Pursuing Multiple Evaders in the Plane

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August 12, 2017

1 Introduction

The pursuit-evasion game represents a virtually limitless problem space. One of a number of unexplored subproblems concerns the pursuit of multiple evaders by a group of pursuers in an unbounded plane with no obstacles. Though similar formulations have been investigated, this particular issue has received little attention. It is the aim of this report to remedy that fact, providing an introduction into the field of open-plane pursuit-evasion games to furnish a basis for future work. Pursuit in a totally unbounded environment represents a unique challenge in that there is no predefined boundary against which to trap evaders; the pursuers themselves must create such a boundary by guarding paths to prevent escape, and if an evader can attain sufficient separation from the pursuers, it can survive indefinitely.

2 Related Work

Extensive research exists with regard to games involving only one evader [1], including a formulation in the open plane [2]. Some forays have been made into multiple-evader arrangments in the plane, but with obstacles [3]. A "framework" has been furnished for this problem in $\mathbb{R}^n$, but again, only with a single evader [4].

3 The Problem

The problem at hand can be stated as follows:
Given a pursuit-evasion game with a set of n pursuers, $P = \{p_1, p_2, \ldots, p_n\}$, and a set of m evaders, $E = \{e_1, e_2, \ldots, e_m\}$, with all players having positions given by points in $\mathbb{R}^2$ and a maximum speed c, determine whether the pursuers can always win.

Some related terms will be helpful in discussing the problem further:

Definition 1. *Capture*: A pursuer p_i with position p_{i_x} is said to capture an evader e_j with position e_{j_x} when $p_{i_x} = e_{j_x}$; when captured, e_j is removed from E.

Definition 2. *Winning/losing*: The pursuers win (and the evaders lose) if, after a finite amount of time, $E = \emptyset$, i.e. all evaders have been captured. The pursuers lose (and the evaders win) if this will never be the case, i.e. the evaders can follow some strategy that will guarantee that $|E| > 0$ after any amount of time.

The distinction between it being the case that one side "can win" or "will win" can be made clearer by assuming that if it is possible for the pursuers to capture all evaders, they will, and inversely that if it is possible for all evaders to escape, they will. Because the pursuers and evaders are modeled as having agency, capable of choosing how to move, scenarios can be imagined in which sufficiently suboptimal stratgies are followed that an arbitrary arrangement of pursuers may lose to a single evader that they are theoretically capable of capturing, or a large number of evaders may fail to escape a single pursuer that they "should" evade; however, these will hereafter be ignored.

Definition 3. *Capture position*: A set of pursuers is said to have, or be in, a capture position relative to a group of evaders if each pursuer's future strategy involves only pursuing a specific, contained evader until it is captured.

Multiple interpretations still exist within this formulation. Here sensing limitations will be ignored; that is, all agents have perfect information about all other agents' states at all times, but a variant of the game with a finite visibility radius can also be considered and has applications to mobile robotics [5][6]. If time is continuous, as is most natural for the continuous-space formalization in question, all players move simultaneously and the speed limit c has a conventional meaning. Time can also be discretized, in which case the "speed" c is interpreted as a maximum step size and the players move in turns.Discrete time is typically employed to ease analysis of the single-evader case [1] and can be useful here as well in evaluating particular strategies, but due to its more general nature, continous time will be assumed throughout.

4 Some Elementary Findings

The following statements concern a group of k pursuers $p_1, p_2, \ldots, p_k$ focused exclusively on a single evader e contained in the pursuers' convex hull (a capture position) and will be useful in stating other results. Denote by C_P the convex hull of a set of pursuers P, and by V_e the Voronoi cell for e in the Voronoi diagram for $p_1, p_2, \ldots, p_k$ and e.

Theorem 1. For some strategy of the pursuers, regardless of e's strategy, V_e at any moment is a subset of V_e at any previous time.

Proof. Jankovič provides a pursuer strategy that maintains the original orientations of the segments connecting e to each surrounding pursuer [2]. Consequently, the edges of V_e also never change orientation. Since by Jankovič's strategy, any distance traveled by e in the direction of the segment connecting it to some p_i is at least offset by the motion of that p_i, the shared Voronoi edge will remain in the same place or advance toward the evader. Thus, e can be confined to its Voronoi cell. □

Theorem 2. Capture of an evader e requires that at least three pursuers converge on e_x and reduce the area of V_e to zero.

Proof. If there exists a point that can be reached by e before it is reached by a pursuer, e can avoid capture by moving to that point. Therefore, capture requires that there not exist any such points. □

In essence, Theorem 2 shows that a number of pursuers must all become co-located with an evader to capture it, assuming the evader's strategy involves maximizing its survival time.

5 Basic Conditions for Capture

Certain degenerate cases in the pursuit-evasion game exist; these are listed here and thereafter assumed not to be the case.

- When $m = 0$, E is initially empty, representing a trivial win for the pursuers.
- When $n = 0$, for arbitrary positive m the evaders have a trivial win, since there is never a risk of capture.
- When an evader is initially co-located with a pursuer, the notion of "pursuit-evasion game" loses meaning, as the evader can be captured with no pursuit necessary; when this is the case for all evaders, the pursuers win without any "effort."

5.1 One evader

The most elementary finding regards a necessary and sufficient condition for the pursuers in P to capture a single evader e: capture can be guaranteed if and only if e is positioned within C_P [2]. As such, there must be at least three pursuers to guarantee capture of one evader, but this case offers little else of interest.

5.2 Two evaders

The situation becomes less clear with the addition of a second evader. As with the one-evader case, capturing a given evader requires that it be located within

C_P, but with multiple evaders this is merely a necessary condition for capture, not a sufficient one. For cases in which both evaders are not captured simultaneously, denote the first evader captured by e_1 and the second by e_2.

5.2.1 Fewer than five pursuers

A cursory analysis shows that the pursuers lose if $n \leq 4$. By Theorem 2, at least three pursuers are required to capture one evader. For $n = 3$, all of the pursuers are required to capture e_1, allowing e_2 to escape, even if both evaders are initially within C_P. Similarly, for $n = 4$, at least three pursuers must converge on e_1, allowing e_2 to escape by moving away from the segment between the fourth pursuer and the capturing group of three.

5.2.2 Five pursuers

For $n = 5$, no general statement can be made about which side wins; instead, the outcome depends on the initial configuration. In this case, a variant of the idea of successive pursuit, which has been analyzed for a single, faster pursuer attempting to capture multiple evaders [7], can be considered. One evader must still be captured first by convergence of three pursuers, but a convex hull can exist for the point at which those three converge and the locations of the remaining two pursuers. As such, if three pursuers can capture e_1 while the other two prevent e_2 from escaping C_P (which will have shifting boundaries), e_2 can then be captured using any winning strategy, and five pursuers will have sufficed for the capture of two evaders.

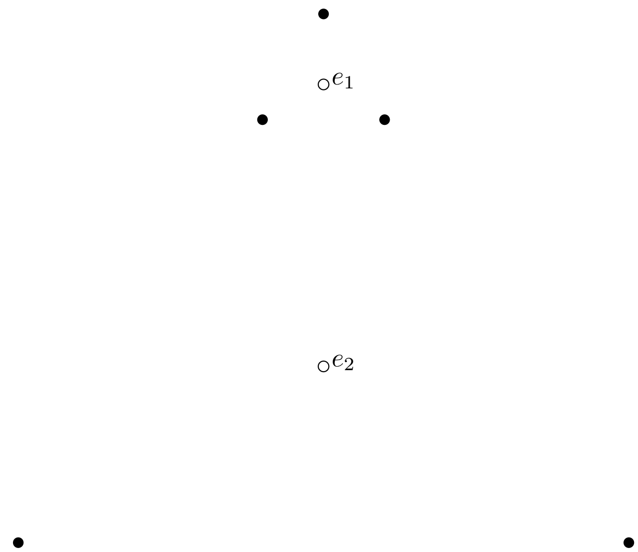

Figure 1: Five pursuers, two evaders

For example, consider the structure shown in Figure 1. Without dealing with tangible numbers, denote by t^* an upper bound for the time taken by the three upper pursuers to capture e_1, and by δ the distance from e_2's initial position to the nearest possible point on an edge of C_P after e_1 has been caught, based on the result of Theorem 1 that e_1 will be caught somewhere within V_{e_1}. It can be seen that if $\delta < ct^*$, e_2 cannot escape the new C_P that will exist once e_1 is captured and can then be captured by the pursuers regardless of its strategy. Other scenarios similar to this one can be constructed, and the lower evaders may be required to move to ensure that e_2 always remains within C_P.

5.2.3 More than five pursuers

When $n = 6$, it is again possible for each evader to be contained within a unique convex hull of pursuers, which can concentrate solely on capturing their respective evaders, and successive pursuit is unnecessary. However, capture cannot be guaranteed in such a straightforward way even if both evaders lie within C_P if such a partition as discussed below cannot be found. In this case, the pursuers must maneuver to establish a capture position, and whether this can always be done is uncertain.

5.3 m evaders

In general, given m evaders, $n = 3m$ pursuers are in a capture position and therefore have a simple winning strategy if, for some partition $\Pi = \{\pi_1, \pi_2, \ldots, \pi_m\}$ of P, there exists a bijection $f : \Pi \to E$ such that $f(\pi_i) = e_j \implies C(\pi_i, e_j)$, where $C(P, e)$ means that e is contained within C_P. While finding Π (and a corresponding f) may represent something of a computational challenge for extremely large m, once f is found, it is a simple matter for the three pursuers in each π_i to capture their "assigned" evader. If the pursuers do not have a capture position, a more complicated strategy is necessary to contain all evaders while a capture position is established.

6 Triangle Madness

Typically, initial existence of a capture position for m evaders requires $3m$ pursuers. However, building on the successive pursuit idea from the two-evader case in which five pursuers suffice for capture, structures can be defined allowing a much smaller ratio of pursuers to evaders. In this section a configuration of pursuers and evaders is outlined in which each pursuer is part of a capture position for a single evader in any given epoch of the game, but the game may be divided into many epochs.

Definition 4. Define a level-n Triangle Madness network (abbreviated $TM(n)$, pronounced "Timn") recursively as follows:
$TM(-1)$: A single pursuer.
$TM(n)$: 3^n evaders co-located at the center of an equilateral triangle with the center of a $TM(n-1)$ at each vertex, for $n \geq 0$.

For example, a $TM(1)$ is shown in Figure 2.

6.1 Numbers of players

Theorem 3. A $TM(n)$ contains $P(n) = 3^{n+1}$ pursuers.

Proof. By the definition of a $TM(n)$,

$$P(n) = 3P(n-1)\,.$$

Induction on k: As a basis, for $n = -1$,

$$3^{n+1} = 3^0 = 1 = P(-1)\,.$$

Now suppose $P(k) = 3^{k+1}$. Then

$$P(k+1) = 3(3^{k+1}) = 3^{k+2}\,.$$

□

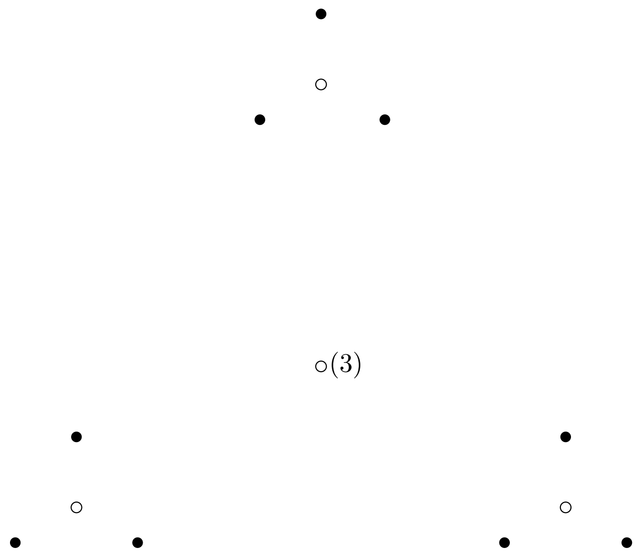

Figure 2: An example of a $TM(1)$

Theorem 4. A $TM(n)$ contains $E(n) = (n+1)3^n$ evaders.

Proof. By the definition of a $TM(n)$,

$$E(n) = 3^n + 3E(n-1)\,.$$

Induction on k: As a basis, for $n = -1$,

$$(n+1)3^n = (0)3^{-1} = 0 = E(-1)\,.$$

Now suppose

$$E(k) = (k+1)3^k\,.$$

Then

$$\begin{aligned} E(k+1) &= 3^{k+1} + 3\left((k+1)3^k\right) \\ &= 3^{k+1} + (k+1)3^{k+1} = (k+2)3^{k+1}\,. \end{aligned}$$

□

For a $TM(n)$,

$$\frac{E(n)}{P(n)} = \frac{(n+1)3^n}{3^{n+1}} = \frac{n+1}{3}\,. \tag{1}$$

Since

$$\lim_{n\to\infty} \frac{n+1}{3} = \infty\,,$$

the ratio of evaders to pursuers is unbounded as the number of levels increases.

6.2 Capturability

The preceding discussion provides for certain numbers of pursuers and evaders, but says nothing about relative sizes of the triangles at different levels or about whether all of the evaders can necessarily be caught. As in the five-pursuer instance of the two-evader case, analyzing upper bounds on the time taken to capture earlier evaders can dictate conditions for capturing later evaders. In a given $TM(n)$, additional parameters can be defined that may vary between different networks of the same level. Constrain all level-i substructures within a $TM(n)$ to be of the same size, with the side length of their convex hulls denoted by d_i, for $i = 0, 1, \ldots, n$.Denote by t_i^* and $d_i^* = ct_i^*$ upper bounds for the time to capture all evaders within a $TM(i)$ and the maximum distance the central evaders of a $TM(i+1)$ can travel in that time, respectively, for $i = 0, 1, \ldots, n$ ($t_{-1}^* = d_{-1}^* = 0$).

Theorem 5. All evaders in a $TM(1)$ can be captured if $\frac{d_1}{d_0} > 14$.

Proof. When a set of pursuers P is pursuing one evader e located within C_P, Janković [2] shows that some pursuer p_i is always able to move toward e at a speed of at least $c \cos \alpha$ if all pursuers follow a strategy that traps e within V_e, where α is half of the largest angle between the segments connecting e to each pursuer. In a $TM(0)$, $\alpha = \pi/3$ since e is at the center of an equilateral triangle with pursuers at the vertices. Consequently, e's total distance from the three pursuers decreases at a rate of at least $c \cos(\pi/3) = c/2$. Further, by simple geometry, e is initially at a distance of $d_0/\sqrt{3}$ from each pursuer, for a total distance of $3d_0/\sqrt{3} = \sqrt{3}d_0$, and so

$$t_0^* = \frac{\sqrt{3}d_0}{c/2} = \frac{2\sqrt{3}d_0}{c}.$$

Thus, $d_0^* = 2\sqrt{3}d_0$, and the center evaders in a $TM(1)$ can be captured if they are initially at a distance greater than d_0^* from the nearest (to the center evaders' initial location) possible side of a C_P that can exist after all level-zero evaders have been captured. Since each e is confined to V_e, which is initially a triangle congruent to and rotated π from C_P for a $TM(0)$, said nearest possible side would be a segment connecting the vertices of two V_e's which are nearest to the center of the $TM(1)$. It is apparent that the length of such a segment would be $d_1 - 2d_0$, and so basic geometry shows that capture can occur if

$$\frac{d_1 - 2d_0}{2\sqrt{3}} > d_0^*,$$

or, substituting for d_0^* and rearranging, $d_1 > 14d_0$, which is equivalent to

$$\frac{d_1}{d_0} > 14.$$

□

The above reasoning establishes a sufficient condition for capture based on a "worse-than-worst-case" analysis; in practice, the center evader of a given $TM(0)$ will be captured more quickly and/or farther from the edge of its Voronoi cell than assumed, but it is clear that capture must occur within the calculated t^*. Upper bounds for capture times are more difficult to ascertain for higher-level networks because movement by the inner evaders disrupts the equilateral triangle property, but qualitative results can still be obtained.

Theorem 6. For arbitrary $n \geq 1$, there exists some minimum finite ratio of $\frac{d_n}{d_{n-1}}$ for which all evaders in a $TM(n)$ can be captured in finite time.

Proof. A proof by induction is straightforward. For a basis, this result has already been proven for $n = 1$ in Theorem 5. Now, assume that the proposition is true for some $n = k$, so that all evaders in a $TM(k)$ with finite side length d_k can be captured in some time t_k^*, and so the center evaders of a $TM(k+1)$ can move no farther than $d_k^* = ct_k^*$.

Consider an equilateral triangle τ with area enclosed by segments parallel to the sides of the $TM(k+1)$ with initial locations translated a distance of $d_k\sqrt{3}/2$ toward the center evaders, so that each segment is part of a line passing through two pursuers from different $TM(k)$ networks, and all pursuers are outside τ; let τ_0 and τ_f denote the triangles formed by segments of lines through the same pursuers before movement of any agents and after all evaders within the outer $TM(k)$ networks have been captured, respectively. Drawing a picture of τ is left as an exercise to the reader. Cursory analysis shows that any evader in the interior of τ can be captured by any set of three pursuers with one pursuer chosen from each composite $TM(k)$ of the $TM(k+1)$; as such, if no center evader can escape τ before the pursuers can turn their attention to said evaders (after capturing the evaders within each outer $TM(k)$), all evaders can be captured. Denote by δ the distance from the center evaders to any side of τ_0. It is clear that each side of τ_f can be translated from its position in τ_0 (with the possibility of rotation, making τ no longer equilateral) at most d_k^* in t_k^*, so if $\delta > 2d_k^*$, a capture position with respect to each central evader will always exist.

The length of each side of τ_0 can be seen to be $d_{k+1} - 3d_k$, and since the center evaders of the $TM(k+1)$ are also at the center of τ_0,

$$\delta = \frac{d_{k+1} - 3d_k}{2\sqrt{3}}.$$

Consequently, if

$$\frac{d_{k+1} - 3d_k}{2\sqrt{3}} > 2d_k^*,$$

or equivalently

$$d_{k+1} > 4d_k^*\sqrt{3} + 3d_k,$$

the center evaders cannot escape. Note that since both d_k^* and d_k are finite (by hypothesis), d_{k+1}—and therefore $\frac{d_{k+1}}{d_k}$—can also be made finite while still guaranteeing capture, and there must exist some lower bound beyond which capture is no longer certain. □

Theorem 6 demonstrates that, with the right parameters, the pursuers can capture all evaders in any level of Triangle Madness network. This result, in combination with Equation 1, shows that a number of pursuers can capture a number of evaders that, in the limit, is infinitely larger, or equivalently that there is no limit to the number of evaders that a single pursuer can successfully capture in one game (each pursuer will aid in the capture of $n+1$ evaders in a $TM(n)$), a somewhat surprising finding.

7 Conclusion and Future Directions

In this report, the problem of pursuing multiple evaders in the plane has been explored, with an important result being that a single pursuer can aid in the capture of an unbounded number of evaders with the right initial configuration. A natural extension of this work would be to provide a tighter bound on capture time in a $TM(1)$ and to use that as a foundation for quantitatively analyzing higher levels. In the same vein, necessary conditions to go along with the sufficient ones discussed above could be supplied both for capture in Triangle Madness networks and for general cases with certain numbers of evaders. Relatedly, the "Holy Grail" of this type of problem would be an algorithm that, given the initial configuration of pursuers and evaders and possible environmental factors (e.g. obstacles), would output whether all evaders can be captured by some pursuer strategy. A result of such generality is unlikely to be entirely produced, but this lofty aspiration provides a guiding path for further research.

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