Singular Perturbations of the Quadratic Map $z \to z^2 + c + \beta/z^2$

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1. Introduction

In this report, an overall summary of the undergraduate research project done by myself with Dr. Bruce Peckham is given. The project was done over the 2014 spring semester. The main goal of the project was to investigate the dynamics of a new family of functions in an area known as complex dynamical systems (although our specific functions are rational maps of $\mathbb{R}^2$ to $\mathbb{R}^2$, they are directly analogous to maps of $\mathbb{C}$ to $\mathbb{C}$).

2. Dynamical Systems Background

Before I move on to more specific details of the project, it is necessary for one to have a brief background on dynamical systems. In dynamical systems, when given a function, for example $z \rightarrow z^2+c+\frac{\beta}{z^2} = f_{c,\beta}(z)$, we let $z_n+1 = f_{c,\beta}(z_n)$, where we fix c and β and choose an initial condition z_0. We then iteratively plug successive z_n back into the equation, defining a sequence called an "orbit" ($\{z_0, z_1, z_2, \ldots\}$). The goal of dynamical systems is to investigate the long-term "fate" of these orbits. A few of the main questions to be answered are: How does the fate depend on the initial condition z_0? How does the fate depend on the parameters c and β? Do orbits diverge to infinity or stay bounded? Do the bounded orbits end up in periodic cycles?

3. Escape Pictures

Perhaps the simplest experiment that can give insight into the answers to these questions is the "escape picture." Below is an example of an escape picture:

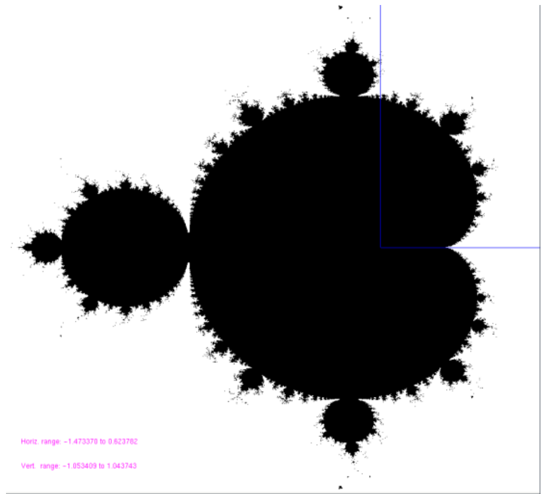


Figure 1. Mandelbrot Set (escape picture).

In the picture above, any areas that are white indicate orbits that diverge to infinity, and areas that are black indicate orbits that stay bounded. All escape pictures in this paper lie in the plane, where the x axis represents the real component of a complex number and the y axis represents the imaginary component of a complex number, however what exactly that complex plane represents can vary between pictures. Figure 1 is an example of an escape picture in what is called "parameter space". In this space, the complex plane represents the parameter c in the map $z \rightarrow z^2+c$. Then, with a fixed initial condition z_0 (analytically determined, known as the "critical point"), each point in the plane, representing a different c value for the family $z \rightarrow z^2+c$, is iterated a predetermined number of times (which means z_0 is plugged into the function, then z_1 is plugged back into the function, etc.). After the iterations are completed, if the final value z_n is outside of a set circle determined via the escape theorem (called the "radius of divergence", essentially a cut-off point to say "outside of this circle, orbits will continue to go off to infinity, inside, they stay bounded",), the point in the complex plane is colored white, and if z_n remains inside the circle, the point is colored black. This same process can also be done in the "phase space". In phase space, the complex plane represents the real and imaginary parts of z , meaning that each point in the plane defines a new initial condition z_0. In phase space, the parameter c is fixed, and the process described above is repeated, except now the escape picture represents the dependence of an orbit's fate on z_0, rather than on c.

4. Project Description

The main focus of our project was to first generate and then understand a set of the escape pictures for the specific function $z \rightarrow z^2+c+\frac{\beta}{z^2} = f_{c,\beta}(z)$. This family of functions is known as "singular perturbations of $z \rightarrow z^2+c$", where the term $\beta/z^\wedge 2$ is known as a "singular perturbation" term. The name singular perturbation comes from the fact that the term changes, or perturbs, the original function and that near $z = 0$, the term sends the orbit off to infinity, behavior known as a "singularity". The first facet of the project was to create a computer program that could generate these pictures for us in a reasonable amount of time. Most of the first half of the project was spent attempting to write computer code in Sage (an open source mathematics software, similar to Mathematica or MATLAB). The thought behind using Sage was that we could write code in a program that could be accessed from any computer for free (Mathematica and MATLAB are very expensive programs). Unfortunately, due to lack of computer coding wizardry, the programs we wrote in Sage were very slow, taking up to a few minutes to generate a single picture, and as we needed the functionality of comparing multiple pictures very frequently, our code just didn't cut the mustard. So, we decided to scrap the coding aspect of the project and use a program that Dr. Peckham had written in years previous. The program worked excellently for our purposes and generated beautifully detailed escape pictures in a few seconds. Once we had a working program to use, we set out to understand what was actually happening mathematically with the family of functions $z \rightarrow z^2+c+\beta/z^2$ where β and c are parameters and $/z$ represents the complex conjugate of z (which means the sign of the imaginary part of z is flipped).

With a working program, the first thing we set out to do was observe some escape pictures in the phase plane (where the x axis is the real part of z and the y axis is the imaginary part of z), and see how these pictures differ from previously observed phase plane pictures. In order to observe these pictures, we decided to fix the parameter β at $\beta = 0.001$, and we chose a number of c values close to the origin in an attempt to understand what kind of changes occur as the parameter c is varied. The following picture is the escape picture for $c = 0$, or $z \rightarrow z^2 + 0.001/z^2$ (ignore the green circles for now; they will be explained later).

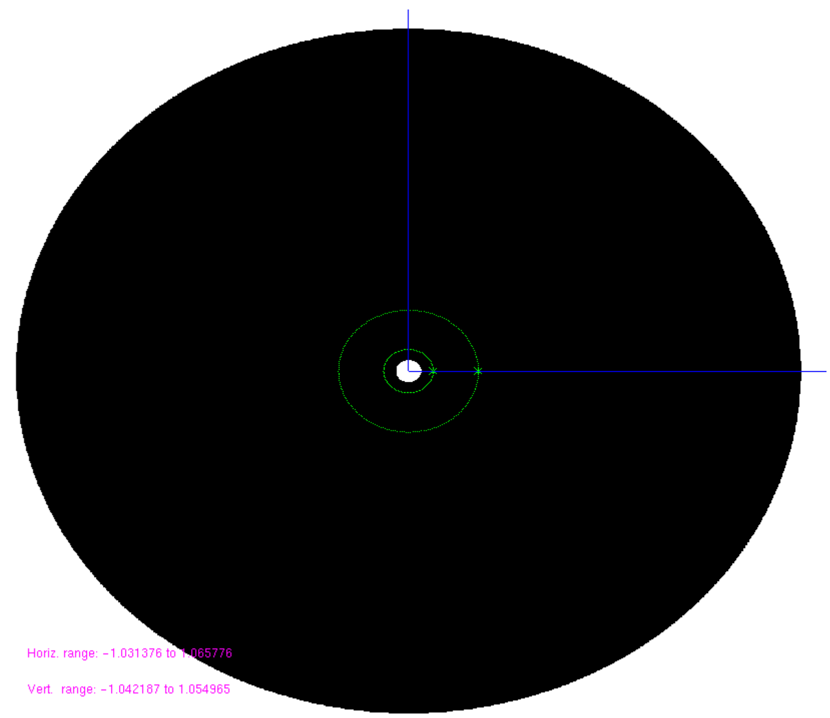


Figure 2. Phase plane, $c = 0$, $\beta = 0.001$

The main difference between this escape picture and the one observed without the singular perturbation term (β/z^2) is that our picture has a white hole around the origin (the escape picture for the non-perturbed function is a closed unit disk). This hole can be easily explained by the (β/z^2) term; when $|z|$ is near 0, (β/z^2) gets extremely large, so the first iterate, $z_1 = z_0^2 + 0.001/z_0^2$

will be very large, then, on the next iterate, the z^2 term will dominate, so z^2 will be even larger. This process will continue, sending the orbit off to infinity and causing the area around the origin to be colored white. This area we call the "trap door". If, as an orbit is being iterated, it ever has a value that lands inside of the trap door, that orbit will be sent off to infinity via the process just described, meaning that whatever the initial condition of that orbit was will be colored white in the escape picture. With this further understanding of the escape pictures in the phase plane, we examined the effect that varying the parameter c had on the pictures (again, ignore the colored circles for now).

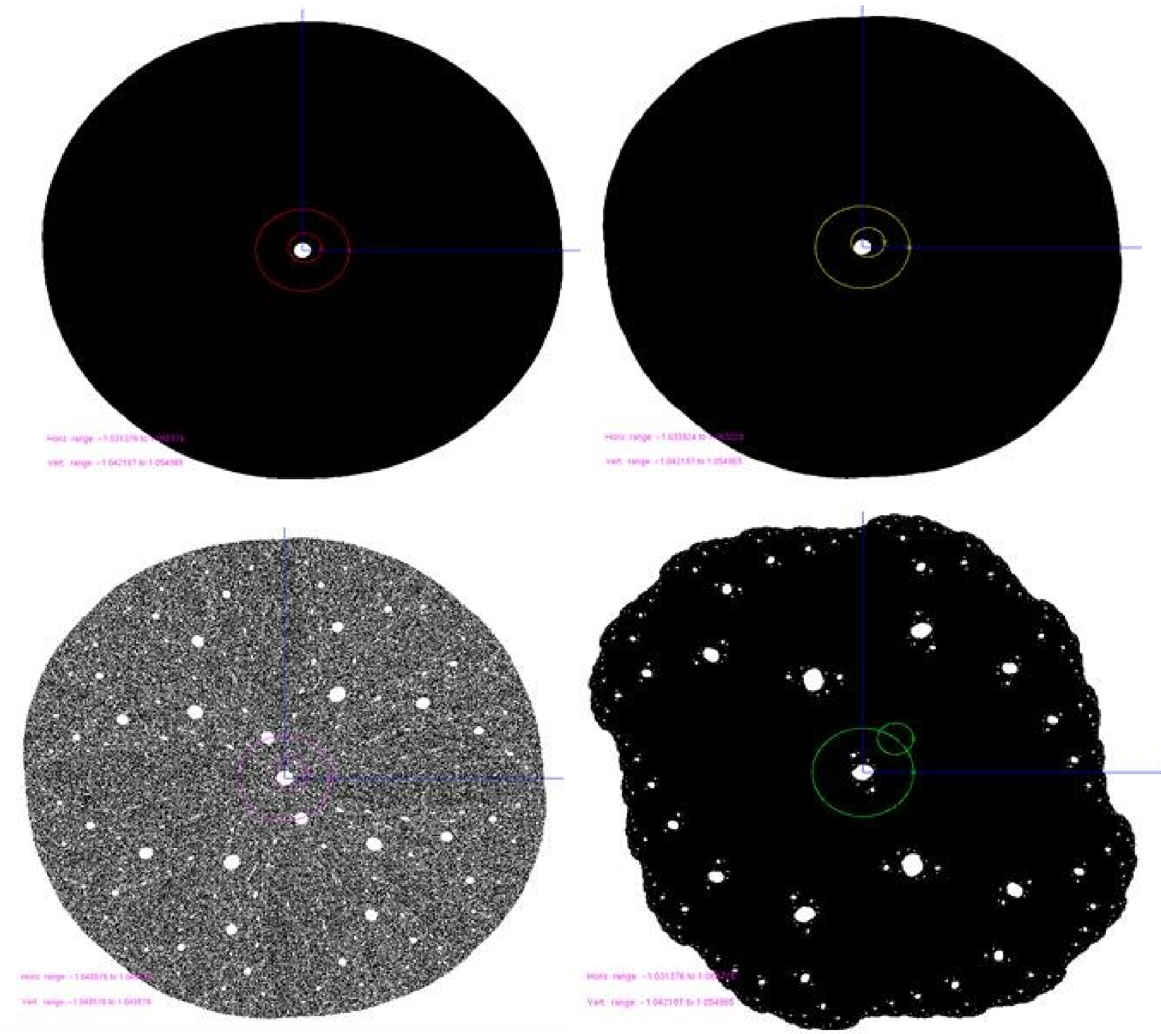


Figure 3. Phase plane pictures for varying c. Top left: c= 0.010125 + 0.010765i. Top right: c = 0.021480 + 0.020747i. Bottom left: c = 0.0248203 + 0.023050i. Bottom right: c = 0.114331 + 0.1359189i.

As is clear from the sequence of images above, there is a transition between solid, connected sets to sets covered with white "holes" around the c value $\approx 0.025 + 0.023i$. In order to investigate what exactly causes this transition, it is necessary that we talk about a concept mentioned briefly earlier in the paper, the critical point. As we know from calculus, a critical point is the z value where the derivative of a function equals zero, usually corresponding to either a local minimum or a local maximum. For the well understood function $z \rightarrow z^2+c$, it is obvious that the derivative of this function is equal to zero at the lone point $z = 0$, so the function has one critical point. However, with our function, $z \rightarrow z^2+c+\beta/z^2$, taking the derivative and finding critical points is much less straightforward. In order to find the critical points we must substitute x and y into the function by representing the complex numbers z and $\bar{z}$ by their real and imaginary components: $z = x + iy$, $\bar{z} = x-iy$. Once this substitution is made, the function is separated into its real and imaginary components and can be viewed as a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ instead of from $\mathbb{C}$ to $\mathbb{C}$. At this point, the Jacobian matrix of the function is set up. The determinant of this matrix is then set to zero; this process is analogous to the process of simply setting the derivative of $z^2 + c$ to zero. When this process is fully carried out, we find that the function $z \rightarrow z^2+c+\beta/z^2$ has not a critical point, but a critical circle, a continuum of critical points within the magnitude $|\beta|^{1/4}$.

Now, looking back at the phase plane pictures above, what the two circles at the center of every picture represent are the critical circle and the first image of the critical circle (meaning that every point on the circle is plugged into the function, and the value returned by the function is plotted). In the pictures, the circle centered at the origin is the critical circle, and the circle centered at the c value is the image of the critical circle. An advantageous way to think of these two circles is to think, in terms of radii, of the big one as the pre-image of a minimum, and the smaller one as the actual minimum value. The situation is analogous to the local minimum at the bottom of a parabola, the radius of the bigger circle is essentially the x value of the minimum, and the radius of the smaller circle is the y value of the minimum (since the domain and range are both two dimensional, we plot both the critical circle and its image in the same pictures). Now, we know nothing will ever be smaller than the y value of the minimum because it is the lowest value the function ever takes; in the same way, nothing will ever map to the inside of the image of the critical circle, (because its radius is the lowest value the function takes). Thus, as we see in the first few pictures, when the image of the critical circle completely encircles the trap door, nothing ever maps into the trap door, and the escape picture is solid black, not spotted with holes. Then, what we discovered is that the transition seen in the third image of Figure 3 occurs exactly at the point where the image of the critical circle crosses into the trap door. This can be very clearly seen in the following image, which is a close-up zoom of the image of the critical circle for the third image from Figure 3.

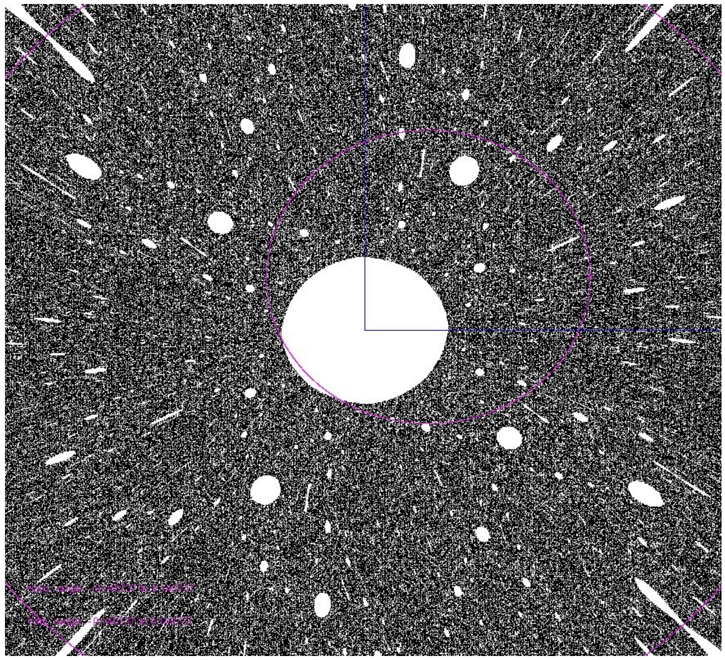


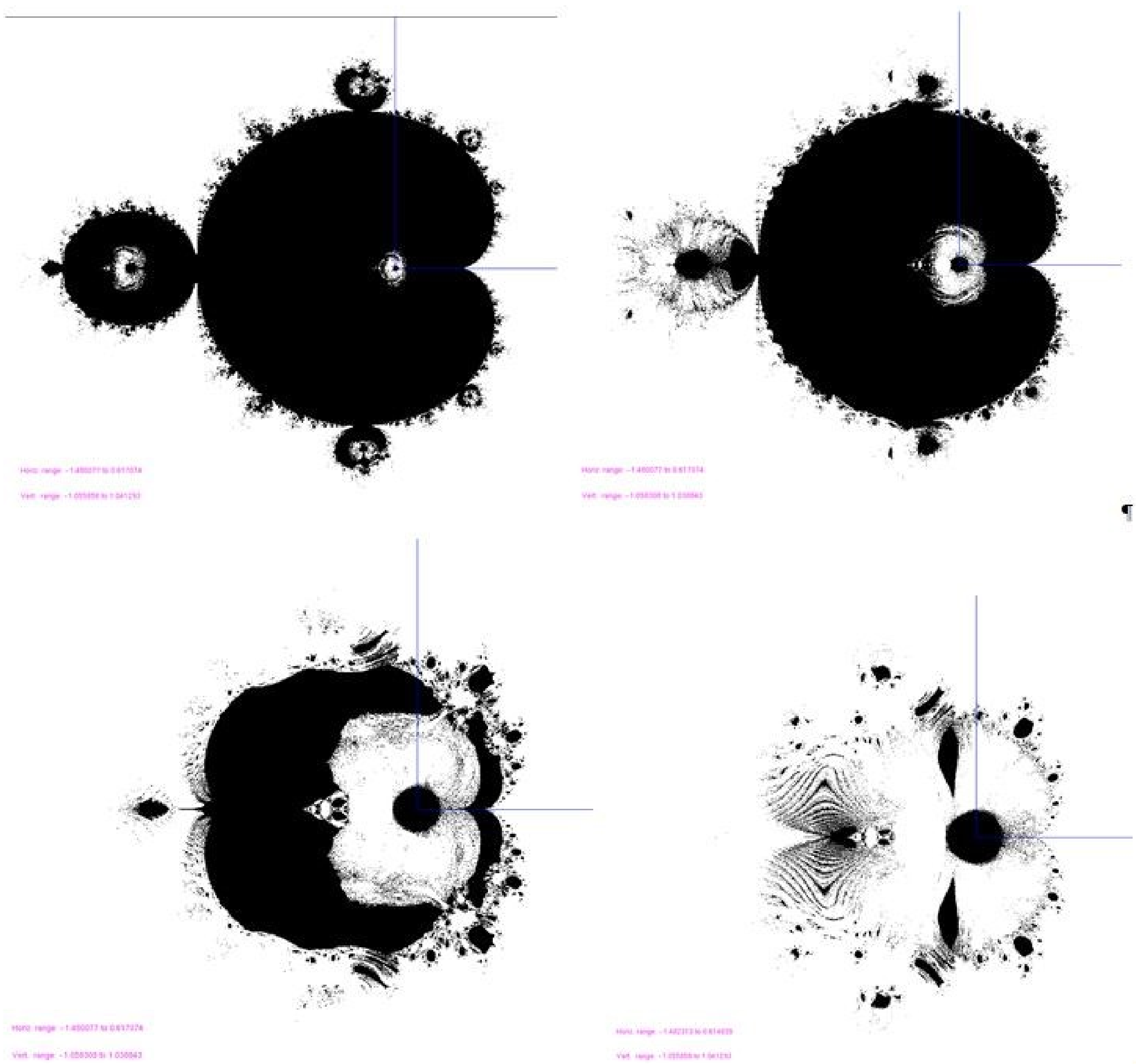
Figure 4. Zoom Around Trap Door for $c = 0.0248203 + 0.023050i$, β = 0.001

Since the image of the critical circle crosses into the trap door, other initial conditions are allowed to map into the lower left slice of the trap door outside of the image of the critical circle, meaning that many other points will be colored white, as their orbits now fall into the trap door and go off to infinity. Also, we found that the difference between the third and fourth pictures in Figure 3 is that in the third picture, the image of the critical circle itself falls into the trap door, whereas in the fourth picture, the images of the critical circle never cross into the trap door, and instead close in to a single fixed point (a point where, when successively plugged into the function, stays exactly the same, and is hence "fixed"). The reason for this transition is still a little unclear to us as we did not have sufficient time to devote to its understanding, but is certainly a topic for further research.

5. Parameter Plane

The second main topic of the project was to gain some kind of understanding of what was occurring in the parameter plane. As mentioned earlier, in the parameter plane, the x and y axes represent the real and imaginary parts of the parameter c , respectively. In parameter plane pictures, the value for β is fixed and an initial condition, z_0 is also fixed. Without digressing too much into the dynamical systems theory for choosing the initial condition, because of a famous theorem in dynamical systems, z_0 is be chosen to be the critical point for the function. In the case of the well understood family $z \to z^2 + c$, choosing the critical point is very easy as there is only one at $z = 0$. Unfortunately, as described in the previous paragraphs, for our family of functions $z \rightarrow z^2+c+\beta/z^2$ there is an entire circle of critical points. In creating parameter plan escape pictures, we arbitrarily chose the point on the critical circle with positive real part, or $z = |\beta|^{1/4}$. This choice made the computations easier. Any other choice of a point on the circle would be equally acceptable and would result in slightly different parameter plane pictures.

With a choice made for z0, we were able to generate some parameter plane escape pictures. We knew that as β approached 0, our parameter plane pictures would approach the Mandelbrot Set, which is the parameter plane escape picture for z → $z^2+c+\beta/z^2$, so we chose several small values for β and saw that as β increases, the singular perturbation term dominates and totally destroys the Mandelbrot Set. The following is a sequence of parameter plane pictures for the β 0.0001, 0.001, 0.007, 0.013, and 0.02:



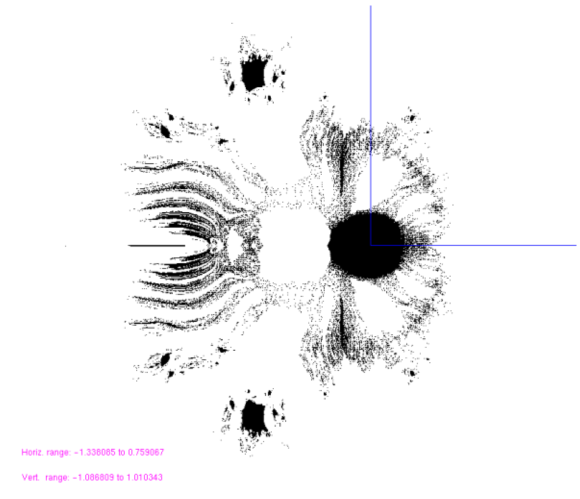


Figure 5. Parameter Plane Escape pictures for $\beta = 0.0001,\ 0.001,\ 0.007,\ 0.013,\ \textit{and}\ 0.02$

As Figure 5 clearly demonstrates, the singular perturbation term very quickly takes over the Mandelbrot Set and morphs into something completely unrecognizable. In an attempt to understand exactly what is happening to the parameter plane, we looked at the picture for $\beta =$ 0.001:

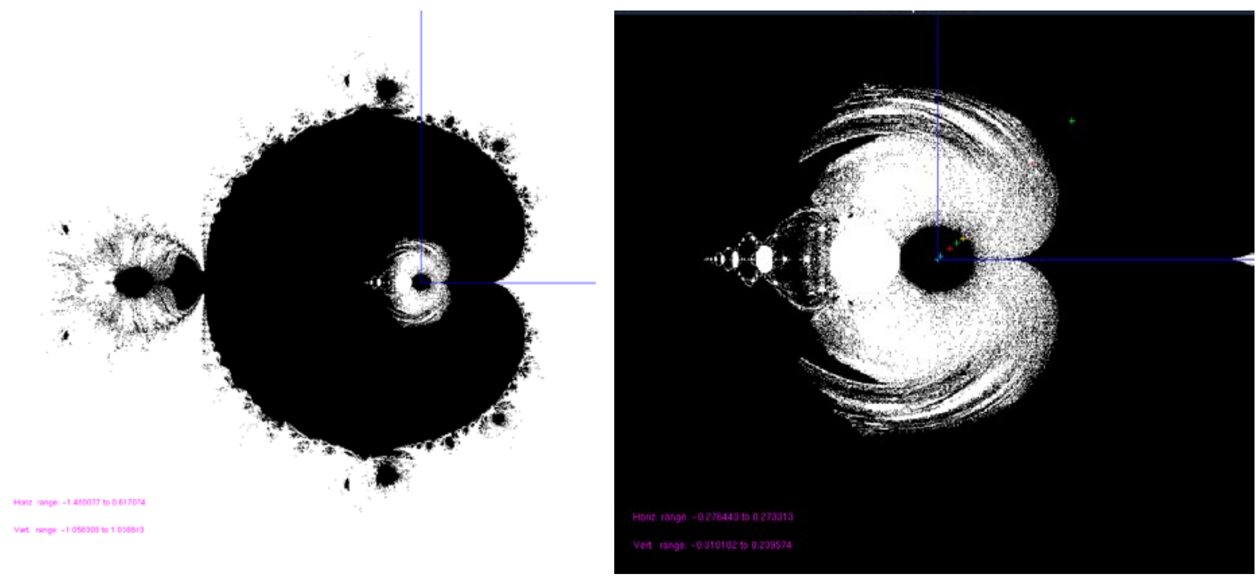


Figure 6. (left) Parameter plane escape picture for $\beta = 0.001$. (right) Zoom of center region in parameter plane.

Our goal was to understand exactly what is going on with this whole picture, but what we explained was the tiny black region around the origin. In order to explain this region, we used the sequence of images presented in Figure 3. Each one of these images correspond to a single point in the parameter plane picture, since the parameter plane is the plane for varying c , when you pick a single point in the parameter plane (fix a specific c), there is a corresponding phase plane picture with that specific c value for varying z. The pictures in Figure 3 correspond to the points shown with plus signs in the right side of Figure 6, transitioning from inside the black area around the origin to outside of it. What we found is that the small black region around the origin corresponds to places where the image of the critical circle still totally encircles the trap door. This means that the point on the critical circle that we chose as our initial condition stays bounded and never falls into the trap door, so that region gets colored black. The white region directly surrounding the black area near origin corresponds to places where the image of the critical circle lies within the trap door, meaning that our chosen initial condition hits the trap door and gets shot off to infinity, so that region gets colored white. Then, the very large black region that makes up most of the main cardioid corresponds to places where the successive images of the critical circle never cross the trap door and close in on a fixed point, meaning that our initial condition closes in on that fixed point and stays bounded, so again, this region gets colored black.

6. Summary

We set out to understand as much as possible about the dynamics of the $z \rightarrow z^2+c+\beta/z^2$ family. We described one transition that takes place around the origin in the parameter plane escape picture. As should be obvious, there is still a plethora of knowledge yet to be discovered about this family. To name just a few, we would like to come up with a way to generate parameter plane pictures that use more of the critical circle instead of just the one point, we would like to understand what is going on in the outer regions of the parameter plane picture as β is slowly increased, we would like to find all fixed points of the family, etc. Hopefully, in the next phase of Dr. Peckham's research, his team will find the answers to these questions.

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