ESTIMATES ON THE \((N - 1)\)-DIMENSIONAL HAUSDORFF MEASURE OF THE BLOW-UP SET FOR A SEMILINEAR HEAT EQUATION

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ESTIMATES ON THE \((N - 1)\)-DIMENSIONAL HAUSDORFF MEASURE OF THE BLOW-UP SET FOR A SEMILINEAR HEAT EQUATION*

J.J.L. VELÁZQUEZ**

Abstract. We consider the following problem

\[ u_t = \Delta u + u^p, \quad x \in \mathbb{R}^N, t > 0 \]
\[ u(x, 0) = u_0(x), \quad x \in \mathbb{R}^N \]

where \(1 < p < \frac{N+2}{N-2}\), and \(u_0(x)\) is a continuous nonnegative and bounded function. It is shown here that for any blowing-up solution which is different from the uniform one \(u_T(x, t) = ((p - 1)(T - t))^{-\frac{1}{p-1}}\) the \((N - 1)\)-dimensional Hausdorff measure of the blow-up set is bounded in compact sets of \(\mathbb{R}^N\).

Key words. Blow-up set, Hausdorff measure, semilinear parabolic equations.

AMS(MOS) subject classifications. 35B40, 35K55, 35K57

1. Introduction. This paper deals with the following problem

\begin{align}
(1.1) & \quad u_t = \Delta u + u^p \quad \text{when} \ x \in \mathbb{R}^N (N \geq 1), t > 0, 1 < p < \frac{N+2}{N-2} \\
(1.2) & \quad u(x, 0) = u_0(x) \quad \text{when} \ x \in \mathbb{R}^N
\end{align}

where \(u_0(x)\) is continuous, nonnegative and bounded. It is well known that solutions of (1.1), (1.2) may develop singularities in a finite time. In particular, a solution \(u(x, t)\) is said to blow-up in a finite time \(T\) if \(u(x, t)\) satisfies (1.1), (1.2) in \(S_T = \mathbb{R}^N \times (0, T)\), and

\[ \limsup_{t \uparrow T, x \in \mathbb{R}^N} u(x, t) = +\infty \]

A point \(x_0\) is called a blow-up point if there exists sequences \(\{x_n\}, \{t_n\}\) such that \(\lim_{n \to \infty} x_n = x_0, \lim_{n \to \infty} t_n = T\) and \(\lim_{n \to \infty} u(x_n, t_n) = +\infty\). The set \(S\) consisting of all the blow-up points is termed the blow-up set. We refer to [BBE], [B], [FM], [L], [W] for earlier results in blow-up problems. Throughout this paper, we shall assume that:

\begin{align}
(1.3a) & \quad u(x, t) \leq M(T - t)^{-\frac{1}{p-1}} \text{ for any } x \in \mathbb{R}^N, t < T \text{ and some constant } M > 0. \\
(1.3b) & \quad \lim_{t \uparrow T} (T - t)^{-\frac{1}{p-1}} u(x_0 + y(T - t)^{1/2}, t) = (p - 1)^{-\frac{1}{p-1}} \text{ uniformly on bounded sets } |y| \leq R \text{ with } R > 0.
\end{align}

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We remark that conditions (1.3) hold under our current assumptions on the initial value $u_0(x)$ if $1 < p < \frac{N+2}{N-2}$ (cf. [GP], [GK1], [GK2], [GK3]).

A question that has deserved great attention is that of the structure of the blow-up set $S$. In the one-dimensional case it was obtained in [CM] that for initial values with compact support every blow-up point is isolated (cf [FM] [W] for earlier results in this direction and [HV3] for a different proof). In the $(N-1)$-dimensional case a natural generalization of the previous result would be to show that the blow-up set is contained in $(N-1)$-dimensional structures. The result that we obtain in this paper asserts that this is true from a measure theoretical point of view. More precisely we have the following:

**Theorem.** Assume that $u(x,t)$ satisfies (1.1), (1.2), (1.3) and $u(x,t) \neq ((p-1)(T-t))^{-\frac{1}{p-1}}$. Suppose that $u_0(\cdot)$ is continuous nonnegative and bounded, then, for any fixed $R > 0$, there holds:

\[(1.4) \quad \mathcal{H}^{N-1}(S \cap B_R(0)) < +\infty\]

where $\mathcal{H}^{N-1}$ is the standard $(N-1)$-dimensional Hausdorff measure, and $B_R(0) = \{x \in \mathbb{R}^N : |x| < R\}$.

It is easy to check that the previous result is sharp. Indeed, if we take an one-dimensional solution $u_1(x,t)$ blowing up in a finite set of points, we can make up a $N$-dimensional solution blowing up in a set of $(N-1)$-dimensional planes by setting, say, $u(x,t) = u_1(x_1,t)$, where $x = (x_1, x_2 \ldots x_N)$. In [GK3] was also obtained the existence of solutions whose blow-up set is a $(N-1)$-dimensional sphere. A problem which is closely related to that of the structure of the blow-up set is the study of the asymptotic behaviour of solutions of (1.1), (1.2) near a blow-up point. Since some results concerning this problem will be used in the sequel, we shall proceed to describe briefly the current state of such question. Following [GP], [GK1], we introduce similarity variables as follows

\[(1.5a) \quad u(x,t) = (T-t)^{-\frac{1}{p-1}} \Phi \left( \frac{x-x_0}{(T-t)^{1/2}}, \log(T-t) ; x_0 \right) \]

\[(1.5b) \quad y = \frac{x-x_0}{(T-t)^{1/2}}, \quad \tau = -\log(T-t)^{1/2}. \]

Then $\Phi(y, \tau)$ satisfies the equation

\[(1.6) \quad \Phi_{\tau} = \Delta \Phi - \frac{y \cdot \nabla(\Phi)}{2} + p \Phi^p - \frac{1}{(p-1) \Phi}. \]

For $k \geq 1, 1 < q < +\infty$ we consider the weighted spaces

\[L^q_\omega(\mathbb{R}^N) = \left\{ f \in L^q_{loc}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |f(s)|^q e^{-s^2/4} ds < +\infty \right\} \]

\[H^k_\omega(\mathbb{R}^N) = \left\{ f \in L^2_\omega(\mathbb{R}^N) : D^\alpha f \in L^2_\omega(\mathbb{R}^N) \text{ for } \alpha = (\alpha_1, \ldots, \alpha_N), |\alpha| \leq k, \alpha_i = 0, 1 \ldots \text{ for } 1 \leq i \leq N \right\} \]
where we use the standard notation \( D^{\alpha} f = \frac{\partial^{\alpha}}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}} f \), and \( |\alpha| = \alpha_1 + \cdots + \alpha_N \). Norms in \( L^2_\omega(\mathbb{R}^N) \), \( H^k_\omega(\mathbb{R}^N) \) are defined in a natural way. In particular
\[
\|f\|_{q,\omega}^{2} = \int_{\mathbb{R}^N} |f(s)|^q e^{-s^2} ds
\]
The \( L^2_\omega \)-norm will be used repeatedly in the sequel, and we shall write it in the form \( \|\|_2,\omega \equiv \|\| \). The scalar product in \( L^2_\omega(\mathbb{R}^N) \) will be denoted as \( , >. \). An operator that will be used very often is
\[
A \psi = \Delta \psi - \frac{y \cdot \nabla \psi}{2} + \psi,
\]
\( A \) is a self-adjoint operator in \( L^2_\omega(\mathbb{R}^N) \) with domain \( H^2_\omega(\mathbb{R}^N) \). Its eigenvalues \( H_\alpha(y) \) are given by \( H_\alpha(y) = H_{\alpha_1}(y_1) \cdots H_{\alpha_N}(y_N) \), where \( \alpha = (\alpha_1, \ldots, \alpha_N) \), \( \alpha_i \) is a nonnegative integer for \( 1 \leq i \leq N \), \( y = (y_1, \ldots, y_N) \), \( \overline{H}_m(y) = c_m \widetilde{H}_m (\frac{y_i}{2}) \), \( c_m = (2^{m/2} (4\pi)^{1/4} (m!)^{1/2})^{-1} \) and \( \widetilde{H}_m(\cdot) \) is the standard \( m \)-th-Hermite polynomial. The corresponding eigenvalues are given by \( \lambda_\alpha = 1 - \frac{|\alpha|}{2} \).

We then define \( \psi = \Phi - (p-1)^{-\frac{1}{p-1}} \). Then \( \psi \) satisfies
\[
\frac{d\psi}{d\tau} = A\psi + f(\psi).
\]
where \( f(\psi) \) is a suitable function \( f(\psi) \), that may be bounded as \( |f(\psi)| \leq C |\psi|^2 \), where \( C \) is a positive constant depending only on \( p \) and \( M \) given in (1.3a). The following result, which will play a key role in our approach, has been proved in [V1].

**Theorem A.** Let \( u(x,t) \) be the solution of (1.1), (1.2) and assume that (1.3) holds. Then, if \( \psi(y,\tau) \neq 0 \) for some \( \tau > 0 \), the following alternative arises. Either there exists an orthogonal transformation of the coordinate axes such that denoting still by \( y \) the new coordinates,
\[
(1.9a) \quad \psi(y,\tau,x_0) = -\frac{C_p}{\tau} \sum_{k=1}^{\ell} H_2(y_k) + o\left(\frac{1}{\tau}\right) \text{ as } \tau \to \infty.
\]
where \( 1 \leq \ell \leq N \), \( C_p = \frac{(4\pi)^{1/4} (p-1)^{-\frac{1}{p-1}}}{\sqrt{2p}} \) and \( y = (y_1, \ldots, y_N) \),
or there exist an even number \( m \), \( m \geq 4 \), and constants \( C_\alpha \) not all zero, such that
\[
(1.9b) \quad \psi(y,\tau,x_0) = -e^{(1-\frac{m}{2})\tau} \sum_{|\alpha|=m} C_\alpha H_\alpha(y) + o(e^{(1-\frac{m}{2})\tau}) \text{ as } \tau \to \infty,
\]
where the homogeneous multilinear form
\[
B(x) = \sum_{|\alpha|=m} \bar{c}_\alpha C_\alpha x^\alpha
\]
is nonnegative. Here \( \bar{c}_\alpha = c_{\alpha_1} \cdots c_{\alpha_N} \),
and \( x^\alpha = x_1^{\alpha_1} \cdots x_N^{\alpha_N} \).
In (1.9a) and (1.9b), convergence takes place in $H^1_\omega(R^N)$ as well as in $C^{k,\gamma}_{loc}(R^N)$ for any $k \geq 0$ and $\gamma \in (0,1)$.

Theorem A was previously shown in [HV1], [HV2] for the case $N = 1$. On the other hand, related results were independently and simultaneously obtained in [FK] and [FL], where the authors proved that, under our current assumptions, (1.9a) must hold unless there is an exponential decay of $\psi(y, \tau)$ as $\tau \to \infty$. It was proved also in [HV2], [HV3] that the behaviours corresponding to (1.6a), (1.6b) with $N = 1$ determine the asymptotic profile of the singularities appearing at $t = T$. In [V2], the results of [HV2], [HV3] were extended to the $N$-dimensional case, and in particular, it was obtained there that for a solution $u$ of (1.1), (1.2) such that $u(x, t) \neq (p - 1)^{-\frac{1}{p-1}}(T - t)^{-\frac{1}{p-1}}$ the Lebesgue measure of the blow-up set is zero, $L^N(S) = 0$. To conclude this Introduction, we point out that we shall keep to the notations in [F] for the rest of the paper.

2. Preliminary Estimates. In this Section we recall some results that will be used in the proof of the Theorem. We shall also obtain some estimates that in many cases are improvements of results previously obtained in [HV1], [V1]. These will be used later in Section 3.

Let $\tau \geq 0$, $1 < q < +\infty$ and $\psi \in L^q_{loc}(R^N)$. Following [V2], we define

\begin{equation}
L^q_\tau(\psi) \equiv \sup_{|\xi| \leq \tau} \left( \int_{R^n} |\psi(y)|^q \exp \left( -\frac{(y - \xi)^2}{4} \right) dy \right)^{1/q}.
\end{equation}

We denote as $S(\tau) \equiv S_A(\tau)$ the semigroup associated to the operator $A$ defined in (1.7). We then have

**Lemma 2.1.** Assume that $1 < q, \beta < +\infty$ and $r, \bar{r} \geq 0$. Set $\beta' = \frac{\bar{r}}{\beta - 1}$. Then, for any $\tau > 0$ and any $\psi$ such that $L^q_\tau(\psi) < +\infty$, there holds

\begin{equation*}
L^q_\tau(S(\tau)\psi) \leq \frac{e^\tau}{(4\pi(1 - e^{-\tau}))^{N/2}} \left( \frac{4\pi\beta(1 - e^{-\tau})}{\beta'(\beta - (1 - e^{-\tau}))} \right)^{\frac{N}{2\beta'}} \cdot \left( \frac{4\pi(\beta - (1 - e^{-\tau}))}{(\beta - 1) - (q - 1)e^{-\tau}} \right)^{\frac{s_+}{4\beta}} \exp \left( \frac{e^{-\tau}(r - \bar{r}e^{\tau}/2)^2}{4((\beta - 1) - (q - 1)e^{-\tau})} \right) L^q_\bar{r}(\psi)
\end{equation*}

where $s_+ = \max\{s, 0\}$.

**Proof.** See [V2], Proposition 2.1. []

For each blow-up point $x_0 \in S$ we can expand $\psi$ in terms of its Fourier-Hermite series

\begin{equation}
\psi(y, \tau; x_0) = \sum_\alpha a_\alpha(\tau; x_0) H_\alpha(y).
\end{equation}

A key point in our approach consists in the following.

\[\]
LEMMA 2.2. Assume that $1 < q < +\infty$, and suppose that $\psi(y, \tau_0, x_0) = \sum a_\alpha(\tau_0, x_0)H_\alpha(y)$. Then there exists $L$ depending on $p, q, M, N$, where $M$ is given in (1.3) such that, for any $\tilde{L} > L$ there exist $C_1, \delta_0$ depending on $p, q, M, \tilde{L}, N$ such that if $||\psi(y, \tau_0, x_0)|| \leq \delta_0$ then

\begin{align*}
(2.4) \quad ||\psi(\tau; x_0) - \sum a_\alpha(\tau_0, x_0)e^{(1 - \frac{1}{2})\tau - \tau_0)}H_\alpha(\cdot)||_{L^q, \omega} + \\
+||\psi(\tau; x_0) - \sum a_\alpha(\tau_0, x_0)e^{(1 - \frac{1}{2})\tau - \tau_0)}H_\alpha(\cdot)||_{L^\infty(\mathbb{R}^N)} \leq \\
\leq C_1 ||\psi(\tau_0; x_0)||^{3/2} \quad \text{for } \tau_0 + L \leq \tau \leq \tau_0 + \tilde{L}.
\end{align*}

Proof. We define

\begin{align*}
(2.5) \quad \bar{\psi}(\tau; x_0) = S(\tau - \tau_0) \left( \sum a_\alpha(\tau_0; x_0)H_\alpha(\cdot) \right) = \sum a_\alpha(\tau_0; x_0)e^{(1 - \frac{1}{2})\tau - \tau_0)}H_\alpha(\cdot).
\end{align*}

Notice that $\bar{\psi}$ satisfies

\begin{align*}
(2.6) \quad \frac{d\bar{\psi}}{d\tau} = A\bar{\psi}
\end{align*}

From (1.8), (1.3) and Kato's inequality, we easily obtain

\begin{align*}
(2.7) \quad \frac{d(|\psi|)}{d\tau} \leq \Delta(|\psi|) - \frac{y\nabla(|\psi|)}{2} + C|\psi|
\end{align*}

where $C$ is a nonnegative constant depending only on $p, M$. From (2.7) and standard regularizing effects, it follows that

\begin{align*}
(2.8) \quad ||\psi(\tau; x_0)|| \leq ||\psi(\tau_0; x_0)||e^{C(\tau - \tau_0)}
\end{align*}

On the other hand, (1.8), (2.6) and Kato's inequality imply

\begin{align*}
\frac{d(|\psi - \bar{\psi}|)}{d\tau} \leq A(|\psi - \bar{\psi}|) + |f(\psi)| \leq A(|\psi - \bar{\psi}|) + C|\psi|^{3/2}
\end{align*}

where $C > 0$ depends only on $M, p$.

By the variation of parameters formula we obtain

\begin{align*}
(2.9) \quad |\psi(\tau; x_0) - \bar{\psi}(\tau; x_0)| \leq C\int_{\tau_0}^{\tau} S(\tau - s)|\psi(s; x_0)|^{3/2}ds
\end{align*}
Taking the $|| \cdot ||_{q, \omega}$ norm, we have that

\begin{equation}
||\psi(\tau; x_0) - \bar{\psi}(\tau; x_0)||_{q, \omega} \leq C \int_{\tau_0}^{\tau_0 + \frac{r - \tau_0}{2}} ||S(\tau - s)||_{L^4/3, L^6}\|\psi(\cdot; s; x_0)\|^{3/2}_{4/3, \omega} ds \\
+ C \int_{\tau_0 + \frac{r - \tau_0}{2}}^{\tau} ||S(\tau - s)||_{L^\infty, L^\infty}\|\psi(\cdot; s; x_0)\|^{3/2}_{3/2N, \omega} ds
\end{equation}

Using Lemma 2.1 with $r = \bar{r} = 0$ and (2.7) we obtain for $\tau - \tau_0 \geq L$ and some $\tilde{C} > 0$ depending on $M, p, N, q$ that

\begin{equation}
||\psi(\cdot; s; x_0)||_{3/2N, \omega} \leq \tilde{C} e^{s - \tau_0} ||\psi(\cdot; \tau_0; x_0)||
\end{equation}

By (2.10), (2.11) and Lemma 2.1 we arrive at

\begin{equation}
||\psi(\tau; x_0) - \bar{\psi}(\tau; x_0)||_{q, \omega} \leq \tilde{C} \int_{\tau_0}^{\tau_0 + \frac{r - \tau_0}{2}} e^{(s - \tau_0)} ||\psi(\cdot; s; x_0)||^{3/2} ds + \\
+ \tilde{C} \left( \int_{\tau_0 + \frac{r - \tau_0}{2}}^{\tau} \frac{e^{(s - \tau_0)} ds}{(1 - e^{r - \tau - s})^{1/2}} \right) ||\psi(\cdot; \tau_0; x_0)||^{3/2}
\end{equation}

and taking into account (2.8) we obtain the desired bound for the first term in the left-hand side of (2.4).

The corresponding estimate for the second term in the left of (2.4) follows in a similar way. Actually, taking the $H^1_\omega(R^N)$ norm on (2.9) we have that

\begin{equation}
||\psi(\tau; x_0) - \bar{\psi}(\tau; x_0)||_{H^1_\omega(R^N)} \leq C \int_{\tau_0}^{\tau_0 + \frac{r - \tau_0}{2}} ||S(\tau - s)|\psi(\cdot; s; x_0)||^{3/2}_{H^1_\omega(R^N)} ds \\
+ C \int_{\tau_0 + \frac{r - \tau_0}{2}}^{\tau} ||S(\tau - S)|\psi(\cdot; s; x_0)||^{3/2}_{H^1_\omega(R^N)} ds \equiv J_1 + J_2
\end{equation}
The regularizing effects for the semigroup \( S(\tau) \) (cf. [HV1]) yields

\[
|J_2| \leq Ce^{(\tau-\tau_0)} \sup \left\{ \|\psi(\cdot, s; x_0)\|^{3/2}_{H^0(L^2_\infty(\mathbb{R}^N))} : \tau_0 + \frac{\tau-\tau_0}{2} \leq s \leq \tau \right\} = 
C e^{(\tau-\tau_0)} \sup \left\{ \|\psi(\cdot, s; x_0)\|^{3/2}_{L^2_\infty(\mathbb{R}^N)} : \tau_0 + \frac{\tau-\tau_0}{2} \leq s \leq \tau \right\} \leq 
C e^{(C+1)(\tau-\tau_0)} \|\psi(\cdot, \tau_0; x_0)\|^{3/2}
\]

if \( L \) is large enough.

On the other hand, if \( \tau - \tau_0 > 2 \), there holds

\[
|J_1| = C \int_{\tau_0}^{\tau_0 + \frac{\tau-\tau_0}{2}} \|S(1)S(\tau - s - 1)\psi(\cdot, s; x_0)\|^{3/2}_{H^0(L^2_\infty(\mathbb{R}^N))} ds
\]

\[
\leq C \int_{\tau_0}^{\tau_0 + \frac{\tau-\tau_0}{2}} \|S(1)\|_{L^2(L^2_\infty(\mathbb{R}^N), H^0(L^2_\infty(\mathbb{R}^N)))} \|S(\tau - s - 1)\|_{L^2(L^2_\infty(\mathbb{R}^N), L^2_\infty(\mathbb{R}^N))} \cdot \|\psi(\cdot, s; x_0)\|^{3/2} ds 
\leq C e^{\frac{\tau-\tau_0}{2}(1 + (\tau - \tau_0))} \|\psi(\cdot, \tau_0; x_0)\|^{3/2}
\]

for \( L \) large enough.

Plugging (2.14), (2.15) into (2.13), the proof of (2.4) is concluded. \( \square \)

The next Lemma improves Proposition 4.1 in [HV1], and Lemma 2.3 in [V1]. The main difference with those results is that we now give an estimate for the lower modes in (2.3) independent of the blow-up point, and depending only on the smallness of the function \( \|\psi\| \).

**Lemma 2.3.** Assume that \( x_0 \in S \). There exist \( \delta_0 > 0 \), and a function \( g : [0, \delta_0) \to \mathbb{R}^+ \) with \( \lim_{r \to 0^+} \frac{g(r)}{r} = 0 \), depending only on \( N, p, M \), but not on \( x_0 \) such that, if \( \|\psi(\cdot, \tau_0; x_0)\| \leq \delta_0 \), then \( \sum_{|\alpha| \leq 1} |a_\alpha(\tau; x_0)| \leq g(\|\psi(\cdot, \tau; x_0)\|) \). Moreover

\[
(2.16) \quad a_0(\tau; x_0) \leq 0
\]

Proof. Without loss of generality we can assume that \( \tau_0 = 0, T = 1 \). Set \( \Phi_0(x) \equiv (p - 1)^{-\frac{1}{p-1}} + \psi(\cdot, 0; x_0) \). We define the function

\[
(2.17) \quad w(x, t) \equiv (\tilde{S}(t)\Phi_0(\cdot))^{-(p-1)} - (p - 1)t^{-\frac{1}{p-1}}
\]

where \( \tilde{S}(t) \) is the semigroup associated to the heat equation. A straightforward calculation shows that \( w \) is a subsolution of (1.1) (cf [HV1], [V1]). On the other hand

\[
(2.18) \quad S_*(t)\Phi_0(\cdot) = (p - 1)^{-\frac{1}{p-1}} + \sum_{\alpha} a_\alpha(a_0 x_0)(1 - t)^{\frac{|\alpha|}{2}} H_\alpha \left( \frac{x}{(1 - t)^{1/2}} \right).
\]
and \( w(x,0) = \Phi_0(x) \).

Notice that \( S_0(1)\Phi_0(\cdot) = (p - 1)^{-\frac{1}{p-1}} + a_0 H_0(y) \). We then observe that the corresponding solution of (1.1), with initial value \( \Phi_0(x)\bar{u}(x,t) \) is defined until \( t = 1 \) only if \( a_0 \leq 0 \), since \( \bar{u}(x,t) \geq w(x,t) \). This proves (2.16). We now obtain the existence of \( g_1 : [0, \delta_0) \to \mathbb{R}^+ \) such that, for \( \delta_0 \) small enough and \( \|\psi(0,0;x_0)\| \leq \delta|a_0(0;x_0)| \leq g(\|\psi(0,0;x_0)\|) \).

Assume on the contrary that there exist initial values \( \psi_n(\cdot,0,x_0) \), such that \( a_0,n(\cdot, x_0) < -\varepsilon \|\psi_n(\cdot,0,x_0)\|, \|\psi_n(\cdot,0,x_0)\| \leq 1/n \), where \( \varepsilon > 0 \) is a positive number, and \( \psi_n(\cdot, \tau, x_0) \to 0 \) as \( n \to \infty \). We will prove that these assumptions give a contradiction. To this end, we argue as in [GK3], (cf. also [FK], [FL]), and define the energy

\[
E(\Phi) = \frac{1}{2} \int_{\mathbb{R}^N} (\nabla \Phi)^2 e^{-y^2/4} dy - \frac{1}{(p+1)} \int_{\mathbb{R}^N} \Phi^{p+1} e^{-y^2/4} dy \\
+ \frac{1}{2} \int_{\mathbb{R}^N} \Phi^2 e^{-y^2/4} dy
\]

It is readily seen that

(2.19) \[ \frac{dE}{d\tau}(\Phi(\cdot, \tau, x_0)) \leq 0 \]

Suppose that \( E(\Phi_0) < E((p - 1)^{-\frac{1}{p-1}}) \). Then (2.19) and the results in [GK1], [GK2], [GK3] imply that \( \psi(\cdot, \tau; x_0) \to -(p - 1)^{-\frac{1}{p-1}} \) as \( \tau \to \infty \) in \( L^2_\omega(\mathbb{R}^N) \). By Lemma (2.2), we readily obtain

\[
\|\psi_n(\cdot, \tau, x_0) - \sum_\alpha a_{\alpha,n}(0;x_0) e^{(1 - \frac{1}{2p})\tau} H_\alpha(\cdot)\| + \\
\|\psi_n(\cdot, \tau, x_0) - \sum_\alpha a_{\alpha,n}(0;x_0) e^{(1 - \frac{1}{2p})\tau} H_\alpha(\cdot)\| + \\
\|\psi_n(\cdot, \tau; x_0) - \sum_\alpha a_{\alpha,n}(0;x_0) e^{(1 - \frac{1}{2p})\tau} H_\alpha(\cdot)\|_{H^s_\omega(\mathbb{R}^N)} \leq \\
\leq C(1/n)^{3/2}
\]

for \( \tilde{L} \leq \tau \leq L \), and some constant \( C \) depending on \( L, N, M, p \).

Set \( \Phi_n = (p - 1)^{-\frac{1}{p-1}} + \psi_n \). Carrying (2.20) into the definition of \( E(\Phi_n) \), we obtain after some straightforward manipulations.

\[
|E(\Phi_n(\cdot, \tau)) - E((p - 1)^{-\frac{1}{p-1}}) - p(p - 1)^{-\frac{1}{p-1}} a_0,n(0;x_0)e^\tau(H_0, 1)| \\
\leq \tilde{C}e^{\tau/2}\|\psi_n(\cdot,0;x_0)\| + \overline{C}\|\psi_n(\cdot,0;x_0)\|^{3/2}
\]

for \( \tilde{L} \leq \tau \leq L \), where \( L \) is arbitrary, and \( \tilde{C} > 0 \) is a constant depending on \( N, p, M \), but not on \( L \), and \( \overline{C} \) depends on \( L \) also. Set now \( \tau = L \). Using the fact that
\[ a_{0,n}(0, x_0) \leq -\varepsilon \| \psi_n(\cdot, 0, x_0) \|, \] we obtain that, if we take \( L \) large enough, and then let \( n \to \infty \), \( E(\Phi_n(\cdot, \tau)) < E((p - 1)^{-\frac{1}{p - 1}}) \), and by (2.19) this gives a contradiction. This proves the existence of \( \delta_{0,1} > 0 \) and \( g_1(0, \delta_{0,1}) \to \mathbb{R}^+ \) such that \( \lim_{r \to 0^+} \frac{g_1(r)}{r} = 0 \) and

\[
(2.21) \quad |a_0(0, x_0)| \leq g_1(\| \psi(\cdot, 0, x_0) \|)
\]

if \( \| \psi(\cdot, 0, x_0) \| \leq \delta_{0,1} \).

We are now ready to conclude the proof of Lemma 2.3. Assume that there exist functions \( \psi_n(\cdot, \tau, x_0) \) such that

\[
\| \psi_n(\cdot, 0; x_0) \| \leq 1/n, \quad n = 0, 1, 2 \quad \| \psi_n(\cdot, \tau; x_0) \| \leq M
\]

\[
\sum_{|\alpha|=1} |a_{\alpha,n}(0; x_0)| \geq \varepsilon \| \psi_n(\cdot, 0; x_0) \|
\]

for some fix \( \varepsilon > 0 \) and \( \psi_n(\cdot, \tau, x_0) \to 0 \) as \( \tau \to \infty \). We then consider the subsolution \( w \) given in (2.17), with \( \Phi_{0,n} = (p - 1)^{-\frac{1}{p - 1}} + \psi_n(\cdot, 0; x_0) \) there. Without loss of generality we can assume (after a suitable rotation of the coordinate axes) that

\[
\sum_{|\alpha|=1} a_{\alpha,n}(0, x_0) H_\alpha(y) = \tilde{a} H_1(y_1) H_0^{N-1}, \quad \text{where} \quad \tilde{a} \geq \varepsilon \| \psi_n(\cdot, 0, x_0) \|.
\]

Set \( x_1 = \eta > 0 \) to be precised, \( x_2 = \cdots x_N = 0 \), and notice that

\[
S_x(t) \Phi_{0,n}(\cdot) = (p - 1)^{-\frac{1}{p - 1}} + \tilde{a} c_1 c_0^{N-1} \eta + a_0(0, x_0) H_0(\cdot) +
\]

\[
+ \sum_{|\alpha| \geq 2} a_{\alpha}(0; x_0)(1-t)^{\frac{1}{2}} H_\alpha \left( \frac{x}{(1-t)^{1/2}} \right) \equiv (p - 1)^{-\frac{1}{p - 1}} +
\]

\[
+ \tilde{a} c_1 c_0^{N-1} \eta + a_0(0; x_0) H_0(\cdot) + \mu(x, t)
\]

Arguing as in [HV1], Lemma 6.1 or in [V1], Lemma 2.7, we arrive at

\[
|\mu(x, t)| \leq C(|x|^2 + |t - 1|) \| \psi_n(\cdot, 0, x_0) \|, \quad \text{for} \quad |x| \leq 1, \quad |t - 1| \leq 1/2.
\]

Then, if \( \eta, (t - 1) \) are small enough, and \( n \) is large enough, we obtain

\[
S_x(t) \Phi_{0,n}(\cdot) \geq (p - 1)^{-\frac{1}{p - 1}} + \frac{K \varepsilon}{2n} \eta,
\]

where \( K \) is a constant depending only on \( N \). Arguing as in the previous case, we deduce that \( \psi(\cdot, \tau, x_0) \) cannot be defined for arbitrarily large \( \tau \). This concludes the proof of Lemma 2.3. \( \square \)

As a next step, we obtain a bound of the coefficients \( a_\alpha \), for \( |\alpha| = 2 \). For any given \( \psi(\cdot, \tau, x_0) \) and any \( \tau \), we can define a matrix \( G(\tau) \in M_N(\mathbb{R}) \) as follows (cf. [V1])

\[
G(\tau) = (G_{i,j}), 1 \leq i, j \leq N, \quad \text{where}
\]

\[
G_{i,j} = \begin{cases} \sqrt{2}(\psi(\cdot, \tau; x_0), H_2(y_i)) H_0^{N-1} & \text{if } i = j \\ (\psi(\cdot, \tau; x_0), H_1(y_i) H_1(y_j)) H_0^{N-2} & \text{if } i \neq j \end{cases}
\]

We then have
Lemma 2.4. Assume that $\psi(\cdot, \tau, x_0) \to 0$ as $\tau \to \infty$. Then there exist $\delta_0 > 0$ and $g : [0, \delta_0) \to \mathbb{R}^+$ depending only on $p, N, M$, satisfying $\lim_{\tau \to 0^+} g(\tau) = 0$, and such that if $\|\psi(\cdot, \tau, x_0)\| \leq \delta_0$, then $G(\tau) \leq g(\|\psi(\cdot, \tau, x_0)\|)I$, (i.e. for any $\xi \in \mathbb{R}^N$
$(\xi, G(\tau)\xi) \leq g(\|\psi(\cdot, \tau, x_0)\|)\|\xi\|^2$).

Proof. We can reduce ourselves to the case $\tau_0 = 0$, $T = 1$. Suppose that there exist a sequence of functions $\psi_n(\cdot, \tau, x_0)$ with $\|\psi_n(\cdot, 0, x_0)\| \leq 1/n$, and $\xi \in \mathbb{R}^N$ such that, for some $\varepsilon \in (0, 1)$.

\begin{equation}
(\xi, G_n(0)\xi) \geq \varepsilon\|\psi_n(\cdot, 0, x_0)\|\|\xi\|^2
\end{equation}

We then recall (2.17) and consider the subsolution $w_n(x, t)$ corresponding to the initial value $\Phi_{0,n}(x) = (p - 1)^{-\frac{1}{p-1}} + \psi_n(\cdot, 0, x_0)$. Let $u_n(x, t)$ be the corresponding solution of (1.1) with initial value $\Phi_{0,n}(x)$. Then

\begin{equation}
w_n(x, t) \leq u_n(x, t).
\end{equation}

On the other hand

\begin{equation}
S_x(t)\Phi_{0,n}(\cdot) = (p - 1)^{-\frac{1}{p-1}} + \sum_{|\alpha| \leq 1} a_{\alpha,n}(0; x_0)(1-t)^\frac{|\alpha|}{2} H_\alpha\left(\frac{x}{(1-t)^{1/2}}\right) + \\
+ \sum_{|\alpha| = 2} a_{\alpha,n}(0; x_0)(1-t)^\frac{|\alpha|}{2} H_\alpha\left(\frac{x}{(1-t)^{1/2}}\right) + \sum_{|\alpha| \geq 3} a_{\alpha,n}(a; x_0)(1-t)^\frac{|\alpha|}{2} H_\alpha\left(\frac{x}{(1-t)^{1/2}}\right) =
\end{equation}

\begin{equation}
(p - 1)^{-\frac{1}{p-1}} + \mu_1(x, t) + \mu_2(x, t) + \mu_3(x, t)
\end{equation}

Arguing as in [HV1] we obtain

\begin{equation}
|\mu_3(x, t)| \leq C(|x|^3 + |1 - t|)\|\psi_n(\cdot, 0, x_0)\|.
\end{equation}

Set $x = \eta \xi$, where $\eta > 0$ is small. It is readily seen that $\sum_{|\alpha| = 2} a_{\alpha,n}(0, x_0)\xi^\alpha = K_N(\xi, G_n(0)\xi)$ where $K_N > 0$. Then, for any fixed $\eta > 0$, we obtain

\begin{equation}
\lim_{t \to 1^-} \left(\sum_{|\alpha| = 2} a_{\alpha,n}(0, x_0)(1-t)^\frac{|\alpha|}{2} H_\alpha\left(\frac{x}{(1-t)^{1/2}}\right)\right) = K_N(\xi, G_n(0)\xi)\eta^2
\end{equation}

uniformly on $n$.

By (2.25), (2.27) and Lemma 2.3 we then deduce that

\begin{equation}
S_x(t)\Phi_{0,n}(\cdot) \geq (p - 1)^{-\frac{1}{p-1}} - g(\|\psi_n(\cdot, 0, x_0)\|) + \frac{K_N}{2}\eta^2\varepsilon\|\psi_n(\cdot, 0, x_0)\| - \\
- C(\eta^3 + (1-t))\|\psi_n(\cdot, 0, x_0)\| \geq (p - 1)^{-\frac{1}{p-1}} + \frac{K_N}{2}\eta^2\varepsilon\|\psi_n(\cdot, 0, x_0)\|
\end{equation}

for $\eta$ small enough and $n$ large enough. Then, by (2.24) $u(x, t)$ blows up for some $\bar{t} < 1$ and this provides a contradiction. ∎

We next show
**Lemma 2.5.** There exist $\tilde{\delta}_0 > 0$, $C_2 > 0$, $\tilde{L} > 0$ depending only on $p, M, N, m$, such that if the following condition is satisfied

\[ (2.28a) \quad \sum_{|\alpha| \leq m} (a_{\alpha}(\tau_0; x_0))^2 \geq 2 \sum_{|\alpha| \geq m+1} (a_{\alpha}(\tau_0; x_0))^2 \]

and

\[ (2.28b) \quad \|\psi(\cdot, \tau_0; x_0)\| \leq \tilde{\delta}_0 \]

then, for any $\tau \geq \tilde{L} + \tau_0$, there holds

\[ (2.29a) \quad \sum_{|\alpha| \leq m} (a_{\alpha}(\tau; x_0))^2 \geq \sum_{|\alpha| \geq m+1} (a_{\alpha}(\tau; x_0))^2, \]

and, moreover

\[ (2.29b) \quad \|\psi(\cdot, \tau; x_0)\| \leq C_2 \|\psi(\cdot, \tau_0; x_0)\| \]

**Proof.** We denote by $\lambda_i(\tau; x_0), i = 1, \ldots, N$ the eigenvalues of $G(\tau)$, which, we can assume to be differentiable (cf. [K]). Define now the auxiliary functions

\[ (2.30a) \quad M(\tau; x_0) = \frac{\sum_{|\alpha| \geq m+1} (a_{\alpha}(\tau; x_0))^2}{\sum_{|\alpha| \leq m} (a_{\alpha}(\tau; x_0))^2} \]

\[ (2.30b) \quad S(\tau; x_0) = \sum_{|\alpha| \geq 3} (a_{\alpha}(\tau; x_0))^2, \]

\[ (2.30c) \quad W(\tau; x_0) = \sum_{\lambda_i(\tau; x_0) < 0} (\lambda_i(\tau; x_0))^2. \]

By Lemma 2.1, we readily get that

\[ (2.31) \quad \|f(\psi(\cdot, \tau; x_0))\| \leq C\|\psi(\cdot, \tau - \theta; x_0)\|^2, \]

where $\theta > 0$ is a fixed number, and $C$ depends only on $N, p$ and $M$. On the other hand, arguing as in [HV1]. Section 4, we have that if $M(s; x_0) \leq 1$ for $s \in [\tau - \theta, \tau]$, then:

\[ (2.32) \quad \|\psi(\cdot, \tau - \theta; x_0)\| \leq C\|\psi(\cdot, \tau; x_0)\| \]

where $C$ depends only on $p, N, M$. Then (2.31), (2.32) show that, if $M(s; x_0) \leq 1$ for $s \in [\tau - \theta, \tau]$, $\|f(\psi(\cdot, \tau; x_0))\| \leq C\|\psi(\cdot, \tau; x_0)\|^2$. 

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Furthermore, by Lemma 2.2 and (2.28a), we derive for $\delta_0$ small enough (depending only on $p, M, N, n$) that $M(\tau; x_0) \leq 1$ for $\tau \in [\tau_0 + L, \tau_0 + L + 2\theta]$. We then may assume that $M(\tau; x_0) \leq 1$, for $\tau = \tau_0 + L + \theta$, and assume $\|f(\psi(\cdot, \tau; x_0))\| \leq C\|\psi(\cdot, \tau; x_0)\|^2$ as long as $M(\tau; x_0) \leq 1$. We now proceed to obtain evolution equations for the functions defined in (2.30). To this end, notice that, as far as $M(\tau; \lambda_0) \leq 1$ there holds:

\begin{equation}
M(\tau; x_0) = 2 \left( \sum_{|\alpha| \leq m} (a_\alpha(\tau; x_0))^2 \right)^{-2} \cdot \\
\cdot \left\{ \left( \sum_{|\alpha| \geq m+1} \left( 1 - \frac{|\alpha|}{2} \right) (a_\alpha(\tau; x_0))^2 \right) \left( \sum_{|\alpha| \leq m} (a_\alpha(\tau; x_0))^2 \right) + \\
+ \left( \sum_{|\alpha| \geq m+1} a_\alpha(\tau; x_0)(f(\psi(\cdot, \tau; x_0)), H_\alpha(\cdot)) \right) \left( \sum_{|\alpha| \leq m} (a_\alpha(\tau; x_0))^2 \right) - \\
- \left( \sum_{|\alpha| \geq m+1} (a_\alpha(\tau; x_0))^2 \right) \left( \sum_{|\alpha| \leq m} \left( 1 - \frac{|\alpha|}{2} \right) (a_\alpha(\tau; x_0))^2 \right) - \\
- \left( \sum_{|\alpha| \geq m+1} (a_\alpha(\tau; x_0))^2 \right) \left( \sum_{|\alpha| \leq m} a_\alpha(\tau; x_0)(f(\psi(\cdot, \tau; x_0)), H_\alpha(\cdot)) \right) \right\} \leq \\
\leq \left( \sum_{|\alpha| \leq m} (a_\alpha(\tau; x_0))^2 \right)^{-3/2} \left( \sum_{|\alpha| \geq m+1} (a_\alpha(\tau; x_0))^2 \right)^{1/2} \cdot \\
\cdot \left\{ - \left( \sum_{|\alpha| \leq m} (a_\alpha(\tau; x_0))^2 \right)^{1/2} \left( \sum_{|\alpha| \geq m+1} (a_\alpha(\tau; x_0))^2 \right)^{1/2} \\
+ C\|\psi(\cdot, \tau; x_0)\|^2 \left[ \left( \sum_{|\alpha| \leq m} (a_\alpha(\tau; x_0))^2 \right)^{1/2} + \left( \sum_{|\alpha| \geq m+1} (a_\alpha(\tau; x_0))^2 \right)^{1/2} \right] \right\}
\end{equation}

Assume that $M(\bar{\tau}; x_0) = 1$ for some $\bar{\tau} \geq \tau_0 + L + \theta$. Then if $\|\psi(\cdot, \tau; x_0)\|$ is small enough (depending only on $M, p, N, m$) (2.33) implies that $\dot{M}(\tau; x_0) < 0$. If $\delta_0$ is small enough, Lemma 2.2 shows that $\|\psi(\cdot, \tau; x_0)\| \leq C_2\|\psi(\cdot, \tau_0; x_0)\|$ for $\tau \in [\tau_0 + L, \tau_0 + L + 2\theta]$, and some $C_2 > 0$ as in the statement of Lemma 2.5. In the rest of the proof we shall obtain that in fact (2.29a) holds for any $\tau > \tau_0 + L + \theta$. Notice that

\begin{equation}
\dot{S}(\tau; x_0) \leq -S(\tau; x_0) + C(S(\tau; x_0))^{1/2}\|\psi(\cdot, \tau; x_0)\|^2
\end{equation}

if $M(s; x_0) \leq 1$ for $s \in [\tau - \theta, \tau]$. 

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Finally, we proceed to estimate the variation of $W(\tau)$. By the results of [K] p. 32, we have that

$$ W(\tau; x_0) = -\frac{1}{2\pi i} \int_{\tilde{\gamma}} Tr(\lambda^2 R(G; \lambda)) d\lambda, $$

where $R(G; \lambda) = (G - \lambda)^{-1}$, and $\tilde{\gamma}$ is a circuit in the complex plane enclosing the eigenvalues of $G$ in the region $\text{Re}\lambda < 0$. Without loss of generality we can assume that

$$ \tilde{\gamma} = \{ \lambda : \text{Im}(\lambda) \in [-1, +1], \text{Re}(\lambda) = 0 \} \cup \{ \lambda : |\lambda| = 1, \text{Re}(\lambda) \leq 0 \} $$

where the line is orientated in the positive sense. It is not difficult to see that the integral in the right-hand side of (2.35) is also well defined if $\lambda = 0$ is a point of the spectrum of $G(\tau)$.

By (1.8), (2.22) and Lemma 2.2, it is readily seen that

$$ ||\dot{G}(\tau) - \sqrt{2}\gamma_N \nu_p G(\tau)||^2 \leq C||G(\tau)||((S(\tau))^{1/2} + C(S(\tau))^2 + h(\|\psi\|) $$

where $h : [0, \delta_0] \to \mathbb{R}$, $\lim_{r \to 0^+} \frac{h(r)}{r^{3/2}} = 0$, $\nu_p = \frac{p(p-1)\pi}{2}, \gamma_N = \sqrt{2}(4\pi)^{-1/2}$.

Notice that

$$ \dot{W}(\tau; x_0) = \frac{1}{2\pi i} \int_{\tilde{\gamma}} Tr(\lambda^2 R(G(\tau); \lambda) \dot{G}(\tau) R(G(\tau); \lambda)) d\lambda $$

This, together with (2.35), yields

$$ \left| \dot{W}(\tau; x_0) - 2\sqrt{2}\gamma_N \nu_p \sum_{\lambda_i(\tau; x_0) < 0} (\lambda_i(\tau; x_0))^2 \right| \leq C(W(\tau; x_0))(S(\tau; x_0))^{1/2} $$

$$ + C(W(\tau; x_0))^{1/2}(S(\tau; x_0)) + h(\|\psi(\cdot, t; x_0)\|) $$

if $||G(\tau)|| \leq 1/2$. Inequality (2.34) and (2.37) hold true as far as $M(\tau; x_0) \leq 1$.

On the other hand Lemmata 2.2–2.3 imply that $||\psi(\cdot, \tau; x_0)|| \leq 2(W(\tau; x_0) + S(\tau; x_0))$ if $||\psi(\cdot, \tau; x_0)||$ is small enough. It then suffices to estimate $W(\tau; x_0) + S(\tau; x_0)$. Notice that

$$ \frac{d}{d\tau} (W(\tau; x_0) + S(\tau; x_0)) \leq -K_{N,p}(W(\tau; x_0))^{3/2} - S(\tau; x_0) + $$

$$ + C(W(\tau; x_0))^{1/2}(S(\tau; x_0) + C(W(\tau; x_0))(S(\tau; x_0)) + C(S(\tau; x_0))^{1/2}||\psi(\cdot, \tau, x_0)||^2 $$

$$ + h(\|\psi(\cdot, \tau; x_0)\|) \leq -K_{N,p}(W(\tau; x_0))^{3/2} - \frac{1}{2}S(\tau; x_0) \leq 0 $$

where $K_{N,p} > 0$ and $||\psi(\cdot, \tau; x_0)||$ is small enough. Then, if $\delta_0$ is sufficiently small, and $M(\tau_0) \leq 1/2$, we have that the same bound holds for $\tau \geq \tau_0 + L + \theta$. This concludes the proof of Lemma 2.5. \]

We will also need the following result that gives some information about points in the neighborhood of a blow-up point.

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LEMMA 2.6. Assume that \( \psi(y, \tau_0; x_0) = \sum_{\alpha} a_{\alpha}(\tau_0; x_0) H_{\alpha}(y) \). Then there exists \( L > 0 \) depending on \( p, M,N \) such that for any \( \tilde{L} > L \) and \( \theta > 0 \) there are \( C_3 > 0, \delta_0 > 0 \) depending on \( p, M, \tilde{L}, N \) such that

\[
L_0^2 \left( \psi(\cdot, \tau; x_0) - \sum_{\alpha} a_{\alpha}(\tau_0; x_0)e^{(1-\frac{\theta}{2})(\tau-\tau_0)}H_{\alpha}(\cdot) \right) \leq
\]

\[
C_3\|\psi(\cdot, \tau_0; x_0)\|^{3/2}, \quad \text{for } \tau_0 + L \leq \tau \leq \tau_0 + \tilde{L},
\]

where \( L_0^2(\cdot) \) is given in (2.1).

Proof. Taking the \( L_0^2 \)-norm in (2.9) we obtain

\[
L_0^2(|\psi(\cdot, \tau; x_0) - \tilde{\psi}(\cdot, \tau; x_0)|) \leq C \int_{\tau_0}^{\tau_0 + \frac{\tau - \tau_0}{2}} e^{(\tau-s)}L_0^2(S(\tau-s)|\psi(\cdot, s; x_0)|^{3/2})ds
\]

\[
+ C \int_{\tau_0 + \frac{\tau - \tau_0}{2}}^{\tau} e^{(\tau-s)}L_0^2(S(\tau-s)|\psi(\cdot, s; x_0)|^{3/2})ds.
\]

(2.39)

A slight modification of the proof in Lemmata 2.1 and 2.2 yields then the desired result. \( \square \)

LEMMA 2.7. For any positive integer \( m \), and any sequence \( \{b_{\alpha}\}_{|\alpha| \leq m} \) there holds

\[
\lim_{\nu \to 0} \sum_{|\gamma| \leq m} \left( \langle H_{\gamma}(\cdot), \sum_{|\alpha| \leq m} b_{\alpha} H_{\alpha}(\cdot + \nu) \rangle - b_{\gamma} \right)^2 \sum_{|\alpha| \leq m} b_{\alpha}^2 = 0
\]

(2.40)

uniformly on the sequence \( \{b_{\alpha}\}_{|\alpha| \leq m} \).

Proof. We can assume that \( \sum_{|\alpha| \leq m} b_{\alpha}^2 = 1 \). Then, the proof of the result follows at once from the uniform continuity of the function

\[
G(\nu; b_{\alpha}) = \sum_{|\gamma| \leq m} \left( \langle H_{\gamma}(\cdot), \sum_{|\alpha| \leq m} b_{\alpha} H_{\alpha}(\cdot + \nu) \rangle - b_{\gamma} \right)^2 \sum_{|\alpha| \leq m} b_{\alpha}^2 = 1, |\nu| \leq 1.
\]

\( \square \)

By Theorem A, for any blow-up point \( x_0 \in S \), we can define an integer \( m(x_0) \) if (1.9a) holds. We will say that \( m(x_0) = 2 \) if (1.9b) holds. Our next result gives an uniform bound of \( m(x_0) \) in bounded sets of \( \mathbb{R}^N \).

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Lemma 2.8. For each fixed $R > 0$ and $\delta > 0$ there exists an integer $\ell_R > 0$ such that, for any $x_0 \in S \cap \overline{B_R(0)}$, and for any $\tau \geq \tau_{R,\delta}$, there holds

\[ (2.41a) \quad \sum_{|\alpha| \leq \ell_R} (a_\alpha(\tau; x_0))^2 \geq \sum_{|\alpha| \geq \ell_R} (a_\alpha(\tau; x_0))^2 \]

and

\[ (2.41b) \quad \|\psi(\cdot, \tau; x_0)\| \leq \delta. \]

Moreover, $m(x_0) \leq \ell_R$, for any $x_0 \in S \cap \overline{B_R(0)}$.

Proof. By Theorem A, for any $\bar{x} \in S \cap \overline{B_R(0)}$, any $\lambda > 0$ say $\delta > 0$ there exists $\bar{r}(\bar{x}, \delta, \lambda)$ such that

\[ (2.42) \quad \sum_{|\alpha| \leq m(\bar{x})} (a_\alpha(\tau; \bar{x}))^2 \geq \lambda \sum_{|\alpha| > m(\bar{x})} (a_\alpha(\tau; \bar{x}))^2 \]

and

\[ (2.43) \quad \|\psi(\cdot, \tau; \bar{x})\| \leq \delta, \quad \text{for } \tau \geq \bar{r}(\bar{x}; \delta, \lambda) = \bar{r} \]

By Lemma 2.6, we have that for $\theta = 1$.

\[ L_\theta^2 \left( \psi(\cdot, \tau; \bar{x}) - \sum_{|\alpha| \leq m(\bar{x})} a_\alpha(\bar{r}, \bar{x})e^{(1-\frac{\lambda|\alpha|}{2})(\tau-\bar{r})}H_\alpha(\cdot) \right) \leq \]

\[ L_\theta^2 \left( \sum_{|\alpha| > m(\bar{x})} a_\alpha(\bar{r}, \bar{x})e^{(1-\frac{\lambda|\alpha|}{2})(\tau-\bar{r})}H_\alpha(\cdot) \right) + C_3 \|\psi(\cdot, \bar{r}; \bar{x})\|^{3/2} \]

for $\bar{r} + L \leq \tau \leq \tau_0 + \bar{L}$.

On the other hand, if $\bar{L}$ is large enough, Lemma 2.1 implies that

\[ (2.45) \quad L_\theta^2 \left( \sum_{|\alpha| > m(\bar{x})} a_\alpha(\tau, \bar{x})e^{(1-\frac{\lambda|\alpha|}{2})(\tau-\bar{r})}H_\alpha(\cdot) \right) \leq \overline{C} \left( \sum_{|\alpha| > m(\bar{x})} (a_\alpha(\bar{r}, \bar{x}))^2 \right)^{1/2} \]

for $\tau_0 + \frac{\bar{L}}{2} \leq \tau \leq \tau_0 + \bar{L}$, and some constant $\overline{C} > 0$ depending only on $N, p, M$.

It is readily seen that

\[ \psi(y, \tau, x_0) = \psi(y + (x_0 - \bar{x})e^{\tau/2}, \tau, \bar{x}) \]
Assume now that \( e^{\tau/2}|x_0 - \bar{x}| \leq \frac{1}{2} \). Then (2.44), (2.45) imply

\[
\left\| \psi(\cdot, \tau; x_0) \sum_{|\alpha| \leq m(\bar{x})} a_\alpha(\bar{\tau}, \bar{x}) e^{(1 - \frac{|\alpha|}{2})(\tau - \bar{\tau})} H_\alpha(\cdot + (x_0 - \bar{x}) e^{\tau/2}) \right\| \\
\leq C \left( \sum_{|\alpha| > m(\bar{x})} (a_\alpha(\bar{\tau}, \bar{x}))^2 \right)^{1/2} + C_3 \| \psi(\cdot, \bar{\tau}, \bar{x}) \|^3/2, \quad \text{for } \bar{\tau} + \frac{\tilde{L}}{2} \leq \tau \leq \bar{\tau} + \bar{L}
\]

Moreover, if \( |x_0 - \bar{x}| e^{\tau/2} \) and \( \delta \) are small enough, Lemma 2.7 yields

\[
\sum_{|\alpha| \leq m(\bar{x})} (a_\alpha(\tau, x_0))^2 \geq \sum_{|\alpha| \leq m(\bar{x})} (a_\alpha(\bar{\tau}, \bar{x}))^2 e^{2(1 - \frac{|\alpha|}{2})(\tau - \bar{\tau})} + \\
- \varepsilon \| \psi(\cdot, \bar{\tau}, \bar{x}) \|^2 - C_2 \sum_{|\alpha| > m(\bar{x})} (a_\alpha(\bar{\tau}, \bar{x}))^2
\]

where \( \varepsilon > 0 \) is arbitrarily small, and \( \bar{\tau} + \frac{\tilde{L}}{2} \leq \tau \leq \bar{\tau} + \bar{L} \). On the other hand, by (2.45), we have that

\[
\sum_{|\alpha| > m(\bar{x})} (a_\alpha(\tau, x_0))^2 \leq C^2 \sum_{|\alpha| > m(\bar{x})} (a_\alpha(\bar{\tau}; \bar{x}))^2 + \varepsilon \| \psi(\cdot, \bar{\tau}; \bar{x}) \|^2
\]

Then, if \( \lambda \) is large enough in (2.42), we deduce that, for \( \bar{\tau} \in [\bar{\tau} + \frac{\tilde{L}}{2}, \bar{\tau} + \bar{L}] \),

\[
2 \sum_{|\alpha| > m(\bar{x})} (a_\alpha(\bar{\tau}, x_0))^2 \leq \sum_{|\alpha| \leq m(\bar{x})} (a_\alpha(\bar{\tau}; x_0))^2,
\]

and

\[
\| \psi(\cdot, \bar{\tau}, x_0) \| \leq \tilde{\delta}_0.
\]

where \( \tilde{\delta}_0 \) is given in as in (2.28b). Then, by Lemma 2.5 we obtain (2.41b) if \( |x_0 - \bar{x}| \) is small enough. Since \( S \cap \overline{B_R(0)} \) is a compact set a standard compactness argument concludes the proof of Lemma 2.8. \( \square \)

As a last result of this Section we can now obtain an improvement of Lemma 2.3.

**Lemma 2.9.** For any fixed \( R > 0 \), there exist \( \tilde{\delta} > 0 \), \( K > 0 \), and \( \mu > 0 \), depending only on \( M, p, N, R \), such that, for any blow-up point \( x_0 \in S \cap \overline{B_R(0)} \), and any \( \tau \geq \mu \) there holds

\[
\sum_{|\alpha| \leq 1} |a_\alpha(\tau; x_0)| \leq K \| \psi(\cdot, \tau; x_0) \|^{3/2}.
\]

**Proof.** Arguing as in Lemma 2.5 (cf. (2.31), (2.32)), and taking into account Lemma 2.8 we can assume that

\[
\| f(\psi(\cdot, \tau; x_0)) \| \leq C \| \psi(\cdot, \tau; x_0) \|^2 \text{ for } \tau \geq \tau_{R, \delta} + \bar{L}.
\]
Set
\[ E(\tau) = \sum_{|\alpha| \leq 1} (a_\alpha(\tau, x_0))^2 - \lambda \|\psi(\cdot, \tau, x_0)\|^4 \]
where \( \lambda > 0 \) will be precised later. A straightforward calculation shows that, for \( \tau \geq \tau_{R, \delta} + \tilde{L} \)
\[
\frac{dE}{d\tau} \geq \sum_{|\alpha| \leq 1} (a_\alpha(\tau; x_0))^2 - C \left( \sum_{|\alpha| \leq 1} (a_\alpha(\tau, x_0))^2 \right)^{1/2} \|\psi(\cdot, \tau, x_0)\|^2 - 
- C\lambda \left( \sum_{\alpha} (a_\alpha(\tau; x_0))^2 \right)^{3/2} \|\psi(\cdot, \tau, x_0)\|^2
\]
Assume that \( E(\tilde{\tau}) = 0 \) for some \( \tilde{\tau} \geq \tau_{R, \delta} + \tilde{L} \). Then
\[
\frac{dE}{d\tau} \geq \left( \sum_{|\alpha| \leq 1} (a_\alpha(\tau, x_0))^2 \right) - \frac{C}{\lambda^{1/2}} \sum_{|\alpha| \leq 1} (a_\alpha(\tau, x_0))^2 - 
- \frac{C}{\lambda^{1/2}} \left( \sum_{|\alpha| \leq 1} (a_\alpha(\tau, x_0))^2 \right)^{5/4} \geq 0
\]
provided that \( \lambda \) is large enough (depending only on \( R, M, N, p \)). On the other hand, by Lemma 2.8 we can always suppose that \( E(\tau_{R, \delta} + \tilde{L}) > 0 \). We then have that \( E(\tau) > 0 \) for \( \tau > \tau_{R, \delta} + \tilde{L} \equiv \mu \). On the other hand, \( E(\tau) > 0 \) for \( \tau > \mu \) implies that \( \psi(\cdot, \tau, x_0) \) does not converge to zero. Indeed, set
\[ S(\tau) = \sum_{|\alpha| \leq 1} (a_\alpha(\tau; x_0))^2 \]
Then, a routine computation yields
\[
\frac{dS}{d\tau} \geq \left( 1 - \frac{C}{\lambda^{1/2}} \right) \sum_{|\alpha| \leq 1} (a_\alpha(\tau; x_0))^2 > 0
\]
if \( \lambda > 0 \) is large enough. this gives a contradiction and concludes the proof of Lemma 2.9. \( \square \)

3. The Main Results. This Section is devoted to obtain some crucial steps in the proof of the Theorem.

Assume that \( V \) is an one-dimensional linear subspace of \( \mathbb{R}^N \) (i.e. \( V \in G(N, 1) \), cf [F], 1.6.2). Fix \( \alpha \in \mathbb{R}^N, 0 < r < +\infty, 0 < s < 1 \). Following [F], 3.3.1, we define
\[
X(\alpha, r, V, s) = \mathbb{R}^N \cap \{ x : s^{-1} \text{dist}(x - \alpha, v) < |x - \alpha| < r \}
\]
The main result of this Section is
PROPOSITION 3.1. Fix $\eta > 0$, and an integer number $m \geq 2$. There exist $\delta > 0$, $0 < \theta < 1$, $K > 0$ depending only on $N, m, p, \eta, M$. such that if $x_0 \in S \cap B_R(0)$, and for some $\tilde{\tau}, -\log(T) < \tilde{\tau} < +\infty$, there holds

$$
\sum_{|\alpha| \leq m} (a_\alpha(\tilde{\tau}, x_0))^2 \geq \eta \sum_{|\alpha| \geq m+1} (a_\alpha(\tilde{\tau}, x_0))^2
$$

and also

$$
||\psi(\cdot, \tilde{\tau}, x_0)|| \leq \delta
$$

then

$$
\mathcal{H}^{N-1}(S \cap B_{\theta^{e^{-\tau/2}}}(x_0)) \leq Ke^{-(N-1)\tilde{\tau}}
$$

The proof of Proposition 3.1 is obtained by induction in $m$. We start with the case $m = 2$.

The proof of Proposition 3.1 for $m = 2$.

LEMMA 3.1. Assume that (3.2) holds for some $\eta > 0$. Fix $\varepsilon > 0$. then there exists $\tilde{Q}$ depending on $\eta, M, p, \varepsilon, N, m$ such that, for any $Q > \tilde{Q}$, there is $\delta_0$ depending on $\eta, M, p, \varepsilon, N, m$ and $Q$ and such that $||\psi(\cdot, \tilde{\tau}, x_0)|| \leq \delta_0$ implies for $\tilde{\tau} + \tilde{Q} \leq \tau \leq \tilde{\tau} + Q$

$$
\sum_{|\alpha| \leq m} (a_\alpha(\tau; x_0))^2 \geq \frac{1}{\varepsilon} \sum_{|\alpha| \geq m+1} (a_\alpha(\tau; x_0))^2.
$$

Proof. It is an easy consequence of Lemma 2.2. □

Throughout this Section, we will denote by $\tilde{Q}$ a generic constant depending on $\eta, M, p, \varepsilon, N, m$, and by $\delta_0$ a constant depending on $\eta, M, \varepsilon, N, m, Q$, where $Q > \tilde{Q}$. Without loss of generality we can assume that such constant is the same in all the results henceforth.

LEMMA 3.2. Assume that (3.2) holds and fix $\varepsilon > 0$. Then there exist $\tilde{Q}$, $\delta_0$, and $\tilde{\theta}$ depending only on $\eta, M, p, \varepsilon, N$, such that if $||\psi(\cdot, \tilde{\tau}, x_0)|| \leq \delta_0$ for any $\tau \geq \tilde{\tau} + \tilde{Q}$ and for any $\tilde{x} \in S \cap B_{\tilde{\theta}^{e^{-\frac{1}{2}\tau}}}(x_0)$ there holds

$$
\sum_{|\alpha|=m} (a_\alpha(\tau, \tilde{x}))^2 \geq \frac{1}{\varepsilon} \sum_{|\alpha| \geq m+1} (a_\alpha(\tau; \tilde{x}))^2.
$$

Proof. By Lemma 3.1 we have for $\tilde{Q} \leq \tau - \tilde{\tau} \leq \tilde{Q} + 1$ that (3.5) is satisfied with $\varepsilon = \varepsilon > 0$ and $\tilde{Q} > 0$ depending on $\eta, M, p, \varepsilon, N$. On the other hand, Lemma 2.6 with
\[ \tau_0 = \tilde{\tau} + \tilde{Q}, \quad \theta = 1 \] implies that for any \( \tilde{x} \in S \) such that \( |\tilde{x} - x_0| \leq e^{-\frac{1}{2}\tilde{Q} - \frac{1}{2}\tilde{L}} e^{-\frac{1}{2}\tilde{r}} \), \( L \leq \tau - (\tilde{\tau} + \tilde{Q}) \leq \tilde{L} \)

\[
\left\| \psi(\cdot, \tau; \tilde{x}) - \sum_{\alpha} a_\alpha(\tilde{\tau} + \tilde{Q}, x_0) e^{(1 - \frac{1}{2}) (\tau - \tilde{\tau} - \tilde{Q})} H_\alpha(\cdot + (\tilde{x} - x_0) e^{\frac{1}{2} \tau}) \right\|
\leq C_3 \left\| \psi(\cdot, \tilde{\tau} + \tilde{Q}; x_0) \right\|^{3/2}
\]

By Lemma 2.7, (3.5) and (3.7) we obtain

\[
\sum_{|\alpha| \leq m} (a_\alpha(\tilde{\tau} + \tilde{Q} + \tilde{L}; \tilde{x}))^2 \geq \tilde{\eta} \sum_{|\alpha| \geq m+1} (a_\alpha(\tilde{\tau} + \tilde{Q} + \tilde{L}; \tilde{x}))^2
\]

for \( \delta_0 \) and \( |x - x_0| e^{\frac{1}{2}(\tau + \tilde{Q} + \tilde{L})} \) small enough – depending only on \( N, M, p \). Applying again Lemma 3.1 we can assume that (2.28a) holds with \( x_0 \) replaced there by \( \tilde{x} \), and \( \tau_0 = \tilde{\tau} + 2\tilde{Q} + \tilde{L} \). Then (2.29a) is satisfied for any \( \tau \geq \tilde{\tau} + \beta \), where \( \beta \) depends only on \( N, N, p, \eta \). Finally using again of Lemma 3.1 the proof of Lemma 3.2 follows. \[ \square \]

**Lemma 3.3.** Assume that (3.2), (3.3) hold with \( m = 2 \) at some blow-up point \( x_0 \in S \cap B_R(0) \). Then, for any \( \tilde{x} \in S \cap B_{\tilde{\theta} - \frac{1}{2} \tau}(x_0) \), where \( \tilde{\theta} \) is given in Lemma 3.2 we have that

\[
(3.8) \quad \left\| \psi(\cdot, \tau, \tilde{x}) \right\| \leq \tilde{C} \left( \frac{\| \psi(\cdot, \tilde{\tau}, x_0) \|}{1 + \beta \| \psi(\cdot, \tilde{\tau}, x_0) \| (\tau - \tilde{\tau} - \tilde{Q})} \right)
\]

for \( \tau \geq \tilde{\tau} + \tilde{Q} \), where \( \tilde{C} \) depends on \( N, p, M, \eta, \epsilon \) but is independent on \( \| \psi(\cdot, \tilde{\tau}; x_0) \| \), and \( \beta \) depends on \( N, p \).

**Proof.** Define \( W(\tau; \tilde{x}) \) as in (2.30c). By (2.30b), (2.37) and Lemma 3.2, we obtain for \( \tau \geq \tilde{\tau} + \tilde{Q} \), that for \( \delta_0 \) small enough.

\[
(3.9) \quad \tilde{W}(\tau; \tilde{x}) \leq -\frac{\sqrt{2}}{2} \gamma N \nu p (W(\tau; \tilde{x}))^{3/2}
\]

Integration of (3.9) and Lemma 3.2 yield then (3.8). \[ \square \]

As a next step, we obtain an improvement of (3.6).

**Lemma 3.4.** Assume that (3.2), (3.3) hold with \( n = 2 \) at \( x_0 \in S \cap \overline{B_R(0)} \). Then, for any \( \tilde{x} \in S \cap B_{\tilde{\theta} - \frac{1}{2} \tau}(x_0) \), where \( \tilde{\theta} \) is as in Lemma 3.2, there holds:

\[
\sum_{|\alpha| \geq 3} (a_\alpha(\tau; \tilde{x}))^2 \leq \max \left\{ \varepsilon e^{-\frac{1}{2} (\tau - \tilde{\tau} - \tilde{Q})}, \tilde{C} \left( \frac{\| \psi(\cdot, \tilde{\tau}, x_0) \|^2}{1 + \beta \| \psi(\cdot, \tilde{\tau}, x_0) \| (\tau - \tilde{\tau} - \tilde{Q})^2} \right) \right\}
\]

\[
(3.10) \quad \left( \sum_{|\alpha| = 2} (a_\alpha(\tau; \tilde{x}))^2 \right)^{1/2}
\]
for $\tau \geq \bar{\tau} + \tilde{Q}$, where $\tilde{C}$ depends on $N, p, M, \eta, \varepsilon$, and $\beta$ is as in Lemma 3.2.

Proof. We define $M(\tau; x_0)$ as in (2.30a) with $m = 2$. Taking into account (2.33) and Lemma 3.2, we obtain

$$
\dot{M}(\tau; x_0) \leq -M(\tau; x_0) + C \left( \sum_{|\alpha| = 2} (a_\alpha(\tau; x_0))^2 \right)^{1/2} M(\tau; x_0) + C \left( \sum_{|\alpha| = 2} (a_\alpha(\tau; x_0))^2 \right)^{3/2} (M(\tau; x_0))^{1/2} \leq -\frac{1}{2} M(\tau; x_0) + C \left( \sum_{|\alpha| = 2} (a_\alpha(\tau; x_0))^2 \right)^{1/2} (M(\tau; x_0))^{1/2}
$$

for $\delta_0$ small enough. By Lemma 3.3, we then deduce that, for $\tau \geq \bar{\tau} + \tilde{Q}$

$$
(3.11) \quad \dot{M}(\tau; x_0) \leq -\frac{1}{2} M(\tau; x_0) + C(M(\tau; x_0))^{1/2} \frac{\|\psi(\cdot, \bar{\tau}; x_0)\|}{(1 + \beta \|\psi(\cdot, \bar{\tau}; x_0)\|(\tau - \bar{\tau} - \tilde{Q}))}
$$

and taking into account (3.6), (3.11) yields (3.10) for $\delta_0$ small enough. \[ \Box \]

The key point in the proof of Proposition 3.1 for $m = 2$ is the following

**Lemma 3.5.** Assume that (3.2) holds with $m = 2$. Thus there exist $\bar{\theta} > 0$, $\bar{\theta} > 0$ depending on $\eta, M, p, N$ but independent on $\delta_0$, and $s > 0$ depending only on $p, N$ such that for any $\bar{x} \in B_{\bar{\theta} - \bar{\tau}}(x) \cap S$ there is a line $V \in G(N, 1)$ (depending on $\bar{x}$), satisfying

$$
S \cap X(\bar{x}, \theta e^{-\frac{1}{2} \bar{\tau}}, V, s) = \emptyset.
$$

Proof. By Lemma 3.2, we can assume that (3.2) holds for any $\bar{x} \in S \cap B_{\bar{\theta} - \bar{\tau}}(x_0) \cap S$ and $\tau = \bar{\tau} + \tilde{Q}$. It is then enough to show the result for a fixed point $x_0$. We define $\gamma(\tau; x_0) = \min\{\lambda_i(\tau; x_0), i = 1, \ldots N\}$, where $\lambda_i(\tau; x_0)$ are defined in the proof of Lemma 2.5.

Arguing as in [V1], Lemma 3.1, we obtain for $\delta_0$ small enough

$$
(3.13) \quad |\dot{\lambda}_j(\tau; x_0) - \sqrt{2} \gamma_N v_p(\lambda_j(\tau; x_0))^2| \leq E(\mu(\tau) + |\gamma(\tau)|^{1/2})|\gamma(\tau)|^2,
$$

where

$$
\mu(\tau) = \max \left\{ e^{1/2} e^{-\frac{1}{2} (\tau - \bar{\tau} - \tilde{Q})}, (\tilde{C})^{1/2} \frac{\|\psi(\cdot, \bar{\tau}; x_0)\|}{1 + \beta \|\psi(\cdot, \bar{\tau}; x_0)\|(\tau - \bar{\tau} - \tilde{Q})} \right\}
$$

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(cf. Lemma 3.4) and \( \lim_{\sigma \to 0^+} E(\tau) = 0 \), and it is readily seen that \( \gamma(\tau) \) is an absolutely continuous function satisfying

\[
|\gamma(\tau; x_0) - \sqrt{2\gamma_0} v_p(\gamma(\tau; x_0))^2| \leq E(\mu(\tau) + |\gamma(\tau)|^{1/2})|\gamma(\tau)|^2
\]

By Lemma 2.4, \( \gamma(\tau; x_0) < 0 \) for \( \tau \geq \hat{\tau} + \tilde{Q} \). Assume that, for some \( \hat{\tau} \geq \tilde{\tau} + \tilde{Q} \), \( \lambda_\ell(\hat{\tau}; x_0) = \frac{i}{2N} \gamma(\hat{\tau}; x_0) \) where \( (2N - 1) \) and \( \ell = 1 \ldots N \). Then (3.13), (3.14) yield \( \frac{d}{d\tau}(\lambda_\ell(\hat{\tau}; x_0)) < \frac{d}{d\tau}(\frac{i}{2N} \gamma(\hat{\tau}; x_0)) \) for \( \varepsilon, \delta_0 \) small enough. This means that any eigenvalue \( \lambda_\ell(\tau; x_0) \) that reaches the value \( \frac{i}{2N} \gamma(\tau; x_0) \) for some \( j = 1, \ldots (2N - 1) \) remains for any larger time being less than \( \frac{i}{2N} \gamma(\tau; x_0) \).

Now, for \( \tau = \hat{\tau} + \tilde{Q} \), we define \( \tilde{j}(\hat{\tau} + \tilde{Q}) \) as the largest integer \( k \) between one and \( 2N - 1 \) such that, there is no eigenvalue in \( [\frac{k+1}{2N} \gamma(\hat{\tau}; x_0), \frac{k}{2N} \gamma(\hat{\tau}; x_0)] \). It is readily seen that such an integer exists. We define \( \tilde{j}(\tau) = \tilde{j}(\hat{\tau} + \tilde{Q}) \) in a similar way for any \( \tau \) such that \( \hat{\tau} + \tilde{Q} \leq \tau < \hat{\tau} \), where

\[
\hat{\tau} = \inf \left\{ \tau > \hat{\tau} + \tilde{Q} : \frac{j(\hat{\tau} + \tilde{Q})}{2N} \gamma(\tau; x_0) = \lambda_\ell(\tau; x_0), \text{ for some } \ell = 1 \ldots N \right\}.
\]

For \( \tau = \hat{\tau} \) we define \( \tilde{j}(\hat{\tau}) \) as the largest positive integer, smaller than \( \tilde{j}(\hat{\tau} + \tilde{Q}) \) such that there is no eigenvalue in \( [\frac{k+1}{2N} \gamma(\hat{\tau}; x_0), \frac{k}{2N} \gamma(\hat{\tau}; x_0)] \).

Arguing inductively, it is readily seen that this defines a decreasing function \( \tilde{j}(\tau) \) for any \( \tau \geq \hat{\tau} + \tilde{Q} \), such that \( \frac{j(\tau)}{2N} \gamma(\tau; x_0) \neq \lambda_\ell(\tau; x_0) \) for any \( \ell = 1 \ldots N \).

Let \( \tilde{C}(\tau) \) be the contour in the complex plane given by

\[
\left\{ z : Re(z) = \frac{j(\tau)}{2N} \gamma(\tau; x_0) \text{ or } Re(z) = 2\gamma(\tau; x_0) \text{ and } |Im(z)| \leq |\gamma(\tau; x_0)| \right\} \cup \\
\left\{ z : |Im(z)| = |\gamma(\tau; x_0)| \text{ and } 2\gamma(\tau; x_0) \leq Re(z) \leq \frac{j(\tau)}{2N} \gamma(\tau; x_0) \right\}
\]

with counterclockwise orientation. Define the valued-projection function \( P(\tau; x_0) \) as

\[
P(\tau; x_0) = -\frac{1}{2\pi i} \int_{\tilde{C}(\tau)} (G(\tau) - Z)^{-1} dz
\]

Notice that the eigenvalues inside \( \tilde{C}(\tau) \) satisfy

\[
\lambda_\ell(\tau; x_0) < \frac{\gamma(\tau; x_0)(j + 1)}{2N}, \text{ so that } \|P(\tau; x_0)(G(\tau) - z)^{-1}\| \leq \frac{2N}{|\gamma(\tau; x_0)|}
\]

for \( z \in \tilde{C}(\tau) \). Notice also that, at the continuity points of \( \tilde{j}(\tau) \) there holds (cf. [K])

\[
\hat{P}(\tau; x_0) = \frac{1}{2\pi i} \int_{\tilde{C}(\tau)} (G - z)^{-1} \hat{G}(\tau)(G - z)^{-1} dz
\]

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On the other hand, (1.8) and Lemma 3.4 imply that

\[
\|\dot{G}(\tau) - \sqrt{2} \gamma N \nu_p (G(\tau))^2 \| \leq C \left( \mu(\tau) \|G(\tau)\|^2 + \|G(\tau)\|^{5/2} \right).
\]

We then obtain at any continuity point of \( \tilde{j}(\tau) \)

\[
(3.16) \quad \|\dot{P}(\tau; x_0)\| \leq C |\gamma(\tau)| (\mu(\tau) + |\gamma(\tau)|^{1/2}).
\]

where \( C \) depends only on \( p, N, M \).

At the discontinuity points of \( \tilde{j}(\tau) \), we see by the definition of \( P(\tau, x_0) \)

\[
P(\tau+, x_0) - P(\tau-, x_0) \geq 0
\]

in the sense of matrices. We then have

\[
(3.17) \quad \|P(\tau; x_0) - (P(\tau + \tilde{Q}; x_0) + Q(\tau; x_0))\| \leq C \int_{\tau + \tilde{Q}}^{+\infty} |\gamma(s)| (\mu(s) + |\gamma(s)|^{1/2}) \, ds \equiv J
\]

where \( Q(\tau; x_0) \geq 0 \). Solving (3.14) and taking into account the definition of \( \mu(\tau) \), we readily obtain

\[
J \leq C \int_{0}^{+\infty} \frac{\|\psi(\cdot; \tau; x_0)\|}{1 + \beta \|\psi(\cdot; \tau; x_0)\|s} \left[ \max \left\{ \varepsilon^{1/2} e^{-\frac{1}{4}s}, \tilde{C}^{1/2} \right\} \frac{\|\psi(\cdot; \tau; x_0)\|}{1 + \beta \|\psi(\cdot; \tau; x_0)\|s} \right] \, ds
\]

It is easily seen that \( J \) may be bounded by an arbitrarily small number if \( \varepsilon \) is small enough, and \( \delta_0 \) is then selected small enough. Notice that, for some \( \xi \in \mathbb{R}^N \) with \( |\xi| = 1 \), there holds \( P(\tau + \tilde{Q})\xi = \xi \). Set \( V = \text{lin} (\xi) \), and fix \( s > 0 \) such that for any \( \xi \in X(0, +\infty, V, s) \)

\[
\langle \xi, P(\tau + \tilde{Q})\xi \rangle \geq \frac{1}{2} |\xi|^2.
\]

Taking into account (3.17), Lemma 2.4, and the definition of \( P(\tau; x_0) \) we obtain that, for any \( \xi \in X(0, +\infty, V, s) \)

\[
(3.18) \quad \langle \xi, G(\tau)\xi \rangle \leq \left( \frac{\gamma(\tau)}{2N} \right) \langle \xi, P(\tau; x_0)\xi \rangle + C g(|\gamma(\tau)|)|\xi|^2,
\]

where \( g(\cdot) \) is given as in Lemma 2.4. On the other hand, by (3.17) we obtain

\[
\langle \xi, P(\tau + \tilde{Q}; x_0)\xi \rangle \geq \langle \xi, P(\tau + Q; x_0)\xi \rangle - J|\xi|^2
\]

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Carrying this expression into (3.18) we arrive at

(3.19) \[ \langle \xi, G(\tau)\xi \rangle \leq \left( \frac{\gamma(\tau)}{\gamma N} \right) |\xi|^2 \text{ for } \tau \geq \bar{\tau} + \bar{Q}, \quad \xi \in X(0, +\infty, V, s), \]

provided that \( J \) is selected small enough (depending only on \( N \)) from the very beginning. Recalling Lemmata 2.1 and 2.6 we now have that for \( \xi \in X(0, +\infty; V, s), \, |\xi| = 1, \)

\[ ||\psi(\cdot, \tau; x_0)|| \text{ small enough, and } \tau \geq \bar{\tau} + \bar{Q} + \bar{L} \]

(3.20) \[ ||\psi(\cdot + \xi, \tau; x_0) - \sum_{|\alpha| \geq 3} a_\alpha(t - \bar{L}; x_0) e^{(1 - \frac{1}{2^n}) \bar{L}} H_\alpha(\cdot + \xi) || \leq h(||\psi(\cdot, \tau - \bar{L}; x_0)||) \]

where \( \lim_{r \to 0} \frac{h(r)}{r} = 0. \) On the other hand, by Lemma 3.1 and Lemma 2.1 we obtain that

\[ || \sum_{|\alpha| \geq 3} a_\alpha(\tau - \bar{L}; x_0) e^{(1 - \frac{1}{2^n}) \bar{L}} H_\alpha(\cdot + \xi) || \leq C \varepsilon ||\psi(\cdot, \tau - \bar{L}; x_0)||. \]

Notice further that

(3.21) \[ \psi(y, \tau, \bar{x}) = \psi(y + (\bar{x} - x_0)e^{\tau/2}, \tau; x_0) \]

From (3.20), (3.21) we obtain

\[ ||\langle H_0(\cdot), \psi(\cdot, \tau, \bar{x}) \rangle || \geq \left| \langle H_0(\cdot), \sum_{|\alpha| = 2} a_\alpha(\tau - \bar{L}; x_0) H_\alpha(\cdot + \xi) \rangle \right| - 2\varepsilon ||\psi(\cdot, \tau - \bar{L}; x_0)|| \]

for \( ||\psi(\cdot, \tau; x_0)|| \text{ small enough and } |\bar{x} - x_0| = e^{-\tau/2} \)

A simple calculation involving the generating function for Hermite’s polynomials shows that

\[ \left\langle H_0(\cdot), \sum_{|\alpha| = 2} a_\alpha(\tau - \bar{L}; x_0) H_\alpha(\cdot + \xi) \right\rangle = K_N \langle \xi, G(\tau - \bar{L})\xi \rangle \]

where \( K_N > 0 \) is a constant depending only on \( N \). Then, if we assume that \( \varepsilon \) is small enough (depending on \( N \)), we obtain for \( \bar{\tau} \) large enough and \( \bar{x} \in X(x_0, +\infty, V, S), \)

\[ |\bar{x} - x_0| = e^{-\frac{\tau}{2}}, \tau \geq \bar{\tau} + \bar{Q} + \bar{L} \text{ that} \]

(3.22) \[ ||\langle H_0(\cdot), \psi(\cdot, \tau, \bar{x}) \rangle || \geq \frac{K_N}{2} ||\psi(\cdot, \tau - \bar{L}; x_0)|| \geq C ||\psi(\cdot, \tau; \bar{x})||. \]

By Lemma 2.2 this implies that \( \bar{x} \notin S \). Then, allowing \( \tau \) to take all the values \( \tau \geq \bar{\tau} + \bar{a} + \bar{L} \), we obtain that \( S \cap X(x_0, \theta e^{-\frac{\tau}{2}}, V, S) = \emptyset. \) This concludes the proof of (3.12). \( \square \)

**End of the proof of Proposition 3.1 for \( m = 2 \).** Since \( G(N, 1) \) a compact manifold, for each \( s > 0 \), we can find a finite collection \( S_s = \{ \tilde{V}_j, j = 1 \ldots \gamma(s, N) : \tilde{V}_j \in G(N, 1) \} \), such that, for any \( V \in G(N, 1) \), there exists \( \tilde{V}_k \in S_s \), satisfying \( X(0, +\infty; \tilde{V}_k, \frac{s}{2}) \subset X(0, +\infty, V, s) \).
By Lemma 3.5, we can assign to each $\bar{x} \in \overline{B}_{\theta e^{-\frac{1}{\theta}}} (x_0) \cap S$ a line $\bar{V}_k(\bar{x}) \in S$, such that

$$X \left( \bar{x}, \theta e^{-\frac{1}{\theta}}, \bar{V}_k; \frac{S}{2} \right) \cap S = \emptyset.$$ 

Define:

$$\mathcal{B}_j = \left\{ \bar{x} \in S \cap B_{\theta e^{-\frac{1}{2}}} (x_0) : \bar{V}_k(\bar{x}) = \bar{v}_j \right\}.$$ 

Notice that $S \cap B_{\theta e^{-\frac{1}{2}}} (x_0) = \bigcup_{j=1}^{\gamma(s, N)} \mathcal{B}_j$. For each $\bar{v}_j \in S$, there exists $p \in 0^*(N, N - 1)$ (cf. [F], 1.7.4) such that $\text{Ker}(p) = \bar{V}_j$. By [F], Lemma 3.3.5, for each subset of $\mathcal{B}_j$ with diameter less than $\theta e^{-\frac{1}{\theta}}$, there exists $f : \mathbb{R}^m \to \mathbb{R}^n$, with $\text{Lip}(f) \leq \frac{1}{s}$ and $p \circ f = 1_{\mathbb{R}^m}$. In particular each such set is rectifiable (cf. [F] 3.2.14). For each $\bar{x} \in \mathcal{B}_j$, we then have, that for some Lipschitz function $f$ depending on $\bar{x}$

$$\mathcal{B}_j \cap B_{\theta e^{-\frac{1}{2}}} (\bar{x}) \subset f(p(B_{\theta e^{-\frac{1}{2}}} (\bar{x})))$$

where $\text{Lip}(f) \leq \frac{1}{s}$. By [F], Corollary 2.10.11, we then obtain that

$$\mathcal{H}^{N-1}(\mathcal{B}_j \cap B_{\theta e^{-\frac{1}{2}}} (\bar{x})) \leq \mathcal{H}^{N-1}(f(p(B_{\theta e^{-\frac{1}{2}}} (\bar{x})))) \leq \left( \frac{1}{s} \right)^{N-1} \mathcal{H}^{N-1}(p(B_{\theta e^{-\frac{1}{2}}} (\bar{x}))) = \omega_{N-1}/s^{N-1}(\theta e^{-\frac{1}{2}})^{N-1}$$

A standard covering argument shows that $\mathcal{H}^{N-1}(\mathcal{B}_j) \leq C(\theta e^{-\frac{1}{2}})^{N-1}$. where $C$ depends on $N$, $\theta$, $\theta$, and finally

$$\mathcal{H}^{N-1}(S \cap B_{\theta e^{-\frac{1}{2}}} (x_0)) \leq \gamma(N, s) C(\theta e^{-\frac{1}{2}})^{N-1}. \square$$

**The proof of Proposition 3.1 for $n$ arbitrary.** Assume that Proposition 3.1 holds for $m = \ell - 1$, with $\ell \geq 3$. We then show that it also holds for $m = \ell$. We need the following preliminary Lemma.

**Lemma 3.6.** For any positive integer $\ell$, there exist positive constants $s_0, \mu_0$ depending only on $N, \ell$, such that for any set of coefficients $\{c_\alpha \in \mathbb{R} : |\alpha| = \ell\}$, there exists $V \in G(N, 1)$ such that

$$\left( \sum_{|\alpha| = \ell} |c_\alpha| \right)^{1/2} \geq \mu_0 \left( \sum_{|\alpha| = \ell} c_\alpha^2 \right)^{1/2}$$

for any $x \in X(0, +\infty, V, s_0) \cap S^{N-1}$.

**Proof.** Assume otherwise that there exists a sequence of coefficients $\{c_\alpha^{(k)} : |\alpha| = \ell\}$, $k = 1, 2, \ldots$ such that, for any $V \in G(N, 1)$ there exists $\bar{x} \in X(0, +\infty, V, \frac{1}{k}) \cap S^{N-1}$ such that

$$\left( \sum_{|\alpha| = \ell} |c_\alpha| \right)^{1/2} \leq \frac{1}{k} \left( \sum_{|\alpha| = \ell} (c_\alpha^{(k)})^2 \right)^{1/2}$$

for any $x \in X(0, +\infty, V, s_0) \cap S^{N-1}$.

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We can assume, after normalization, that \( \sum_{|\alpha|=\ell} (c^{(k)}_\alpha)^2 = 1 \). Set \( \Phi_k(x) \equiv \sum_{|\alpha|=\ell} \tilde{c}_\alpha c^{(k)}_\alpha(x) \). It is readily seen that, for \( |x| = 1 \), \( |\nabla \Phi_k(x)| \leq C_{N,\ell} \). Our assumptions on \( \{c^{(k)}_\alpha, |\alpha| = \ell\} \) imply that
\[
\sum_{|\alpha|=\ell} \tilde{c}_\alpha c^{(k)}_\alpha(x)^\alpha \leq \frac{C_{N,\ell}}{k}
\]
where we use the fact that \( \Phi_k(-x) = (-1)^\ell \Phi_k(x) \).

A standard compactness argument shows then that a subsequence \( \{c^{(k)}_\alpha\} \) exists which converges to \( \{\tilde{C}_\alpha\} \), where \( \sum_{|\alpha|=\ell} (\tilde{C}_\alpha)^2 = 1 \) and \( \sum_{|\alpha|=\ell} \tilde{c}_\alpha \tilde{C}_\alpha(x)^\alpha = 0 \) for any \( x \in \mathbb{R}^N \). This gives a contradiction. \( \Box \)

Our next Lemma avoids the possibility of dominance of modes associated to an odd integer \( m \).

**Lemma 3.7.** Let \( m \) be an odd number. Then there exists \( \varepsilon_0 \) depending on \( M, N, p, m \) such that, for any \( \varepsilon < \varepsilon_0 \), and \( \|\psi(\cdot, \tilde{r}; x_0)\| \leq \delta_0 \) (where \( \delta_0 \) depends on \( M, N, p, M, \varepsilon \)), if

\[
\sum_{|\alpha| \geq m+1} (a_\alpha(\tilde{r}; x_0))^2 \leq \varepsilon \sum_{|\alpha| \leq m} (a_\alpha(\tilde{r}; x_0))^2
\]

then

\[
\sum_{|\alpha| = m} (a_\alpha(\tilde{r}; x_0))^2 \leq \frac{1}{\varepsilon} \sum_{|\alpha| \leq m-1} (a_\alpha(\tilde{r}; x_0))^2.
\]

**Proof.** Assume that \( \varepsilon \sum_{|\alpha| = m} (a_\alpha(\tilde{r}; x_0))^2 \geq \sum_{|\alpha| \leq m-1} (a_\alpha(\tilde{r}; x_0))^2 \). By Lemma 2.6 with \( \theta = 2 \) we obtain for \( L \geq \tilde{L} \)

\[
\|\psi(\cdot + \beta, \tilde{r} + L; x_0) - \sum_\alpha a_\alpha(\tilde{r}; x_0)e^{(1-\frac{1}{2}L)}H_\alpha(\cdot + \beta)\| \leq C\|\psi(\cdot, \tilde{r}; x_0)\|^{3/2}.
\]

where \( C > 0 \) depends only on \( p, M, N \). On the other hand, by Lemma 2.1 if \( L \) is large enough depending only on \( N \)

\[
\|\sum_{|\alpha| \neq m} a_\alpha(\tilde{r}, x_0)e^{(1-\frac{1}{2}L)}H_\alpha(\cdot + \beta)\| \leq Ce\|\psi(\cdot, \tilde{r}, x_0)\|
\]

Notice that

\[
\langle H_0(\cdot), \sum_{|\alpha| = m} a_\alpha(\tilde{r}; x_0)e^{(1-\frac{1}{2}L)}H_\alpha(\cdot + \beta) \rangle = \left( \sum_{|\alpha| = m} a_\alpha(\tilde{r}; x_0)\tilde{c}_\alpha \beta^\alpha \right)e^{(1-\frac{m}{2})L}
\]

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Taking into account Lemma 3.6 and the fact that \( m \) is an odd number, there exists \( \tilde{\beta} \in \mathbb{R}^N \) (\( |\tilde{\beta}| = 1 \)), such that

\[
\sum_{|\alpha|=m} a_\alpha(\tilde{\tau}; x_0) \tilde{c}_\alpha(\tilde{\beta})^\alpha \geq \mu_0 \left( \sum_{|\alpha|=m} (a_\alpha(\tilde{\tau}; x_0))^2 \right)^{1/2}
\]

By (3.23), (3.25), (3.26), (3.27) and (3.28) we have

\[
\left\langle H_0(\cdot), \psi(\cdot, \tilde{\tau} + L; x_0 - \tilde{\beta}e^{-\frac{\tau + L}{2}}) \right\rangle \geq e^{(1 - \frac{\eta}{2})L} \mu_0 \left( \sum_{|\alpha|=m} (a_\alpha(\tilde{\tau}; x_0))^2 \right)^{1/2} - C\varepsilon^{1/2} \left( \sum_{|\alpha|=m} (a_\alpha(\tilde{\tau}; x_0))^2 \right)^{1/2} - C\|\psi(\cdot, \tilde{\tau}; x_0)\|^3 > 0
\]

if \( \varepsilon \) and \( \delta_0 \) are small enough, which contradicts (2.16). \( \square \)

**The proof of Proposition 3.1 for odd \( \ell \).** Assume that (3.2) holds for \( m = \ell \). By Lemma 3.1 we have that for \( \tau = \tilde{\tau} + \tilde{Q} \) (3.5) is satisfied, where \( \varepsilon > 0 \) is selected as in Lemma 3.7. Then, if \( \delta > 0 \) is small enough, (3.24) holds. Combining (3.5) and (3.24) we obtain (3.2) with \( m = \ell - 1, \eta = \varepsilon + \frac{1}{\varepsilon} \). Taking into account (2.29b) the hypothesis of induction and Proposition 3.1 with \( m = \ell - 1 \), the proof is obtained for \( m = \ell \). \( \square \)

We now conclude the proof of Proposition 3.1 for even \( \ell \). The key step is the following

**Lemma 3.8.** Let \( m \) be an even number. Then, there exist constants \( R, \varepsilon_0, \theta_1, \delta_0 \) depending only on \( N, M, p, m \), such that if

\[
\sum_{|\alpha| \neq m} (a_\alpha(\tau; x_0))^2 \leq \varepsilon_0 \sum_{|\alpha|=m} (a_\alpha(\tau; x_0))^2
\]

for \( \tilde{\tau}_1 \leq \tau \leq \tilde{\tau}_2 \), where \( \tilde{\tau}_2 \geq \tilde{\tau}_1 + R \) and \( \|\psi(\cdot, \tilde{\tau}_1; x_0)\| \leq \delta_0 \), then there exists \( V \in G(N, 1) \) depending on \( x_0 \) such that

\[
[X(x_0, \theta_1 e^{-\tau_1/2}, V, s_0) \setminus X(x_0, e^{-\tau_2/2}, V, s_0)] \cap S = \emptyset
\]

where \( s_0 \) is given in Lemma 3.6.

**Proof.** Taking \( R \) large enough, depending only on \( N \), and using Lemmata 2.1 and 2.5, we can assume for \( \tau \geq \tilde{\tau}_1 + R \) that (2.31) holds. Set now

\[
S(\tau) = \sum_{|\alpha|=m} (a_\alpha(\tau; x_0))^2.
\]
It is readily checked that

\[ \hat{S}(\tau) \leq 2 \left( 1 - \frac{m}{2} \right) (S(\tau)) + C(S(\tau))^{3/2}. \]

where (3.29) has been taken into account. This inequality readily implies that

\[ S(\tau) \leq S(\tau + R) \exp \left( 2 \left( 1 - \frac{m}{2} + \sigma \right) (\tau - \tau_1 - R) \right), \]

where \( 0 < \sigma < 1/4 \) if \( \delta_0 \) is small enough. Notice that

\[ |a_\alpha(\tau; x_0) - a_\alpha(\tau_1 + R) e^{(1 - \frac{m}{2})(\tau - \tau_1 - R)}| \leq \int_{\tau_1 + R}^\tau e^{(1 - \frac{m}{2})(\tau - s)} \langle \psi(\cdot, s; x_0), H_\alpha \rangle ds \]

\[ \leq C \int_{\tau_1 + R}^\tau e^{(1 - \frac{m}{2})(\tau - s)} \| \psi(\cdot, s; x_0) \|^2 ds \leq \]

\[ C \left( \sum_{|\alpha| = m} (a_\alpha(\tau_1 + R, x_0))^2 \right)^{1/2} + \infty \int_{\tau_1 + R}^\infty e^{(1 - \frac{m}{2})(\tau - s)} e^{2(1 - \frac{m}{2} + \sigma)(s - \tau_1 - R)} ds \leq \]

\[ \leq C \left( \sum_{|\alpha| = m} (a_\alpha(\tau_1 + R; x_0))^2 \right)^{1/2} e^{(1 - \frac{m}{2})(\tau_1 - R)}, \quad \tau_1 + R \leq \tau \leq \tau_2 \]

where \( C > 0 \) is a constant depending on \( N, m, p, M \) that may change from line to line. Arguing as in Lemma 3.7 (cf. (3.25), (3.26), (3.27)), taking \( R \geq \tilde{L} \) (cf. Lemma 2.5) and taking into account Lemma 3.6, we deduce that

\[ \|(H_0, \psi(\cdot, \tau; x_0 + \beta e^{-\tau/2}))\| \geq C \| \psi(\cdot, \tau; x_0 + \beta e^{-\tau/2}) \| \]

where \( C > 0 \) depends only on \( p, M, N, m, \varepsilon_0, \delta_0 \) are small enough and \( \beta \in V \), where \( V \in G(N, 1) \) is the one associated to the multilinear form \( \sum_{|\alpha| = m} (a_\alpha(\tau_1 + R))(x)^\alpha \) in Lemma 3.6. Notice that we can assume that the multilinear form is fixed for \( \tau_1 + R \leq \tau \leq \tau_2 \) by (3.32). By Lemma 2.3, we obtain that, for small \( \delta_0 \), \( x_0 + \beta e^{-\tau/2} \) is not a blowup point for \( \tau_1 + R \leq \tau \leq \tau_2 \), whence Lemma 3.8 follows.

We conclude the proof of Proposition 3.1 by means of a geometrical argument.

**End of the proof of Proposition 3.1.** Assume that \( m = \ell \) is an even number. Fix \( \varepsilon < \varepsilon_0 \), where \( \varepsilon_0 \) is as in Lemma 3.8. By Lemma 3.2 we can assume that

\[ \sum_{|\alpha| \leq m} (a_\alpha(\tau; \bar{x}))^2 \geq \frac{1}{2\varepsilon} \sum_{|\alpha| \geq m + 1} (a_\alpha(\tau; \bar{x}))^2 \]

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for \( \tau = \tilde{\tau} + \tilde{Q} \), and \( \bar{x} \in B_{\theta e^{-\frac{1}{2}\tau}}(x_0) \). On the other hand, a straightforward modification of Lemma 2.5 yields

\[
\sum_{|\alpha| \leq m} (a_\alpha(\tau; \bar{x}))^2 \geq \frac{1}{\varepsilon} \sum_{|\alpha| \geq m+1} (a_\alpha(\tau; \bar{x}))^2,
\]

for \( \tau \geq \tau + \tilde{Q} + \tilde{L} \) and \( \delta_0 \) small enough. Set \( \tilde{\tau}_1 = \tilde{\tau} + \tilde{Q} + \tilde{L} \). We define the following sets

\[
A = \left\{ \bar{x} \in B_{\theta e^{-\frac{1}{2}\tau}}(x_0) : (3.29) \text{ holds for } \tilde{\tau}_1 \leq \tau \leq \tilde{\tau}_1 + R \right\}
\]

\[
B = \left\{ \bar{x} \in B_{\theta e^{-\frac{1}{2}\tau}}(x_0) : (3.29) \text{ does not hold for some } \tau \text{ in } [\tilde{\tau}_1, \tilde{\tau}_1 + R] \right\}.
\]

For any point \( \bar{x} \in B \), we can apply the hypothesis of induction. Then, there exists a ball \( B_{\theta e^{-\frac{1}{2}\tau}}(\bar{x}) \) such that \( \mathcal{H}^{N-1}(B_{\theta e^{-\frac{1}{2}\tau}}(\bar{x}) \cap S) \leq C(e^{-\frac{1}{2}\tau})^{N-1} \). We can cover \( B_{\theta e^{-\frac{1}{2}\tau}}(x_0) \cap B \) with a finite number of this balls, and such number is bounded by a fixed constant depending on \( \theta, \tilde{\theta}, N \), but not on \( \delta_0 \). Then

\[
\mathcal{H}^{N-1}(B_{\theta e^{-\frac{1}{2}\tau}}(x_0) \cap B) \leq C(e^{-\frac{1}{2}\tau})^{N-1}
\]

We now proceed to bound the set \( A \). For each \( \bar{x} \in A \), Lemma 3.8 gives a \( V(\bar{x}) \in G(N, 1) \). Arguing as in the proof of Proposition 3.1 for \( m = 2 \), we can assume that there is only a finite number of such directions so that we can restrict our attention, without loss of generality to the case where \( V(\bar{x}) \equiv V \in G(N, 1) \) is a fixed line independent on \( \bar{x} \). Set now

\[
\Gamma = \{ \bar{x} \in A : \text{ for any } \bar{x} \in A, \bar{x} \neq \bar{x}, \bar{x} \notin X(\bar{x}, \theta_1 e^{-\tau_1/2}, V, s_0) \}.
\]

Notice that \( \Gamma \) can be dealt with as in the case \( m = 2 \). Actually, by [F], Lemma 3.3.5 \( \Gamma \) is a rectifiable set, and arguing as in the case \( m = 2 \) we obtain

\[
\mathcal{H}^{N-1}(B_{\theta e^{-\frac{1}{2}\tau}}(x_0) \cap A) \leq C(e^{-\frac{1}{2}\tau})^{N-1}.
\]

We finally estimate \( (A \setminus \Gamma) \). Assume that \( \bar{x} \in (A \setminus \Gamma) \). Then, by definition there exists \( \bar{x} \neq \bar{x}, \bar{x} \in A \cap B_{\theta_1 e^{-\tau_1/2}}(x_0) \), and such that

\[
\bar{x} \in X(\bar{x}, \theta_1 e^{-\tau_1/2}, V, s_0),
\]

then

\[
\bar{x} \in X(\bar{x}, \theta_1 e^{-\tau_1/2}, V, s_0) \cap A \equiv D(\bar{x})
\]

Clearly, there exists \( \bar{x} \in D(\bar{x}) \) such that \( |\bar{x} - \bar{x}| \geq \frac{\text{diam}(D(\bar{x}))}{2} \). Set \( \tilde{\tau}(\bar{x}) = -\log(|\bar{x} - \bar{x}|) \). Since \( \bar{x} \in D(\bar{x}) \), Lemma 3.8 implies that there exists \( \tau, \tilde{\tau}_1 + R \leq \tau \leq \tilde{\tau}(\bar{x}) \) such that

\[
\sum_{|\alpha| \leq m-1} (a_\alpha(\tau; \bar{x}))^2 \geq \varepsilon_0 \sum_{|\alpha| = m} (a_\alpha(\tau; \bar{x}))^2
\]
We can apply now the hypothesis of induction, thus obtaining that for some \( \tilde{\theta} > 0 \) depending only on \( N, p, M, m - 1 \),

\[
\mathcal{H}^{N-1}(B_{\tilde{\theta}e^{1/2}}(x_0) \cap A) \leq \mathcal{H}^{N-1}(B_{\tilde{\theta}e^{1/2}}(x_0) \cap S) \leq C(e^{-1/\tilde{r}(x)})^{N-1}.
\]

We can cover now the set of points \( \tilde{x} \in D(\tilde{x}) \) such that \( |\tilde{x} - \tilde{x}| \geq \frac{\text{diam}(D(\tilde{x}))}{2} \) by a finite number of balls (depending only on \( N, \theta \)), we then obtain

\[
\mathcal{H}^{N-1}(D(\tilde{x}) \cap \left\{ x \in \mathbb{R}^N : |x - \tilde{x}| \geq \frac{\text{diam}(D(\tilde{x}))}{2}(1 - \tilde{\theta}) \right\}) \leq C(\text{diam}(D(\tilde{x})))^{N-1}
\]

Without loss of generality we can assume \( \theta < 1 \). Then, iterating the previous argument we obtain

\[
\mathcal{H}^{N-1}(D(\tilde{x})) \leq C(\text{diam}(D(\tilde{x})))^{N-1}(1 + \tilde{\theta}^{N-1} + \tilde{\theta}^{2(N-1)} + \cdots) \leq C(\text{diam}(D(\tilde{x})))^{N-1} \frac{1}{(1 - \tilde{\theta})}.
\]

(3.37)

On the other hand, if we exchange the roles of \( x, \tilde{x} \) in the previous argument, we obtain the existence of a ball \( B_{\tilde{\theta}e^{1/2}}(\tilde{x}) \) such that:

\[
\mathcal{H}^{N-1}(B_{\tilde{\theta}e^{1/2}}(\tilde{x}) \cap A) \leq C(\text{diam} D(\tilde{x}))^{N-1}.
\]

(3.38)

Set \( p \in 0^*(N, N - 1) \) such that \( \text{Ker}(p) = V \). We then define a cylinder \( Q(\tilde{x}) \) such that its axis has the direction of \( V \). \( p(Q(\tilde{x})) \subset p(B_{\tilde{\theta}e^{1/2}}(\tilde{x})) \), its center is \( \tilde{x} \), and

\[
p^{-1}(p(Q(\tilde{x}))) \cap A \subset Q(\tilde{x})
\]

(3.38)

By our previous estimates we can construct \( Q(\tilde{x}) \) in such a way that \( \text{diam}(Q(\tilde{x})) \) is of the same order as \( \text{diam}(D(\tilde{x})) \), and

\[
\mathcal{H}^{N-1}(A \cap Q(\tilde{x})) \subset \mathcal{H}^{N-1}(A \cap (D(\tilde{x}) \cup B_{\tilde{\theta}e^{1/2}}(x_0))) \leq C(\text{diam} Q(\tilde{x}))^{N-1}
\]

(3.39)

We can also repeat this construction for every \( \tilde{x} \in A \). We can cover \( p(A) \) by \( \{p(Q(\tilde{x})) : \tilde{x} \in A\} \) as well. A standard covering argument yields then a covering \( \mathcal{F} \) with the finite overlapping property. Then

\[
\sum_{p(Q) \in \mathcal{F}} (\text{diam}(p(Q)))^{N-1} \leq \mathcal{L}^{N-1}(p(B_{\tilde{\theta}e^{1/2}}(x_0))) \leq C(e^{-1/\tilde{r}_1})^{N-1}
\]

(3.40)

Notice that, by (3.38)

\[
A \subset \bigcup_{Q \in \mathcal{F}} p^{-1}(p(Q)) \cap A \subset \bigcup_{p(Q) \in \mathcal{F}} (Q \cup A)
\]

and, by (3.39), (3.40)

\[
\mathcal{H}^{N-1}(A) \leq \sum_{p(Q) \in \mathcal{F}} \mathcal{H}^{N-1}(Q \cup A) \leq C \sum_{p(Q) \in \mathcal{F}} (\text{diam} Q)^{N-1} \leq C \sum_{p(Q) \in \mathcal{F}} (\text{diam} p(Q))^{N-1} \leq C(e^{-1/\tilde{r}_1})^{N-1}.
\]

This concludes the proof of Proposition 3.1. □
End of the proof of the Theorem. Lemmata 2.6 and 2.5 show that for each $\delta > 0$ and for any $\bar{x} \in \overline{B_R(0) \cap S}$ we can find $\bar{r}(\bar{x})$, and $p(\bar{x})$ such that for any $\bar{x} \in \overline{B_R(0) \cap S \cap B_p(\bar{x})}$ and $\tau \geq \bar{r}(\bar{x})$, $\|\psi(\cdot, \tau; \bar{x})\| \leq \delta$ and (3.2) holds. A standard compactness argument shows that for any $\delta > 0$ there exists $\tau_R$ such that for any $\tau \geq \tau_R$, $\bar{x} \in \overline{B(0) \cap S}$ $\|\psi(\cdot, \tau, \bar{x})\| \leq \delta$. We then choose $\delta \leq \delta$, where $\delta$ is as in Proposition 3.1. We can then, cover $\overline{B_R(0) \cap S}$ by a family of balls $\{B_{\theta e^{-\frac{1}{2}\tau_R}}(\bar{x}) : \bar{x} \in \overline{B_R(0) \cap S}\}$, and such that

$$\mathcal{H}^{N-1}(B_{\theta e^{-\frac{1}{2}\tau_R}}(\bar{x}) \cap S) \leq C(e^{-\frac{1}{2}\tau_R})^{N-1}$$

Notice that $\overline{B_R(0) \cap S}$ can be covered by a finite family of these balls, because all of them have the same radius. This gives (1.4). $\square$

Remark. The proof that was given here shows in fact that $\overline{B_R(0) \cap S}$ is a $(N-1, \mathcal{H}^{N-1})$ countable rectifiable set (cf. [F], 3.2.14). This fact is interesting by itself and has important consequences, for instance, all the $N-1$ dimensional measures defined in [F] have the same value on $S$.

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