Warren Goldfarb

Russell's Reasons for Ramification

I

Russell introduced a form of ramification in his 1906 paper "On 'Insolubilia' and Their Solution by Symbolic Logic."[1] There it is applied to propositions. Extended and somewhat modified, ramification is the central component of the theory of types as it is presented in "Mathematical Logic as Based on the Theory of Types" in 1908 and in *Principia Mathematica*.[2] That is, Russell did not separate the theory of orders, which embodies the ramification of propositional functions, from the theory of types. A disentanglement of these two notions was first urged by Ramsey in 1925, when he formulated a simple theory of types.[3] Ramification could then, and only then, be seen as a superposition of a theory of orders on a basic type-theoretic structure.

For over sixty years now, ramification has elicited an intriguing bipolar reaction. Those who have most shared Russell's motivations in the philosophy of mathematics have thought it confused or misguided (thus Ramsey, and also Carnap, Gödel, and Quine).[4] Yet many whose basic outlook is rather more distant from Russell's have found the notion alluring, have continued to study its nature and consequences (although often in settings removed from the theory of types), and have argued for its importance in foundational studies.[5]

Ramification of a domain of abstract entities is the result of requiring that legitimate specifications of such entities be predicative. Briefly put, a specification is predicative if it contains no quantifier that ranges over a universe to which the entity specified belongs.[6] (Obviously, we speak of specifications in an interpreted language.) The predicativity requirement allows a specification to license an existence claim only if the entity whose existence is inferred lies outside the universes of the quantifications in the specification. Thus, the requirement will yield a hierarchy of entities: those at any given level of the hierarchy are just those that are specifiable using quantifiers that range only over lower levels of the hierarchy. Ramification is just this division of entities into levels.

There is a particular philosophical cast that, it seems, has to be put on the nature of the entities under discussion in order for the predicativity requirement, and hence for ramification, to be justified. This philosophical cast, roughly put, is nonrealist and in a sense constructivist: these entities do not subsist independently of us, but are created or legitimized by our being able to specify them. (I have said "in a sense constructivist" because the specifications need not be constructive in the ordinary sense. There is no constraint of effectiveness, and no prohibition of quantifiers with infinite ranges.) It is, to be sure, not an easy matter to spell out such a constructivist view, particularly if the legitimacy of classical (truth-functional) logic is also to be supported. Yet it does seem clear that some such view will entail the predicativity requirement. Since it is first the specification that legitimizes the entity specified, that specification can in no way depend on the existence of the entity. Therefore, the ranges of the quantifiers in the specification cannot include the entity.[7]

My interest, however, is in the converse claim, namely, that ramification is justified *only* on such a constructivist view (and hence that, implicitly at least, Russell held such a view). This claim was forwarded by Gödel in "Russell's Mathematical Logic":

> If, however, it is a question of objects that exist independently of our constructions, there is nothing in the least absurd in the existence of totalities containing members which can be. . . uniquely characterized only by reference to this totality. (p. 136)

The point is echoed by Quine in *Set Theory and Logic*, speaking of classes as the entities in question:

> For we are not to view classes literally as created through being specified. . . . The doctrine of classes is rather that they are there from the start. This being so, there is no evident fallacy in impredicative specification. It is reasonable to single out a desired class by citing any trait of it, even though we chance thereby to quantify over it along with everything else in the universe. (p. 243)

Given this analysis, by now enshrined as the common wisdom, the bipolar attitude toward ramification that I have mentioned becomes most understandable. The sort of constructivism that Gödel and Quine impute to Russell is, it seems, simply out of place in a logicist. For constructivism bespeaks a shift in the very conception of existence, a shift away from realism; whereas one point of logicism, and other classical theories of mathematics – including that which Gödel espoused – is to vindicate, in one way or another, our full-bloodedly realistic talk about mathematical entities. What work and attention ramification has received over the past sixty years has issued from authors with markedly proof-theoretic leanings.

Now the claim that a constructivist view must underlie ramification is not implausible, and it is even possible to read some of Russell's remarks as pointing to such a view. Yet constructivism does seem inconsonant with Russell's usual overarching manner of talking about existence.[8] In the period of the *Principles of Mathematics*, Russell espoused a strong variety of realism.[9] Subsequently, he became ontologically more and more parsimonious. This parsimony is achieved by elimination and reduction, using the devices of incomplete expressions and logical constructions. That is, statements apparently about certain entities are systematically paraphrased, so that they can be held true without any commitment to those entities. Thus Russell is able to shrink the class of things whose existence must be assumed. But, throughout, his conception of what it is to be an entity does not change. There is no notion of a sort of existence, different from the full-blooded kind in being reliant on our specifications.

Thus, it seems to me that the imputation of constructivism to Russell stands in need of refinement. Gödel and Quine make Russell out to have a general vision of what the existence of abstract entities comes to, and thus to be adopting constructivism as a fundamental stance toward ontology. That does not seem accurate to Russell. Rather, the justification for ramification rests on the particular sorts of entities to which it is applied, namely, propositions and propositional functions. To understand this, we must see more clearly why these entities are central to Russell's logical enterprise and what special features of their structure Russell exploits. The results might have the appearance of constructivism, but Russell's most basic reasons for ramification are not the outgrowth of such a general position; rather, they are far more particular to the nature of the entities he treats.

Attention to the importance of propositions in Russell's conception of logic will also clarify the other widely recognized root of the predicativity requirement. Russell's logical theorizing always proceeds in response to the paradoxes, both set-theoretic and semantic. Now, Ramsey pointed out that the former are blocked by simple type theory alone. The others are not, but they involve notions like truth, expression, and definability (indeed, that is why we call them "semantic"). Therefore, Ramsey argued (as had Peano before him), their solution need not come from the logical theory itself. Thus, it appears, it was a misguided desire on Russell's part that the logical system do more than is appropriate that led to ramification. However, it seems to me that Russell's treating the two sorts of paradoxes as one is not a gratuitous blunder: his view of the aims of logic precludes any sharp separation of them.

In what follows, I shall examine Russell's conception of logic and the theories of propositions that it spawns. The general themes of this conception, canvassed in section II, explain the centrality of propositions and propositional functions to logic and support Russell's view that logical structures must preclude the possibility of semantic paradox. In section III, I examine those features of propositions and propositional functions—features that arise from their intensionality—that undercut any immediate link between ramification and constructivism. Russell's particular theories of propositions, sketched in section IV, exhibit the mechanisms through which ramification is generated, and cast some light on various oddities in *Principia Mathematica*.

Russell's reasons for ramification rest on a wealth of rather intricate views; they cannot be accurately summarized by a label or a quick diagnosis. I hope here to be making a first step towards the fuller treatment that they demand.

II

Russell took logic to be completely universal. It embodies all-encompassing principles of correct reasoning. Logic is constituted by the most general laws about the logical furniture of the universe: laws to which all reasoning is subject. The logical system provides a universal language; it is the framework inside of which all rational discourse proceeds.

For Russell, then, there is no stance outside of logic: anything that can be communicated must lie within it. Thus there is no room for what we would call metatheoretic considerations about logic. Logic is not a system of signs that can be disinterpreted and for which alternative interpretations may be investigated: such talk of interpretations would presuppose just the sort of exterior stance that Russell's conception precludes. In particular, the range of the quantified variables in the laws of logic is not subject to change or restriction. These ranges must be fixed, once for all, and fixed as the most general ranges possible.

The conception of the universality of logic that I have just outlined is intrinsic to Russell's logicism.[10] Although prior to his coming under the influence of Wittgenstein Russell did not have much to say about the status of the laws of logic, he did draw strong philosophical consequences from the reduction of mathematics to logic. These consequences rest on the complete generality that Russell took logic to have. For logic, on his conception, is not a special science. It invokes no concepts or principles peculiar to one or another particular area of knowledge. Rather, it rests only on assumptions that are involved in any thinking or reasoning at all. In this way, Russell could take the logicist reduction to show that no special faculties (such as Kantian intuition) need be postulated in order to account for mathematics.

This conception points also to what has to figure among the "logical furniture of the universe" whose laws are at issue. Logic is the universal framework of rational discourse; this suggests that its primary objects of study will be the vehicles of judgment, that is, the entities to which a person who judges is primitively related. For Russell (in his earlier period), the vehicles of judgment are propositions.[11] Logic will provide laws that govern all propositions, and will thus exhibit the bounds of discourse: the bounds, so to speak, of sense. Now, that there are general laws of propositions depends essentially on the fact that propositions are complex. Hence part of the task of logic is to display what this complexity consists in. The branch of logic that Russell sometimes calls "philosophical logic" provides the general framework for the analysis of propositions: the categories of building blocks from which propositions are made, and the ways in which they are fitted together. In this way propositional functions—functions whose values are propositions—also come to figure centrally among the entities that logic treats.

The centrality of propositions underlies Russell's view that the logical system must treat the semantic paradoxes. Now Russell did recognize a distinction between these and the set-theoretic paradoxes. In "On 'Insolubilia,' " he presents a "simple substitutional theory" that eliminates classes, relations, and propositional functions. He goes on to say:

> The above doctrine solves, as far as I can discover, all paradoxes concerning classes and relations; but in order to solve the *Epimenides* we seem to need a similar doctrine as regards propositions. (p. 204)

(This "similar doctrine" is his earliest form of ramification, discussed in section IV.) The distinction, then, is simply a distinction of subject matter. Just as the set-theoretic paradoxes are about classes and relations, and to solve them logic must inquire into the nature of these entities, the Epimenides paradox and its ilk are about propositions and propositional functions, and logic must inquire here too. Indeed, given Russell's conception of logic, the semantic paradoxes are more important to it. Since the structure of propositions is the very center of logic's attention, the semantic paradoxes pose a greater threat.

Although Ramsey shared Russell's view of the centrality of propositions to logic, he denied that logic bears the responsibility for solving the semantic paradoxes. In "The Foundations of Mathematics," Ramsey gives an account of propositions, using infinitary truth-functions, that supports the simple—rather than ramified—theory of types. He claims that this system need not concern itself with the semantic paradoxes, insofar as

> [they] are not purely logical, and cannot be stated in logical terms alone; for they all contain some reference to thought, language, or symbolism, which are not formal but empirical terms. So they may be due not to faulty logic or mathematics, but to faulty ideas concerning thought and language. (p. 21)[12]

Thus the semantic paradoxes are to be blocked not by laws about the structure of propositions but rather by special features of the notions like truth and definability that are invoked in them.

A brief look at Ramsey's solution of the Grelling paradox (that of the adjective "heterological") will show why his position would be unacceptable to Russell. Ramsey notes that the paradox depends upon a relation of expression between a word (an adjective) and a propositional function. He then urges that "expression" is ambiguous, and that there is a hierarchy of expression relations. Once this is taken into account, the definition of "heterological" leads to no contradiction.

Now Russell, I think, would ask why there is no relation that sums up (is the union of) the different expression relations that Ramsey postulates. (Such a union would reintroduce the paradox.) Since in the absence of ramification the propositional functions expressed are all of the same type, nothing in the nature of the relata of these relations would preclude such a union. Particularly given his acceptance of arbitrary infinitary truth-functions, Ramsey must take the impossibility of summing the relations to be merely factual, perhaps of a natural or empirical sort. Given his notion of proposition, this position is of doubtful coherence—particularly in view of the a priori reasoning that engenders it.

In fact, the ramified theory of types, even with the axiom of reducibility, does prevent precisely this summation. As Church has rigorously substantiated, ramification—in particular, the differing orders of the propositional functions involved—precludes the existence of a single expression relation that could generate Grelling's paradox.[13] The impossibility here flows from the nature of the entities at issue, not from any ad hoc restriction. Thus the hierarchical structure of semantic relations arises from purely logical considerations; it is not a fact of a special science. No conclusion of a special science could block a union of levels of a hierarchy; only a logical impossibility could do so.

There is another criticism that Russell could make of Ramsey's diagnosis of the semantic paradoxes, one specific to the Epimenides. In this paradox, Ramsey would presumably take the notion of truth to be the culprit, and claim that special features of that notion—perhaps some ambiguity in the phrase "is true"—forestall the reflexivity that yields contradiction. (This would be similar to a Tarskian approach.) However, Russell's view of propositions enables one to dispense with all explicit mention of truth in logic. The proposition, e.g., that no proposition having property φ is true can be expressed as

(1) $(p)(\varphi p \supset \neg p)$.

Hence this proposition does not contain the notion of truth. Now, the acceptability of this manner of expression rests on the construal of truth as a property of propositions (rather than of sentences, as in Tarski), and, to some extent, on Russell's view that propositions are both nameable and assertible by sentences; i.e., propositions both are complex entities on all fours with other objects and are the objects of judgment. As a result of these views, the notion of truth simply disappears, through formulations like (1). Moreover, Ramsey agrees with Russell, at least to the extent of accepting formulations like (1); this forms the basis of his redundancy theory of truth.[14]

The Epimenides paradox can then be generated with no use of semantic notions. All that is required is a value for the propositional function φ in (1) that is uniquely satisfied by the proposition expressed by (1). Such a propositional function, it seems, would not be hard to imagine. Thus, it appears, a Ramseyan solution to the paradox is simply not available. The weakness of the extralogical assumptions needed for the paradox makes it clearer why for Russell the nature of propositions themselves must figure in any solution. (In fact, ramification blocks the paradox by precluding the proposition expressed by (1) from being a member of the range of the quantified variable in (1).)[15]

I have been talking of the central role of propositions in Russell's logic, and have so far ignored the other principal sort of entity Russell considers, namely, classes. Now the system of *Principia* makes no class-existence assumptions; rather, Russell eliminates classes in favor of propositional functions. He sees this "no-class theory" as the solution to the problems that had vexed him for many years. For mathematics, it seems, requires classes; yet the paradoxes had raised pervasive questions about their existence. Such questions could not be solved merely by devising a consistent theory of classes and leaving it at that. For this would make the theory a special science, whereas Russell sought to justify classes as *logical* objects: as objects guaranteed by principles implicit in all reasoning. Only thus could the reduction of mathematics to class theory be a logicist reduction, and bear more weight than a mere interpretation of one branch of mathematics in another.[16]

For this reason Russell worries not just about which classes exist but about the very nature of a class. In the *Principles of Mathematics* a central issue is that of how a class is a unity: what logical operation binds the members together into a single object, so that the class can serve as a logical subject. In the *Principles* theory there is a general feature of discourse that is of help here, namely, the quantifier-words. For at that time, Russell took *all* φ's to be a denoting concept that denotes the class of φ's. Since "all" is a logical word if anything is, the logical nature of classes is thereby vindicated. However, this route is not open to Russell after the demise of the *Principles* theory of denoting concepts in 1905 (for more on this theory, see Hylton's contribution to this volume).

Given all this, the attractions of the no-class theory for Russell are obvious. The paradoxes show that the straightforward naive views fail for both propositional functions and classes. But whereas Russell's fundamental conception of logic demands an account of propositions and propositional functions, classes are additional, and pose further questions. The no-class theory enables Russell to be agnostic about the existence of classes and to say nothing about their nature.

III

Important consequences about the nature of ramification follow from the fact that Russell's logic treats propositions and propositional functions rather than classes. Indeed, Russell never envisaged a ramification of classes. When in "Mathematical Logic as Based on the Theory of Types" or in *Principia* he speculates about the existence of classes, he treats them as all of the same order, and he asserts that the existence of classes would imply the axiom of reducibility. Now the claim that ramification necessarily bespeaks constructivism is, it seems to me, most plausible if ramification of classes is at issue. If all the members of two classes are of like logical category, then the intrinsic nature of these classes can provide no ground for a difference in *their* logical category. To support any such distinction, some feature extrinsic to the identity of the classes would have to be brought in. To support ramification in particular, this extrinsic feature would have to reflect something about how the classes are characterized. But to justify in turn the role that a feature extrinsic to the identity of the class is now taken to play, it would have to be maintained that the feature is essential to the existence of the class. The constructivism implicit in such a view is clear.[17]

Thus it appears that the close connection between ramification with respect to classes and constructivism depends on the fact that we have a conception of the nature of the class that works "from below." The identity of a class is determined by its membership; yet ramification would demand that categorical distinctions be made among classes on grounds over and above membership.

Matters are different, however, with propositions and propositional functions. For these are intensional entities. The identity of a propositional function is not determined by the objects of which it is true, nor is the identity of a proposition determined by its truth-value. This opens the possibility that intrinsic features of propositions and propositional functions could support the categorical distinctions that ramification demands. If so, the immediate link between ramification and constructivism that exists for classes would be severed.

Our conception of the nature of a proposition or propositional function comes not "from below" but rather by way of the manner in which the proposition or propositional function is expressed. This points to a distinction in the notion of specification when applied to propositions and propositional functions, as opposed to classes. To specify a class is to give a propositional function that is true of all and only the members of that class. The specification can be understood on its own; given such an understanding, it is a further question whether or not a given class is the one specified. That is, the specification by itself does not tell us which entity *is* the one specified. This space between the specification and the entity specified does not exist in the case of propositions and propositional functions. Here, to understand the specification is to understand the proposition or propositional function specified. That is, the specification immediately tells us which entity is the one meant.

For Russell, understanding is – like judging – a relation to the propositions and propositional functions expressed by our words. To understand a sentence is to grasp the proposition expressed by the sentence.[18] Thus, for propositions and propositional functions, it makes no sense to speak of first understanding a specification and then going on to investigate which entity is specified; for that entity is given in the understanding of the specification. With classes, clearly, matters are different: a specification can be understood, that is, a propositional function can be grasped, yet there can still be a question as to which class is specified.

Indeed, for this reason I find the terminology of "specification" completely misleading in the realm of propositions and propositional functions. The closest analogue to class specifications here would be descriptions like "the last proposition conjectured by Fermat," descriptions the understanding of which is independent of the particular propositions specified. Such "indirect" specifications do not figure in Russell's logical system. Rather, the comprehension axioms for propositions and propositional functions that are implicit in the system involve not so much the specification of these entities as the presentation of them. One is not characterizing a proposition or propositional function: one is giving it. I shall therefore use "presentation" rather than "specification" in this connection.

This helps explain, I think, why no comprehension axioms are explicit in *Principia*. When Russell talks about the existence of classes, he clearly recognizes that the issue concerns which principles of the form

$$(\exists\alpha)(x)(x\epsilon\alpha \equiv Fx)$$

are to be accepted.[19] From a modern perspective, a theory of propositions and propositional functions would have to answer the analogous questions: for which formulas F and Gx (and for what orders of the variables p and φ) are

$$(\exists p)(p = F)$$

and

$$(\exists\varphi)(\varphi\hat{x} = G\hat{x})$$

to be accepted?[20] But Russell never comes close to formulating this or to having axioms of this form. The mathematical power of such axioms is obtained by means of generalization and instantiation, as in

$$\frac{H(G\hat{x})}{(\exists\varphi)H(\varphi\hat{x})} \quad \text{and} \quad \frac{(\varphi)H(\varphi\hat{x})}{H(G\hat{x}).}$$

The absence of such axioms springs from the idea that, once the presentation $G\hat{x}$ is given, one needs no special principle to yield the existence of the propositional function. Rather, in using the formula $G\hat{x}$ in our sentences at all, we are using that propositional function in our propositions. Thus, inferences like the preceding appear to be purely logical, and not the result of special existence assumptions.

The role of presentations of propositions and propositional functions also lies behind Russell's not distinguishing among the various formulations of the vicious-circle principle. One formulation is "no totality may contain members that are definable only in terms of that totality"; the others use "presuppose" and "involve" instead of "are definable only in terms of." Gödel points out that, prima facie, these are three distinct principles, and he claims that only the first yields ramification, whereas only the second and third are plausible without recourse to constructivism ("Russell's Mathematical Logic," p. 135). But if definitions are not external to the entities under consideration, as they are to classes but are not – if they are presentations in my sense – to propositions and propositional functions, then the distinction among these formulations seems to collapse. For, in that case, to say that a totality is necessarily involved in any presentation of a proposition is to say that the totality is involved in the proposition.

In pointing to what I have called the lack of space between the intrinsic nature of a proposition or propositional function and the manner in which it is presented, I do not mean to be suggesting that the existence of such an entity is dependent on its having been presented. That would lead to an extreme constructivism involving a notion of temporality, of the sort to which Quine alludes (*Set Theory and Its Logic*, p. 243). It may appear, however, that every such entity must be in some sense presentable. Indeed, Ramsey and Gödel speak of Russell's system as treating only "nameable" entities. Yet the terms "presentable" and "nameable" must be approached with caution; they cannot be used here in any ordinary sense. Certainly, presentability construed as nameability in the language of *Principia* is not at issue. Indeed, Russell never tells us – nor does he think he needs to tell us, in the logical writings – what the basic building blocks of presentations are. The logical vocabulary of *Principia* is a mere skeleton: any notions the special sciences (or epistemology) might arrive at are to be added to its vocabulary. Logic does not set any limits on which predicates and relations (i.e., elementary propositional functions) exist. Even to speak of propositions as "in principle presentable" is somewhat misleading. This is shown by a claim Russell makes concerning a consequence of the hypothesis that classes exist: he says that if α is a class, then $\hat{x} \epsilon \alpha$ is an elementary propositional function (*Principia*, p. 166). Now, if classes exist, there are too many of them to be even potentially nameable; thus these elementary propositions also fail to be nameable. In short, the notion of "presentability" that is at issue does not involve even the notion of a possible language. It is, so to speak, an ontological rather than a linguistic notion. Thus it provides no ground for the sort of criticisms that Ramsey and Gödel make.

Since logic sets no limits to the elementary functions, it may appear that it can provide no information at all about presentations of propositions. This appearance is deceptive. Consider, for example, the Epimenides paradox. Here we present (or so it seems) a proposition like:

> Every proposition hereby asserted is false.

Now our understanding of this proposition is *as* a universal one, with a quantifier over propositions. That is, the complexity represented by the quantifier is built into the proposition. Since logic does tell us what sorts of complexity can figure in the structure of propositions, this proposition and ones like it can be treated by logic, even in ignorance of what the basic building blocks of propositions generally—that is, what the elementary propositions and propositional functions—are.

I have been arguing that the intensional nature of propositions and propositional functions makes it in principle possible to justify ramification without reliance on a constructivist view of these entities. This does not yet provide any justification. The substantive constraints on the complex structures of propositions and propositional functions that comprise ramification rely on the details of the theory of these entities that Russell adopts. To those details I now turn.

IV

Unfortunately, in *Principia* and the writings leading up to it, Russell never lays out any comprehensive theory of propositions. (This is in marked contrast to the *Principles*, in which the nature of propositions is investigated at length.) One clear reason for this absence is that, during this period, Russell's views frequently shift. Often, several different theories seem to be espoused simultaneously. Hence I shall here be engaging in somewhat speculative reconstruction.

One rather startling view that Russell expresses in *Principia* is that propositions simply do not exist. On page 44 he writes,

> What we call a "proposition"... is not a single entity at all. That is to say, the phrase which expresses a proposition is what we call an "incomplete" symbol; it does not have meaning in itself, but requires some supplementation... when I judge "Socrates is human" the meaning is completed by the act of judging.

This view is of a piece with Russell's "multiple-relation" theory of judgment, about which Russell subsequently writes at length. It does not appear that this view is consistent with the logic of *Principia* (see Church, "Comparison," p. 748). Luckily, the view seems to play no real role in Russell's explanations of his logical system. In what follows, I shall take the charitable course of ignoring it.

As background, let me recall Russell's early view of propositions. A proposition is a complex of entities; ordinarily, it is that complex which, if true, is identical with the fact that makes it true. Thus the constituents of a proposition are, ordinarily, the entities that the proposition is about; for example, the proposition that Mont Blanc is higher than the Zugspitze literally contains two mountains, as well as the relation *higher than*. I have said "ordinarily" since, in the *Principles of Mathematics*, there are exceptions that arise in connection with denoting concepts. As a result of "On Denoting," the exceptions disappear.[21]

The theory of descriptions of "On Denoting," and, indeed, the general notion of incomplete expression there introduced, puts into prominence the role of quantified variables. In line with the early view of propositions, Russell takes variables to be actual constituents of propositions, that is, to be entities underlying the use of a letter like "x" or "φ" in the same way that an object underlies the use of its name. To the obvious question of what these entities are, Russell has no answer, but does not seem overly perturbed by this. In "On Denoting" he says, "I take the notion of the *variable* as fundamental," and explains no further (p. 104). Note that Russell speaks of "the variable" as though there were only one. In this period Russell took variables to be completely unrestricted: all variables range over everything. Since every variable confers the same generality, the particular identity of a variable is needed only for its role in cross-referencing. Thus, in a sense, there is only *the* variable—that is, the notion of unrestricted variation—to explain.

Indeed, during this period Russell stresses the complete generality of variables. He takes the lack of restrictions on range to arise from the universality of logic, especially from the idea that anything expressible at all can be expressed inside its framework. (See, e.g., "On 'Insolubilia,'" p. 206. Such reasons also seem to underlie his dissatisfaction with the primitive theory of types discussed in Appendix B of the *Principles of Mathematics*.) The retention of unrestricted generality is a great advantage he sees in the "substitutional theory" he develops in 1906. In that theory, classes and propositional functions are eliminated in favor of propositions and a primitive notion of substitution of one entity for another in propositions.[22] The set-theoretic paradoxes are avoided, even though the basic language contains no type restrictions, because of the form of the contextual definitions through which class variables can be introduced. Were the basic language to be extended by means of these definitions, the class variables so introduced *would* have to be restricted as to type. In other words, any putative formula in the extended language that violates type restrictions could not be reduced by means of the contextual definitions to a formula of the basic language.

However, Russell recognizes in the third section of "On 'Insolubilia'" that the theory as he had so far laid it out does not block the semantic paradoxes. Hence he takes it a step further and denies that any but elementary propositions exist: that is, he suggests, there are no propositions that contain bound variables. Thus paradox-engendering forms like $(p)(\varphi p \supset \neg p)$ do not express propositions; a fortiori, there is nothing they express that can instantiate the universal variable, and paradox is blocked. Of course, Russell then has to show how to make sense of quantified statements at all. He does this by talking of ambiguous statements:

> "For all values of x, $x = x$" I take to be an ambiguous statement of any one of the various propositions of the form "$x = x$." There is thus not a new proposition, but merely an unlimited undetermined choice among a number of propositions. (p. 207)

It is hard to extract a coherent theory from Russell's sketchy remarks here, but I take it that what he has in mind is, roughly, that judgment, assertion, truth, and so on relate only to elementary propositions. The force of applying such notions to quantified statements is captured by more complex ways in which the notions can be related to elementary propositions. Thus, he says:

> If we want to say what is equivalent to "I am making a false statement containing n apparent variables," we must say something like: "There is a propositional function $\varphi(x_1, x_2, \ldots, x_n)$ such that I assert that $\varphi(x_1, x_2, \ldots, x_n)$ is true for any values of $x_1, x_2, \ldots, x_n$ and this is in fact false." (p. 208)

Statements are logical fictions, and "there is no way of speaking of statements *in general*: we can speak of statements of propositions, or statements containing one, two, three, . . . apparent variables, but not of statements in general." (p. 207)

Russell does not give the contextual definitions needed here (and there may well be serious problems in this, particularly with regard to alternating quantifiers and nonprenex statements). But it seems clear that what he envisages still preserves the virtues of the substitutional theory. In the basic language, all variables range over the whole universe; now, though, the universe contains none but elementary propositions. If the contextual definitions are used to extend the language by introducing variables for quantified propositions, the form of those definitions would require a hierarchy of such variables. Indeed, his remarks suggest that to quantify over n-quantifier statements is really to use $n + 1$ quantifiers in the basic language; thus, one cannot quantify over all statements, and, more particularly, one cannot in an n-quantifier statement quantify over n-quantifier statements. A ramified structure is thereby induced by the contextual definitions, although none exists in the basic language.[23]

The sort of ramification that results, however, is more finely grained than that which would arise from a constraint of predicativity. *Any* additional quantifier affects the order of a statement; that is, order is determined by the number of quantifiers, not by their range. To be sure, the constraint of predicativity is obeyed, but as a special case. No statement can contain a quantifier that ranges over a domain of which that statement is a member, since that would require a statement to contain simultaneously n and $n + 1$ quantifiers.

Russell seems to jettison this theory: there is no hint of it in "Mathematical Logic as Based on the Theory of Types," and no subsequent assertion that elementary propositions are real while others are logical fictions. (This may be due to difficulties in giving the appropriate contextual definitions or in finding a satisfactory account of judgment and other propositional attitudes; more likely, Russell may feel that his later official view that *no* propositions exist supersedes it.) Nonetheless, as we shall see, remnants of the 1906 view persist in *Principia*.

In "Mathematical Logic," Russell's view is quite different from that exhibited in the substitutional theory. Propositional functions are now taken as legitimate entities; with their reappearance, the realistic view of propositions as complexes also returns. Moreover, variables no longer have unlimited generality; rather, a variable ranges only over the entities of one order. These two points are closely related. Once propositional functions are allowed, with the concomitant idea that every well-formed open formula of the language expresses one, paradoxes like the Russell paradox will immediately arise unless some restrictions are put on the variables. Fortunately, the admission of propositional functions allows Russell to think that restrictions in the ranges of variables are consonant with the universality of logic. For, he argues, those restrictions come from the ranges of significance of the propositional functions. That is, inherent in a propositional function is a range of arguments to which the propositional function can be applied with sense. In this way, the restrictions on variables are intrinsic rather than stipulated ad hoc; they do display the bounds of sense.

I shall not examine Russell's tenuous arguments for the coincidence of ranges of significance and orders. I wish only to point to a consequence of having variables that lack complete generality. Once such variables are used, the question of the nature of the variable (as an entity) becomes far more urgent. Different variables can have different ranges; it then appears that our understanding of a proposition or a propositional function that contains quantified variables will depend quite heavily on an understanding of what those ranges are. The variable must carry with it some definite information; it must in some way represent its range of variation. Therefore, I would speculate, Russell takes a variable to presuppose the full extent of its range. Now, since the identity of a proposition or propositional function depends on the identity of the variables it contains, and the variables presuppose their range of variation, even the weakest form of the vicious-circle principle suffices to yield ramification. Indeed, Russell may perhaps even think of the variable as *containing* all the entities over which it ranges; in that case, the only principle needed is that a complex entity cannot contain itself as a proper part. In any case, the ramification engendered is just the sort required by the constraint of predicativity. In short, in this theory ramification springs from the strongly realistic picture Russell has of propositions and propositional functions, and in particular of the nature of variables.

In *Principia Mathematica* it is by and large this theory of 1908 that is put to work. However, in both the philosophical discussions and the technical work there are remnants of the 1906 view. For example, in the Introduction, chapter II, part III, Russell distinguishes between first truth, which elementary propositions can have, and second truth, the truth of propositions that contain one quantified variable. He takes second truth to be parasitic on first truth. This leads him to a distinction of orders of propositions on the basis of the number of quantifiers they contain, a clear shadow of the 1906 view. Yet in part IV of the same chapter, he distinguishes orders of propositions on the basis of the sorts of variables they contain, in the standard ramified manner. Both views are reflected in subsequent technical work. The more finely grained ramification that stems from 1906 motivates his grounding quantificational logic in *9 by *defining* the truth-functions of quantified propositions in terms of quantifications of truth-functions of elementary ones. This is unnecessary on the 1908 view; and in *10, he starts quantificational logic over again, giving what he calls an "alternative method." This method is a more standard axiomatic approach, consonant with the fully realistic view of quantified propositions of 1908. There are other curiosities in the text that exhibit the presence of two distinct theories.[24] Thus, it seems to me that the murky beginning sections of *Principia* will be illuminated if the influence of these competing theories is traced through.

Conclusion

I have been emphasizing how the issues surrounding ramification look when we bear in mind the conception of logic that focuses on propositions and propositional functions. Russell's logicism comes down to the idea of trying to obtain mathematics from the laws of these intensional entities; in this way, his basic conception of logic is less mathematical than Frege's. Russell's logicist enterprise fails, as is shown by the need for the axiom of reducibility (which cannot be justified on any grounds but expediency); this failure may indeed show, as Gödel says, that there is irreducibly mathematical content in mathematics.

Many features of the project give rise to ramification. Russell is no constructivist, although a semblance of constructivism can arise from the tight connection between how these intensional entities are presented and what they are. In fact, Russell tends toward realism, but a realism modified by his concern to investigate how intelligible claims on reality could function. Criticisms of ramification from a "Platonist" point of view simply ignore this concern. However muddled Russell sometimes gets, he seems to have grasped the extremely subtle points that such a concern can unearth.[25]

Notes

1. B. Russell, "On 'Insolubilia' and Their Solution by Symbolic Logic," in Russell, *Essays in Analysis*, ed. D. Lackey (New York: Braziller, 1973), pp. 190–214.

2. B. Russell, "Mathematical Logic as Based on the Theory of Types," (1908), in J. van Heijenoort (ed.), *From Frege to Gödel: A Sourcebook in Mathematical Logic* (Cambridge, MA: Harvard University Press, 1967), pp. 150-82. A. N. Whitehead and B. Russell, *Principia Mathematica*, vol. 1 (Cambridge: Cambridge University Press, 1910).

3. F. P. Ramsey, "The Foundations of Mathematics" (1925), in F. P. Ramsey, *The Foundations of Mathematics and Other Logical Essays* (London: Routledge & Kegan Paul, 1931), pp. 1-61.

4. R. Carnap, "On the Logicist Foundations of Mathematics" (1931), in P. Benacerraf and H. Putnam (eds.), *Philosophy of Mathematics: Selected Readings* (Cambridge: Cambridge University Press, 1983), pp. 41-52. K. Gödel, "Russell's Mathematical Logic," in P. A. Schilpp (ed.), *The Philosophy of Bertrand Russell* (LaSalle, IL: Open Court, 1944), pp. 125-53. W. V. Quine, *Set Theory and Its Logic* (Cambridge, MA: Harvard University Press, 1969).

5. See, e.g., H. Weyl, *Das Kontinuum* (Leipzig: Veit, 1918), S. Feferman, "Systems of Predicative Analysis," *Journal of Symbolic Logic*, 29 (1964), pp. 1-30, and I. Hacking, "What Is Logic?" *Journal of Philosophy*, 76 (1979), pp. 285-319.

6. In the Russellian setting, the notion of predicativity has to be amplified to preclude quantifiers that range over universes containing anything that "presupposes" the entity specified. For the amusing history of the word "predicativity," see Quine, *Set Theory and Its Logic*, p. 242.

7. Clearly, this argument relies upon a premise to the effect that the sense of a specification depends in some way on the ranges of the quantifiers it contains. Interestingly enough, it is this

premise—rather than the thesis that certain entities are first legitimized by their specifications—that Carnap denies in "Logicist Foundations."

8. I do not use Russell's ontological terminology. For him, existence means existence in space and time. Abstract entities thus do not exist; they have Being (they *are*). Since I prefer to avoid the grammatical awkwardness, and am not concerned with any contrast between concreta and abstracta, I use "existence" for Russell's "Being."

9. B. Russell, *The Principles of Mathematics* (London: Allen & Unwin, 1903).

10. Fuller accounts of this conception can be found in J. van Heijenoort, "Logic as Calculus and Logic as Language," *Synthèse*, 32 (1967), pp. 324-30; W. Goldfarb, "Logic in the Twenties: The Nature of the Quantifier," *Journal of Symbolic Logic*, 44 (1979), pp. 351-68; and P. Hylton, "Russell's Substitutional Theory," *Synthèse*, 45 (1980), pp. 1-31. Hylton also discusses in more detail the relation between this conception and the philosophical upshot Russell took logicism to have.

11. This move, although natural, is not inevitable. Although Frege had a similar conception of logic, he distinguished between the vehicles of judgment (thoughts) and the entities that logical laws are about (objects, including the truth-values, and functions). Russell, of course, rejected the basic points on which Frege's distinction rests: the theory of sense and reference and the notion that sentences refer to truth-values.

12. Ramsey continues, "If so, they would not be relevant to. . . logic, if by 'logic' we mean a symbolic system, though of course they would be relevant to logic in the sense of the analysis of thought"; and, in a footnote: "These two meanings of 'logic' are frequently confused." This rather glib distinction suggests that Ramsey is not, in the end, in agreement with key features of Russell's conception of logic. Perhaps it explains how Ramsey could introduce infinitary truth-functions so cavalierly, a step justly criticized by Gödel as undercutting the whole point of a logicist reduction ("Russell's Mathematical Logic," p. 142).

13. A. Church, "Comparison of Russell's Resolution of the Semantical Antinomies with That of Tarski," *Journal of Symbolic Logic*, 41 (1976), pp. 747-60.

14. See F. P. Ramsey, "Facts and Propositions" (1927), in *Foundations of Mathematics and Other Logic Essays*, p.143.

15. Cf. *The Principles of Mathematics*, p. 527, where Russell considers a paradox that, as far as I know, does not reappear in his writings. This paradox would be classified as semantic, from our point of view; but it is engendered by means that, for Russell, involve no nonlogical notions. (I am grateful to Leonard Linsky and to C. Anthony Anderson for calling my attention to this passage.)

16. An interest in showing classes to be logical objects also lies behind Frege's criticizing Cantor at the same time as adopting the notion of extension (and, later, course-of-values) of a concept in order to obtain classes. Of course, Frege and Russell have different conceptions of what it is to be a logical entity.

17. Even the simple theory of types looks unnatural if taken directly as a theory of classes. While there might be arguments from our conception of classes to the need for a cumulative hierarchy, I see none for a noncumulative hierarchy of classes, one that precludes all mixed-level classes.

18. Sentences and, in general, words and symbols play no role in Russell's characterizations of understanding, judgment, assertion, and inference. That is why such use-mention errors as exist in his writings rarely turn out to have vicious effect.

19. See B. Russell, "On Some Difficulties in the Theory of Transfinite Numbers and Order Types" (1906), in *Essays in Analysis*, pp. 135-64, especially section II.

20. Russell uses the circumflex as an abstraction operator: if Fx is an open sentence, then $F\hat{x}$ expresses the propositional function that, for each argument a, yields as value the proposition expressed by Fa. He discusses this operator in the Introduction to *Principia*, p. 15; but in the body of that work it is introduced almost on the sly, in *9.131. Cf. Hylton, "Russell's Substitutional Theory," p. 27.

21. B. Russell, "On Denoting" (1905), in *Essays in Analysis*, pp. 103-19.

22. Details of this theory are given in Hylton, "Russell's Substitutional Theory." Note that it is entities that are substituted one for another, not words or names. Hence this theory is quite distinct from what is currently called "the substitutional theory of quantification."

23. Here I disagree sharply with Hylton's reading of this section of "On 'Insolubilia' " ("Russell's Substitutional Theory," pp. 22-26).

24. A more minor example is his definition of predicative function. In the Introduction (p. 51),

Russell identifies the predicative functions with the first-order ones. Thus a predicative function of individuals is one whose quantifiers, if any, range only over individuals. This is in keeping with standard ramification. But in *12 he calls a propositional function predicative only if it is elementary, a definition that the more finely grained ramification of 1906 would support.

25. I am greatly indebted to Peter Hylton for many illuminating conversations and much useful advice. I would also like to thank Burton Dreben, Leonard Linsky, and Thomas Ricketts for helpful discussions.