Applications of geometric techniques in Coxeter-Catalan combinatorics

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Expressions of Gratitude

It is often the case that overbearing declarations saturate this section. What's worse, honest feelings of respect and admiration (towards a mentor) might appear pompous, or pretentious. I have, however, enjoyed five deeply fulfilling years in the University of Minnesota, and I choose to express this emphatically:

I would like to first thank my advisor Vic Reiner, who has been the most important champion of this work. It was a privilege to enjoy his guidance, his sheer competence as a mathematician, but also his all around congeniality. Above all, he has, in a sense, allowed me to fulfill perhaps the only explicit (mathematical) dream I have had; to work on things I truly consider beautiful.

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Dedication

To the dark and cold Minnesotan winters,
to the warmth of friends that came with them.

Abstract

In the seminal work [Bes15], Bessis gave a geometric interpretation of the noncrossing lattice $NC(W)$ associated to a well-generated complex reflection group W. He used it as a combinatorial recipe to construct the universal covering space of the arrangement complement $V \setminus \bigcup H$, and to show that it is contractible, hence proving the $K(\pi, 1)$ conjecture.

Bessis' work however relies on a few properties of $NC(W)$ that are only known via case by case verification. In particular, it depends on the numerological coincidence between the number of chains in $NC(W)$ and the degree of a finite morphism, the LL map.

We propose a (partially conjectural) approach that refines Bessis' work and transforms the numerological coincidence into a corollary. Furthermore, we consider a variant of the LL map and apply it to the study of finer enumerative properties of $NC(W)$. In particular, we extend work of Bessis and Ripoll and enumerate the so-called "primitive factorizations" of the Coxeter element c. That is, length additive factorizations of the form $c = w \cdot t_1 \cdots t_k$, where w belongs to a prescribed conjugacy class and the t_i's are reflections.

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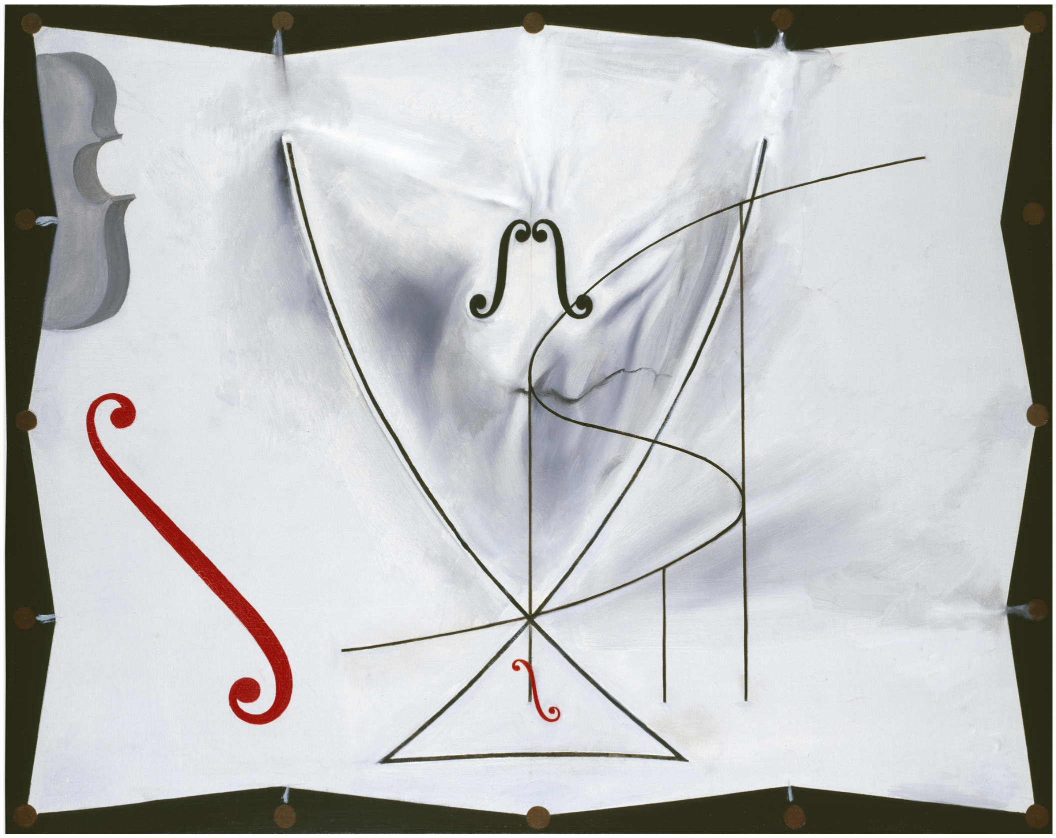


Figure 1: The Swallowtail (1983, Oil on canvas, 73 x 92,2 cm). This is Dali's last work and belongs to the series of catastrophe paintings.

Chapter 1

Introduction

In this Thesis, we are happy to work with questions that lie in the overlap of many modern areas of mathematics, and have received much attention before us. To a combinatorially leaning audience we might present this work as a study of enumerative and structural properties of the noncrossing lattice $NC(W)$ of a (well-generated, complex) reflection group W. The techniques that we use to tackle this problem, however, rely on deep connections with geometric group theory, invariant theory of reflection groups, and the theory of simple singularites.

1.1 The story

In some sense, Hurwitz's formula (1.2) from the 1890's is the *starting point* of our work. However, most of the ideas, mathematical objects, and open questions, that form the *natural context* of this Thesis, first appear in the late 60's and early 70's.

Our (mathematical) heroes of the era include Brieskorn, Arnol'd, and Deligne. It is not unfair to start with Arnol'd's contributions. It might surprise the reader that the origin of much of what follows is in Hilbert's thirteenth problem.

1.1.1 Complements of discriminants and the $K(\pi, 1)$-conjecture

Hilbert [Hil90, Problem XIII] asked for a proof that a solution of the seventh-degree equation

$$z^7 + az^3 + bz^2 + cz + 1 = 0,$$

considered as a function $z(a, b, c)$ of the coefficients, cannot be represented as a superposition[1] of continuous functions of *two* variables. A young Arnol'd (then, only 19 years old) extending work of Kolmogorov, shows [Giv+09b] that, in fact, Hilbert's hypothesis is incorrect.

Arnol'd however, thinks[2] that the *genuine* Hilbert problem is to acknowledge that $z(a, b, c)$ is an *algebraic function*, and thus ask whether it can be represented as a superposition of, again, algebraic functions in two variables. In the late 60's he returns to this problem and studies the algebraic function $z(a_1, \cdots, a_n)$ defined by the equation

$$z^n + a_1 z^{n-1} + \cdots + a_n = 0.$$

At this point, Arnol'd has the (truly novel) idea to use the topology of the complement of the discriminant of $z(\mathbf{a})$ in order to produce an obstruction to the representation of $z(\mathbf{a})$ in terms of algebraic functions of a smaller number of variables. Recall that the discriminant Σ of $z(\mathbf{a})$ is defined as the set of tuples $\mathbf{a} := (a_1, \cdots, a_n)$ such that $z(\mathbf{a})$ has multiple roots.

This work of his [Arn14b] is, as Vassiliev [Vas95] mentions, the initiation of the topological study of discriminant complements. Arnol'd notices first that the complement $\mathbb{C}^n \setminus \Sigma$ is the space $K(\pi, 1)$ for the group $B(n)$ of braids with n strings. He employs this relation in both directions; he investigates the cohomologies of braid groups [Arn14b; Arn14a; Arn14c], but also uses these findings to confirm non-representability results of algebraic functions [Arn14d] (although he does *not* solve this version of the Hilbert problem).

Around the same time Brieskorn considers the fundamental groups of the spaces of regular orbits of finite reflection groups W. We write $V^{\mathrm{reg}} := V \setminus \bigcup H$, where $V \cong \mathbb{C}^n$, for the complement of the reflection arrangement of W, and denote the group by $B(W) := \pi_1(W \backslash V^{\mathrm{reg}})$. Indeed, for the symmetric group $W = \mathfrak{S}_n$, $B(W)$ is the braid group B_n on n strings.

In fact, these regular orbit spaces may also be naturally identified as complements of the discriminant hypersurfaces $\mathcal{H} \subset \mathbb{C}[V]^W \cong \mathbb{C}^n$ of W (see Section 3.2). After

[1] This is a tricky concept; see [Giv+09a] for an explicit definition.
[2] The reader might consult Arnol'd's own recollection in [Arn06].

Tits' suggestion[3] , Brieskorn proves [Bri71] that the *generalized braid groups* $B(W) = \pi_1(\mathbb{C}^n \setminus \mathcal{H})$ have Artin-like presentations.

He then [Bri73] tries to apply Arnol'd's methods towards computing the cohomologies of these newly defined generalized braid groups. "We hope to be able to compute the cohomologies of these groups" he says, "because of the following conjecture:"

The $K(\pi, 1)$ Conjecture. *[Bri73] The spaces V^{reg} and $W\backslash V^{\mathrm{reg}}$ are Eilenberg-MacLane spaces.*

Brieskorn has already proven the conjecture, in a case by case fashion, for all types but H_3, H_4, E_6, E_7 and E_8. Almost immediately however, Deligne [Del72] destroys the problem by in fact proving uniformly that the complexified complements of *any central simplicial* arrangement are $K(\pi, 1)$.

Deligne considers a certain braid monoid whose combinatorial structure is analogous to the weak Bruhat order lattice, the chains of which correspond to reduced factorizations of the longest element w_0, but in *simple* reflections. He uses that as a combinatorial recipe to construct the universal covering space of V^{reg}. The lattice property, which means that the monoid is Garside, forces the contractibility of the covering space and hence completes the proof.

1.1.2 The Lyashko-Looijenga morphism

Back in the world of simple singularities, Looijenga [Loo74] and Lyashko[4] independently consider the complement of another special variety, namely the *bifurcation locus* $\mathcal{B}_f \subset \mathbb{C}^n$ of (the semiuniversal deformation of) a simple singularity f (of Milnor number n).

This is the set of parameters $\mathbf{u}$ in a deformation space of the function f, for which the deformed function $f_{\mathbf{u}}$ has colliding critical values. It shouldn't be confused with the *discriminant locus* $\mathcal{D}_f$ of f, which is the set of parameters $\mathbf{u}$ for which $f_{\mathbf{u}}$ is still singular at the origin.

Lyashko and Looijenga define a quasi-homogeneous morphism (the *LL* map) that assigns to each parameter $\mathbf{u}$, the *multiset* of critical values of $f_{\mathbf{u}}$. It will be important for us

[3] Tits had in fact first considered [Tit66] these generalized Braid groups as extensions of Coxeter groups.

[4] Although Lyashko didn't publish the work, it was presented by Arnol'd in [Arn74].

later, to note a different geometric interpretation: The LL map records the intersection points of a "vertical" line $L_{\mathbf{u}}$, indexed by the parameter space, with the discriminant $\mathcal{D}_f$ (see Defn. 44).

Its restriction over the complement of the bifurcation locus is a covering map onto the configuration space $\mathrm{Conf}_n(\mathbb{C})$. This shows that the complement $\mathbb{C}^n \setminus \mathcal{B}$ is a $K(\pi, 1)$ space, where π is a subgroup of finite index $\nu = \deg(LL)$ in the braid group B_n with n strings.

Although Looijenga computes this index for all simple singularities (types A,D,E), it is Arnol'd [Arn75, Thm. 11] who first[5] observes that it has a uniform expression in terms of the associated reflection group:

$$\nu = \deg(LL) = \frac{h^n n!}{|W|}.$$

Looijenga realizes that the fibers of the LL map produce geometric generators of the group W which are related to factorizations of the Coxeter element c (although in his context, the Coxeter element corresponds to the monodromy of the function f at the isolated singularity). He conjectures that the subgroup π associated to the covering is the same as the stabilizer, under the Hurwitz action of B_n, of (any) given reduced reflection factorization of c.

He shows, and again we are slightly reinterpreting his work, that this is equivalent to proving that the set of reduced reflection factorizations of c, denoted $\mathrm{Red}_W(c)$, is enumerated by the same number ν:

Looijenga's Conjecture. *The number of reduced reflection factorizations of c equals the degree of the LL map. That is,*

$$|\mathrm{Red}_W(c)| = \deg(LL) = \frac{h^n n!}{|W|}. \tag{1.1}$$

Deligne[6], as Deus ex Machina, appears again [Del] to immediately solve the problem. The proof, however, is case by case. In fact, already in 1974 they use a computer for E_8.

[5] However, Arnol'd is presenting Lyashko's work, so we cannot tell whether it is the latter who first observed the uniform version of the formula.

[6] Although he credits Tits and Zagier for the work.

1.1.3 Bessis' work

The next few decades are no less exciting. The theory of hyperplane arrangements is strengthened [see OT92], and the study of *complex* reflection groups becomes popular [see OS80]. Eventually, the $K(\pi, 1)$-conjecture is phrased for the generality of all complex reflection groups; perhaps first in the Orlik-Terao book as Bessis notes [Bes15, Conj. 0.1].

In the early 2000's, Bessis [Bes06b] introduces the dual braid monoid. Its construction is motivated by the existence of certain finite order automorphisms of the generalized braid groups $B(W)$, induced by the Springer-regular elements of W [Bes06b, Prop. 3.5.1]. Bessis considers presentations of $B(W)$ with *many* generators that are more symmetric (and thus explain the automorphisms).

The dual braid monoid arises then in association with a special Springer-regular element; namely the Coxeter element c. In fact, it further inspires an intrinsic, uniform definition for the noncrossing lattice $NC(W)$, as the interval $[1, c]_{\leq_R}$ in the absolute reflection order $\leq_R$ (see Section 2.3). The noncrossing lattice was first introduced by Reiner [Rei97] as a pictorial generalization of the noncrossing partitions of Kreweras [Kre72], to types B and D[7] . It is enumerated by the W-Catalan numbers (see § 9.4.1).

In the different work [Bes01], but at the same time, Bessis proves the existence[8] of presentations for the braid groups $B(W)$, with special *geometric* generators, also associated to Springer-regular elements. These generators are loops around smooth points of the discriminant $\mathcal{H}$, that all lie in a single complex line L, transverse to $\mathcal{H}$ (it is the *direction* of L that relates this construction with different regular elements).

These two ideas appear to be the point of departure for Bessis' proof of the generalized $K(\pi, 1)$ conjecture [Bes15]. First, he realizes that the dual braid monoid may be defined for well-generated groups W, where the lattice property of $NC(W)$ would force the contractibility of the universal covering space in the same way as in Deligne's proof.

He then decides to construct the universal covering space of the quotient space $W\backslash V^{\text{reg}} = \mathbb{C}^n\backslash\mathcal{H}$ (as opposed to the reflection arrangement complement V^{reg}), where he can use the geometry of the discriminant $\mathcal{H}$. He credits conversations with Kyoji Saito and Frédéric

[7] Although the *correct* definition for type D appears later in [AR04].

[8] This was meant to give an *a priori* justification for the Coxeter-like diagrams for complex reflection groups from [BMR98].

Chapoton (see [Bes15, Acknowledgments]) for this choice. In particular, Chapoton pointed out that the number of chains in the noncrossing lattice $NC(W)$ equals the degree of the LL map, which as we mentioned, produces geometric generators of the group W in a similar way to Bessis' work [Bes01].

1.2 Our contribution

Although Bessis' proof deals with the various combinatorial objects in a uniform way, it relies on some of their properties that are still known to be true only via case by case verification. These are the following:

1. The transitivity of the Hurwitz action of B_n on $\mathrm{Red}_W(c)$ (Prop. 63).
2. The fact that parabolic subgroups of well-generated groups are also well-generated (Prop. 5).
3. The numerological coincidence $|\,\mathrm{Red}_W(c)| = \deg(LL) = \dfrac{h^n n!}{|W|}$ (Corol. 69).
4. The lattice property of $NC(W)$.

All the properties but the third have uniform proofs in the context of *real* reflection groups. Recently, a uniform proof for Weyl groups of the enumeration of $\mathrm{Red}_W(c)$ appeared in [Mic16]; it is built on deep results from the theory of Deligne-Lusztig representations.

The first three of these properties are also necessary for Bessis' Trivialization Theorem (Section 7.3), which is the main object of study for this thesis (and the key ingredient in the proof of the $K(\pi,1)$ conjecture). We postpone the complete statement until §7.3 and provide the reader with a lighter version for now.

Trivialization Theorem. *The elements in the generic fiber of the LL map are in a natural* 1-1 *correspondence with the set* $\mathrm{Red}_W(c)$ *of reduced reflection factorizations of the Coxeter element c.*

Our main *structural* contribution to the theory is an approach towards the proof of the Trivialization Theorem that does not rely on the numerological coincidence above. It starts with a geometric analysis of the behavior of the LL map over 1-dimensional flats, and an analogous combinatorial analysis of noncrossing lines L. The latter borrows ideas heavily from the proof of the strong parking space conjecture for lines (see [ARR15, Prop. 2.13:(ii)]).

Then, we produce an inductive argument, that however still depends on a geometric lemma for the local multiplicities of the LL map (see Section 8.2) for which we have no uniform proof. Notice that our approach, but for this geometric lemma, would provide the first uniform proof of the enumeration formula for $\mathrm{Red}_W(c)$, for all real reflection groups.

Along the way, we recreate a big part of Bessis' proof of the Trivialization Theorem and at times fill in missing details (see Prop. 39 and Remark 40, as well as Corol. 66), while on one occasion we correct a faulty geometric argument (see Prop. 37, Remark 38, and Thm. 51). Hopefully, this might help in making Bessis' work more accessible, as it is our belief that it has not been explored to its full potential.

1.2.1 Enumerative applications of the Trivialization Theorem

Hurwitz was one of the early disciples of Riemann surface theory. Already in 1891 [Hur91] he set forth to classify all geometrically distinct n-sheeted Riemann surfaces over the sphere, with a finite number of branch points. Hurwitz translated this into the following enumeration problem in the symmetric group and solved it:

Theorem. *[Hur91] The set or minimal length factorizations $t_1 \cdots t_{n-1} = (12 \cdots n)$, of the long cycle $c = (12 \cdots n)$ into transpositions t_i, is denoted by $\mathrm{Red}_{\mathfrak{S}_n}(c)$ and enumerated by:*

$$|\,\mathrm{Red}_{\mathfrak{S}_n}(c)| = n^{n-2}. \tag{1.2}$$

More than 80 years later Looijenga was, as we mentioned, aware of the connection between the LL map and factorizations of the Coxeter element c. For that matter, he explains [Loo74, Section 3.7] how one can compare the degree of the LL map with the number of trees with n numbered vertices.

However, it is Arnol'd again, and another 20 years later [Arn96], who first sees the LL map as a possible means to tackling further enumeration questions.[9] After him,

[9] In fact, Arnol'd states [Arn96] that *"The simplest way to prove this theorem of Cayley* [the tree enumeration] *is perhaps to count the multiplicity of the quasi-homogeneous Lyashko-Looijenga mapping"*.

there is a profusion of results where variants of the LL map are considered, in order to enumerate combinatorial objects usually associated with factorizations in the symmetric group $\mathfrak{S}_n$ (see Section 9.5).

In this thesis, we similarly extend Bessis' version of the LL map and apply the Trivialization Theorem to enumerate the so called *primitive factorizations* of the Coxeter element:

Theorem (Section 9)**.** *Let W be a well-generated complex reflection group, acting irreducibly on the space V, and let Z be one of its flats. Then, the number $\mathrm{Fact}_W(Z)$ of reduced block factorizations of c of the form $w \cdot t_1 \cdots t_k = c$ where $V^w = Z$, the t_i's are reflections, and $k = \dim(Z)$, is given by:*

$$\mathrm{Fact}_W(Z) = \frac{h^{\dim(Z)} \cdot \big(\dim(Z)\big)!}{[N_W(Z) : W_Z]}.$$

Here h is the Coxeter number of W, and $N_W(Z)$ and W_Z the setwise and pointwise stabilizers of Z respectively.

Our formula can easily be seen to generalize the formulas of Bessis (1.1) and Hurwitz (1.2), by setting $Z = V$ (see also Remark 101) and by further setting $W = \mathfrak{S}_n$ respectively. It should also be considered as a further generalization of [Rip12, Thm. 4.1], although we use a different approach. In particular, Ripoll's formula only allows Z of codimension 2, and is in a sense less explicit than ours.

To prove the theorem, we first lift the LL morphism to a map $\widehat{LL}$ with domain Z and target a *decorated* configuration space. Then, we compare the degree of the new map $\widehat{LL}$ with the number of primitive factorizations via the Trivialization Theorem. The index $[N_W(Z) : W_Z]$ appears naturally as an overcounting factor.

1.2.2 Overview

In Sections 2 and 3, we build general background on complex reflection groups and generalized braid groups. In Sections 4 and 5, we reproduce Bessis' geometric construction of the Coxeter element and define the labeling map rlbl. The proof of the transversality of the line L_y on the discriminant $\mathcal{H}$ is new (see Remark 38) and fundamental to the theory.

In Sections 6 and 7, we define the LL map and establish its properties, leading to Bessis' proof of the Trivialization Theorem. The presentation follows Bessis' work but includes new Lemmas that we use later on (Corol. 66, Corol. 67), and specifically remarks on the occasions where we fill in missing details (when important) of the original arguments (see above Thm. 51, Corol. 65, and Prop. 79).

In Section 8, we present our approach for a proof of the Trivialization Theorem that does not rely on the numerological coincidence (1.1). The missing geometric lemma is described in §8.2. In Section 9, we present our proof of the formula for primitive factorization of the Coxeter element c.

Finally, each of the Sections 5, 6, and 7, has a special subsection (geometric interludes) which introduces a lot of the necessary algebraic or complex-analytic geometry that is later used. Our purpose for those is to make this work accessible to a Combinatorics audience, that might not be familiar with some of these techniques.

Chapter 2

Complex reflection groups and their invariant theory

Complex reflection groups are *finite* subgroups W of $\mathrm{GL}_n(\mathbb{C})$ that are generated by pseudo-reflections. These are $\mathbb{C}$-linear maps of finite order that fix a codimension 1 subspace of $\mathbb{C}^n$. In other words, the matrix of a pseudo-reflection is similar to a diagonal $n \times n$ matrix that has a single non-identity entry ζ, for which $\zeta^k = 1$ for some k. We say that a complex reflection group W is *irreducible* if there is no non-trivial linear subspace of the ambient space $V \cong \mathbb{C}^n$, stable under the action of W.

Shephard and Todd [ST54] completed the classification of irreducible complex reflection groups W into an infinite three-parameter family $G(m, r, n)$ and 34 exceptional cases. Although there exist abstract group-theoretic isomorphisms between members of this classification, they are of no importance to us. In fact, when we consider such a group W, we are always given a particular embedding $W \leqslant \mathrm{GL}_n(\mathbb{C})$ (up to conjugacy in $\mathrm{GL}_n(\mathbb{C})$).

To any complex reflection group W we associate its reflection arrangement $\mathcal{A}_W$, the elements of which are the fixed spaces of (all) the pseudo-reflections of W (i.e. the reflecting hyperplanes of W). The intersection lattice $\mathcal{L}_W := \mathcal{L}(\mathcal{A}_W)$ contains all possible intersections of hyperplanes in $\mathcal{A}_W$. Its elements are linear subspaces that we call *flats* and denote by $X \in \mathcal{L}_W$.

Complex reflection groups have a beautiful invariant theory. The action of such a group W on the ambient space $V \cong \mathbb{C}^n$ induces an action on the polynomial ring $\mathbb{C}[V] \cong \mathrm{Sym}(V^*)$ (via $(w * f)(v) := f(w^{-1}(v))$). One might then consider the algebra of invariant polynomials:

$$\mathbb{C}[V]^W := \left\{ f \in \mathbb{C}[V] : \ w * f = f, \ \forall w \in W \right\}$$

In the case of the symmetric group $W = \mathfrak{S}_n$, the invariant algebra $\mathbb{C}[V]^{\mathfrak{S}_n}$ is the algebra of symmetric polynomials on n variables. It was already known to Gauss (see [Neu07]) that this algebra is generated by the elementary symmetric polynomials, and that they are algebraically independent. Shephard and Todd [ST54] observed, and later Chevalley [Che55] gave a uniform proof of, the following remarkable generalization:

Theorem (Shephard-Todd-Chevalley)**.** *A finite subgroup W of $GL_n(\mathbb{C})$ is a complex reflection group (i.e. it is generated by pseudo-reflections) if and only if its invariant algebra $\mathbb{C}[V]^W$ is a polynomial algebra. That is, it may be generated by algebraically independent polynomials f_i, and in fact by n-many such polynomials.*

The invariant polynomials f_i are called the *fundamental invariants* of W. They may be chosen homogeneous, and are indexed in increasing degree order[1] ($\deg f_i \leq \deg f_{i+1}$). The corresponding degrees $d_i := \deg f_i$ are independent of the choice of the f_i's and are called the *fundamental degrees* of W; their product equals the size of the group $\prod_i^n d_i = |W|$ (for these and for more numerological properties of the d_i's, [see Hum90, Chapter 3]).

2.1 Parabolic subgroups and reducible reflection groups.

The pointwise stabilisers W_S of any subset S of V are called *parabolic subgroups*. It is a theorem of Steinberg [see Bro10, Sec. 4.2.3] that W_S is generated by the reflections that fix the set S. This implies that all parabolic subgroups are of the form W_X, where X is some flat of $\mathcal{A}_W$.

A reducible complex reflection group W decomposes the ambient space V into a sum of W-stable subspaces V_i that carry irreducible representations of W. It is a subtle consequence of the fact that W is generated by pseudo-reflections on V, that the V_i's

[1] When W is reducible however, we will index the fundamental invariants of each irreducible subgroup separately.

can only be either trivial or reflection representations [see Kan01, Appendix B, Sec. 4]. We gather the trivial representations together, thus forming a subspace Z, and write $V = Z \oplus V_1 \oplus \cdots \oplus V_j$.

This in turn implies that we can express $W = W_1 \times \cdots \times W_j$, where each W_i acts on V_i as an irreducible complex reflection group. Furthermore, we can choose the fundamental invariants $\mathbf{g}$ so that there is a partition of them:

$$\mathbf{g} = \big(\mathbf{z}, \mathbf{g^1}, \cdots, \mathbf{g^j}\big) = \Big(\ (z_1, \cdots, z_l), (g^1_1, \cdots, g^1_{r_1}),\ \cdots\ , (g^j_1, \cdots, g^j_{r_j})\ \Big),$$

where r_i is the dimension of V_i, $\mathbf{z}$ is a basis of Z^* and $l = \dim Z$, and each $\mathbf{g^i}$ is a system of fundamental invariants for W_i. In particular, each g^i_k belongs to $\mathbb{C}[V_i]$ (for more details, see Example 21). We note that the W_i's are parabolic subgroups as they fix the subspaces $Z \oplus V_1 \oplus \cdots \oplus \hat{V}_i \oplus \cdots \oplus V_j$.

Given an irreducible reflection group W and a flat X, the parabolic subgroup W_X decomposes V into a sum $V = X \oplus V_1 \oplus \cdots \oplus V_j$. Here X carries the trivial representation with multiplicity $\dim X$, and the V_i's are reflection representations. The fundamental invariants can be chosen as above, starting with a basis for X; we will use this decomposition in the proof of Prop. 20.

We will often consider the natural action of W_X to be on the space $V_1 \oplus \cdots \oplus V_j$ (even though $W_X \subset \mathrm{GL}(V)$). In that case, we say that W_X is irreducible if $j = 1$ (that is, if W_X cannot be decomposed as the product of two non-trivial reflection groups). The particular flats $X \in \mathcal{L}_W$ for which W_X is irreducible, are called *irreducible flats* and are very important in the theory of wonderful compactifications [see DP95], but we will not deal with them here.

2.2 Well-generated complex reflection groups and their Coxeter elements.

It was observed after the classification that all complex reflection groups can be generated by $\dim V + 1$ reflections. The groups W that are generated by exactly $\dim V$-many reflections are called *well-generated*. This includes all real and Shephard groups, as well as most of the exceptional ones.

There are in fact many equivalent characterizations of well-generated groups [see Bes01, Thm. 5.5, Prop. 4.2; and Bes15, Thm. 2.4] and in what follows we will use the one given in Theorem 13. Well-generated groups have good analogs of Coxeter elements; in fact, Defn. 3 is yet another characterization of them (this is essentially [Bes01, Prop. 4.2] and well known facts about regular numbers). We need some auxiliary definitions before that though:

Definition 1. [Spr74] If W is a complex reflection group acting on the space V, then $v \in V$ is called a *regular vector* if no non-identity element of W fixes v. An element $w \in W$ is called a *Springer regular element* if it has a regular eigenvector v. In particular, if $w \cdot v = \zeta v$, we say that w is *ζ-regular*. We call the order of a Springer regular element a *regular number*.

Definition 2. [GG12] Let W be a complex reflection group of rank n. We write N for the number of reflecting hyperplanes of W and N^* for the number of (pseudo-)reflections of W. We define the *Coxeter number* h to be the ratio:

$$h := \frac{N + N^*}{n}.$$

When W is well-generated, the Coxeter number h equals the highest fundamental degree d_n and $d_n \gneqq d_{n-1}$.

We are now ready to define Coxeter elements for the context of this work:

Definition 3. [RRS17] Every well-generated group W has a ζ_h-regular element, in the sense of Springer, where $\zeta_h = e^{2\pi i/h}$ is the primitive h^{th} root of unity of smallest argument, and h is the Coxeter number as defined above. We call such elements *Coxeter elements*.

Remark 4. The theory of (finite) *real reflection groups* is built largely on the fact that their reflecting hyperplanes divide the ambient space into chambers which are fundamental domains for the group action [see Cox34]. In this context, the product of the reflections across the walls of any given chamber is called a *Coxeter element* of the group and is denoted by c, after Coxeter, who studied many of its numerological and structural properties [see Cox51].

In particular, Coxeter observed, and then Steinberg [Ste59] gave a uniform proof of, the fact that the order h of c satisfies the relation $2N = nh$. To do that, Steinberg showed [see also Hum90, Sec. 3.17] that c acts as a rotation of order h on a special 2-dimensional plane P which is transversal to all reflecting hyperplanes. Indeed, the previous definitions model precisely these properties of Coxeter elements in the real case.

The following Proposition will be necessary for some of our geometric arguments (see in particular Lemma 34). In the real case it is trivial, but generally the proof goes through the classification. It would be interesting to have a uniform proof for any of the equivalent characterizations of well-generated groups that their class is closed under taking parabolic subgroups.

Proposition 5. *[Bes15, Lemma 2.7] Parabolic subgroups of well-generated groups are also well-generated.*

2.3 The non-crossing lattice $NC(W)$.

Recall that the *absolute reflection length* $l_R(w)$ of an element $w \in W$, is the smallest number s of (pseudo-)reflections t_i needed to factor $w = t_1 \cdots t_s$. This length function determines a partial order $\leq_R$ (the *absolute order*) on the elements of W:

$$u \leq_R v \iff l_R(u) + l_R(u^{-1}v) = l_R(v).$$

Definition 6. We define the *noncrossing lattice $NC(W)$* to be the interval $[1, c]_{\leq_R}$ in the absolute order $\leq_R$ between the identity 1 and an arbitrary Coxeter element c. We say that an element c_i is *noncrossing with respect to* c, for some Coxeter element c, if $c_i \leq_R c$.

Remark 7. All Coxeter elements according to our Defn. 3 are conjugate [see LT09, Corol. 11.25] and conjugation respects the set of reflections, so that the intervals $[1, c]_{\leq_R}$ are isomorphic for the various c.
There is however a more general definition of a Coxeter element, where ζ_h is only required to be any primitive h^{th} root of unity. In that case, there might be multiple conjugacy classes of Coxeter elements, but they can still be mapped to each other by

reflection preserving (outer)-automorphisms of W. Hence, the poset-isomorphism type of the non-crossing lattice is again independent of c [see RRS17].

Definition 8. We say that an expression $c = w_1 {\cdot} w_2 \cdots w_k$ is a *reduced block factorization* of c, if it is length additive. That is, if $l_R(c) = l_R(w_1) + \cdots l_R(w_k)$. If all the w_i's are moreover pseudo-reflections, we call it a *reduced reflection factorization* of c.
We write $\mathrm{Red}_W(c)$ for the set of all reduced reflection factorizations of c. It is in bijection with the set of maximal chains of $NC(W)$.

2.4 Geometric Invariant Theory (GIT).

One of the main goals of GIT is the study of the quotient $G\backslash X$, when a group G acts "algebraically" on a variety (or in general scheme) X. A well known GIT mantra is that *"the best* ***candidates*** *for coordinate functions of the quotient $G\backslash X$ are precisely the coordinate functions of X that are invariant under G."*
The situation is particularly simple when X is an affine variety and the group G is finite. In that case, the invariant subalgebra $A[X]^G$ of the coordinate ring $A[X]$ of X, is finitely generated. It therefore determines an affine variety Y and a natural map $\rho : X \to Y$ induced by the inclusion $A[X]^G \hookrightarrow A[X]$.
Furthermore, it is easy to see that the map ρ is finite and hence surjective [see Sha13, Ex. 1.29 and Thm. 1.12, Sec. 5.3, p. 61], and that the fibers $\rho^{-1}(y)$ are precisely the orbits of the points of X under G [see Eis95, Exer. 13.2-4 and Sec. 1.7].
In this context, the Shephard-Todd-Chevalley theorem tells us that the GIT quotient $G\backslash X$ is the simplest possible (i.e. an affine *space*) exactly when G is a complex reflection group W (and X is itself also an affine space $V \cong \mathbb{C}^n$). Moreover, it gives us an explicit description of the quotient map $\rho : V \to W\backslash V \cong \mathbb{C}^n$ via the fundamental invariants:

$$\mathbb{C}^n \cong V \ni \mathbf{x} := (x_1, \cdots, x_n) \xrightarrow{\rho} \mathbf{f}(\mathbf{x}) := \big(f_1(\mathbf{x}), \cdots, f_n(\mathbf{x})\big) \in W\backslash V \cong \mathbb{C}^n \tag{2.1}$$

Remark 9. The fact that the quotient $W\backslash V$ is (even just topologically) homeomorphic to the affine space $\mathbb{C}^n$ is not trivial already in the case $W = \mathfrak{S}_2$. Here, $\mathfrak{S}_2 := \{1, s\}$ acts on $V \cong \mathbb{C}$ via $s \cdot x = -x$ and the quotient is naturally identified with a cone, which is homeomorphic to the plane.

The surjectivity of ρ is also not superficial. In fact, we know of no elementary way to show that the system of equations $f_i(\mathbf{x}) = b_i$ always has solutions, for any complex numbers $b_i \in \mathbb{C}$.

Chapter 3

Braid groups and the discriminant hypersurface

Emil Artin [Art25] introduced the braid group on n strands B_n already in 1925. In that paper, he gave a constructive proof of the (now celebrated) presentation:

$$B_n := \langle s_1, \cdots, s_{n-1} \mid s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1},\ s_i s_j = s_j s_i,\ j \neq i \pm 1 \rangle$$

He casually observed[1] that by adding the relations $s_i^2 = 1$, one gets a presentation for the symmetric group $\mathfrak{S}_n$. He couldn't have known (at the time) that this is precisely the presentation of $\mathfrak{S}_n$ as a reflection group; Coxeter would only classify finite reflection groups ten years later [Cox34].

3.1 Braid groups and Artin groups

In 1962, Fox and Neuwirth [FN62b] gave a new derivation of the previous presentation by interpreting B_n as the fundamental group of $\mathrm{Conf}_n(\mathbb{C})$, the configuration space of n *distinct* points in the plane (but see also Remark 43). To that end they (essentially) further regarded $\mathrm{Conf}_n(\mathbb{C})$ as the space of regular orbits of the natural action of $\mathfrak{S}_n$ on $\mathbb{C}^n$.

Each of these three interpretations of the braid group (the Coxeter-like presentation, the configuration space, and as the fundamental group of the regular orbits space) have

[1] Although he failed to mention that this was already known to Moore [see Moo97].

analogs for other reflection groups. The latter one however, which is also the only one that *a priori* makes sense for all complex reflection groups, provides the natural environment for our geometric study of Coxeter elements and their factorizations:

Definition 10. Let $W \leq \mathrm{GL}(V)$ be a complex reflection group and V^{reg} the set of points in V that have a trivial stabilizer under the W-action. We define the *braid group* $B(W)$ to be the fundamental group of the space of regular orbits of W:

$$B(W) := \pi_1(W \backslash V^{\mathrm{reg}})$$

In fact, the previously mentioned generalizations of the braid group B_n for other reflection groups agree (when they are defined). Brieskorn proved [Bri71] that $B(W)$ has an Artin-like presentation for real W, and Allcock [All02] realized $B(B_n)$ and $B(D_n)$ as fundamental groups (or subgroups thereof) of configuration spaces on certain (very mild) orbifolds. For a (somewhat older) survey of braid groups, [see Mag74].

3.1.1 The short exact sequence $1 \hookrightarrow P(W) \to B(W) \twoheadrightarrow W \to 1$.

As we mentioned earlier, Steinberg proved that the (pointwise) W-stabilizer of any $x \in V$ is generated by the reflections across the hyperplanes $H \in \mathcal{R}$ that contain x. This implies that W acts freely precisely on the complement of the hyperplane arrangement (i.e. it implies that $V^{\mathrm{reg}} = V - \bigcup_{H \in \mathcal{R}} H$). In other words, the quotient map $V^{\mathrm{reg}} \xrightarrow{\rho} W \backslash V^{\mathrm{reg}}$ is a Galois covering.

Now, the following short exact sequence, where we call $P(W)$ the *pure braid group* of W, is an immediate corollary of covering space theory [see Hat02, Prop. 1.40]:

$$\begin{array}{ccccccccc} 1 \hookrightarrow & \pi_1(V^{\mathrm{reg}}) & \xrightarrow{\rho_*} & \pi_1(W \backslash V^{\mathrm{reg}}) & \xrightarrow{\pi} & W \to 1 \\ & \| & & \| & & \\ & P(W) & & B(W) & & \end{array} \tag{3.1}$$

Its significance lies in that it gives a topological interpretation of W as the group of deck transformations of a covering map $V^{\mathrm{reg}} \xrightarrow{\rho} W \backslash V^{\mathrm{reg}}$, for which we have an explicit algebraic formula (2.1).

Notice that the surjection $\pi : B(W) \twoheadrightarrow W$ is well-defined up to conjugation. Indeed, given a choice of a basepoint $v \in V^{\mathrm{reg}}$, a loop $b \in B(W)$ lifts to a path that connects v to $b_*(v)$ (we call this the *Galois action* of b). Then, we define $w := \pi(b)$ to be the *unique* element $w \in W$ such that $w \cdot v = b_*(v)$.

3.1.2 Centered Configuration Spaces.

A slightly modified version of the configuration space $\mathrm{Conf}_n(\mathbb{C})$ plays a very significant role in what follows as the natural target of the LL map (see Defn. 44). We provide the necessary definitions here:

Definition 11. We denote by E_n the set of centered configurations of n unordered, not necessarily distinct[2] points in $\mathbb{C}$, i.e.,

$$E_n := \mathfrak{S}_n \backslash H_0, \text{ where } H_0 = \big\{ \mathbf{x} := (x_1, \cdots, x_n) \in \mathbb{C}^n \mid \sum_{i=1}^{n} x_i = 0 \big\} \cong \mathbb{C}^{n-1}.$$

Notice that we may identify $E_n \cong \mathbb{C}^{n-1}$ via the map $\mathbf{x} \to \big(e_2(\mathbf{x}), \cdots, e_n(\mathbf{x})\big)$, where e_i is the i^{th} elementary symmetric polynomial (since $e_1(\mathbf{x}) = 0$ is the *centered* condition). Indeed, each configuration is determined by the coefficients of the monic polynomial that has the x_i's as its roots. Those, in turn, are by the Vieta formulas the evaluations $e_i(\mathbf{x})$. We write E_n^{reg} for those centered configurations where the points x_i are distinct.

Remark 12. It is easy to see that $E_n^{\text{reg}} \cong \mathrm{Conf}_n(\mathbb{C})$ and hence $\pi_1(E_n^{\text{reg}}) = B_n$. In § 6.2.1 we will describe, for each well-generated W, an associated finite covering of E_n^{reg}. We warn the reader for the double appearance of braid groups: For each (generalized) braid group $B(W)$, we will produce a finite index subgroup of *the* (type A_{n-1}) braid group B_n on n strands.

3.2 Discriminant hypersurface

A great amount of the combinatorics and topology of reflection groups is built on the study of the reflection arrangement $\mathcal{A}_W$. As we mentioned earlier, one of the gifts of GIT to the subject is that the quotient map to the orbit space $\rho : V \to W\backslash V^{\text{reg}}$ is a finite morphism with a very simple form (2.1). In this section we describe the discriminant hypersurface $\mathcal{H}$ of W; it is the image under ρ of the arrangement $\mathcal{A}_W$.

For each hyperplane $H \in \mathcal{A}_W$, consider a linear form α_H such that $H = \ker(\alpha_H)$. The product $Q(\mathcal{A}_W) = \prod_{H \in \mathcal{A}_W} \alpha_H$ is called the *defining polynomial* of $\mathcal{A}_W$; it is well-defined

[2] This would be a subset of what topologists usually call the n^{th} *symmetric product* of $\mathbb{C}$, as usually $\mathrm{Conf}_n(X)$ is meant to assume that points in the configuration are distinct.

up to a scalar multiple. The action of W respects the set of hyperplanes, but might scale the defining polynomial by a root of unity. In fact, $Q(\mathcal{A}_W)$ is a *relative invariant* in the sense of [OT92, p. 228]; more precisely, w scales $Q(\mathcal{A}_W)$ by $\det(w)^{-1}$.

Now, for each hyperplane H, let e_H be the order of the cyclic group W_H generated by all the (quasi)-reflections that fix H. The product $\Delta(W) = \prod_{H\in\mathcal{A}_W} \alpha_H^{e_H}$ is called the *discriminant* of the group W and is invariant under the action of W [see OT92, Defn. 6.44].

The discriminant is therefore an element of the algebra $\mathbb{C}[V]^W$ and can be written as a polynomial in the basic invariants f_i. We will denote it by $\Delta(W;\mathbf{f})$ to indicate its dependence on a choice of $\mathbf{f}$. Its zero set in $\mathbb{C}^n \cong W\backslash V$ is called the *discriminant hypersurface* of W and is denoted by $\mathcal{H}(W)$ (or simply $\mathcal{H}$). The following is of fundamental importance in what follows:

Theorem 13. *[for real W see Sai93, Sec. 3; for the general case Bes15, Thm. 2.4]* *Let W be an irreducible complex reflection group. Then W is well-generated if and only if for any system of basic invariants $\mathbf{f}$, we have $\left(\frac{\partial}{\partial f_n}\right)^n \Delta(W;\mathbf{f}) \in \mathbb{C}^\times$. That is, $\Delta(W;\mathbf{f})$, viewed as a polynomial in the highest degree invariant f_n and with coefficients in $\mathbb{C}[f_1,\cdots,f_{n-1}]$, is* **monic and of degree n**.

Remark 14. In fact Bessis proves a much stronger statement in [Bes01, Lemma 1.6]: If a degree d_i is a regular number in the sense of Springer, then there exists a system of basic invariants $\mathbf{f}$ such that the discriminant is monic with respect to f_i.

Example 15. The (classical) discriminant of a monic polynomial of degree n is nothing more than the discriminant $\Delta(\mathfrak{S}_n;\mathbf{e})$ of the symmetric group $\mathfrak{S}_n$, where $\mathbf{e}$ denotes the elementary symmetric polynomials. The reader might recognize the following formula

$$\mathrm{Disc}_x(x^3+ax^2+bx+c) = -27c^2+18abc-4a^3c-4b^3+a^2b^2,$$

and notice that it is monic and of degree 2 (the rank of $\mathfrak{S}_3 = A_2$) with respect to c.

As a monic polynomial of degree n in f_n, the discriminant $\Delta(W;\mathbf{f})$ has n coefficients $\alpha_i \in \mathbb{C}[f_1,\cdots,f_{n-1}]$ that depend on $n-1$ parameters. As such, the α_i's are algebraically dependent. It will be important for us later to make the number of parameters and coefficients agree, so we start by observing that we can always get rid of the coefficient of f_n^{n-1}.

Recall first that the *total degree* of a monomial $z_1^{m_1}\cdots z_n^{m_n}$, where the z_i's are weighted variables of degree $\deg(z_i)=d_i$, is defined to be the sum $m=d_1m_1+\cdots d_nm_n$. Now, we will say that a polynomial $p(z_1,\cdots,z_n)$ is *quasi-homogeneous* and of weighted degree m, if each term of p is a monomial in the z_i's of total degree m.

Corollary 16. *When W is well-generated, we can write the discriminant as*

$$\Delta(W,\mathbf{f})=f_n^n+\alpha_2 f_n^{n-2}+\cdots+\alpha_n$$

where $\alpha_i\in\mathbb{C}[f_1,\cdots,f_{n-1}]$ are quasi-homogeneous polynomials of weighted degree hi. When we are referring to the discriminant in this form, we will use the symbol $\big(\Delta(W,\mathbf{f});f_n\big)$.

Proof. This can be achieved by the usual cubic equation trick, by setting $f_n'=f_n-\dfrac{\alpha_1}{n}$. The weighted homogeneity is because $\Delta(W)$ is already homogeneous in the x_i's, of degree $N+N^*=hn$. Finally, recall that when W is well-generated, we have $\deg(f_n)=:d_n=h$. □

Remark 17. The situation is very similar when $W=W_1\times\cdots\times W_j$ is reducible and the ambient space decomposes under its action as $V=Z\oplus V_1\oplus\cdots\oplus V_j$ (see Section 2.1). In this case, the discriminant factors as $\Delta(W)=\prod_{H_1\in\mathcal{A}_{W_1}}\alpha_{H_1}^{e_{H_1}}\cdots\prod_{H_j\in\mathcal{A}_{W_j}}\alpha_{H_j}^{e_{H_j}}$, where we view the linear forms α_{H_i} as elements of $V^*\supset V_i^*$. With respect to the fundamental invariants $\mathbf{g}$, it can be written as:

$$\Delta(W,\mathbf{g})=\Big(\big(g_{r_1}^1\big)^{r_1}+\alpha_2^1\,\big(g_{r_1}^1\big)^{r_1-2}+\cdots+\alpha_{r_1}^1\Big)\cdots\Big(\big(g_{r_j}^j\big)^{r_j}+\alpha_2^j\,\big(g_{r_j}^j\big)^{r_j-2}+\cdots+\alpha_{r_j}^j\Big).\tag{3.2}$$

This expression will be used in the proof of Lemma 34 and Prop. 20.

We also record the following simple Corollary:

Corollary 18. *The natural projection of the discriminant hypersurface $\mathcal{H}$ on the first $n-1$ coordinates f_i is surjective.*

Proof. Indeed, for any choice of $\mathbf{y}=(f_1,\cdots,f_{n-1})$, the equation

$$t^n+\alpha_2(\mathbf{y})t^{n-2}+\cdots+\alpha_n(\mathbf{y})=0$$

will have solutions (in fact, exactly n-many counted with multiplicity). This might not happen if t^n also had a factor $\alpha_0(\mathbf{y})$; in that case, the equation could demand $0 = \alpha_n(\mathbf{y})$. $\square$

Remark 19. This surjectivity property is important in the recent work [FRS16]. Our previous Remark 14 applies here as well: Whenever $\Delta(W; \mathbf{f})$ is monic with respect to some f_i, the natural projection of $\mathcal{H}$ on the $n-1$ coordinates $(f_1, \cdots, \widehat{f_i}, \cdots, f_n)$ is surjective.

3.2.1 The orbit stratification of the discriminant hypersurface.

The ambient space V is stratified by the reflection arrangement $\mathcal{A}_W$, the strata being the flats $X \in \mathcal{L}_W$ in the intersection lattice. We will use the symbol X^{reg} to indicate the *regular* part of X, that is, $X^{\text{reg}} := X \setminus \bigcup_{H \not\supset X} H$.

The quotient map $\rho : V \to W \backslash V^{\text{reg}}$, induces then the orbit stratification of the discriminant hypersurface $\mathcal{H}$, the strata of which are the W-orbits of the flats X. We denote them by $[X] \in W \backslash \mathcal{L}_W$. As before, we will use the symbol $[X^{\text{reg}}]$ for the *regular* part of $[X]$.

The local topology of the reflection arrangement is very well understood. Around a point $p \in X^{\text{reg}}$, $\mathcal{A}_W$ looks like the direct product of the flat X and the reflection arrangement $\mathcal{A}_{W_X}$ of the parabolic subgroup W_X. The same behavior is exhibited by the discriminant hypersurface $\mathcal{H}$: Locally at a point $[p] \in [X]$, it looks like the product of X and the discriminant hypersurface $\mathcal{H}(W_X)$.[3]

This local behavior induces an embedding of the corresponding braid groups $B(W_X) \hookrightarrow B(W)$, which is well defined up to conjugation. The following proposition is known but we sketch a proof, as we will need, in Section 5.3, the semi-explicit morphism between $W_X \backslash V$ and $W \backslash V$ that induces the embedding.

Proposition 20. *[BMR98, Prop. 2.29] For every point $[p] \in [X^{\text{reg}}]$, there is an embedding of the braid groups $B(W_X) \hookrightarrow B(W)$ induced by the restriction of the quotient map $\tau_X : W_X \backslash V \to W \backslash V$ locally at the W_X-orbit of p.*

[3] Notice that here we consider the natural action of W_X on the orthogonal complement of X, as in Section 2.1.

Proof. By Steinberg's theorem, the parabolic subgroup W_X is a complex reflection group, hence the invariant subalgebra $\mathbb{C}[V]^{W_X}$ is a polynomial subalgebra. The inclusion of algebras $\mathbb{C}[V] \supset \mathbb{C}[V]^{W_X} \supset \mathbb{C}[V]^W$ induces (again, because of GIT) surjective morphisms that realize the quotient maps:

$$\begin{array}{ccccc} V & \xrightarrow{\rho_X} & W_X\backslash V & \xrightarrow{\tau_X} & W\backslash V \\ (v_1, \cdots, v_n) & \longrightarrow & (\mathbf{x}, \mathbf{g^1}, \cdots, \mathbf{g^j}) & \longrightarrow & (f_1, \cdots, f_n) \end{array} \tag{3.3}$$

Here $\mathbf{x}$ is a basis for X and the $\mathbf{g_i}$'s are as in Section 2.1 associated with the decomposition $V = X \oplus V_1 \oplus \cdots \oplus V_j$ induced by W_X. Note that the fundamental invariants f_i are also W_X-invariants, and hence are polynomials in $\mathbf{x}$ and the $\mathbf{g_i}$'s. That is, τ_X is a polynomial map. Clearly, $\rho = \tau_X \circ \rho_X$.

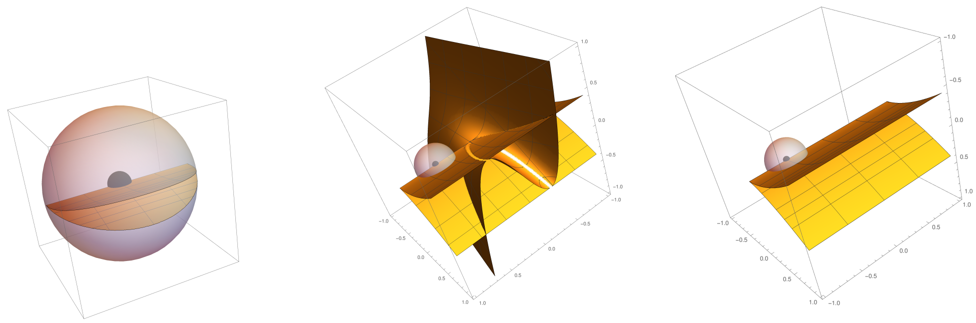
Let $p \in X^{\text{reg}}$ be a regular point on X. We write $[p]_X$ for the W_X-orbit of p and consider a small ball $B_{[p]_X}$ centered at $[p]_X$. Since $p \in X^{\text{reg}}$, if the ball is small enough it will only intersect orbits whose W-stabilizers are contained in W_X. In particular, no two points in the ball will be identified by the quotient map τ_X. This means [see also OT92, Lemma 6.107] that τ_X is a homeomorphism (in fact, a biholomorphism) locally at $[p]_X$. Consider now the regular part of the ball $B^{\text{reg}}_{[p]_X} := B_{[p]_X} - W_X\backslash \bigcup_{H\supset X} H$. We assumed $B_{[p]_X}$ small enough so that it avoids the W_X-orbit of hyperplanes H that do not contain X; that is, $B^{\text{reg}}_{[p]_X} = B_{[p]_X} - W_X\backslash \bigcup H$. The two inclusions below are portrayed in Fig. 3.1 for the case $\mathfrak{S}_{\{1,2,3\}} \leq \mathfrak{S}_4$.

$$\begin{array}{ccccc} B^{\text{reg}}_{[p]_X} & \subset & W_X\backslash V^{\text{reg}} & \xrightarrow{\tau_X} & W\backslash V^{\text{reg}} \\ & & \cap & & \\ & & W_X\backslash\big(V - \cup_{H\supset X} H\big) & & \end{array}$$

It is easy to see that the composition of inclusions $B^{\text{reg}}_{[p]_X} \subset W_X\backslash\big(V - \cup_{H\supset X} H\big)$ is in fact a homeomorphism. Indeed, this depends on the quasi-homogeneity of the discriminant equation; the ball is a weighted-scaling of the second space (compare Fig. 3.1a and Fig. 3.1c).

This however implies that the first inclusion $B^{\text{reg}}_{[p]_X} \subset W_X\backslash V^{\text{reg}}$ induces an embedding of fundamental groups. Since moreover τ_X is a covering map over $W\backslash V^{\text{reg}}$ (and thus also induces an embedding of fundamental groups), we have the following chain:

$$B(W_X) \cong \pi_1\big(B^{\text{reg}}_{[p]_X}\big) \hookrightarrow \pi_1\big(W_X\backslash V^{\text{reg}}\big) \hookrightarrow \pi_1\big(W\backslash V^{\text{reg}}\big) = B(W) \tag{3.4}$$



(a) The ball $B^{\mathrm{reg}}_{[p]_X}$ is naturally embedded ...

(b) ... into $W_X \backslash V^{\mathrm{reg}}$ but does not intersect $\cup_{H \not\supset X} H$, so...

(c) ... it is also a subset of $W_X \backslash \big(V - \cup_{H \supset X} H\big)$.

Figure 3.1: The (first) embedding $B(W_X) \hookrightarrow \pi_1(W_X \backslash V^{\mathrm{reg}})$ from (3.4), for $\mathfrak{S}_3 \leq \mathfrak{S}_4$.

It is clear that the resulting embedding $B(W_X) \hookrightarrow B(W)$ is induced by (the restriction locally at $[p]_X$ of) τ_X. $\square$

The explicit form of the morphism τ is easy to produce with a computer algebra system, but might otherwise be unfamiliar. Since it is going to be important in Section 5.3 (see also Remark 40), we present here the case $\mathfrak{S}_3 \leq \mathfrak{S}_4$ (which is also depicted in Fig. 3.1):

Example 21. The complex reflection group $\mathfrak{S}_4$ acts irreducibly on the 3-dimensional subspace $V := \mathbb{C}^4/\langle x_1 + \cdots + x_4 = 0\rangle \subset \mathbb{C}^4$. Consider the basis

$$z_1 = x_1 - x_2, \quad z_2 = x_2 - x_3, \quad z_3 = x_1 + x_2 + x_3, \quad z_4 = x_1 + x_2 + x_3 + x_4$$

of the dual space $(\mathbb{C}^4)^*$ and notice that $\{z_1, z_2, z_3\}$ is a basis of V^* such that z_3 is invariant under $\mathfrak{S}_3 \leq \mathfrak{S}_4$. Here are matrices for the actions of the generators of $\mathfrak{S}_4$ on V^*, in the $\{z_1, z_2, z_3\}$-basis:

$$(12) : \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (23) : \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (34) : \begin{bmatrix} 1 & -1/2 & -1/3 \\ 0 & 1/3 & -2/3 \\ 0 & 4/3 & 1/3 \end{bmatrix}$$

In order to find fundamental invariants for the action of $\mathfrak{S}_4$ on $\mathbb{C}[V]$, we may start with the elementary polynomials on $\mathbb{C}^4$ and eliminate the form $z_4 = x_1 + x_2 + x_3 + x_4$. We produce the following polynomials (here the indices of the f_i's indicate their degrees):

$$3f_2 = -z_1^2 - z_1z_2 - z_2^2 - 2z_3^2, \quad 27f_3 = z_1^2(2z_1 + 3z_2 + 6z_3) - 3z_1z_2(z_2 - 2z_3) + 2(z_2 - 2z_3)^2(z_2 + z_3),$$

$$27f_4 = z_3\big(3z_1z_2(z_2 + z_3) + (2z_2 - z_3)(z_2 + z_3)^2 - 2z_1^3 - 3z_1^2(z_2 - z_3)\big)$$

Similarly, to construct fundamental invariants for the action of $\mathfrak{S}_3$ on $\mathbb{C}[z_1, z_2]$, we eliminate the form $z_3 = x_1 + x_2 + x_3$ from the elementary symmetric polynomials on $\{x_1, x_2, x_3\}$. We produce the following polynomials:

$$3g_2 = -z_1^2 - z_1z_2 - z_2^2, \quad 27g_3 = z_1^2(2z_1 + 3z_2) - z_2^2(3z_1 + 2z_2)$$

Finally, we may express the fundamental invariants of $\mathfrak{S}_4$ in terms of the fundamental invariants of $\mathfrak{S}_3$ and the form z_3 (since z_3 is the dual basis of the fixed flat of $\mathfrak{S}_3$ in V):

$$\begin{aligned} 3f_2 &= 3g_2 - 2z_3^2 \\ 27f_3 &= 27g_3 - (9 + 18z_3)g_2 - 8z_3^3 \\ 27f_4 &= -27g_3z_3 - z_3^4 - g_2z_3(9z_3 - 9) \end{aligned} \tag{3.5}$$

The above describe explicitly the map $\tau : (z_3, g_2, g_3) \to (f_2, f_3, f_4)$. It is easy to see that τ is locally invertible when $z_3 \neq 0$.

Remark 22. The combinatorics and local behavior of the discriminant hypersurface and its complement (hence $B(W)$) replace the combinatorics and local behavior of the hyperplane arrangement and its complement (chamber decomposition). This is how the classical theory transitions to the complex case.

Chapter 4

Geometric factorizations of the Coxeter element

As we mentioned in the introduction, Bessis [in Bes15] used the lattice $[1,c]_R$ as a combinatorial recipe to construct the universal covering space of V^{reg} (and show that it is contractible, thereby proving the $K(\pi,1)$ conjecture). In this section, we are going to review one of his main tools: a new, geometric interpretation of the Coxeter element and its reduced factorizations.

4.1 Geometric construction of the Coxeter element.

We will construct an element c of W that satisfies the characterization we gave for Coxeter elements in Defn. 3. In particular, c will be a Springer regular element with a (regular) eigenvalue equal to $e^{2\pi i/h}$. We will do that by first producing a loop in $B(W)$, and then proving that it maps to an appropriate element c via the fixed surjection $\pi : B(W) \twoheadrightarrow W$ from §3.1.1. In fact, this will be the spirit of our later constructions as well.

Recall that we have identified $W\backslash V$ with $\operatorname{Spec}\mathbb{C}[f_1, f_2, \cdots, f_n]$. Let us now define the *base space* $Y := \operatorname{Spec}\mathbb{C}[f_1, f_2, \cdots, f_{n-1}]$ so that $W\backslash V \cong Y \times \mathbb{C}$ with coordinates written (y, x), or sometimes (y, f_n). This projection on the first $n-1$ invariants was first

introduced by Saito [Sai04, Section 2.1] but was also implicit in singularity theory.
We consider the slice $L_0 := \mathbf{0} \times \mathbb{C}$, given by $f_1 = \cdots = f_{n-1} = 0$ and f_n arbitrary, in the orbit space $W\backslash V$. It is clear, by the monicity of the discriminant equation (Corol. 16), that the only intersection between L_0 and $\mathcal{H}$ is the origin $(\mathbf{0}, 0)$.
Therefore, the loop inside L_0 given by $f_n(t) = e^{2\pi i t}$ (with $t \in [0, 1]$ and $f_i = 0,\ i < n$) is an element of the braid group $B(W) = \pi_1(W\backslash V^{\text{reg}})$ (with basepoint $(\mathbf{0}, 1)$ which we will be suppressing throughout). We call this loop δ; it corresponds to Deligne's element Δ [see Bes15, Defn. 6.11].
We wish to understand the image of δ in W under the fixed surjection π. As the covering map $\rho : V^{\text{reg}} \to W\backslash V^{\text{reg}}$ is given explicitly with respect to the fundamental invariants f_i, this is easy to do:
Indeed, we pick a basepoint $v \in V^{\text{reg}}$ such that $f_1(v) = \cdots = f_{n-1}(v) = 0$ and $f_n(v) = 1$ (we have $|W|$-many options). Now, the path $e^{(2\pi i/h)\cdot t} \cdot v$, for $t \in [0, 1]$, which is merely a rotation of v by $2\pi/h$ around the origin, is precisely the lift of our element δ (because of the homogeneity of the f_i's and because $\deg f_n = h$).
The fact that the Galois action of δ sends v to $\delta_*(v) = e^{2\pi i/h} \cdot v$ means that the deck transformation $c := \pi(\delta)$ acts on v via multiplication by $e^{2\pi i/h}$. As v is also an element of V^{reg}, we have that c is a Springer regular element with an eigenvalue equal to $e^{2\pi i/h}$.

Definition 23. We call $c := \pi(\delta)$ the *geometric Coxeter element* of the group W.
Different choices of basepoints in V^{reg} give all elements conjugate to c. This is of course the same conjugacy class as in our Defn. 3; we use the word "geometric" to draw attention on this special construction of c.

The following Corollary (essentially of the definition) of the Coxeter element is fundamental in the proof of the transversality of the line L_y (Prop. 37). It is the analog of a theorem of A'Campo [ACa73] that the trace of the monodromy operator for an isolated singularity is equal to -1 [for the connection, see Ent97, Thm. 2 and Prop. 6.1].

Corollary 24. *[Bes15, Lemma 7.2] The fixed space of any Coxeter element c is trivial; it consists of only the origin* $\mathbf{0}$.

Proof. As part of his theory of regular elements, Springer [Spr74] showed that the eigenvalues of a ζ-regular element are $\zeta^{1-d_1}, \cdots, \zeta^{1-d_n}$. Applying this to $\zeta = e^{2\pi i/h}$ and noting that $2 \leq d_1 \leq \cdots < d_n = h$, we see that $1 = \zeta^0$ never appears as an eigenvalue of c. $\square$

Remark 25 (**Independence of the construction**)**.** Bessis' construction of the Coxeter element does not depend on the GIT realization of the orbit space ($W\backslash V \cong \mathbb{C}^n$). The slice $\mathbf{0}\times\mathbb{C}$ is special, in that it is unchanged under a different choice of fundamental invariants. If $\mathbf{f}' = (f_1', \cdots, f_n')$ is such a selection and F is an algebra isomorphism that sends $\mathbf{f}'$ to $\mathbf{f}$, then F respects the slice $\mathbf{0}\times\mathbb{C}$. That is, in the new coordinates, the slice is also given by $f_1' = \cdots = f_{n-1}' = 0$ and f_n' arbitrary.
Indeed, because $\deg f_n \gneq \deg f_i,\ i < n$, the new invariants $\mathbf{f}'$ are given via $f_i' = g_i(f_1, \cdots, f_{n-1}),\ i < n$, and $f_n' = g_n(f_1, \cdots, f_n)$ for some weighted-homogeneous polynomials g_i. That is, only f_n' may depend on f_n and, in fact, it must depend linearly on it (i.e $\dfrac{\partial g_n}{\partial f_n} = \lambda$, for some constant λ). The claim follows immediately.
The uniqueness of the slice L_0 is essentially the same as the uniqueness of Saito's primitive form [see Sai04, Section 1.6]. The only choice left in the construction is that of a lift of the basepoint $(\mathbf{0}, 1)$. This provides us with a whole conjugacy class of Coxeter elements.

Remark 26. In the case of a reducible reflection group $W = W_1 \times \cdots \times W_j$, we define the Coxeter element to be the image $\pi(\delta_1 \cdots \delta_j)$ of the product of the loops $\delta_i \in B(W_i)$. It is easy to see that the concatenation of these loops is homotopic to the one given by $g_s^i(t) = 0,\ s < r_i$, and $g_{r_i}^i(t) = e^{2\pi i\cdot t}$ with $t \in [0,1]$ (see §5.3 for the notation).

Remark 27. A further generalization of the Coxeter element, is any element w that is a Springer regular element of order d_i, for some fundamental degree d_i. The geometric construction still makes sense here, since by Remark 14 the discriminant equation $\Delta(W;\mathbf{f})$ will be monic with respect to f_i.
There is as of yet no clear way, however, to generalize the structural results of the next sections for regular elements.

4.2 Factorizations via the moving slice

The previous section described the construction of the Coxeter element in W, via the lifting of a loop that is contained in the special slice L_0. If we consider other nearby

"vertical" lines, they might intersect the discriminant at multiple points.
Bessis defines appropriate loops around these points, and shows that they lift to factorizations of the Coxeter element. Our presentation is slightly different than [Bes15, Section 6] in that we avoid defining his "tunnels", an object more suited to people better versed in categorical methods for topology.
We pick a point $y \in Y$ and consider the slice $L_y := y \times \mathbb{C}$ in the orbit space $W \backslash V \cong Y \times \mathbb{C}$. It will intersect $\mathcal{H}$ at n points, counted with multiplicity. We order them *complex-lexicographically* (that is, by increasing real part first and increasing imaginary part to cut ties) and denote them (y, x_i). These are precisely the roots of the equation

$$\big(\Delta(W, \mathbf{f}); (y,t)\big) := t^n + \alpha_2(y)t^{n-2} + \cdots + \alpha_n(y) = 0. \tag{4.1}$$

This is the discriminant as given in Corol. 16, where we view the fundamental invariant f_n as the unknown t, while the α_i's depend on the parameter $y = (f_1, \cdots, f_{n-1})$.
Now choose a special point x_∞ in L_y that lies above all points x_i, and think of it as the infinity in L_y (i.e. $\mathrm{Im}(x_\infty) \gg \mathrm{Im}(x_i)\ \forall\ i$). Of course, x_∞ depends on y, but we will not use the cumbersome notation $x_\infty(y)$. From x_∞ construct paths β_i, in L_y, to the points x_i, such that:

1. They never cross each other (or themselves) and move only downwards.

2. Their order as they leave x_∞ is given by their indices (i.e. β_1 is the leftmost one, β_2 is the second leftmost, etc.).

Notice that such a system of paths is uniquely determined up to homotopy (these are usually called *distinguished paths* in singularity theory). In Figure 4.1, we see a collection of four paths that satisfy the above properties.

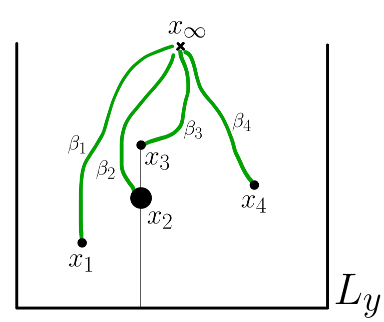


Figure 4.1: A distinguished system of paths.

In order to eventually construct elements of $B(W)$, we need to connect these paths β_i to the basepoint $(\mathbf{0}, 1) \in Y \times \mathbb{C}$. For this purpose however, the position of the slice L_y (given by the point y) is not *a priori* sufficient; we need a path θ in Y that connects $\mathbf{0}$ with y (see Figure 4.2). For any such θ, consider indeed a path β_θ in $W \backslash V \cong Y \times \mathbb{C}$ that projects down to θ and satisfies the following:

1. It starts at $(\mathbf{0}, 1)$,

2. it always stays above the intersections $L_{y'} \cap \mathcal{H}$ (for any $y' \in \theta$),

3. and it ends at (y, x_∞).

Such a path always exists. As it happens, the coefficients $\alpha_i(y)$ are bounded (since y traces a path) which implies that the roots x_i of the polynomial (4.1) are bounded too.

Remark 28. Notice that this relies on the monicity of the discriminant (Thm. 13). In fact, when the equation is not monic, there are certain paths in Y (that end up at points where the coefficient of t^n becomes zero), over which the roots explode to infinity.

More importantly, all such paths β_θ as above are, after all, homotopic. This relies on Bessis' fat basepoint trick [Bes15, Appendix A]: There is a dense contructible subset of $W \backslash V^{\text{reg}}$, that may act as basepoint for $B(W)$, and which contains all paths β_θ as above. To indicate that, we drop θ from the notation and write β_y instead.
Given this information, we can now easily construct elements $b_{(y,x_i)}$ of $B(W)$: First, we follow the path β_y from the basepoint $(\mathbf{0}, 1)$ to (y, x_∞), then we go down β_i but before we reach its end, we trace a small counterclockwise circle around x_i, and finally we return by the same route (see Figure 4.2).
The product $\delta_y = b_{(y,x_1)} \cdots b_{(y,x_k)}$ (where k is the number of *geometrically distinct* points x_i) surrounds $\mathcal{H}$ and is, of course, homotopic to the loop δ from the previous section. We have finally made it; from each slice L_y, we have constructed a geometric factorization of δ. The fixed surjection $\pi : B(W) \twoheadrightarrow W$ allows us to turn this into a factorization of the Coxeter element c:

Definition 29. There is a map[1] $\operatorname{rlbl} : \mathcal{H} \to W$ given by $\operatorname{rlbl}(y, x_i) = c_i := \pi(b_{(y,x_i)})$, where π and $b_{(y,x_i)}$ are as in the previous discussion. If we only specify the position y of the slice L_y, the "reduced label" map gives us the whole associated factorization of c in the form of a tuple, i.e. $\operatorname{rlbl}(y) := (c_1, c_2, \cdots, c_k)$.

[1] Here we have merged Bessis' and Ripoll's notation from [Bes15] and [Rip12]. Notice that Bessis' rlbl map refers to factorizations both in the braid group $B(W)$ and in W itself, as they are eventually shown to be equivalent.

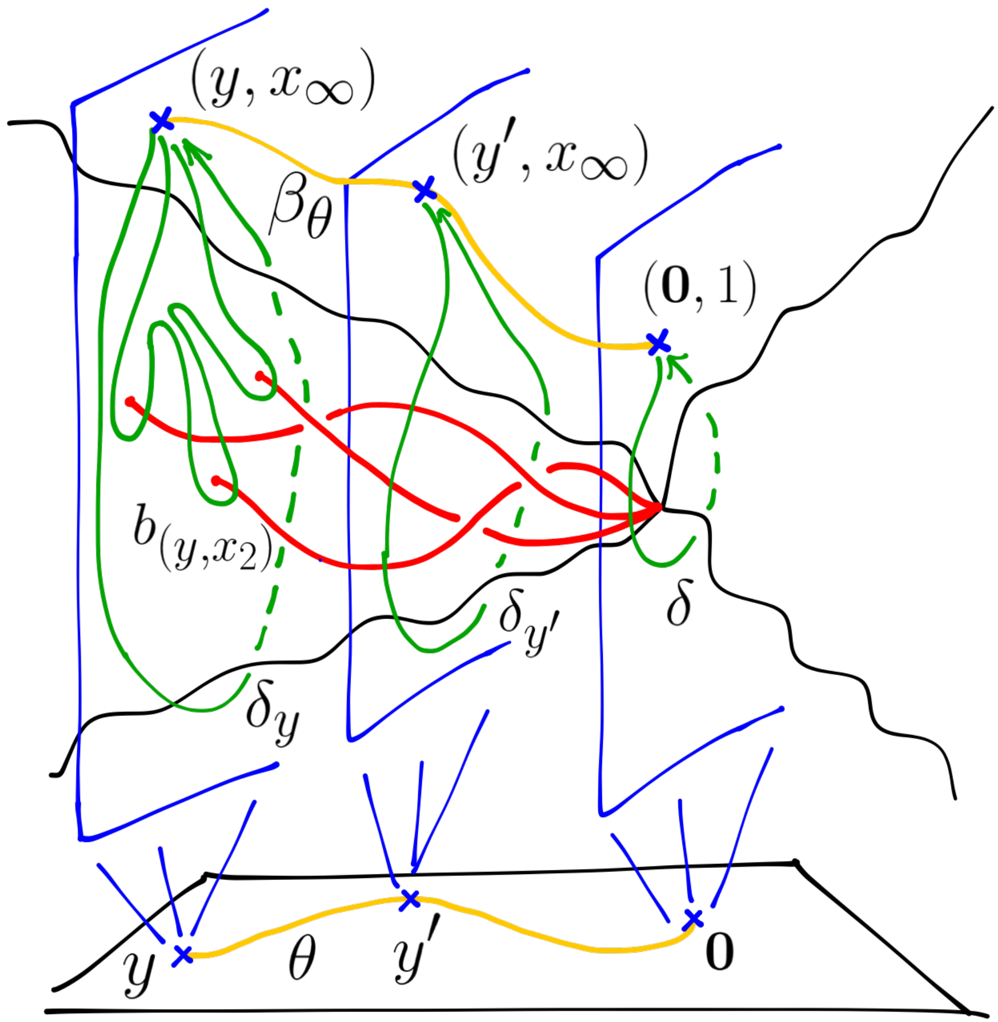


Figure 4.2: Geometric factorizations via the moving slice.

Chapter 5

Structural properties of the rlbl map and compatibility with the discriminant stratification.

The (reduced) labeling map that we introduced in the previous section is in a very strong way compatible with the orbit stratification of the discriminant hypersurface $\mathcal{H}$. The main result of this section is Prop. 39, which states that all labels are parabolic Coxeter elements.

Our analysis uses some techniques from algebraic geometry that might be unfamiliar to a reader with more of a combinatorial background. For this reason, we interject in our exposition a brief presentation of the few facts and definitions we will be using. The reader might choose to skip it at their own risk.

5.1 Geometric Interlude, No. 1

The main geometric object that we will employ in the local analysis of the discriminant hypersurface $\mathcal{H}$, is the tangent cone. Quoting [EH00, Section III.2.4], it is *"a more accurate* [than the tangent space] *reflection of the tangential behavior of a scheme X at a point $p \in X$"*.

We use the tangent cone to study the local behavior of a scheme X at a *singular* point

p. In our case, these would be points $p \in \mathcal{H}$ that belong to the regular part $[X^{\text{reg}}]$ of a flat X, such that $\operatorname{codim}(X) > 1$. We give the following definition:

Definition 30. [EH00, Exer. III.-29] Let X be a subscheme of the affine space over a field K, that is, $X \subset \operatorname{Spec} K[x_1, \cdots, x_n]$. Assume moreover, that the point $p \in X$ is the origin $(x_1, \cdots, x_n) \in \mathbb{A}^n_K$ (or translate the origin to p).
The *tangent cone* $TC_p(X)$ of the scheme X at the point p is the subscheme defined as the zero locus of the leading terms of all elements $f \in I$, where $I = I(X)$ is the ideal of X. (The leading term of a polynomial f is also known as its initial form; it is the sum of the smallest degree monomials in f).

Even though we will only deal with hypersurfaces, where I is generated by a single polynomial f, it is necessary to work in the context of schemes, as our tangent cones are not reduced:

Example 31. We wish to describe the tangent cone of the discriminant hypersurface $\mathcal{H}$ of a (well-generated) irreducible complex reflection group W at the origin $\mathbf{0}$. For this, we need to compute the initial form of $\Delta(W; \mathbf{f}) = f_n^n + \alpha_2 f_n^{n-2} + \cdots + \alpha_n$ (see Cor. 16). Since $\deg(f_n) \gneq \deg(f_i)$, $i \leq n-1$, and since $\alpha_i \in \mathbb{C}[f_1, \cdots, f_{n-1}]$, the quasi-homogeneity of $\Delta(W; \mathbf{f})$ implies that the leading term is just f_n^n.
Thus, the equation of the tangent cone $TC_{\mathbf{0}}\mathcal{H}$ is $f_n^n = 0$; its zero set is the coordinate hyperplane orthogonal to f_n, but taken with multiplicity n.

In fact, considering the scheme-theoretic tangent cone allows us to define a very important invariant of a scheme, that will be fundamental throughout our analysis:

Definition 32 (ibid)**.** We define the *multiplicity* of a scheme X at p to be the degree of the (projectivized) tangent cone $\mathbb{P}TC_p(X)$ and denote it by $\operatorname{mult}_p(X)$. In the case that X is a hypersurface cut by the polynomial f at p, the degree of $\mathbb{P}TC_p(X)$ agrees with the degree of the initial form of f.

As with tangent spaces, invertible algebraic maps respect tangent cones. We will phrase the following proposition in the analytic category to provide a more exact reference, but the reader should not worry: We will only apply it to polynomial maps $f : \mathbb{C}^n \to \mathbb{C}^n$, in particular to the maps τ_X of Prop. 20.

Proposition 33. *[see the much stronger Whi65, Thm. 4.7] Suppose $p \in V \subset \mathbb{C}^n$, $p' \in V' \subset \mathbb{C}^n$ for analytic sets V and V', and let $\Phi : \mathbb{C}^n \to \mathbb{C}^n$ be a holomorphic map such that $\Phi(V,p) = (V',p')$. Suppose, furthermore, that Φ is a biholomorphic homeomorphism in a neighborhood of p. Then $\mathrm{d}\,\Phi(p)$ is a biholomorphism between the two tangent cones; that is,*

$$\mathrm{d}\,\Phi(p)\big(TC_p(V)\big) = TC_{p'}(V').$$

5.2 Transversality of the line L_y[1]

The most important geometric property of the line L_y is that it is transverse to the discriminant hypersurface $\mathcal{H}$ (Prop. 37). This result is fundamental, as a lot of the theory (nontrivially) relies on it. In particular, it is used in the proof of the finiteness of the LL map (Thm. 51) and in the main result of this section (Prop. 39), two statements that are themselves essential to the rest of this work.[2]

Before we proceed with Prop. 37, we need a few supporting Lemmas. The following is a corollary of Prop. 33; it relates the local geometry of the discriminant $\mathcal{H}$ with the combinatorics of the hyperplane arrangement (see Remark 22).

Lemma 34. *[Bes15, Lemma 5.4] Let W be a well-generated complex reflection group, (y,x) a point in the discriminant hypersurface $\mathcal{H}$, and let X be a flat such that $(y,x) \in [X^{\mathrm{reg}}]$. Then,*

$$\mathrm{mult}_{(y,x)}(\mathcal{H}) = \mathrm{codim}(X).$$

Proof. For an arbitrary point $[p] = (y,x)$, consider as in Prop. 20 the quotient map $\tau_X : W_X\backslash V \to W\backslash V$. We showed that τ_X is biholomorphic in a neighborhood of $[p]$, which implies by Prop. 33 (see also Defn. 32), that

$$\mathrm{mult}_{[p]}(\mathcal{H}) = \mathrm{mult}_{[p]_X}\big(\mathcal{H}(W_X)\big).$$

To compute the latter quantity, we need to first consider the decomposition into irreducible subgroups $W_X = W_1 \times \cdots \times W_j$ as in Section 2.1. From Remark 17, we can see

[1] **Although transversity might be the correct term [see Wik].**

[2] Moreover, it is deeply related to the theory of Frobenius manifolds where the unit field e is everywhere transverse to the discriminant $\mathcal{D}$ [see Her02, Remark 4.2].

that the initial form of $\Delta(W_X;(\mathbf{x},\mathbf{g}))$ is the product $(g^1_{r_1})^{r_1}\cdots(g^j_{r_j})^{r_j}$, of total degree $r_1+\cdots+r_j=\sum_{i=1}^j \operatorname{rank}(W_i)=\operatorname{codim}(X)$. $\square$

Notice that the explicit form of the discriminant $\Delta(W_X)$ relies on the fact that parabolic subgroups of well-generated groups are themselves well-generated (Prop.5) which is proven uniformly only for real W. See also [Bes15, Remark 5.5]; the statement of Lemma 34 is in fact a characterization of well-generated groups W.

Remark 35. Another corollary of Prop. 33 and the special form of the leading term of $\Delta(W_X;(\mathbf{x},\mathbf{g}))$ is that the tangent cone of $\mathcal{H}$ at (y,x) looks like the union of j-many tranversally intersecting hyperplanes, where j is the number of irreducible components of W_X. This might prove interesting in the context of wonderful compactifications [DP95], where irreducible flats are building blocks for the exceptional divisor.

As we mentioned earlier, the main result of this section is that the map rlbl produces parabolic Coxeter elements (Prop. 39). The next lemma is a weaker intermediate result that is necessary for the proof of the transversality property of L_Y (which in turn is the main ingredient for Prop. 39).

Lemma 36. *For any point (y,x_i) in $\mathcal{H}$, the label $c_i:=\operatorname{rlbl}(y,x_i)$ belongs to some parabolic subgroup W_{X_i} such that $(y,x_i)\in[X_i^{\text{reg}}]$.*

Proof. Consider the loop $b_{(y,x_i)}$ that surrounds (y, x_i), as in §4.2. Its Galois action sends the basepoint $v \in V$ to $c_i \cdot v$. Now, again using the notation from §4.2, let v' be the endpoint of the lift (in V) of the first part of the path β_i (just until we start tracing the counterclockwise circle around x_i).

Because of the homotopy lifting property (essentially), the Galois action of the small circle around x_i also sends v' to $c_i \cdot v'$ (see picture on the right). Now by shrinking the radius of the circle around x_i to zero, we can see that c_i must fix the corresponding limit point $\hat{v}$ of v', which is such that $\rho(\hat{V}) = (y, x_i)$ where $\rho : V \to W\backslash V$ is the quotient map.

Let X_i be the unique flat that contains $\hat{v}$ in its regular part (which, of course, implies that $[X_i^{\text{reg}}]$ contains (y, x_i)). By Steinberg's theorem we have that $V^{c_i} \supset \bigcap_{H \ni \hat{v}} H = X_i$; in other words, $c_i \in W_{X_i}$. □

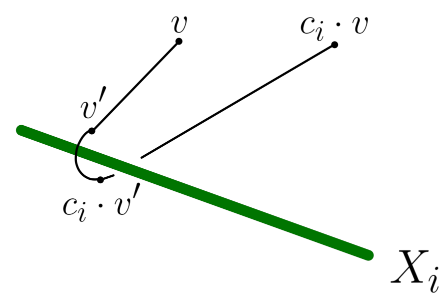


Figure 5.1: If $v'v$ is a lift of the path $-\beta_i$ starting at v', then $(c_i \cdot v')(c_i \cdot v)$ is the lift of the same path starting at $c_i \cdot v'$.

Proposition 37. *The line $L_y := y \times \mathbb{C} \subset Y \times \mathbb{C} \cong W\backslash V$ is transverse to the discriminant hypersurface $\mathcal{H}$ for all y.*

Proof. Assume on the contrary that for some point $y \in Y$, the line L_y is not transversal to $\mathcal{H}$. This means that there is a point $(y, x_i) \in L_y \cap \mathcal{H}$ such that the multiplicity of x_i as a root of $\big(\Delta(W, \mathbf{f}); f_n\big)|_y = 0$ is greater than the multiplicity of $\mathcal{H}$ at (y, x_i). We will show that this implies that the product of the labels of the (y, x_i)'s must have fixed points, and hence it cannot be the Coxeter element c.

Indeed, consider the reduced label $\text{rlbl}(y) = (c_1, c_2, \cdots, c_k)$. By Lemma 36, c_i belongs to a parabolic subgroup W_{X_i} such that $(y, x_i) \in [X_i^{\text{reg}}]$. Now, since by Lemma 34, we have $\text{mult}_{(y,x_i)}(\mathcal{H}) = \text{codim}(X_i)$, the first paragraph implies that $\sum_{i=1}^{k} \text{codim}(X_i) < n$, since n is the degree of $\Delta(W, \mathbf{f})$ as a polynomial in f_n.

In other words, this forces that $\bigcap X_i \neq \{0\}$, in fact that $\bigcap V^{c_i} \neq \{0\}$, since $V^{c_i} \supset X_i$. On the other hand $V^c \supset \bigcap V^{c_i}$, but Corollary 24 states that the Coxeter element $c = c_1 \cdots c_k$ can have no nontrivial fixed point; a contradiction.

□

Remark 38. As we mentioned earlier, the previous Proposition is critical in the proof of the finiteness of the LL map (Thm. 51). In [Bes15, in the proof of Lemma 5.6], the argument that is provided for it is flawed; it assumes that the tangent cone of the discriminant hypersurface is a closed subvariety of the tangent bundle. This is not always true for arbitrary hypersurfaces; the simplest counterexample is Whitney's umbrella $y^2 - zx^2$ [see also Whi65, Remark 5.5].
In our situation, the above argument can be extended to indeed show that the tangent cone of $\mathcal{H}(W)$ is the same as its Whitney cone $C4$ [for the definition, see Whi65, Section 3] which is always a closed subvariety. We will not pursue this direction however.

5.3 Parabolic Coxeter elements

The next Proposition is the principal component in the proof of Corollary 65 which relates parabolic Coxeter elements with noncrossing elements. Bessis had originally proved Cor. 65 uniformly for the real case [Bes06b, Lemma 1.4.3] and in a case-by-case fashion for well-generated groups [see earlier version 2 Bes06a, Lemma 7.4]. Both proofs moreover were combinatorial.
Afterwards he sketched a uniform geometric argument [Bes15, Lemma 7.4] but with few details (see also Remark 40). Our proof below should be read as an elaboration of Bessis' reasoning.

Proposition 39. *For any point $(y, x_i) \in \mathcal{H}$ the label $c_i := \mathrm{rlbl}(y, x_i)$ is a parabolic Coxeter element. In fact, it is a Coxeter element of a parabolic subgroup W_X for which $(y, x_i) \in [X^{\mathrm{reg}}]$. Furthermore, every parabolic Coxeter element is realizable as the label of some point in $\mathcal{H}$, after a suitable choice of basepoint in V^{reg}.*

Proof. Consider the loop $\delta_X \in B(W_X)$ as in Remark 26. It defines a parabolic Coxeter element c_X after the surjection $\pi : B(W_X) \twoheadrightarrow W_X$. Now Prop. 20 states that the map τ_X induces (locally at some representative of (y, x_i) in $W_X \backslash V$) an embedding $B(W_X) \hookrightarrow B(W)$. We will show that, under this embedding, the element δ_X is mapped to a loop homotopic to the $b_{(y,x_i)}$ (the loop that defines the label $\mathrm{rlbl}(y, x_i)$ as in Defn. 29).

We write as in Prop. 20 the map $\tau_X : W_X\backslash V \to W\backslash V$:

$$\tau_X : (\mathbf{x}, \mathbf{g^1}, \cdots, \mathbf{g^j}) \to (f_1, \cdots, f_n).$$

Now, we pick a preimage $[p]_X$ under τ_X of the point (y, x_i). Its coordinates will be given by $(x_1, \cdots, x_k, 0, \cdots, 0) =: (\mathbf{x}, \mathbf{0})$ for some $\mathbf{x} \in X^{\mathrm{reg}}$. As in Remark 26, we may represent the element δ_X via the loop given by

$$\mathbf{x} = \text{fixed}, \quad g^i_t = 0,\ t < r_i, \quad g^i_{r_i} = e^{2\pi i\theta},\ \theta \in [0, 1],$$

where r_i is the rank of W_i if $W_X = W_1 \times \cdots \times W_j$ (see Section 2.1).
The tangent cone of $\Delta(W_X) \times X$ at $[p]_X$ is given by the equation $(g^1_{r_1})^{r_1} \cdots (g^j_{r_j})^{r_j} = 0$ (as in the proof of Lemma 34). Since τ_X is locally invertible at $[p]$, Prop. 33 implies that the tangent cone of $\mathcal{H}$ at $(y, x_i) = \tau_X([p]_X)$ will be the product of j-many linear forms each raised to the r_i^{th} power:

$$TC_{(y,x_i)}(\mathcal{H}) :\ (\lambda^1_1 s_1 + \cdots + \lambda^1_n s_n)^{r_1} \cdots (\lambda^j_1 s_1 + \cdots \lambda^j_n s_n)^{r_j} = 0 \tag{5.1}$$

Here, the s_i's are local coordinates of $W\backslash V$ at (y, x_i) (in the direction of the f_i's) and the λ^i_t's are scalars given by $\mathrm{d}\,\tau_X|_{[p]_X}$.
On the other hand, since τ_X acts locally as a linear map, the image $\tau_X(\delta_X)$ would be homotopic to the loop given in local coordinates by $s_i = \rho_i e^{2\pi i\theta}$ for some constants ρ_i. We only need a homotopy from this loop to the one given by $\{s_i = 0 \text{ when } i \neq n, \text{ and } s_n = e^{2\pi i\theta}\}$ (which is essentially $b_{(y,x_i)}$).
This is always possible, just by scaling the coefficients $\rho_i,\ i \neq n$ by a complex variable z that goes to 0 and avoids the (j-many distinct) roots of the equation:

$$(\lambda^1_1 z\rho_1 + \cdots + \lambda^1_{n-1} z\rho_{n-1} + \lambda^1_n \rho_n)^{r_1} \cdots (\lambda^j_1 z\rho_1 + \cdots \lambda^j_{n-1} z\rho_{n-1} + \lambda^j_n \rho_n)^{r_j} = 0.$$

It is necessary however that $\lambda^i_n \neq 0$ or our homotopy would rotate the loop $\tau_X(\delta_X)$ onto one that lives inside the tangent cone $TC_{(y,x_i)}(\mathcal{H})$ (in which case, we couldn't be certain it defines an element in $B(W)$). This is in fact a consequence of the transversality of the line L_y on $\mathcal{H}$ (Prop. 37). Indeed, if any $\lambda^i_n = 0$, then the line L_y (given by $s_i = 0,\ i \neq n$ in local coordinates) satisfies (5.1) which means it is not transversal to $\mathcal{H}$, a contradiction.

The last statement of the proposition is only meant to indicate that the Coxeter element associated with the resulting label, will depend on our choice of basepoint in $P(W_X)$.

□

Remark 40. Bessis' geometric argument for the previous statement [Bes15, Lemma 7.4] didn't explain why the image $\tau_X(\delta_X)$ should really be homotopic to a label. Already in our Example 21, in formula (3.5), one can see that the image of δ_X does not live in a fixed line L_y (as not only f_4 but also f_3 depend on the highest degree invariant g_3 of the parabolic $\mathfrak{S}_3$).
For that matter, not all "straight" loops (that is, loops that are embedded in straight complex lines of any direction) around a point are homotopic. Already at the origin, a straight loop around direction f_i will map to a regular element of order d_i (if d_i is a regular number).
Finally we want to point out the significant (but also very subtle) reliance of yet another fact (Prop. 39), on the transversality of the line L_y on $\mathcal{H}$.

5.4 The Hurwitz rule

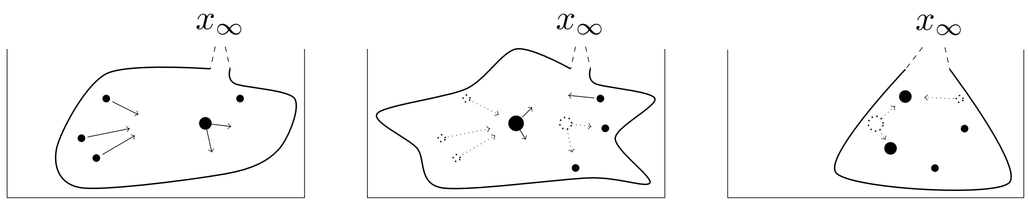
Up to now we have only described static properties of the labeling map; we would like to know how $\mathrm{rlbl}(y)$ is affected as y varies in Y. As a first step we record a criterion for checking that two loops are homotopic if they live in different lines L_y. It is illustrated in Figure 5.2, where each picture is a line L_y and the black dots are the points $L_y \bigcap \mathcal{H}$.

Lemma 41. *[Bes15, Lemma 6.15 (The Hurwitz rule)] A continuous family of loops $\{\gamma_t\}$ that each lie in a single line L_{y_t}, and that are all based at the corresponding x_∞*

$$\gamma_t = (y_t, x_t) : [0,1] \to W\backslash V^{\mathrm{reg}} \subset Y \times \mathbb{C}$$
$$s \mapsto (y_t, x_t(s))$$

is a homotopy if the points in $L_{y_t} \bigcap \mathcal{H}$ never intersect the loop $x_t : [0,1] \to \mathbb{C}$.

The next step would be some sort of explicit description of the *action* of a given path in Y on the label $\mathrm{rlbl}(y)$. An initial difficulty is that we have no a-priori control on what happens to the intersections $L_y \bigcap \mathcal{H}$. That is, we may not *assume* that there is always a path in Y that can rearrange the points (y, x_i) in any prescribed way.



(a) As y varies, ...

(b) the points in $L_y \bigcap \mathcal{H}$ might merge and unmerge

(c) ...and the loop might change shape as well.

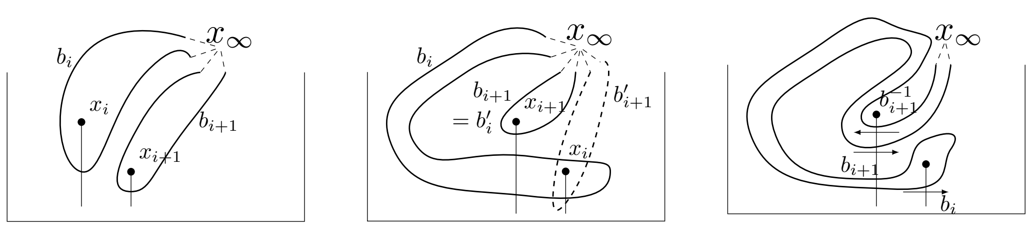
Figure 5.2: A pictorial description of a homotopy.

As it happens this is true (see Corol. 53), but it relies on the geometric analysis of the LL map in the next Section. For now we will phrase the following statement, conditionally on the existence of a suitable path:

Corollary 42. *[The Hurwitz action; see Bes15, Corol. 6.20] Assume there is a path θ in Y that connects two points y_0 and y_1 and that along this path, the only change affected on the intersections $L_y \bigcap \mathcal{H}$ is that x_i and x_{i+1} move counter-clockwise around each other. That is, no other points x_j move and in the end, the relative (complex lexicographic) order of x_i and x_{i+1} has changed.*
Then, the labels of the points y_0 and y_1 are related as follows:

$$\textit{If } \operatorname{rlbl}(y_0) = (c_1, \cdots, c_i, c_{i+1}, \cdots, c_k), \quad \textit{then } \operatorname{rlbl}(y_1) = (c_1, \cdots, c_{i-1}, c_{i+1}, c_{i+1}^{-1} c_i c_{i+1}, \cdots, c_k).$$

Proof. This is a striaght forward application of the previous Lemma. The following Figure 5.3 describes the loops that give the homotopy. □



(a) As x_i and x_{i+1} move around each other...

(b) ...the loop b_i streches to avoid b_{i+1}.

(c) The loop $b_{i+1}^{-1} b_i b_{i+1}$ is homotopic to b'_{i+1}.

Figure 5.3: The Hurwitz action.

Remark 43. Already in 1891, as Hurwitz [Hur91] was studying the newly created Riemann surfaces, he produced the above argument. In his Section 2 [Abschnitt II], the monodromy groups A and B are actions of the braid and pure braid groups B_n and PB_n respectively.

In fact, it is clear that these actions are induced by continuously deforming point configurations. That is, even though neither the (concept of a) fundamental group nor the Braid group had been introduced at the time, Hurwitz (essentially) interpreted B_n as the fundamental group of the configuration space of n points in $\mathbb{C}$. It would take 71 years for Fox and Neuwirth [FN62b] to rediscover that argument!

Chapter 6

The Lyashko-Looijenga morphism.

In the previous two sections we constructed, in a geometric way, factorizations of the Coxeter element c and proved a few structural properties for them. In Section 7 we will show that this geometric procedure is sufficient to produce *all* reduced block factorizations of c (see Defn. 8).

We will do that by studying the geometry and algebraic properties of a morphism that we introduce in this Section (Defn. 44), which records the intersection points of the line L_y with the disciminant hypersurface $\mathcal{H}$ as y varies.

Recall our decomposition of the orbit space into the product $W\backslash V \cong Y \times \mathbb{C}$ from § 4.1 and the notation E_n for the space of centered configurations from § 3.1.2. Also, the equation (4.1) associated to the discriminant, where we view f_n as an unknown t and $y = (f_1, \cdots, f_{n-1})$ as the parameter:

Definition 44. For an irreducible well-generated complex reflection group W, we define the *Lyashko-Looijenga* map by:

$$\begin{array}{ccc} Y & \xrightarrow{LL} & E_n \\ y = (f_1, \cdots, f_{n-1}) & \longrightarrow & \text{multiset of roots of } \big(\Delta(W, \mathbf{f}); (y, t)\big) = 0 \end{array}$$

and denote it by LL, or $LL(W)$ when there are multiple groups in question.

We will write $LL(y) = \{x_1, \cdots, x_k\}$ to indicate that the natural target of LL is an unordered configuration space, but we will always index the x_i's with complex lexicographical order, so as to be compatible with the rlbl map. As the notation suppresses the multiset data, we will define $\text{mult}_{x_i}\big(LL(y)\big)$ to be the multiplicity of x_i in the multiset $LL(y)$.

Notice that there is a simple description of LL as an algebraic morphism. Indeed the (multiset of) roots of a polynomial is completely determined by its coefficients, therefore we can express LL as the map:

$$\begin{array}{ccc} Y \cong \mathbb{C}^{n-1} & \xrightarrow{LL} & E_n \cong \mathbb{C}^{n-1} \\ y = (f_1, \cdots, f_{n-1}) & \longrightarrow & \big(\alpha_2(f_1, \cdots, f_{n-1}), \cdots, \alpha_n(f_1, \cdots, f_{n-1})\big) \end{array}$$

where the α_i's are as in Corollary 16.

6.1 Geometric Interlude, No. 2

In our second Geometric intermission, we introduce the fundamental notions of finiteness and flatness for a map and provide criteria for them. We will state the various results in different degrees of generality, trying to provide the most accessible references. Be assured however, that we will only apply them in the simplest of cases, namely a quasi-homogeneous morphism $F : \mathbb{C}^n \to \mathbb{C}^n$.

Definition 45. [Eis95, Corol. 9.3] A morphism of affine algebraic sets $F : X \to Y$ such that the coordinate ring $A[X]$ of X is a finitely generated $A[Y]$-module, is called a *finite* map.

In the case of a quasi-homogeneous morphism (i.e. a morphism that is given by quasi-homogeneous polynomials on $\mathbb{C}^n$), there is a simple criterion for finiteness. It appears explicitly in the literature, for instance as [LZ04, Thm. 5.1.5], but the proof there is mainly a sketch since the book does not assume familiarity with geometry. Since it is essential for our work, we provide a more complete argument piecing together known results from commutative algebra:

Proposition 46. *A quasi-homogeneous map $F : \mathbb{C}^n \to \mathbb{C}^n$ is finite, if and only if $F^{-1}(\mathbf{0}) = \mathbf{0}$.*

Proof. Indeed, assume that F has the form

$$\mathbb{C}^n \ni \mathbf{x} := (x_1, \cdots, x_n) \xrightarrow{F} \mathbf{g} := \big(g_1(\mathbf{x}), \cdots, g_n(\mathbf{x})\big),$$

where $g_i(\mathbf{x})$ are quasi-homogeneous polynomials. Then, notice that:

$$F^{-1}(\mathbf{0}) = \mathbf{0} \iff \sqrt{(g_1, \cdots, g_n)} = (x_1, \cdots, x_n) \iff (x_1, \cdots, x_n)^N \subset (g_1, \cdots, g_n)$$

for suitably large N, which is precisely the condition for $\{g_1, \cdots, g_n\}$ to be a (quasi-)homogeneous system of parameters [see BH93, below Defn. 1.5.13]. This is, in turn, equivalent to $\mathbb{C}[x_1, \cdots, x_n]$ being a finite module over $\mathbb{C}[g_1, \cdots, g_n]$ [see BH93, Thm. 1.5.17]. $\square$

Another notion that is important in the geometry of the LL map, is that of flatness. Recall that a homeomorphism of rings $F : A \to B$ is called *flat* if it makes B a flat A-module.

The concept of flatness for a map usually means that its fibers are well-behaved, see for instance Prop. 73. A finite morphism is not always flat, the easiest examples appearing when the image is not Cohen-Macaulay [see Kov]. In our case however, we have the following:

Proposition 47. *[see the stronger Eis95, Corol. 18.17] Finite morphisms between regular (smooth) varieties are flat.*

The notions of finiteness and flatness have similar meanings in the context of local analytical geometry. In fact, in that setting, they force very useful properties on the maps in terms of the *complex* topology of the ambient space. Since these will be necessary for various arguments involving the local behavior of the LL map, we present them here briefly. We further explain why, in our context, it is safe to move from the algebraic to the analytic category and vice versa.

Recall that a *germ* of an analytic space X at x, denoted (X, x), is an equivalence class of analytic spaces that agree with X at some neighborhood of x [see JP00, Section 3.4]. Similarly a germ of an analytic map $f : (X, x) \to (Y, y)$ is an equivalence class of maps that agree on a neighborhood of x. We denote by $\mathcal{O}_{X,x}$ the space of germs of analytic functions of X at x.

We are now ready to define the concepts of finiteness and flatness in the (local) analytic category. Notice that we have interchanged the definition and theorem from the reference to make the analogy to the algebraic category better.

Definition 48. Let $f : (X, x) \to (Y, y)$ be a map between germs of analytic spaces. Then,

1. f is called *finite* if $\mathcal{O}_{X,x}$ is a finitely-generated $\mathcal{O}_{Y,y}$-module. A finite map f is always closed and has finite fibers in a neighborhood of x [compare JP00, Thm. 3.4.24 and Defn. 3.4.9 and 3.4.7].

2. f is called *flat* if $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Y,y}$-module [Fis76, Section 3.11].

Another topological consequence of the finiteness of f as described above, is that it is *proper* (see below) in a neighborhood of x. Indeed, it is not difficult to see that a closed map with finite fibers satisfies the following more common definition:

Definition 49. [Chi89, Section 3.1] A continuous map $f : X \to Y$ between topological spaces is called *proper* if the preimage of every compact set $K \subset Y$ is a compact set in X.

Remark 50. For a quasi-homogeneous map $F : \mathbb{C}^n \to \mathbb{C}^n$, it is indeed equivalent to check finiteness or flatness in the algebraic or the analytic setting. This comes from the fact that both the polynomial ring $\mathbb{C}[x_1, \cdots, x_n]$ and the analytic ring $\mathcal{O}_{\mathbb{C}^n,\mathbf{0}} := \mathbb{C}\{x_1, \cdots, x_n\}$ are local (homogeneous) Noetherian rings and have the same completion (the ring of formal power series $\mathbb{C}[[x_1, \cdots, x_n]]$). Indeed, the completion of a local Noetherian ring is faithfully flat [see Stacks, Tag 00MC], and faithfully flat ring maps respect finiteness and flatness [see Stacks, Tag 00HJ].

6.2 Finiteness of the LL map.

The most important property of the LL map is that it is a finite morphism. The argument below, which Bessis credits to Looijenga, appears completely in [Bes15]. However, the presentation there is faulty as it relies on a wrong proof of the transversality property of the slice L_y (see Prop. 37 and Remark 38). For completeness, we reproduce it here:

Theorem 51. *[Bes15, Thm. 5.3] The LL map is a finite morphism.*

Proof. Because the LL map is quasi-homogeneous, by Prop. 46 it is enough to show that $LL^{-1}(\mathbf{0}) = \mathbf{0}$.

Assume that there is some point $y \neq 0$ in Y such that $LL(y) = \mathbf{0}$. This means that $\alpha_i(y) = 0$ for all i, and 0 is a root of multiplicity n for the equation $\big(\Delta(w, \mathbf{f}); f_n\big)|_y = 0$. By definition, this is also the intersection multiplicity $i\big((y, 0), L_y \cdot \mathcal{H}; \mathbb{C}^n\big)$ (the multiplicity of the point $(y, 0)$ in the intersection of the slice L_y and the discriminant $\mathcal{H}$).

Now, by Prop. 37 the line L_y is transversal to $\mathcal{H}$ for all y. This implies by [Ful98, Corol. 12.4] that the intersection multiplicity of (y, x) in $L_y \cdot \mathcal{H}$ equals the multiplicity of $\mathcal{H}$ at (y, x) (this is true for *all* (y, x)). In fact, we have

$$\operatorname{mult}_x \big(LL(y)\big) := i\big((y, x), L_y \cdot \mathcal{H}; \mathbb{C}^n\big) = \operatorname{mult}_{(y,x)}(\mathcal{H}), \tag{6.1}$$

where $\operatorname{mult}_x \big(LL(y)\big)$ denotes the multiplicity of x in the configuration $LL(y)$ and the first equality is essentially by definition.

Finally the condition $\operatorname{mult}_{(y,x)}(\mathcal{H}) = n$ can only be satisfied at the origin (i.e. when $y = \mathbf{0},\ x = 0$). Indeed, by Lemma 34 we have that if $[X]$ is the (unique) stratum of $\mathcal{H}$ that contains (y, x) in its regular part $[X^{\text{reg}}]$, then $\operatorname{mult}_{(y,x)}(\mathcal{H}) = \operatorname{codim}(X)$. Now, the only flat X for which $\operatorname{codim}(X) = n$ is the origin $\mathbf{0}$. $\square$

We will state the following propositions, including Corol. 59, as Corollaries of the finiteness of the LL map. This is not because they are *trivial* consequences of it, but because we would like to **warn the reader** that they depend on it; often in a subtle way!

Corollary 52. *The LL map, as a topological map between two copies of $\mathbb{C}^n$, is proper.*

Proof. After the discussion in §6.1, finiteness of the LL map implies (see Defn. 48) that it is closed (in the complex topology) and has finite fibers in a neighborhood of the origin. Since, it is also quasi-homogeneous, the same holds for all of $\mathbb{C}^n$. Now, as we mentioned earlier, this easily implies that LL is proper. $\square$

Corollary 53. *The LL map is surjective. Furthermore, any path (continuous movement) in the centered configuration space E_n can be lifted to a (not necessarily unique) path in Y.*

Proof. Indeed, finiteness of the LL map implies that the image $LL(Y)$ is a Zariski-closed subset of E_n, of dimension n [see Eis95, Corol. 9.3:(2)]. That is, $LL(Y) = E_n$.
The second statement, which the reader should associate with general "continuity of roots" arguments, is an elementary (but not trivial) consequence of properness and the algebraicity of LL. A detailed argument may be found in [Iva]. □

Remark 54. In fact, finiteness (or at least generic finiteness) seems necessary to prove even that *there exists* at least one point y such that $LL(y)$ is a configuration of n *distinct* points.

Corollary 55. *The LL map is flat.*

Proof. Indeed, the LL morphism defined from $\mathbb{C}^{n-1}$ to $\mathbb{C}^{n-1}$ is a finite map between regular varieties, so Prop. 47 applies. □

Corollary 56. *[Bes15, Thm. 5.3] The degree of the LL map is equal to the Hurwitz number*

$$\deg(LL) = \frac{h^n n!}{|W|}.$$

Proof. Because the map $LL : \ \mathbf{f}' := (f_1, \cdots, f_{n-1}) \to \big(\alpha_2(\mathbf{f}'), \cdots, \alpha_n(\mathbf{f}')\big)$ is finite and quasi-homogeneous, its degree is given by the formula:

$$\deg(LL) = \frac{\prod_{i=2}^{n} \deg(\alpha_i)}{\prod_{i=1}^{n-1} \deg(f_i)}.$$

This is a version of Bezout's theorem for quasi-homogeneous polynomials [see LZ04, Thm. 5.1.5]. For the proof, see [Chi89, Thm. 2', p.114] for the analytic category, and [Bes15, Thm. 5.3] for an algebraic version (a Hilbert series calculation).
In our case, recall from Corollary 16 that $\deg(\alpha_i) = ih$, so that we have:

$$\deg(LL) = \frac{\prod_{i=2}^{n} ih}{\prod_{i=1}^{n-1} d_i} = \frac{h^{n-1} n!}{\frac{|W|}{h}} = \frac{h^n n!}{|W|}.$$

□

The following Corollary says that the LL map is compatible with the absolute length function l_R. It is essentially [Bes15, Lemma 7.7] and is explicitly stated as [Rip12, Prop. 3.4 and Property (P2)].

Corollary 57. *[Bes15] Let* $LL(y) = \{x_1, \cdots, x_k\}$ *and* $\mathrm{rlbl}(y) = (c_1, \cdots, c_k)$*, and choose flats* X_i *such that* $(y, x_i) \in [X_i^{\mathrm{reg}}]$*. Then*

$$\mathrm{codim}(X_i) = l_R(c_i) = \mathrm{mult}_{x_i}\big(LL(y)\big).$$

Proof. Consider a path in E_n that separates the "multiple" point x_i into $n_i := \mathrm{mult}_{x_i}\big(LL(y)\big)$ many distinct ones, named $x_i^1, \cdots, x_i^{n_i}$, all arbitrarily close to x_i. By Corollary 53, we may lift this to a path in Y which will terminate at a point y' such that

$$LL(y') = \{x_1, \cdots x_{i-1}, x_i^1, \cdots, x_i^{n_i}, x_{i+1}, \cdots, x_k\}$$

, and by the Hurwitz rule (Lemma 41) we will have that $c_i^1 \cdots c_i^{n_i} = c_i$.
Now, all the c_i^j's are reflections. Indeed, by Prop. 39, there are corresponding flats X_i^j, such that c_i^j is a coxeter element for $W_{X_i^j}$. In particular, $V^{c_i^j} = X_i^j$ is a hyperplane because of Lemma 34 and (6.1).
This in turn implies that $l_R(c_i) \leq n_i$. Additionally, it is known that $\mathrm{codim}(X_i) \leq l_R(c_i)$ [see for instance HLR17, Prop. 2.11], which gives

$$\mathrm{codim}(X_i) \leq l_R(c_i) \leq \mathrm{mult}_{x_i}\big(LL(y)\big).$$

Finally, the first and last terms are equal, because of (6.1) and Lemma 34, which completes the argument. □

Remark 58. It is a theorem of Carter [Car72] that for real reflection groups W, $\mathrm{codim}(V^w) = l_R(w)$ for all $w \in W$. This statement however, is no longer true for complex reflection groups [see Fos14]. Therefore, one can interpret Corollary 57 as a partial generalization of Carter's theorem, for the case of parabolic Coxeter elements.

6.2.1 The covering map property

We define the *bifurcation locus* $\mathcal{K}$ to be the set of points y in Y, such that $LL(y)$ has less than n *distinct* elements. We refer to its complement as the regular part of Y, and write $Y^{\mathrm{reg}} := Y \setminus \mathcal{K} = LL^{-1}(E_n^{\mathrm{reg}})$.

Corollary 59. *[Bes15, Lemma 5.7] The restriction* $LL : Y^{\mathrm{reg}} \to E_n^{\mathrm{reg}}$ *is a topological covering map.*

Proof. A finite map over $\mathbb{C}$, between nonsingular varieties, is always a topological covering map when restricted to its unramified part (this is a consequence of the inverse function theorem).
The ramification locus of a finite map is given by the zero set of its Jacobian (find explicit reference for this?). The calculation that $\mathcal{K} \subset Z(J_{LL})$ is done in [Bes15, Lemma 5.7] and some missing details appear in [Rip12, Prop. 4.1]. See also Remark 81. $\square$

Corollary 60. *[Bes15, Cor. 5.8] The space* Y^{reg} *is a* $K(\pi, 1)$*.*

Proof. This is immediate, since $\mathrm{Conf}_n(\mathbb{C}) \cong E_n^{\mathrm{reg}}$ is a $K(B_n, 1)$ [see FN62a, Corol. 2.2]. Notice that this is *not* the main result of [Bes15]. $\square$

Chapter 7

Geometry of the LL map via Combinatorics

The geometry of the LL map, both locally and generically, is very much related to the combinatorics of block factorizations of the Coxeter element c. The connection is mostly encoded in the Trivialization Theorem (but see also Prop. 79), the highlight of this Section and the whole Thesis.
In this Section we reproduce Bessis' proof of the Trivialization Theorem, which relies on the numerological coincidence of Corol. 69. Along the way, we elaborate on its dependence on various geometric facts about LL, fill in some missing details (for instance, Corol. 66), and build up a few new results (including Corol. 67) that we will use in the next Section.

7.1 Transitivity of the Hurwitz action

The driving (combinatorial) force behind the results in this section is Prop. 63. Before we state it, we recall the definition of the Hurwitz action of the braid group B_n:

Definition 61. For *any* group G, there is a natural action of the braid group B_n on the set of n-tuples of elements of G. The generator s_i acts via:

$$s_i * (g_1, \cdots, g_i, g_{i+1}, \cdots, g_n) = (g_1, \cdots, g_{i-1}, g_{i+1}, g_{i+1}^{-1} g_i g_{i+1}, \cdots, g_n).$$

We call this the (right) *Hurwitz action* of B_n on G_n.

It is clear that the Hurwitz action respects the product of the g_i's. Therefore, we may restrict it on tuples that encode fixed length factorizations of elements of G. In our context, we consider the action of $B_{l_R(c)}$ on the set of reduced reflection factorizations of c, $\mathrm{Red}_W(c)$.

The Hurwitz action determines how the label $\mathrm{rlbl}(y)$ is affected as y varies in the base space Y. Recall first that after a choice of basepoint in E_n^{reg}, we may identify elements $b \in B_n$ with loops β in E_n^{reg}. We may lift those to paths in Y^{reg} via the covering map $LL : Y^{\mathrm{reg}} \to E_n^{\mathrm{reg}}$ (see Corol. 59) and we write $\beta \cdot y$ for the Galois action:

Lemma 62. *[Bes15, Corol. 6.20] The labelling map is equivariant with respect to the Hurwitz action and the Galois action. That is,*

$$\mathrm{rlbl}(\beta \cdot y) = b * \mathrm{rlbl}(y),$$

where β and b are as above.

Proof. This is an immediate consequence of Corol. 42, which states that the rlbl map is equivariant with respect to the Hurwitz and Galois action of the generators s_i of B_n. □

The driving force behind the results in this section is the following Proposition. The proof is uniform (via the combinatorics of chromatic pairs) for real reflection groups [see Bes06b, Prop. 1.6.1], and case-by-case for well generated W.[1]

Proposition 63. *[Bes15, Prop. 7.6] The Hurwitz action is transitive on* $\mathrm{Red}_W(c)$*.*

As with the finiteness of the LL map, we choose to list here, as Corollaries, a few consequences of the previous Proposition. Again, they don't follow trivially from it, but they unequivocally rely on it.

Corollary 64. *All reduced reflection factorizations of c appear as labels of points $y \in Y^{\mathrm{reg}}$. That is, the map* $\mathrm{rlbl} : Y^{\mathrm{reg}} \to \mathrm{Red}_W(c)$ *is surjective.*

Proof. This is just a rephrasing of the transitivity of the Hurwitz action (Prop. 63) in terms of the Galois action (see Lemma 62). □

[1] There is however a conjectural approach by Bessis [see Bes04, Conj. 6.2] towards a topological proof for the general case.

The following Corollary is important for the study of the ramification properties of the LL map. Bessis had given a uniform (combinatorial) proof for real W in [Bes06b, Lemma 1.4.3] and originally had checked it case by case for well-generated W. The proof below relies on Prop. 39, see Remark 40.

Corollary 65. *[for real W Bes06b, Lemma 1.4.3] For a well-generated group W, the set of parabolic Coxeter elements and the set of elements that are noncrossing, with respect to any Coxeter element c, coincide.*

Proof. The fact that parabolic Coxeter elements are noncrossing for some c is precisely Prop. 39. For a given parabolic subgroup W_X, different parabolic Coxeter elements will be noncrossing with respect to different Coxeter elements. This is just a consequence of the choice of basepoint in the embedding $B(W_X) \hookrightarrow B(W)$.
For the other direction, pick a noncrossing element $c_i \in [1, c]_{\leq_R}$ and recall that it can be written as a product $c_i = t_1 \cdots t_j$, where the t_i's are an initial string in a reduced reflection factorization $c = t_1 \cdots t_j \cdot t_{j+1} \cdots t_n$. By the previous Corol. 64, we know that there exist a point y such that $\operatorname{rlbl}(y) = (t_1, \cdots, t_n)$.
Consider now the configuration $LL(y) = \{x_1, \cdots, x_j, x_{j+1}, \cdots, x_n\}$ and a path in E_n^{reg} along which the first j points radially collide into their center of gravity. By Corol. 53, we may lift this to a path in Y which would terminate at a point y' in the ramification locus $\mathcal{K}$. By the Hurwitz rule (Lemma 41), its label will be $\operatorname{rlbl}(y') = (c_i, t_{j+1}, \cdots, t_n)$. That is, the arbitrary noncrossing element c_i might be represented as the label of a point $(y', x) \in \mathcal{H}$ after a suitable choice of basepoint. Finally, Prop. 39 states that all labels are parabolic Coxeter elements. This completes the proof. $\square$

The following Proposition is (also) necessary for the study of the ramification properties of the LL map (see Prop. 79). For real reflection groups W, this statement is true for any $w \in W$ [see Bes06b, above Lemma 1.4.2]. We write $\operatorname{Red}_{W_X}(w)$ for the set of reduced reflection factorizations $w = t_1 \cdots t_j$, where all t_i's belong to W_X.

Corollary 66. *Let c_X be a coxeter element of the parabolic subgroup W_X. Then,*

$$\operatorname{Red}_W(c_X) = \operatorname{Red}_{W_X}(c_X).$$

Proof. Consider a reduced factorization $c_X = t_1 \cdots t_k$, where the t_i's are arbitrary reflections in W. Since, by the previous Corol. 65, c_X is a noncrossing element (for *some* c), the t_i's form an initial string of some (reduced reflection-) factorization $c = t_1 \cdots t_n$. Now, by Corol. 64, all such factorizations appear as labels. That is, there is a $y_0 \in Y$ such that $\mathrm{rlbl}(y_0) = (t_1, \cdots, t_n)$. We may follow a path in E_n that crushes the first (lexicographically) k points $\{x_1, \cdots, x_k\}$ of $LL(y_0)$ into a single point x.

Again, we may lift this path to Y (via Corol. 53), and get an endpoint y_1, such that $LL(y_1) = \{x, x_{k+1}, \cdots, x_n\}$ where x has multiplicity k. If, as in the previous Corollary, we choose the path in E_n to be a radial collision, it is once more clear by the Hurwitz Rule (Lemma 41) that $\mathrm{rlbl}(y_1, x) = c_X$. As y moves from y_0 to y_1, we wish to understand the family of paths $\beta_y \cdot \beta_i$, $1 \leq i \leq k$, that give us the labels t_i (see Section 4.2). In particular, we want to understand their lifts $\rho_*(\beta_y \cdot \beta_i)$ in V via the covering map $\rho : V^{\mathrm{reg}} \to W \backslash V^{\mathrm{reg}}$. Since the deck transformation associated to $b_{(y,x)}$ is multiplication by c_X, the lift $\rho_*(\beta_{y_1} \cdot \beta_1)$ must terminate in X^{reg}.

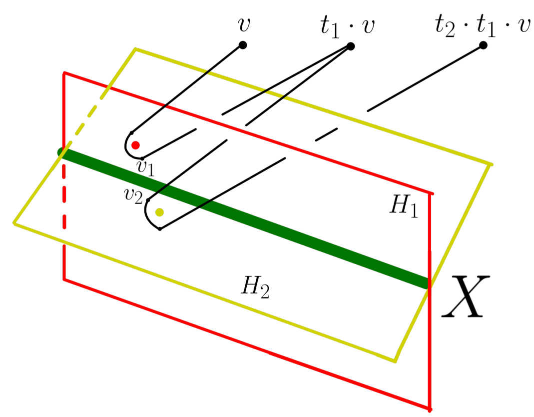


Figure 7.1: Notice that the paths $(t_1 \cdot v)v_1$ and $(t_1 \cdot v)v_2$ lift the paths β_1 and β_2, which may be assumed identical until very close to x.

Therefore, as y approaches y_1, the lifts $\rho_*(\beta_y \cdot \beta_i)$, $1 \leq i \leq k$, can only approach some of the hyperplanes that contain X (see Fig. 7.1). That is $V^{t_i} \supset X$, or $t_i \in W_X$ which is what we need.

□

Corollary 67. *The degree of the LL map must be a multiple of the number of reduced* *reflection factorizations of c. In fact, there is a number k such that the map:*

$$LL \times \mathrm{rlbl} : Y^{\mathrm{reg}} \to E_n^{\mathrm{reg}} \times \mathrm{Red}_W(c)$$

is k -to- 1.

Proof. This is a simple consequence of the homotopy lifting property of the covering map $LL : Y^{\mathrm{reg}} \to E_n^{\mathrm{reg}}$ (Corol. 59). It will be however very important in the next Section 8.

By Corol. 64 we know that all elements of $\mathrm{Red}_W(c)$ are realizable as labels. Let k be the maximal number such that for a given configuration $e \in E_n^{\mathrm{reg}}$, there are k-many points $\{y_1, \cdots, y_k\}$ in the fiber $LL^{-1}(e)$, with the same label $\mathrm{rlbl}(y_i)$.

Now, consider an arbitrary factorization $c = t_1 \cdots t_n$ in $\mathrm{Red}_W(c)$ and an element b of the braid group B_n, such that $(t_1, \cdots, t_n) = b * \mathrm{rlbl}(y_i)$. By Lemma 62, the Hurwitz action on the factorization can be realized via the Galois action of a loop $\beta \in \pi_1(E_n^{\mathrm{reg}}) = B_n$ on any point y_i.

Finally, by the homotopy lifting property, we may lift the loop β to k *distinct* paths in Y^{reg}, each starting at a different y_i. This means that there are at least k-many separate points y_i' such that $\mathrm{rlbl}(y_i') = (t_1, \cdots, t_n)$. The statement follows from our assumption on the maximality of k.

□

In fact, as the next Proposition shows, k has to equal 1. However, there is as of yet no uniform proof of this statement, the largest family simultaneously covered by a single argument being the Weyl groups [see Mic16]. In the next Section 8, we will describe a (partially conjectural) approach for a proof of a stronger assertion (the **Trivialization Theorem for** $\mathbf{Y}^{\mathrm{reg}}$), that only depends on the transitivity of the Hurwitz action.

Proposition 68. *[Bes15, Prop. 7.6] The number of reduced reflection factorizations of the Coxeter element c is given by the Hurwitz number:*

$$|\,\mathrm{Red}_{\mathrm{W}}(c)| = \frac{h^n n!}{|W|}.$$

Corollary 69 (Numerological coincidence)**.** *The degree of the LL map equals the number of reduced reflection factorizations of the coxeter element c:*

$$\deg(LL) = |\,\mathrm{Red}_W(c)|$$

We are now ready to state and prove the generic version of the Trivialization Theorem:

Trivialization Theorem for $\mathbf{Y}^{\mathrm{reg}}$. *The map* $LL \times \mathrm{rlbl} : Y^{\mathrm{reg}} \to E_n^{\mathrm{reg}} \times \mathrm{Red}_W(c)$ *is a bijection.*

Proof. By Corollary 67, there exists a number k such that

$$\deg(LL) = k \cdot |\,\mathrm{Red}_W(c)|,$$

and the map $LL \times \mathrm{rlbl}$ is k-to-1. By the previous Corollary, k must be equal to 1 and the result follows. □

7.2 Geometric Interlude, No. 3

We recall here the concept of multiplicity of a morphism at a point, which we will need to be able to phrase certain results on the local geometry of the LL map. We will work in the analytic category.

Definition 70. [Dou68, Section 5, Defn. 1] Let $f : X \to S$ be a finite morphism of analytic spaces, and $s \in S$. For each point x in the fiber $f^{-1}(s)$, the stalk $\mathcal{O}_{f^{-1}(s),x} = \mathbb{C} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{X,x}$ is a finite dimensional vector space over $\mathbb{C}$ (by $\mathcal{O}_{X,x}$ we denote the space of germs of holomorphic functions of X at x). We define the *algebraic multiplicity* of f at x to be the dimension of this vector space:

$$\mathrm{mult}_x(f) := \dim_{\mathbb{C}} \mathbb{C} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{X,x}.$$

In the case that X and S are complex manifolds (so they have to be smooth, etc.), there is a more elementary definition of multiplicity:

Definition 71. [Chi89, Section 10.2] Let $f : X \to S$ be a finite morphism of complex manifolds, let $\mathcal{B} \subset S$ be its branch locus. Pick a point $s \in S$ and a point x in the fiber $f^{-1}(s)$. Then, there are coordinate neighborhoods $x \in U$ and $s \in V$ such that the restriction $f|U \to V$ is also finite. In particular, for points $s' \in V \setminus \mathcal{B}$, the number of preimages of s' in U is constant; we define the *geometric multiplicity* of f at s to be this quantity:

$$\mathrm{mult}_x(f) := \lim_{s' \to s} \# f^{-1}(s') \cap U.$$

Remark 72. Of course, these two definitions are equivalent. The existence of the neighborhoods U and V above, which indeed explains the equivalence, is in [Chi89, Section 3.1].
The reason that we mention both of them is that the former is most often used in the *proof* of the ramification formula (7.1), but the latter is the one we will use when *applying* (7.1).

As we touched upon in §6.1, one of the important geometric properties of the LL map is that it is flat. For a finite morphism in general, flatness is equivalent to having equinumerous fibers, when their elements are counted with multiplicity:

Proposition 73. *[Dou68, Section 5, Thm. 1] Let $f : X \to S$ be a finite morphism between analytic spaces and let S be reduced. Then, f is flat if and only if the following holds for all $s \in S$:*

$$\sum_{x\in f^{-1}(s)} \mathrm{mult}_x(f) = \textit{constant} \quad (= \deg(f)) \tag{7.1}$$

Definition 74. The above expression is usually called the *ramification formula* for f. For a (stronger) algebraic analog [see Ful98, Example 4.3.7].

Corollary 75. *The LL map satisfies the ramification formula* (7.1)*.*

We note that, although it might seem that this definition of multiplicity and the one we gave in §5.1 have little in common, they are actually the same! The notion of multiplicity of a scheme at a point can be generalized to include the case of multiplicity at a subscheme [see Ful98, Example 4.3.4]. For us, the subscheme would be the (scheme-theoretic) fiber of the morphism LL.

7.3 The trivialization theorem.

Following Bessis, we introduce notation for block factorizations of the Coxeter element, with a prescribed number of blocks:

Definition 76. [Simplicial Hurwitz structures, see Bes15, Defn. 7.10]

Let k be a positive integer. We set

$$D_k(c) := \{(w_1, \cdots, w_k) \ \in W^k | \ c = w_1 \cdots w_k \text{ and } l_R(c) = \sum_i l_R(w_i)\},$$

$$D_\bullet(c) := (D_k(c))_{k \in \mathbb{Z}_{\geq 0}}.$$

The space E_n is similarly stratified by the multiplicities of the points in an arbitrary configuration. We need a notion of compatibility between configurations in E_n and factorizations in $D_\bullet(c)$:

Definition 77. [Bes15, Defn. 7.17]
Let $e \in E_n$ be some configuration $e = \{x_1, \cdots, x_k\}$, indexed lexicographically, and let $n_i := \mathrm{mult}_{x_i}(e)$ be the corresponding multiplicities. Let $\sigma = (w_1, \cdots, w_l) \in D_l(c)$ be a block factorization of c.
We denote by $E_n \boxtimes D_\bullet(c)$, the set of compatible pairs (e, σ). That is,

$$E_n \boxtimes D_\bullet(c) := \Big\{ \big(\underbrace{(x_1, \cdots, x_k)}_{e \in E_n}, \underbrace{(w_1, \cdots, w_l)}_{\sigma \in D_\bullet(c)} \big) \in E_n \times D_\bullet(c) \mid k = l \text{ and } n_i = l_R(w_i) \ \Big\}.$$

We are ready to state the Trivialization Theorem:

Trivialization Theorem. *The map* $LL \times \mathrm{rlbl} : Y \to E_n \boxtimes D_\bullet(c)$ *is a bijection.*

We will delay the proof a bit, as we first need to study the ramification properties of the LL map. We begin by defining the set F_σ of reduced reflection factorizations that refine a given block factorization σ:

Definition 78. [Bes15, Defn. 7.17] For $\sigma \in D_\bullet(c)$ and $\rho \in \mathrm{Red}_W(c)$, we write

$$\rho \vdash \sigma$$

if σ is a concatenation of terms of ρ. That is, if $\sigma = (w_1, \cdots, w_k)$ and $\rho = (t_1, \cdots, t_n)$, we must have $w_1 = t_1 \cdots t_{l_R(w_1)}$, etc.
For all $\sigma \in D_\bullet(c)$, we set

$$F_\sigma := \{\rho \in \mathrm{Red}_W(c) | \rho \vdash \sigma\}.$$

The following Proposition provides a fantastic relation between the local geometry of the LL map and the combinatorics of the non-crossing lattice. The argument appears already in [Bes15, Thm. 7.20] but the presentation there fails to acknowledge the subtle reliance on Corollary 66. We produce it here again for completeness.

Proposition 79. *[in the proof of Bes15, Thm. 7.20] Let $y \in Y$ with* $\mathrm{rlbl}(y) = \sigma$*. Then, the multiplicity of LL at y, in the sense of Defn. 71, is equal to $|F_\sigma|$.*

Proof. Let $e = LL(y) = \{x_1, \cdots, x_k\}$ be the resulting configuration and $(n_1, \cdots, n_k)$ the corresponding multiplicites. We pick small balls $y \in U \subset Y$ and $e \in V \subset E_n$ and wish to compute the limit:

$$\mathrm{mult}_y(LL) := \lim_{e' \to e} \#\big\{LL^{-1}(e') \cap U\big\},$$

with $e' \in V \cap E_n^{\mathrm{reg}}$.

Now, in the configurations e' the points must be arranged in k groups, each containing n_i distinct elements $\{x'^1_i, \cdots, x'^{n_i}_i\}$ all of which are located very close to the former point $x_i \in e$ (see figure draw the figure). For each $y' \in LL^{-1}(e') \cap U$, we must have $\mathrm{rlbl}(y') \vdash \sigma$ (indeed, the loop around the point x_i and the one that surrounds the n_i points x'^j_i are homotopic).

However, the Trivialization Theorem for Y^{reg}, which currently relies on the Numerological Coincidence, does not permit the same reduced reflection factorization to appear twice as a label. That is, there can be *at most* $|F_\sigma|$ such preimages y' in U.

As we move e' in the small ball V, we cannot construct representatives for *all* elements of the braid group B_n, but we may indeed construct (see Fig. 7.2) all of the image of the inclusion $B_{n_1} \times \cdots \times B_{n_k} \hookrightarrow B_n$[2] . The Galois action of these braids on the point $y' \in LL^{-1}(e') \cap V$ agrees with the Hurwitz action (Lemma 62):

$$B_{n_1} \times \cdots \times B_{n_k} * \Big((t_1^1, \cdots, t_1^{n_1}), \cdots, (t_k^1, \cdots, t_k^{n_k})\Big),$$

where $t_i^j := \mathrm{rlbl}(y', x'^j_i)$.

[2] This is precisely the Lemma about $B(W_I) \hookrightarrow B(W)$ for type A.

Now, if $\sigma = (w_1, \cdots, w_k)$, Prop. 39 implies that the w_i's are parabolic Coxeter elements (and as such, products of irreducible Coxeter elements). Therefore, the transitivity of the Hurwitz action implies that there are *at least* as many preimages y' in $LL^{-1}(e') \cap V$, as there are elements in the set $\mathrm{Red}_{W_1}(w_1) \times \cdots \times \mathrm{Red}_{W_k}(w_k)$.
On the other hand, Corollary 66 implies that

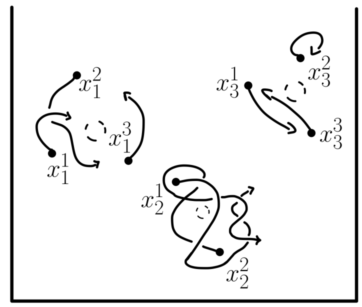


Figure 7.2: This is what a path in $V \cap E_n^{\mathrm{reg}}$ might look like.

$$\mathrm{Red}_{W_1}(w_1) \times \cdots \times \mathrm{Red}_{W_k}(w_k) = \mathrm{Red}_W(w_1) \times \cdots \times \mathrm{Red}_W(w_k),$$

and this completes the argument as the right hand side is by definition equal to F_σ.

$\square$

Remark 80. One might worry that as we move e' in V, creating complicated braids, some of the pre-images in $LL^{-1}(e')$ might escape the ball U. That this does not occur, is precisely the point of Defn. 71. In fact, this is because the map LL is open (see Defn. 48), so that we may assume $f(U) = V$.

Remark 81. The previous proposition suggests that LL is ramified over the whole preimage of the branch locus $E_n \setminus E_n^{\mathrm{reg}}$. Indeed, this is the case because any parabolic Coxeter element that is not a reflection has at least two distinct reflection factorizations. That is, $\mathrm{mult}_y(LL) = |F_\sigma| \geq 2$ unless $LL(y) \in E_n^{\mathrm{reg}}$.
Of course, this is not always true for arbitrary finite maps. In fact, any polynomial map $p : \mathbb{C} \to \mathbb{C}$ with at least two different (not including ∞) critical values fails this property.
Finite morphisms $F : X \to Y$ that indeed achieve this relation between the ramification and branch loci define what Ripoll calls "well-ramified" extensions [Rip10, Prop. 3.2]. The fact that LL is well-ramified plays a key role in Ripoll's factorization of the Jacobian of the LL map [see Rip12, Thm. 4.5].
As it happens, the resulting factorization had been very important in Saito's work [see Sai04, formula (2.2.3)], even though at the time he had no conceptual proof of it.

We are now ready to prove the Trivialization Theorem. The following argument is due to Bessis [Bes15, Thm. 7.20] and we provide it here for completeness:

*Proof of the **Trivialization Theorem**.* That the image lies in the set of compatible configurations and factorizations is Corol. 57. We will show injectivity and surjectivity separately:

1. Surjectivity of $LL \times \mathrm{rlbl} : Y \to E_n \boxtimes D_\bullet$: This is mostly the same argument as in Corol. 65. Let $\sigma = (w_1, \cdots, w_k) \in D_\bullet$ be a block factorization of c and pick a refinement $\rho \in \mathrm{Red}_W(c)$ ($\rho \vdash \sigma$).

   By Corol. 64, there exists a point y such that $\mathrm{rlbl}(y) = \rho$. Again, we may divide the points of $LL(y)$ in k-many groups so that the label of each group is w_i. Consider a path in E_n, along which the points of each of the above groups collide in a single point of multiplicity $l_R(w_i)$. By Corol. 53, we may lift this to a path in Y, whose endpoint y' satisfies via the Hurwitz rule (Lemma 41) that $\mathrm{rlbl}(y') = \sigma$.

   Notice that we can further follow a path to some y'' to achieve any compatible configuration $e \in E_n$. Indeed, as long as the relative (complex lexicographic) positions of the elements of $LL(y'')$ do not change, the label $\mathrm{rlbl}(y'') = \mathrm{rlbl}(y') = \sigma$ is not affected either.

2. Injectivity of $LL \times \mathrm{rlbl} : Y \to E_n \boxtimes D_\bullet$: This depends on Prop. 79:

   Pick a multiset $e \in E_n$. Since LL is flat, the ramification formula (7.1) states that

$$\sum_{y \in LL^{-1}(e)} \mathrm{mult}_y(LL) = \deg(LL) = |\,\mathrm{Red}_W(c)|, \tag{7.2}$$

   where the second equality is the numerological coincidence (Corol. 69). Now, if we group the summation in terms of the labels of the points $y \in LL^{-1}(e)$ and apply Prop. 79, we get

$$\sum_{y \in LL^{-1}(e)} \mathrm{mult}_y(LL) = \sum_{\substack{\sigma \in D_\bullet(c) \\ \sigma \text{ compatible with } e}} |LL^{-1}(e) \cap \mathrm{rlbl}^{-1}(\sigma)| \cdot |F_\sigma|. \tag{7.3}$$

   On the other hand, it is clear by Defn. 78 that

$$|\,\mathrm{Red}_W(c)| = \sum_{\substack{\sigma \in D_\bullet(c) \\ \sigma \text{ compatible with } e}} |F_\sigma|, \tag{7.4}$$

as all reduced reflection factorizations of c refine some block factorization with the same length statistic as σ.

Now, surjectivity of $LL \times \mathrm{rlbl}$ implies that $|LL^{-1}(e) \cap \mathrm{rlbl}^{-1}(\sigma)| \geq 1$. This, and comparison of the three equations above forces that, in fact, $|LL^{-1}(e) \cap \mathrm{rlbl}^{-1}(\sigma)| = 1$, which completes the proof.

$\square$

Chapter 8

Combinatorics via the geometry of the LL map.

In the previous Section, we presented Bessis' proof of the Trivialization Theorem, which relied on the numerological coincidence that $\deg(LL) = |\operatorname{Red}_W(c)|$ (Corol. 69). Here we will develop an approach (that is still partially conjectural) for a proof that relies only on the transitivity of the Hurwitz action (and Prop. 5).

The key idea lies in the following comparison, of the ramification formula (7.1) for LL, and the partition of $\operatorname{Red}_W(c)$ we saw in (7.4). As before, pick a multiset $e \in E_n$ and assume (for the sake of this illustration) that there is a geometric reason so that $\operatorname{mult}_y(LL)$ is constant when $\operatorname{rlbl}(y)$ stays fixed. Then, we will have:

$$\deg(LL) = \sum_{\substack{\sigma=(c_1,\cdots,c_k)\in D_\bullet(c)\\ \sigma \text{ compatible with } e}} |LL^{-1}(e)\cap \operatorname{rlbl}^{-1}(\sigma)|\cdot \operatorname{mult}_{y(\sigma)}(LL), \qquad \text{and} \tag{8.1}$$

$$|\operatorname{Red}_W(c)| = \sum_{\substack{\sigma=(c_1,\cdots,c_k)\in D_\bullet(c)\\ \sigma \text{ compatible with } e}} 1 \cdot \prod_{i=1}^{k} |\operatorname{Red}_W(c_i)|, \tag{8.2}$$

where $y(\sigma)$ is any y such that $\operatorname{rlbl}(y) = \sigma$.

It is clear (perhaps after recalling Corol. 66) that (8.2) gives a (parabolically) recursive condition for $\operatorname{Red}_W(c)$; in fact, it gives a new one for each possible tuple of multiplicities in e. If now, for some special multiset e, we had an *a priori* explanation of why $|LL^{-1}(e)\cap \operatorname{rlbl}^{-1}(\sigma)| = 1$, and if we could further *geometrically* relate $\operatorname{mult}_y(LL)$ with $\deg(LL)$

in a (parabolically) recursive way, we might be able to *inductively* force the equality $\deg(LL) = |\operatorname{Red}_W(c)|$.

This is precisely the premise of our first meta-conjecture in §8.2. However, we can refine the argument by using Corol. 67 and only require that we have a *geometric control* of the multiplicity of LL for even a single point in $LL^{-1}(e)$ (for our special multiset e). Our second meta-conjecture in §8.2 applies here.

Indeed, in §8.1 we find such a special multiset e. In §8.2, we ask for a geometric proof of a simple statement for the multiplicity of the LL map (this is the partially conjectural part), and finally in §8.3 we show how to put these together and prove the **Trivialization Theorem for** Y^{reg}, which in turn implies the general case as we showed in Section 7.

8.1 The LL map over lines

Here, we will prove (Prop. 85) the **Trivialization Theorem** for the restriction of the LL map on a special subset $Y_{n-1,1}$ of Y. We will start by defining its image in E_n:

Definition 82. Let $E_{n-1,1}$ be the set of configurations with two *distinct* points such that the first has multiplicity $n-1$:[1]

$$E_{n-1,1} := \{e \in E_n |\ e = \{x_L, x_H\} \text{ with } \operatorname{mult}_{x_L}(e) = n-1 \text{ and } \operatorname{mult}_{x_H}(e) = 1\}$$

Now, we define the preimage $Y_{n-1,1} := LL^{-1}(E_{n-1,1})$. One easily sees via Corollary 57 that $\overline{Y_{n-1,1}}$ is the projection of the 1-dimensional strata of $\mathcal{H}$ onto Y. We will denote the 1-dimensional flats (lines) of $\mathcal{A}_W$ by L, their images on $\mathcal{H}$ by $[L]$ and their projections on Y by $[L]_Y$.

In the following Proposition, we show an *a priori* relation between the combinatorics of *lines* in $\mathcal{L}_W$ and the geometry of the LL map. The reason is that the quasi-homogeneity of LL almost completely determines its formula over the 1-dimensional strata $[L]_Y$.

Proposition 83. *Let W be an irreducible complex reflection group of rank $n \geq 3$ and let L be a line in $\mathcal{L}_W$. Then, the restriction of the LL map on the stratum $[L]_Y$ has*

[1] It is easy to see that under the realization of E_n with coordinates given by symmetric polynomials on the points in the configuration, $E_{n-1,1}$ is embedded as a straight line.

degree

$$\deg\left(LL|_{[L]_Y}\right) = \frac{h}{|N_W(L) : W_L|}.$$

Remark. Here we are considering the projection $[L]_Y$ of the stratum $[L] \subset \mathcal{H}$ on Y, as a reduced analytic space and not as a subscheme of Y. In fact, we might as well define $\deg(LL|_{[L]_Y})$ to be the number of *distinct* preimages of any (nonzero) point of $E_{n-1,1}$ in $[L]_Y$.

Proof of Prop. 83. Since L is 1-dimensional, we may choose a vector $\mathbf{v} \in L$ and parametrize the whole flat as $L := \{\lambda \cdot \mathbf{v} \mid \lambda \in \mathbb{C}\}$. Now, as is clear by (8.3) below, the LL map has a simple formula over $[L]_Y$.

However, we still cannot easily compute the degree of LL over Y, as we get different answers if some of the $f_i(\mathbf{v})$'s are 0. For a uniform approach, we are going to lift the LL map all the way to the flat L:

$$\begin{array}{ccccccc} L & \xrightarrow{\quad\rho\quad} & [L] & \xrightarrow{\quad\pi_Y\quad} & [L]_Y & \xrightarrow{\quad LL\quad} & E_{n-1,1} \\ \lambda \cdot \mathbf{v} & \longrightarrow & \left(\lambda^{d_1} f_1(\mathbf{v}), \cdots, \lambda^h f_n(\mathbf{v})\right) & \longrightarrow & \left(\lambda^{d_1} f_1(\mathbf{v}), \cdots, \lambda^{d_{n-1}} f_{n-1}(\mathbf{v})\right) & \longrightarrow & \lambda^h f_n(\mathbf{v}) \end{array} \tag{8.3}$$

Notice that we have parametrized $E_{n-1,1}$ only with the coordinate f_n. Indeed, if $LL(y) = \{x_L, x_H\} \in E_{n-1,1}$, then x_L completely determines the configuration as $(n-1)x_L + x_H = 0$.

Now, we are going to calculate the degree of LL by computing the degrees of ρ, π_Y and the composition $LL \circ \pi_Y \circ \rho$. Notice that we will be using the assumption $\mathrm{rank}(W) \geq 3$ for the case of π_Y.

1. Recall that ρ is just the quotient map $V \xrightarrow{\rho} W\backslash V$ sending points in L to their W-orbits. Its degree over L is therefore the (maximal) number of points in the flat L that are in the same W-orbit. By definition, this will be the index of the pointwise stabilizer of L inside the setwise stabilizer:

$$\deg(\rho) = |N_W(L) : W_L|.$$

2. The projection map π_Y is a bijection. Indeed, the arbitrary preimage of $\pi_Y : \mathcal{H} \to Y$, for some point y, is precisely the configuration $LL(y)$. Now, if some point y

had two preimages (y, x_L) and (y, x'_L) in $[L]$, then the configuration $LL(y)$ would have at least $2n-2$ points accounting for multiplicity. But this cannot happen, as for $n \geq 3$ we have that $2n-2 > n$, which is the total number of points in $LL(y)$. We write:

$$\deg(\pi_Y) = 1.$$

3. The composition of the three maps has the very simple formula $\lambda \mathbf{v} \to \lambda^h f_n(\mathbf{v})$, where $\mathbf{v}$ is the fixed vector and λ the parameter. Notice that $f_n(\mathbf{v}) \neq 0$, or otherwise the fiber $LL^{-1}(\mathbf{0})$ would be 1-dimensional which contradicts the finiteness of the LL map (Thm. 51). We write:

$$\deg(LL \circ \pi_Y \circ \rho) = h.$$

Now, the degree of the LL map, restricted to $[L]_Y$, is given by:

$$\deg(LL|_{[L]_Y}) = \frac{\deg(LL \circ \pi_Y \circ \rho)}{\deg(\pi_Y) \cdot \deg(\rho)} = \frac{h}{|N_W(L) : W_L|}.$$

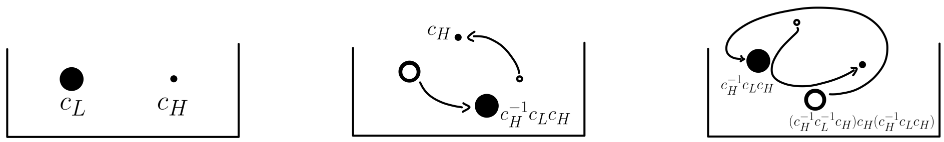
□

We are now going to relate the degree of the restriction on $LL_{[L]_Y}$ with the combinatorics of the noncrossing lattice (in particular the size of certain Kreweras orbits):

Corollary 84. *Let W be as above and let c_L be a parabolic Coxeter element such that $l_R(c_L) = n-1$ (i.e. $L := V^{c_L}$ is a noncrossing line). Now, if d is the smallest number such that $c^{-d} c_L c^d = c_L$, then the restriction of the LL map on the stratum $[L]_Y$ has degree d.*

Proof. Recall first by Prop. 39 that c_L must appear as the label of a point in $[L]$. Pick a point $y \in [L]_Y$ with label $\operatorname{rlbl}(y) = (c_L, c_H)$ and let $LL(y) = \{x_L, x_H\}$ be the associated configuration.
We would like now, to understand the Galois action of the fundamental group $\pi_1(E_{n-1,1})$ on the point y (equation (8.3) shows that the restriction $LL : [L]_Y \to E_{n-1,1}$ is a covering map). Because the unequal multiplicities of x_L and x_H distinguish them inside the configurations of $E_{n-1,1}$, the fundamental group is naturally identified with the *pure braid group* PB_2 (as opposed to B_2).

If we let σ_1 be the generator of B_2 and σ_1^2 the generator of PB_2, then the Hurwitz rule implies that $\mathrm{rlbl}(\sigma_1^2 * y) = (c^{-1}c_L c, c^{-1}c_H c)$ (see Fig.8.1). Indeed, need only see that $c_H^{-1}c_L c_H = c_H^{-1}c_L^{-1}c_L c_L c_H = c^{-1}c_L c$ (and similarly for $c^{-1}c_H c$).



(a) We start with $\mathrm{rlbl}(y) = (c_L, c_H)$...

(b) ... and apply the Hurwitz rule as x_L and x_H move ...

(c) ... around each other, to get $\mathrm{rlbl}(\sigma_1^2 * y) = (c^{-1}c_L c, c^{-1}c_H c)$.

Figure 8.1: The effect of rotation x_L and x_H around eachother on $\mathrm{rlbl}(y)$.

Since our assumption implies that there exist precisely d-many distinct labels of the form $c^{-i}c_L c^i$, the LL map must also have *at least* d-many preimages over $\{x_L, x_H\}$. That is,

$$\deg(LL) \geq d.$$

On the other hand, $c^{-d}c_L c^d = c_L$ implies that $c^d \cdot L = L$ or, in other words, that $c^d \in N_W(L)$. Moreover, by Prop. 87 the order of c^d inside $N_W(L)/W_L$ is h/d which in turn forces

$$|N_W(L) : W_L| \geq h/d.$$

Finally, the last two inequalities along with Prop. 83 force the desideratum. $\square$

We are now ready to give the first uniform proof for a special case of the Trivialization Theorem:

Proposition 85 (***Trivialization Theorem*** for Lines)**.**
The map $LL \times \mathrm{rlbl} : Y_{n-1,1} \to E_{n-1,1} \boxtimes D_2(c)$ *is a bijection.*

Proof. The surjectivity is the same as in the proof of the Trivialization Theorem in Section 7. It only needs the transitivity of the Hurwitz action and does not rely on the Numerological Coincidence. The injectivity follows from Corollary 84:

Indeed, assume that there are two points y and y' in $Y_{n-1,1}$ with the same image. Let $LL(y) = LL(y') = \{x_L, x_H\}$ and let $\mathrm{rlbl}(y, x_L) = \mathrm{rlbl}(y', x_L) = c_L$. Now, by Prop. 39, both points (y, x_L) and (y', x_L) must belong to the stratum $[L]$, where $L := V^{c_L}$.
This cannot happen as Corol. 84 implies that all preimages (under LL and in the flat L) of $\{x_L, x_H\}$ must have different labels. to see this, note of course that all labels of the form $(c^{-i}c_L c^i, c^{-i}c_H c^i)$ must occur, and since their number equals the degree of LL over $[L]_Y$, they must occur exactly once. □

8.2 Speculation on the local geometry of the LL map

Here, we present two statements about the local multiplicity of the LL map. We use the term *meta-conjectures*, as what we are asking for is a *uniform* geometric proof of them. For that matter, it is in fact easy to see, after Section 7, that the statements are true.
Notice that these are much simpler (and more geometric) statements than the numerological coincidence (Corol. 69). Furthermore, we are only going to need the first one below, which seems less challenging.
It is clear however, that whichever geometric argument could be used to tackle the first case should be extendable to the second one as well. That is, of the two statements, the true Lemma is the latter one, although the necessary (and possibly easier one) is the former.

Meta-Conjecture I. Let $L \in \mathcal{L}_W$ be an *irreducible line* (as in §2.1) and let $y \in [L^{\mathrm{reg}}] \subset Y$. Then, the multiplicity of the LL map at y equals the multiplicity at the origin (i.e. the degree) of a different LL map; the one associated with the irreducible complex reflection group W_L. That is,

$$\mathrm{mult}_y(LL) = \mathrm{mult}_{\mathbf{0}}\big(LL(W_L)\big) = \deg\big(LL(W_L)\big).$$

Meta-Conjecture II. There is a way to extend the definition of the LL map for *reducible,* well-generated, complex reflection groups, so that it is still true that $\deg(LL) = \mathrm{Red}_W(c)$. Furthermore, we will now have that for *any* flat $X \in \mathcal{L}_W$, and for any point $y \in [X^{\mathrm{reg}}] \subset Y$, the following is again true:

$$\mathrm{mult}_y(LL) = \mathrm{mult}_{\mathbf{0}}\big(LL(W_X)\big) = \deg\big(LL(W_X)\big).$$

The reasoning behind these meta-conjectures is essentially the local geometry of the discriminant hypersurface. As we discussed in Section 3.2.1, locally at any point $p \in [X^{\text{reg}}]$, the discriminant $\mathcal{H}(W)$ looks like the product $X \times \mathcal{H}(W_X)$.

We would essentially like to say that the LL map may be defined for a germ (H, x) of a hypersurface H, and that germs which are isomorphic (for instance via the map τ_X from Prop. 20) have LL maps with the same geometry. For that matter, the LL map should not depend on the direction of the line L_y (as long as it is transverse to the tangent cone).

8.3 The Trivialization Theorem revisited

We recall here the statement of the Trivialization Theorem for the regular part of Y:

Trivialization Theorem for $\mathbf{Y}^{\text{reg}}$. *The map* $LL \times \text{rlbl} : Y^{\text{reg}} \to E_n^{\text{reg}} \times \text{Red}_W(c)$ *is a bijection.*

Uniform Proof of Trivialization Theorem for Y^{reg}. We will prove the theorem by induction on the rank of W. The case $n = 1$ (where W is a cyclic group) is easy and we omit it. The base case $n = 2$ is covered in Prop. 85. For $n \geq 3$, we proceed as follows:

We already know from Corollary 67 that there exists a number k such that the map $LL \times \text{rlbl} : Y^{\text{reg}} \to E_n^{\text{reg}} \times \text{Red}_W(c)$ is k-to-1; we wish to show that $k = 1$. To start, pick an irreducible noncrossing line L (one always exists by Prop. 88) and consider the associated block factorization $\sigma = (c_L, c_H)$. By Prop. 39 there exists a point $y \in [L]_Y$ such that $\text{rlbl}(y) = (c_L, c_H)$ and by Prop. 85, if we fix the configuration $LL(y) = \{x_L, x_H\}$, the point y is unique.

Now consider a configuration $e' \in E_n^{\text{reg}}$ close to $\{x_L, x_H\}$, such that the last (lexicographically) point in e' is x_H. The number of reflection factorizations of c that refine σ is equal to $\text{Red}_W(c_L)$, or by Corollary 66 equal to $\text{Red}_{W_L}(c_L)$. For each one of them, there exist k-many preimages y' of e'.

Consider a path in E_n from e' to e, along which the (lexicographically) first $n-1$ points of the multiset collide (Have figure here?). By the Hurwitz rule (Lemma 41), all the $\big(k \cdot \text{Red}_{W_L}(c_L)\big)$-many preimages we discussed in the previous paragraph will have to collide with the (unique) point y. That is,

$$\mathrm{mult}_y(LL) = k \cdot \mathrm{Red}_{W_L}(c_L).$$

This is exactly the setting of our Meta-Conjectures. We have a uniform combinatorial *control* of the multiplicity of LL at the particular point y and we can compare it with our geometric description (conjectural) of the multiplicity there.
Indeed, by our **Meta-Conjecture I**, we must have $\mathrm{mult}_y(LL) = \deg(LL(W_L))$. Our inductive assumption on the other hand (recall that L was taken to be irreducible), implies that $\deg(LL(W_L)) = \mathrm{Red}_{W_L}(c_L)$. That is, $k = 1$. □

The proof of the **Trivialization Theorem** for lines (Prop. 85) relies on the following direct generalization of [ARR15, Lemma 5.4]. The argument goes through almost verbatim, but we include it here partially because we think this fact about the geometry of noncrossing lines has not been mined enough for applications.
We will need to discuss orthogonality of vectors in the complex setting. Consider for this a W-invariant non-degenerate Hermitian form $(\cdot,\cdot)$ on V, and say that two vectors are orthogonal if they pair to 0 through it.

Proposition 86. *There is no noncrossing line L orthogonal to the $e^{2\pi i/h}$-eigenspace of c.*

Proof. Let $c_L \in [1,c]_{\leq_R}$ be the noncrossing element such that $V^{c_L} = L$ and consider the factorization $c_H c_L = c$. Let H be the reflecting hyperplane for c_H and v a basis vector for L. Assume that v is perpendicular to Z, the (1-dimensional) $e^{2\pi i/h}$-eigenspace of c. We have that $c_H(v) = c_H(c_L(v)) = c(v)$, since $c_L(v) = v$. Since, by Cor. 24 $c(v) \neq v$, we must also have $c_H(v) \neq v$. Now, notice that c is an orthogonal transformation (since W is unitary) which respects Z (i.e. $c(Z) = Z$). Therefore, the vector $c_H(v) = c(v)$ is perpendicular to Z and hence the difference $c_H(v) - v$ is also (nonzero and) perpendicular to Z.
Now, notice [Kan01, Section 14.1, Example 2] that we may write

$$c_H(v) - v = (\xi - 1)\frac{(\alpha, v)}{(\alpha, \alpha)}\alpha,$$

where ξ is the unique non-identity eigenvalue of c_H and α is the corresponding eigenvector ($c_H(\alpha) = \xi \cdot \alpha$). This implies that $c_H(v) - v$ is perpendicular to H.

The previous two paragraphs force that $Z \subset H$. This is a contradiction as we have constructed c to be a $e^{2\pi i/h}$-regular element, which means that the (1-dimensional) eigenspace Z cannot intersect any hyperplane away from the origin.

□

The following proposition also appears almost entirely [in the proof of ARR15, Prop. 2.13] and we copy it here for completion:

Proposition 87. *For a noncrossing line L,*

$$c^d \in W_L \iff d = h.$$

Proof. Consider the orthogonal projection $p : V \to Z$ on the $e^{2\pi i/h}$-eigenspace Z of c. If c^d fixes L for some d, then since c^d acts orthogonally, it will fix the projection $p(L)$ as well. By Prop. 86, L is not orthogonal to Z, so $p(L) = Z$ since Z is 1-dimensional. Finally, c^d can only fix the projection $p(L) = Z$ when $d = h$; this being the order by which c acts on Z. □

Proposition 88. *There always exists an irreducible noncrossing line.*

Proof. In the case that W is real, an irreducible parabolic subgroup is one that corresponds to a connected subdiagram of the Coxeter diagram of W. Since the latter is a tree, we can always remove one of its leaves and produce an irreducible parabolic of rank $n-1$.

In fact, the same argument works in the well-generated case. Let $W \leq \mathrm{GL}(V)$ be a complex reflection group generated by n reflections $t_1, \cdots, t_n$, where $n = \dim_{\mathbb{C}}(V)$, and let e_i be their corresponding non-trivial eigenvectors. Following [Kan01, Section 24-3, proof of Theorem A], we may represent the orthogonality relations between the vectors $\{e_1, \cdots, e_n\}$ by a graph Γ consisting of n vertices, labelled e_i, with an edge between i and j if $(e_i, e_j) \neq 0$.

Kane [ibid] later shows that $W \leq \mathrm{GL}(V)$ is irreducible if and only if the corresponding graph Γ is connected. We proceed by removing a vertex (which we may assume to be e_n) such that the resulting graph is still connected.[2] Now, we have a reflection

[2] It is an easy exercise to (inductively) show that every non-trivial connected graph has *at least two* vertices that are not cut-vertices.

subgroup H generated by the reflections $s_1, \cdots, s_{n-1}$ that acts irreducibly on the space V_1, where $V = L \oplus V_1$ and L is the 1-dimensional intersection of the fixed hyperplanes of the s_i's.

Notice that H is not *a priori* parabolic, but it is clear that $H \leq W_L$. Furthermore since H acts irreducibly on V_1, so must W_L. That is, W_L is an irreducible parabolic of rank $n-1$.

Notice that we have only produced irreducible *lines*, while we needed irreducible *non-crossing lines*. This is hardly an issue though, as by Prop. 39 all orbits $[X] \in W \backslash \mathcal{L}_W$ appear in the noncrossing lattice. In other words, there is an element $c_{L'} \in [1, c]_{\leq_R}$ such that $[V^{c_{L'}}] = [L]$. □

The following proposition provides the base case for the induction in the proof of the Trivialization Theorem. The proof is uniform if one assumes the transitivity of the Hurwitz action.

Proposition 89 (***Trivialization Theorem*** for rank 2)**.** *Let W be a well-generated, irreducible complex reflection group of rank 2. Then, the map $LL \times \mathrm{rlbl} : Y^{\mathrm{reg}} \to E_2^{\mathrm{reg}} \times \mathrm{Red}_W(c)$ is a bijection.*

Proof. Again, surjectivity is already done uniformly in Section 7.3. Now, notice that since W is of rank 2, quasi-homogeneity forces a very simple form on the discriminant equation:

$$\big(\Delta(W, \mathbf{f}); f_2\big) = f_2^2 - f_1^{2h/d_1}. \tag{8.4}$$

This and the transitivity of the Hurwitz action imply that for injectivity we need only show that $|\mathrm{Red}_W(c)| = 2h/d_1$.

Let $c_{H_1} c_{H_2} = c$ be a reduced reflection factorization of c. By the transitivity of the Hurwitz action, all elements of $\mathrm{Red}_W(c)$ are of the form

$$c^{-d} c_{H_1} c^d \cdot c^{-d} c_{H_2} c^d = c \qquad \text{or} \qquad c^{-d} c_{H_2} c^d \cdot c^{-d-1} c_{H_1} c^{d+1} = c.$$

We are interested in the smallest number d such that $c^{-d} c_{H_1} c^d = c_{H_1}$, or equivalently, such that $c^d \cdot H_1 = H_1$. Notice that, since W is of rank 2, the eigenspaces of c^d are either the same as those of c, or c^d acts by scalar multipication on the whole ambient space V. Recall that the eigenspaces of c are cut by the equation $f_1 = 0$, so they only intersect the discriminant $\mathcal{H}$ at the origin. That is, H_1 cannot be an eigenspace of c.

Therefore, the smallest number d such that $c^{-d}c_{H_1}c^d = c$ is also the smallest number d for which c^d has a single eigenvalue, or by Springer's characterization of those (see proof of Corol. 24), the smallest number d such that $\zeta^d = \zeta^{d-dd_1}$ where $\zeta = e^{2\pi i/h}$. In fact, d is the smallest number such that $h|dd_1$, or

$$d = \frac{h}{\gcd(d_1, h)}.$$

Now, we are going to consider cases for $\gcd(d_1, h)$. Since, by the form of the discriminant in (8.4), $2h/d_1$ is an integer, we only have two options:

1. $\gcd(d_1, h) = d_1$: Then, the discriminant polynomial $f_2^2 - f_1^{2h/d_1}$ factors into two irreducible components, which in turn means that there are two W-orbits of hyperplanes. Now, since $c_{H_1}c_{H_2} = c$, the hyperplanes H_1 and H_2 must be on different orbits, which means that c_{H_1} and c_{H_2} cannot be conjugate.

   That is, $d = \dfrac{h}{\gcd(d_1, h)} = h/d_1$ counts only *half* of the factorizations and hence $|\operatorname{Red}_W(c)| = 2h/d_1$.

2. $\gcd(d_1, h) = d_1/2$: Then, the discriminant polynomial is irreducible, which means that there is only one orbit of hyperplanes. It is possible however that c_{H_1} and c_{H_2} are not conjugate via c (i.e. that there is no m such that $c^{-m}c_{H_1}c^m = c_{H_2}$.

   In that case, there would be exactly $2d = \dfrac{2h}{\gcd(d_1, h)} = 4h/d_1$ elements of $\operatorname{Red}_W(c)$ which contradicts the fact that the degree of LL must divide the size of $\operatorname{Red}_W(c)$ (Corol. 67). That is, we are forced to have $|\operatorname{Red}_W(c)| = d = 2h/d_1$.

$\square$

Chapter 9

Primitive factorizations of a Coxeter element

This section is devoted to the proof of the main new enumerative results of the thesis, culminating in our formula for the number of primitive factorizations of c (Thm. 99). Its derivation follows a pattern of results (see §9.5), where variations of the LL map from singularity theory have been used to enumerate combinatorial objects associated with coverings of the complex sphere. The idea has been to lift the LL morphism to suitable spaces where the geometry of the new map reflects finer combinatorial data.

In our case, we will restrict our attention to factorizations with prescribed conjugacy classes of factors. The simplest of those are the so called *primitive factorizations*, that is, block factorizations of the form

$$c = c_1 \cdot t_1 \cdots t_{n-k},$$

where c_1 is a parabolic Coxeter element of length k, and the t_i's are reflections.

By Corol. 65, we know that all factors in a block factorization of c have to be parabolic Coxeter elements. That is, not all conjugacy classes may appear in a block factorization of c, and those that do appear can be indexed by data associated to a flat $Z \in \mathcal{L}_W$. We recall here the concept of "type", which is essentially due to [AR04, above Thm. 6.3], and generalizes the "block sizes" of a partition:

Definition 90. We say that c_i is a *parabolic Coxeter element of type* $[Z]$, for an orbit $[Z] \in W\backslash\mathcal{L}_W$, if c_i is a Coxeter element of some parabolic subgroup $W_{Z'}$ such that

$[Z] = [Z']$. Notice that all such elements form a single conjugacy class in W; we write $c_{[Z]}$ for an arbitrary representative.

Similarly, we say that $c = c_1 \cdot t_1 \cdots t_{n-k}$ is a *primitive factorization of type* $[Z]$ and we write

$$c = c_{[Z]} \cdot t_1 \cdots t_{n-k},$$

if $c_1 = c_{[Z]}$ is a parabolic Coxeter element of type $[Z]$.

9.1 Lifting the Lyashko-Looijenga morphism

By Prop. 39, we know that a point $(y, x_i) \in \mathcal{H}$ may be labeled by a parabolic Coxeter element of type $[Z]$ (i.e. $\mathrm{rlbl}(y, x_i) = c_{[Z]}$) if and only if $(y, x_i) \in [Z^{\mathrm{reg}}]$. Therefore, if we would like to understand the points $y \in Y$ whose labels contain a factor of type $[Z]$, we must study the restriction of the LL map on the set

$$[Z^{\mathrm{reg}}]_Y := \{y \in Y : \ L_y \cap [Z^{\mathrm{reg}}] \neq \emptyset\},$$

which is the projection on Y of the stratum $[Z^{\mathrm{reg}}]$.

This might be difficult to do: In general, $[Z^{\mathrm{reg}}]_Y$ is only a constructible set and we have little control on the ideal of its (Zariski)-closure. Instead we will consider a variant $\widehat{LL}$ of the Lyashko-Looijenga morphism, whose domain is the flat Z, and which has a much simpler geometry. We first introduce a generalization of our configuration space E_n that is going to be the natural target of our lifted $\widehat{LL}$ map:

Definition 91. We define the *decorated (centered) configuration space* $E_{(k,1^{n-k})}$ to be the set of centered configurations of n points in $\mathbb{C}$, that are further required to include a special (decorated) point of multiplicity (at least) k. That is,

$$E_{(k,1^{n-k})} := \mathfrak{S}_{n-k}\backslash H_{(k,1^{n-k})} = \Big\{(\underbrace{x_0, \cdots, x_0}_{k\text{-times}}, x_1, \cdots, x_{n-k}) \in \mathbb{C}^n \mid k \cdot x_0 + \sum_{i=1}^{n-k} x_i = 0\Big\},$$

where the action of $\mathfrak{S}_{n-k}$ is on the last $n-k$ coordinates.

It is easy to see, via the Vieta formulas again, that $E_{(k,1^{n-k})} \cong \mathbb{C}^{n-k}$. Indeed, the coefficients of the polynomial $(t - x_1) \cdots (t - x_{n-k})$ completely determine the unordered configuration $\{x_1, \cdots, x_{n-k}\}$ and the *centered* condition gives x_0.

We will denote its elements by $\{\widehat{x_0}, x_1, \cdots, x_{n-k}\}$. Notice that we are not assuming x_0 to be different from the x_i's. As with E_n, we will write $E^{\text{reg}}_{(k,1^{n-k})}$ for those decorated configurations where $\widehat{x_0} \neq x_i \neq x_j,\ \forall i, j$.

Given an $(n-k)$-dimensional flat $Z \in \mathcal{L}_W$, we may easily express the restrictions of the fundamental invariants f_i on Z as polynomials in a basis $\mathbf{z} := (z_1, \cdots, z_{n-k})$ of Z. Indeed, we can choose a basis of V that extends $\mathbf{z}$ and write the f_i's with respect to that basis. Then their restrictions on Z will involve no other variables but the z_i's.
We may therefore parametrize $[Z]_Y$ via $y(\mathbf{z}) := \big(f_1(\mathbf{z}), \cdots, f_{n-1}(\mathbf{z})\big)$. Notice that by treating points in Y as images of points $\mathbf{z}$, we gain information about the multiset $LL\big(y(\mathbf{z})\big)$. In particular, we know that it contains the point $f_n(\mathbf{z})$ and (Corol. 57) with multiplicity at least $\operatorname{codim}(Z) = k$. We are now ready to introduce the following *lift* of the LL map:

Definition 92. For an irreducible well-generated complex reflection group W and a flat $Z \in \mathcal{L}_W$, we define the *lifted Lyashko-Looijenga* map, denoted[1] $\widehat{LL}$, by:

$$\begin{array}{ccc} Z & \xrightarrow{\widehat{LL}} & E_{(k,1^{n-k})} \\ \mathbf{z} := (z_1, \cdots, z_{n-k}) & \longrightarrow & \text{multiset } LL\big(y(\mathbf{z})\big) \text{ with the decorated point } \widehat{f_n(\mathbf{z})}. \end{array}$$

[1] In what follows, the dependence on the flat Z will be suppressed for ease of notation.

The diagram on the right describes the relation between LL and $\widehat{LL}$. It is immediate by the definition, that if F is the forgetful map that sends the decorated multiset $\{\widehat{x_0}, \cdots, x_{n-k}\}$ to the undecorated one $\{x_0, \cdots, x_{n-k}\}$ (respecting the multiplicity of x_0), the diagram commutes. That is, we have

$$F \circ \widehat{LL} = \big(LL \circ \mathrm{pr}_Y \circ \rho\big)|_Z. \tag{9.1}$$

$$\begin{tikzcd}
V \arrow[d, "\rho"'] & \supset & Z \arrow[d, "\rho"] \arrow[rr, "\widehat{LL}"] & & E_{(k,1^{n-k})} \arrow[dd, "F"] \\
W\backslash V \arrow[d, "\mathrm{pr}_Y"'] & \supset & {[Z]} \arrow[d, "\mathrm{pr}_Y"] & & \\
Y & \supset & {[Z]_Y} \arrow[rr, "LL"] & & E_n
\end{tikzcd}$$

Figure 9.1: The lifted Lyashko-Looijenga morphism.

Remark 93. Notice that F is not in general invertible. If there are several points of multiplicity greater than or equal to k, there is no way to know which one was decorated. In fact, we should think of $E_{(k,1^{n-k})}$ as a desingularization (since it is isomorphic to $\mathbb{C}^{n-k}$) of its image in E_n.

9.2 The geometry of the lifted $\widehat{LL}$ map

Our first step towards understanding the geometry of the $\widehat{LL}$ map will be to give it an explicit description in terms of polynomials. We will need to study the restriction of the discriminant $\big(\Delta(W, \mathbf{f}); (y,t)\big)$ on $[Z]_Y$.

As before, we may view the α_i's as polynomials in $\mathbf{z}$. The fact that $f_n(\mathbf{z})$ is always a root of the discriminant at $y = y(\mathbf{z})$, and of multiplicity at least k, implies that we can factor the latter

$$\big(\Delta(W, \mathbf{f}); (y(\mathbf{z}), t)\big) = t^n + \alpha_2\big(y(\mathbf{z})\big) \cdot t^{n-2} + \cdots + \alpha_n\big(y(\mathbf{z})\big), \tag{9.2}$$

as

$$\big(\Delta(W, \mathbf{f}); (\mathbf{z}, t)\big) = \big(t - f_n(\mathbf{z})\big)^k \big(t^{n-k} + b_1(\mathbf{z}) \cdot t^{n-k-1} + \cdots + b_{n-k}(\mathbf{z})\big), \tag{9.3}$$

where the b_i's are *a priori* functions of $\mathbf{z}$. Our first task will be to show that they are in fact polynomials in $\mathbf{z}$:

Lemma 94. *The coefficients $b_i(\mathbf{z})$ that appear in the previous factorization of the discriminant $\big(\Delta(W,\mathbf{f});(\mathbf{z},t)\big)$ are homogeneous polynomials in the z_i's, of degree hi.*

Proof. Indeed, by comparing coefficients on the right hand sides of (9.2) and (9.3), we get the following equations:

$$\begin{aligned}
0 &= kf_n - b_1,\\
\alpha_2 &= b_2 - kf_nb_1 + \binom{k}{2}f_n^2,\\
\alpha_3 &= b_3 - kf_nb_2 + \binom{k}{2}f_n^2b_1 - \binom{k}{3}f_n^3,\\
\alpha_4 &= \cdots,
\end{aligned}$$

where α_i, b_i and f_n are all considered as functions on $\mathbf{z}$. Now, by definition, the α_i's and f_n are polynomials in $\mathbf{z}$. Moreover, the above equations can be used to inductively express b_i as a polynomial in the α_j's (with $j \leq i$) and f_n; therefore as a polynomial in the z_i's.

The homogeneity is also an immediate consequence of the previous argument. Indeed, the α_i's are weighted-homogeneous in the f_i's, of weighted-degree hi. This means precisely that they are *homogeneous in the z_i's* and of the same degree. Along with the fact that $\deg(f_n) = h$, this forces (inductively) all monomials that appear in the i^{th} equation to be homogeneous and of degree hi. $\square$

Corollary 95. *The lifted $\widehat{LL}$ map is an algebraic morphism, given explicitly as:*

$$\begin{array}{ccc}
Z \cong \mathbb{C}^{n-k} & \xrightarrow{\widehat{LL}} & E_{(k,1^{n-k})} \cong \mathbb{C}^{n-k}\\
\mathbf{z} := (z_1,\cdots,z_{n-k}) & \longrightarrow & \big(b_1(z_1,\cdots,z_{n-k}),\cdots,b_{n-k}(z_1,\cdots,z_{n-k})\big).
\end{array}$$

Proof. It is clear by the definition of the b_i's, and our choice of parametrization for the space $E_{(k,1^{n-k})}$ as described in Defn. 91, that the tuple $(b_1,\cdots,b_{n-k})$ represents the decorated multiset $\widehat{LL}(\mathbf{z})$. The algebraicity of the map $\widehat{LL}$ is precisely the previous lemma. $\square$

Remark 96. Notice that in the same way, we may show that the forgetful map F is algebraic. Indeed, it is precisely given as $\mathbf{b} := (b_1,\cdots,b_{n-k}) \xrightarrow{F} \big(\alpha_2(\mathbf{b}),\cdots,\alpha_n(\mathbf{b})\big)$, where the α_i's are given in terms of $\mathbf{b}$ according to the equations in the proof of Lemma 94.

We are now in a similar situation as in Section 6. Instead of attempting to reproduce all of its statements in the context of the lifted $\widehat{LL}$ map, we focus only on those that are pertinent to the enumerative questions; namely the finiteness and the degree calculation.

Proposition 97. *The lifted $\widehat{LL}$ map is a finite morphism and its degree is given by*

$$\deg(\widehat{LL}) = h^{\dim(Z)} \cdot \big(\dim(Z)\big)! \ .$$

Proof. As $\widehat{LL}$ is homogeneous, we may apply Prop. 46. In order to show that $\big(\widehat{LL}\big)^{-1}(\mathbf{0}) = \mathbf{0}$, we rely on the connection with the LL map, as described in Fig. 9.1.
To begin with, notice that $\mathbf{0} \in E_{(k,1^{n-k})}$ represents the multiset with n copies of 0, where the (unique) element 0 is decorated. Of course, $\mathbf{0} \in E_n$ is the same multiset without the decoration. Now, it is easy to see that $F^{-1}(\mathbf{0}) = \mathbf{0}$ (this just says that a multiset with a single element can only be decorated in one way).
Therefore, if $\big(\widehat{LL}\big)^{-1}(\mathbf{0}) \neq \mathbf{0}$, this would imply that $\big(\widehat{LL}\big)^{-1} \circ F^{-1}(\mathbf{0}) \neq \mathbf{0}$, which is the same as

$$\rho^{-1} \circ \mathrm{pr}_Y^{-1} \circ LL^{-1}(\mathbf{0}) \neq \mathbf{0},$$

according to (9.1) (here the notation is again as in Fig. 9.1). But we have shown already (see proof of Thm. 51) that $LL^{-1}(\mathbf{0}) = \mathbf{0}$, and then $\mathrm{pr}_Y^{-1}(\mathbf{0}) = \mathbf{0}$, because $L_{\mathbf{0}}$ intersects $\mathcal{H}$ only at the origin, and finally $\rho^{-1}(\mathbf{0}) = \mathbf{0}$, because $\mathbf{0}$ is the unique point in Z fixed by the whole group W. That is, we must have $\big(\widehat{LL}\big)^{-1}(\mathbf{0}) = \mathbf{0}$.
Our degree calculation is the same as in Corol. 56. Bezout's theorem gives us the formula:

$$\deg(\widehat{LL}) = \prod_{i=1}^{n-k} \deg(b_i) = \prod_{i=1}^{n-k} hi = h^{n-k} \cdot (n-k)! \ ,$$

and since $\dim(Z) = n - k$, the proof is complete. □

9.3 Enumeration of Primitive factorizations

We would like now to apply this geometric analysis of the $\widehat{LL}$ map to our enumerative problem. First we will use the Trivialization Theorem to phrase the question in terms of the local geometry of the LL map, which then we will reduce to a simpler problem for the $\widehat{LL}$ map.

Recall how Bessis' **Trivialization Theorem** relates the fibers of the LL map with compatible block factorizations. To enumerate *primitive* factorizations, we consider a special multiset $e \in E_n$ whose leftmost point is of multiplicity k and whose other points are simple. For instance, pick

$$e := \big\{ \underbrace{\frac{-1}{k}\binom{n-k}{2}, \cdots, \frac{-1}{k}\binom{n-k}{2}}_{k-\text{times}}, 1, \cdots, n-k \big\}. \tag{9.4}$$

Now the Trivialization Theorem implies the following Lemma:

Lemma 98. *Let W be a well-generated complex reflection group, let Z be one of its flats, and let e be as above. Then the number of primitive factorizations of type $[Z]$, denoted $\mathrm{FACT}_W(Z)$, is equal to*

$$\#\big\{LL^{-1}(e) \cap [Z]_Y\big\}.$$

Proof. Indeed, the bijectivity of the map $LL \times \mathrm{rlbl} : Y \to E_n \boxtimes D_\bullet$ implies that the number of *all* primitive factorizations $c = c_1 \cdot t_1 \cdots t_{n-k}$ (i.e. with $l_R(c_1) = k$) is equal to the size of (number of distinct points in) the fiber $LL^{-1}(e)$.
In addition, as we discussed at the beginning of §9.1, primitive factorizations of type $[Z]$ may only appear as labels of points in $[Z^{\mathrm{reg}}]_Y$. Finally notice that $LL^{-1}(e)$ intersects $[Z]_Y$ only at $[Z^{\mathrm{reg}}]_Y$. This is a consequence of Lemma 34 and concludes the proof. $\square$

In the following theorem we relate the size of the fiber $LL^{-1}(e) \cap [Z]_Y$ with the degree of the $\widehat{LL}$ map, to obtain a closed, uniform enumeration formula:

Theorem 99. *Let W be a well-generated group and let Z be one of its flats. Then, the number of primitive factorizations of type $[Z]$ is given by the formula*

$$\mathrm{FACT}_W(Z) = \frac{h^{\dim(Z)} \cdot \big(\dim(Z)\big)!}{[N_W(Z) : W_Z]},$$

where $N_W(Z)$ and W_Z are, respectively, the setwise and pointwise stabilizers of Z.

Proof. To prove the theorem, we will lift the special multiset $e \in E_n$ (as in (9.4)) all the way to the flat Z, following both paths described in Fig. 9.1, and then we will compare the two fibers. The argument should be reminiscent of the one for Prop. 83.

1. $\big(\rho|_Z\big)^{-1} \circ \big(\operatorname{pr}_Y|_Z\big)^{-1} \circ \big(LL|_Z\big)^{-1}(e)$: The application of the first inverse map gives us precisely the set on the right hand side of Lemma 98 above. For the second map, recall first that $\operatorname{pr}_Y^{-1}(y)$ is by definition equal to the intersection $L_y \cap \mathcal{H}$, whose f_n coordinates are recorded in $LL(y)$.

   In our case, if y is such that $LL(y) = e$ and $\big(y, \frac{-1}{k}\binom{n-k}{2}\big) \in [Z]$, we must have that $\operatorname{mult}_x\big(LL(y)\big) = \operatorname{codim}(Z) = k$ (by Corol. 57). This forces the restricted preimage $\big(\operatorname{pr}_Y|_Z\big)^{-1}(y)$ to be the single point $\big(y, \frac{-1}{k}\binom{n-k}{2}\big)$, since $\frac{-1}{k}\binom{n-k}{2}$ is the *unique* element in e of multiplicity k.

   For the quotient map ρ, notice to begin with that any $(y, x) \in [Z]$ such that $LL(y) = e$ must belong to $[Z^{\text{reg}}]$. Indeed, if that were not the case, there would be a point in $LL(y)$ of multiplicity greater than k. Now, by definition, the fiber of ρ over $[Z^{\text{reg}}]$ is the set of points in Z^{reg} that are in the same W-orbit. This is precisely the index $[N_W(Z) : W_Z]$.

   Putting the previous three paragraphs together, we have the following relation between fibers:

$$\#\big\{\big(\rho|_Z\big)^{-1} \circ \big(\operatorname{pr}_Y|_Z\big)^{-1} \circ \big(LL|_Z\big)^{-1}(e)\big\} = [N_W(Z) : W_Z] \cdot \#\big\{LL^{-1}(e) \cap [Z]_Y\big\}.$$

2. $\big(\widehat{LL}\big)^{-1} \circ F^{-1}(e)$: Since there is a single point of multiplicity k in e, this must have been the one that was decorated. That is, $F^{-1}(e)$ is a singleton; we call its unique element $\widehat{e}$.

   For $\widehat{LL}$, recall that a finite morphism achieves its degree (i.e. the number of preimages of a point equals the degree of the map) over a (Zariski)-open set. In our case, it is clear that $E^{\text{reg}}_{(k,1^{n-k})}$ is open in $E_{(k,1^{n-k})}$ (actually both in the Zariski and the complex topology).

   Therefore, it will intersect any other open set (since $E_{(k,1^{n-k})}$ is irreducible) and, in fact, it will intersect the set where $\widehat{LL}$ achieves its degree (is unramified). Without loss of generality, we may assume that $\widehat{e}$ is in that intersection (since in dealing with e, we haven't relied on anything but the multiplicities of its elements). This means that:

$$\#\big\{\big(\widehat{LL}\big)^{-1} \circ F^{-1}(e)\big\} = \deg(\widehat{LL}) = h^{\dim(Z)} \cdot \big(\dim(Z)\big)! \ .$$

Putting together the last two equalities and combining them with (9.1), we immediately get

$$\#\big\{LL^{-1}(e)\cap[Z]_Y\big\}=\frac{h^{\dim(Z)}\cdot\big(\dim(Z)\big)!}{[N_W(Z):W_Z]},$$

which, after Lemma 98, is exactly what we need. □

Remark 100. One can compute the numbers $[N_W(Z):W_Z]$ for the exceptional groups, using the tables in [OT92, Appendix C]. They also appear explicitly in literature; for the real case see [OS83, Tables III-VIII], while for the complex ones see [OS82, Tables 3-11].

Remark 101. Notice that this proof essentially works for the ambient flat $Z=V$ as well. In that case, we will have a centered configuration of n-many points that is decorated at the point $f_n(\mathbf{v})$ which might not belong to them. In fact, the first coordinate of the $\widehat{LL}$ map will be f_n now, since the relation with b_1 is $0\cdot f_n=b_1$.
More interesting, is what happens if we set $Z=H$, for some reflecting hyperplane H. There are at most two W-orbits of hyperplanes and we may consider representatives H and H'. Then, it is clear that

$$\mathrm{FACT}_W(H)+\mathrm{FACT}_W(H')=\mathrm{Red}_W(c)=\frac{h^n n!}{|W|}.$$

Comparing this with the formula of Thm. 99, we get the equation

$$hn=\frac{|W|}{[N_W(H):W_H]}+\frac{|W|}{[N_W(H'):W_{H'}]}=[W:N_W(H)]\cdot|W_H|+[W:N_W(H')]\cdot|W_{H'}|,$$

which easily gives the well-known formula $hn=N+N^*$ as in Defn. 2.

9.4 A geometric interpretation of Kreweras numbers

The concept of type as described in Defn. 90 determines a meaningful partition of the noncrossing lattice $NC(W)$. Kreweras [Kre72] was the first to compute the block-sizes of this partition for the symmetric group.

Definition 102. We define the *Kreweras numbers* for W, to be the numbers

$$\mathrm{Krew}_W(Z):=\#\big\{c_i\in NC(W)\mid c_i\text{ is of type }[Z]\big\}.$$

As is the case for the total size of the noncrossing lattice, we have uniform formulas for the Kreweras numbers, but no uniform *proofs* of these formulas. Recall that the characteristic polynomial of a hyperplane arrangement $\mathcal{A}$ on V is defined by

$$\chi(\mathcal{A},t) := \sum_{Z\in\mathcal{L}_{\mathcal{A}}} \mu(V,Z)\cdot t^{\dim(Z)},$$

where $\mu(V,Z)$ is the Möbius function on the intersection lattice $\mathcal{L}_{\mathcal{A}}$, and that $\mathcal{A}^Z$ denotes the restriction of $\mathcal{A}$ on one of its flats Z.

Proposition 103. *[essentially AR04, Thm. 6.3] If W is a Weyl group and Z is one of its flats, then we have*

$$\mathrm{Krew}_W(Z) = \frac{\chi(\mathcal{A}_W^Z, h+1)}{[N_W(Z):W_Z]}.$$

Sketch: In [AR04, Thm. 6.3], Athanasiadis and Reiner prove that the Kreweras numbers are equal to the corresponding statistics for nonnesting partitions. Those had already been uniformly shown to adhere to the above formula; see for instance [Som05, Prop. 6.6:(1)]. $\square$

In a similar fashion to our Lemma 98, we may relate the number of noncrossing elements of type $[Z]$ (i.e. the Kreweras numbers) with the size of a particular fiber of the lifted $\widehat{LL}$ map. Let $\widehat{e}_{(k,n-k)}$ be the decorated multiset $\{-\widehat{(n-k)},\ \underbrace{k,\cdots,k}_{(n-k)\text{-times}}\}$ (see Defn. 91).

Lemma 104. *The number* $\mathrm{Krew}_W(Z)$ *of noncrossing elements of type* $[Z]$ *is equal to:*

$$\frac{\#\big\{\big(\widehat{LL}\big)^{-1}(\widehat{e}_{(k,n-k)})\big\}}{[N_X(Z):W_Z]}.$$

Proof. The proof is analogous to the ones in Lemma 98 and Thm.99, so we present it more compactly.

By the Trivialization Theorem and the definition of $\widehat{LL}$, the points $\mathbf{z}$ that lie in the fiber $\big(\widehat{LL}\big)^{-1}(\widehat{e}_{(k,n-k)})$ are such that $\mathrm{rlbl}\big(y(\mathbf{z})\big) = (c_1,c_2)$ where c_1 is of type $[Z]$. Since the multiplicity of the decorated point in $\widehat{e}_{(k,n-k)}$ is equal to $k=\mathrm{codim}(Z)$, the points $\mathbf{z}$ will further belong to the regular part Z^{reg}.

That is, for each block factorization $c_1\cdot c_2 = c$, with c_1 of type $[Z]$, we have $[N_W(Z):W_Z]$-many preimages in the fiber. On the other hand, the number of such block factorizations is precisely the number of noncrossing elements of type $[Z]$ (since each c_1 has a unique *Kreweras complement* $c_2 = c_1^{-1}c$). This completes the argument. $\square$

9.4.1 Speculation towards a uniform enumeration of the noncrossing lattice

The previous Lemma 104 suggests that we would get a uniform proof of the Kreweras formulas (Prop. 103) if we could show in a *geometric* way that

$$\#\big\{\big(\widehat{LL}\big)^{-1}(\widehat{e}_{(k,n-k)})\big\} = \chi(\mathcal{A}_W^Z, h+1).$$

The difficulty here is in the fact that the fiber is not reduced. Our combinatorial description of the local multiplicities of the LL map can be translated of course to the lifted case, but apparently is not sufficient. We need a geometric reason for this local behavior of $\widehat{LL}$.

We would like to further note that if this is successful, the formulas for the Kreweras numbers imply (in a uniform way) the formula for the size of the noncrossing lattice (which is the usual definition of the W-Catalan numbers):

$$\mathrm{Cat}(W) := |NC(W)| = \frac{1}{|W|}\prod_{i=1}^{n}(h+d_i).$$

Indeed, the following calculation, which is true for all complex reflection groups W, is definitely known to the experts (and is applied to our case by setting $t = h+1$). We provide it here as we couldn't find an explicit reference.

Proposition 105. *Consider the* **Kreweras polynomials** $\mathrm{Krew}_W^Z(t) := \dfrac{\chi(\mathcal{A}_W^Z, t)}{[N_W(Z) : W_Z]}$. *Then, we have*

$$\sum_{[Z]\in W\backslash\mathcal{L}_W} \mathrm{Krew}_W^Z(t) = \frac{1}{|W|}\prod_{i=1}^{n}(t+d_i-1).$$

Proof. Notice that we may write

$$\sum_{[Z]\in W\backslash\mathcal{L}_W} |W|\cdot\mathrm{Krew}_W^Z(t) = \sum_{[Z]\in W\backslash\mathcal{L}_W} \frac{|W|}{|N_W(Z)|}\cdot|W_Z|\cdot\chi(\mathcal{A}_W^Z, t) = \sum_{Z\in\mathcal{L}_W} |W_Z|\cdot\chi(\mathcal{A}_W^Z, t),$$

where the change in the summation index for the third equality is justified as the quotient $\frac{|W|}{|N_W(Z)|}$ counts the number of flats in the orbit $[Z]$.

Expanding the characteristic polynomial, we have

$$\sum_{Z\in\mathcal{L}_W} |W_Z|\cdot\chi(\mathcal{A}_W^Z, t) = \sum_{Z\in\mathcal{L}_W} |W_Z|\cdot\sum_{Z\supseteq Y}\mu(Z,Y)\cdot t^{\dim(Y)},$$

and reordering the summations we have

$$\sum_{Z\in\mathcal{L}_W} |W_Z| \cdot \sum_{Z\supseteq Y} \mu(Z,Y)\cdot t^{\dim(Y)} = \sum_{Y\in\mathcal{L}_W} t^{\dim(Y)} \cdot \sum_{Z\supseteq Y} |W_Z|\cdot\mu(Z,Y).$$

Finally, it is easy to see that the sum $\sum_{Z\supseteq Y}|W_Z|\cdot\mu(Z,Y)$ counts precisely those elements in W that fix exactly Y. That is, our quantity becomes

$$\sum_{Y\in\mathcal{L}_W} t^{\dim(Y)} \cdot \sum_{\substack{w\in W\\ V^w=Y}} 1 = \sum_{w\in W} t^{\dim(V^w)} = \prod_{i=1}^{n}(t+d_i-1),$$

where the last equality, a known formula of Shephard and Todd, is a corollary of the uniform [OS80, Thm. 3.3]. □

Remark 106. Although, one would prefer a proof of the enumeration of the noncrossing lattice *first*, and the Kreweras numbers as a corollary of that, this approach is not immediately compatible with the geometry of the LL map. In fact, the size of $NC(W)$ would have to be interpreted as the preimage of n *distinct* multisets in E_n and we would still miss the identity and the Coxeter element.
Of course, we might be able to overcome this by considering yet another variant of LL. We still think this idea is worth pursuing though. Already in Section 8.1 we proved the formulas for the Kreweras numbers in the case that Z is a line.

9.5 Past applications of the LL map in enumeration

As we mentioned in the introduction, Arnol'd [Arn96] was first to see the enumerative applications of the LL map. We list here the classes of factorizations that have been enumerated by using some lift or otherwise variant of the Lyashko-Looijenga morphism:

1. Reduced reflection factorizations in $\mathfrak{S}_n$ of a product of two cycles [Arn96].
2. Reduced reflection factorizations in $\mathfrak{S}_n$ of any element[2] [GL99].
3. *Any* reflection factorization in $\mathfrak{S}_n$ of any element (although the answer is in terms of integrals of Chern classes over the Deligne-Mumford compactification $\overline{\mathcal{M}_{g;n}}$ of the moduli space $\mathcal{M}_{g;n}$ of genus g curves with n marked points) [Eke+01].

[2] Although, there might be a mistake in the proof, see [after the bibliography Eke+01, Notes added in proof].

4. Reduced block factorizations in $\mathfrak{S}_n$, with prescribed conjugacy classes, of the long cycle $(12\cdots n)$ [ZL99].

5. Reduced block and *symmetric* factorizations in $\mathfrak{S}_n$, with prescribed conjugacy classes, of the long cycle [Bai99].

One important note is that all of the above results have uniform proofs, since they only deal with the symmetric group $\mathfrak{S}_n$. Indeed, in that case, Corol. 69 is no coincidence at all. The Trivialization Theorem (including the transitivity of the Hurwitz action) appears as a corollary of the Riemann existence theorem [LZ04, Thm. 1.8.14].

Our work is particularly related to the fourth case above. Lando and Zvonkine [ZL99] gave a geometric derivation of formulas that were already proven via combinatorial means by Goulden and Jackson [GJ92]. They did that by first covering the case of *primitive factorizations.*

In particular, they express the number of block factorizations in terms of the degrees of the transverse intersections of some varieties that are associated to our strata $[Z]_Y$. It would be very interesting if we can extend their work in the context of complex reflection groups.

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