Toward a stochastic description of reheating

Marcos A. G. García

Rice University

Based on 1705.????, with

M. Amin, O. Wen (Rice U.)
H. Xie (U. Wisconsin)

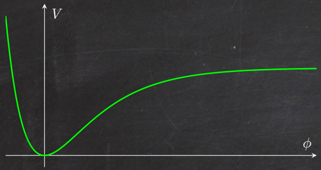
Congratulations!

(and thank you for the invitation)

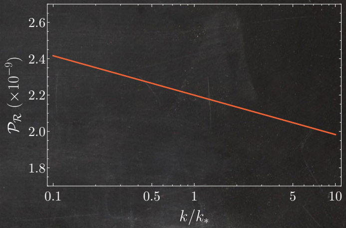
- Motivation
- Stochastic Particle Production
- Exact Results
- Conclusion

Motivation

All you need to drive inflation ($a \sim e^{Ht}$) is a scalar field,



$$\xrightarrow{\substack{\phi \to \phi + \delta\phi \\ g \to g + \delta g}}$$



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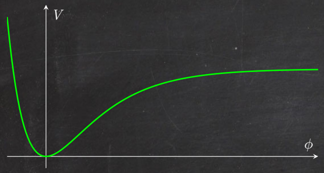

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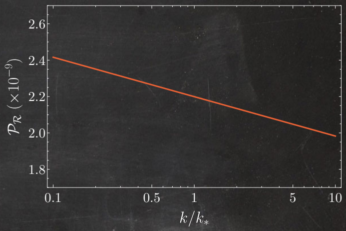
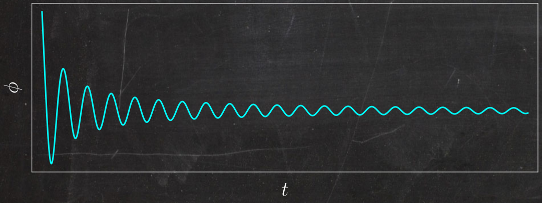


When inflation ends, the Universe reheats…



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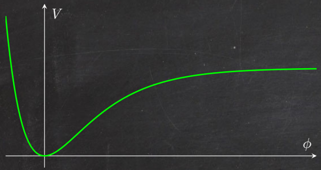

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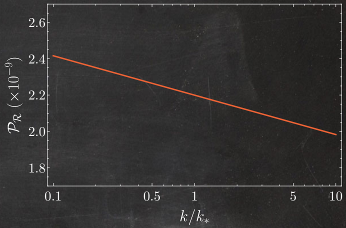

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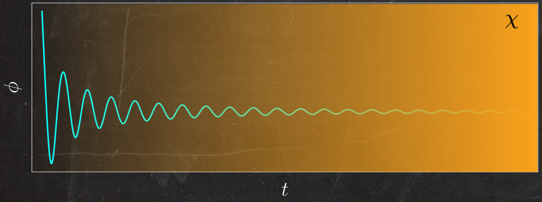

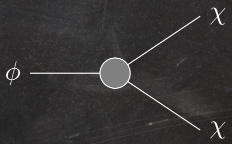

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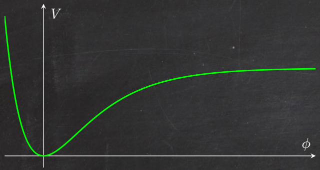

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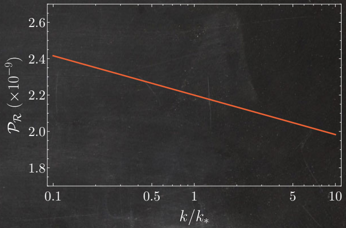
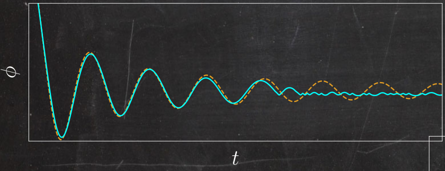
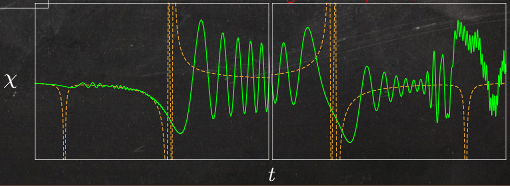


When inflation ends, the Universe reheats…in a potentially complicated way



Ellis, G, Nagata, Nanopoulos, Olive 2017



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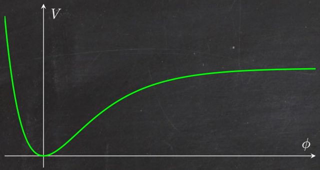

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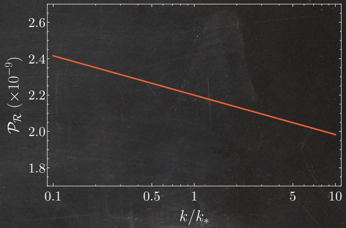

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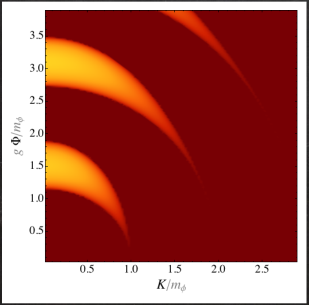
Amin, Hertzberg, Kaiser, Karouby 2015



$$\left[\mathbb{1}\left(\partial_\tau^2 + (k/a)^2\right) + \mathbf{p}(\tau)\partial_\tau + \mathbf{m}(\tau)\right]\cdot\delta\boldsymbol{\chi} = 0$$

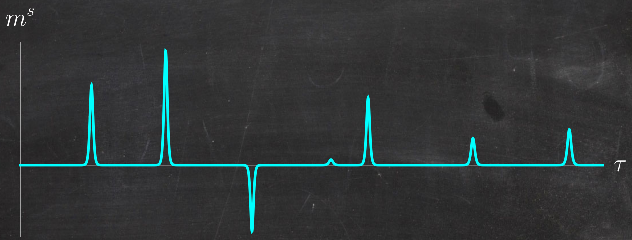
$$(\mathbf{p})^a_{\ b} = 2\Gamma^a_{\ bc}\phi^{c\,\prime} + 3H\delta^a_b$$

$$(\mathbf{m})^a_{\ b} = (G^{ac}V_{,c})_{,b} + \Gamma^a_{\ cd,b}\phi^{c\,\prime}\phi^{d\,\prime}$$

Stochastic Particle Production

Consider N_f coupled (scalar) fields. Assume the evolution of fluctuations contains localized non-adiabatic events with random strengths at random intervals, and that the fields are otherwise free

$$\left[\mathbb{1}\,\partial_\tau^2 + \boldsymbol{\omega}^2 + \mathbf{m}^s(\tau)\right]\cdot\boldsymbol{\chi}(\tau,\mathbf{k}) = 0\,, \qquad \omega_a^2 = k^2 + m_a^2$$



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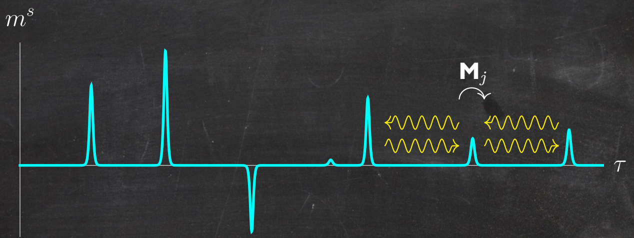

After the j-th event,

$$\chi_j^a(\tau) \equiv \frac{1}{\sqrt{2\omega_a}}\left[\beta_j^a e^{i\omega_a\tau} + \alpha_j^a e^{-i\omega_a\tau}\right]\,, \qquad \begin{pmatrix}\boldsymbol{\beta}_j\\ \boldsymbol{\alpha}_j\end{pmatrix} = \mathbf{M}_j\begin{pmatrix}\boldsymbol{\beta}_{j-1}\\ \boldsymbol{\alpha}_{j-1}\end{pmatrix}$$

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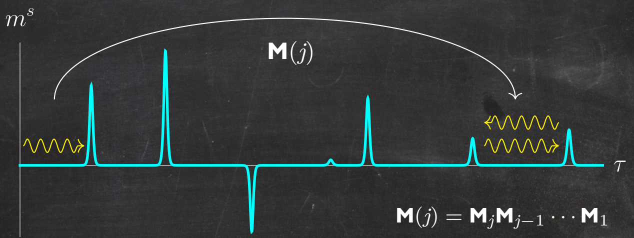

$$\mathbf{M}(j) = \mathbf{M}_j\mathbf{M}_{j-1}\cdots\mathbf{M}_1$$

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Stochastic Particle Production

Consider $N_{\rm f}$ coupled (scalar) fields. Assume the evolution of fluctuations contains localized non-adiabatic events with random strengths at random intervals, and that the fields are otherwise free

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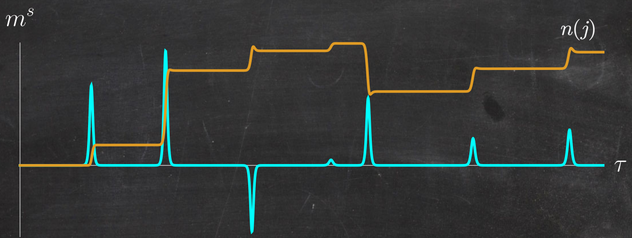
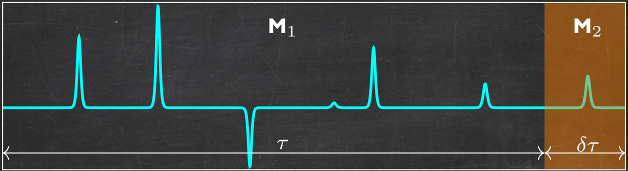


A random walk (with drift) for the occupation number

$$n_a(j) = \frac{1}{2\omega_a}\left(|\dot{\chi}_j^a|^2 + \omega_a^2|\chi_j^a|^2\right) - \frac{1}{2} = |\beta_j^a|^2$$

Randomness and non-adiabaticity are encoded in $\mathbf{M}$.
A probability distribution can be defined, $P(\mathbf{M};\tau)$



If the processes in the τ and $\delta\tau$ strips are uncorrelated, the distribution for $\mathbf{M}\equiv\mathbf{M}_2\mathbf{M}_1$ satisfies

$$P(\mathbf{M};\tau+\delta\tau)=\int d\mu(\mathbf{M}_2)\,P(\mathbf{M}_2^{-1}\mathbf{M};\tau)P(\mathbf{M}_2;\delta\tau)$$

$\Rightarrow$ Markovian evolution.

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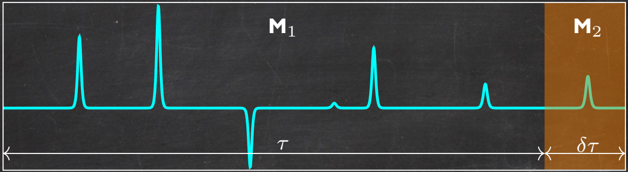

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$\Rightarrow$ Markovian evolution. Moreover, as $\delta\tau/\tau\longrightarrow 0$,

$$\partial_\tau P(\mathbf{M};\tau)=-\partial_{\mathbf{M}}\left[\frac{\langle\delta\mathbf{M}\rangle_{\mathbf{M}_2}}{\delta\tau}P(\mathbf{M};\tau)\right]+\frac{1}{2!}\partial^2_{\mathbf{M}}\left[\frac{\langle\delta\mathbf{M}^2\rangle_{\mathbf{M}_2}}{\delta\tau}P(\mathbf{M};\tau)\right]+\dots$$

(Fokker-Planck)

A general transfer matrix can be parametrized as

$$\mathbf{M} = \begin{pmatrix} \mathbf{u} & 0 \\ 0 & \mathbf{u}^* \end{pmatrix} \begin{pmatrix} \sqrt{1+\mathbf{n}} & \sqrt{\mathbf{n}} \\ \sqrt{\mathbf{n}} & \sqrt{1+\mathbf{n}} \end{pmatrix} \begin{pmatrix} \mathbf{v} & 0 \\ 0 & \mathbf{v}^* \end{pmatrix}$$

where $\mathbf{u}, \mathbf{v} \in U(N_\mathrm{f})$, and $\mathbf{n} = \mathrm{diag}(n_1, n_2, \cdots) \quad \Rightarrow \quad N_\mathrm{f}(2N_\mathrm{f}+1)$ variables in FP equation!

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Late-time equilibrium $\Rightarrow$ maximal entropy

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The Maximum Entropy Ansatz

Assume the building block P maximizes the Shannon entropy

$$S[P] = -\int P(\mathbf{M}_2;\delta\tau)\,\ln P(\mathbf{M}_2;\delta\tau)\,d\mu(\mathbf{M}_2)$$

subject to the constraints:

- The local mean particle production rate is known and fixed, $\mu_j \equiv \dfrac{1}{N_\mathrm{f}}\dfrac{\langle n_j\rangle_{\delta\tau}}{\delta\tau}$
- Coarse-grained continuity, $\mathbf{M}_{\tau+\delta\tau} \xrightarrow{\delta\tau\to 0} \mathbf{M}_\tau$

Consequences (Mello, Pereyra, Kumar 1988; Amin, Baumann 2016):

- P is independent of $\mathbf{u}$,

$$dP(\{\mathbf{u},\mathbf{n},\mathbf{v}\}) = P(\{\mathbf{n},\mathbf{v}\})\, d\mu(\mathbf{u})$$

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2. The marginal distribution $\int P(\{\mathbf{n},\mathbf{v}\})\, d\mu(\mathbf{v})$ satisfies the Fokker-Planck equation

$$\frac{1}{\mu}\frac{\partial}{\partial\tau}P(n_a;\tau) = \sum_{a=1}^{N_\mathrm{f}}\left[(1+2n_a)+\frac{1}{N_\mathrm{f}+1}\sum_{b\neq a}\frac{n_a+n_b+2n_an_b}{n_a-n_b}\right]\frac{\partial P}{\partial n_a}+\frac{2}{N_\mathrm{f}+1}\sum_{a=1}^{N_\mathrm{f}}n_a(1+n_a)\frac{\partial^2 P}{\partial n_a^2}$$

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3. A closed set of equations for the moments of $n=\sum_a n_a$ can be obtained. It implies

$$\partial_\tau\langle\ln(1+n)\rangle \xrightarrow{\tau\to\infty} \frac{2N_{\rm f}}{N_{\rm f}+1}\,\mu$$

$$\partial_\tau \mathrm{Var}[\ln(1+n)] \xrightarrow{\tau\to\infty} \frac{4}{N_{\rm f}+1}\,\mu$$

i.e. exponential growth for the occupation number

Exact Results

Consider the approximation

$$m^{\mathrm{s}}_{ab}(\tau) = 2\sqrt{\omega_a\omega_b}\sum_{j=1}^{N_{\mathrm{s}}}\Lambda_{ab}(\tau_j)\delta(\tau-\tau_j)\,,$$

where τ_j are uniformly distributed, and

$$\langle\Lambda_{ab}\rangle = 0, \qquad \langle\Lambda_{ab}\Lambda_{cd}\rangle = \sigma^2_{ab}(\delta_{ac}\delta_{bd}+\delta_{ad}\delta_{bc})$$

The transfer matrix takes the form

$$\mathbf{M}_j = \mathbb{1} + i\underbrace{\begin{pmatrix}\mathbf{a}_j^* & 0\\ 0 & \mathbf{a}_j\end{pmatrix}\begin{pmatrix}\mathbf{\Lambda}_j & \mathbf{\Lambda}_j\\ -\mathbf{\Lambda}_j & -\mathbf{\Lambda}_j\end{pmatrix}\begin{pmatrix}\mathbf{a}_j & 0\\ 0 & \mathbf{a}_j^*\end{pmatrix}}_{\mathbf{m}_j}, \qquad \mathbf{a}_j \equiv \mathrm{diag}(e^{i\omega_1\tau_j}, e^{i\omega_2\tau_j}, \cdots)$$

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Will focus on the total occupation number. Define $\mathbf{R} = \mathbf{M}\mathbf{M}^\dagger$:

$$n(j) \;=\; \frac{1}{4}\mathrm{Tr}\left[\mathbf{M}(j)\mathbf{M}^\dagger(j) - \mathbb{1}\right] \;\equiv\; \frac{1}{4}\mathrm{Tr}\left[\mathbf{R}(j) - \mathbb{1}\right]$$

Algorithm:

1. Parametrize $\mathbf{R}$

$$\mathbf{R} = \begin{pmatrix} \mathbf{u} & 0 \\ 0 & \mathbf{u}^* \end{pmatrix} \begin{pmatrix} \mathbf{f} & \tilde{\mathbf{f}} \\ \tilde{\mathbf{f}} & \mathbf{f} \end{pmatrix} \begin{pmatrix} \mathbf{u}^\dagger & 0 \\ 0 & \mathbf{u}^\top \end{pmatrix}, \qquad \begin{aligned} \mathbf{f} &= 2\mathbf{n} + \mathbb{1} \\ \tilde{\mathbf{f}} &= \sqrt{\mathbf{f}^2 - \mathbb{1}} \end{aligned}$$

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2. Solve for first and second order perturbations in terms of previous value of **R**

$$\mathbf{R}(j+1) = \mathbf{R}(j) + \delta\mathbf{R}\,,$$
$$\delta\mathbf{R} = \mathbf{R}(j)\mathbf{m}^\dagger_{j+1} + \mathrm{h.c.} + \mathbf{m}_{j+1}\mathbf{R}(j)\mathbf{m}^\dagger_{j+1}\,.$$

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3. Calculate correlators

$$\langle \delta f_a^{(1)} \delta f_b^{(1)} \rangle_{\delta\tau} = 2\tilde{f}_a\tilde{f}_b \sum_{c,d} \sigma^2_{cd} \left(u^\dagger_{ac} u^\dagger_{ad} u_{cb} u_{db} + u^\dagger_{bc} u^\dagger_{bd} u_{ca} u_{da} \right)$$

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4. Write and solve the FP equation

Single field

Only two parameters, f and $u = e^{i\phi}$. Computation is straightforward,

$$\langle \delta f^{(1)} \delta f^{(1)} \rangle = 2\tilde{f}^2 \sigma^2$$

$$\langle \delta f^{(1)} \delta \phi^{(1)} \rangle = 0$$

$$\langle \delta \phi^{(1)} \delta \phi^{(1)} \rangle = \frac{\sigma^2}{2\tilde{f}^2}(2\tilde{f}^2 + f^2)$$

$$\langle \delta f^{(2)} \rangle = 2f\sigma^2$$

$$\langle \delta \phi^{(2)} \rangle = 0$$

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No ϕ dependence! $\quad\Rightarrow\quad$ maximum entropy

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$$\langle \delta f^{(2)} \rangle = 2f\sigma^2$$
$$\langle \delta \phi^{(2)} \rangle = 0$$

No ϕ dependence! $\Rightarrow$ maximum entropy

The FP equation is

$$\frac{1}{\sigma^2}\frac{\partial}{\partial \tau}P(f;\tau) = \frac{\partial}{\partial f}\left[(f^2-1)\frac{\partial}{\partial f}P(f;\tau)\right]$$

Single field

Only two parameters, f and $u = e^{i\phi}$. Computation is straightforward,

$$\langle \delta f^{(1)} \delta f^{(1)} \rangle = 2\tilde{f}^2 \sigma^2$$
$$\langle \delta f^{(1)} \delta \phi^{(1)} \rangle = 0$$
$$\langle \delta \phi^{(1)} \delta \phi^{(1)} \rangle = \frac{\sigma^2}{2\tilde{f}^2}(2\tilde{f}^2 + f^2)$$

$$\langle \delta f^{(2)} \rangle = 2f\sigma^2$$
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$$\frac{1}{\sigma^2}\frac{\partial}{\partial \tau} P(n;\tau) = \frac{\partial}{\partial n}\left[n^2 \frac{\partial}{\partial n} P(n;\tau)\right]$$

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with solution

$$P(n;\tau)\, dn = \frac{1}{\sqrt{4\pi\sigma^2\tau}} \exp\left[-\frac{(\ln n - \sigma^2\tau)^2}{4\sigma^2\tau} \right] d\ln n$$

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$$\langle \delta \phi^{(1)} \delta \phi^{(1)} \rangle = \frac{\sigma^2}{2\tilde{f}^2}(2\tilde{f}^2 + f^2)$$

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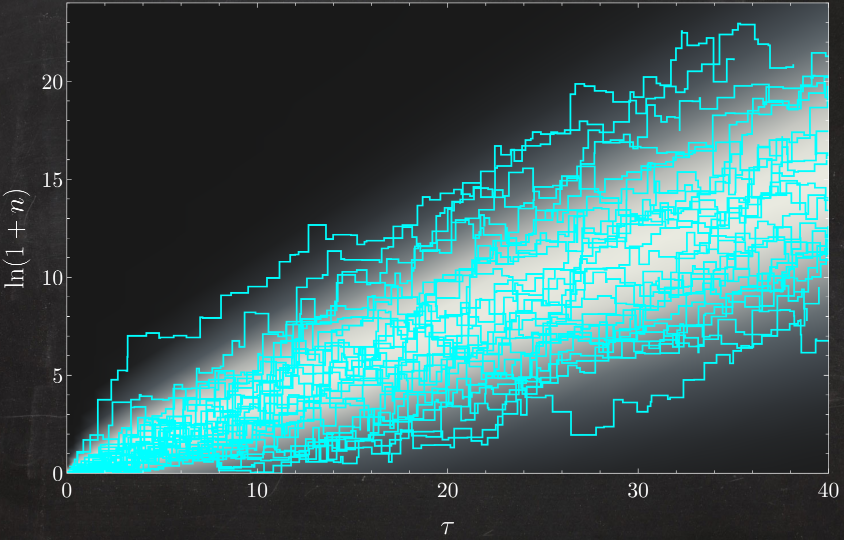
with solution

$$P(n;\tau)\, dn = \frac{1}{\sqrt{4\pi\sigma^2\tau}} \exp\left[-\frac{(\ln n - \sigma^2\tau)^2}{4\sigma^2\tau}\right] d\ln n$$

$$\Rightarrow \quad n = e^{\sigma^2\tau} - 1$$



0
5
10
15
20
0
10
20
30
40
$\ln(1+n)$
τ

Two fields

Six parameters now, f_1, f_2 and

$$\mathbf{u}(\phi,\theta,\psi,\varphi) = e^{-\frac{i}{2}\phi}\begin{bmatrix} \cos\frac{\theta}{2}\, e^{-\frac{i}{2}(\varphi+\psi)} & -\sin\frac{\theta}{2}\, e^{-\frac{i}{2}(\varphi-\psi)} \\ \sin\frac{\theta}{2}\, e^{\frac{i}{2}(\varphi-\psi)} & \cos\frac{\theta}{2}\, e^{\frac{i}{2}(\varphi+\psi)} \end{bmatrix}$$

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$\Rightarrow$ need 27 correlators

Two fields

Six parameters now, f_1, f_2 and

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$\Rightarrow$ need 27 correlators

Let

$$\langle(\Lambda_{11})^2\rangle_{\delta\tau} = \sigma_1^2\,, \qquad \langle(\Lambda_{22})^2\rangle_{\delta\tau} = \sigma_2^2\,, \qquad \langle(\Lambda_{12})^2\rangle_{\delta\tau} = \langle(\Lambda_{12})^2\rangle_{\delta\tau} = \sigma_\perp^2\,.$$

Two fields

Six parameters now, f_1, f_2 and

$$\mathbf{u}(\phi,\theta,\psi,\varphi) = e^{-\frac{i}{2}\phi}\begin{bmatrix} \cos\frac{\theta}{2}\,e^{-\frac{i}{2}(\varphi+\psi)} & -\sin\frac{\theta}{2}\,e^{-\frac{i}{2}(\varphi-\psi)} \\ \sin\frac{\theta}{2}\,e^{\frac{i}{2}(\varphi-\psi)} & \cos\frac{\theta}{2}\,e^{\frac{i}{2}(\varphi+\psi)} \end{bmatrix}$$

$\Rightarrow$ need 27 correlators

$$\langle\delta f_1^{(1)}\delta f_1^{(1)}\rangle = \tilde f_1^2\,\gamma_1(\theta),$$
$$\langle\delta f_1^{(1)}\delta f_2^{(1)}\rangle = 2\tilde f_1\tilde f_2\cos(2\psi)\,\gamma_3(\theta),$$
$$\langle\delta f_1^{(1)}\delta\theta^{(1)}\rangle = -\frac{2\tilde f_1}{\Delta f}\left[\tilde f_1+\tilde f_2\cos(2\psi)\right]\gamma_4(\theta),$$
$$\langle\delta f_1^{(1)}\delta\psi^{(1)}\rangle = -\tilde f_1\sin(2\psi)\left(\frac{f_2}{\tilde f_2}\gamma_3(\theta)-2\frac{\tilde f_2}{\Delta f}\gamma_4(\theta)\cot\theta\right),$$
$$\langle\delta f_1^{(1)}\delta\varphi^{(1)}\rangle = -2\frac{\tilde f_1\tilde f_2}{\Delta f}\sin(2\psi)\gamma_4(\theta)\csc\theta,$$
$$\langle\delta f_1^{(1)}\delta\phi^{(1)}\rangle = \tilde f_1\frac{f_2}{\tilde f_2}\sin(2\psi)\gamma_3(\theta),$$
$$\langle\delta f_2^{(1)}\delta f_2^{(1)}\rangle = \tilde f_2^2\,\gamma_2(\theta),$$
$$\langle\delta f_2^{(1)}\delta\theta^{(1)}\rangle = -\frac{2\tilde f_2}{\Delta f}\left[\tilde f_2+\tilde f_1\cos(2\psi)\right]\gamma_5(\theta),$$
$$\langle\delta f_2^{(1)}\delta\psi^{(1)}\rangle = -\tilde f_2\sin(2\psi)\left(\frac{f_1}{\tilde f_1}\gamma_3(\theta)-2\frac{\tilde f_1}{\Delta f}\gamma_5(\theta)\cot\right.$$ [redacted]
$$\langle\delta f_2^{(1)}\delta\varphi^{(1)}\rangle = -2\frac{\tilde f_1\tilde f_2}{\Delta f}\sin(2\psi)\gamma_5(\theta)\csc\theta,$$
$$\langle\delta f_2^{(1)}\delta\phi^{(1)}\rangle = -\tilde f_2\frac{f_1}{\tilde f_1}\sin(2\psi)\gamma_3(\theta),$$
$$\langle\delta\theta^{(1)}\delta\theta^{(1)}\rangle = 2\sigma_\perp^2+\frac{1}{\Delta f^2}\left(\tilde f_1^2+\tilde f_2^2+2\tilde f_1\tilde f_2\cos(2\psi)\right)\gamma_6(\theta),$$
$$\langle\delta\theta^{(1)}\delta\psi^{(1)}\rangle = \frac{1}{\Delta f}\left[\frac{f_1\tilde f_2}{\tilde f_1}\gamma_4(\theta)+\frac{\tilde f_1 f_2}{\tilde f_2}\gamma_5(\theta)\right]\sin(2\psi)-\cos\theta\,\langle\delta\theta^{(1)}\delta\varphi^{(1)}\rangle,$$
$$\langle\delta\theta^{(1)}\delta\varphi^{(1)}\rangle = \frac{2\tilde f_1\tilde f_2\sin(2\psi)}{\Delta f^2}\gamma_6(\theta)\csc\theta,$$
$$\langle\delta f_1^{(2)}\rangle = \frac{1}{\Delta f}\left[\tilde f_1^2 l_1(\theta)-(f_1f_2-1)l_3(\theta)\right],$$

$$\langle\delta\theta^{(1)}\delta\phi^{(1)}\rangle = \frac{1}{4\Delta f}\left[\left(f_1\frac{\tilde f_2}{\tilde f_1}-f_2\frac{\tilde f_1}{\tilde f_2}\right)(\sigma_1^2-\sigma_2^2)\sin\theta+4\left(f_1\frac{\tilde f_2}{\tilde f_1}+f_2\frac{\tilde f_1}{\ldots}\right.\right.$$ [redacted] $$\left.\right]\sin(2\psi)$$
$$\langle\delta\psi^{(1)}\delta\psi^{(1)}\rangle = \frac{1}{4}\left[\left(\frac{f_1}{\tilde f_1}\right)^2\gamma_1(\theta)+\left(\frac{f_2}{\tilde f_2}\right)^2\gamma_2(\theta)\right]-$$ [redacted] $$\sigma_2^2\big)+2\sin^2\theta\sigma_\perp^2-\cos\theta\left(2\langle\delta\psi^{(1)}\delta\varphi^{(1)}\rangle+\cos\theta\,\langle\delta\varphi^{(1)}\delta\varphi^{(1)}\rangle\right),$$
$$\langle\delta\psi^{(1)}\delta\varphi^{(1)}\rangle = -\frac{1}{\Delta f}\left(\frac{f_1\tilde f_2}{\tilde f_1}\gamma_4(\theta)\right.$$ [redacted] $$\left(\sigma_1^2+\sigma_2^2\right)-3\sigma_\perp^2\Big]\cos\theta-\cos\theta\,\langle\delta\varphi^{(1)}\delta\varphi^{(1)}\rangle,$$
$$\langle\delta\psi^{(1)}\delta\phi^{(1)}\rangle$$ [redacted] $$\theta-\langle\delta\varphi^{(1)}\delta\phi^{(1)}\rangle\cos\theta,$$

[redacted]

CENSORED

$$-\sigma_2^2)\left[5(f_1-f_2)+\left(\tilde f_1\frac{f_2}{\tilde f_2}-\tilde f_2\frac{f_1}{\tilde f_1}\right)\cos(2\psi)\right],$$
$$4\left(\frac{f_1^2}{\tilde f_1^2}-\frac{f_2^2}{\tilde f_2^2}\right)\cos\theta-\frac{1}{2}\left[\frac{f_1^2}{\tilde f_1^2}+\frac{f_2^2}{\tilde f_2^2}-2\frac{f_1}{\tilde f_1}\frac{f_2}{\tilde f_2}\cos(2\psi)\right]\gamma_3(\theta),$$
$$\Big],$$
$$-\sigma_2^2\big)\sin\theta-\frac{1}{2}\sin(2\theta)\left(\sigma_1^2+\sigma_2^2-3\sigma_\perp^2\right)+\frac{1}{\Delta f}\left(\langle\delta f_2^{(1)}\delta\theta^{(1)}\rangle-\langle\delta f_1^{(1)}\delta\theta^{(1)}\rangle\right)+\frac{1}{4}\sin(2\theta)\langle\delta\varphi^{(1)}\delta\varphi^{(1)}\rangle,$$
$$\frac{}{8}\left(\frac{f_1}{\tilde f_2}+\frac{\tilde f_2}{\tilde f_1}\right)\left(\sigma_1^2+\sigma_2^2-2\sigma_\perp^2\right)\sin^2\theta\sin(2\psi)-\frac{1}{2}\left(\frac{f_1}{\tilde f_1^2}\langle\delta f_1^{(1)}\delta\phi^{(1)}\rangle-\frac{f_2}{\tilde f_2^2}\langle\delta f_2^{(1)}\delta\phi^{(1)}\rangle\right)-\frac{1}{2}\left(\frac{f_1}{\tilde f_1^2}\langle\delta f_1^{(1)}\delta\psi^{(1)}\rangle+\frac{f_2}{\tilde f_2^2}\langle\delta f_2^{(1)}\delta\psi^{(1)}\rangle\right)$$
$$-\frac{1}{2}\left(\frac{f_1}{\tilde f_1^2}\langle\delta f_1^{(1)}\delta\varphi^{(1)}\rangle+\frac{f_2}{\tilde f_2^2}\langle\delta f_2^{(1)}\delta\varphi^{(1)}\rangle\right)\cos\theta+\frac{1}{2}\sin\theta\,\langle\delta\theta^{(1)}\delta\varphi^{(1)}\rangle-\cos\theta\langle\delta\varphi^{(2)}\rangle$$
$$-\frac{1}{2}\left(\frac{\tilde f_1}{\tilde f_2}-\frac{\tilde f_2}{\tilde f_1}\right)\left[\frac{1}{4}\left(\langle\delta\varphi^{(1)}\delta\varphi^{(1)}\rangle\sin^2\theta-\langle\delta\theta^{(1)}\delta\theta^{(1)}\rangle\right)\sin2\psi+\frac{1}{2}\langle\delta\varphi^{(1)}\delta\theta^{(1)}\rangle\sin\theta\cos2\psi\right],$$
$$\langle\delta\varphi^{(2)}\rangle = \frac{1}{\Delta f}\left(\langle\delta f_2^{(1)}\delta\varphi^{(1)}\rangle-\langle\delta f_1^{(1)}\delta\varphi^{(1)}\rangle\right)-\cot\theta\,\langle\delta\theta^{(1)}\delta\varphi^{(1)}\rangle,$$
$$\langle\delta\phi^{(2)}\rangle = 0$$

Two fields

Six parameters now, f_1, f_2 and

$$\mathbf{u}(\phi,\theta,\psi,\varphi) = e^{-\frac{i}{2}\phi}\begin{bmatrix} \cos\frac{\theta}{2}\,e^{-\frac{i}{2}(\varphi+\psi)} & -\sin\frac{\theta}{2}\,e^{-\frac{i}{2}(\varphi-\psi)} \\ \sin\frac{\theta}{2}\,e^{\frac{i}{2}(\varphi-\psi)} & \cos\frac{\theta}{2}\,e^{\frac{i}{2}(\varphi+\psi)} \end{bmatrix}$$

Solution of full FP equation can be bypassed,

$$\partial_\tau\langle\ln(1+n)\rangle = \left\langle \frac{1}{2(1+n)}\sum_{a=1}^{N_\mathrm{f}}\frac{\langle\delta f_a\rangle_{\delta\tau}}{\delta\tau} - \frac{1}{8(1+n)^2}\sum_{a,b=1}^{N_\mathrm{f}}\frac{\langle\delta f_a\delta f_b\rangle_{\delta\tau}}{\delta\tau}\right\rangle$$

$$\xrightarrow{\tau\to\infty}\left\langle l(\theta)-\frac{1}{2}\gamma(\theta)\right\rangle$$

with

$$\gamma(\theta) = 2\left[\cos^4\left(\tfrac{\theta}{2}\right)\sigma_1^2 + \sin^4\left(\tfrac{\theta}{2}\right)\sigma_2^2 + 4\sin^2\left(\tfrac{\theta}{2}\right)\cos^2\left(\frac{\theta}{2}\right)\sigma_\perp^2\right]$$

$$l(\theta) = 2\left[\cos^2\left(\tfrac{\theta}{2}\right)\sigma_1^2 + \sin^2\left(\tfrac{\theta}{2}\right)\sigma_2^2 + \sigma_\perp^2\right]$$

Two fields

Six parameters now, f_1, f_2 and

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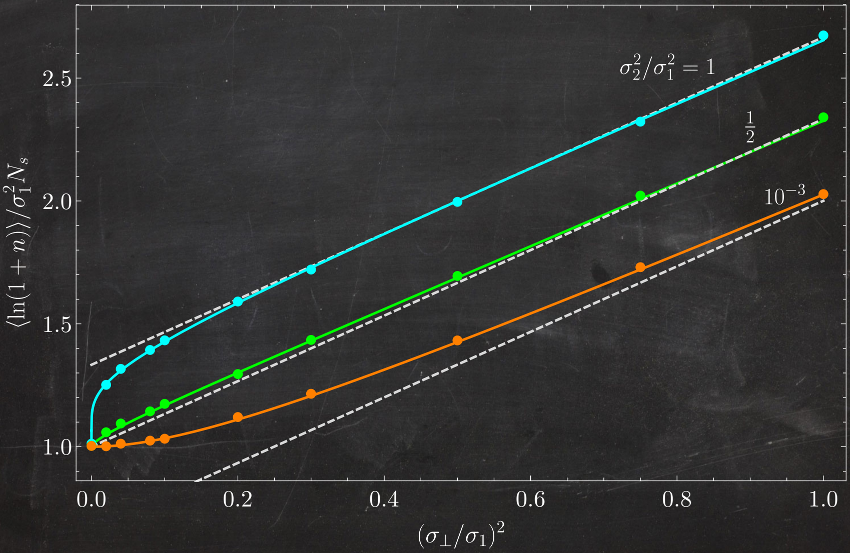
$$\xrightarrow{\tau\to\infty} \left\langle l(\theta) - \frac{1}{2}\gamma(\theta)\right\rangle$$

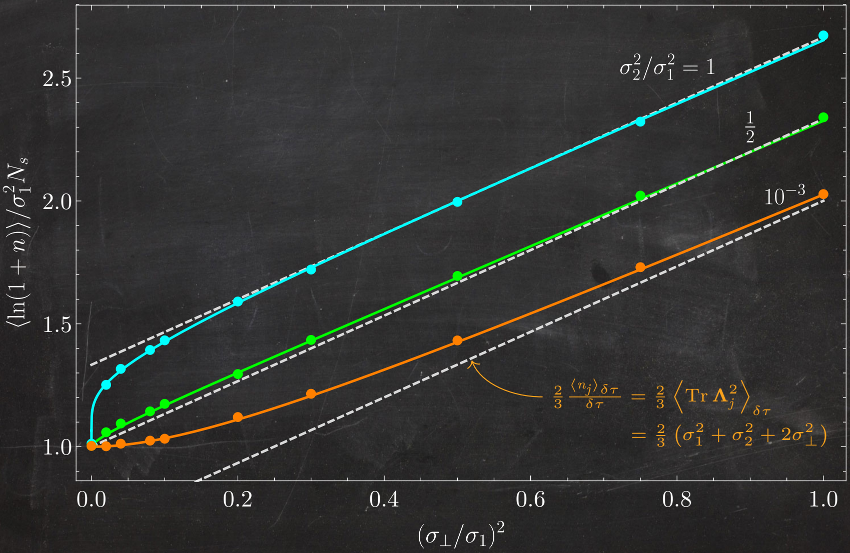
with

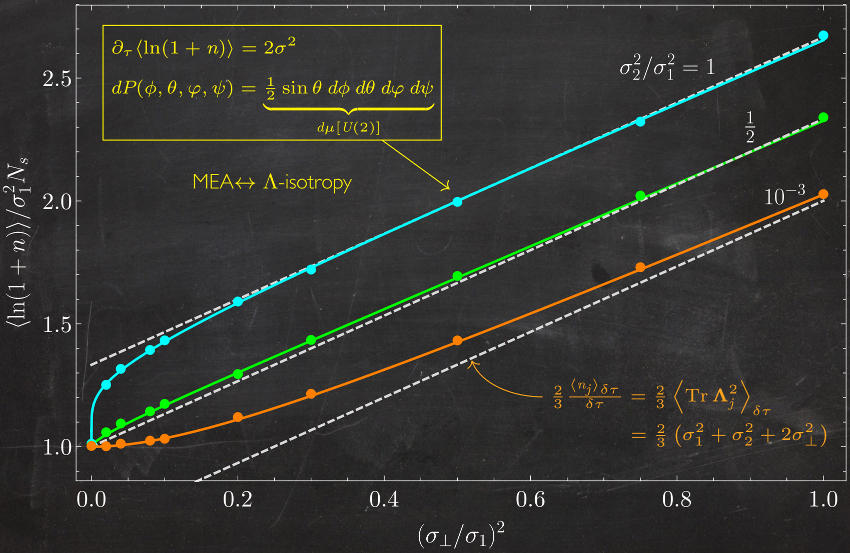
$$\int P\, d\mathbf{f}\, d\phi\, d\varphi\, d\psi = \frac{1}{\mathcal{N}}\frac{\sin\theta}{Q\sin^2\theta+1}\exp\left[2\nu\,\mathrm{arctanh}\left(\sqrt{\frac{Q}{1+Q}}\cos\theta\right)\right]$$

$$Q = \frac{\sigma_1^2+\sigma_2^2-4\sigma_\perp^2}{8\sigma_\perp^2}, \qquad \nu = \frac{|\sigma_1^2-\sigma_2^2|}{8\sigma_\perp^2\sqrt{|Q|(1+Q)}}$$



$\sigma_2^2/\sigma_1^2 = 1$
$\frac{1}{2}$
10^{-3}
$\langle \ln(1+n) \rangle / \sigma_1^2 N_s$
$(\sigma_\perp/\sigma_1)^2$
2.5
2.0
1.5
1.0
0.0
0.2
0.4
0.6
0.8
1.0


$\sigma_2^2/\sigma_1^2 = 1$
$\frac{1}{2}$
10^{-3}
$\frac{2}{3}\frac{\langle n_j\rangle_{\delta\tau}}{\delta\tau} = \frac{2}{3}\left\langle \mathrm{Tr}\,\mathbf{\Lambda}_j^2\right\rangle_{\delta\tau}$
$= \frac{2}{3}\left(\sigma_1^2+\sigma_2^2+2\sigma_\perp^2\right)$
2.5
2.0
1.5
1.0
0.0
0.2
0.4
0.6
0.8
1.0
$\langle \ln(1+n)\rangle/\sigma_1^2 N_s$
$(\sigma_\perp/\sigma_1)^2$



$\partial_\tau \langle \ln(1+n) \rangle = 2\sigma^2$
$dP(\phi,\theta,\varphi,\psi) = \underbrace{\frac{1}{2}\sin\theta\, d\phi\, d\theta\, d\varphi\, d\psi}_{d\mu[U(2)]}$
MEA$\leftrightarrow$ Λ-isotropy
$\sigma_2^2/\sigma_1^2 = 1$
$\frac{1}{2}$
10^{-3}
$\frac{2}{3}\frac{\langle n_j \rangle_{\delta\tau}}{\delta\tau} = \frac{2}{3}\left\langle \mathrm{Tr}\, \mathbf{\Lambda}_j^2 \right\rangle_{\delta\tau}$
$= \frac{2}{3}\left(\sigma_1^2 + \sigma_2^2 + 2\sigma_\perp^2\right)$
$\langle \ln(1+n) \rangle / \sigma_1^2 N_s$
1.0
1.5
2.0
2.5
0.0
0.2
0.4
0.6
0.8
1.0
$(\sigma_\perp/\sigma_1)^2$

$N_{\rm f}$ fields

$N_{\rm f}(N_{\rm f}+1)$ parameters now, $f_1, f_2, \ldots, f_{N_{\rm f}}$ and (Tilma, Sudarshan 2002)

$$\mathbf{u} = \left(\prod_{2\leqslant k\leqslant N} \mathbf{A}(k) \right) \cdot [SU(N-1)] \cdot e^{i\boldsymbol{\lambda}_{N^2-1}\alpha_{N^2-1}} \,, \qquad \mathbf{A}(k) = e^{i\boldsymbol{\lambda}_3\alpha_{(2k-3)}}\, e^{i\boldsymbol{\lambda}_{(k-1)^2+1}\alpha_{2(k-1)}}$$

N_f fields

$N_\mathrm{f}(N_\mathrm{f}+1)$ parameters now, $f_1, f_2, \ldots, f_{N_\mathrm{f}}$ and (Tilma, Sudarshan 2002)

$$\mathbf{u} = \left(\prod_{2\leqslant k\leqslant N} \mathbf{A}(k) \right) \cdot [SU(N-1)] \cdot e^{i\boldsymbol{\lambda}_{N^2-1}\alpha_{N^2-1}}\,, \qquad \mathbf{A}(k) = e^{i\boldsymbol{\lambda}_3\alpha_{(2k-3)}}\, e^{i\boldsymbol{\lambda}_{(k-1)^2+1}\alpha_{2(k-1)}}$$

$\Rightarrow$ need $\mathcal{O}({N_\mathrm{f}}^4)$ correlators!

N_f fields

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$\Rightarrow$ need $\mathcal{O}(N_\mathrm{f}{}^4)$ correlators! Assume

$$\boldsymbol{\sigma}^2 = \begin{pmatrix} \sigma_1^2/2 & \sigma_\perp^2 & \cdots & \sigma_\perp^2 \\ \sigma_\perp^2 & \sigma_2^2/2 & \cdots & \sigma_\perp^2 \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_\perp^2 & \sigma_\perp^2 & \cdots & \sigma_2^2/2 \end{pmatrix}$$

$\Rightarrow$ Results depend only on the angle $\theta = \alpha_{2(N_\mathrm{f}-1)}$ and

$$\mathcal{F}(\Omega_{N_\mathrm{f}}) = \mathcal{F}(\Omega_{N_\mathrm{f}-1})\cos^4(\alpha_{2(N_\mathrm{f}-2)}/2) + \sin^4(\alpha_{2(N_\mathrm{f}-2)}/2)$$

with $\mathcal{F}(\Omega_2) = 1$

N_f fields

$N_\mathrm{f}(N_\mathrm{f}+1)$ parameters now, $f_1, f_2, \ldots, f_{N_\mathrm{f}}$ and (Tilma, Sudarshan 2002)

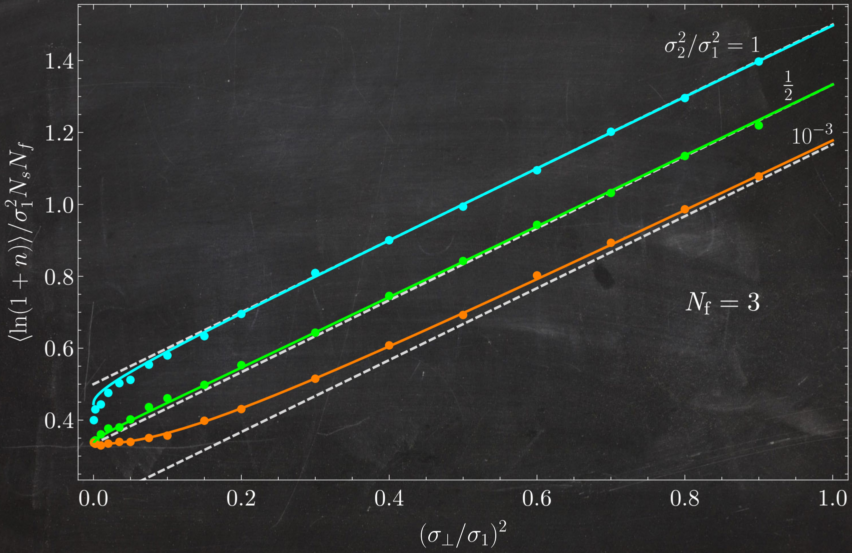
$$\mathbf{u} = \left(\prod_{2 \leqslant k \leqslant N} \mathbf{A}(k) \right) \cdot [SU(N-1)] \cdot e^{i\boldsymbol{\lambda}_{N^2-1}\alpha_{N^2-1}}\,, \qquad \mathbf{A}(k) = e^{i\boldsymbol{\lambda}_3\alpha_{(2k-3)}}\, e^{i\boldsymbol{\lambda}_{(k-1)^2+1}\alpha_{2(k-1)}}$$

$\Rightarrow$ need $\mathcal{O}({N_\mathrm{f}}^4)$ correlators! Assume

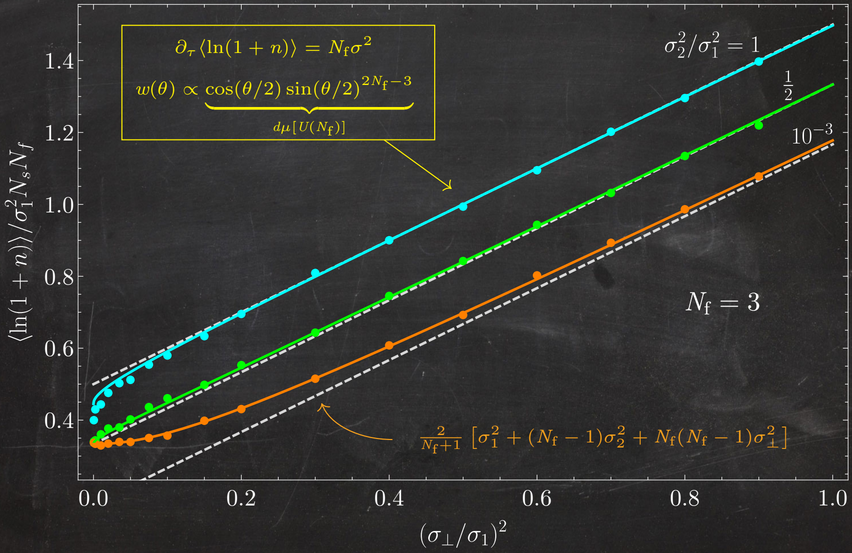
$$\boldsymbol{\sigma}^2 = \begin{pmatrix} \sigma_1^2/2 & \sigma_\perp^2 & \cdots & \sigma_\perp^2 \\ \sigma_\perp^2 & \sigma_2^2/2 & \cdots & \sigma_\perp^2 \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_\perp^2 & \sigma_\perp^2 & \cdots & \sigma_2^2/2 \end{pmatrix}$$

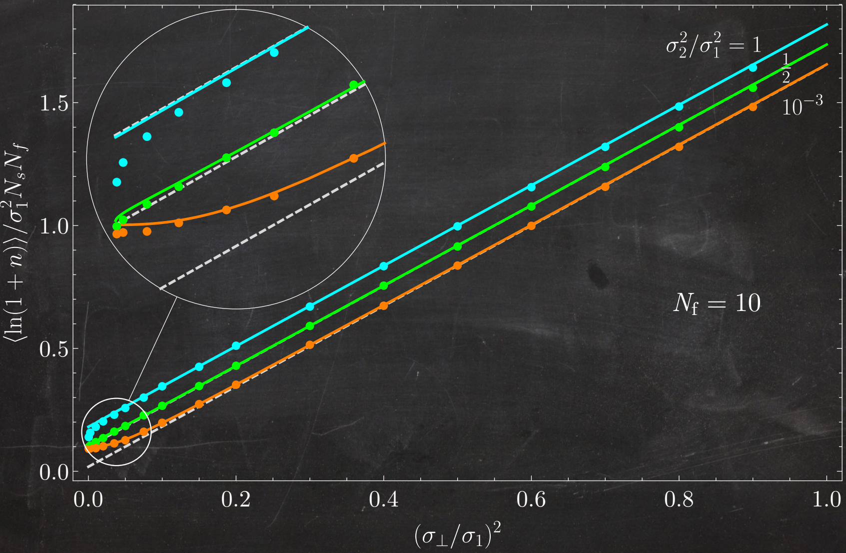
$$\partial_\tau \langle \ln(1+n) \rangle \;=\; \left\langle l(\theta) - \frac{1}{2}\gamma(\theta, \Omega_{N_\mathrm{f}}) \right\rangle$$

$$l(\theta) = 2\left[\sigma_1^2 \cos^2(\theta/2) + \sigma_2^2 \sin^2(\theta/2) + \sigma_\perp^2 (N_\mathrm{f}-1)\right]$$
$$\gamma(\theta, \Omega_{N_\mathrm{f}}) = 2\Big[\sigma_1^2 \cos^4(\theta/2) + \sigma_2^2 \sin^4(\theta/2)\, \mathcal{F}_\Omega + 2\sigma_\perp^2 \left(1 - \cos^4(\theta/2) - \sin^4(\theta/2)\, \mathcal{F}_\Omega\right)\Big]$$

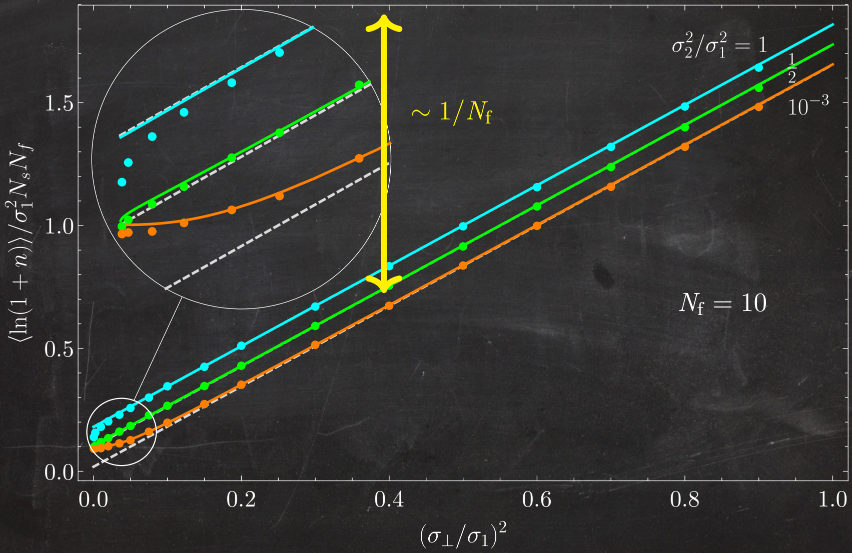


$\sigma_2^2/\sigma_1^2 = 1$
$\frac{1}{2}$
10^{-3}
$N_{\rm f} = 3$
$\langle \ln(1+n) \rangle / \sigma_1^2 N_s N_f$
$(\sigma_\perp/\sigma_1)^2$
0.0
0.2
0.4
0.6
0.8
1.0
0.4
0.6
0.8
1.0
1.2
1.4


$\partial_\tau \langle \ln(1+n) \rangle = N_f \sigma^2$
$w(\theta) \propto \underbrace{\cos(\theta/2) \sin(\theta/2)^{2N_f-3}}_{d\mu[U(N_f)]}$
$\sigma_2^2/\sigma_1^2 = 1$
$\frac{1}{2}$
10^{-3}
$N_f = 3$
$\frac{2}{N_f+1} [\sigma_1^2 + (N_f-1)\sigma_2^2 + N_f(N_f-1)\sigma_\perp^2]$
$\langle \ln(1+n) \rangle / \sigma_1^2 N_s N_f$
$(\sigma_\perp/\sigma_1)^2$
0.0
0.2
0.4
0.6
0.8
1.0
1.4
1.2
1.0
0.8
0.6
0.4



$\sigma_2^2/\sigma_1^2 = 1$
$\frac{1}{2}$
10^{-3}
$N_\mathrm{f} = 10$
$\langle \ln(1+n)\rangle/\sigma_1^2 N_s N_f$
$(\sigma_\perp/\sigma_1)^2$
0.0
0.5
1.0
1.5
0.0
0.2
0.4
0.6
0.8
1.0


$\sigma_2^2/\sigma_1^2 = 1$
$\frac{1}{2}$
10^{-3}
$\sim 1/N_{\rm f}$
$N_{\rm f} = 10$
$\langle \ln(1+n) \rangle / \sigma_1^2 N_s N_f$
$(\sigma_\perp/\sigma_1)^2$
0.0
0.5
1.0
1.5
0.0
0.2
0.4
0.6
0.8
1.0

Conclusion

- Avoid relying on detailed model building, and take a coarse grained approach to the particle production in the early universe
- MEA captures the universal features arising from a Central Limit Threorem (concentration of measure)…
- …as long as there's no hierarchy of couplings
- Break from weak scattering limit $\Rightarrow$ Random Matrix Theory?
- Next: include expansion and metric perturbations

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Thank you