

Supplementary material

Constrained clamped-clamped Elastica of constant length

We consider next the case of a constrained elastica clamped at both ends. There are two important differences between this case and the pinned-pinned elastica analyzed above. First, there exist two distinct roots for the first buckling mode of a clamped-clamped elastica, with the smaller root corresponding to a stable symmetric configuration and the larger root to an unstable anti-symmetric configuration. Second, the first symmetric buckling mode configuration with continuous contact is unstable, while it was stable for a pinned-pinned elastica. We analyze separately the symmetric, anti-symmetric, and asymmetric configurations.

0.1 First Buckling Mode (Symmetric Case)

0.1.1 Unconstrained Buckling

Referring to Figure 1, the unconstrained elastica is divided into four identical segments for the first symmetric buckling mode. More generally, there are $m = 4k$ with $n_p = 2k$ for the symmetric k -buckling mode, with the inclination $\theta_1(s)$ and the horizontal force \mathcal{R}_1 of the first segment given by

$$\theta_1(s) = \theta_{[2]} \left\{ \cos \left[2k\pi (s - s_{[2]}) \right] + \frac{\theta_{[2]}^2}{192} \left(\cos \left[2k\pi (s - s_{[2]}) \right] - \cos \left[6k\pi (s - s_{[2]}) \right] \right) \right\}$$

$$\mathcal{R}_1 = 4(k\pi)^2 \left[1 + \frac{\theta_{[2]}^2}{8} \right] \tag{1}$$

with $\theta_{[2]}$ viewed here as the loading parameter. Using symmetry considerations, the solution is trivially extended to the other segments.

0.1.2 Constrained Buckling with Discrete Contact

This first contact event is characterized by contact force $F_{[j]} = 0$, $j \in \mathcal{I}_w$ and $\Delta y_i = c/2$, $i \in \mathcal{I}_\ell$ for each segment. Past this first contact event, increasing $\theta_{[2]}$ leads to the build-up of contact force $F_{[j]} > 0$, $j \in \mathcal{I}_w$ at the discrete contacts and the progressive decrease of the moment at the location of the contacts. The derivation is similar to the pinned-pinned case and is thus omitted.

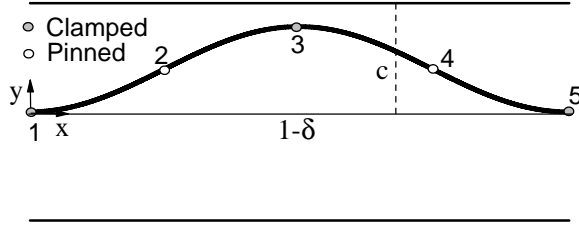


Figure 1: Symmetrical first buckling mode of clamped-clamped elastica

0.1.3 Secondary Buckling

With increasing $\theta_{[2]}$, the moment at the discrete contact eventually vanishes. In contrast to the pinned-pinned elastica, this condition does not mark the onset of continuous contact, as this configuration of the elastica become unstable ($n_p^{(l)} = 2m_e^{(l)}$). In addition the same holds for the secondary buckling described below, for which $n_p^{(d)} = m_e^{(d)} + 2$. Due to vanishing moments of both ends, the secondary buckling corresponds to a change in the sign of the moment at the clamped ends, as reflected in the deformed configuration shown in Fig. 2.

To construct the solution for the secondary buckling with one discrete contact, only one half of the elastica needs to be considered. As shown in Fig. 2, there are two distinct clamped-clamped segments with an inflection point at the midpoint of each segment. Two distinct canonical problems have thus to be solved, corresponding to segments 2 (bounded by nodes 2 and 3) and 3 (bounded by nodes 3 and 4), with a combined length $\ell_2 + \ell_3 = 1/4$. The segments have relative inclinations $\psi_2(s) = \theta_2(s) + \alpha_2$ and $\psi_3(s) = \theta_3(s) - \alpha_3$, respectively with $\alpha_2 \geq 0$, $\alpha_3 > 0$. As the resultant force \mathcal{R}_i , $i \in \mathcal{I}_\ell$ is uniform, $\alpha_i = \alpha_{i+1}$, $i \in \mathcal{I}_\ell$. Due to continuity of the bending moment at node 3, $\beta_2 = -\beta_3$, and a relation between length ℓ_i and angle β_i is derived

$$\ell_i = \frac{1}{4\pi} \arccos \beta_i \quad (2)$$

Hence, $\mathcal{R}_i = 16\pi^2$, $i \in \mathcal{I}_\ell$.

The contact constraints for the segments are $\Delta y_1 = \Delta y_2 = -\gamma c/2$ and $\Delta y_3 = \Delta y_4 = c(1 + \gamma)/2$ with $0 \leq \gamma < 1$. Number γ increases with end-displacement δ , starting at 0 at the onset of secondary buckling up to 1 when the elastica contacts the opposite wall at two symmetric points. With three discrete contacts, the segmentation remains the same. However the relative inclination for the second segment is modified according to $\psi_2(s) = \theta_2(s) - \alpha_2$ and the transverse components of the force transmitted by the segments are related by

$$\mathcal{R}_2 \sin \alpha_2 = \mathcal{R}_3 \sin \alpha_3 + F_{[3]} \quad (3)$$

This means that the inclination angles $\alpha_2 \neq \alpha_3$ are different. Using the continuity of the bending moment at node 3 and uniformity of horizontal force, the complete solution can be easily derived.

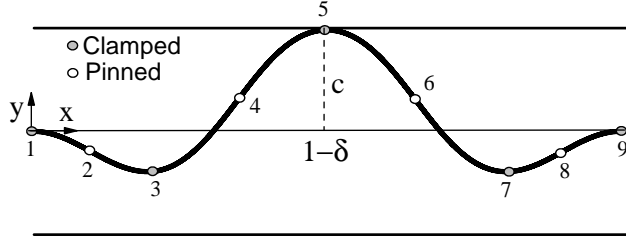


Figure 2: Secondary buckling of clamped-clamped elastica.

0.2 First Buckling Mode (Anti-symmetric Case)

The anti-symmetric case of the second root differs remarkably from the symmetric one. It is a double clamped-pinned case, which requires some special consideration. An asymmetrical deformation shape similar to the anti-symmetric case is also discussed.

0.2.1 Unconstrained Buckling

The second root of the first buckling mode of a clamped-clamped elastica corresponds to the anti-symmetric configuration illustrated in Fig. 3. Thus only one-half of the elastica has to be solved. Using the segmentation technique, one half of the elastica is divided into three distinct segments, with segments 1 (bounded by nodes 1 and 2) and 2 (bounded by nodes 2 and 3) being identical; the combined lengths $2\ell_2 + \ell_3 = 0.5$. Considering that the node 4 lies along the centerline, the sum of the vertical offsets of the segments satisfy

$$\sum_{i=1}^3 \Delta y_i = 0 \quad (4)$$

In the absence of contact, the force in the elastica is uniform. The solution can be obtained by applying continuity of the bending moment at node 3.

0.2.2 Constrained Buckling with Discrete Contact

The contact event occurs when the contact force $F_{[3]} = 0$ and $\Delta y_3 = c$. The deflection of the segments is then constrained by $2\Delta y_1 = 2\Delta y_2 = \Delta y_3 = c$ and the segment lengths by $2\ell_2 + \ell_3 = 0.5$. Further increase of δ leads to $F_{[3]} > 0$ and thus to a jump in the transverse component of the force transmitted by segments 2 and 3.

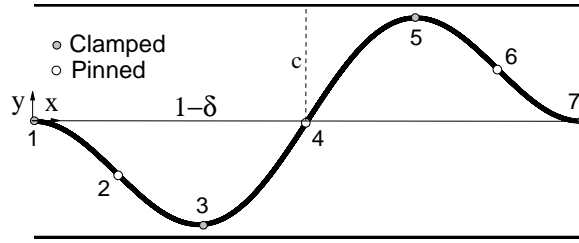


Figure 3: Asymmetrical first buckling mode of clamped-clamped elastica

0.2.3 Asymmetrical configuration (displacement controlled)

For a displacement-controlled loading, an asymmetrical configuration of the elastica is observed beyond the point of vanishing moment at the discrete contact 3, as shown in Figure 4. It is similar to the second root of first buckling mode but in an asymmetrical manner and with a single discrete contact. With increasing δ an abrupt decrease of the horizontal force R_o occurs. The position of the discrete contact varies (i.e., $s_{[3]} \leq 0.5$), while the vertical displacement of the two last segments (i.e., $\Delta y_5 + \Delta y_6$) increases until a second discrete contact develops at node 5 .

To derive the solution for the asymmetrical configuration, the elastica should be divided into three clamped-clamped segments with an inflection point at the midpoint of each segment. Three canonical problems should be solved, e.g., segments 2 (bounded by nodes 2 and 3), 4 (bounded by nodes 4 and 5) and 5 (bounded by nodes 5 and 6) with combined length $\ell_2 + \ell_4 + \ell_5 = 1/2$. The segments have relative inclinations $\psi_2(s) = \theta_2(s) - \alpha_2$, $\psi_4(s) = \theta_4(s) - \alpha_4$ and $\psi_5(s) = \theta_5(s) + \alpha_5$, respectively, assuming positive inclinations. The horizontal applied force $R_i \cos \alpha_i$, $i \in \mathcal{I}_\ell$ is uniform. In addition, $\mathbf{R}_4 = \mathbf{R}_5$ and thus $\alpha_4 = \alpha_5$. Due to continuity of the bending moment at node 5, $\beta_4 = -\beta_5$ and a relation between lengths ℓ_4, ℓ_5 and their resultant force is obtained

$$\ell_4 + \ell_5 = \frac{\pi}{\sqrt{\mathcal{R}_4}} \quad (5)$$

Continuity of bending moment at the discrete contact at node 3 has also to be satisfied. The contact constraints for the segments are $\Delta y_1 = \Delta y_2 = -c/2$, $\Delta y_3 = \Delta y_4 = c(1 + \gamma)/2$ and $\Delta y_5 = \Delta y_6 = \gamma c/2$ with $0 \leq \gamma < 1$. For $\gamma = 1$ the deformation configuration is characterized by two discrete contactst and the elastica should be divided differently as explained in Section 0.2.2.

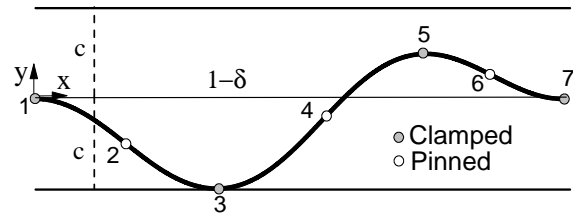


Figure 4: Asymmetrical configuration of clamped-clamped elastica