Fastest scrambler

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Dedication

To those who held me up over the years

Abstract

Scrambling is a unitary Hamiltonian evolution that evolves known initial states such that at time t^*, called the scrambling time, it is not possible to distinguish different initial states without measuring a large fraction of the degrees of freedom of the system. The variable that describes how far a system is from an idealistic scrambled system is called the order of scrambling. For a quantum system, we can compare the scrambling time of well-normalized Hamiltonians to the same order of scrambling and find the fastest scrambler. For normalization of Hamiltonian, we assume the energy of the system grows extensively with the number of qubits [1]. Because the fastest scrambler is different for various orders of scrambling, we choose the smallest possible order of scrambling in what follows. The questions that we answered are finding the Hamiltonian of the fastest scrambler and its scrambling time, especially for large numbers of qubits. Because scrambling and entanglement have a close correspondence, we were able to find a diagram that produces the Hamiltonian which evolves any computational basis to a highly symmetric entangled state. Such states are highly useful in quantum information processing, for example in quantum error correction codes.

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Introduction

1.1 Scrambling and entanglement

Recently a connection between quantum information theory and string theory has been recognized [2] [3] [4]. The link of this connection is entanglement, which happens between at least two subsets of degrees of freedom, such that those subsets interact in a way that quantum state of one subset can not be described independently of the other subset(s), instead, a quantum state should describe the whole system. Scrambling, which is roughly thermalisation in quantum systems, has been shown to be equivalent to the unitary Hamiltonian evolution of known initial states of the system to highly entangled states [1]; so it is important to understand entanglement to investigate scrambling. Although in chapter 2 we will explain entanglement in details, to get clear picture lets follow a famous hypothetical experiment.

1.1.1 Entanglemet

Assume a quantum system of two spin half fermions, such that their spin is the only degree of freedom. The outcome of the measurement of spin in any arbitrary direction, for each fermion, can get two possible values $+\frac{1}{2}$, spin up, and $-\frac{1}{2}$, spin down. Imagine the state of quantum system is $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|+\frac{1}{2}, -\frac{1}{2}\rangle + |-\frac{1}{2}, +\frac{1}{2}\rangle\right)$, this state is called Bell state and is a maximally entangled state for a system of two spin half fermions, maximally entangled state means measure of entanglement for any bipartition of the system gets maximum possible value. In this particular case, there is only one way to

bipartite the system, we can label one fermion with A, and the other fermion with B. For this state if we measure spin of both fermions in z direction there is 50% probability that A and B have spin $+\frac{1}{2}$ and $-\frac{1}{2}$ in the z direction respectively, and 50% of probability that A and B have spin $-\frac{1}{2}$ and $+\frac{1}{2}$ in the z direction respectively. According to quantum mechanics laws, for this Bell state as soon as one measures the spin of one fermion in z direction with measurement Z_1, the state will collapse, and measurement of the other fermion in the same direction, Z_2, results in exactly the opposite value of measurement Z_1. On the other hand, denote eigenstates of the measurement operator in x direction as $|up\rangle$ and $|down\rangle$, state $|\psi\rangle$ in the basis of measurement operator in x direction is $|\psi\rangle = \frac{1}{\sqrt{(2)}}(|up\ up\rangle - |down\ down\rangle)$. For this state if we measure the spin of the first fermion in x direction with measurement X_1, it collapses the original state and measurement of the second fermion in the same direction, X_2, results in the same value of measurement X_1 and vice versa. Now set up the experiment to perform measurement X_1 on A and measurement Z_2 on B. Measurement X_1 completely specify the spin of A (so spin of B) in x direction, measurement Z_2 specify the spin of B (so the spin of A) in z direction. If performing measurement X_1 on A does not effect on B we can find spin of each fermion in both z and x direction simultaneously through the setup mentioned above, which is the violation of uncertainty principle. Uncertainty principle tells us it is impossible to measure the spin of a fermion in two orthogonal directions simultaneously. There is no way to prevent violation of uncertainty principle, but there exist an interaction between A and B such that measurement of A effects on B. This case will hold even if two fermions are space like with each other, space like fermions means the local observer in the frame of one fermion can not interact with the local observer in the frame of the other fermion. This case violates causality. Causality means an effect can not occur if the cause is space like with the event. Einstein, Podolsky, and Rosen came up with such hypothetical experiment in 1935 to show quantum mechanics violates causality, so a paradox called EPR paradox raised. They claimed that quantum mechanics is correct but it is not a complete theory to provide a complete description of reality, and their solution to restore causality to quantum mechanics was a theory called hidden variables. They stated there must exist unknown (hidden) variables, to supplement quantum mechanics and restore causality to quantum mechanics. In 1964 John Bell formulated their idea of hidden variable mathematically and showed to be

incompatible with the statistical prediction of quantum mechanics [5]. Nowadays there is a wide consent that quantum mechanics is a correct and complete theory, entanglement simply does not correspond to classical intuition.

Research on entanglement got boosted science early 1980 when quantum computation proposed by Feynman to simulate quantum systems. Entanglement found to be a post in quantum computation, in most of the quantum computation tasks like parallel transport, and error correction codes one can locate the trace of entanglement. Many people investigated its characteristics, categorization, and provided different measures for different categorize to quantify entanglement. For the purpose of scrambling we will deal with a classification of entanglement called, the bipartite entanglement of pure quantum state, and among different measures of entanglement, we will use entanglement entropy. Although we will review bipartite entanglement of pure quantum state and entanglement entropy in chapter 2, we will provide a brief introduction to these two key subjects in next few paragraphs and explain their connection with scrambling.

1.1.2 Bipartite entanglement of pure quantum state

To understand bipartite entanglement of pure quantum state, imagine a closed quantum system, Q. A closed quantum system is a system that does not interact with the environment. A unitary Hamiltonian evolution can describe the evolution of a closed quantum system. Assume physical quantities of Q are spin $\frac{1}{2}$ fermions, and spin is the only degree of freedom for each fermion. In quantum information theory, such fermions are known as qubits because a qubit is a degree of freedom of a quantum system such that its measurement can get two possible value. Spin half fermions are not the only qubits, qubit refers to any degree of freedom like a special setup for polarization of light with two possible outcomes of measurement, in this case, horizontal and vertical. To describe this degree of freedom, people use the notation 0 and 1 instead of spin up, $+\frac{1}{2}$, and spin down, $-\frac{1}{2}$, (horizontal and vertical polarization) respectively, now on we will use quantum information notation. Let's say the quantum system Q has n qubits, which means it has n degrees of freedom. Because each degree of freedom can only get two values, 0 or 1, there exist 2^n different possible outcome for measurement of all degrees of freedom. Each possible result of measurements of all degrees of freedom in quantum information notation is a computational basis, for example for $n = 2$ there are $2^n = 4$

different computational basis $\{\hat{e_1} = |00\rangle, \hat{e_2} = |01\rangle, \hat{e_3} = |10\rangle, \hat{e_4} = |11\rangle\}$. Assume S is in a pure state $|\psi\rangle$, a pure state is a normalized linear summation of computational basis.

$$|\psi\rangle = \sum_{i=1}^{2^n} \alpha_i \hat{e_i} \quad , \quad \sum_{i=1}^{2^n} |\alpha_i|^2 = 1$$

Here α_is are complex numbers. For the case of quantum system Q that has n qubits in pure state $|\psi\rangle$, bi-partitioned into part A having m qubits and part B having $n-m$ qubits, Bipartite entanglement of the pure state $|\psi\rangle$ is the entanglement between A and its complement B while the whole system is in a pure state. Now on, whenever we talk about entanglement, we mean bipartite entanglement of a pure state. In scrambling, we will deal with closed quantum systems, initially in a computational basis which is a pure state. Because it is a closed quantum system, unitary Hamiltonian evolution warranty the whole system remains in a pure state. So at each time t we look at bipartite entanglement of a pure state. It worth to mention the whole system is not entangled with any other systems, because it is in a pure state and it will remain in a pure state during evolution, but its subsystems can entangle with their complement, we are looking at entanglement between subsystems and their complements.

1.1.3 Entanglement entropy and mutual information

Different measures such as the entanglement entropy [6], the geometric measure of entanglement [7], and the concurrence monotones [8] have been introduced to quantify entanglement of different categorize, for bipartite entanglement of a pure state, which we are interested in, entanglement entropy is a well understood and popular measure of entanglement. To walk through entanglement entropy, we need to have a glance at reduced density matrice. Density matrice is defined as summation over outer product of pure states multiplied by the probability of being in those states, $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, where p_i is the probability of being in a pure state ψ_i, $\sum_i p_i = 1$. Density matrice of a pure state $\psi_i = \psi$, with probability $p_i = 1$, is $\rho = |\psi\rangle\langle\psi|$. Reduced density matrice of partition A, $\rho^A = Tr_B(\rho)$, where trace over B of ρ , $Tr_B(\rho)$, is a summation of all possible collapsed density matrices when B is measured, $Tr_B(\rho) = \sum_k {}_B\langle k|\rho|k\rangle_B$, where $|k\rangle_B$s are possible results of measurement of subsystem B. Entanglement entropy of A, S^A, is defined as Von Neumann entropy of reduced density matrice of A, $S^A = -tr\rho^A log\rho^A$ [6], it can

quantify bipartite entanglement of a pure state between A and its complement B. There are two important properties of entanglement entropy for bipartite entanglement of a pure state which is useful in scrambling. First there is no difference between tracing over A or B to calculate entanglement entropy of a bipartite entanglement of a pure state, $S^A = S^B$; second, the maximum possible value for entanglement entropy is the dimension of the smaller subsystem. If $|A| = m$, $|B| = n - m$, and $m < n - m$ then maximum possible entanglement entropy between A and B is m. We will review the exact and detailed definition of entanglement entropy and reduced density matrices in chapter 2.

1.1.4 Relation of scrambling and entanglement entropy

Scrambling of a quantum system M by definition is a unitary Hamiltonian evolution that results in the indistinguishability of different initial states, where initial states are the computational basis, by measuring an insufficient number of degrees of freedom. First time t^* after which we are not able to distinguish different initial states is called scrambling time. In other words "A closed system never forgets its initial state, but over time it might become impossible to distinguish different initial states without measuring a large fraction of all the system degrees of freedom. The minimum time required for the information about the initial state gets lost called scrambling time" [1]. As mentioned it has been shown scrambling is equivalent to the evolution of the computational basis to highly entangled states [1]. Although we will review their work in 2 their result is as follow: If for all bi-partitions average of entanglement entropy over all initial computational basis gets within a small distance of maximum possible entanglement entropy, m, the system gets scrambled.

$$m - Avg(S^A_\psi) < \delta$$

where average is overall computational basis ψ and δ is a small number, "A standard continuity result implies that δ can be chosen to be $3\epsilon n + f(\epsilon)$ where $f(\epsilon)$ goes to zero with ϵ and is independent of n" [1] [9]. The aforementioned small distance δ is called order of scrambling.

1.2 Fastest scramblers and black holes

Our focus is on fastest scramblers. To compare the scrambling time of different Hamiltonians, H, and talk about fastest scramblers we should have, first an absolute scale of time, and second a fixed order of scrambling. To have a fixed scale of time we need to normalize Hamiltonian because Hamiltonian has the dimensionality of energy, which is the inverse of time. In other words, due to unitary Hamiltonian evolution, we can always absorb time into Hamiltonian, so we will not have an absolute scale for time unless we normalize Hamiltonian. For the Hamiltonian normalization, it is widely accepted that energy of Hamiltonian should grow extensively with the number of degrees of freedom [1], and we will use this consensus:

$$\frac{tr\left(H^2\right)}{2^n} = n^2$$

Different order of scrambling will results in different scrambling time for the same Hamiltonian, as mentioned δ is a small number. A small δ means a high average of entanglement entropy over different initial states which means the system is closer to the idealistic scrambled system. For the purpose of scrambling we need the average of entanglement entropy for any bi-partition stay within 1 qubit of information, this means $\delta \leq 1$ is a good choice.

Fastest scrambler is important in the study of black holes. Bekenstein provided a thermodynamic approach to black hole physics using similarities between the behavior of black hole area and entropy [2]. Using quantum information, he was successful in describing the concept of black hole entropy as the measure of shared information between black hole degrees of freedom. The measure of shared information between two subsystems A and B, also known as mutual information between A and B, quantifies how much we can learn about A if we measure B. It is defined as $I(A:B) \equiv S^A + S^B - S^{A,B}$, for the case of bipartite entanglement of a pure state, $S^{A,B} = 0$, and $S^A = S^B$ so $I(A:B) = 2S^A$. So maximum possible shared information is $Max(I(A:B)) = 2|A|$. To have a better understanding of measure of shared information recall the hypothetical experiment that we described at the beginning of the introduction. When two spin half fermions are completely entangled, and someone measures the spin of one fermion in the z direction, Z_1, the state collapses, and measurement of the spin of the other fermion in

the same direction, Z_2 result in the different value than Z_1. For that state, $I(A:B) = 2$, which is the maximum shared information between two qubits in a pure state.

If a black hole degrees of freedom are highly entangled then Hawking radiation is highly entangled with the black hole and shared information between Hawking radiation and the black hole is maximum. So an observer outside the horizon who can measure all emitted Hawking radiation recover n qubits of information that fall into the black hole as soon as a few more than n qubits of information emits in Hawking radiation [10]. It is expected for a black hole that radiated at least half of initial entropy to be maximally entangled; then Hawking radiation will reveal additional information that falls into the black hole very rapidly. Although it is known that time for a black hole to emit half of its entropy is long, a black hole can evolve under unitary Hamiltonian evolution and reach a highly entangled state through dynamics in a much more faster time [10]. Sekino and Suskind conjectured that black holes are the fastest scramblers in nature, and its scrambling time is logarithmic in the number of degrees of freedom [11]. Many attempts have been established to evaluate this conjecture using both numerical and analytical methods. This conjecture is important because if a quantum system can scramble faster than logarithmic time, the complementary principle of black holes will get questioned. The complementary principle of blackholes roughly states no observer inside the horizon of black hole can send information out of horizon and physics for observers outside of horizon is not affected by evolution inside horizon. Among these attempts the Brownian quantum circuit and the antiferromagnetic Ising model on a sparse random graph, both scramble in logarithmic time; but both fail to truly be the fastest scramblers because the Brownian quantum circuit uses time-dependent Hamiltonian and the antiferromagnetic Ising model on a sparse random graph fails to fully scramble [1].

1.2.1 Our approach to fastest scrambler

As mentioned fastest scrambling to the order of $\delta \simeq 1$ is not only important in quantum information processing but also in black hole physics. In this thesis, we answered to the question of finding fastest scrambler Hamiltonian for small systems and their scrambling time. Also, we introduced Hamiltonians that scramble large quantum systems in logarithmic time. Lashkari et.al [1] used Lieb-Robinson techniques to prove a logarithmic lower band on the scrambling time of systems with finite norm terms in

Hamiltonian, so the Hamiltonians that we found are the fastest scramblers for large systems. Meanwhile, we introduced a diagram that produces Hamiltonians which evolve a computational basis to maximally symmetric entangled states, maximally symmetric entangled states are useful states in quantum information processing for example in the perfect quantum error correction code [12].

Fastest scramblers for the small systems of n qubits are the Hamiltonian produced by the aforementioned diagram, which unitary evolves any initial state to a maximally symmetric entangled state. We found this evolution will scramble the system to the order of $\delta \propto \frac{n}{4}$. Figure 3.3 is an example of the diagram for $n = 15$ qubits. In this diagram vertical direction is not the time step but it is for clarification of components in Hamiltonian. Each row is a part of the Hamiltonian, and the entire Hamiltonian should act on the system through a unitary Hamiltonian evolution. Basic rules of the diagram is the fallowing

- start with $H = 0$ and draw n points represents n qubits.
- Divide qubits into pairs and have two body interactions between two qubits of each pair added to Hamiltonian H, if one qubit remained unpaired then combine it with one of the pairs and have a three body interaction added.
- Divide previous pairs of qubits into different sets of two pairs (quads). Again have a two-body interaction between one qubit of each pair with one qubit of the other pair in the same quad added to Hamiltonian H, if one pair does not fit in a quad then combine it with one of the quads and have a three-body interaction.
- Keep pairing previous divisions, and do the same until everything gets connected with two or three body interactions.

each l body interaction is n tensor product of $n-l$ Identity, I, and l pauli matrice in x direction, σ_x. For example having $n = 4$, a two body interaction between first qubit and second qubit is $\sigma_x \otimes \sigma_x \otimes I \otimes I$, a three body interaction between first, third, and fourth qubit is $\sigma_x \otimes I \otimes \sigma_x \otimes \sigma_x$.

Although having $\delta \propto \frac{n}{4}$ is good for scrambling small systems and some quantum processing operations like error correction codes, it is not good enough for scrambling large systems, so we utilized probability theory and what we learned from small systems

to tackle large systems. Don Page showed for large systems of n qubits the average entanglement entropy between arbitrary subsystem A with m qubits and its complement B, where $m \leq \frac{n}{2}$, is $m - 2^{2m-n-1}$ [13]. This average is over Haar-random states, which is a uniformly random distribution of all pure states. Thus for a Haar-random state, average entanglement entropy of all bi-partitions will be within $\frac{1}{2}$ distance of the maximum possible entanglement entropy. Assume we have a random Hamiltonian, if this random Hamiltonian evolves any initial computational basis to a Haar-random state at time t^*, then entanglement entropy of all bipartitions for any final state will be within $\frac{1}{2}$ distance of the maximum possible entanglement entropy. It means the system is scrambled to the order of $\frac{1}{2}$, this small order of scrambling is what we look for.

As mentioned beside probability theory, the lesson from small systems provided lights toward large systems, to explain this lesson we will discuss two important key subjects in next paragraph, one is initial states, and the other is Pauli expansion of Hamiltonian.

Initial states are computational bases. In a computational basis each qubit is either in state $|0\rangle$ or $|1\rangle$. State of the whole system is the n tensor product of the state of those n qubits, for example in a system of n spin half fermions, the computational basis is the possible result of spin measurement of all fermions in z direction. Because in quantum mechanics result of a measurement is an eigenstate of both the operator of the measured quantity, and the Identity operator, action of any arbitrary n tensor product of l spin operator in z direction ,σ_z, and $n - l$ identity operator I on computational basis result in the basis up to a coefficient. but action of arbitrary n tensor product of l (σ_z or I) and $n - l$ Pauli matrices in x or y direction, (σ_x or σ_y), will result in spin flip of the $n - l$ qubits that (σ_x or σ_y) has been acted on. These arbitrary n tensor products that we described is called Pauli basis Ξ_i, because there are 4 Pauli matrices, $\sigma_0 = I, \sigma_1 = \sigma_x, \sigma_2 = \sigma_y, \sigma_3 = \sigma_z$, we can generate 4^n Pauli basis, so $i \in \{1, 2, 3, ..., 4^n\}$. To understand Pauli expansion of Hamiltonian, remember Hamiltonian is a Hermitian operator and a Hermitian operator acting on n qubit system can be described by 4^n real numbers. The number of Pauli basis is the same as the number of real numbers that we need to describe Hamiltonian so we can write each Hamiltonian as a summation over Pauli basis weighted by real numbers α_i, this summation is called Pauli expansion of

Hamiltonian.

$$H = \sum_{j=1}^{4^n} \alpha_j \Xi_j$$

We learned from small systems that to have the fastest scrambler, Pauli basis in Pauli expansion of fastest scrambler is a n tensor product of $n-m$ Identity I, and m spin flip operator σ_x. There are 2^n different Pauli basis that can be formed by I and σ_x, so 2^n real random coefficients of Pauli basis will describe a random Hamiltonian of the anzats. Scrambling time is proportional to number of non-zero Pauli basis in Pauli expansion of Hamiltonian, f_n, to have the fastest scrambler not only the random distribution of coefficients should Haar randomize initial states but also f_n should be minimized to assure minimum scrambling time.

The relationship between α_i and state of the system at time t can be established by utilizing Taylor series expansion of unitary operator and Pauli expansion of Hamiltonian. For anzats mentioned above different initial states will only rearrange components of final states, it means if a Hamiltonian produces a Haar-random state initiating with only one of the computational basis, any computational basis also will be Haar randomized. At chapter 4 we provided eligible probability distributions of α_i that match the first moment of the state of the system at time t^* and the first moment of Haar-random distribution, adjusting f_n matches other moments to its equivalent Haar-random distribution. To find scrambling time as a function of the probability distribution of α, n, and f_n we used normalization of Hamiltonian. We found scrambling time cubed is proportional to f_n, probability distribution of α, and $\frac{1}{n^2}$. We showed numerically for a simple probability distribution, and $f_n = n^2 log_2^2(n)$ we will get a Haar-random state, so indeed scrambling time for this Hamiltonian is logarithmic in the number of qubits.

Chapter 2

Preliminaries

2.1 State of qubits

In quantum information theory, qubits are the quantum analog of classical bits. They are degrees of freedom such that the outcome of the measurement of each degree of freedom can get two possible values, which is called two state degree of freedom. For example, a spin half fermion where its spin is the only degree of freedom is a qubit because the outcome of measurement of the spin in an arbitrary direction is either spin up or spin down. Assume an isolated qubit, for example, a fermion in an infinite potential well that does not interact with the environment. State of this qubit can be described by $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, where α and β are complex numbers, $|0\rangle$ and $|1\rangle$ are two possible outcome of the measurement of degree of freedom. For this fermion, $|0\rangle$ means the fermion is spin up, and $|1\rangle$ means it is spin down. $|\alpha|^2$ is the probability of being at state $|0\rangle$ and $|\beta|^2$ is the probability of being at state $|1\rangle$, summation of probabilities should be one so we have the constraint $|\alpha|^2 + |\beta|^2 = 1$. We describe $|\psi\rangle$ with two complex numbers α and β, totally four real numbers can describe $|\psi\rangle$, but there is one constraint such that $|\alpha|^2 + |\beta|^2 = 1$, so $4 - 1 = 3$ independent real numbers can describe the state of one qubit.

Imagine a quantum system Q of n qubits that does not interact with environment. Q is called a closed quantum system and evolution of a closed quantum system is described by unitary Hamiltonian evolution. Because each qubit is a two state degree of freedom, measurement of n qubits has 2^n possible out come, a possible outcome is

a computational basis for Q. For example measurement of all degrees of freedom in a system of 2 qubits can have four possible outcome $\{|0\rangle_1 \otimes |0\rangle_2, |0\rangle_1 \otimes |1\rangle_2, |1\rangle_1 \otimes |0\rangle_2, |1\rangle_1 \otimes |1\rangle_2\}$. It is consensus for simplicity to discard tensor product and abbreviate them to $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, these abbreviations are binary form of non negative integers so one can also use decimal representation of these binary state $\{|0\rangle = |00\rangle, |1\rangle = |01\rangle, |2\rangle = |10\rangle, |3\rangle = |11\rangle\}$. State of closed quantum system Q of n qubits is a linear summation of 2^n computational basis multiplied by complex numbers such that square of complex numbers, which is the probability of being in corresponding computational basis, add up to one. These states are called pure states.

$$|\Psi\rangle = \sum_{i=0}^{2^n-1} \alpha_i |i\rangle \quad , \quad \sum_{i=0}^{2^n-1} |\alpha_i|^2 = 1$$

2^n complex numbers, equivalent to 2^{n+1} real numbers describe state of n qubits in a pure state, also there is a constrain such that summation of square of all those numbers should add up to 1, so $2^{n+1}-1$ independent real numbers describe a pure state. Because $\sum_{i=0}^{2^n-1} |\alpha_i|^2 = 1$, we can think about each pure state as a vector pointing on the unit sphere $S^{2^{n+1}-1}$.

Although these n qubits, Q, do not interact with the environment, they interact with each other due to unitary Hamiltonian evolution. Any partition of Q, called A such that $A \notin \{\varnothing, Q\}$, interacts with its complement B. An open quantum system is a partition of a larger closed system that interacts with its complement, so A is an open quantum system [14]. The action of dividing a closed quantum system Q to partition A, which is an open quantum system, and its complement B is called bi-partitioning. In this case of bi-partitioning, there is no preference between partition A and B, so There are $\frac{2^n-2}{2} = 2^{n-1} - 1$ different ways to bipartite Q. To distinguish different bi-partitions employ label $l = \{1, 2, 3, ..., 2^{n-1} - 1\}$, such that A_l and its complement B_l determine which bi-partition we are talking about, We will assume number of qubits in A_l is $|A_l| = m$, so $|B_l| = n - m$. Although pure states are good enough to represent closed quantum systems, they are not able to represent open quantum systems like A_l. To generalize the description to open quantum systems we should use density matrix ρ. The density matrix is a summation over the outer product of pure states $|\psi_j\rangle$ multiplied

by the probability of being in that pure state p_j.

$$\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j|$$

If the system is in a pure state $|\psi\rangle$, it means the probability of being in that state is one, so the density matrix of a pure state is $\rho = 1 * |\psi\rangle\langle\psi|$. If the system is not in a pure state with probability one, it is in a mixed state. A density matrix should describe aforementioned open quantum system A_l. Knowing ρ, the state of the system, we can find the density matrix that describes partition A_l, ρ^{A_l}, this density matrix is called the reduced density matrix of A_l. To obtain the reduce density matrix of A_l we need to trace over its complement B_l. Tracing over a subsystem B_l is the summation over all possible reminder of ρ if all qubits of B_l get measured. Measurement of each qubit in B_l can have two possible values, 0 or 1, so there are $2^{|B_l|}$ possible outcome for measurement of all qubits in B_l, where $|B_l|$ is the number of qubits in subsystem B_l, in other words there are $2^{|B_l|}$ computational basis for B_l. Let's show k^{th} computational basis of B_l with $|k\rangle_{B_l}$, then ${}_{B_l}\langle k|\rho|k\rangle_{B_l}$ gives us the reminder of ρ if measurement of B_l results in $|k\rangle_{B_l}$. There are $2^{|B_l|}$ possible $|k\rangle_{B_l}$, a summation over all possible reminders will provide us the reduced density matrix of A_l.

$$\rho^{A_l} = Tr_{B_l}(\rho) = \sum_{k=0}^{2^{|B_l|-1}} {}_{B_l}\langle k|\rho|k\rangle_{B_l}$$

2.1.1 Evolution of a closed system of qubits

Assume Q is a closed system and initial state of system is a computational basis.

$$|\psi\rangle \in \{|00...0\rangle, |00...1\rangle, ..., |11...1\rangle\}$$

Unitary Hamiltonian evolution of a system in a pure state will keep the system in a pure state, starting with a normalized pure state at time $t = 0$ will end up to a normalized pure state at time t. To see this point lets have Hamiltonian H evolves Q through unitary operator $U(t) = e^{-iHt}$

$$|\psi(t)\rangle = U(t)|\psi\rangle = e^{-iHt}|\psi\rangle \tag{2.1}$$

so

$$\langle\psi(t)|\psi(t)\rangle = \langle\psi|e^{+iHt}e^{-iHt}|\psi\rangle = \langle\psi|\psi\rangle = 1$$

Initial state $|\psi\rangle$ is computational basis, which is a normalized pure state, so the system Q will always remain in a normalized pure state. Because of evolution although Q remains in a normalized pure state, subsystem A_l will get mixed with its complement, so we should describe A_l with reduced density matrix of A_l. Here we want to find reduced density matrix of A_l at time t, for this purpose expand $|\psi(t)\rangle$ in computational basis, having $|A_l| = m$

$$|\psi(t)\rangle = \sum_{i=1}^{2^n} \beta_i(t)|i\rangle = \sum_{i=1}^{2^m}\sum_{j=1}^{2^{n-m}} \beta_{i,j}(t)|i\rangle_{A_l}|j\rangle_{B_l}$$

reduced density matrix of system Q at time t starting with initial state $|\psi\rangle$ is simply outer product of $|\psi(t)\rangle$ multiplied by probability one, because we want to investigate scrambling we need to keep track of initial state, so we use initial state ψ as subscript of reduced density matrix.

$$\rho_\psi(t) = |\psi(t)\rangle\langle\psi(t)|$$

$$\rho_\psi(t) = \sum_{i,i'=1}^{2^m}\sum_{j,j'=1}^{2^{n-m}} \beta_{i,j}(t)\beta^*_{i',j'}(t)\left(|i\rangle_{A_l}|j\rangle_{B_l}\right)\left({}_{A_l}\langle i'|_{B_l}\langle j'|\right)$$

Last step is to find reduced density matrice of subsystem A_l, $\rho_\psi^{A_l}(t)$, by tracing over subsystem B_l.

$$\rho_\psi^{A_l}(t) = Tr_{B_l}(\rho_\psi(t)) = \sum_{k=1}^{n-m_l} {}_{B_l}\langle k|\rho_\psi(t)|k\rangle_{B_l}$$

$$\rho_\psi^{A_l}(t) = \sum_{i,i'=1}^{2^m}\sum_{j,j',k=1}^{2^{n-m}} \beta_{i,j}(t)\beta^*_{i',j'}(t){}_{B_l}\langle k|\left(|i\rangle_{A_l}|j\rangle_{B_l}{}_{A_l}\langle i'|_{B_l}\langle j'|\right)|k\rangle_{B_l}$$

Because $|k\rangle_{B_l}$ is orthonormal to $|j\rangle_{B_l}$, ${}_{B_l}\langle k|j\rangle_{B_l} = \delta^k_j$ we can simplify above equation

$$\rho_\psi^{A_l}(t) = \sum_{i,i'=1}^{2^m}\sum_{j,j',k=1}^{2^{n-m_l}} \beta_{i,j}(t)\beta^*_{i',j'}(t)|i\rangle_{A_l}\delta^j_k\ {}_{A_l}\langle i'|\delta^{j'}_k$$

$$\rho_\psi^{A_l}(t) = \sum_{i,i'=1}^{2^{m_l}}\sum_{k=1}^{2^{n-m_l}} \beta_{i,k}(t)\beta^*_{i',k}(t)|i\rangle_{A_l}\ {}_{A_l}\langle i'|$$

We will use the reduce density matrix in next section to find entanglement entropy.

2.2 Relation between entanglement entropy and scrambling

As mentioned in introduction, scrambling is highly related to entanglement, to have a good understanding of scrambling it is necessary to investigate bipartite entanglement. During unitary Hamiltonian evolution of closed system Q, some or all partitions A_l can get entangled with their complement B_l. To have a quantitative description of entanglement in a bi-partition we chose entanglement entropy to be the measure of entanglement. Entanglement entropy between subsystem A_l and its complement B_l is defined as Von Numann entropy of reduced density matrix of A_l.

$$S^{A_l} = -tr\ \rho^{A_l} log_2\left(\rho^{A_l}\right)$$

Considering evolution of system from initial state ψ, the entanglement entropy between subsystem A_l and its complement B_l at time t is

$$S_\psi^{A_l}(t) = -tr\ \rho_\psi^{A_l}(t) log_2\left(\rho_\psi^{A_l}(t)\right)$$

Because initial state is a computational basis and a computational basis is a n tensor product, at time $t = 0$ the reduced density matrix of any arbitrary bi-partition A_l, $\rho_\psi^{A_l}(0)$, is a pure state, and entanglement entropy of a pure state is zero. So at time $t = 0$ entanglement entropy between any arbitrary subsystem A_l and its complement B_l is zero, no subsystem is entangled. On the other hand maximum entanglement entropy between A_l and its complement B_l happens for a completely mixed reduced density matrix, $\rho_\psi^{A_l} = \frac{1}{2^{m_l}} I$. This maximum entanglement entropy is equal to the size of smaller subsystem, without losing generality assume smaller subsystem is A_l, so $m \leq n - m = m \leq \frac{n}{2}$. It is possible at some time $t = t_0$ for some bi-partitions A_l, $\rho_\psi^{A_l}(t_0)$ be completely mixed then $S_\psi^{A_l} = m$. Disscussion of different categorize, aspects and measures of entanglement takes a life time, but provided information on entanglement is sufficient for the purpose of scrambling, in the fallowing paragraphs we will review the relation between scrambling and entanglement that has been explained in [1].

Scrambling is a unitary Hamiltonian evolution of closed quantum system initially in a computational basis such that at time t^* which is called scrambling time, it is

not possible to distinguish different initial states without measuring large fraction of degrees of freedom. At scrambling time t^* the system Q is in the pure state $\psi(t^*)$ with corresponding density matrix $\rho_\psi(t^*)$. Assume an auxiliary Hilbert space $\Re$, the same size as Hilbert space of the system Q, lets name Q's Hilbert space $\aleph$, and consider a density matrix for combined $\Re\aleph$ such that each orthonormal basis of $\Re$ records in initial state in $\aleph$ at time t^*.

$$\rho^{\Re\aleph}(t^*) = \frac{1}{dim(\aleph)} \sum_{\psi} |\psi\rangle_{\Re\Re}\langle\psi| \otimes \rho_\psi(t^*)$$

Because subsystem A_l is scrambled, for any arbitrary ψ and ϕ, $\rho_\psi^{A_l}(t^*) = tr_{B_l}\left(\rho_\psi(t^*)\right)$ should be indistinguishable from $\rho_\phi^{A_l}(t^*) = tr_{B_l}\left(\rho_\phi(t^*)\right)$, recall that subscript of density matrix was introduced to keep track of initial states. As mentioned $\Re$ recorded in initial state of Q, so for not being able to distinguish $\rho_\phi^{A_l}(t^*)$ from $\rho_\psi^{A_l}(t^*)$, there should be no significant correlation between $\Re$ and A_l, quantitatively speaking, the mutual information between $\Re$ and A_l at time t^* should be less than small number δ, $I(\Re : A_l) < \delta$. Using the relation between mutual information and entanglement entropy

$$I(\Re : A_l) = S^{A_l}(t^*) + S^{\Re}(t^*) - S^{\Re A_l}(t^*) < \delta$$

Because $\rho_\psi(t^*)$ forms an orthonormal basis for $\aleph$, and we constructed $\rho^{\Re\aleph}(t^*)$ to be an equal mixture of all othonormal basis of $\aleph$, essentially $\rho^{\aleph} = tr_{\Re}\left(\rho^{\Re\aleph}(t^*)\right)$ is maximally mixed on $\aleph$. Any subset of $\aleph$ such as A_l, is also maximally mixed, therefor $S^{A_l}(t^*) = m$. substituting this in above equation

$$m - \left(S^{\Re A_l}(t^*) - S^{\Re}(t^*)\right) < \delta \tag{2.2}$$

In the above equation, $S^{\Re A_l}(t^*) - S^{\Re}(t^*)$ is known as conditional entropy, it is the uncertainty remaining in A_l once $\Re$ is known. In this case it is the average of entanglement entropy over different initial states at time t^*.

$$S^{\Re A_l}(t^*) - S^{\Re}(t^*) = \frac{1}{dim(\aleph)} \sum_{\psi} S_\psi^{A_l}(t^*)$$

substituting this in the 2.2 will give us a condition known as scrambling condition:

$$m - \frac{1}{dim(\aleph)} \sum_{\psi} S_\psi^{A_l}(t^*) < \delta \tag{2.3}$$

We assumed A_l is an arbitrary bi-partition with size $m \leq \frac{n}{2}$, so above condition should be correct at scrambling time t^* for any arbitrary A_l, where $|A_l| = m \leq \frac{n}{2}$.

Here δ is called the order of scrambling, it has been mentioned in [1]"A standard continuity result implies that δ can be chosen to be $3\epsilon dim(\aleph) + f(\epsilon)$, where $f(\epsilon)$ goes to zero with ϵ and is independent of n [9]". Scrambling to the order of small δ can happen when the average entanglement entropy of bipartition A_l over all possible initial states gets close to maximum possible entanglement entropy, m. The small order of scrambling is what we are interested in, for large systems we will choose $\delta \simeq 1$, for $\delta \simeq 1$ scrambling is equivalent to a unitary Hamiltonian evolution of the system initially at computational basis to highly entangled state at time $t = t^*$.

2.3 Basics of fastest scramblers

2.3.1 Assumptions

Scrambler is referred to the Hamiltonian that scrambles the system. We want to compare the scrambling time of different Hamiltonians acting on a system of n qubits and determine the fastest scrambler. Comparison of scrambling time requires defined scale of time, the scale of time is proportional to the inverse of energy, so normalization of Hamiltonian defines the scale of time. For a better understanding assume a unitary Hamiltonian evolution with unitary operator$U = exp(-iHt)$, we can rewrite U as $U = exp(-iaH\frac{t}{a}) = exp(-iH't')$, where a is a real number greater than one, Hamiltonian H' is multiplication of H by a, $H' \equiv aH$, and $t' = \frac{t}{a}$. This multiplication of a by H results in more energy in H' than H and reduce the time needed to scramble the system by a factor of $\frac{1}{a}$, $t' = \frac{t}{a}$, so one can always find a faster scrambler by increasing energy of Hamiltonian. To normalize the Hamiltonian, we will assume energy of Hamiltonian grows extensively with the size of the system [1].

$$\frac{tr(H^2)}{2^n} = n^2 \tag{2.4}$$

Although the Hamiltonian is normalized, it is not fair to compare the scrambling time of different Hamiltonians with various orders of scrambling. Because of any Hamiltonian scramble faster for a larger order of scrambling, so we have to fix the order of scrambling before comparing the scrambling time of different Hamiltonians. Also if one Hamiltonian

is the fastest scrambler for a small order of scrambling, δ_0, it does not mean it will be the fastest scrambler for a larger order of scrambling, δ_1. For example assume H_0 scramble to the order of δ_0 in a finite time, while H_1 does not scramble to the order δ_0 at all. It means H_0 scramble faster than H_1. Meanwhile it is possible that H_1 scramble the system to the order of δ_1, where $\delta_1 > \delta_0$, faster than H_0. All reasons above shows the importance of fixing the order of scrambling, δ. Our attention is on the order of scrambling $\delta \simeq 1$, this small order of scrambling is interesting not only as the point of view of black holes but also provide equivalency of scrambling with highly entangled states. Fallowing is the synopsis of our assumptions to tackle the question of finding fastest scrambler and their scrambling time.

- Q is a closed system of n qubits.
- Initial states of Q are computational bases.
- Energy grows extensively with number of qubits, $\frac{tr(H^2)}{2^n} = n^2$.
- Order of scrambling is small compared to size of system, $\delta \simeq 1$.

2.3.2 Pauli expansion of Hamiltonian

The substantial key for investigating fastest scrambler is Pauli expansion of Hamiltonian. We are going to use this expansion to come up with an anzats for fastest scrambler of large systems. In chapter 3 we introduce the fastest scrambler for small systems and find common properties in their Pauli expansion. Also in chapter 4, we utilized Pauli expansion of Hamiltonian to come up with the fastest scrambler for large systems, so it is essential to review Pauli expansion.

A Hamiltonian acting on n qubits system is a Hermitian operator, and its matrix representation is a 2^n dimensional hermitian matrix. Each diagonal component of hermitian matrix is a real number and non diagonal components are complex numbers, so one can describe aforementioned hermitian matrix with 2^n real numbers and $\frac{2^n * 2^n - 2^n}{2}$ complex numbers, complex numbers can describe by two real numbers so totally $\frac{2^n * 2^n - 2^n}{2} * 2 + 2^n = 4^n$ real numbers describe a Hamiltonian. Pauli expansion of Hamiltonian code these 4^n real numbers in 4^n coefficients of Pauli basis. Pauli basis, Ξ is nothing but $n-1$ tensor product of regular Pauli matrices $\{\sigma_0 = I, \sigma_1 = \sigma_x, \sigma_2 =$

$\sigma_y, \sigma_3 = \sigma_z\}$, for example in a system of 2 qubits, Pauli basis is a tensor product of Pauli matrices

$$\Xi_{00} = \sigma_0 \otimes \sigma_0, \Xi_{01} = \sigma_0 \otimes \sigma_1, \Xi_{02} = \sigma_0 \otimes \sigma_2, ..., \Xi_{32} = \sigma_3 \otimes \sigma_2, \Xi_{33} = \sigma_3 \otimes \sigma_3$$

There are 4^n possible subscripts of Ξ and each of these Quaternary representation can be represented by equivalent decimal number, for $n = 2$, $\Xi_0 = \Xi_{00}, \Xi_1 = \Xi_{01}, ..., \Xi_{15} = \Xi_{33}$. A summation of all these 4^n Pauli basis, each of them multiplied by a real number covers any hermitian Hamiltonian.

$$H = \sum_{k=0}^{4^n-1} \alpha_k \Xi_k \tag{2.5}$$

There are some special Hamiltonians that when you decompose them in Pauli expansion, their Pauli basis are commutable, $\forall i, j : [\Xi_i, \Xi_j] = 0$, for unitary operator of these Hamiltonians, $U(t) = e^{-iHt}$, it is easy to do the Pauli expansion .

$$U(t) = e^{-i\sum\limits_k \alpha_k \Xi_k t} = \prod_k e^{-i\alpha_k \Xi_k t} = \prod_k (Cos(\alpha_k t)\Xi_0 - iSin(\alpha_k t)\Xi_k) \tag{2.6}$$

2.3.3 Probability theory in scrambling

Another tool that we use to find the fastest scrambler of large systems is the probability theory. Recall that we can imagine a pure state of Q as a vector pointing on a unit sphere $S^{2^{n+1}-1}$, a Haar-random pure state is a pure state chosen from a uniform probability distribution on the aforementioned unit sphere. Don Page showed if a large quantum system of dimension 2^n is in a random pure state, the average entanglement entropy of a 2^m dimension subsystem A_l is, $\langle S^{A_l} \rangle = m - 2^{2m-n-1}$ [13], where the average is over Haar-random pure states. Assume $m' < m$, the $2^{2m'-n-1} < 2^{2m-n-1}$, this means average entanglement entropy of smaller subsystems are closer to maximum possible value. Maximum distance of $\langle S^{A_l} \rangle$ from maximum possible value m is $\frac{1}{2}$ and corresponds to $|A_l| = \frac{n}{2}$. This means for a Haar-random pure state of large systems, the expectation value of entanglement entropy for arbitrary subsystem A_l will be within $\frac{1}{2}$ distance of m. So if a random unitary Hamiltonian evolution of any computational basis results in a Haar-random pure state at time $t = t^*$, then scrambling condition will satisfy to the order of $\delta = \frac{1}{2}$. Time t^* that meets this condition is the scrambling time. As mentioned we are looking for this small order of scrambling.

We can tackle the problem of finding fastest scrambler for a large system by using the probability approach for finding a random Hamiltonian that evolves any computational basis to a Haar-random pure state at scrambling time t^*, order of scrambling for this method is $\delta = \frac{1}{2}$. In [1] they used Lieb-Robinson techniques to prove there is a logarithmic lower band for scrambling time with finite norm terms in Hamiltonian, so it is sufficient to show a random Hamiltonian scramble the system in logarithmic time to call it the fastest scrambler. Our focus in section 4 is to find such random Hamiltonian.

Random real coefficients of Pauli basis in Pauli expansion of Hamiltonian produces a random Hamiltonian, this means for a random Hamiltonian H

$$H = \sum_{k=0}^{4^n-1} \alpha_k \Xi_k$$

α_k are random numbers. Using the results for fastest scrambler of small systems, we came up with an educated guess that Pauli basis of fastest scrambler is $n-1$ tensor product of $\{\sigma_0 = I, \sigma_1 = \sigma_x\}$. The anzats of random Hamiltonian that we are interested in is

$$H = \sum_{k=1}^{f_n} \alpha_k \Xi_k$$

where f_n is the number of random pauli basis in Hamiltonian H, subscript k is the k^{th} random Pauli basis chosen from a uniform random distribution of all possible Pauli bases , it should not be mistaken by decimal representation of a binary sequence, for example $n = 2$, $\Xi_1 \in \{\Xi_{00}, \Xi_{01}, \Xi_{10}, \Xi_{11}\}$.

As number of Pauli bases in the Hamiltonian increase scrambling time decrease, to see this, using normalization of Hamiltonian we have

$$\frac{trH^2}{2^n} = \frac{tr \sum_{k=1}^{f_n} \alpha_k^2 I}{2^n} = \sum_{k=1}^{f_n} \alpha_k^2 = n^2$$

α s are independent real random numbers chosen from same probability distribution so

$$f_n \langle \alpha^2 \rangle = n^2 \Rightarrow \sqrt{\langle \alpha^2 \rangle} = \sqrt{\frac{n^2}{f_n}}$$

scrambling time is proportional to inverse of energy, $t^* \propto \frac{1}{\sqrt{\langle \alpha^2 \rangle}} = \frac{\sqrt{f_n}}{n}$, so more Pauli bases in Hamiltonian means higher scrambling time. Because we are looking for the

fastest scrambler, we will keep f_n as a variable, minimum f_n that satisfies scrambling condition for this anzats should results in the fastest scrambler. At the end of chapter 4 we will come up with probability distribution of α such that unitary Hamiltonian evolution of any computational basis results into a pure random state for which at time t^* first moment of the random state is equal to first moment of Haar-random state. Numerically we will find minimum f_n such that at time t^* other moments of random state get equal to corresponding moments of Haar-random state. Using normalization of Hamiltonian we will find time t^* as a function of probability distributions of α s, and f_n in the random Hamiltonian. having all variables we will show scrambling time is logarithmic in number of qubits. As mentioned above in [1] they used Lieb-Robinson techniques to prove there is a logarithmic lower band for scrambling time with finite norm terms in Hamiltonian, so the random Hamiltonian that we found is the fastest scrambler.

Chapter 3

Fastest scrambler for small n

Although many attempts have been established to find the fastest scrambler, none of them was successful to provide a time-independent Hamiltonian that scrambles the system in logarithmic time, some of these attempts are analytic, and some are numerical. Most of the analytic failures caused by complex nature of scrambling, different approaches to finding distance of reduced density matrices have been introduced to reduce the complexity. Numerical failures are due to lack of computation resources, memory and time. For example, a system of fifteen qubits, $n = 15$, has 2^{15} different initial state, the Hamiltonian is a 2^{15} dimensional Hermitian matrix which is enormous and a conventional RAM can not handle such a Hamiltonian. Besides lack of memory capacity, it takes a long time to find scrambling time of a random Hamiltonian with standard algorithms, checking scrambling condition for a constant time by averaging over 2^{15} initial states takes double exponential time, it is not even efficient to check if the scrambling condition is satisfied. All these reasons come together to find the fastest scrambler for small systems and hopefully find a pattern for the fastest scrambler of larger systems.

Small systems are not as complicated as large systems, and it is easy to simulate their evolution and find the scrambling time of an arbitrary Hamiltonian acting on small systems using personal computers. It takes a short time to find the scrambling time of an arbitrary Hamiltonian and minimize scrambling time using different minimization techniques. In this chapter, we will show using numerical methods we found the fastest scrambler to the order of $\delta = 0$ for a system of two and three qubits. We came up

with a well engineered Hamiltonian that is the fastest scrambler to the order of $\frac{n}{4}$ by investigation of fastest scrambler of two and three qubits. We used the Hamiltonian to find the fastest scrambler for a system of four and five qubits to the order of $\delta = 1$, Using search algorithm we found no scrambler faster than our engineered Hamiltonian. For small systems, the order of scrambling $\delta = \frac{n}{4}$ is small although for large systems it is not. We will tackle large systems with probability approach in next chapter but to do so we need a deep understanding of fastest scrambler for small systems which has been provided in this chapter.

3.1 Fastest scrambler for two qubits

A system of two qubits has four computational basis, $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, which are the four different initial states of scrambling. Unitary Hamiltonian evolution of computational basis, $U(t) = e^{-iHt}$, is a function of both the time t, and the Hamiltonian H. A four-dimensional Hermitian matrix can represent H. There is only one way to bipartite this system, label the first qubit A_1, and its complement (second qubit) B_1. At an anytime t state of the system is a pure state but subsystem A can be in a mixed state so the entanglement entropy between A and B can be non-zero. Let's plot $S_\psi^{A_l}$ as a function of time when a random normalized Hamiltonian is acting on a two-qubit system initially at a computational basis. Figure 3.1 shows this random evolution of entanglement entropy for all initial states with different colors. The cyan line is the average of entanglement entropy over all four initial states. Recall the scrambling condition 2.3, in the case of two qubits system it is

$$1 - \frac{1}{4}\sum_{\psi} S_\psi^{A_l}(t^*) < \delta$$

according to the graph for this random Hamiltonian, the system is scrambled to the order of $\delta = 0.4$ at time $t^* \simeq 1.5$ and to the order of $\delta = 0.18$ at time $t^* \simeq 9.6$. Such graphs of entanglement entropy evolution are not only provided for the purpose of clarification of scrambling condition in a system of two qubits, but also it is useful to analyze the fastest scrambler of a two-qubit system.

To find the fastest scrambler, the simplest approach is to provide many random Hamiltonian and compare their scrambling time to the same order of scrambling and find a fast scrambler $H'_{approximate}$, which is an approximation of the fastest scrambler

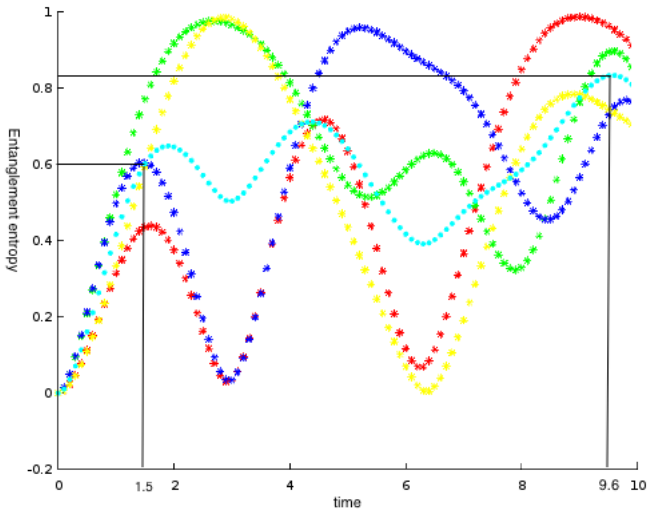

Figure 3.1: Entanglement entropy evolution for a random unitary Hamiltonian acting on two qubits
Each non-cyan line corresponds to one initial state. The cyan line is the average over all initial states.

H', then minimize scrambling time with respect to variables of Hamiltonian around vicinity of $H'_{approximate}$ and find the fastest scrambler H'. Because this approach use both random search and local minimization, it is possible to get trapped in a local fastest scrambler but further analysis provided sufficient evidences that actually the H' we found with this approach is not a local fastest scrambler. This approach is efficient in time and computation memory for small system of two or three qubits, but for $n \geq 4$ it is not efficient. Using program in appendix A written in octave language, one can find scrambling time of any Hamiltonian for n qubits to the order of δ, combination of this program and a search algorithm results in $H'_{approximate}$. To find H' we tried to vary parameters of Hamiltonian in the vicinity of obtained fast scrambler and minimize scrambling time, we used Nelder minimization approach which can be find in appendix B. Fallowing Hamiltonian is the result of random search and Nelder minimization to

find fastest scrambler H' to the order of $\delta \simeq 0.01$.

$$H' \simeq 4 \begin{pmatrix} 0.009 & -0.005 & 0.009+0.019i & 0.49+0.091i \\ -0.005 & -0.038 & 0.467-0.170i & -0.027-0.009i \\ 0.009-0.019i & 0.467+0.170i & 0.005 & -0.016-0.009i \\ 0.49-0.091i & -0.027+0.009i & -0.016+0.009i & -0.036 \end{pmatrix}$$

As mentioned earlier graph of entanglement entropy evolution reveals properties of the fastest scrambler, for aforementioned H', this evolution has been plotted in 3.1. It seems fastest scrambler acting on two qubits provide the same entanglement entropy of the only possible bi-partition at time t for all initial states, in other words, entanglement entropy of different initial states evolve simultaneously for fastest scrambler of two qubits. We need to find properties of eigenvalues and eigenstates of H' to investigate this evidence.

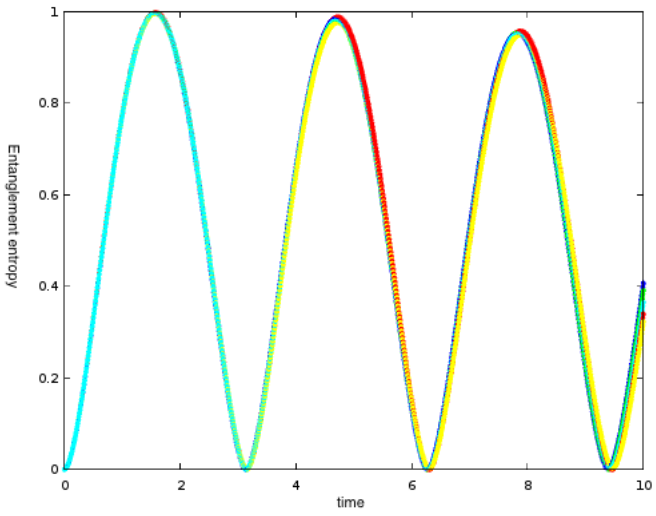

Figure 3.2: Entanglement entropy evolution for H' acting on two qubits
Each non-cyan line corresponds to one initial state. The cyan line is the average over all initial states.

Consider Hamiltonian H acting on a two qubit system, it can be represented as a four by four Hermitian matrix, with four orthonormal eigenstates , $\{|E_1\rangle, |E_2\rangle, |E_3\rangle, |E_4\rangle\}$

$$H|E_i\rangle = E_i|E_i\rangle$$

Define the matrix P such that each column of P is an eigenstate of Hamiltonian

$$P \equiv \begin{pmatrix} |E_1\rangle & |E_2\rangle & |E_3\rangle & |E_4\rangle \end{pmatrix}$$

Using the fact that eigenstates are orthonormal we can easily find p^{-1} to be

$$P^{-1} = \begin{pmatrix} \langle E_1| \\ \langle E_2| \\ \langle E_3| \\ \langle E_4| \end{pmatrix}$$

to obtain diagonal form of H, calculate $P^{-1}HP$

$$D = \begin{pmatrix} \langle E_1| \\ \langle E_2| \\ \langle E_3| \\ \langle E_4| \end{pmatrix} H \begin{pmatrix} |E_1\rangle & |E_2\rangle & |E_3\rangle & |E_4\rangle \end{pmatrix} = \begin{pmatrix} E_1 & 0 & 0 & 0 \\ 0 & E_2 & 0 & 0 \\ 0 & 0 & E_3 & 0 \\ 0 & 0 & 0 & E_4 \end{pmatrix}$$

each diagonal element of matrix D is an eigenvalue of Hamiltonian H. investigating eigenvalues and eigenstates of Hamiltonian H' can illuminate properties of fastest scrambler H'. Diagonal form of matrix H' is D'

$$D' = 4 \begin{pmatrix} -0.52706 & 0 & 0 & 0 \\ 0 & -0.50182 & 0 & 0 \\ 0 & 0 & 0.45649 & 0 \\ 0 & 0 & 0 & 0.51185 \end{pmatrix}$$

also the matrix P', which has been formed by eigenstates of H' is

$$P' \simeq \begin{pmatrix} 0.470 + 0.078i & -0.488 - 0.101i & -0.493 - 0.113i & -0.511 - 0.077i \\ 0.006 - 0.523i & 0.038 - 0.496i & -0.375 - 0.328i & 0.379 + 0.290i \\ -0.200 + 0.465i & -0.173 + 0.437i & -0.263 - 0.454i & 0.255 + 0.424i \\ -0.491 & 0.530 & -0.467 & -0.508 \end{pmatrix}$$

There are some properties for eigenstates of H' that we discuss in section 3.1, also in section 3.1 we show eigenvalues of fastest scrambler acting on two qubit system should be $\{-2, -2, +2, +2\}$, which is the case in H' up to some precision. We realized any Hamiltonian with these eigenvalues, which satisfies eigenstates properties of H', is also fastest scrambler, and provide the simplest form for the fastest scrambler which is $H = 2\Xi_{11} = 2(\sigma_x \otimes \sigma_x)$.

Eigenvalues

As we saw in numerical analysis simultaneous evolution of entanglement entropy is an evidence of having fastest scrambler, so it is logical to find eigenvalues of the Hamiltonian that evolves entanglement entropy of all initial states simultaneously. Assuming H evolves entanglement entropy of all initial states simultaneously, we will show that eigenvalues should be $\{-2,-2,+2,+2\}$. For this purpose write Hamiltonian H in its eigenstate basis, $H = PDP^{-1}$, where D is the diagonal form of H, and P is the matrix of eigenstates of H. Any function of Hamiltonian $f(H) = P\ f(D)\ P^{-1}$, so

$$|\psi(t)\rangle = e^{-iHt}|\psi\rangle = Pe^{-iDt}P^{-1}|\psi\rangle$$

C is the co-factor of P, using $P^{-1} = \frac{1}{det(P)}Adj(P) = \frac{1}{det(P)}C^{\top}$

$$|\psi(t)\rangle = \frac{1}{detP}Pe^{-iDt}C^{\top}|\psi\rangle$$

Lets say $|\psi\rangle$ is the m^{th} computational basis, in the matrix form of $|\psi\rangle = |m\rangle$ all the components are zero but the m^{th} component which is one, using $P^{-1}_{ij} = \frac{1}{detP}C^{\top}_{ij} = \frac{1}{detP}C_{ji}$ and matrix form of $|m\rangle$

$$|m(t)\rangle = \frac{1}{detP}\sum_{k=1}^{4}\sum_{j=1}^{4}e^{-iE_jt}P_{kj}C_{mj}|k\rangle$$

lets define $m_k(t)$ as

$$m_k(t) \equiv \frac{1}{detP}\sum_{j=1}^{4}e^{-iE_jt}P_{kj}C_{mj} \tag{3.1}$$

so

$$|m(t)\rangle = \sum_{k=1}^{4}m_k(t)|k\rangle$$

$$\rho^m(t) = |m(t)\rangle\langle m(t)|$$

$$\rho^m(t) = \frac{1}{(detP)^2}\begin{pmatrix} m_1(t)m_1^*(t) & m_1(t)m_2^*(t) & m_1(t)m_3^*(t) & m_1(t)m_4^*(t) \\ m_2(t)m_1^*(t) & m_2(t)m_2^*(t) & m_2(t)m_3^*(t) & m_2(t)m_4^*(t) \\ m_3(t)m_1^*(t) & m_3(t)m_2^*(t) & m_3(t)m_3^*(t) & m_3(t)m_4^*(t) \\ m_4(t)m_1^*(t) & m_4(t)m_2^*(t) & m_4(t)m_3^*(t) & m_4(t)m_4^*(t) \end{pmatrix}$$

tracing over the first qubit, reduced density matrix of second qubit $\rho_m^A(t)$ is

$$\rho_m^A(t) = \frac{1}{(detP)^2}\begin{pmatrix} m_1(t)m_1^*(t) + m_3(t)m_3^*(t) & m_1(t)m_2^*(t) + m_3(t)m_4^*(t) \\ m_2(t)m_1^*(t) + m_4(t)m_3^*(t) & m_2(t)m_2^*(t) + m_4(t)m_4^*(t) \end{pmatrix}$$

$$= \frac{1}{(detP)^2}\begin{pmatrix} a_m & \alpha_m \\ \alpha_m^* & b_m \end{pmatrix} \tag{3.2}$$

$\rho_m^A(t)$ is a Hermitian matrix, its diagonal for is $\rho_m^{'A}(t)$

$$\rho_m^{'A} = \begin{pmatrix} \frac{1+\sqrt{4\alpha_m^2+(a_m-b_m)^2}}{2} & 0 \\ 0 & \frac{1-\sqrt{4\alpha_m^2+(a_m-b_m)^2}}{2} \end{pmatrix} = \begin{pmatrix} \beta_m^+ & 0 \\ 0 & \beta_m^- \end{pmatrix} \tag{3.3}$$

now that we have diagonal reduced density matrix at time t, we can find entanglement entropy.

$$S_m^A = -tr\left(\rho_m^A(t)\ log_2\left(\rho_m^A(t)\right)\right) = -tr\left(B\rho_m^{'A}(t)B^{-1}B\ log_2\left(\rho_m^{'A}(t)\right)B^{-1}\right) = -tr\left(\rho_m^{'A}(t)\ log_2\rho_m^{'A}(t)\right)$$

from (3.3)

$$\rho_m^{'A}(t)\ log_2\left(\rho_m^{'A}(t)\right) = \begin{pmatrix} log_2\left((\beta_m^+)^{\beta_m^+}\right) & 0 \\ 0 & log_2\left((\beta_m^-)^{\beta_m^-}\right) \end{pmatrix}$$

$$S_m^A = -log_2\left((\beta_m^+)^{\beta_m^+}(\beta_m^-)^{\beta_m^-}\right)$$

Using the assumption that fastest scrambler H evolves entanglement entropy of all initial states simultaneously we want $\forall\{m, m'\} : S_m^A(t) - S_{m'}^A(t) = 0$

$$S_m^A(t) - S_{m'}^A(t) = 0 \longrightarrow log_2\left(\frac{(\beta_m^+)^{\beta_m^+}(\beta_m^-)^{\beta_m^+}}{(\beta_{m'}^+)^{\beta_{m'}^+}(\beta_{m'}^-)^{\beta_{m'}^-}}\right) = 0$$

$$\frac{(\beta_m^+)^{\beta_m^+}(\beta_m^-)^{\beta_m^-}}{(\beta_{m'}^+)^{\beta_{m'}^+}(\beta_{m'}^-)^{\beta_{m'}^-}} = 1 \longrightarrow (\beta_m^+)^{\beta_m^+}(\beta_m^-)^{\beta_m^-} = (\beta_{m'}^+)^{\beta_{m'}^+}(\beta_{m'}^-)^{\beta_{m'}^-}$$

looking back to the equation (3.3), we can see

$$\beta_m^- = 1 - \beta_m^+$$

so

$$(\beta_m^+)^{\beta_m^+}(1-\beta_m^+)^{1-\beta_m^+} = (\beta_{m'}^+)^{\beta_{m'}^+}(1-\beta_{m'}^+)^{1-\beta_{m'}^+}$$

β_m^+ is a real number, and because m and m' are arbitrary different initial states, we can choose m and m' such that $\beta_m^+ \neq \beta_{m'}^+$. for this case the only way that above equation satisfies for the real number, β_m^+, is when

$$\beta_m^+ = 1 - \beta_{m'}^+ \tag{3.4}$$

Using (3.3) and (3.4)

$$\frac{1 + \sqrt{4\alpha_m^2 + (a_m - b_m)^2}}{2} + \frac{1 + \sqrt{4\alpha_{m'}^2 + (a_{m'} - b_{m'})^2}}{2} = 1$$

$$\sqrt{4\alpha_m^2 + (a_m - b_m)^2} + \sqrt{4\alpha_{m'}^2 + (a_{m'} - b_{m'})^2} = 0$$

this means for any initial state $|m\rangle$

$$\sqrt{4\alpha_m^2 + (a_m - b_m)^2} = 0 \rightarrow \alpha_m = 0 \;\; \& \;\; a_m = b_m$$

Using definition of $m_k(t)$ at (3.1)

$$m_r(t)m_s^*(t) = \sum_{j=1}^{4}\sum_{k=1}^{4} e^{i(E_k - E_j)t} P_{rj} C_{mj} P_{sk}^* C_{mk}^* \tag{3.5}$$

Consider above equation (3.5), and (3.2) to calculate α_m, which should satisfy $\alpha_m = 0$

$$\alpha_m = \sum_{j=1}^{4}\sum_{k=1}^{4} e^{i(E_k - E_j)t}(p_{1j} C_{mj} p_{3k}^* C_{mk}^* + p_{2j} C_{mj} p_{4k}^* C_{mk}^*)$$

$$\sum_{j=1}^{4}\sum_{k=1}^{4} e^{i(E_k - E_j)t} C_{mj} C_{mk}^* (p_{1j} p_{3k}^* + p_{2j} p_{4k}^*) = 0 \tag{3.6}$$

Using above equation we want to show there must be a degeneracy in each eigenvalue. To do so we will use proof by contradiction. Assume for any arbitrary k and j, such that $k \neq j$, there is no degeneracy in eigenvalues, $E_k - E_j \neq 0$ so $e^{i(E_k - E_j)t} \neq 1$. Equation 3.6 should be correct at any arbitrary time t.

$$\forall\ t : \sum_{j=1}^{2^n}\sum_{k=1}^{2^n} e^{i(E_k - E_j)t} C_{mj} C_{mk}^* (p_{1j} p_{3k}^* + p_{2j} p_{4k}^*) = 0$$

Because this equation is correct and the term in the summation, $e^{i(E_k-E_j)t}$, is a function of time, its coefficient must be zero.

$$C_{mj}C^*_{mk}(p_{1j}p^*_{3k} + p'_{2j}p'^*_{4k}) = 0$$

$C_{mj}C^*_{mk}$ can not be zero for all m, the reason is if $\forall\ m : C_{mj}C^*_{mk} = 0$ then a row of C is zero, which is in contrast with invertibility of P. So there exist at least one m such that $C_{mj}C^*_{mk} \neq 0$, choose that m. For this case

$$p_{1j}p^*_{3k} + p_{2j}p^*_{4k} = 0$$

$$p_{1j} = -\frac{p^*_{4k}}{p^*_{3k}}p_{2j} = Cp_{2j}$$

again it means P is not invertible. All above states our assumption was wrong and we should have degeneracy in each eigenvalue of Hamiltonian H, in other words eigenvalues are pairs.

Using normalization of Hamiltonian (2.4) and demanding $\forall j, k \neq j : E_j^2 = E_k^2$ we can write D as fallowing which is compatible with numeric results.

$$D = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Now that we know about eigenvalues of the fastest scrambler for two qubits system, we have to investigate its eigenstates in next section. The combination of the results in this section and next section illuminate a simple form of the fastest scrambler for two-qubit system which is useful for investigation of larger systems.

Eigenstate

To find the properties of eigenstates we will use H' that we obtained using numerical methods. $P' = (|E_1\rangle, |E_2\rangle, |E_3\rangle, |E_4\rangle)$, according to P' in 3.1 we can see the fallowing relations between eigenstates of H'.

$$|E_1\rangle + |E_2\rangle = \alpha_1|1\rangle + \alpha_2|2\rangle$$

$$|E_1\rangle - |E_2\rangle = \alpha_0|0\rangle + \alpha_3|3\rangle$$

$$|E_3\rangle + |E_4\rangle = \beta_0|0\rangle + \beta_3|3\rangle$$

$$|E_3\rangle - |E_4\rangle = \beta_1|1\rangle + \beta_2|2\rangle$$

Using $\alpha_i' \equiv \frac{\alpha_i}{2}$ and $\beta_i' \equiv \frac{\beta_i}{2}$, find each eigenstate in terms of computational basis $\{|0\rangle, |1\rangle, |2\rangle, |3\rangle\}$.

$$|E_1\rangle = \alpha_0'|0\rangle + \alpha_1'|1\rangle + \alpha_2'|2\rangle + \alpha_3'|3\rangle$$

$$|E_2\rangle = -\alpha_0'|0\rangle + \alpha_1'|1\rangle + \alpha_2'|2\rangle - \alpha_3'|3\rangle$$

$$|E_3\rangle = \beta_0'|0\rangle + \beta_1'|1\rangle + \beta_2'|2\rangle + \beta_3'|3\rangle$$

$$|E_4\rangle = \beta_0'|0\rangle - \beta_1'|1\rangle - \beta_2'|2\rangle + \beta_3'|3\rangle$$

inner product of these eigenstates should satisfy fallowing conditions

$$\langle E_1|E_1\rangle = \langle E_2|E_2\rangle = 1 \longrightarrow \sum_{i=0}^{3} \alpha_i^{'2} = 1$$

$$\langle E_1|E_2\rangle = 0 \longrightarrow -|\alpha_0'|^2 + |\alpha_1'|^2 + |\alpha_2'|^2 - |\alpha_3'|^2 = 0$$

combining both above equations

$$|\alpha_0'|^2 + |\alpha_3'|^2 = |\alpha_1'|^2 + |\alpha_2'|^2 = \frac{1}{2} \tag{3.7}$$

Same calculation for $|E_3\rangle$ and $|E_4\rangle$ shows

$$\langle E_3|E_3\rangle = \langle E_4|E_4\rangle = \sum_{i=0}^{3} \beta_i^{'2} = 1$$

$$\langle E_3|E_4\rangle = 0 \longrightarrow |\beta_0'|^2 - |\beta_1'|^2 - |\beta_2'|^2 + |\beta_3'|^2 = 0$$

combining both above equations

$$|\beta_0'|^2 + |\beta_3'|^2 = |\beta_1'|^2 + |\beta_2'|^2 = \frac{1}{2} \tag{3.8}$$

Also other inner products of eigenstates should satisfy fallowing equations

$$\langle E_1|E_3\rangle = \langle E_2|E_4\rangle = 0 \longrightarrow \alpha_0^{'*}\beta_0' + \alpha_1^{'*}\beta_1' + \alpha_2^{'*}\beta_2' + \alpha_3^{'*}\beta_3' = 0$$

$$\langle E_1|E_4\rangle = \langle E_2|E_3\rangle = 0 \longrightarrow \alpha_0^{'*}\beta_0' - \alpha_1^{'*}\beta_1' - \alpha_2^{'*}\beta_2' + \alpha_3^{'*}\beta_3' = 0$$

Combining these two equations we will get the fallowing condition.

$$\alpha_0^{'*}\beta_0' + \alpha_3^{'*}\beta_3' = \alpha_1^{'*}\beta_1' + \alpha_2^{'*}\beta_2' = 0 \tag{3.9}$$

Conditions (3.7), (3.8), and (3.9) completely determine properties of eigenstates for fastest scrambler H'. It is logical to expect other fastest scramblers also satisfy these conditions, we will show that this a correct conjecture using both numeric and analytic methods. Fallowing matrix satisfy above conditions for any arbitrary complex number c

$$P = \begin{pmatrix} c & -c & -c & -c \\ c^* & c^* & -c & c \\ -c^* & -c^* & -c & c \\ -c & -c & -c & -c \end{pmatrix}$$

Using the D that we found in section 3.1, and above P, which is the matrix of eigenstates of fastest scrambler, we can find $H = PDP^{-1}$. Normalization of H will give us the fallowing matrix.

$$H = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix}$$

One can easily see this H is nothing but $2\Xi_{11} = 2\sigma_x \otimes \sigma_x$. To check correctness of our assumptions and calculation we used this H and our numerical method to calculate scrambling time to the same order of 0.01, it was the fastest scrambler as we expected. Also we searched for other Hamiltonians that scramble the system to other order of scrambling, $\delta = 0.02, 0.05, 0.1, 0.3$, faster than H, but we found no faster scrambler.

Picture 3.1 shows how entanglement entropy of different initial states evolve with time for H', entanglement entropy of the various initial states evolve together as we wanted.

In the fallowing section, we will use Pauli expansion of H and unitary operator e^{-iHt} to investigate why this Hamiltonian is the fastest scrambler analytically. This investigation is useful to find fastest scramblers in a large system and a small order of scrambling. Also as mentioned earlier this analysis will give us satisfaction results

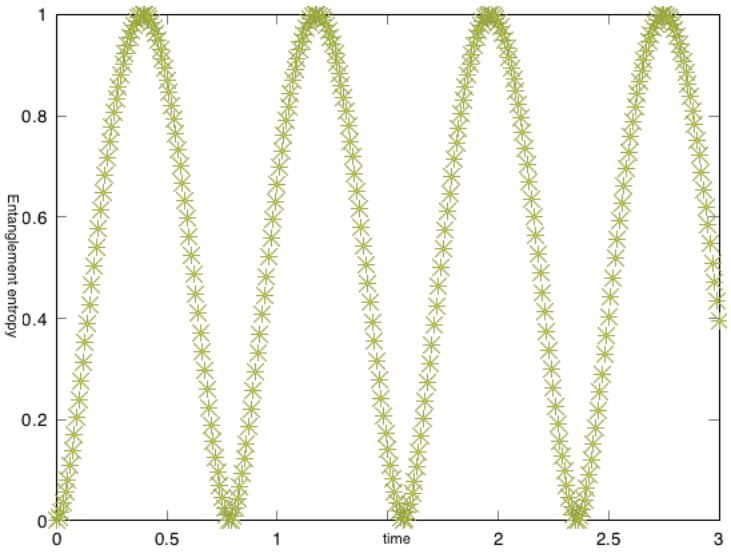

Figure 3.3: Evolution of entanglement entropy for $H = \Xi_{11}$
Although there are 5 colored lines, four lines corresponds to different initial states and one line is the average over them, but because they overlap we only see olive color.

that the fastest scrambler we found is not a local fastest scrambler. In conclusion, next section is an essential step towards fastest scrambler of large systems.

3.1.1 Pauli expansion of fastest scrambler for two qubits system

As we showed earlier, numerical results promise that $H = 2\Xi_{11}$ is the fastest scrambler for a system of two qubits. In this section, we reason analytically that this Hamiltonian should be the fastest scrambler. We provide a relation between the fastest scrambler and initial states, using that relationship we will show $2\Xi_{22}$ is also the fastest scrambler. Using results of this section one can obtain an in-depth understanding of fastest scrambling and its relation with both initial states, which are the computational basis, and desired states that we want to reach at time t^*, which are highly entangled states. In the case of two qubits, we will see the desired states are maximally entangled states or Bell states.

Consider Hamiltonian $H = 2\Xi_{11}$, where $\Xi_{11} \equiv \sigma_1 \otimes \sigma_1 = \sigma_x \otimes \sigma_x$. There exist only one term in Pauli expansion of the Hamiltonian H, which is Ξ_{11}, so $\forall i, j : [\Xi_i, \Xi_j] = 0$. Unitary operator of this Hamiltonian is $U(t) = e^{-iHt} = e^{-2i\Xi_{11}t}$, use equation 2.6 to

expand $U(t)$

$$U(t) = Cos(2t)\Xi_{00} - iSin(2t)\Xi_{11}$$

Action of this Unitary operator on initial state will evolve the system to the pure state $|\psi(t)\rangle$

$$|\psi(t)\rangle = (Cos(2t)\Xi_{00} - iSin(2t)\Xi_{11})\,|\psi\rangle$$

Consider action of above Pauli matrices, Ξ_i, on a computational basis. Ξ is a $n = 2$ tensor product of Pauli matrices $\sigma_0 = I$ and $\sigma_1 = \sigma_x$, so when it acts on computational basis, each pauli matrix acts on its corresponding qubit. The qubits which has been acted on by $\sigma_0 = I$ will not change, $\sigma_1|0\rangle = |0\rangle$ and $\sigma_1|1\rangle = |1\rangle$, but the ones that has been acted by σ_1 flips, $\sigma_1|0\rangle = |\hat{0}\rangle = |1\rangle$ and $\sigma_1|1\rangle = |\hat{1}\rangle = |0\rangle$. So one can consider Pauli basis as an operator that flip state of some qubits, in quantum information such operators are called spin flip operator. keep in mind the name of spin flip operator does not confine the operators such that they should act on qubits which are spin half fermions, but as mentioned qubit is a general term and spin flip operators act on all kind of qubits. Look back at $|\psi(t)\rangle = (Cos(2t)\Xi_{00} - iSin(2t)\Xi_{11})\,|\psi\rangle$, distribution of $|\psi\rangle$ on the operators shows that first operator Ξ_{00} will not change the state $|\psi\rangle$ and second operator Ξ_{11} spin flip both qubits of $|\psi\rangle$. No matter what is the initial state, this operator evolve the system to a bell state when $Sin(2t) = Cos(2t) = \frac{1}{\sqrt{2}}$. We have 4 initial states, so

$$(Cos(2t)\Xi_{00} - iSin(2t)\Xi_{11})\,|00\rangle = Cos(2t)|00\rangle - iSin(2t)|11\rangle$$

$$(Cos(2t)\Xi_{00} - iSin(2t)\Xi_{11})\,|01\rangle = Cos(2t)|01\rangle - iSin(2t)|10\rangle$$

$$(Cos(2t)\Xi_{00} - iSin(2t)\Xi_{11})\,|10\rangle = Cos(2t)|10\rangle - iSin(2t)|01\rangle$$

$$(Cos(2t)\Xi_{00} - iSin(2t)\Xi_{11})\,|11\rangle = Cos(2t)|11\rangle - iSin(2t)|00\rangle$$

at time $2t = \frac{\pi}{4}$ we will get bell states. A bell state is a maximally entangled state for system of two qubit so at this time the scrambling condition 2.3 satisfies, $t^* = \frac{\pi}{8}$. Although these steps was sufficient to understand how this Hamiltonian scramble the system, but finding entanglement entropy through matrix representation of reduced density matrices provide a deeper understanding that we need. Using $|\psi(t)\rangle$, find $|\rho_\psi(t)\rangle$

$$|\rho_\psi(t)\rangle = (Cos(2t)\Xi_{00} - iSin(2t)\Xi_{11})\,|\psi\rangle\langle\psi|\,(Cos(2t)\Xi_{00} + iSin(2t)\Xi_{11})$$

There are four different initial states, we only refer to one initial state $|\psi\rangle = |00\rangle$ and the reader can do the same calculations for other initial states, he must get the same entanglement entropy that we are going to calculate for $|\psi\rangle = |0\rangle$.

$$\rho_0(t) = \begin{pmatrix} Cos^2(2t) & 0 & 0 & iSin(2t)Cos(2t) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -iSin(2t)Cos(2t) & 0 & 0 & Sin^2(2t) \end{pmatrix}$$

So reduced density matrix is

$$\rho_0^A(t) = \begin{pmatrix} Cos^2(2t) & 0 \\ 0 & Sin^2(2t) \end{pmatrix}$$

that results into the fallowing entanglement entropy

$$S_0^A = -tr \begin{pmatrix} Cos^2(2t) & 0 \\ 0 & Sin^2(2t) \end{pmatrix} log_2 \begin{pmatrix} Cos^2(2t) & 0 \\ 0 & Sin^2(2t) \end{pmatrix}$$

$$= -log_2 \left(\left(Cos^2(2t)\right)^{Cos^2(2t)} \left(Sin^2(2t)\right)^{Sin^2(2t)} \right)$$

As mentioned it doesn't matter what the initial state is. One should always get the same entanglement entropy as above. At time $2t = \frac{\pi}{4}$, the system evolves to maximally entangled states, $|\psi(t^*)\rangle$, so scrambling condition satisfies at this time, $t^* = \frac{\pi}{8}$, to the order of 0. All calculation mentioned above convince us this type of scramblers should be the fastest scramblers for a system of two qubits. A careful reader realize σ_x is not the only spin flip Pauli matrix, and this reasoning should also work with σ_y which do the same action but with different phases; this point is correct, and it is what we expect because of symmetry. Look at initial states, which are the computational basis; they are n tensor products of $|0\rangle$ and $|1\rangle$, which are eigenstates of σ_z. There is no preference between the action of Pauli matrix in x direction on the eigenstate of σ_x, and Pauli matrix in y direction on the same eigenstate. So if we change all σ_x with σ_y, one will see no difference in entanglement entropy at time t and Ξ_{22} is also the fastest scrambler for two qubits system.

Although aforementioned calculation conviced us to consider Ξ_{11}, as the fastest scrambler for two qubits, we wrote a program in appendix B, to search for fastest

scrambler of n qubits based on another definition of scrambling in [1], which says to have an scramble system distance of reduced density matrices of smaller system for any arbitrary initial states and all bi-partitions should be less than a small number ϵ. This definition is called distance definition of scrambling. definition of scrambling based on distance for small ϵ is equivalent to the definition of scrambling based on entanglement entropy for small δ. In the definition of scrambling based on distance, if the fallowing condition satisfies for some small number ϵ, then the system is scrambled. This ϵ can be compared to the order of scrambling δ when both approach to zero.

$$\forall A_l, \psi, \phi : ||\rho_{\psi}^{A_l}(t^*) - \rho_{\phi}^{A_l}(t^*)|| < \epsilon$$

Where

$$||\rho_{\psi}^{A_l}(t) - \rho_{\phi}^{A_l}(t)|| = tr\sqrt{\left(\rho_{\psi}^{A_l}(t) - \rho_{\phi}^{A_l}(t)\right)^{\dagger}\left(\rho_{\psi}^{A_l}(t) - \rho_{\phi}^{A_l}(t)\right)}$$

To simplify this definition recall $\rho_{\psi}^{A_l}(t)$ is Hermitian, so $\left(\rho_{\psi}^{A_l}(t) - \rho_{\phi}^{A_l}(t)\right)^{\dagger} = \left(\rho_{\psi}^{A_l}(t) - \rho_{\phi}^{A_l}(t)\right)$

$$||\rho_{\psi}^{A_l}(t) - \rho_{\phi}^{A_l}(t)|| = tr\sqrt{\left(\rho_{\psi}^{A_l}(t) - \rho_{\phi}^{A_l}(t)\right)^2} = tr|\rho_{\psi}^{A_l}(t) - \rho_{\phi}^{A_l}(t)|$$

It is not surprising the program we wrote using definition of scrambling based on distance found no Hamiltonian scramble faster than Ξ_{11}.

In this paragraph, we will analyze the system using the definition of scrambling based on distance. Not only analysis of the system using this definition assures that Ξ_{11} is the fastest scrambler, but also it will provide us an insight on fastest scrambler for $n \geq 4$. In the case of $H = \Xi_{11}$, using the same calculation that we did in previous paragraphs to obtain $\rho_0^A(t)$, we can find any $\rho_{\psi}^A(t)$

For $|\psi\rangle \in \{|00\rangle, |01\rangle\}$

$$\rho_{\psi}^A(t) = Cos^2(2t)|0\rangle\langle 0| + Sin^2(2t)|1\rangle\langle 1| = \begin{pmatrix} Cos^2(2t) & 0 \\ 0 & Sin^2(2t) \end{pmatrix}$$

and for $|\psi\rangle \in \{|10\rangle, |11\rangle\}$

$$\rho_{\psi}^A(t) = Sin^2(2t)|0\rangle\langle 0| + Cos^2(2t)|1\rangle\langle 1| = \begin{pmatrix} Sin^2(2t) & 0 \\ 0 & Cos^2(2t) \end{pmatrix}$$

at time $t^* = \frac{\pi}{8}$, $\rho_\psi^A(t^*) = \frac{1}{2}I$ so $\forall A_l, \psi, \phi : ||\rho_\psi^{A_l}(t^*) - \rho_\phi^{A_l}(t^*)|| = 0$, which means system gets scramble to the order of $\epsilon = 0$ at time $t^* = \frac{\pi}{8}$, It is compatible with other definitions and previous results, which is what we expected.

At this step one may guess, to the order of zero, the fastest scrambler of three qubits system should be Ξ_{111}, This conjecture is correct and in the fallowing section we will show that. Also, we will provide the results of the numerical analysis which support calculations. But this conjecture can not be generalized to $n \geq 4$, in the upcoming chapters we show why this is the case and provide a time independent diagram that provides the fastest scrambler to the order of $\frac{n}{4}$. Desired states of evolution with this time independent diagram are the states known as maximally entangled symmetric states.

3.2 Fastest scrambler for three qubits

As mentioned earlier, the Hamiltonian $H = \Xi_{111}$ is a logical conjecture for the fastest scrambler of a three-qubit system. This conjecture is reasonable because of common characteristic between three qubits and two qubits; first, all three different way of bi-partitioning the system of three qubits results in a subsystem $|A_l| = 1$, so again the reduced density matrix for the smaller subsystem is a two-dimensional density matrix. Also maximally entangled states in three-qubit systems, like $|GHZ\rangle = \frac{1}{\sqrt{2}}|000\rangle + |111\rangle$, are quite similar to the two-qubit system $|B\rangle = \frac{1}{\sqrt{2}}|00\rangle + |11\rangle$. One can demand these maximally entangled states for three qubits to be the desired states and find the fastest scrambler that evolves the system to these states. We will go through all above reasoning in details step by step. These steps are convincing that we have the fastest scrambler for three qubits, but to further assure $H = \Xi_{111}$ is the fastest scrambler for a three-qubit system, we follow the numerical method. Using the programs in Appendices A, and B we searched for a Hamiltonian that scrambles the system faster than $H = \Xi_{111}$ but found no faster scrambler.

Let's follow the same analysis that we had done for two qubits, but for three qubits system. In the two qubits system, the fastest scrambler evolves entanglement entropy of different initial states together. Recall simultaneous evolution of entanglement entropy is an evidence of having the fastest scrambler. For the case of three qubits, we have the

same evidence as you can see in 3.2, $H = 3\Xi_{111}$ evolve all initial entanglement entropy simultaneously.

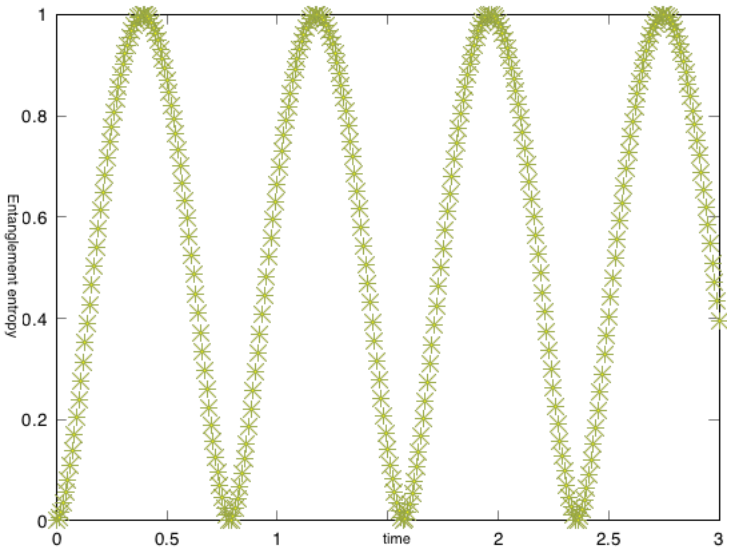

Figure 3.4: Evolution of entanglement entropy for $H = \Xi_{111}$
Although there are 9 colored lines, 8 lines correspond to different initial states and one line is the average over them, but because they overlap we only see olive color.

Besides numerical evidence that suggests scrambling between two and three qubits are similar, analyzing the Hamiltonian evolution of a three qubits system initially in a computational basis under the action of $H = \Xi_{111}$ shows the similarity between scrambling of two qubits and three qubits. We can bipartite a three qubits system to three different bipartitions; the number of qubits in smaller partition $|A_l| = m$ is esentially $m = 1$. Recall for two qubits system also the number of qubits in the smaller subsystem is $m = 1$. So all we need is to maximally entangle each qubit with the other two qubits at minimum scrambling time t^*.

Normalization of Hamiltonian, $\frac{a^2 tr\Xi_{111}^2}{8} = 9$, tells us$a = 3$. Using the normalized Hamiltonian to evolve the system the unitary operator is

$$U(t) = e^{-iHt} = e^{-3i\Xi_{111}t}$$

Again we can use Taylor series expansion that has been mentioned in 2.6

$$U(t) = Cos(3t)\Xi_{000} - iSin(3t)\Xi_{111}$$

as mentioned in the two qubit section, Ξ acts like a spin flip operator that flip some of the qubits, here Ξ_{111} flips all three qubits and Ξ_{000} leave them untouched, so at time t, state of the system is

$$|\psi(t)\rangle = (Cos(3t)\sigma_{000} - iSin(3t)\sigma_{111})\,|\psi\rangle = Cos(3t)|\psi\rangle - iSin(3t)|\tilde{\psi}\rangle$$

This shows at time $t^* = \frac{\pi}{12}$, any computational basis evolves to a maximally entangled state for three qubit system. Scrambling time of this Hamiltonian is less than its equivalent Hamiltonian for two qubit system while the order of scrambling is the same. Not only $H = 3\Xi_{111}$ is the fastest scrambler to the order of $\delta = 0$, but also it is the fastest scrambler to the order $\delta \geq 0$, which is the same as fastest scrambler of two qubit system.

Also, we employed the program in Appendix C, which uses the definition of scrambling based on distance, to search for a Hamiltonian that scrambles faster than $H = 3\Xi_{111}$, not surprising we found no Hamiltonian scramble faster. Analyzing scrambling of $H = 3\Xi_{111}$ provides an insight toward the fastest scrambler for $n > 4$. For this analysis consider the system in the state of $|\psi(t)\rangle$ at time t.

$$|\psi(t)\rangle = Cos(3t)\Xi_{000}|\psi\rangle - iSin(3t)\Xi_{111}|\psi\rangle = Cos(3t)|\psi\rangle - iSin(3t)|\tilde{\psi}\rangle$$

The density matrix is outer product of $|\psi(t)\rangle$ times probability of being in the state $|\psi(t)\rangle$, which is one.

$$\rho_\psi(t) = Cos^2(3t)|\psi\rangle\langle\psi| + iSin(3t)Cos(3t)|\psi\rangle\langle\tilde{\psi}| - iSin(3t)Cos(3t)|\tilde{\psi}\rangle\langle\psi| + Sin^2(3t)|\tilde{\psi}\rangle\langle\tilde{\psi}|$$

Reduced density matrices of different bipartitions A for some initial states are the same. Reduced density matrices of choosing A to be the first qubit with the initial states $|\psi\rangle \in \{|000\rangle, |001\rangle, |010\rangle, |011\rangle\}$, and reduced density matrices of choosing A to be the second qubit with the initial states $|\psi\rangle \in \{|000\rangle, |001\rangle, |100\rangle, |101\rangle\}$, and reduced density matrices of choosing A to be the third qubit with the initial states $|\psi\rangle \in \{|000\rangle, |010\rangle, |100\rangle, |110\rangle\}$ are the same which is

$$\rho_\psi^A(t) = Cos^2(2t)|0\rangle\langle 0| + Sin^2(2t)|1\rangle\langle 1| = \begin{pmatrix} Cos^2(2t) & 0 \\ 0 & Sin^2(2t) \end{pmatrix}$$

Also reduced density matrices of choosing A to be the first qubit with the initial states $|\psi\rangle \in \{|100\rangle, |101\rangle, |110\rangle, |111\rangle\}$, and reduced density matrices of choosing A to be the second qubit with the initial states $|\psi\rangle \in \{|010\rangle, |011\rangle, |110\rangle, |111\rangle\}$, and reduced density matrices of choosing A to be the third qubit with the initial states $|\psi\rangle \in \{|001\rangle, |011\rangle, |101\rangle, |111\rangle\}$ are the same which is

$$\rho_{\psi}^{A}(t) = Sin^2(2t)|0\rangle\langle 0| + Cos^2(2t)|1\rangle\langle 1| = \begin{pmatrix} Sin^2(2t) & 0 \\ 0 & Cos^2(2t) \end{pmatrix}$$

As we can see these reduced density matrices have the same matrix form that we found for two qubits system, at time $t^* = \frac{\pi}{12}$, $\rho_{\psi}^{A}(t^*) = \frac{1}{2}I$ so $\forall A_l, \psi, \phi : ||\rho_{\psi}^{A_l}(t^*) - \rho_{\phi}^{A_l}(t^*)|| = 0$, which means system gets scramble to the order of $\epsilon = 0$ at time $t^* = \frac{\pi}{12}$, Again it is compatible with other definitions and previous results, which is what we expected. If we compare scrambling time for three qubits system, it is $\frac{2}{3}$ of scrambling time for two qubits system for the same order of scrambling. Do not prejudge that fastest scrambling time should decrease as the number of qubits increase. In [1], they used Lieb-Robinson techniques to prove a logarithmic lower band on the scrambling time of systems with finite norm terms in Hamiltonian, so this logarithmic band should hold. We will show for the order of scrambling $\delta = \frac{n}{4}$, fastest scrambling time of system will decrease as the number of qubits increase, but for $\delta = 0$ the aforementioned logarithmic band holds.

Find fastest scramblers for three and two qubits are easy to follow, but for four qubits not only it requires to check more initial states in the scrambling condition, but also the number of bipartitions increases, this means numerical methods to search for fastest scrambler is inefficient, and we have to use analytic methods. Toward this objective, we will use the fastest scrambler for three and two qubits to come up with a great diagram that provides the fastest scrambler for n qubits to the order of $\delta \simeq \frac{n}{4}$. For these fastest scramblers at scrambling times any computational basis evolves to symmetric maximally entangled states which are very useful in quantum information. This diagram can be used to find fastest scrambler of small systems where $\delta = \frac{n}{4}$ is small, remember we are looking for a small order of scrambling δ. We will take another approach for finding the fastest scrambler to the order of $\delta = 1$ in next section. It is important to follow this diagram because the desired states of this diagram are maximally symmetric entangled

states, which are important in quantum information and gives us a profound insight, which we can take advantage of in next chapter.

3.3 The diagram

As the number of qubits exceeds three, bipartition A_l can have $|A_l| = m > 1$ qubits which add complexity to the problem. In the case of two and three qubits it is sufficient to have each qubit highly entangled with its complement at time t^*; because there was no bipartition such that $|A_l| > 1$. In the case of $n \geq 4$ not only we need each qubit to get highly entangled with its complement but also each bipartition of two, three, ... , and $\frac{n}{2}$ qubits should get highly entangled with its complement at time t^* for all initial states. In the case of two and three qubits reduced density matrices at scrambling time, when $Sin^2(at) = Cos^2(at)$, is proportional to a two-dimensional Identity matrix, that is why in the definition of scrambling based on distance $\forall A_l, \psi, \phi : ||\rho_\psi^{A_l}(t^*) - \rho_\phi^{A_l}(t^*)|| = 0$, but according to [15] in the case of four qubits it is not possible to get all reduced density matrices proportional to Identity simultaneously. The argument is that the system evolves from a computational state which is a pure state to the desired state at time t which is another pure state, but there is no pure stat for four qubits, such that

$$\forall l : \rho_\psi^{A_l} = \frac{1}{2^{m_l}} I$$

. Because of this, it seems hopeless to search for fastest scrambler to the order of $\epsilon = 0$ using the definition of scrambling based on distance. As mentioned at the limit of small numbers both ϵ and δ are equivalent, so It also seems hopeless to search for fastest scrambler to the order of $\delta = 0$ using the definition of scrambling based on entanglement entropy. This means there is no desired pure state for $n \geq 4$ such that entanglement entropy of all possible bipartitions, A_l, be $|A_l| = m$. We have to look for other reasonable desired states. We found it useful to select maximally symmetric entangled states as the desired states. We will come up with a diagram that produces Hamiltonian, which evolves any initial computational basis into a maximally entangled state. We will show for large number of qubits, this Hamiltonian scramble the system to the order of $\frac{n}{4}$, and scrambling time for that is $t^* \frac{\pi}{4} \sqrt{\frac{n-1}{n^2}}$. As mentioned earlier this order of scrambling is not what we are looking for, so proportionality of scrambling time with the inverse of $\sqrt{n}$ is not surprising.

We have learned from fastest scrambler of two and three qubits that Pauli expansion of fastest scrambler consist of merely spin flip Pauli matrices, this also should be the case here. Without loosing generality we decide to use $\sigma_1 = \sigma_x$ and $\sigma_0 = I$ as the components of spin flip Pauli basis.

$$\Xi = \{\Xi_{0...00}, \Xi_{0...01}, \Xi_{0...10}, \Xi_{0...11}, ..., \Xi_{1...111}\}$$

Ξ_k is an spin flip operator so if it acts on a computational basis it changes into a pure state which is again a member of computational basis Consider two operator Ξ_j, Ξ_k act on computational basis respectively, $\Xi_{j,k} \equiv \Xi_j \Xi_k$, again the computational basis changes into another computational basis under the action of $\Xi_{j,k}$. Considering that Ξ is a complete set in the sense that it consist of all the members that we need to change any arbitrary computational basis $|\psi\rangle$ into any arbitrary computational basis $|\psi'\rangle$, $|\psi'\rangle = \Xi_k|\psi\rangle$ we can conclude $\Xi_{j,k}$ is also a member of Ξ, in other word Ξ is close under product of two members,$\Xi_{j,k} = \Xi_k \Xi_j \in \Xi$. This properties of Ξ can be easily investigated, to do so assume Ξ_k is a spin flip operator that flip state of l^{th}, m^{th}, and n^{th} qubit, and Ξ_j is a spin flip operator that spin flip state of n^{th} and p^{th} qubits. Action of both of these together spin flip state of l^{th}, m^{th}, p^{th} qubit once, and spin flip state of n^{th} qubit two times, which is equivalent to not changing state of n^{th} qubit at all. So $\Xi_{j,k}$ spin flip states of l^{th}, m^{th}, and p^{th} qubit, for example $\Xi_{01001}\Xi_{11000} = \Xi_{10001}$.

This properties of Ξ_k results in a very simple approach for evolving the system from computational basis to desired states, which is symmetric maximally entangled states. Because $[\Xi_l, \Xi_k] = 0$, we can use 2.6 to write unitary operator, combining 2.6 with the fact that Ξ is close under product we can have

$$U(t) = \prod_{j=1}^{2^n} Cos(\alpha_j t)I - iSin(\alpha_j t)\Xi_j = \sum \sum_{j=1}^{2^n} \beta_j(t)\Xi_j$$

We are going to engineer the Hamiltonian such that when the unitary operator act on initial computational basis, we get symmetric maximally entangled states. Look closer to Pauli expansion of unitary operator, it is a product of 2^n parts, $U(t) = \prod_{j=1}^{2^n} Cos(\alpha_j t)I - iSin(\alpha_j t)\Xi_j$. Because $[\Xi_j, \Xi_k] = 0$ we can simply write

$$U(t) = \prod_{j=1, j\neq k}^{2^n} \left(Cos(\alpha_j t)I - iSin(\alpha_j t)\Xi_j\right)\left(Cos(\alpha_k t)I - iSin(\alpha_k t)\Xi_k\right)$$

Consider the case that $\alpha_k = 0$ then $Cos(\alpha_k t) = 1$ and $Sin(\alpha_k t) = 0$, for this case any pure state will evolve to itself, If there are f_n nonzero α_k, then we can write unitary operator as

$$U(t) = \prod_{j=1}^{f_n} Cos(\alpha_j t)I - iSin(\alpha_j t)\Xi_j$$

where j in Ξ_j does not show decimal representation of a binary sequence, but it shows that Ξ_j is a member of Ξ. Now consider $\alpha_k \neq 0$, action of $Cos(\alpha_k t)I - iSin(\alpha_k t)\Xi_k$ on $|\psi\rangle$ is

$$|\psi_k\rangle = (Cos(\alpha_k t)I - iSin(\alpha_k t)\Xi_k)\,|\psi\rangle = Cos(\alpha_k t)|\psi\rangle - iSin(\alpha_k t)\Xi_k|\psi\rangle$$

It produced a linear combination of untouched initial pure state through $Cos(\alpha_k t)I$ and spin flipped initial pure state through $-iSin(\alpha_k t)\Xi_k$. The action of $Cos(\alpha_k t)I - iSin(\alpha_k t)\Xi_k$ on one computational basis results in a linear combination of two computational basis. With the same approach one can act next term in product form of $U(t)$, $Cos(\alpha_{k'} t)I - iSin(\alpha_{k'} t)\Xi_{k'}$, on $|\psi_k\rangle$ and find

$$|\psi_{k'}\rangle = (Cos(\alpha_{k'} t)Cos(\alpha_k t)I - iSin(\alpha_{k'} t)Cos(\alpha_k t)\Xi_{k'}$$

$$-iCos(\alpha_{k'} t)Sin(\alpha_k t)\Xi_k - Sin(\alpha_{k'} t)Sin(\alpha_k t)\Xi_{k'}\Xi_k)|\psi\rangle$$

if action of $|\psi\rangle \neq \Xi_k|\psi\rangle \neq \Xi_{k'}|\psi\rangle \neq \Xi_k\Xi_{k'}|\psi\rangle$ on $|\psi_k\rangle$ then we will get four computational basis. Keep doing this approach for all f_n terms to find $|\psi(t)\rangle$, maximum number of computational basis in $|\psi(t)\rangle$ can be 2^{f_n}. We demand $|\psi(t)\rangle$ to be symmetric maximally entangled states. Examples of symmetric maximally entangled states for two, three, four, and five qubits are

$$|M_2\rangle = \frac{1}{\sqrt{2}}\,(|00\rangle + |11\rangle)$$

$$|M_3\rangle = \frac{1}{\sqrt{2}}\,(|000\rangle + |111\rangle)$$

$$|M_4\rangle = \frac{1}{\sqrt{8}}\,(|0000\rangle - |0011\rangle + |1100\rangle + |1111\rangle + |0101\rangle - |0110\rangle + |1001\rangle + |1010\rangle)$$

$$|M_5\rangle = \frac{1}{\sqrt{8}}(|00000\rangle - |00011\rangle + |11000\rangle + |11011\rangle$$
$$+|01101\rangle - |01110\rangle + |10101\rangle + |10110\rangle)$$

We can look at these state as a linear combination of spin flip operator, which is a unitary operator U_{SME}, acted on $|0\rangle$. Other examples of these symmetric maximally entangled states can be obtained by flipping some of the qubits, for example if we flip the second qubit in one of the above states, like $|M_3\rangle$, we will get $|M_3'\rangle = \frac{1}{\sqrt{2}}(|010\rangle + |101\rangle)$ which is another symmetric maximally entangled state that have entanglement as same as $|M_3\rangle$. The fact that flipping some of qubits will again results into another symmetric maximally entangled state suggest action of unitary operator U_{SME} on any computational basis $|\psi\rangle$ results in a symmetric maximally entangled state. Considering this, the question can be solved by choosing one of these symmetric maximally entangled states, write it as a linear combination of spin flip operators acted on $|0\rangle$ to find the unitary operator U_{SME}, and find the Hamiltonian that produce such unitary operator based on what we just learned. For example M_4 is

$$|M_4\rangle = \frac{1}{\sqrt{8}}(\Xi_{0000} - \Xi_{0011} + \Xi_{1100} + \Xi_{1111} + \Xi_{0101} - \Xi_{0110} + \Xi_{1001} + \Xi_{1010})\,|0000\rangle$$

$$U_{SME} = \Xi_{0000} - \Xi_{0011} + \Xi_{1100} + \Xi_{1111} + \Xi_{0101} - \Xi_{0110} + \Xi_{1001} + \Xi_{1010}$$

compare this to what we learned in Pauli expansion of unitary operator to find the H which produce these eight terms. At least there must exist three non zero pauli basis in the Hamiltonian so we can get eight terms in the unitary operator, as we discussed in previous paragraph. Each $\Xi_j \neq \Xi_{0000}$ can be one of these Pauli basis of Hamiltonian, for example get Ξ_{0011} to be one pauli basis, $e^{-i\Xi_{0011}}$ consist of Ξ_{0000}, and Ξ_{0011}, choose the next pauli basis to be Ξ_{1100}, then $e^{-i\Xi_{1100}}e^{-i\Xi_{0011}}$ consist of Ξ_{0000}, Ξ_{0011}, Ξ_{1100}, and Ξ_{1111}. At the end consider Ξ_{1001} to be the last component, $e^{-i\Xi_{1001}}e^{-i\Xi_{1100}}e^{-i\Xi_{0011}}$ consist of Ξ_{0000}, Ξ_{0011}, Ξ_{1100}, Ξ_{1111}, Ξ_{1001}, Ξ_{1010}, Ξ_{0101}, and Ξ_{0110}, which is the unitary operator we are looking for. Although number of negative terms in $|M_4\rangle$ was two out of eight and here it is six out of eight, or some of these spin flip matrices are multiplied by imaginary number i, but again these changes will result in another symmetric maximally entangled state that have entanglement as same as $|M_4\rangle$. So the Hamiltonian $H = \alpha(\Xi_{1100} + \Xi_{0011} + \Xi_{1001})$ evolve any computational basis into Symmetric maximally entangled state. We will get maximally entangled state when $Sin(\alpha t) = Cos(\alpha t) = \frac{1}{\sqrt{2}}$ so the scrambling time for this Hamiltonian is $t^* = \frac{\pi}{4\alpha}$ using normalization of Hamiltonian, $\alpha = \frac{4}{\sqrt{3}}$, so $t^* = \frac{\pi\sqrt{3}}{16}$.

One can do the same analysis for any n to determine the Hamiltonian that evolves any computational basis into symmetric maximally entangled states, but here we came up with a diagram that does the same job as analysis and determine different terms in Pauli expansion of Hamiltonian that evolve any computational basis into symmetric maximally entangled states. Please remember that the Hamiltonian is time-independent, so the y axis of the diagram does not show any time evolution. The Hamiltonian H that we are looking for is in the form of

$$H = \alpha \sum_k \Xi_k \tag{3.10}$$

Where α is the normalization constant, and Ξ_k is a spin flip operator that flip the state of only two or three qubits. The diagram can specify the qubits that it should act on. General rules to produce the diagram is as fallowing

- start with $H = 0$ and draw n points, which represents n qubits.
- Divide qubits into pairs and have two body interactions between two qubits of each pair added to Hamiltonian H, if one qubit remained unpaired then combine it with one of the pairs and have a three body interaction added.
- Divide previous pairs of qubits into different sets of two pairs (quads). Again have a two-body interaction between one qubit of each pair with one qubit of the other pair in the same quad added to Hamiltonian H, if one pair doesn't fit in a quad then combine it with one of the quads and have a three-body interaction.
- Keep pairing previous divisions, and do the same until everything gets connected with two or three body interactions.

For example a diagram for $n = 15$ has been shown in figure 3.3. In this diagram each cup represents one Ξ_k, as mentioned Ξ_k is a two or three spin flip Pauli matrix, the dots under legs of each cup determine which qubits should flip, for example in figure 3.3 second cup from left in the second row represents $\Xi_{000001100000000}$ and the last row is $\Xi_{001001000100000}$. The Hamiltonian provided by this diagram evolve any computational basis into symmetric maximally entangled states, to have these states at time t^* we demand $Cos(\alpha t^*) = Sin(\alpha t^*) = \frac{1}{\sqrt{2}}$ so $t^* = \frac{\pi}{4\alpha}$. We can obtain the constant

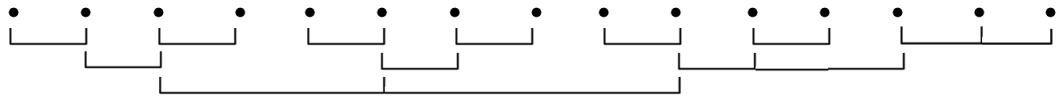

Figure 3.5: Fast scrambler Pauli basis

α using normalization of Hamiltonian,

$$\frac{trH^2}{2^n} = \frac{tr\left(\sum_{k=1}^{f_n}\sum_{l=1}^{f_n}\alpha^2\Xi_k\Xi_l\right)}{2^n} = \frac{tr\left(\sum_{k=1}^{f_n}\sum_{l=1}^{f_n}(1-\delta_k^l)\alpha^2\Xi_k\Xi_l + \delta_k^l\alpha^2\Xi_k\Xi_l\right)}{2^n}$$

$tr\left((1-\delta_k^l)\Xi_k\Xi_l\right) = 0$ so

$$\frac{trH^2}{2^n} = \frac{\sum_{k=1}^{f_n}\alpha^2 tr\left(\Xi_k\Xi_k\right)}{2^n} = \frac{f_n\alpha^2 tr\left(\Xi_{0\ldots00}\right)}{2^n} = f_n\alpha^2$$

Using Normalization condition (2.4), $f_n\alpha^2 = n^2$ which means $\alpha = \frac{n}{\sqrt{f_n}}$. To find f_n for the diagram add number of terms in each row of diagram, for $n = 2^a$ where a is a natural number $\frac{n}{2} + \frac{n}{4} + ... + 1 = n - 1$. Substituting f_n and α in t^* results in $t^* = \frac{\pi\sqrt{n-1}}{4n}$. We conjecture that the Hamiltonian which is produced with the diagram should be the fastest scrambler up to the order of scrambling $\delta = \frac{n}{4}$, which we found numerically. For large systems this δ is not a small order of scrambling that we are looking for, so its scrambling time does not represent trend of scrambling time for fastest scrambler. We are looking for $\delta \simeq 1$ and its scrambling time should be logarithmic in number of qubits according to [1], who used Lieb-Robinson techniques to prove a logarithmic lower band on the scrambling time of systems with finite norm terms in Hamiltonian, In next chapter we will provide another approach to come up with a scrambler for large systems based on probability theory that scramble in logarithmic time.

3.3.1 Fastest scrambler for four and five qubits

Now that we have the diagram that can provide the fastest scrambler for small systems, we can go through four and five qubits systems. All we have to do is to generate

the Hamiltonian through the diagram, and act it on the computational basis at time $t^* = \frac{\pi}{4}\frac{\sqrt{f_n}}{n}$ to get maximally symmetric entangled states. For four-qubit system

$$H = \alpha\left(\Xi_{1100} + \Xi_{0011} + \Xi_{0110}\right)$$

Using normalization of Hamiltonian that we solved in previous section, $\alpha = \frac{4}{sqrt3}$. The Unitary operator of this Hamiltonian is

$$U(t) = e^{-iHt} = e^{-i\alpha(\Xi_{1100}+\Xi_{0011}+\Xi_{0110})t} = e^{-i\alpha\Xi_{1100}t}e^{-i\alpha\Xi_{0011}t}e^{-i\alpha\Xi_{0110}t}$$

$$= \left(Cos(\alpha t)\Xi_{0000} - iSin(\alpha t)\Xi_{1100}\right)\left(Cos(\alpha t)\Xi_{0000} - iSin(\alpha t)\Xi_{0011}\right)$$

$$\left(Cos(\alpha t)\Xi_{0000} - iSin(\alpha t)\Xi_{0110}\right)$$

$$U(t) = Cos^3(\alpha t)\Xi_{0000} - iCos^2(\alpha t)Sin(\alpha t)\sigma_{1100} - iCos^2(\alpha t)Sin(\alpha t)\Xi_{0011}$$

$$-iCos^2(\alpha t)Sin(\alpha t)\Xi_{0110} - Cos(\alpha t)Sin^2(\alpha t)\Xi_{1111} - Cos(\alpha t)Sin^2(\alpha t)\Xi_{1010}$$

$$-Cos(\alpha t)Sin^2(\alpha t)\Xi_{0101} + iSin^3(\alpha t)\Xi_{1001}$$

As you can see $H \propto \Xi_{1100}+\Xi_{0011}+\Xi_{0110}$ produced a linear combination of all spin flip operators that we need to evolve initial states from computational basis into symmetric maximally entangled state at time $t^* = \frac{\pi\sqrt{3}}{16}$. A closer investigation of these spin flip operators shows not only we have all required operators to evolve the system into the symmetric maximally entangled state, but also the number of spin flip operators with negative coefficient is six out of eight. The result is compatible with the symmetric maximally entangled state of four qubits multiplied by a minus sign. We used our program to search for a faster scrambler to the order of $\delta = 1$ but found no Hamiltonian to do so, in fact scrambling time of those random Hamiltonian was much larger than$t^* = \frac{\pi\sqrt{3}}{16}$, this Hamiltonian should be the fastest scrambler to the order of $\delta = 1$.

To investigate another example consider $n = 5$ qubits

$$H = \alpha\left(\sigma_{00110} + \sigma_{01001} + \sigma_{10011}\right)$$

Using normalization of Hamiltonian $\alpha = \frac{5}{\sqrt{3}}$. The unitary operator of H is

$$U(t) = e^{-iHt} = e^{-i\alpha(\Xi_{00110}+\Xi_{01001}+\Xi_{10011})t} = e^{-i\alpha\Xi_{00110}t}e^{-i\alpha\Xi_{01001}t}e^{-i\alpha\Xi_{10011}t}$$

$$= (Cos(\alpha t)\Xi_{0000} - iSin(\alpha t)\Xi_{00110})\,(Cos(\alpha t)\Xi_{0000} - iSin(\alpha t)\Xi_{01001})$$
$$(Cos(\alpha t)\Xi_{0000} - iSin(\alpha t)\Xi_{10011})$$

$$U(t) = Cos^3(\alpha t)\Xi_{0000} - iCos^2(\alpha t)Sin(\alpha t)\Xi_{00110} - iCos^2(\alpha t)Sin(\alpha t)\Xi_{01001}$$
$$-iCos^2(\alpha t)Sin(\alpha t)\Xi_{10011} - Cos(\alpha t)Sin^2(\alpha t)\Xi_{01111} - Cos(\alpha t)Sin^2(\alpha t)\Xi_{10101}$$
$$-Cos(\alpha t)Sin^2(\alpha t)\Xi_{11010} + iSin^3(\alpha t)\Xi_{11100}$$

At time $t^* = \frac{\sqrt{3}\pi}{20}$ the system evolve to symmetric maximally entangled state for five qubits which has been mentioned in [12]. Again we used our program to search for a faster scrambler but found nothing. This Hamiltonian is the fastest scrambler to the order of $\delta = 1$.

$\delta = \frac{n}{4}$ is the limit of the order of scrambling for large systems scrambled by the Hamiltonian produced by the diagram. Because δ is large for large systems, we come up with a new approach using probability theory for finding the fastest scrambler of a large system. In next chapter, we will show that this method results in the Hamiltonians that scramble system to the order of $\delta = \frac{1}{2}$ and the scrambling time of those Hamiltonians is logarithmic in the size of the system, n. So according to Lashkari et al. [1] they are the fastest scrambler.

Chapter 4

Fast Scrambler for large n

4.1 Probabilty theory and the fastest scrambler

As mentioned earlier in chapter 3 the order of scrambling for the fastest scramblers that we found using the diagram is large $\delta = \frac{n}{4}$, we are looking for the order of scrambling smaller than $\delta = 1$. Our approach for finding the fastest scrambler to the small order of scrambling uses probability theory. The probability theory is a powerful tool that can be used along with our results of the fastest scrambler for small systems to find the fastest scrambler for large systems. In this section, we provide a random Hamiltonian H, that scramble the system to the order of $\delta = \frac{1}{2}$ in logarithmic time. Lashkari et al. [1] used Lieb-Robinson techniques to prove a logarithmic lower band on the scrambling time of systems with finite norm terms in Hamiltonian, this shows the Hamiltonian H is the fastest scrambler.

Again the same as scrambling for small systems we can demand initial states evolve to specific pure states, in order to have desired order of scrambling. Recall that we can think about a pure state as a vector pointing on the unit sphere $S^{2^{n+1}-1}$, a Haar random pure state is a random state that has been chosen from a uniform distribution on the $S^{2^{n+1}-1}$. The aforementioned desired state is a Haar random state, if a random unitary Hamiltonian evolution evolves any arbitrary computational basis, which is the initial state, to a Haar random state at time $t = t^*$ the system scrambles to the order of $\delta = \frac{1}{2}$ at this time. To understand why a Haar random state satisfy scrambling condition to the order of half imagine a random unitary Hamiltonian evolution of the system using

a random Hamiltonian H. Any computational basis evolve to a random state through this evolution, scrambling condition 2.3 for this random unitary Hamiltonian evolution is

$$m - \frac{1}{2^n} \sum_{\psi} \langle S_{\psi}^{A_l}(t^*) \rangle < \delta \tag{4.1}$$

If the unitary Hamiltonian evolution evolve any initial state into a Haar random state at time t^*, then $\langle S_{\psi}^{A_l}(t^*) \rangle = \langle S^m \rangle$, where $\langle S^m \rangle$ is the average entropy of subsystem A_l of dimension m over Haar random distribution. Substituting this in equation 4.1 results in the fallowing scrambling condition

$$m - \langle S^m \rangle < \delta \tag{4.2}$$

The average entanglement entropy over Haar random states, $\langle S^m \rangle$, between arbitrary subsystem A and its complement B has been calculated in [13] by Don Page, he showed for large systems of n qubits the average entanglement entropy between arbitrary subsystem A with m qubits and its complement B, where $m \leq \frac{n}{2}$, is $\langle S^m \rangle = m - 2^{2m-n-1}$ [13], so $m - \langle S^m \rangle = 2^{2m-n-1}$. Maximum of 2^{2m-n-1} is half and corresponds to $m = \frac{n}{2}$, thus on average for a Haar random state entanglement entropy of all bi-partitions will be within half distance of m, $\forall A : m - \langle S^A \rangle < \frac{1}{2}$. Matching this result with the scrambling condition that we obtained for Haar random states 4.2 shows a random unitary Hamiltonian evolution of the system that evolves any initial computational basis to a Haar random pure state at time $t = t^*$, scramble the system to the order of $\delta = \frac{1}{2}$.

An important point is that the random unitary Hamiltonian evolution must Haar randomize not only some but all initial states at time $t = t^*$. We will assume the anzats obtained from scrambling for small systems also give us the fastest scrambler for large systems, for this anzats we can show all initial computational basis get Haar randomized at scrambling time $t = t^*$. The anzats is that Pauli expansion of fastest scrambler is a linear combination of spin flip pauli matrices, we chose to work with the complete set of spin flip Pauli matrices $\Xi = \{\Xi_{0...00}, \Xi_{0...01}, ..., \Xi_{1...11}\}$, this set is complete in the sense that action of all 2^n members of this set on an arbitrary computational basis results in a complete set of all 2^n computational basis, so $H = \sum_{i=1}^{2^n} \alpha_i \Xi_i$, where α_i is a real number. A rigorous calculation of scrambling time in section 4.3 shows that more $\alpha_i \neq 0$ in the Hamiltonian results in the longer scrambling time, lets have variable f_n to

determine number of non zero α_i in Hamiltonian, it is essential to have minimum f_n to get the fastest scrambler. Assume the probability distribution of having a Pauli basis in the Hamiltonian is uniform, choose f_n Pauli bases Ξ^i using a uniform distribution and choose f_n real random number α_i to produce the Hamiltonian, so

$$H = \sum_{i=1}^{f_n} = \alpha_i \Xi^i$$

please note that we used a superscript i for Ξ^i, this superscript shows i is not a decimal representation of a binary sequence but it is merely the i^{th} random spin flip Pauli basis chosen from a uniform distribution on the set Ξ. When ever we want to represent a decimal representation for a member of Ξ we will use subscript Ξ_i, while if we want to work with i^{th} random Pauli basis we will choose superscript Ξ^i. Using equation 2.6 and $[\Xi_j, \Xi_k] = 0$ the product expansion of unitary operator is

$$U(t) = \prod_k \Big(Cos(\alpha_k t)\Xi_0 - iSin(\alpha_k t)\Xi^k \Big)$$

remember Ξ is close under product, $\Xi_j \Xi_k = \Xi_l$, using this property and assuming there exist atleast n spin flip Pauli basis in the Hamiltonian such that $\prod_{m=1}^{n} (I + \Xi^m)$ produces a linear combination of all 2^n spin flip Pauli bases, we can write the unitary operator as

$$U(t) = \prod_{k=1}^{f_n} Cos(\alpha_j t)I - iSin(\alpha_k t)\Xi^k = \sum_{j=1}^{2^n} \beta_j(t)\Xi_j$$

As mentioned Ξ is a complete set in the sense that action of all members of this set on an arbitrary computational basis results in a complete set of computational basis, so action of $U(t)$, which is a linear combination of all spin flip pauli basis, on different computational basis merely match $\beta_j(t)$ with different computational basis. For example in a system of two qubits $U(t) = \beta_0(t)\Xi_{00} + \beta_1(t)\Xi_{01} + \beta_2(t)\Xi_{10} + \beta_3(t)\Xi_{11}$ then

$$U(t)|00\rangle = \beta_0(t)|00\rangle + \beta_1(t)|01\rangle + \beta_2(t)|10\rangle + \beta_3(t)|11\rangle$$

$$U(t)|01\rangle = \beta_0(t)|01\rangle + \beta_1(t)|00\rangle + \beta_2(t)|11\rangle + \beta_3(t)|10\rangle$$

$$U(t)|10\rangle = \beta_0(t)|10\rangle + \beta_1(t)|11\rangle + \beta_2(t)|00\rangle + \beta_3(t)|01\rangle$$

$$U(t)|11\rangle = \beta_0(t)|11\rangle + \beta_1(t)|10\rangle + \beta_2(t)|01\rangle + \beta_3(t)|00\rangle$$

this shows if a random Hamiltonian results in a Haar random unitary operator at time $t = t^*$ for only one of the computational basis, then not only one but all computational basis evolve to a Haar random state at time $t = t^*$.

To determine $\beta_j(t)$ one can directly product all terms in $\prod_{j=1}^{f_n} Cos(\alpha_j t)I - iSin(\alpha_j t)\Xi_j$, but a more useful way of thinking is to have binary matrix J^j, corresponds to β_j such that its elements determine β_j. This matrix J^j will simplify our calculation because we will show what ever it is, it does not effect on the first moment of random states. Production of all terms in product expansion of unitary operator results in the fallowing $\beta_j(t)$

$$\beta_j(t) = \sum_{l=1}^{2^{f_n - n}} \prod_{m=1}^{f_n} \Big(J^j_{lm} Cos(\alpha_m t) - i(1 - J^j_{lm}) Sin(\alpha_m t) \Big) \tag{4.3}$$

where J^j_{lm} is an element of binary matrix J^j, It is called a binary matrix because each element can be either zero or one. As mentioned having exact matrix J^j is not important to match the first moment of random state at time t^* but it is essential for other moments. Now that we have $\beta_j(t)$ as a function of α_m we can find probability distribution for α such that first moment of $\beta_j(t^*)$ match the first moment of a Haar random state. Having appropriate probability distribution and using numerical methods we will find the minimum f_n such that the probability distribution of $\beta_j(t^*)$ be the Haar random distribution. At last we show that the Hamiltonian that produced by appropriate distribution of α minimum f_n scramble the system to the order of $\delta = \frac{1}{2}$ at scrambling time $t^* \propto log_2(n)$.

4.2 Distribution of α

The ultimate goal is to come up with a probability distribution of α such that it Haar randomize $\beta_j(t^*)$. Imposing constraints on the probability distribution of α by matching different moments of $\beta_j(t^*)$ with moments of Haar-random state assure probability distribution of α results in a Haar-random state at time t^*. Although it is possible to match the first moment of $\beta_j(t^*)$ with the first moment of a Haar-random state without knowing J_j, to match the higher moments we need J_j which is unknown unless we are aware of exact terms of Pauli basis that we use in the Hamiltonian. For example in the diagram that we produced we know the exact Pauli basis in Hamiltonian, they are spin

flip operators that flip the spin of only two or three qubits, so it is easy to obtain $\beta_j(t)$ for that diagram. While here using the probability method we are choosing Ξ^k from a uniform distribution, so it is not possible to determine J_j unless we know definite matrices Ξ^k. Although we have a lack of information about J_j which reduce maneuverability to match higher moments of $\beta_j(t^*)$ with Haar-random distribution, having variable f_n provide us sufficient flexibility to compensate the lack of knowledge about J_j. In this section, we find the probability distribution of α that matches the first moment of $\beta_j(t^*)$ with the first moment of Haar-random distribution. In next section, we will find the scrambling time as a function of probability distribution and f_n. In the end, using numerical methods, we will find the minimum number of terms in Hamiltonian, f_n, that Haar randomizes the system at scrambling time t^* and will show t^* is logarithmic in the number of qubits.

Product moments of Haar random distribution plays an important role, so it worth to rewrite them here. In the Fangs book called symmetric multivariate and related distributions [16] one can fallow steps of the fallowing results. If $Z = (z_1, z_2, ..., z_p)$ is Haar random vector then for any integers $\{k_1, k_2, ...k_p\}$ where $k = \sum_{i=1}^{p} k_i$, the product moment is given by

$$\langle \prod_{i=1}^{p} z_i^{k_i} \rangle = \begin{cases} 0 & if\ \exists\ k_i : k_i\ is\ odd \\ \frac{\Gamma(\frac{p}{2})}{2^k \Gamma(\frac{k+p}{2})} \prod_{i=1}^{p} \frac{k_i!}{(\frac{k_i}{2})!} & if\ \forall\ k_i : k_i\ is\ even \end{cases} \tag{4.4}$$

First moment of a Haar random state is $\langle z_i \rangle = 0$, so here we want to find a probability distribution for α such that $\langle \beta_j(t^*) rangle = 0$. In previous section we found $\beta_j(t) = \sum_{l=1}^{2^{f_n-n}} \prod_{m=1}^{f_n} \Big(J_{lm}^j Cos(\alpha_m t) - i(1 - J_{lm}^j) Sin(\alpha_m t) \Big)$, so

$$\langle \beta_j(t^*) \rangle = \langle \sum_{l=1}^{2^{f_n-n}} \prod_{m=1}^{f_n} J_{lm}^j Cos(\alpha_m t^*) - i(1 - J_{lm}^j) Sin(\alpha_m t^*) \rangle = 0$$

expectation value of summation is equal to sum of expectation values, $\langle a+b \rangle = \langle a \rangle + \langle b \rangle$, using this property we will get

$$\sum_{l=1}^{2^{f_n-n}} \langle \prod_{m=1}^{f_n} J_{lm}^j Cos(\alpha_m t^*) - i(1 - J_{lm}^j) Sin(\alpha_m t^*) \rangle = 0$$

α_m are independent variables chosen from the same probability distribution and for independent variables a and b, expectation value of a function of a, $f(a)$, multiply by

a function of b, $f(b)$, is equal to expectation value of $f(a)$ multiply by expectation value of $f(b)$, $\langle f(a)f(b)\rangle = \langle f(a)\rangle\langle f(b)\rangle$, this property of random variables along with $\langle f(a)+f(b)\rangle = \langle f(a)\rangle + \langle f(b)\rangle$ results in

$$\sum_{l=1}^{2^{fn-n}} \prod_{m=1}^{f_n} \langle j_{lm} Cos(\alpha_m t^*)\rangle - i\langle (1-j_{lm}) Sin(\alpha_m t^*)\rangle = 0$$

Please not j_{lm} is either zero or one, if $j_{lm}=0$ then $\langle 0Cos(\alpha_m t^*)\rangle = 0$ but $\langle 1Sin(\alpha_m t^*)\rangle$ is not necessarily zero, we demand $\langle 1Sin(\alpha_m t^*)\rangle$ to be zero, also if $j_{lm}=1$ then $\langle 0Sin(\alpha_m t^*)\rangle = 0$ but $\langle 1Cos(\alpha_m t^*)\rangle$ is not necessarily zero, we demand $\langle 1Cos(\alpha_m t^*)\rangle$ to be zero.

$$\langle Sin(\alpha t^*)\rangle = 0 \tag{4.5}$$

and

$$\langle Cos(\alpha t^*)\rangle = 0 \tag{4.6}$$

this is not the only way to have $\langle \beta_j(t^*)\rangle = 0$ but it is one of the ways that impose constrain on probability distribution of α. If any probability distribution of α satisfy both of these conditions then first moment of $\beta_j(t^*)$ matches with the first moment of Haar random distribution which is zero. In both equations 4.5, and 4.6 we are dealing with expectation value of a function of random variable α, this expectation value is the integral of the function multiply by probability distribution of the function over α. To relate probability distribution of these functions, $Cos(\alpha t^*)$ and $Sin(\alpha t^*)$, to probability distribution of α lets define $Y \equiv \alpha t^*$, $-\infty < \alpha < \infty \rightarrow -\infty < y < \infty$, $g(y)$ is probability distribution of y and $G(y)$ is the probability function of y. We can find the relation between probability distribution of y and probability distribution of α

$$g(y) = \frac{d}{dy}G(Y<y) = \frac{d}{dy}P(\alpha < \frac{y}{t^*})$$

$$= \frac{d}{dy}\int_{-\infty}^{\frac{y}{t^*}} p(\alpha)d\alpha = \frac{d}{dy}[P(\frac{y}{t^*}) - P(-\infty)] = \frac{1}{t^*}p(\frac{y}{t^*}) \tag{4.7}$$

Lets first find a condition on probability distribution of α such that equation (4.5) satisfies. We obtained the probability distribution for $y = \alpha t^*$, now we can find the probability distribution of $Sin(y) = Sin(\alpha t^*)$ having the probability distribution of y.

Lets define $W \equiv Sin(y)$, $-\infty < y < \infty \rightarrow -1 \leq W \leq 1$, choose $-\frac{\pi}{2} < Sin^{-1}(W) < \frac{\pi}{2}$. $f(w)$ is probability distribution of w and $F(w)$ is the probability function of w. To find the relation between probability distribution of w and probability distribution of y we have

$$f(w) = \frac{d}{dw}F(W < w) = \frac{d}{dw}G(y < sin^{-1}(w))$$

$$= \frac{d}{dw}\sum_{k=-\infty}^{+\infty} \int_{2k\pi+Sin^{-1}(w)}^{(2k+1)\pi-Sin^{-1}(w)} g(y)dy$$

$$= \sum_{k=-\infty}^{+\infty} \frac{d}{dw}[G((2k+1)\pi - Sin^{-1}(w)) - G(2k\pi + Sin^{-1}(w))]$$

$$= -\frac{1}{\sqrt{1-w^2}} \sum_{k=-\infty}^{+\infty} g((2k+1)\pi - Sin^{-1}(w)) + g(2k\pi + Sin^{-1}(w)) \qquad (4.8)$$

Having the relation between probability distribution of w and y and the relation probability distribution of y and α enable us to the constrain on distribution of α such that $\langle w \rangle = 0$. Demand condition 4.5 to be satisfied

$$\langle w \rangle = \int_{-\infty}^{+\infty} wf(w)dw = \int_{-\infty}^{0} wf(w) + \int_{0}^{+\infty} wf(w)dw = \int_{0}^{+\infty} w(f(w) - f(-w))dw = 0$$

Above condition will satisfy if $f(w) - f(-w) = 0$, We have the relation between probability distribution of w and probability distribution of y, using this relation we have

$$\sum_{k=-\infty}^{+\infty} g((2k+1)\pi - Sin^{-1}(w)) + g(2k\pi + Sin^{-1}(w))$$

$$-g((2k+1)\pi - Sin^{-1}(-w)) - g(2k\pi + Sin^{-1}(-w)) = 0$$

k is a free parameter summed over $-\infty$ to $+\infty$, so we can change $2k+1$ to $-(2k+1)$ in the first component and $2k$ to $-2k$ in the last component of summation, also considering $Sin^{-1}(-w) = -Sin^{-1}(w)$

$$\sum_{k=-\infty}^{+\infty} g(-(2k+1)\pi - Sin^{-1}(w)) - g((2k+1)\pi + Sin^{-1}(w))$$

$$+g(2k\pi + Sin^{-1}(w)) - g(-2k\pi - Sin^{-1}(w)) = 0$$

Above condition can be satisfied when $g(y) - g(-y) = 0$, so we found the condition on $g(y)$ such that $\langle Sin(\alpha t^*)\rangle = 0$, again using the same method we can find the condition on $p(\alpha)$ so $\rangle Sin(\alpha t^*)\langle = 0$. We have the relation between probability distribution of y and probability distribution of α, using this relation we can find the constrain on $p(\alpha)$ such that $\rangle Sin(\alpha t^*)\langle = 0$. $\frac{1}{t^*}p(\alpha) - \frac{1}{t^*}p(-\alpha) = 0$ so $p(\alpha)$ also should be even to satisfy 4.5, $p(\alpha) - p(-\alpha) = 0$.

We found any even probability distribution satisfies $\langle Sin(\alpha t^*)\rangle = 0$ 4.5, now we want to put more constrain on $p(\alpha)$ such that the other condition, $\langle \cos(\alpha t^*)\rangle = 0$ 4.6 also satisfies. We already defined $y = \alpha t^*$ and found the relation between probability distribution of y and α, lets define $Z \equiv Cos(y)$, $-\infty < y < \infty \rightarrow -1 \leq Z \leq 1$, we need inverse of Z so we choose $0 < Cos^{-1}(Z) < \pi$. $h(z)$ is probability distribution of z and $H(z)$ is the probability function of z.

$$h(z) = \frac{d}{dz}H(Z < z) = \frac{d}{dz}G(y < Cos^{-1}(z))$$

$$= \frac{d}{dz}\sum_{k=-\infty}^{+\infty} \int_{2k\pi + Cos^{-1}(z)}^{(2k+2)\pi - Cos^{-1}(z)} g(y)dy =$$

$$\sum_{k=-\infty}^{+\infty} \frac{d}{dz}[G((2k+2)\pi - Cos^{-1}(z)) - G(2k\pi + Cos^{-1}(z))]$$

$$= \frac{1}{\sqrt{1-z^2}} \sum_{k=-\infty}^{+\infty} g((2k+2)\pi - Cos^{-1}(z)) + g(2k\pi + Cos^{-1}(z)) \tag{4.9}$$

k is a free parameter summed over $-\infty$ to $+\infty$, so we can change $2k+2$ with $-2k$ in the first component of summation, also recall that we found $g(y)$ should be even, so $g(-2k\pi - Cos^{-1}(z)) = g(2k\pi + Cos^{-1}(z))$, using these the relation between probability distribution of z and y simplifies to

$$h(z) = \frac{2}{\sqrt{1-z^2}} \sum_{k=-\infty}^{+\infty} g(2k\pi + Cos^{-1}(z)) \tag{4.10}$$

Having the relation between probability distribution of z and y and the relation probability distribution of y and α enable us to implement more constrains on the distribution of α such that $\langle z\rangle = 0$. Lets demand condition 4.6, $\langle z\rangle = 0$, to be satisfied, z is an odd

function, so $h(z)$ should be even to have $\int_{-\infty}^{+\infty} zh(z) = 0$, $h(z) - h(-z) = 0$. We have the relation between probability distribution of z and probability distribution of y 4.10, using this relation and the constrain on $h(z)$, we can find the constrain on $g(y)$.

$$\sum_{k=-\infty}^{+\infty} g(2k\pi + Cos^{-1}(z)) - g(2k\pi + Cos^{-1}(-z)) = 0$$

Recall $Cos^{-1}(-z) = \pi - Cos^{-1}(z)$, also again as k is a free parameter summed over $-\infty$ to $+\infty$, we can change $(2k+1)$ to $-(2k+1)$, remember $g(y)$ is an even function $g(y) = g(-y)$, all these simplify above condition such that

$$\sum_{k=-\infty}^{+\infty} g\left(\frac{2k\pi + Cos^{-1}(z)}{t^*}\right) - g\left(\frac{2k\pi + (\pi + Cos^{-1}(z))}{t^*}\right) = 0$$

again using the same method we can find the condition on $p(\alpha)$ so $\rangle Cos(\alpha t^*)\langle = 0$. We have the relation between probability distribution of y and probability distribution of α, using this relation

$$\frac{1}{t^*}\sum_{k=-\infty}^{+\infty} p\left(\frac{2k\pi + Cos^{-1}(z)}{t^*}\right) - p\left(\frac{2k\pi + (\pi + Cos^{-1}(z))}{t^*}\right) = 0 \qquad (4.11)$$

Divide domain of $P(\alpha)$ into equal parts of $\frac{2\pi}{t^*}$ and label the divisions by $k \in \{0, \pm 1, ...\}$, these divisions are exactly the same divisions that we summed over in the above equation. Write a Fourier series expansion of $P(\alpha)$ for each division, which has been labeled by k, $P(\alpha)$ is an even function so Fourier series expansion of $p(\frac{2k\pi}{t^*} + x)$, where $0 \leq x \leq \frac{2\pi}{t^*}$ is

$$p(\frac{2k\pi}{t^*} + x) = \frac{a_0^k}{2} + \sum_{m=1}^{\infty} a_m^k Cos(mt^* x) \quad , \quad a_m^k = \frac{t^*}{2\pi} \int_{\frac{2\pi k}{t^*}}^{\frac{2\pi(k+1)}{t^*}} Cos(mt^* x) p\,(x)\, dx \qquad (4.12)$$

Recall we defined Cos^{-1} to be between 0 and π, so the arguments of $p(\alpha)$ in equation 4.11satisfy these conditions $\frac{2\pi k}{t^*} \leq \frac{2k\pi + Cos^{-1}(z)}{t^*} \leq \frac{2\pi(k+1)}{t^*}$, and also $\frac{2\pi k}{t^*} \leq \frac{2k\pi + (\pi + Cos^{-1}(z))}{t^*} \leq \frac{2\pi(k+1)}{t^*}$. Because both arguments are in the same domain that we choose for Fourier series expansion, we will have

$$\sum_{k=-\infty}^{+\infty} \left[\frac{a_0^k}{2} + \sum_{m=1}^{\infty} a_m^k Cos(mt^* \frac{Cos^{-1}(z)}{t^*}) - \frac{a_0^k}{2} - \sum_{m=1}^{\infty} a_m^k Cos(mt^* \frac{\pi + Cos^{-1}(z)}{t^*})\right] = 0$$

$$\sum_{k=-\infty}^{+\infty}\sum_{m=1}^{\infty} a_m^k \left[Cos(mCos^{-1}(z)) - Cos(m\pi + mCos^{-1}(z))\right] = 0$$

$$\sum_{k=-\infty}^{+\infty}\sum_{m=1}^{\infty} -2a_m^k Sin(\frac{m(2Cos^{-1}(z)+\pi)}{2})Sin(\frac{m\pi}{2}) = 0$$

One of the ways that this summation be zero is to have each term equal to zero. For even m, $Sin(\frac{m\pi}{2}) = 0$ so a_m^k does not have to be zero, but for odd m $Sin(\frac{m\pi}{2}) = 1$, so a_m^k should be zero. So (4.6) is satisfied if Fourier expansion of $p(\frac{2k\pi}{t^*} + x)$ be a summation over even m.

$$p(\frac{2k\pi}{t^*} + x) = \frac{a_0^k}{2} + \sum_{m=1}^{\infty} a_{2m}^k Cos(2t^* mx) \tag{4.13}$$

In conclusion any even probability distribution $P(\alpha)$ that satisfies equation 4.13, results in $\langle Cos(\alpha t^*)\rangle = 0$ and $\langle Sin(\alpha t^*)\rangle = 0$. Recall these conditions promise to match the first moment of state at time t^* with the first moment of Haar-random state. For our objective, it is essential to match higher product moments of random state with the similar moment of Haar-random distribution. We do not know exact binary matrices J^j because Pauli basis is random, so we do not know the exact relation of $\beta_j(t^*)$ and $\alpha_k(t^*)$ which means we can not match other moments analytically.

To overcome this problem, instead of analytic approach we take a numerical approach. We choose f_n random Pauli basis using a uniform distribution, and f_n real number α_k using $P(\alpha)$. We find scrambling time t^* (which is a function of probability distribution $P(\alpha)$, number of terms in Hamiltonian f_n, and size of the system), and evolve initial state to final state at time t^*. Then we check if the higher moments of final state matches with similar moments of Haar-random distribution at time t^*. We will keep changing f_n and repeat the aforementioned process until we get a Haar-random state. An important block of the numerical analysis is to find t^* analytically so we can determine the final state. Next section is dedicated to this purpose, using the result of next section in the last section we can find the minimum number of terms in Hamiltonian f_n that Haar randomize any initial state at time t^*. We show that t^* is logarithmic in the number of qubits for minimum f_n. According to Lashkari et al. [1] there is a logarithmic lower band on the scrambling time of systems with finite norm terms in Hamiltonian, this shows our Hamiltonian is the fastest scrambler.

4.3 Scrambling time for a special $P(\alpha)$

As mentioned having a probability distribution that matches first moment of random state with Haar random distribution at time t^* does not promise to give us a Haar random distribution for any arbitrary f_n; to find minimum f_n that provide us with a Haar random state we can numerically vary it until we get a Haar random distribution at scrambling time t^*. To fallow this approach we need scrambling time t^*, which can be obtained by using Hamiltonian normalization. Recall $H = \sum_{k=1}^{f_n} \alpha_k \Xi_k$, normalization of this Hamiltonian results in

$$\frac{trH^2}{2^n} = \frac{tr\left(\sum_{k=1}^{f_n}\sum_{l=1}^{f_n}\alpha_k\alpha_l\Xi_k\Xi_l\right)}{2^n} = \frac{tr\left(\sum_{k=1}^{f_n}\sum_{l=1}^{f_n}(1-\delta_k^l)\alpha_k\alpha_l\Xi_k\Xi_l + \delta_k^l\alpha_k\alpha_l\Xi_k\Xi_l\right)}{2^n}$$

$$= \frac{tr\left(\sum_{k=1}^{f_n}\alpha_k^2\Xi_{0\ldots00} + \sum_{k=1}^{f_n}\sum_{l=1}^{f_n}(1-\delta_k^l)\alpha_k\alpha_l\Xi_k\Xi_l\right)}{2^n}$$

Because we are working with spin flip Pauli basis $tr\left((1-\delta_k^l)\Xi_k\Xi_l\right) = 0$ so

$$\frac{\sum_{k=1}^{f_n}\alpha_k^2 tr\left(\Xi_{0\ldots00}\right)}{2^n} = \sum_{k=1}^{f_n}\alpha_k^2 = f_n\langle\alpha^2\rangle$$

Using normalization condition (2.4)

$$f_n\langle\alpha^2\rangle = n^2 \tag{4.14}$$

In the previous section we found the probability distribution for α, $P(\alpha)$, using this probability distribution we can find $\langle\alpha^2\rangle$

$$f_n\int_{-\infty}^{+\infty}\alpha^2 p(\alpha)d\alpha = n^2$$

Recall that we found $P(\alpha)$ should be even function, $p(\alpha) = p(-\alpha)$, so

$$\int_{0}^{+\infty}\alpha^2 p(\alpha)d\alpha = \frac{n^2}{2f_n}$$

In previous section we found the constrain on $p(\alpha)$, the Fourier series expansion of $P(\alpha)$ for $\frac{2k\pi}{t^*} < \alpha < \frac{2(k+1)\pi}{t^*}$ is

$$p(\frac{2k\pi}{t^*} + x) = \frac{a_0^k}{2} + \sum_{m=1}^{\infty} a_{2m}^k Cos(2t^* mx)$$

Choose the simplest possible probability distribution, where all a_{2m}^k and a_0^k are zero but a_0^0. This means the probability distribution is $P(\alpha) = \frac{a_0^0}{2}$, where $-\frac{2\pi}{t^*} < \alpha < \frac{2\pi}{t^*}$, and $P(\alpha) = 0$, where $\alpha < -\frac{2\pi}{t^*}$ and $\frac{2\pi}{t^*} < \alpha$. Using this simple probability distribution and Hamiltonian normalization that we found in previous paragraph find scrambling time

$$\int_0^{+\infty} \alpha^2 p(\alpha) d\alpha = \int_0^{\frac{2\pi}{t^*}} \alpha^2 \frac{a_0^0}{2} d\alpha = \frac{1}{3} \left(\frac{2\pi}{t^*}\right)^3 \frac{a_0^0}{2} = \frac{n^2}{2f_n} \tag{4.15}$$

use normalization of probability distribution to find a_0^0

$$\int_{-\infty}^{+\infty} p(\alpha) d\alpha = 2 \int_0^{\frac{2\pi}{t^*}} \frac{a_0^0}{2} d\alpha = 1$$

so $a_0^0 = \frac{t^*}{2\pi}$. Substituting a_0^0 in equation 4.15 gives us the scrambling time as a function of number of terms in the Hamiltonian for a simple probability distribution.

$$\frac{1}{3} \left(\frac{2\pi}{t^*}\right)^3 \frac{t^*}{2\pi} = \frac{n^2}{f_n}$$

$$t^* = \frac{2\pi}{n} \sqrt{\frac{f_n}{3}} \tag{4.16}$$

In next section we will provide numerical results that shows having $f_n = n^2 \left(log_2(n)\right)^2$ give us a Haar random state at time $t^* = \frac{2\pi}{\sqrt{3}} log_2(n)$. Because "there is a logarithmic lower band on the scrambling time of systems with finite norm terms in Hamiltonian" [1], our Hamiltonian is the fastest scrambler.

4.4 Logarithmic scrambling time

Up to this point we provided all necessary building blocks to come up with a random Hamiltonian that scramble in logarithmic time, we have a probability distribution

$P(\alpha) = \frac{t^*}{4\pi}$ that matches first moment of $|\psi(t^*)\rangle$ with Haar random distribution, also we found for this probability distribution scrambling time should be $t^* = \frac{2\pi}{n}\sqrt{\frac{f_n}{3}}$. The last step is to show that the higher moments of final state $|\psi(t^*)\rangle$ matches with equivalent moments of Haar random distribution by having $f_n = n^2\left(log_2(n)\right)^2$ terms in Hamiltonian. Showing this prove that the random Hamiltonian with $P(\alpha) = \frac{2\pi}{\sqrt{12}}log_2(n)$, and $f_n = n^2\left(log_2(n)\right)^2$ terms scramble the system to the order of $\delta = \frac{1}{2}$ at scrambling time $t^* = \frac{2\pi}{\sqrt{3}}log_2(n)$.

We show the product moments of final state $|\psi(t^*)\rangle$ matches with product moments of Haar random distribution using numerical methods by writing a program that generates random Hamiltonian with aforementioned number of terms and probability distribution, and evolves an arbitrary initial state (without loosing generality we chose $|\psi\rangle = |00...0\rangle$) to the final state $|\psi(t^*)\rangle$, where $t^* = \frac{2\pi}{\sqrt{3}}log_2(n)$ appendix D . For $|\psi(t^*)\rangle$ we calculated $P\left(k_1, k_2, ..., k_{2^{n+1}}\right)$, which is the expectation value of product moments of $|\psi(t^*)\rangle$

$$P\left(k_1, k_2, ..., k_{2^{n+1}}\right) \equiv \langle \prod_{i=1}^{2^n} \left(Re\left(\psi_i(t^*)\right)\right)^{k_{2i-1}} \left(Im\left(\psi_i(t^*)\right)\right)^{k_{2i}} \rangle$$

To measure the Haar randomness of $|\psi(t^*)\rangle$, we compared $P\left(k_1, k_2, ..., k_{2^{n+1}}\right)$, with equivalent moments of Haar random distribution in equation 4.17. For comparison find distance $D\left(k_1, k_2, ..., k_{2^{n+1}}\right)$, which is the distance of $P\left(k_1, k_2, ..., k_{2^{n+1}}\right)$ from equivalent Haar random product moment.

$$\begin{cases} D\left(k_1, k_2, ..., k_{2^{n+1}}\right) = P\left(k_1, k_2, ..., k_{2^{n+1}}\right) & if\ \exists\ k_i : k_i\ is\ odd \\ D\left(k_1, k_2, ..., k_{2^{n+1}}\right) = P\left(k_1, k_2, ..., k_{2^{n+1}}\right) - \frac{\Gamma(2^n)}{2^k\Gamma\left(\frac{k}{2}+2^n\right)} \prod_{i=1}^{2^{n+1}} \frac{k_i!}{\left(\frac{k_i}{2}\right)!} & if\ \forall\ k_i : k_i\ is\ even \end{cases} \tag{4.17}$$

There are infinite $D(k_1, k_2, ..., k_{2^{n+1}})$, but it is sufficient to check small k where k is the summation of all k_i, $k = k_1 + k_2 + ... + k_{2^{n+1}}$. Notice that changing k_l with k_m change the power ofl^{th} vector component to k_m, and the power of m^{th} vector component to k_l but the summation $k = k_1 + k_2 + ... + k_{2^{n+1}}$ remain the same. Number of permutations like this is exponential in number of qubits, to consider all permutations average over all of them. These permutations can be specified with permutation set $\{k_i, k_j, ...\}$, which is a set of positive integers such that their summation is equal to k, so the average can be represented by permutation set, $D\left(\{k_i, k_j, ...\}\right)$. For example having $k = 1$ there

is only one set $\{k_i = 1\}$, subscript i can have 2^{n+1} value so there are 2^{n+1} different permutations of $k_i = 1$ that we should average over, namely we should average over $\{D(1,0,0,...,0), D(0,1,0,...,0), ..., D(0,0,0...,1)\}$

$$D(\{1\}) = \frac{1}{2^{n+1}} \left(D(1,0,...,0) + D(0,1,...,0) + ... + D(0,0,...,1)\right.$$

To have another example lets take $k = 2$ there are two sets $\{k_i = 1, k_j = 1\}$, and $\{k_i = 2\}$. Subscript i can have 2^{n+1} value, and subscript j can have $2^{n+1} - 1$ value so we can have $\frac{2^{n+1}(2^{n+1}-1)}{2}$ permutations for the first set which we should average over, and 2^{n+1} permutations for later set that we should average over.

$$D(\{1,1\}) = \frac{1}{2^n(2^{n+1}-1)} \left(D(1,1,0,...,0) + D(1,0,1,...,0) + ... + D(0,0,...,1,1)\right.$$

$$D(\{2\}) = \frac{1}{2^{n+1}} \left(D(2,0,...,0) + D(0,2,...,0) + ... + D(0,0,...,2)\right.$$

We computed $D\left(\{k_i, k_j, ...\}\right)$ for $k = \{2,3,4\}$ using our program which can be find in appendix D. Table 4.4 is the result for $n = \{2,3,4,5,6,7\}$.

	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$
$D(\{1,1\})$	$3.9e-3$	$2.2e-3$	$2.3e-4$	$7.7e-5$	$2.9e-5$	$1.5e-6$
$D(\{2\})$	0	0	0	0	0	0
$D(\{1,1,1\})$	$5.3e-4$	$1.0e-4$	$1.4e-6$	$2.6e-7$	$3.7e-7$	$5.8e-11$
$D(\{1,2\})$	$3.5e-4$	$6.7e-4$	$1.4e-4$	$1.7e-6$	$3.8e-5$	$2.5e-6$
$D(\{3\})$	$2.3e-2$	$3.0e-4$	$1.0e-3$	$4.2e-5$	$1.1e-4$	$2.4e-7$
$D(\{1,1,1,1\})$	$2.2e-4$	$5.0e-5$	$3.9e-7$	$2.4e-8$	$1.1e-8$	$4.6e-10$
$D(\{1,1,2\})$	$8.6e-5$	$8.1e-5$	$7.9e-6$	$1.0e-6$	$2.1e-7$	$6.0e-9$
$D(\{2,2\})$	$2.3e-3$	$1.0e-4$	$7.2e-7$	$8.1e-7$	$6.1e-8$	$1.0e-8$
$D(\{1,3\})$	$2.2e-3$	$4.9e-4$	$7.2e-6$	$5.2e-6$	$7.6e-7$	$5.0e-9$
$D(\{4\})$	$1.6e-2$	$1.5e-3$	$2.2e-5$	$5.1e-5$	$7.7e-6$	$2.7e-6$

The first and second moments of random state exactly matches with their similar moment of Haar-random distribution, $D(\{2\}) = 0$. Even higher distances $D(k_1, k_2, ..., k_{2^{n+1}})$ approaches zero as number of qubits increases. With this result we successfully showed that a random Hamiltonian with $f_n = n^2 \left(log_2(n)\right)^2$ random Pauli basis in Pauli expansion such that coefficients of Pauli basis are real random numbers obtained from

$P(\alpha) = \frac{1}{2\sqrt{3}} log_2(n)$ evolves a system initially in a computational basis to a Haar-random state at time $t^* = \frac{2\pi}{\sqrt{3}} log_2(n)$. Also, we already proved that for a large number of qubits if any computational basis evolves to a Haar-random state the system scramble to the order of $\delta = \frac{1}{2}$, so we find the scrambler that scrambles the system to the order of half in logarithmic time. According to Lashkari et al. [1] there is a logarithmic lower band on the scrambling time of systems with finite norm terms in Hamiltonian, so the Hamiltonian that we engineered is the fastest scrambler for a large number of qubits.

Chapter 5

Conclusion and Discussion

When the system gets scrambled, it is not possible to distinguish different initial states without measuring a large fraction of degrees of freedom. δ is the order of scrambling which describe how far a system is from the idealistic scrambled system. For a quantum system with fixed size, n, we can compare the scrambling time of well-normalized Hamiltonians to the same order of scrambling and find the fastest scrambler. For normalization of Hamiltonian, we assumed energy of system grows extensively with the number of qubits $\frac{Tr\ H^2}{2^n} = n^2$ [1]. We found the fastest scrambler to the order of $\delta = 0$ for $n = 2$, and $n = 3$ are Ξ_{xx} and Ξ_{xxx} respectively. Also, we found the fastest scrambler for a system of n qubits to the order of $\delta = \frac{n}{4}$ is the Hamiltonian produced by the diagram that we described in section 3.3. These Hamiltonians evolve any computational basis to a highly symmetric entangled state. The diagram rules is as fallowing

- start with $H = 0$ and draw n points, which represents n qubits.
- Divide qubits into pairs and have two body interactions between two qubits of each pair added to Hamiltonian H, if one qubit remained unpaired then combine it with one of the pairs and have a three body interaction added.
- Divide previous pairs of qubits into different sets of two pairs (quads). Again have a two-body interaction between one qubit of each pair with one qubit of the other pair in the same quad added to Hamiltonian H, if one pair doesn't fit in a quad then combine it with one of the quads and have a three-body interaction.

- Keep pairing previous divisions, and do the same until everything gets connected with two or three body interactions.

Utilizng probability theory and based on Don Page [13] and lashkari et al. [1] works, we found the fastest scrambler of large systems to the order of $\delta = \frac{1}{2}$. These Fastest scramblers are random Hamiltonians $H = \sum_{i=1}^{n^2(log_2(n))^2} \alpha_i \Xi^i$ where α_i are real random numbers obtained from the probability distribution $P(\alpha) = \frac{1}{2\sqrt{3}} log_2(n)$, and Ξ^i are random Pauli basis obtained from a uniform probability distribution. We showed H scramble large systems to the order of $\delta = \frac{1}{2}$ at scrambling time $t^* = \frac{2\pi}{\sqrt{3}} log_2(n)$. According to Lashkari et al [1] there is a logarithmic lower band on the scrambling time of systems with finite norm terms in Hamiltonian, so H is the fastest scrambler for large number of qubits.

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Appendix A

Appendix

```
%this function evaluate  entanglement entropy of m qubits of an evolved state of n
    qubits and calculate scrambling time.
function TimeOfAvg=STHvecSearch(Hvec,n,epsilon);
starttime=0;timestep=0.01;endtime=10000;
R=0;phai=0;ro=0;i=complex(0,1);
Savg=0.9999;
SME=SMaxEntangled(n);
TimeOfAvg=99999;
H=HvecToH(Hvec,n);
sai=reshape(zeros(1,(4^n)),(2^n),1,(2^n));
for j=1:1:(2^n);
   sai(j ,:, j)=1;
endfor;
[V,lambda]=eig(H);
t=starttime;
while t<=endtime;
   check=1;
   checkH=1;
   SOavgtotal=0;
   z=1;
   while z<=2^n;
```

```
phai=0;ro=0;SOavg=zeros(1,n-1);counter=0;
if checkH == 1;
    H=lambda;
    for k=1:1:2^n;
        H(k,k)=exp(-i*lambda(k,k)*t);
    endfor;
    H=V*H*inverse(V);
    checkH=0;
endif;
phai=H*sai(:,:,z);
ro=phai*conj(transpose(phai));
#Here we want to trace over m quibits of rho
m=n-1;
while m>=(n/2);
    A2=[1:m];
    c=(factorial(n)/(( factorial (m)*factorial(n-m))));
    cnt=1;
    while cnt<=c;
        rocomp=zeros(2^(n-m));
        #here we check position of qubit ," A(2)" does not exceed n.
        s=1;
        while s<=m;
            if A2(s)>n;
                A2(s-1)=A2(s-1)+1;
                #here we make sure after A2(s-1) we get bigger numbers.
                for r=1:1:(m-s+1);
                    A2(s+r-1)=A2(s-1)+r;
                endfor;
                s=0;
            endif;
            s=s+1;
        endwhile;
```

```
#here we are tracing over qubits  labeled  by A2
for p=1:1:(2^n);
   for q=p:1:(2^n);
      #OP is row and Oq is column s of rho,which shows the states of
          n qubits
      Op=dec2bin(p-1);
      while columns(Op)<n;
         Op=["0",Op];
      endwhile;
      Oq=dec2bin(q-1);
      while columns(Oq)<n;
         Oq=["0",Oq];
      endwhile;
      #here we are mapping Op,Oq of n qubits to OP,Oq of n-m
          qubits.
      delta=1;
      r=1;
      for r=1:1:m;
         if Op((A2(r)-r+1))!=Oq((A2(r)-r+1));
            delta=0;
         endif;
         Op(:,[(A2(r)-r+1)]) = [];
         Oq(:,[(A2(r)-r+1)]) = [];
      endfor;
      #here we check ifwe chose right columns and vectors to add
          their components
      if delta==1;
         p1=bin2dec(Op)+1;
         q1=bin2dec(Oq)+1;
         if p1==q1;
            rocomp(p1,q1)=rocomp(p1,q1)+ro(p,q);
         else;
```

```
                                        rocomp(p1,q1)=rocomp(p1,q1)+ro(p
                                            ,q);
                                        rocomp(q1,p1)=rocomp(q1,p1)+
                                            conj(ro(p,q));
                    endif;
                endif;
            endfor;
        endfor;
        #here we find entanglement entropy of m qubits labeled by A2
        [U,gama]=eig(rocomp);
        lgama=gama;
        for k=1:1:(2^(n-m));
            if gama(k,k)!=0;
                lgama(k,k)=log2(gama(k,k));
            endif;
        endfor;
        gama=U*gama*inverse(U);
        lgama=U*lgama*inverse(U);
        SO=-(real(trace(gama*lgama)))
        m
        pause(0.0001)
        SOavg(1,m)=SOavg(1,m)+SO;
        counter=counter+1;
        A2(1,m)=A2(1,m)+1;
        cnt=cnt+1;
    endwhile;
    #here we find the average entanglement entropy of m qubits.
    SOavg(1,m)=SOavg(1,m)/c;
    if SOavg(1,m)<SME(m)-epsilon;
        check=0;
        m=1;
        z=2^n;
```

```
            endif;
            m=m−1;
        endwhile;
                                z=z+1;
    endwhile;
    if check==1;
        TimeOfAvg=t;
        t=endtime;
    endif;
    t=t+timestep;
endwhile;
endfunction;
```

```
function H=HvecToH(Hvec,n);
cc=0;
for k=1:1:2^n;
    cc=cc+1;
    H(k,k)=Hvec(cc,1);
    for m=k+1:1:2^n;
        cc=cc+1;
        a1=Hvec(cc,1);
        cc=cc+1;
        a2=Hvec(cc,1);
        H(k,m)=complex(a1,a2);
        H(m,k)=conj(H(k,m));
    endfor;
endfor;
[V,lambda]=eig(H);
lambda=lambda*(2^(n/2)*n/sqrt(sum(sum(lambda.*conj(lambda)))));
H=V*lambda*inverse(V);
endfunction;
```

Appendix B

Appendix

```
%this function evaluate distance of reduced density matrix of two different evolved
    state with same hamiltonian.
function STAvg=STDSearch(Hvec,n,epsilon);
starttime=0;timestep=0.01;endtime=1;
i=complex(0,1);
STAvg=9999;
H=HvecToH(Hvec,n);
sai=reshape(zeros(1,(4^n)),(2^n),1,(2^n));
for j=1:1:(2^n);
   sai(j ,:, j)=1;
endfor;
[V,lambda]=eig(H);
t=starttime;
while t<=endtime;
   check=1;checkH=1;
   m=n-1;
   while m>=(n/2);
      A2=[1:m];
      c=(factorial(n)/(( factorial (m)*factorial(n-m))));
      cnt=1;
      while cnt<=c;
```

```
#here we want to find evolved state for each initial state.
phai=reshape(zeros(1,(4^n)),(2^n),1,(2^n));
ro=reshape(zeros(1,(8^n)),(2^n),(2^n),(2^n));
rocomp=reshape(zeros(1,(4^(n-m)*2^n)),(2^(n-m)),(2^(n-m)),(2^n));
z=1;
while z<=(2^n);
  if checkH==1;
    H=lambda;
    for k=1:1:(2^n);
      H(k,k)=exp(-i*lambda(k,k)*t);
    endfor;
    H=V*H*inverse(V);
    checkH=0;
  endif;
  phai(:,:,z)=H*sai(:,:,z);
  ro(:,:,z)=phai(:,:,z)*conj(transpose(phai(:,:,z)));
  #here we check position of qubit ," A(2)" does not exceed n.
  s=1;
  while s<=m;
    if A2(s)>n;
      A2(s-1)=A2(s-1)+1;
      for r=1:1:(m-s+1);
        A2(s+r-1)=A2(s-1)+r;
      endfor;
      s=0;
    endif;
    s=s+1;
  endwhile;
  #here we are tracing over qubits labeled by A2
  for p=1:1:(2^n);
    for q=p:1:(2^n);
```

```
#OP is row and Oq is column s of rho,which shows the states of n
    qubits
Op=dec2bin(p-1);
while columns(Op)<n;
   Op=["0",Op];
endwhile;
Oq=dec2bin(q-1);
while columns(Oq)<n;
   Oq=["0",Oq];
endwhile;
#here we are mapping Op,Oq of n qubits to OP,Oq of n-m qubits.
delta=1;
r=1;
for r=1:1:m;
   if Op((A2(r)-r+1))!=Oq((A2(r)-r+1));
      delta=0;
   endif;
   Op(:,[(A2(r)-r+1)]) = [];
   Oq(:,[(A2(r)-r+1)]) = [];
endfor;
#here we check if we chose right columns and vectors to add their
    components
if delta==1;
   p1=bin2dec(Op)+1;
   q1=bin2dec(Oq)+1;
   if p1==q1;
      rocomp(p1,q1,z)=rocomp(p1,q1,z)+ro(p,q,z);
   else;
                              rocomp(p1,q1,z)=rocomp(p1,q1,z)+ro(p
                                  ,q,z);
                              rocomp(q1,p1,z)=rocomp(q1,p1,z)+
                                  conj(ro(p,q,z));
```

```
            endif;
          endif;
        endfor;
      endfor;
      x=z-1;
      #here we check to see if the distance is less than epsilon.
      while x>0;
        Dis=conj(transpose(rocomp(:,:,z)-rocomp(:,:,x)))*(rocomp(:,:,z)-
            rocomp(:,:,x));
        [W,kesi]=eig(Dis);
        for y=1:1:(2^(n-m));
          kesi(y,y)=sqrt(kesi(y,y));
        endfor;
        Dis=W*kesi*inverse(W);
        dis=real(trace(Dis));
 if dis>1.8;
 dis;
 endif;
 if dis > epsilon;
                              x=0;
                              z=2^n+1;
                              cnt=c+1;
                              m=0;
                              check=0;
        endif;
 x=x-1;
    endwhile;
    z=z+1;
endwhile;
cnt=cnt+1;
if m!=0;
A2(1,m)=A2(1,m)+1;
```

```
            endif;
        endwhile;
        m=m−1;
    endwhile;
    if check==1 ;
        STAvg=t;
        t=endtime;
    endif;
        t=t+timestep;
endwhile;
```

Appendix C

Appendix

```
clc;
n=4;

HvecminR=0;STmin=99999999;
for y=1:1:1000;
    Hvec=stdnormal_rnd(4^n,1);
    ST=STHvec(Hvec,n);
    if ST<STmin;
        STmin=ST;
        HvecminR=Hvec;
    endif;
endfor;
save HveminREntropy4q HvecminR;
load HveminREntropy4q;
f=@(Hvecmin) STHvec(Hvecmin,n);
[Hvec,STmin]=fminsearch(f,HvecminR);
save FSH4E Hvec;
```

```
%Distance Definition.

%{
HvecminR=0;STmin=99999999;
for y=1:1:200;
   Hvec=stdnormal_rnd(4^n,1);
   ST=STD(Hvec,n);
   if ST<STmin;
      STmin=ST;
      HvecminR=Hvec;
   endif;
endfor;
save HveminR HvecminR;
load HvecminR;
f=@(Hvec) STD(Hvec,n);
[Hvecmin,STmin]=fminsearch(f,HvecminR);
save FSD4a Hvecmin;
%}
```

Appendix D

Appendix

```
clc;
cla;
n=7;
%here we find expectation value product moments
AvgProdMoment2=0;
AvgProdMoment11=0;
AvgProdMoment3=0;
AvgProdMoment12=0;
AvgProdMoment111=0;
AvgProdMoment4=0;
AvgProdMoment13=0;
AvgProdMoment112=0;
AvgProdMoment1111=0;
AvgProdMoment22=0;
hold on;
for j=1:1:10;
 FState=HaarFinalstate(n);
 for k=1:1:2^n;
  k2=2*k;
  z(k2-1,1)=real(FState(k,1));
  z(k2,1)=imag(FState(k,1));
```

```
endfor;
%k=2
%second moment k=(2,0,0,...)/k=(0,2,0,...)/...
SecndMoment=0;
for l=1:1:2^(n+1);
 SecndMoment=SecndMoment+z(l)^2;
endfor;
AvgProdMoment2=AvgProdMoment2+SecndMoment/2^(n+1);
%first produt moment k=(1,1,0,0,...)/k=(1,0,1,0,0,...) /...
ProdMoment11=0;
for l=1:1:2^(n+1);
 for m=l+1:1:2^(n+1);
   ProdMoment11=ProdMoment11+z(l)*z(m);
 endfor;
endfor;
 AvgProdMoment11=AvgProdMoment11+ProdMoment11/(2^(n)*(2^(n+1)-1));
 %k=3
  %third moment k=(3,0,0,...)/k=(0,3,0,...) /...
thirdMoment=0;
for l=1:1:2^(n+1);
 thirdMoment=thirdMoment+z(l)^3;
endfor;
AvgProdMoment3=AvgProdMoment3+thirdMoment/2^(n+1);
%produt moment k=(1,2,0,0,...)/k=(1,0,2,0,0,...) /...
ProdMoment12=0;
for l=1:1:2^(n+1);
 for m=l+1:1:2^(n+1);
   ProdMoment12=ProdMoment12+z(l)^1*z(m)^2;
   ProdMoment12=ProdMoment12+z(l)^2*z(m)^1;
 endfor;
endfor;
 AvgProdMoment12=AvgProdMoment12+ProdMoment12/(2^(n+1)*(2^(n+1)-1));
```

```
%produt moment k=(1,1,1,0,...)/k=(1,0,1,1,0,...) /...
ProdMoment111=0;
for l=1:1:2^(n+1);
 for m=l+1:1:2^(n+1);
   for o=m+1:1:2^(n+1);
     ProdMoment111=ProdMoment111+z(l)*z(m)*z(o);
   endfor;
 endfor;
endfor;
 AvgProdMoment111=AvgProdMoment111+ProdMoment111/(2^(n+1)*(2^(n+1)
     -1)*(2^(n+1)-2)/6);
   %k=4
  %fourth moment k=(4,0,0,...)/k=(0,4,0,...) /...
fourthMoment=0;
for l=1:1:2^(n+1);
 fourthMoment=fourthMoment+z(l)^4;
endfor;
AvgProdMoment4=AvgProdMoment4+fourthMoment/2^(n+1);
%produt moment k=(1,3,0,0,...)/k=(1,0,3,0,0,...) /...
ProdMoment13=0;
for l=1:1:2^(n+1);
 for m=l+1:1:2^(n+1);
   ProdMoment13=ProdMoment13+z(l)*z(m)^3;
   ProdMoment13=ProdMoment13+z(l)^3*z(m);
 endfor;
endfor;
 AvgProdMoment13=AvgProdMoment13+ProdMoment13/(2^(n+1)*(2^(n+1)-1));
%produt moment k=(1,1,2,0,...)/k=(1,0,1,2,0,...) /...
ProdMoment112=0;
for l=1:1:2^(n+1);
 for m=l+1:1:2^(n+1);
   for o=m+1:1:2^(n+1);
```

```
        ProdMoment112=ProdMoment112+z(l)*z(m)*z(o)^2;
        ProdMoment112=ProdMoment112+z(l)*z(m)^2*z(o);
        ProdMoment112=ProdMoment112+z(l)^2*z(m)*z(o);
      endfor;
    endfor;
  endfor;
    AvgProdMoment112=AvgProdMoment112+ProdMoment112/(2^(n)*(2^(n+1)-1)
        *(2^(n+1)-1));
  %produt moment k=(1,1,1,1,...)/k=(1,0,1,1,1,...) /...
  ProdMoment1111=0;
  for l=1:1:2^(n+1);
    for m=l+1:1:2^(n+1);
      for o=m+1:1:2^(n+1);
        for p=o+1:1:2^(n+1);
          ProdMoment1111=ProdMoment1111+z(l)*z(m)*z(o)*z(p);
        endfor;
      endfor;
    endfor;
  endfor;
    AvgProdMoment1111=AvgProdMoment1111+ProdMoment1111/(2^(n+1)*(2^(n
        +1)-1)*(2^(n+1)-2)*(2^(n+1)-3)/24);
    %produt moment k=(2,2,0,0,...)/k=(2,0,2,0,0,...) /...
  ProdMoment22=0;
  for l=1:1:2^(n+1);
    for m=l+1:1:2^(n+1);
      ProdMoment22=ProdMoment22+z(l)^2*z(m)^2;
    endfor;
  endfor;
    AvgProdMoment22=AvgProdMoment22+ProdMoment22/(2^n*(2^(n+1)-1));
endfor;
AvgProdMoment2=AvgProdMoment2/10;
AvgProdMoment11=AvgProdMoment11/10;
```

```
AvgProdMoment3=AvgProdMoment3/10;
AvgProdMoment12=AvgProdMoment12/10;
AvgProdMoment111=AvgProdMoment111/10;
AvgProdMoment4=AvgProdMoment4/10;
AvgProdMoment13=AvgProdMoment13/10;
AvgProdMoment112=AvgProdMoment112/10;
AvgProdMoment1111=AvgProdMoment1111/10;
AvgProdMoment22=AvgProdMoment22/10;
%Calculation of D(k1,k2,...)
D11=AvgProdMoment11
D2=(AvgProdMoment2-gamma(2^(n))/gamma(2^(n)+1)/(2^2)*2)%/(gamma(2^(n
    ))/gamma(2^(n)+1)/(2^2)*2)
D111=AvgProdMoment111
D12=AvgProdMoment12
D3=AvgProdMoment3
D1111=AvgProdMoment1111
D112=AvgProdMoment112
D22=(AvgProdMoment22-gamma(2^(n))/gamma(2^(n)+2)/(2^4)*2*2)%/(gamma
    (2^(n))/gamma(2^(n)+2)/(2^4)*2*2)
D13=AvgProdMoment13
D4=(AvgProdMoment4-gamma(2^(n))/gamma(2^(n)+2)/(2^4)*4*3)%/(gamma
    (2^(n))/gamma(2^(n)+2)/(2^4)*4*3)
```

```
%This function will give us The final state at scrambling time from a random
    Hamiltonian
function FState=HaarFinalstate(n)
%fn=floor(n^2*log2(n)^2);
fn=n^2*log2(n)^2;
sctime=2*pi/sqrt(3)*log2(n);
i=complex(0,1);
sigma (:,:,1) =[1,0;0,1];
sigma (:,:,2) =[0,1;1,0];
```

```
sai=zeros(2^n,1);
sai(1,1)=1;
% this part generate a random Hamiltonian
H=zeros(2^n,2^n);
for k=1:1:fn;
    j=floor(unifrnd(0,2^n));
    binn=dec2bin(j);
    c=columns(binn);
     while c<n;
        binn=["0",binn];
      c=c+1;
     endwhile;
    numb=bin2dec(binn(n));
    Xi=sigma(:,:,numb+1);
    for m=1:1:n-1;
      numb=bin2dec(binn(n-m));
      Xi=TensorProduct(sigma(:,:,numb+1),Xi);
    endfor;
    alpha(k,1)=unifrnd(-2*pi/sctime,2*pi/sctime);
    H=H+alpha(k,1)*Xi;
 endfor;
 %Hamiltonian Normalization
[V,lambda]=eig(H);
lambda=lambda*(2^(n/2)*n/sqrt(sum(sum(lambda.*conj(lambda)))));
% This part calculate the Unitary operator of Random Hamiltonian at scrambling
    time
for k=1:1:2^n;
   lambda(k,k)=exp(-i*lambda(k,k)*sctime);
endfor;
U=V*lambda*inverse(V);
FState=U*sai;
```