MAX-MIN PROPERTIES OF MATRIX FACTOR NORMS

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Abstract

Given a set of real matrices $C_0, C_1, ..., C_k$, we consider conditions under which the equality

$$\min_{\alpha_1,...,\alpha_k} \max_{||w||=1} ||(C_0+\sum_{i=1}^{k} \alpha_i C_i)w|| = \max_{||w||=1} \min_{\alpha_1,...,\alpha_k} ||(C_0+\sum_{i=1}^{k} \alpha_i C_i)w||$$

holds. It is shown that if the matrices C_i, $i = 0, 1, ..., k$ are normal and commute with one another then the equality holds. In particular, this implies that if $C_i = A^i$ or $C_i = A^{k-i}$, where A is a normal matrix, then the equality holds. An example is given to show that the equality may fail for non-commuting matrices, when $k > 1$. It is shown that the equality holds for arbitrary matrices if $k = 1$.

1 Introduction

The following problem arises in the analysis of iterative methods for solving linear systems and computing eigenvalues. To solve a linear system, $Ax = b$,

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given an initial guess x^0 for the solution, the GMRES method [5] generates approximate solutions x^k, $k = 1, 2, \ldots$ of the form

$$x^k = x^0 + \sum_{i=1}^{k} \alpha_{ik} A^{i-1} r^0,$$

where $r^0 \equiv b - Ax^0$ is the initial residual. The residual vectors $r^k \equiv b - Ax^k$ are of the form

$$r^k = r^0 - \sum_{i=1}^{k} \alpha_{ik} A^i r^0,$$

and the coefficients $\alpha_{1k}, \ldots, \alpha_{kk}$ are chosen to make the 2-norm of r^k as small as possible. A bound on the 2-norm of the residual at any step k is given by

$$\|r^k\| \leq \min_{\alpha_1, \ldots, \alpha_k} \|I - \sum_{i=1}^{k} \alpha_i A^i\| \cdot \|r^0\|$$

The question arises as to whether this bound is ever attained; that is, whether there is an initial residual r^0 such that

$$\min_{\alpha_1, \ldots, \alpha_k} \|(I - \sum_{i=1}^{k} \alpha_i A^i) r^0\| = \min_{\alpha_1, \ldots, \alpha_k} \|I - \sum_{i=1}^{k} \alpha_i A^i\| \cdot \|r^0\|.$$

In other words, we have the following max-min problem: Is the inequality

$$\max_{\|r^0\|=1} \min_{\alpha_1, \ldots, \alpha_k} \|(I - \sum_{i=1}^{k} \alpha_i A^i) r^0\| \leq \min_{\alpha_1, \ldots, \alpha_k} \max_{\|r^0\|=1} \|(I - \sum_{i=1}^{k} \alpha_i A^i) r^0\| \tag{1}$$

actually an equality?

A similar question arises in analyzing the Arnoldi method [1] for computing eigenvalues. Given an initial vector q with $\|q\| = 1$, the Arnoldi iteration constructs a sequence of monic polynomials P_k, $k = 1, 2, \ldots$ whose coefficients are chosen to minimize $\|p_k(A)q\|$ over all monic polynomials p_k of degree k. The roots of these polynomials are taken as approximate eigenvalues of the matrix A. The question arises as to whether, for each k, there is an initial vector q such that the monic polynomial P_k constructed by the Arnoldi process also minimizes $\|p_k(A)\|$. A similar max-min statement of the problem asks if the inequality

$$\max_{\|q\|=1} \min_{\alpha_1,\ldots,\alpha_k} \|(A^k - \sum_{i=1}^{k} \alpha_i A^{k-i})q\| \leq \min_{\alpha_1,\ldots,\alpha_k} \max_{\|q\|=1} \|(A^k - \sum_{i=1}^{k} \alpha_i A^{k-i})q\| \quad (2)$$

is actually an equality.

In this paper, we consider a somewhat more general question: Given an arbitrary sequence of real matrices $C_0, C_1, ..., C_k$, under what circumstances will the equality

$$\min_{\alpha_1,\ldots,\alpha_k} \max_{\|w\|=1} \|(C_0 + \sum_{i=1}^{k} \alpha_i C_i)w\| = \max_{\|w\|=1} \min_{\alpha_1,\ldots,\alpha_k} \|(C_0 + \sum_{i=1}^{k} \alpha_i C_i)w\| \quad (3)$$

hold? It is shown that if the matrices C_i, $i = 0, 1, ..., k$ are normal and commute with one another then (3) holds. In particular, this implies that if $C_i = A^i$ or $C_i = A^{k-i}$, where A is a normal matrix, then the equality holds. This generalizes some known results showing that equality holds in (1) and (2) when the matrix A is normal [2,3,4]. An example is given to show that (3) may fail for non-commuting matrices, when $k > 1$. It is shown that the equality (3) holds for arbitrary matrices if $k = 1$. The question of whether equality holds in (1) and (2) when $k > 1$ remains open, as does the question of more general conditions on the matrices $C_0, C_1, ..., C_k$ that would ensure that (3) holds.

Throughout this paper, we assume that the matrices and vectors appearing in our max-min statements and related theorems are real (though, of course, the eigenvalues and eigenvectors of these matrices may be complex). We will use the notation $A > 0$ to mean that the symmetric matrix A is positive definite. For a vector w, $\|w\|$ will always denote the 2-norm, and for a matrix A, $\|A\|$ will denote the corresponding matrix norm, $\max_{\|w\|=1} \|Aw\|$.

The next section gives the main theorems and examples.

2 Main Theorems

The first theorem gives conditions under which some linear combination of given symmetric matrices is positive definite.

Theorem 1. Let $A_1, ..., A_k$ be real n by n symmetric matrices that commute: $A_iA_j = A_jA_i$, $i, j = 1, ..., k$. There exist scalars $\alpha_1, ..., \alpha_k$ such that

$$\sum_{i=1}^{k} \alpha_i A_i > 0 \tag{4}$$

if and only if for every nonzero n-vector w, we have

$$\langle A_i w, w\rangle \neq 0 \tag{5}$$

for some i.

Proof: The necessity of (5) is clear since if, for some w, $\langle A_i w, w\rangle = 0$ for all i, then

$$\langle \sum_{i=1}^{k} \alpha_i A_i w, w\rangle = 0,$$

for any $\alpha_1, ..., \alpha_k$. To prove that (5) is sufficient, we must use the fact that the symmetric matrices A_i commute and hence can be diagonalized by a single orthogonal matrix Q:

$$A_i = Q\Lambda_i Q^T, \quad QQ^T = Q^TQ = I, \quad \Lambda_i = diag(\lambda_{i1}, ..., \lambda_{in}).$$

It therefore suffices to consider the diagonal matrices Λ_i, $i = 1, ..., k$. Suppose no linear combination of these matrices is positive definite. Then the linear subspace spanned by Λ_i, $i = 1, ..., k$ and the convex cone of positive definite diagonal matrices have empty intersection and so they can be separated. That is, there is a diagonal matrix $D = diag(d_1, ..., d_n)$ such that

$$tr(D\sum_{i=1}^{k} \alpha_i \Lambda_i) = 0, \tag{6}$$

for all $\alpha_1, ..., \alpha_k$ and

$$tr(DP) > 0, \tag{7}$$

for all positive definite diagonal matrices P. From (7), it follows that the diagonal elements of D are nonnegative, with at least one d_j being positive. Define w to be the vector $(\sqrt{d_1}, ..., \sqrt{d_n})^T$. ¿From (6) we have for each i,

$$tr(D\Lambda_i) = \sum_{j=1}^{n} d_j \lambda_{ij} = \langle \Lambda_i w, w \rangle = 0,$$

which contradicts (5). □

We now use Theorem 1 to establish conditions under which the optimal coefficients $\alpha_1, ..., \alpha_k$ on both sides of equality (3) are zero.

Theorem 2. Let $C_1, ..., C_k$ be real square matrices such that each pair $(C_i + C_i^T)$ and $(C_j + C_j^T)$ commute. Suppose

$$\min_{\alpha_1,...,\alpha_k} \max_{||w||=1} ||(I + \sum_{i=1}^{k} \alpha_i C_i)w|| = 1. \tag{8}$$

Then

$$\max_{||w||=1} \min_{\alpha_1,...,\alpha_k} ||(I + \sum_{i=1}^{k} \alpha_i C_i)w|| = 1. \tag{9}$$

Proof: For a given vector w, we have

$$\min_{\alpha_1,...,\alpha_k} ||(I + \sum_{i=1}^{k} \alpha_i C_i)w|| = 1$$

if and only if $\langle C_i w, w \rangle = 0$ for all i; i.e., if and only if $\langle (C_i + C_i^T)w, w \rangle = 0$ for all i. Suppose (9) does not hold. Then for any vector $w \neq 0$ there is an i such that

$$\langle (C_i + C_i^T)w, w \rangle \neq 0.$$

¿From Theorem 1, there is a linear combination

$$\sum_{i=1}^{k} \alpha_i (C_i + C_i^T)$$

that is positive definite. For ϵ sufficiently small, then,

$$\|I - \epsilon\sum_{i=1}^{k}\alpha_i C_i\|^2 = \|I - \epsilon\sum_{i=1}^{k}\alpha_i(C_i + C_i^T)\| + \mathcal{O}(\epsilon^2) < 1,$$

which contradicts the assumption (8). □

Theorem 2 is now used to establish certain conditions under which equality (3) holds.

Theorem 3. Let $C_0, C_1, ..., C_k$ be nonsingular normal matrices that commute. Then

$$\min_{\alpha_1,...,\alpha_k}\max_{\|w\|=1}\|(C_0 + \sum_{i=1}^{k}\alpha_i C_i)w\| = \max_{\|w\|=1}\min_{\alpha_1,...,\alpha_k}\|(C_0 + \sum_{i=1}^{k}\alpha_i C_i)w\|. \quad (10)$$

Proof: Suppose $\hat{\alpha}_1, ..., \hat{\alpha}_k$ minimize $\|C_0+\sum_{i=1}^{k}\alpha_i C_i\|$. We can assume without loss of generality that this minimal norm is 1. We will consider two cases:

1. First, suppose all singular values of $U \equiv C_0 + \sum_{i=1}^{k}\hat{\alpha}_i C_i$ are equal. Then U is a real orthogonal matrix and it commutes with each matrix C_i. The same holds for the inverse matrix U^T. We can write

$$\min_{\alpha_1,...,\alpha_k}\|C_0 + \sum_{i=1}^{k}\alpha_i C_i\| = \min_{\beta_1,...,\beta_k}\|U^T(C_0 + \sum_{i=1}^{k}(\hat{\alpha}_i + \beta_i)C_i)\|$$

$$= \min_{\beta_1,...,\beta_k}\|I + \sum_{i=1}^{k}\beta_i U^T C_i\| = 1.$$

Because the matrices C_i are normal and commute with each other and with U and U^T, each pair $(U^T C_i + C_i^T U)$ and $(U^T C_j + C_j^T U)$ commute. Therefore from Theorem 2 we have

$$\max_{\|w\|=1}\min_{\beta_1,...,\beta_k}\|(I + \sum_{i=1}^{k}\beta_i U^T C_i)w\| = 1,$$

and from this the desired result follows.

2. Now suppose some singular values are less than 1. We can write the real Schur decomposition of each matrix C_i in the form

$$C_i = QD_iQ^T, \quad QQ^T = Q^TQ = I$$

where each D_i is a block diagonal matrix with 1 by 1 or 2 by 2 blocks on the main diagonal. It suffices to consider the block diagonal matrices D_i. Define $\hat{D} \equiv D_0 + \sum_{i=1}^{k} \hat{\alpha}_i D_i$, and order the elements so that $\hat{D}$ is of the form

$$\hat{D} = \begin{pmatrix} U & 0 \\ 0 & K \end{pmatrix}$$

where U is, say, a t by t matrix whose eigenvalues are all equal to 1 in magnitude and K is an $n-t$ by $n-t$ matrix whose eigenvalues are all less than 1 in magnitude. Note that each matrix D_i has this same block structure

$$D_i = \begin{pmatrix} D_{i1} & 0 \\ 0 & D_{i2} \end{pmatrix},$$

since any off-diagonal block X in D_i would have to satisfy the homogeneous Sylvester equation: $UX - XK = 0$, in order that D_i and $\hat{D}$ commute. Since the spectrum of U does not intersect the spectrum of K, this equation has only the trivial solution $X = 0$. We can write

$$\min_{\alpha_1,\ldots,\alpha_k} \|D_0 + \sum_{i=1}^{k} \alpha_i D_i\| = \min_{\beta_1,\ldots,\beta_k} \|D_0 + \sum_{i=1}^{k} (\hat{\alpha}_i + \beta_i) D_i\|$$

$$= \min_{\beta_1,\ldots,\beta_k} \| \begin{pmatrix} U & 0 \\ 0 & K \end{pmatrix} + \sum_{i=1}^{k} \beta_i \begin{pmatrix} D_{i1} & 0 \\ 0 & D_{i2} \end{pmatrix} \|. \tag{11}$$

Since $\|U\| > \|K\|$, the same coefficients $\beta_1, \ldots, \beta_k$ that minimize the norm of the matrix in (11) (namely, $\beta_i = 0$, $i = 1, \ldots, k$) also minimize the norm of the upper left t by t block, and we have

$$\min_{\beta_1,\ldots,\beta_k} \|U - \sum_{i=1}^{k} \beta_i D_{i1}\| = 1.$$

Since the singular values of the minimal norm matrix of this form are all equal to one and since the matrices D_{i1} are normal and commute with each other, it now follows from part 1 that there is a t-vector $\hat{w}$ with $\|\hat{w}\| = 1$ such that

$$\min_{\beta_1,\ldots,\beta_k} \|U - \sum_{i=1}^{k} \beta_i D_{i1}\| = \min_{\beta_1,\ldots,\beta_k} \|(U - \sum_{i=1}^{k} \beta_i D_{i1})\hat{w}\|.$$

Defining $\tilde{w}$ to be the n-vector whose first t elements are equal to those of $\hat{w}$ and whose remaining elements are zero, we find

$$\min_{\alpha_1,\ldots,\alpha_k} \|D_0 + \sum_{i=1}^{k} \alpha_i D_i\| = \min_{\alpha_1,\ldots,\alpha_k} \|(D_0 + \sum_{i=1}^{k} \alpha_i D_i)\tilde{w}\|,$$

from which the desired result follows. □

Note that the assumption of commutativity in Theorem 1, and hence in Theorems 2 and 3, is necessary. Consider, for example, the symmetric matrices

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{12}$$

For any vector w, we have

$$\langle A_1 w, w\rangle = w_1^2 - w_2^2, \quad \langle A_2 w, w\rangle = w_1^2 + 2w_1 w_2 - w_2^2,$$

and if the first inner product is zero and w_1 and w_2 are not both zero, then the second inner product cannot be zero. Yet there is no linear combination of A_1 and A_2 that is positive definite. We have

$$\min_{\alpha_1,\alpha_2} \|I + \alpha_1 A_1 + \alpha_2 A_2\| = 1,$$

but for any vector w,

$$\min_{\alpha_1,\alpha_2} \|(I + \alpha_1 A_1 + \alpha_2 A_2)w\| = 0.$$

In the next lemma we consider general real m by n matrices $C_0, C_1, \ldots, C_k$. Suppose $\hat{\alpha}_1, \ldots, \hat{\alpha}_k$ minimize

$$\|C_0 + \sum_{i=1}^{k} \alpha_i C_i\|.$$

Define $\hat{C} \equiv C_0 + \sum_{i=1}^{k} \hat{\alpha}_i C_i$ and write the singular value decomposition of $\hat{C}$ as

$$\hat{C} = U \Sigma V^T,$$

where U is an m by m real orthogonal matrix, V is an n by n real orthogonal matrix, and Σ is an m by n matrix of the form

$$\Sigma = \begin{pmatrix} \sigma_1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \sigma_n \\ 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & 0 \end{pmatrix} \quad or \quad \Sigma = \begin{pmatrix} \sigma_1 & & & 0 & \cdot & 0 \\ & \cdot & & \cdot & \cdot & \cdot \\ & & \cdot & \cdot & \cdot & \cdot \\ & & \sigma_m & 0 & \cdot & 0 \end{pmatrix}$$

according as $m \geq n$ or $m \leq n$. Assume that the singular values σ_i, $i = 1, ..., min\{m, n\}$ satisfy

$$\sigma_1 = ... = \sigma_t > \sigma_{t+1} \geq ...,$$

and let V_t be the n by t matrix consisting of the first t columns of V, while $V_{t+1:n}$ is the n by $n - t$ matrix consisting of columns $t + 1$ through n of V.

The following lemma is used to prove Theorem 4, but it is also of significant interest in itself.

Lemma. Using the above notation, the coefficients $\hat{\alpha}_1, ..., \hat{\alpha}_k$ that minimize

$$\|C_0 + \sum_{i=1}^{k} \alpha_i C_i\| \tag{13}$$

also minimize

$$\|(C_0 + \sum_{i=1}^{k} \alpha_i C_i) V_t\|. \tag{14}$$

Proof: Suppose $\hat{\alpha}_1, ..., \hat{\alpha}_k$ do not minimize (14). Then there are coefficients $\tilde{\alpha}_1, ..., \tilde{\alpha}_k$ such that

$$\|(C_0 + \sum_{i=1}^{k} \tilde{\alpha}_i C_i) V_t\| < \|(C_0 + \sum_{i=1}^{k} \hat{\alpha}_i C_i) V_t\|.$$

We will show that for sufficiently small values of ϵ the coefficients $(1-\epsilon)\hat{\alpha}_i + \epsilon\tilde{\alpha}_i$ satisfy

$$\|C_0 + \sum_{i=1}^{k}((1-\epsilon)\hat{\alpha}_i + \epsilon\tilde{\alpha}_i) C_i\| < \|C_0 + \sum_{i=1}^{k} \hat{\alpha}_i C_i\|,$$

which contradicts the assumption that $\hat{\alpha}_1, ..., \hat{\alpha}_k$ minimize (13).

For any ϵ in $(0, 1)$ we have

$$\begin{aligned} \|(C_0 + \sum_{i=1}^{k}((1-\epsilon)\hat{\alpha}_i + \epsilon\tilde{\alpha}_i) C_i) V_t\| \;&\leq\; (1-\epsilon)\|(C_0 + \sum_{i=1}^{k} \hat{\alpha}_i C_i) V_t\| \\ +\; \epsilon\|(C_0 + \sum_{i=1}^{k} \tilde{\alpha}_i C_i) V_t\| \;&<\; \sigma_1 - \mathcal{O}(\epsilon). \end{aligned} \tag{15}$$

For sufficiently small ϵ we also have

$$\begin{aligned} \|(C_0 + \sum_{i=1}^{k}((1-\epsilon)\hat{\alpha}_i + \epsilon\tilde{\alpha}_i) C_i) V_{t+1:n}\| \;&\leq\; (1-\epsilon)\|(C_0 + \sum_{i=1}^{k} \hat{\alpha}_i C_i) V_{t+1:n}\| \\ +\; \epsilon\|(C_0 + \sum_{i=1}^{k} \tilde{\alpha}_i C_i) V_{t+1:n}\| &< \sigma_{t+1} + \frac{1}{2}(\sigma_1 - \sigma_{t+1}). \end{aligned} \tag{16}$$

Define the matrix $K \equiv (K_1, K_2)$ by

$$K_1 = U^T \left(C_0 + \sum_{i=1}^{k}((1-\epsilon)\hat{\alpha}_i + \epsilon\tilde{\alpha}_i) C_i \right) V_t =$$

$$(1-\epsilon)\sigma_1 \begin{pmatrix} I \\ 0 \end{pmatrix} + \epsilon U^T (C_0 + \sum_{i=1}^{k} \tilde{\alpha}_i C_i) V_t$$

$$K_2 = U^T \left(C_0 + \sum_{i=1}^{k}((1-\epsilon)\hat{\alpha}_i + \epsilon\tilde{\alpha}_i) C_i \right) V_{t+1:n} =$$

$$(1-\epsilon)\begin{pmatrix} 0 \\ \Sigma_{t+1:n} \end{pmatrix} + \epsilon U^T(C_0 + \sum_{i=1}^{k} \tilde{\alpha}_i C_i)V_{t+1:n}, \quad \Sigma_{t+1:n} \equiv \begin{pmatrix} \sigma_{t+1} & & \\ & \cdot & \\ & & \sigma_n \end{pmatrix}.$$

We would like to show that $\|K\| < \sigma_1$, or, equivalently, that the matrix

$$\sigma_1^2 I - K^T K = \begin{pmatrix} \sigma_1^2 I - K_1^T K_1 & -K_1^T K_2 \\ -K_2^T K_1 & \sigma_1^2 I - K_2^T K_2 \end{pmatrix} \tag{17}$$

is positive definite. From (15) and (16) it follows that the diagonal blocks are positive definite, so it suffices to show

$$(\sigma_1^2 I - K_1^T K_1) - K_1^T K_2 (\sigma_1^2 I - K_2^T K_2)^{-1} K_2^T K_1 > 0.$$

It is easy to check that

$$\|K_1^T K_2 (\sigma_1^2 I - K_2^T K_2)^{-1} K_2^T K_1\| = \mathcal{O}(\epsilon^2),$$

while the eigenvalues of $\sigma_1^2 I - K_1^T K_1$ are of order ϵ. For sufficiently small ϵ, then, the matrix (17) is positive definite, and this gives the desired contradiction. □

Using this lemma and Theorem 2 we can now prove equality (3) for general matrices, when $k = 1$.

Theorem 4. Let C_0 and C_1 be arbitrary real m by n matrices. Then

$$\min_{\alpha} \max_{\|w\|=1} \|(C_0 + \alpha C_1)w\| = \max_{\|w\|=1} \min_{\alpha} \|(C_0 + \alpha C_1)w\|. \tag{18}$$

Proof: We will use induction on the number of columns n. If $n = 1$, the result is clearly true. Assume it is true for matrices with $n-1$ columns, and now consider matrices with n columns. Suppose $\hat{\alpha}$ minimizes $\|C_0 + \alpha C_1\|$. Define $\hat{C} \equiv C_0 + \hat{\alpha} C_1$ and write the singular value decomposition of $\hat{C}$ as

$$\hat{C} = U \Sigma V^T,$$

where U, V, and Σ are as defined earlier. Assume that the singular values σ_i, $i = 1, ..., min\{m, n\}$ satisfy

$$\sigma_1 = \ldots = \sigma_t > \sigma_{t+1} \geq \ldots,$$

and let V_t be the n by t matrix consisting of the first t columns of V, while $V_{t+1:n}$ is the n by $n-t$ matrix consisting of columns $t+1$ through n of V.

We will consider two cases. In the first case, assume that $t < n$. According to the lemma, $\hat{\alpha}$ minimizes

$$\|(C_0 + \alpha C_1)V_t\|,$$

and so, by the induction hypothesis,

$$\|(C_0 + \hat{\alpha} C_1)V_t\| = \max_{\|w\|=1} \min_{\alpha} \|(C_0 + \alpha C_1)V_t w\|.$$

If $\hat{w}$ is the t-vector for which this maximum is attained and if we define the n-vector $\tilde{w}$ to be $V_t \hat{w}$, then we have the desired result

$$\|C_0 + \hat{\alpha} C_1\| = \min_{\alpha} \|(C_0 + \alpha C_1)\tilde{w}\|.$$

Now suppose $t = n$. We can assume without loss of generality that $\sigma_1 = 1$, and then we have

$$\min_{\alpha} \|C_0 + \alpha C_1\| = \min_{\beta} \|U^T(C_0 + (\beta + \hat{\alpha})C_1)V\|$$

$$= \min_{\beta} \| \begin{pmatrix} I \\ 0 \end{pmatrix} + \beta \begin{pmatrix} U_n^T C_1 V \\ U_{n+1:m}^T C_1 V \end{pmatrix} \| = 1,$$

where U_n consists of the first n columns of U and $U_{n+1:m}$ consists of columns $n+1$ through m of U. The same coefficient β that minimizes the norm of the entire matrix also minimizes the norm of the top n by n block of this matrix and so we have

$$\min_{\beta} \|I + \beta U_n^T C_1 V\| = 1.$$

¿From Theorem 2 it now follows that

$$\max_{\|w\|=1} \min_{\beta} \|(I + \beta U_n^T C_1 V)w\| = 1$$

and hence that (18) holds. □

3 Further Discussion

Extensive numerical testing of the inequalities in (1) and (2) for a variety of matrices suggests that they are, indeed, equalities. Theorem 4 proves this is so for $k = 1$, but we have been unable to prove (or disprove) this result for $k > 1$. The example (12) shows that the proof must rely on special properties of polynomials, since the result is not true for arbitrary non-commuting matrices, even if they are normal.

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