Legendrian knot $L \subset (Y, \xi)$ we say that (Y', ξ') is obtained by *Legendrian surgery* on L if

$$
\begin{array}{ccc}
 & & \text{Tight} \\
 & & \bigcup \not\bigcap \\
\text{Strongly symplectically semi-fillable} & \subset & \text{Weakly symplectically semi-fillable} \\
\bigcup & & \bigcup \\
\text{Strongly symplectically fillable} & \subset & \text{Weakly symplectically fillable} \\
\bigcup & & \\
\text{Holomorphically fillable} & &
\end{array}
$$

Table 5.1: Table of Inclusions

- Y' is obtained by surgery on Y along L with coefficient -1 with respect to the contact framing
- ξ' is the unique extension (up to isotopy) of $\xi_{|_{Y \setminus L}}$) so that it is tight in a neighborhood of the surgery.

Remark. Eliashberg showed that Legendrian surgery preserves holomorphically fillable contact structures [**El2**]. Weinstein proved the above theorem for strongly symplectically fillable contact structures. (cf. [**Wei**]).

Theorem 5.2.4. *Legendrian surgey is a category-preserving for each of the category in the diagram of inclusions in Figure 5.1 [*__EH__*], with the possible exception of the category of tight contact structures.*

We refer the reader to section 2.5 in [**EH**] for a proof of Theorem 5.2.4.

Remark. Recently, it was shown by A. Wand that Legendrian surgery also preserves tightness. (cf. [**Wa**])

5.3 Construction

Let M_g be the 3-manifold obtained by 0-framed surgery on $(2, 2g+1)$-torus knot in $\mathbb{S}^3$, which we denote as $T_{(2,2g+1)}$ (see Figure 5.2). M_g is obtained from $\mathbb{S}^3$ by removing a solid torus neighborhood of $T_{(2,2g+1)}$ and replacing it with another solid torus in such a way that the longitude of $T_{(2,2g+1)}$ bounds the meridian $\{pt\} \times \partial D^2$ of the new solid torus.

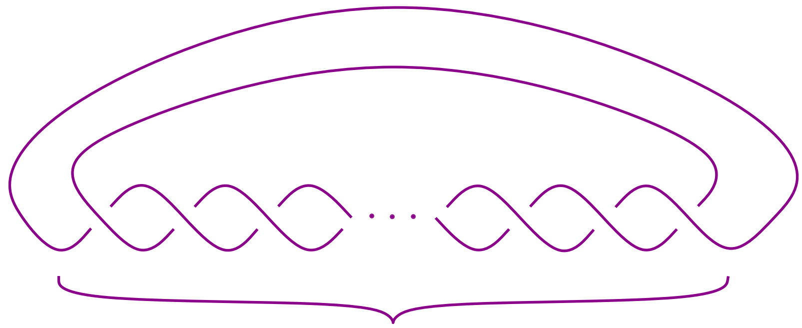


Figure 5.2: $(2, 2g+1)$- torus knot $T_{(2,2g+1)}$

$$M_g = \left(\mathbb{S}^3 \setminus \nu T_{(2,2g+1)}\right) \bigcup \left(\mathbb{S}^1 \times D^2\right)$$

It is well-known that a (p, q)-torus knot is a fibered knot with fiber being its minimal surface. This gives an open book decomposition of $\mathbb{S}^3$, where the fiber is a surface of genus $\frac{1}{2}(p-1)(q-1)$ with one boundary component and the monodromy is a product of $(p-1)(q-1)$ right-handed Dehn twists along nonseparating (i.e., homologically essential) curves. (cf. [**AO**, **BZ**, **OzSt**])

Thus, M_g admits a presentation of Σ_g-bundle over $\mathbb{S}^1$ with monodromy map

$$\varphi : \Sigma_g \times \{1\} \longrightarrow \Sigma_g \times \{0\}.$$

To be more explicit, (M_g, ξ_g) is supported by an open book $(M_g, B, \Sigma, \varphi)$, where the fiber is a surface of genus g with one boundary component and the the monodromy map is $\varphi = t_{a_1} t_{a_2} \cdots t_{a_{2g-1}} t_{a_{2g}}$, where t_{a_i} is the right-handed Dehn twist along the vanishing cycle a_i as shown in Figure 5.3.

There is a contact form $\alpha \in \Omega^1(M_g)$ for ξ_g such that $\alpha|_{TB} > 0$ and $d\alpha|_{\pi^{-1}(\theta)} > 0$, for each $\theta \in \mathbb{S}^1$, where

- $B = \partial\Sigma$ is an oriented link in M_g and called *the binding* of the open book,
- $\pi : M_g \setminus B \to \mathbb{S}^1$ is a fibration of the complement of B such that $\pi^{-1}(\theta)$ is the

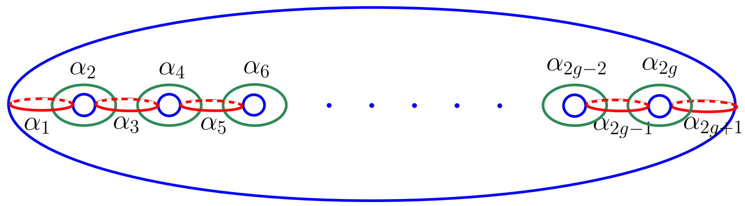


Figure 5.3: vanishing cycles

interior of a compact surface $\Sigma_\theta \subset M_g$ and $\partial\Sigma_\theta = B$ for all $\theta \in \mathbb{S}^1$.

- The surface $\Sigma = \Sigma_\theta$ for any $\theta \in \mathbb{S}^1$, is called *the page* of the open book, which are genus g surfaces.
- $\varphi : \Sigma \to \Sigma$ is a diffeomorphism such that φ is identity in a neighborhood of $\partial\Sigma$, which is called *monodromy*.

$$(M_g)_\varphi = \Sigma_\varphi \bigcup_\psi \left(\coprod_{|\partial\Sigma|} \mathbb{S}^1 \times D^2 \right)$$

where $|\partial\Sigma|$ denotes the number of boundary components of Σ and Σ_φ is the mapping torus of φ, i.e. $\Sigma \times [0,1]/(x,1) \sim (\varphi(x),0)$ for all $x \in \Sigma$. Finaly, $\bigcup_\psi$ means that the diffeomorphism ψ is used to identify the boundaries of the two manifolds. For each boundary component l of Σ, the map $\psi : \partial(\mathbb{S}^1 \times D^2) \to l \times \mathbb{S}^1 \subset \Sigma_\varphi$ is defined to be the unique (up to isotopy) diffeomorphism that takes $\mathbb{S}^1 \times \{p\}$ to l where $p \in \partial D^2$ and $\{q\} \times \partial D^2$ to $(\{q'\} \times [0,1])/\sim = \mathbb{S}^1$, where $q \in \mathbb{S}^1$ and $q' \in \partial\Sigma$.

Thus, it is easy to see that $M_g \setminus B$ is a Σ bundle over $\mathbb{S}^1$.

Theorem 5.3.1. *M_g admit a contact structure ξ_g for all $g \geq 2$ which are in fact weakly symplectically fillable.*

Proof. Attach 2-handle to $\Sigma_g \times D^2$ along the vanishing cycles a_1, a_2,..., a_{2g-1}, a_{2g} with framing -1 relative to the product framing on $\partial\left(\Sigma_g \times D^2\right) = \Sigma_g \times \mathbb{S}^1$. In this way we get a 4-manifold X_g which admits a genus g Lefschetz fibration over D^2. Hence, X_g admits a symplectic structure, say ω. ([**Go2**]) The Lefschetz fibration on X_g induces an open book decomposition and hence a contact structure ξ_g on $\partial\left(X_g\right) = (M_g, \xi_g)$. Thus,

(X, ω) is a symplectic filling of (M_g, ξ_g). Therefore, (M_g, ξ_g) is weakly symplectically fillable. (cf. [**Et3**], [**EF**]) $\square$

5.3.1 Legendrian surgery on (M_g, ξ_g)

Since $T_{(2,2g+1)}$ is a fibered knot, its monodromy $f_g : \Sigma_g^o \to \Sigma_g^o$ has a fixed point. Therefore, $M_g \setminus B$ is a Σ_g bundle over $\mathbb{S}^1$ and admits a section S. Since $\{pt\} \times D^2$ is fixed, $\partial(\{pt\} \times D^2) = \{pt\} \times \partial(D^2) = \{pt\} \times \mathbb{S}^1 = S$. Therefore, S is an unknot. We can think of S as it is Legendrian with respect to the contact structure ξ_g for all g, since any knot in a contact 3-manifold can be C^0-approximated by Legendrian knot isotopic to it. (cf. [**Ge**], Thm. 3.3.1, pg. 101). Moreover, it can be approximated by a Legendrian knot with Thurston-Bennequin invariant $tb = -2g + 1$.

We can define a framing by choosing a diffeomorphism

$$\varphi : \mathbb{S}^3 \setminus \nu(T_{(2,2g+1)}) \to M_g \setminus \nu(S)$$

such that the meridian of $T_{(2,2g+1)}$ is mapped to a longitude of S. Thus, we can define a twisting number for S using the framing defined in this way.

Definition 35. Let L be a Legendrian knot. Let's first pick an orientation of L. L has a natural framing $\mathcal{N}$ called *normal framing* which is induced from ξ by taking $v_p \in \xi_p$ so that $(v_p, L(p))$ form an oriented basis for ξ_p. Then *the twistig number of* L with respect to a given framing $\mathcal{F}$ is defined to be the integer diference of number of twists in $\mathcal{N}$ and number of twists in $\mathcal{F}$. By convention, clockwise twists are counted as 1 and counterclockwise twists are counted as -1. More precisely,

- Use $\mathcal{F}$ to identify $N(L)$ with $\mathbb{S}^1 \times D^2$
- Make an oriented identification $\partial(\mathbb{S}^1 \times D^2) \simeq \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ as follows: map the meridian $\{pt\} \times \partial D^2$ to $(1, 0)$ and $\mathbb{S}^1 \times \{pt\}$ to $(0, 1)$.

If the closed curve on $\mathbb{T}^2$ corresponding to the normal framing is $(n, 1)$ then $tn(L, \mathcal{F}) = n$. (See section 3.3 in [**Ho**] for more details)

Lemma 5.3.2. *The twisting number of ξ_g along the Legendrian curve S is $tn\,(S, \xi_g) = -n + 1$.*

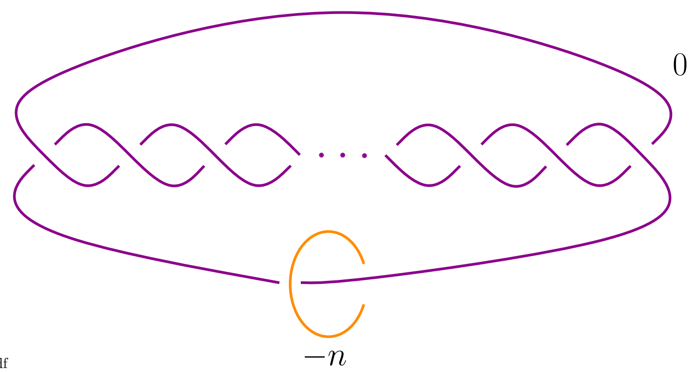


Figure 5.4: The surgery diagram of $-\Sigma\,(2, 2g+1, 2(2g+1)n-1)$

Proof. Using the definition above and the framing we choose, one can conclude that $tn\,(S, \xi_g) = -n+1$ for $-\Sigma\,(2, 2g+1, 2(2g+1)n-1)$. $\square$

Now, Perform Legendrian surgery on (M_g, ξ_g) along S.

Lemma 5.3.3. *The Legendrian surgery on (M_g, ξ_g) along S is smoothly equivalent to the surgery described by the Figure 5.4 which produces the manifold $-\Sigma\,(2, 2g+1, 2(2g+1)n-1)$.*

Proof. Let a, b be coprime integers. Let $S^3_r(K)$ denote the 3-manifold obtained by *Dehn surgery* along a knot K in $\mathbb{S}^3$ with *surgery coefficient* $r = a/b \in \mathbb{Q} \cup \{\infty\}$ such that the gluing map g sends the meridian $\nu = \{pt\} \times \partial D^2$ to $a\nu + b\lambda$, where $\lambda = \mathbb{S}^1 \times \{pt\}$ is a longitude (equivalently a framed pushoff of the knot) ([**Sa2**], section 1.1.5).

$$S^3_r(K) = \overline{(\mathbb{S}^3 \setminus \nu K)} \bigcup_g \left(\mathbb{S}^1 \times D^2\right)$$

We start with a 0-framed surgery along the $(2, 2g+1)$-torus knot $T_{(2,2g+1)}$ in which case we understand $a = 1$ and $b = 0$ and it results in $\mathbb{S}^3$ back. Such a surgery is called *trivial*. In particular when $a = 1$, $S^3_r(T_{(p,q)})$ is an integral homology sphere since $H(S^3_r(K), \mathbb{Z}) \cong \mathbb{Z}_{|a|}$.

Then, we do Legendrian surgery along a Legendrian knot on this manifold. Recall that the Legendrian surgery along a Legendrian knot K in the standard tight contact

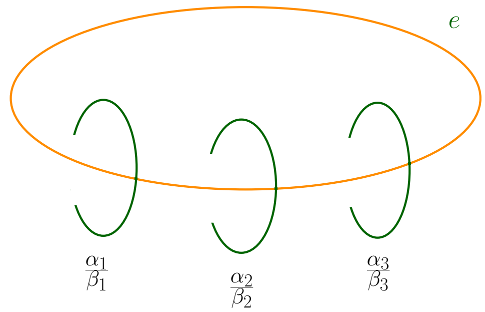


Figure 5.5: $Y\left(e; \dfrac{\alpha_1}{\beta_1}, \dfrac{\alpha_2}{\beta_2}, \dfrac{\alpha_3}{\beta_3}\right)$

structure on $\mathbb{S}^3$ is topologically the Dehn surgery along K with the contact -1 (i.e. $tb(K)-1$) framing, and any integral Dehn surgery along a knot K in $\mathbb{S}^3$ with framing less than $tb(K)$ can be realized as a Legendrian surgery along a Legendrian representative of K in the standard contact structure (cf. [**Ge**], [**Ya**]).

Let $Y\left(e; \dfrac{\alpha_1}{\beta_1}, \dfrac{\alpha_2}{\beta_2}, \dfrac{\alpha_3}{\beta_3}\right)$ denote the 3-manifold that results by performing surgeries with the listed fractional coefficients on disjoint fibers of degree e on $\mathbb{S}^1$-bundle over $\mathbb{S}^2$ as in Figure 5.5.

For any rational number r,

$$S^3_r(T_{(p,q)}) = Y\left(2; \frac{p}{q^*}, \frac{q}{p^*}, \frac{pq-r}{pq-r-1}\right)$$

where $qq^* \equiv 1 \pmod p$, $1 \leq q^* < p$ and $pp^* \equiv 1 \pmod q$, $1 \leq p^* < q$ (cf. [**OwSt**] Lemma 4.4 or [**Mos**]). In our case $p = 2$ and $q = 2g+1$ and hence $q^* = 1$ and $p^* = g+1$. In other words,

$$S^3_{1/n}\left(T_{(2,2g+1)}\right) = Y\left(2; \frac{2}{1}, \frac{2g+1}{g+1}, \frac{2(2g+1)-1/n}{2(2g+1)-1/n-1}\right)$$

.

Then performing two Rolfsen twists (see for example [**GS**] §5.3) to the unknots in the Dehn surgery diagram for $Y\left(2; \dfrac{2}{1}, \dfrac{2g+1}{g+1}, \dfrac{2(2g+1)-1/n}{2(2g+1)-1/n-1}\right)$, we obtain the three

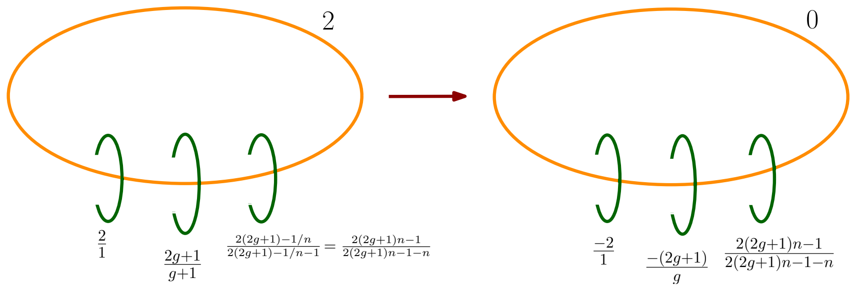


Figure 5.6: Rolfsen twist

manifold $Y\left(0; \dfrac{-2}{1}, \dfrac{-(2g+1)}{g}, \dfrac{2(2g+1)n-1}{2n^2-2n+1}\right)$ (See Figure 5.6). These two surgery diagrams produce the same 3-manifold ([**Ro2**]) up to orientation preserving homeomorphisms. Now, given pairwise coprime numbers $\alpha_1, \alpha_2, \alpha_3$, there exists a unique Seifert manifold $Y\left(e; \dfrac{\alpha_1}{\beta_1}, \dfrac{\alpha_2}{\beta_2}, \dfrac{\alpha_3}{\beta_3}\right) = \Sigma(\alpha_1, \alpha_2, \alpha_3)$ with the α_i' s representing exceptional fibers and $e\left(\Sigma(\alpha_1, \alpha_2, \alpha_3) \to \mathbb{S}^2\right) = \dfrac{-1}{\alpha_1 \cdot \alpha_2 \cdot \alpha_3}$ is a homology sphere where $e\left(Y\left(e; \dfrac{\alpha_1}{\beta_1}, \dfrac{\alpha_2}{\beta_2}, \dfrac{\alpha_3}{\beta_3}\right) \to \mathbb{S}^2\right) = -\Sigma_{i=1}^{3} \dfrac{\alpha_i}{\beta_i}$ (See [**JN**] , Thm. 6.4). Therefore,

$$-\Sigma\left(2, 2g+1, 2(2g+1)n-1\right) = \Sigma\left(-2, -(2g+1), -(2(2g+1)n-1)\right)$$

has Seifert invariant

$$e = \frac{-1}{(-2)\cdot(-(2g+1))\cdot(-(2(2g+1)n-1))} = \frac{1}{2\cdot(2g+1)\cdot(2(2g+1)n-1)}.$$

So, $\dfrac{\beta_1}{-2} + \dfrac{\beta_2}{-(2g+1)} + \dfrac{\beta_3}{-(2(2g+1)n-1)} = \dfrac{-1}{2\cdot(2g+1)\cdot(2(2g+1)n-1)}$ which implies that $\beta_1 = 1$, $\beta_2 = g$ and $\beta_3 = -(2(2g+1)n-1-n)$. Hence,

$$\begin{aligned}
-\Sigma\left(2, 2g+1, 2(2g+1)n-1\right) &= \Sigma\left(-2, -(2g+1), -(2(2g+1)n-1)\right) \\
&= Y\left(0; \frac{-2}{1}, \frac{-(2g+1)}{g}, \frac{-(2(2g+1)n-1)}{-(2(2g+1)n-1-n)}\right) \\
&= Y\left(0; \frac{-2}{1}, \frac{-(2g+1)}{g}, \frac{2(2g+1)n-1}{2(2g+1)n-1-n}\right)
\end{aligned}$$

Another way to prove this could be studying the handlebody diagrams. Fuller [**Fu**] examined the handlebody description for a family $Y_g(n)$ of complex surfaces which admit a singular fibration over $\mathbb{S}^2$ by complex curves of genus g. Using this description, he proved that these complex surfaces can be smoothly decomposed as the Milnor fiber of the Brieskorn homology 3-sphere $\Sigma\left(2, 2g+1, 2(2g+1)n-1\right)$ union a small submanifold $N_g(n)$ called *nucleus*. (Theorem 3 in [**Fu**]). i.e.

$$Y_g(n) = B(2, 2g+1, 2(2g+1)n-1) \bigcup N_g(n)$$

where $B\left(2, 2g+1, 2(2g+1)n-1\right)$ is the Milnor fiber of $\Sigma\left(2, 2g+1, 2(2g+1)n-1\right)$. The nucleus $N_g(n) \subset Y_g(n)$ is defined to be a regular neighborhood of a cusp fiber union a section, so that it looks like Figure 18 in [**Fu**]. So, the boundary of the manifold in Figure 18 [**Fu**] , which is $\Sigma\left(2, 2g+1, 2(2g+1)n-1\right)$, is exactly the manifold defined by Figure 5.4. □

5.4 Proof of Strong Fillability of $\left(-\Sigma\left(2, 2g+1, 2(2g+1)n-1, \mu_0\right)\right)$

Let μ_0 denote the tight contact structure on $-\Sigma\left(2, 2g+1, 2(2g+1)n-1\right)$ obtained by the Legendrian surgery on (M_g, ξ_g) along S.

Theorem 5.4.1. *The contact manifolds* $\left(-\Sigma\left(2, 2g+1, 2(2g+1)n-1\right), \mu_0\right)$ *are strongly symplectically fillable for any* $n, g \geq 1$.

We will state the following theorems without proofs.

Theorem 5.4.2. *[**El1**] Suppose that a symplectic manifold* (W, ω) *weakly fills a contact manifold* (V, ξ)*. If the form* ω *is exact near* $\partial W = V$ *then it can be modified into a symplectic form* $\tilde{\omega}$ *such that* $(W, \tilde{\omega})$ *is a strong symplectic filling of* (V, ξ).

Theorem 5.4.3. (Eliashberg, 1991, [**El6**]; Ohta and Ono, 1999, [**OO**])

If M *is a rational homology sphere then any weak filling of* (M, ξ) *can be deformed into a strong filling.*

Now, we can prove the theorem we stated at the beginning of this section.

Proof. **(proof of Theorem 5.4.1)**

The contact manifolds (M_g, ξ_g) are weakly symplectically fillable by Theorem 5.3.1. $(-\Sigma(2, 2g+1, 2(2g+1)n-1), \mu_0)$ is also weakly symplectically fillable since Legendrian surgery preserves weak symplectic fillability by Theorem 5.2.4. Since the 3-manifolds $(-\Sigma(2, 2g+1, 2(2g+1)n-1)$ are homology spheres, the symplectic form on the filling can be modified in a neighborhood of the boundary so that the filling becomes strong by Theorem 5.4.3. □

5.5 A Brief Review of Heegaard-Floer Homology and Contact Invariant

5.5.1 Heegard-Floer Homology

For a closed oriented 3-manifold Y and a *spin*$^{\mathbb{C}}$ structure $\mathbf{t}$ on Y, four versions of *Heegard Floer homology groups* $\widehat{HF}(Y, \mathbf{t})$, $HF^+(Y, \mathbf{t})$, $HF^-(Y, \mathbf{t})$, and $HF^\infty(Y, \mathbf{t})$ described by Ozsváth and Szabó. (cf. [**OS1**, **OS2**, **OS3**]). They associate

- the vector spaces $\widehat{HF}(Y, \mathbf{t})$ and $HF^+(Y, \mathbf{t})$ over $\mathbb{Z}/2\mathbb{Z}$ to any closed oriented *Spin*$^{\mathbb{C}}$ 3-manifold $(Y, \mathbf{t})$.
- the homomorphisms $F^o_{W,\mathbf{s}} : HF^o(Y_1, \mathbf{t}_1) \to HF^o(Y_2, \mathbf{t}_2)$ to any oriented *Spin*$^{\mathbb{C}}$-cobordism $(W, \mathbf{s})$ between two *Spin*$^{\mathbb{C}}$-manifolds $(Y_1, \mathbf{t}_1)$ and $(Y_2, \mathbf{t}_2)$ such that $\mathbf{s}_{Y_i} = \mathbf{t}_i$.

These groups are all smooth invariants of $(Y, \mathbf{t})$ and are also $\mathbb{Z}[U]$-modules, where multiplication by U decreases degree by 2 (cf. [**Ka**]).

Here HF^o denotes either $\widehat{HF}$ or HF^+.

The exact triangle is a key calculational tool in Heegaard Floer homology. It relates the Heegaard Floer homology groups of three-manifolds obtained by surgeries along a framed knot in a closed, oriented three-manifold by the following long exact sequence.

$$\cdots \longrightarrow \widehat{HF}(Y, \mathbf{t}) \longrightarrow HF^+(Y, \mathbf{t}) \xrightarrow{U} HF^+(Y, \mathbf{t}) \longrightarrow \cdots \tag{5.1}$$

If $c_1(\mathbf{t})$ is a torsion cohomology class, so $HF^0(Y,\mathbf{t})$ is relatively $\mathbb{Z}$-graded, there is an absolute $\mathbb{Q}$ grading on $HF^o(Y,\mathbf{t})$. So, for a torsion *spin*$^{\mathbb{C}}$ structure $\mathbf{t}$ on Y, the Heegaard Floer Homology groups split as (cf. [**OS1**])

$$HF^o(Y,\mathbf{t}) = \bigoplus_{d\in\mathbb{Q}} HF^o_{(d)}(Y,\mathbf{t}).$$

Recall that the set of *Spin*$^{\mathbb{C}}$-structures comes equipped with a natural involution called *conjugation*, which we denote by $\mathbf{t} \mapsto \bar{\mathbf{t}}$: if v is a nonvanishing vector field which represents $\mathbf{t}$, then $-v$ represents $\bar{\mathbf{t}}$ (cf. [**OS2**]). The homology groups are symmetric under this involution:

Theorem 5.5.1. ([**OS2**], Thm.2.4) *There are* $\mathbb{Z}[U]\bigotimes_{\mathbb{Z}}\bigwedge^* H_1(Y;Z)/Tors$*-module isomorphisms*

$HF^{\pm}(Y,\boldsymbol{t}) \cong HF^{\pm}(Y,\bar{\boldsymbol{t}})$, $HF^{\infty}(Y,\boldsymbol{t}) \cong HF^{\infty}(Y,\bar{\boldsymbol{t}})$, $\widehat{HF}(Y,\boldsymbol{t}) \cong \widehat{HF}(Y,\bar{\boldsymbol{t}})$.

There is also a natural map, the first Chern class, $c_1 : Spin^{\mathbb{C}}(Y) \longrightarrow H^2(Y;\mathbb{Z})$, which is defined by $c_1(\mathbf{t}) = \mathbf{t} - \bar{\mathbf{t}}$. Thus, we have $c_1(\bar{\mathbf{t}}) = -c_1(\mathbf{t})$ for any given *Spin*$^{\mathbb{C}}$-structure $\mathbf{t}$, where $\bar{\mathbf{t}}$ denotes the conjugation of *Spin*$^{\mathbb{C}}$-structure $\mathbf{t}$. Recall that if $c_1(\mathbf{t})$ is torsion cohomology class then the $\mathbb{Q}$-grading of the Heegaard-Floer homology groups is preserved by the isomorphism $\mathcal{J}$ defined above in Theorem 5.5.1.

Proposition 5.5.2. ([**OS1**], Thm 3.6) *Let* $(W,\boldsymbol{t})$ *be a* $Spin^{\mathbb{C}}$*-cobordism between* $(Y_1,\boldsymbol{t}_1)$ *and* $(Y_2,\boldsymbol{t}_2)$. *Then the following diagram commutes.*

$$\begin{tikzcd}
HF^o(Y_1,\boldsymbol{t}_1) \arrow[r, "F^o_{W,s}"] \arrow[d, "\mathcal{J}"'] & HF^o(Y_2,\boldsymbol{t}_2) \arrow[d, "\mathcal{J}"] \\
HF^o(Y_1,\bar{\boldsymbol{t}}_1) \arrow[r, "F^o_{W,\bar{s}}"'] & HF^o(Y_2,\bar{\boldsymbol{t}}_2)
\end{tikzcd}$$

The isomorphism $\mathcal{J}$ commutes also with the maps in the exact triangle (1).

5.5.2 Contact Invariant

If $H^2(Y;\mathbb{Z}) \cong H_1(Y;\mathbb{Z})$ has no 2-torsion, then $c_1(\xi)$ determines the $Spin^{\mathbb{C}}$-structure $\mathbf{t}_\xi$ induced by ξ. A second cohomology class uniquely extends from $Y - D^3$ to Y on a 3-dimensional manifold. Therefore, $c_1(\xi) = c_1(t_\xi)$ (cf. [**OzSt**]). For any contact manifold (Y, ξ) there is an associated element $c(\xi) \in \widehat{HF}(-Y, t_\xi)$ which is an isotopy invariant of ξ. ([**OS4**])

Theorem 5.5.3. ([**OS4**], Thm.1.4 and Thm. 1.5) *Let ξ be a contact structure on a closed, oriented 3-manifold Y.*

- *If ξ is an overtwisted contact structure, then $c(\xi) = 0$ in $\widehat{HF}(-Y)/(\pm 1)$.*
- *If ξ is a Stein fillable contact structure, then $c(\xi) \neq 0$ in $\widehat{HF}(-Y)/(\pm 1)$.*

Proposition 5.5.4. ([**OS4**], Prop.4.6) *If $c_1(\boldsymbol{t}(\xi))$ is a torsion homology class, then the absolute grading of $c(\xi)$ agrees with the Hopf invariant*

$$h(t) = \frac{c_1(W,J)^2 - 2\chi(W) - 3\sigma(W) + 2}{4}.$$

i.e. $c(\xi)$ is a homogeneous element of degree $-d_3(\xi) - \frac{1}{2}$, where $d_3(\xi)$ denotes the 3-dimensional homotopy invariant introduced by Gompf.

$$d_3(\mu_i) = \frac{c_1^2(J_i) - (2\chi(X) + 3\sigma(X))}{4}$$

Theorem 5.5.5. ([**Pl**], Thm 4) *Let W be a smooth compact 4-manifold with boundary $Y = \partial W$. Let J_1, J_2 be two Stein structures on W that induce $Spin^{\mathbb{C}}$-structures $\boldsymbol{s}_1$, $\boldsymbol{s}_2$ on W and contact structures ξ_1, ξ_2 on Y. We puncture W and regard it as a cobordism from $-Y$ to $\mathbb{S}^3$. Suppose that $\boldsymbol{s}_1 \mid_Y$ a isotopic to $\boldsymbol{s}_2 \mid_Y$, i.e. $\boldsymbol{s}_1 \mid_Y = \boldsymbol{s}_2 \mid_Y$, but the $Spin^{\mathbb{C}}$-structures $\boldsymbol{s}_1$, $\boldsymbol{s}_2$ are not isomorphic. Then*

(1) $F^+_{W,\boldsymbol{s}_i}(c(\xi_j)) = 0$ for $i \neq j$;

(2) $F^+_{W,\boldsymbol{s}_i}(c(\xi_i))$ is a generator of $HF^+(\mathbb{S}^3)$.

Theorem 5.5.6. ([**Pl**], Thm 2) *Let W be a smooth compact 4-manifold with boundary $Y = \partial W$. Let J_1, J_2 be two Stein structures on W that induce $Spin^{\mathbb{C}}$-structures $\boldsymbol{s}_1$,*

$\boldsymbol{s}_2$ on W and contact structures ξ_1, ξ_2 on Y. If the $Spin^{\mathbb{C}}$-structures $\boldsymbol{s}_1$, $\boldsymbol{s}_2$ are not isomorphic, then the contact invariants ξ_1, ξ_2 are distinct elements of $\widehat{HF}(-Y)$.

OR

Theorem 5.5.7. ([**LM**], Thm 1.2) *Let W be a smooth compact 4-manifold with boundary $Y = \partial W$. Let J_1, J_2 be two Stein structures on W with associated $Spin^{\mathbb{C}}$-structures $\boldsymbol{s}_1$, $\boldsymbol{s}_2$. If the induced contact structures ξ_1, ξ_2 on $Y = \partial W$ are isotopic then $\boldsymbol{s}_1$, $\boldsymbol{s}_2$ are isomorphic and in particular have the same c_1.*

Note that Proposition 5.5.5 implies Proposition 5.5.6.

Let $\bar{\xi}$ denote the contact structure on Y obtained from ξ by inverting the orientation of the planes. This operation is compatible with the conjugation of the *Spin*$^{\mathbb{C}}$-structures, in fact $\mathbf{t}_{\bar{\xi}} = \bar{\mathbf{t}_{\xi}}$ (cf. [**Gh1**]).

Theorem 5.5.8. ([**Gh2**], Thm 2.10) *Let (Y, ξ) be a contact three-manifold, then $c(\bar{\xi}) = \mathcal{J}(c(\xi))$.*

5.6 Proof of Non Stein Fillability of $(-\Sigma\,(2, 2g+1, 2(2g+1)n-1)\,, \mu_0)$

In this section we prove the following last part of our main theorem.

Recall that μ_0 denote the tight contact structure on $-\Sigma\,(2, 2g+1, 2(2g+1)n-1)$ obtained by the Legendrian surgery on (M_g, ξ_g) along S.

Theorem 5.6.1. *The contact manifolds $(-\Sigma\,(2, 2g+1, 2(2g+1)n-1)\,, \mu_0)$ are not Stein fillable for any $n > 1$ and odd.*

Proposition 5.6.2. *Let ξ be a contact structure on a 3-manifold Y which is isotopic to its conjugate $\bar{\xi}$. If (W, J) is a Stein filling of ξ and $\boldsymbol{s}$ is its canonical Spin$^{\mathbb{C}}$-structure, then $\boldsymbol{s}$ is isomorphic to its conjugate $\bar{\mathbf{s}}$.*

Proof. ([**Gh1**], Prop. 4.1) If (W, J) is a Stein filling of ξ then $(W, -J)$ is a Stein filling of $\bar{\xi}$, and $\bar{\mathbf{s}}$ is the canonical *Spin*$^{\mathbb{C}}$-structure of $(W, -J)$. Since ξ is isotopic to its conjugate $\bar{\xi}$, $\mathbf{s}\,|_Y$ is isotopic to $\bar{\mathbf{s}}\,|_Y$. Assume $\mathbf{s}$ is not isomorphic to its conjugate $\bar{\mathbf{s}}$. Puncture W and regard it as a cobordism between $-Y$ and $\mathbb{S}^3$.

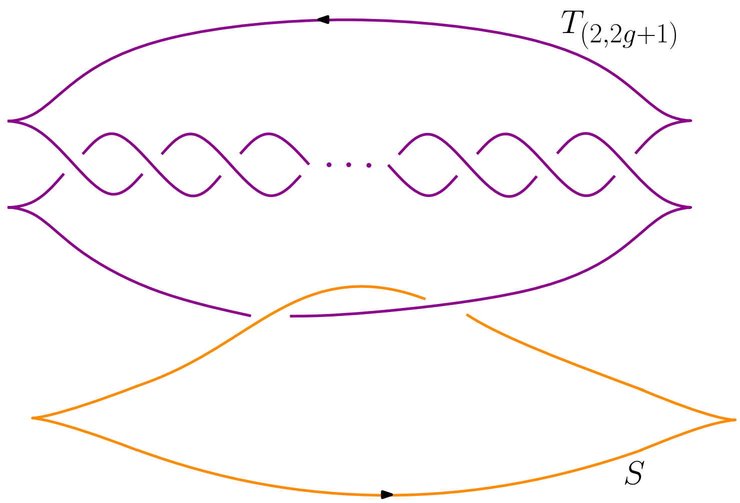


Figure 5.7: Contact structure μ_0 on the three-manifold $-\Sigma\left(2, 2g+1, 2(2g+1)n-1\right)$

Since ξ is isotopic to its conjugate $\bar{\xi}$, we have

$$F_{W,\mathbf{s}}\left(c(\xi)\right) = F_{W,\mathbf{s}}\left(c(\bar{\xi})\right) = F_{W,\mathbf{s}}\left(\mathcal{J}\left(c(\xi)\right)\right) = \mathcal{J}\left(F_{W,\bar{\mathbf{s}}}\left(c(\xi)\right)\right).$$

On the other hand, $F_{W,\mathbf{s}}\left(c(\xi)\right) \neq 0$ by Theorem 5.5.5. Thus, $F_{W,\bar{\mathbf{s}}}\left(c(\xi)\right) \neq 0$, which contradict first part of Theorem 5.5.5. Therefore, $\mathbf{s}$ is isomorphic to its conjugate $\bar{\mathbf{s}}$. $\square$

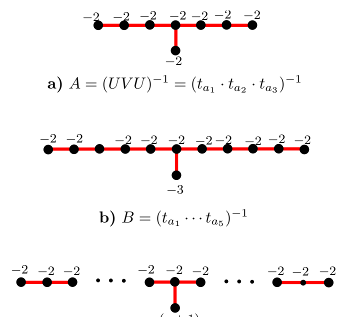
Lemma 5.6.3. *The* 3*-dimensional homotopy invariant of* μ_0 *is* $d_3\left(\mu_0\right) = \dfrac{-(2g+1)}{2}$ *and therefore the contact invariant* $c\left(\mu_0\right)$ *belongs to* $\widehat{HF}_{(+g)}\left(\Sigma\left(2, 2g+1, 2(2g+1)n-1\right)\right)$.

Proof. First, we will compute $d_3\left(\mu_0\right)$ using the symplectic filling described by Figure 5.7, say $\left(W_0, J_0\right)$.

We can see by Figure 5.7 that $\left\langle c_1\left(W_0, J_0\right), T_{(2,2g+1)}\right\rangle = rot\left(T_{(2,2g+1)}\right) = 0$ and $\left\langle c_1\left(W_0, J_0\right), S\right\rangle = rot(S) = 0$, where S is the section described in section 3.1. Hence, $c_1\left(W_0, J_0\right) = PD\left(0 \cdot [T_{(2,2g+1)}]\right) = 0$.

Euler characteristic of W_0 is computed to be $\chi\left(W_0\right) = 2g+1$, since it has only one 0-handle and $2g$ 2-handles.

Next, we will prove that $\sigma\left(W_0\right) = 0$. Consider the well known Lefschetz fibration with



a) $A = (UVU)^{-1} = (t_{a_1} \cdot t_{a_2} \cdot t_{a_3})^{-1}$

b) $B = (t_{a_1} \cdots t_{a_5})^{-1}$

c) $C = (t_{a_1} \cdots t_{a_{2g+1}})^{-1}$

Figure 5.8: plumbing diagrams

total space $\mathbb{CP}^2 \# (4g+5)\overline{\mathbb{CP}}^2$ and monodromy

$$\left(t_{a_1} \cdots t_{a_{2g-1}} t_{a_{2g}} t_{a_{2g+1}}^2 t_{a_{2g}} t_{a_{2g-1}} \cdots t_{a_1}\right)^2 = 1.$$

Note that this monodromy consists of two copies of $t_{a_{2g+1}} t_{a_{2g}} \cdots t_{a_1}$ and two copies of $t_{a_1} \cdots t_{a_{2g}} \cdot t_{a_{2g+1}}$, which is the monodromy of the Lefschetz fibration we described in section 3 with one additional Dehn twist $t_{a_{2g+1}}$.

$\mathbf{g = 1}$ **case:** $\mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2$ Observe that monodromy of this Lefschetz fibration can be think of as

$(t_{a_1} t_{a_2} t_{a_3})\,(t_{a_3} t_{a_2} t_{a_1})\,(t_{a_1} t_{a_2} t_{a_3})\,(t_{a_3} t_{a_2} t_{a_1}) = (t_{a_1} t_{a_2} t_{a_3}) \cdot A = 1.$

The plumbing diagram of $A = (t_{a_1} t_{a_2} t_{a_3})^{-1} = (UVU)^{-1}$ is shown as in Figure 5.8a which is of type III^* ([**KM2**]). Clearly $\sigma(A) = -8$, since it consists of 8 sphere of self intersection -2. Hence, $\sigma\,(t_{a_1} t_{a_2} t_{a_3}) = 0$ since $\sigma(\mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2) = -8$.

$\mathbf{g = 2}$ **case:** $\mathbb{CP}^2 \# 13\overline{\mathbb{CP}}^2$ Observe that monodromy of this Lefschetz fibration can be written as $(t_{a_1} t_{a_2} \cdots t_{a_5})\,(B) = 1$. Figure 5.8b is the plumbing diagram of $B = (t_{a_1} \cdots t_{a_5})^{-1}$ which is of type B and 19 in the table given in [**Ogg**]. Clearly $\sigma(B) = -12$, since it consists of 12 spheres. Hence, $\sigma\,(t_{a_1} t_{a_2} t_{a_3}) = 0$, since $\sigma(\mathbb{CP}^2 \# 13\overline{\mathbb{CP}}^2) = -12$.

Similarly we can generalize this for $g \geq 3$ as well.

$\sigma(C) = \sigma\left((t_{a_1} \cdots t_{a_{2g+1}})^{-1}\right) = -4(g+1)$, since it consists of $4g+4$ spheres (see Figure 5.8c). Hence, $\sigma\left(t_{a_1} \cdot t_{a_2} \cdots t_{a_{2g+1}}\right) = 0$, since $\sigma\left(\mathbb{CP}^2 \# (4g+5)\overline{\mathbb{CP}}^2\right) = -4(g+1)$.

Hence, we conclude that

$$d_3(\mu_0) = \frac{c_1^2(J_0) - (2\chi(W_0, J_0) + 3\sigma(W_0, J_0))}{4} = \frac{-(2g+1)}{2} = -g - \frac{1}{2}.$$

It is a well known fact that for a closed, connected, oriented 3-manifold Y the Heegaard Floer homology groups of Y and the Seiberg-Witten Floer homology groups of Y are isomorphic. Moreover, the relative grading is preserved on both side by this isomorphism (see KLT Main Theorem or [**CGH**]). The Seibert-Witten Heegaard Floer homology groups are graded with the homotopy classes on the 3-manifold Y which corresponds to the 3-dimensional homotopy invariant d_3 in the Heegaard Floer Homology. Therefore, the contact invariant $c(\mu_0)$ is an element of $\widehat{HF}_{(+g)}(\Sigma(2, 2g+1, 2(2g+1)n-1))$, since the degree of $c(\mu_0)$ is $-d_3(\xi) - \frac{1}{2} = -\left(\frac{-(2g+1)}{2}\right) - \frac{1}{2} = g.$ □

Lemma 5.6.4. *For any odd number $n > 1$,*

$$\widehat{HF}_{(+g)}(\Sigma(2, 2g+1, 2(2g+1)n-1)) \cong \begin{cases} (\mathbb{Z}_2)^{n-1} & \textit{if} \quad g = \textit{odd}; \\ (\mathbb{Z}_2)^{n+1} & \textit{if} \quad g = \textit{even}. \end{cases}$$

Proof. Tweedy have computed the groups $HF^+(-\Sigma(2, 2g+1, 2(2g+1)n-1))$ in [**Tw**]. Looking at the graded roots of these Heegaard Floer Homology groups (see Figure 5.9 and Figure 5.10 for the cases $g = 2$ and $g = 3$ respectively), we can compute the kernel $ker(U)$ and $Im(U)$.

Then using the formulas

$$Coker(U) = HF^+/Im(U)$$

and

$$\widehat{HF}(-\Sigma) = Ker(U) \oplus Coker(U)[-1]$$

(cf. [**KL**]), we can compute $\widehat{HF}(-\Sigma(2, 2g+1, 2(2g+1)n-1))$ and we conclude that

$$\widehat{HF}_{(-g)}(-\Sigma(2, 2g+1, 2(2g+1)n-1)) \cong \begin{cases} (\mathbb{Z}_2)^{n-1} \text{ if } g = \textit{odd} \\ (\mathbb{Z}_2)^{n+1} \text{ if } g = \textit{even} \end{cases}$$

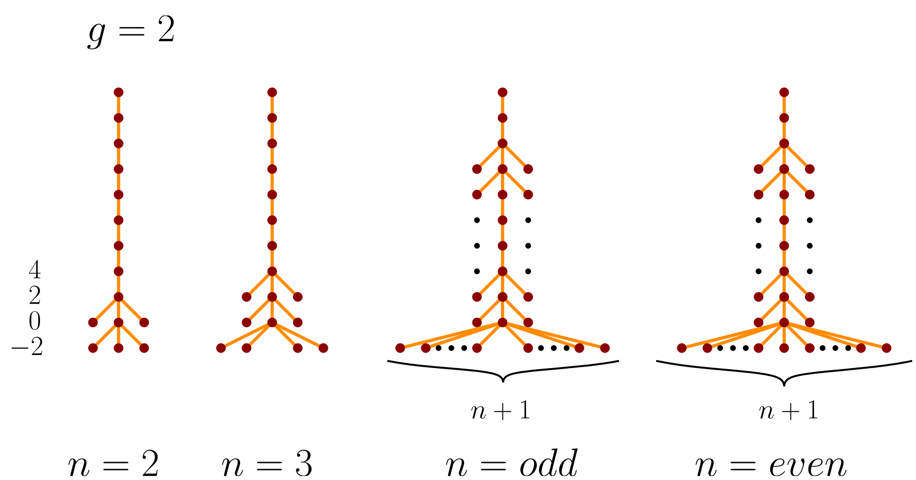


Figure 5.9: graded roots for $HF^+_{(-2)}(-\Sigma(2,5,10n-1))$

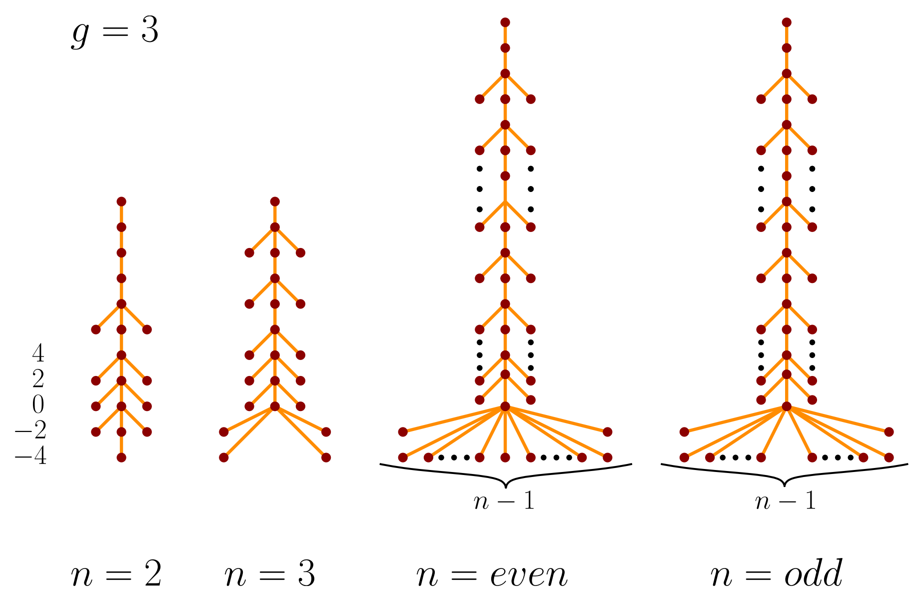


Figure 5.10: graded roots for $HF^+_{(-4)}(-\Sigma(2,7,14n-1))$

Then, we can finish the proof using the isomorphism (cf. Proposition 2.2, [**LS3**])

$$\widehat{HF}_{(d)}(Y) \cong \widehat{HF}_{(-d)}(-Y),$$

where $d(-\Sigma(2, 2g+1, 2(2g+1)n-1)) = -2\left\lceil \frac{g}{2} \right\rceil$. □

Remark. In this example we will compute $\widehat{HF}_{(+g)}(\Sigma(2, 2g+1, 2(2g+1)n-1))$ for $g = 3$ and $n = 3$. So, we are using the $(2,7)$-torus knot $T_{(2,7)}$ and $c(\mu_0) \in \widehat{HF}_{(+3)}(\Sigma(2,7,41))$ since it has grading $-d_3(c(\mu_0)) - \dfrac{1}{2} = g = 3$.

$$S_{(2,7)} = \{s \in \mathbb{N} \mid s = 2a + 7b \ for\ some\ a, b \in \mathbb{N}\}$$

$$\alpha_i = number\ of\ elements\ in\ \{s \notin S \mid s > i\}$$

$$\alpha_0 = 3 > \alpha_1 = \alpha_2 = 2 > \alpha_3 = \alpha_4 = 1 > \alpha_k = 0 \ for\ all\ k \geq 5$$

$$\delta = \frac{(p-1)(q-1)}{2} = \frac{(2-1)(7-1)}{2} = 3\ and$$

$$d(-\Sigma(2,7,41)) = -2\alpha_{\delta-1} = -2\alpha_2 = -4.$$

Now, we will first compute $HF^+(-\Sigma(2,7,41))$ using Tweedy's computation [**Tw**].
$d(3,i) = \left\lceil \frac{i}{3} \right\rceil \left(\{\frac{i-1}{3}\}3 + i - 1\right)$ where $1 < i < 6$ and $\{x\} = x - \left\lfloor x \right\rfloor$.
Hence, we get $d(3,1) = 0$, $d(3,2) = 2$, $d(3,3) = 4$, $d(3,4) = 6$, $d(3,5) = 10$, $d(3,6) = 14$ and

$$HF^+_{odd}(-\Sigma(2,7,41)) = 0$$

$$HF^+_{even}(-\Sigma(2,7,41)) = \mathcal{T}_{-4}(2)^{\oplus 2} \oplus \mathcal{T}_{-2}(1)^{\oplus 2} \oplus \mathcal{T}_0(1)^{\oplus 2} \oplus \mathcal{T}_2(1)^{\oplus 2} \oplus \mathcal{T}_4(1)^{\oplus 2} \oplus \mathcal{T}_8(1)^{\oplus 2} \oplus \mathcal{T}_{12}(1)^{\oplus 2}.$$

$$Ker(U) = (\mathbb{Z}_{(-4)})^2 \oplus (\mathbb{Z}_{(-2)})^2 \oplus (\mathbb{Z}_{(0)})^2 \oplus (\mathbb{Z}_{(2)})^2 \oplus (\mathbb{Z}_{(4)})^2 \oplus (\mathbb{Z}_{(8)})^2 \oplus (\mathbb{Z}_{(12)})^2$$

$$Im(U) = \mathcal{T}^+_{-4}$$

$$\begin{aligned} coker(U) &= HF^+(-\Sigma(2,7,41))/Im(U) \\ &= \mathbb{Z}_{(-4)} \oplus (\mathbb{Z}_{(-2)})^2 \oplus (\mathbb{Z}_{(0)})^2 \oplus (\mathbb{Z}_{(2)})^2 \oplus (\mathbb{Z}_{(4)})^2 \oplus (\mathbb{Z}_{(8)})^2 \oplus (\mathbb{Z}_{(12)})^2 \end{aligned}$$

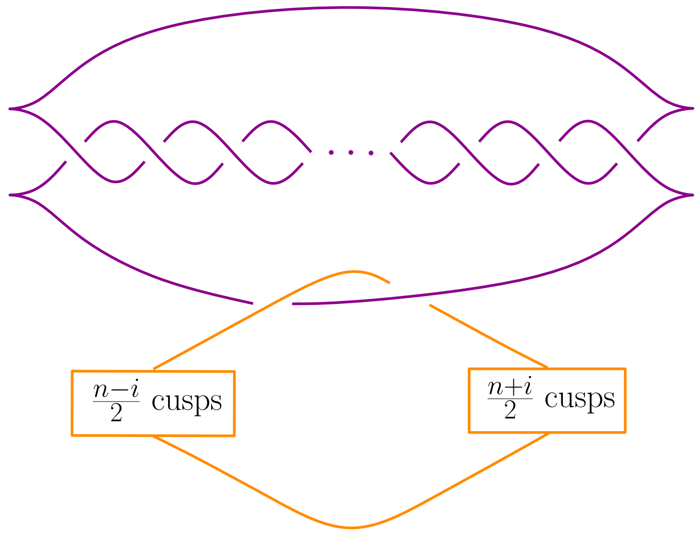


Figure 5.11: Legendrian surgery presentation of $-\Sigma\,(2, 2g+1, 2(2g+1)n-1)$ for $i \in \mathcal{P}_n^*$

$$\begin{aligned}
\widehat{HF}\,(-\Sigma(2,7,41)) &= Ker(U) \oplus Coker(U)[-1] \\
&= (\mathbb{Z}_{(-4)})^2 \oplus (\mathbb{Z}_{(-2)})^2 \oplus (\mathbb{Z}_{(0)})^2 \oplus (\mathbb{Z}_{(2)})^2 \oplus (\mathbb{Z}_{(4)})^2 \oplus (\mathbb{Z}_{(8)})^2 \oplus (\mathbb{Z}_{(12)})^2 \\
&\oplus \mathbb{Z}_{(-5)} \oplus (\mathbb{Z}_{(-3)})^2 \oplus (\mathbb{Z}_{(-1)})^2 \oplus (\mathbb{Z}_{(1)})^2 \oplus (\mathbb{Z}_{(3)})^2 \oplus (\mathbb{Z}_{(7)})^2 \oplus (\mathbb{Z}_{(11)})^2
\end{aligned}$$

So, $\widehat{HF}_{(-4)}\,(-\Sigma(2,7,41)) \cong (\mathbb{Z}_2)^2$.

Then using the orientation reversing formula $\widehat{HF}_{(d)}\,(Y) \cong \widehat{HF}_{(-d)}\,(-Y)$, we conclude that $\widehat{HF}_{(4)}\,(\Sigma(2,7,41)) \cong (\mathbb{Z}_2)^2$

Remark. In the proofs and computations for the rest of the paper, we will consider g is odd case but they can be modified very easily for g is even case as well.

Now, consider the set $\mathcal{P}_n^* = \{-n+2, -n+4, \cdots, n-4, n-2\}$, where $n \in \mathbb{N}$ is odd and $n > 1$. Note that, if n is odd, then $0 \notin \mathcal{P}_n^*$. Let μ_i, $i \in \mathcal{P}_n^*$, denote the contact structure on $-\Sigma\,(2, 2g+1, 2(2g+1)n-1)$ obtained by the Legendrian surgery on the Legendrian link in the standard $\mathbb{S}^3$ shown in Figure 5.11.

Remark. Here, we will always think of $\dfrac{n-i}{2}$ and $\dfrac{n+i}{2}$ as integers.

Assume g is even so that we don't confuse μ_0 defined in Section 2 with μ_i where $i \in \mathcal{P}_n^*$.

Lemma 5.6.5. *μ_i, $i \in \mathcal{P}_n^*$, are all Stein fillable.*

Proof. (cf. [**El2**] or [**El4**], Theorem 6.1) Recall that an almost complex $2n$-manifold admits a Stein structure if and only if

- it admits a handlebody decomposition without any handles of index $> n$.
- when $n = 2$, the 2-handles must be attached along Legendrian knots, with framing one negative twist more than the contact framing.

(cf. [**Go1**], thm 0.1, thm 1.3) A 4-manifold with boundary admits a Stein structure if and only if it is given by a 2-handlebody on a Legendrian link in a standard form, with framing coefficients $tb - 1$ where $tb(K) = w(K) - \#\{$ left cusps $\}$. □

Lemma 5.6.6. *The contact structures μ_i, $i \in \mathcal{P}_n^*$, are pairwise homotopic with 3-dimensional homotopy invariant $d_3(\mu_i) = \dfrac{-(2g+1)}{2}$.*

Proof. (cf. [**GS**]) Take the Stein surface (W_i, J_i) defined by the link diagram in Figure 5.11. Recall that $rot(K) = \frac{1}{2}(D - U)$, where D is the number of down cusps and U is the number of up cusps in the front projection. Looking at Figure 5.12, we can compute that

$$rot\left(T_{(2,2g+1)}\right) = \frac{1}{2}(2-2) = 0$$

$$rot(\sigma) = \frac{1}{2}\left((n-i+n-i-1) - (n+i+n+i-1)\right) = \frac{1}{2}(-4i) = -2i.$$

Consequently, $W_i \approx N_g(n)$ for all $1 \leq i \leq n-2$ $(n \geq 3)$. Now,

$$\left\langle c_1(W_i, J_i), T_{(2,2g+1)}\right\rangle = rot\left(T_{(2,2g+1)}\right) = 0$$

where $T_{(2,2g+1)}$ denotes the homology class of the fiber in $N_g(n)$, and

$$\langle c_1(W_i, J_i), \sigma\rangle = rot(\sigma) = -2i.$$

Hence,

$$c_1(W_i, J_i) = PD\left((-2i)[T_{(2,2g+1)}]\right).$$

Therefore, $c_1^2(J_i) = 4i^2[T_{(2,2g+1)}]^2 = 0$.

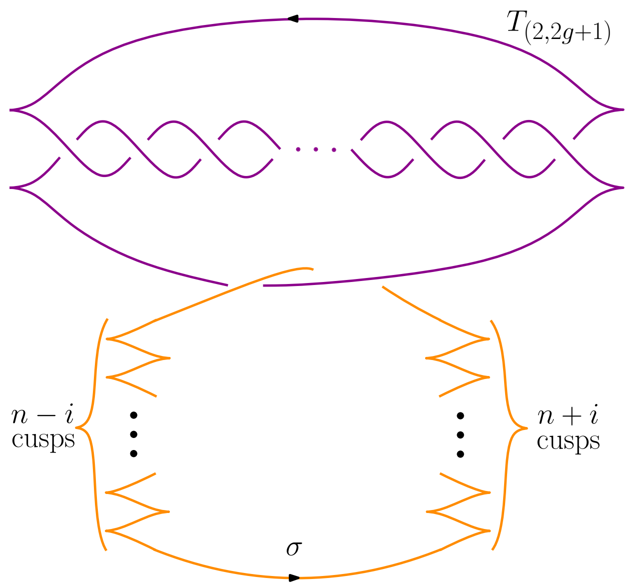


Figure 5.12: Contact structures μ_i on the three-manifold $-\Sigma\,(2, 2g+1, 2(2g+1)n-1)$ for $i \in \mathcal{P}_n^*$

Now, note that the Stein fillings W_i all have the same topological invariants with W_0 and we have already computed its signature and euler characteristic as $\sigma\,(W_0) = 0$ and $\chi\,(W_0) = 2g+1$ in the proof of Lemma 5.6.3. Hence,

$$d_3\,(\mu_i) = \frac{c_1^2\,(J_i) - (2\chi(W_0) + 3\sigma(W_0))}{4} = \frac{-(2g+1)}{2}, \text{ for all } i \in \mathcal{P}_n^*.$$

□

For the proof of the following theorem, the reader is referred to Section 5 in [**LS3**].

Theorem 5.6.7. *The contact invariants $c(\mu_1)$, $c(\mu_3)$, ... , $c(\mu_{n-2})$ are linearly independent.*

Proposition 5.6.8. *The contact invariants $c(\mu_i)$ generate $\widehat{HF}_{(+g)}\,(\Sigma\,(2, 2g+1, 2(2g+1)n-1))$ where $i \in \mathcal{P}_n^*$.*

Proof. Let $\mathbf{s}_i$ be the $Spin^{\mathbb{C}}$-structure corresponding to μ_i. If $H_1\,(M;\mathbb{Z})$ has no 2-torsion, and in particular, if M is simply connected, then for each $x \in H^2\,(M;\mathbb{Z})$, there is a unique $Spin^{\mathbb{C}}$-structure $\mathbf{s}_i$ satisfying $PD\left(c_1\left(\mathbf{s}_i\right)\right) = x$ (see [**FS4**]). (In particular, for all $x \in H^2(M;\mathbb{Z})$, $c_1\langle x, \mathbf{s}_i\rangle = c_1(\mathbf{s}_i) + 2x$.) On the other hand, in the proof of Lemma 5.6.6, we have computed $c_1(W, J_i) = PD\left((-2i)[T_{(2,2g+1)}]\right)$. Therefore, the $Spin^{\mathbb{C}}$-structures are not isomorphic. Hence, the contact invariants $c(\mu_i)$ are distinct by Theorem 5.5.6. Moreover, by Theorem 5.5.5 the contact invariants $c(\mu_i)$ are primitive elements of $\widehat{HF}_{(+g)}(\Sigma(2, 2g+1, 2(2g+1)n-1))$ and $c(\mu_i)$ span $\widehat{HF}_{(+g)}(\Sigma(2, 2g+1, 2(2g+1)n-1))$. □

Proposition 5.6.9. *The contact structure $\bar{\mu}_i$ obtained from μ_i by conjugation is isotopic to μ_{-i} for $i \in \mathcal{P}_n^*$.*

Proof. Let S_+ and S_- denote the operations of positive and negative stabilization ([**Et1**]. Let μ_i, $i \in \mathcal{P}_n^*$, denote the contact structure on $-\Sigma(2, 2g+1, 2(2g+1)n-1)$ obtained by Legendrian surgery on (M_g, ξ_g) along the Legendrian knot $S_+^{(n-1+i)/2} S_-^{(n-1-i)/2}(S)$. Recall that performing Legendrian surgery on a stabilization of S is equivalent to performing Legendrian surgery on a stabilization of a meridian of $(2, 2g+1)$ torus knot $T_{(2,2g+1)}$. (cf. [**Gh1**], pg. 171)

For any $g \in \mathbb{N}^+$ (M_g, ξ_g) is isotopic to $\left(M_g, \overline{\xi}_g\right)$ with the isotopy induced by translation in the t direction in the cover $\Sigma_g \times \mathbb{R}$. Therefore it fixes S. Consider $S_+^{(n-1+i)/2} S_-^{(n-1-i)/2}(S)$ as a Legendrian knot $\overline{S_+^{(n-1+i)/2} S_-^{(n-1-i)/2}(S)}$ in $\left(M_g, \overline{\xi}_g\right)$. Then, $\overline{S_+^{(n-1+i)/2} S_-^{(n-1-i)/2}(S)}$ is Legendrian isotopic to $S_+^{(n-1-i)/2} S_-^{(n-1+i)/2}(S)$, since changing the orientation of the planes changes the positive stabilizations into negative ones and vice versa. Therefore, inverting an orientation of planes transforms Legendrian surgery on $S_+^{(n-1+i)/2} S_-^{(n-1-i)/2}(S)$ into Legendrian surgery on $S_+^{(n-1-i)/2} S_-^{(n-1+i)/2}(S)$ (cf. [**Gh1**], Prop.3.8, Pg. 171). □

Claim. μ_0 is isotopic to its conjugate $\bar{\mu}_0$

In the rest of this section, we assume this claim is true.

Lemma 5.6.10. *The contact structures μ_i, $i \in \{1, 3, \cdots n-2\}$, are pairwise non-isotopic.*

Proof. Assume n is odd.

Case 1: Let $1 \leq k, k^{'} \leq n-2$. Assume that μ_k is isotopic $\mu_{k'}$. Then, by Theorem 4.2 in [**LM**], either $k = k^{'}$ or $k = n-1-k^{'}$. However, in the latter case they have different 3-dimensional invariants. i.e. $d_3\left(\mu_k\right) \neq d_3\left(\mu_{k'}\right)$. Therefore, μ_k is not isotopic to $\mu_{k'}$ if $k \neq k^{'}$.

Case 2: Now consider the couple μ_k and μ_{-k} , where $k \in \mathcal{P}_n^*$. Note first that $k \neq -k$ and $k \neq n-1-(-k)$ (since $n \neq 1$). Hence, ξ_g^k is not isomorphic to ξ_g^{-k} by Theorem 4.2 in [**LM**]. Thus, μ_k is not isotopic to μ_{-k}.

Case 3: Finally, we will show that μ_0 is not isotopic to μ_k for any $k \in \mathcal{P}_n^*$. Assume μ_0 is isotopic to μ_k for some $k \in \mathcal{P}_n^*$. Inverting the orientation of the contact planes, we obtain that $\bar{\mu}_0$ is isotopic to $\bar{\mu}_k$. Applying the previous proposition, we get μ_0 is isotopic to μ_{-k}, which implies μ_k is isotopic to μ_{-k}. However, this contradicts the Case 2.

See [**LS3**] Corollary 5.2 for an alternative proof. □

Lemma 5.6.11. *If n is odd, then $Fix\left(\mathcal{J}\right) \subset \widehat{HF}_{(+g)}\left(\Sigma\left(2, 2g+1, 2(2g+1)n-1\right)\right)$ is generated by $c\left(\mu_i\right) + c\left(\mu_{-i}\right)$, $i = 1, 3, \cdots, n-2$.*

Proof. Let $x \in Fix\left(\mathcal{J}\right)$ be a fixed point. Then $x \in \widehat{HF}_{(+g)}\left(\Sigma(2, 2g+1, 2(2g+1)n-1)\right)$. We also know by Theorem 5.6.8 that $\widehat{HF}_{(+g)}\left(\Sigma(2, 2g+1, 2(2g+1)n-1)\right) = \langle c(\mu_i)\rangle$. Therefore, we can express x as a linear combination like

$$x = \sum_{i \in \mathcal{P}_n^*} \alpha_i c(\mu_i) \qquad where \qquad \alpha_i = 0, 1.$$

Acting $\mathcal{J}$ on both side of this expression, we get

$$\mathcal{J}x = \sum_{i \in \mathcal{P}_n^*} \alpha_i \mathcal{J} c(\mu_i) = \sum_{i \in \mathcal{P}_n^*} \alpha_i c(\bar{\mu}_i) = \sum_{i \in \mathcal{P}_n^*} \alpha_i c(\mu_{-i}).$$

So, we conclude that

$$\sum_{i \in \mathcal{P}_n^*} \alpha_i c(\mu_i) = \sum_{i \in \mathcal{P}_n^*} \alpha_i c(\mu_{-i})$$

since $x = \mathcal{J}x$ being x a fixed point.

Then, $\alpha_{-i}c(\mu_{-i}) = \alpha_i c(\mu_{-i})$ which implies $\alpha_{-i} = \alpha_i$. Thus,

$$\begin{aligned} x &= \alpha_{-n+2}c(\mu_{-n+2}) + \cdots + \alpha_{-1}c(\mu_{-1}) + \alpha_1 c(\mu_1) + \cdots + \alpha_{n-4}c(\mu_{n-4}) + \alpha_{n-2}c(\mu_{n-2}) \\ &= \alpha_{n-2}c(\mu_{-n+2}) + \cdots + \alpha_1 c(\mu_{-1}) + \alpha_1 c(\mu_1) + \cdots + \alpha_{n-4}c(\mu_{n-4}) + \alpha_{n-2}c(\mu_{n-2}) \\ &= \alpha_{n-2}\left(c(\mu_{-n+2}) + c(\mu_{n-2})\right) + \cdots + \alpha_1\left(c(\mu_{-1}) + c(\mu_1)\right) \\ &= \sum_{i=1}^{n-2} \alpha_i \left(c(\mu_{-i}) + c(\mu_i)\right) \end{aligned}$$

□

Proof. **(proof of Theorem 5.6.1)**

Assume that (W, J) is a Stein filling of $(-\Sigma(2, 2g+1, 2(2g+1)n-1), \mu_0)$ with the canonical $Spin^c$-structure $\mathbf{s}$. Then, by Proposition 5.6.2, $\mathbf{s}$ is isomorphic to its conjugate $\bar{\mathbf{s}}$ since μ_0 is isotopic to its conjugate $\bar{\mu}_0$ by Proposition 5.6.9. So, $c(\mu_0) = c(\bar{\mu}_0)$. Thus, $\mu_0 \in Fix(\mathcal{J})$ since $c(\bar{\mu}_0) = \mathcal{J}(\mu_0)$ by Theorem 5.5.8. Therefore;

$$c(\mu_0) = \sum_{i=1}^{n-2} \alpha_i \left(c(\mu_{-i}) + c(\mu_i)\right)$$

Since the map

$$\widehat{HF}\left(\Sigma(2, 2g+1, 2(2g+1)n-1)\right) \longrightarrow HF^+\left(\Sigma(2, 2g+1, 2(2g+1)n-1)\right)$$

sends $c(\mu_i)$ to $c^+(\mu_i)$, we get

$$c^+(\mu_0) = \sum_{i=1}^{n-2} \alpha_i \left(c^+(\mu_i) + c^+(\mu_{-i})\right).$$

Now, consider W as a cobordism from $\Sigma(2, 2g+1, 2(2g+1)n-1)$ to $\mathbf{S}^3$ by puncturing it. Then,

$$F^+_{W,s}\left(c^+(\mu_{-i}) + c^+(\mu_i)\right) = F^+_{W,s}\left(c^+(\mu_{-i})\right) + F^+_{W,s}\left(c^+(\mu_i)\right).$$

Also note that,

$$\begin{aligned}
F^+_{W,s}\left(c^+(\mu_{-i})\right) &= F^+_{W,s}\left(c^+(\bar{\mu}_i)\right) && \textit{by Propositon } 5.6.9 \\
&= F^+_{W,s}\left(\mathcal{J}c^+(\mu_i)\right) && \textit{by Propositon } 5.5.8 \\
&= \mathcal{J}\left(F^+_{W,\bar{s}}(c^+(\mu_i))\right) && \textit{by naturality of } \mathcal{J} \\
&= F^+_{W,\bar{s}}\left(c^+(\mu_i)\right) && \textit{by triviality of the } \mathcal{J} \textit{ action on } HF^+\left(\mathbf{S}^3\right) \\
&= F^+_{W,s}\left(c^+(\mu_i)\right) && \textit{since } \mathbf{s} \textit{ is isomorphic to } \bar{\mathbf{s}}
\end{aligned}$$

Therefore,

$$F^+_{W,s}\left(c^+(\mu_{-i}) + c^+(\mu_i)\right) = 2F^+_{W,s}\left(c^+(\mu_i)\right) = 0.$$

Hence,

$$F^+_{W,s}\left(c^+(\mu_0)\right) = \sum_{i=1}^{n-2} \alpha_i F^+_{W,s}\left(c^+(\mu_{-i}) + c^+(\mu_i)\right) = 0$$

which contradict Theorem 5.5.5(2). $\square$

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