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Poincaré against the Logicists

Although the great French mathematician Henri Poincaré wrote on topics in the philosophy of mathematics from as early as 1893, he did not come to consider the subject of modern logic until 1905. The attitude he then expressed toward the new logic was one of hostility. He emphatically denied that its development over the previous quarter century represented any advance whatsoever, and he dismissed as specious both the tools devised by the early logicians and the foundational programs they urged. His attack was broad: Cantor, Peano, Russell, Zermelo, and Hilbert all figure among its objects. Indeed, his first writing on the subject is extremely polemical and is laced with ridicule and derogation. Poincaré's tone subsequently became more reasonable but his opposition to logic and its foundational claims remained constant.

Poincaré's first paper on logic (1905) is a response to a series of articles published the previous year by Couturat, who was the purveyor of logicism to the French. Poincaré seems particularly outraged at the logicists' claim to have conclusively refuted Kant's philosophy of mathematics. Thus Poincaré is moved to write in the area by a purely philosophical animus.[1] No mathematical issues are on his mind; his concern is with general, philosophical claims about the nature of mathematics, and he is reacting to claims of this sort made by the partisans of the new logic. Apparently, then, to questions in the foundations of mathematics Poincaré wishes to provide answers opposed to those of the logicists. Such a description of the situation, however, is simplistic. Rather, as I hope to make clear, Poincaré's conception of what the questions in the foundations of mathematics *are* differs considerably from that held by the logicists. An examination of the differences that operate at this level can, I believe, illuminate both logicism and the rejection of logicism.

A second reason for interest in Poincaré's writings on foundations lies in the fact that two of his points continue to figure in contemporary discussions. The first is what I shall call Poincaré's *petitio* argument. Poincaré alleges that any program of grounding number theory in something else will invariably beg the question; for if that foundation is to be carried out, number theory (and in particular mathematical induction) will have to be presupposed. The second is Poincaré's *vicious circle principle*, which precludes mathematical definitions that are impredicative, that is, that involve quantified variables ranging over a universe that contains the defined entity. Indeed, Poincaré's introduction of the notion of predicativity is his most influential contribution to foundational studies.

I shall treat these two points in turn. An analysis of the *petitio* argument can provide helpful clues both about Poincaré's view of foundational questions and about the construal of logicism that is needed to turn the objection aside. Matters are less straightforward with the vicious circle principle, and my treatment will be more purely historical. I shall trace how Poincaré arrives at the principle and how, after seeing its force, he begins to argue for it more philosophically. Those arguments, together with his remarks on the technical import of the principle, provide further insight into his central motivations.

My discussion is limited to Poincaré's explicit writing on logic and foundations. It might well be most interesting and fruitful to investigate relations between his foundational views and his mathematical work in other areas, but unfortunately there is little go on. As I have mentioned, Poincaré comes to discuss foundations solely for philosophical reasons. Unlike Hilbert, Cantor, and others, no technical factors drive him on. As a result, his papers on foundations are disconnected from his positive work in mathematics. (Again, this contrasts with his philosophical writings on geometry, in which, for example, analysis situs is explicitly discussed.) Of course, it may be noted that Poincaré's style of mathematics generally was more constructive than, say, Hilbert's—Poincaré was a *French* analyst. The question is whether anything specific can be drawn from this that would illuminate his position on foundations. Such an investigation is beyond the scope of this paper. My aim below is to sort out what the foundational issues are; it is a subsequent task to find out whether and how Poincaré's stand on these issues can be clarified by attention to his mathematical work.

I

Poincaré begins his attack on the new logic with the avowed aim of showing that the new logicians have not eliminated the need for intuition in mathematics. By showing this, he says, he is vindicating Kant (1905, 815). This avowal is misleading, for in Poincaré's hands the notion of intuition has little in common with the Kantian one. The surrounding Kantian structure is completely lacking; there is no mention, for instance, of sensibility or of the categories. Indeed, in (1900), his address to the Paris Congress, Poincaré explicitly separates mathematical intuition from imagination and the sensible. Thus there is no hint of an epistemological framework in which the notion functions. For Poincaré, to assert that a mathematical truth is given to us by intuition amounts to nothing more than that we recognize its truth and do not need, or do not feel a need, to argue for it. Intuition, in this sense, is a psychological term; it might just as well be called "immediate conviction."[2] As a result of the unstructured nature of the notion, Poincaré's argument in (1905) can be purely negative. If a mathematical proposition is convincing, that is, it seems self-evident to us, and the purported logical proofs of it are insufficient, then, tautologously, intuition in Poincaré's sense is what is at work.

Such a sense of "intuition" underlies another of Poincaré's criticisms, namely, that the logicists have done nothing but rename the chapter in which certain truths are to be listed: what was called "mathematics" is now called "logic." These truths, he says, "have not changed in nature, they have changed only in place" (1905, 829). This is merely a renaming, he claims, for since the axioms the logicists use are not merely "disguised definitions," they must rest on intuition. He emphasizes this even with respect to truth-functional logic. Here Poincaré is simply ignoring a central philosophical point of logicism. Frege and Russell take logic to consist of those general principles that underlie all rational discourse and all rational inference, in every area, about matters mathematical or empirical, sensible or nonsensible. Consequently, the truth of these principles could not stem from any faculty with a specialized purview; certainly, their truth does not play any particular role in constituting the world of experience. If one persists, as Poincaré does, in claiming that intuition is needed, then one has in fact robbed "intuition" of all content.[3]

Poincaré also complains that logicism errs insofar as "in reducing mathematical thought to an empty form. . . one mutilates it" (1905, 817).

He sees logicists as saying something analogous to "all the art of playing chess reduces to rules for the movement of the pieces," whereas, of course, one does not understand chess just by knowing those rules. In his Paris Congress address, he stresses how intuition is needed to gain an understanding of proofs in analysis, even after the correctness of each step in the proof has been recognized; and in (1905) he talks of intuition as that which guides the choice of conventions to adopt and of routes to take toward a proof. Intuition thus becomes merged with a notion of what might be called "mathematical sagacity." For here Poincaré's concern is explicitly with the psychology of mathematical thinking, with a faculty or skill that enables a person to do mathematics, or to do it well.[4]

A similar concern is exhibited in Poincaré's discussion of the continuum in (1893), amplified in (1902). His remarks amount to a speculation about what could be called the psychogenesis of the idea of the continuum, that is, how we came to think or how we come to think of this mathematical structure. By 1903 Poincaré was acquainted with Dedekind's construction of the real numbers. He applauds it for *reflecting* the "origin of the continuum": we come to "admit as an intuitive truth that if a straight line is cut into two rays their common frontier is a point" (1902, 40). Nothing could be further from Dedekind's outlook; Dedekind, after all, explicitly offers his construction to show how to *rid* analysis of any geometrical elements.

In invoking this sense of intuition, in (1905), Poincaré raises the charge that logicism does not accurately portray the psychology of mathematical thinking. He then claims to forswear the charge: he admits that there is a distinction between context of discovery (or invention) and context of justification, and that intuition in the broader sense pertains to the former whereas the logicists are concerned with the latter (1905, 817). His more serious antilogicist arguments are not of this sort, on the surface. However, closer scrutiny reveals a continued dependence on a psychologistic conception of the foundational enterprise. This can be seen in his *petitio* argument.

There are, in fact, several forms of this argument in Poincaré's writing. The clearest is directed against the notion that mathematical induction is not a principle with content, but is just an implicit definition of the natural numbers. Poincaré notes that such a definition must be justified by showing that it does not lead to contradiction; yet any such demonstration would have to rely on mathematical induction. The argument is a good one but it has force only against a naive version of Hilbert's position. Early on, Hilbert saw that if axioms are to endow talk of their objects with warrant, a consistency proof is needed. He was unclear about the status of such a proof at first. Subsequently, he came to see that one cannot hope to ground mathematics *without residue* in this way; rather, the metamathematical reasoning employed has to be taken for granted. His formalist program rests on the idea that the residue would be finitary mathematics, a small part of number theory. Finitary mathematics includes some amount of induction—although Hilbert was never explicit about the matter—namely, induction on decidable number-theorectic predicates (that is, quantifier-free induction). There is no *petitio* since there is no claim that all number-theoretic principles are to be legitimized from scratch.

To be sure, Poincaré's arguing in this manner is explicable. Hilbert's sketch in (1904) of the application of his early ideas of implicit definition to number theory is quite unclear. The distinction between mathematics and metamathematics does not become rigorous until nearly two decades later, and the question of limitation of means available in the metatheory does not exist in 1904, even in embryo. More to the historical point, Couturat (1904), to which Poincaré is responding, does not distinguish Hilbert's notion of implicit definition from the Frege-Dedekind-Russell strategy for explicit definition of the numbers. Hence it is not surprising that Poincaré thinks of the project of reducing arithmetic to logic as being carried out by implicit definition. In point of fact, the logicists objected strongly to the idea of implicit definition. Frege and Russell (and, later on, Couturat) insist that existence is not proved by consistency; rather, consistency is vouchsafed by showing existence. Hence this form of the argument does not touch logicism.

Another form is more relevant. In a sarcastic passage (1905, 823), Poincaré examines definitions of particular numbers given in Peano-writing by Burali-Forti (1897). He notices, in the definition of zero, the use of the concept "in no case," and, in the definition of one, the concept that would be rendered in ordinary language "a class that has only one member." He concludes, "I do not see that the progress is considerable." Similarly, he cites phrases used in introducing logical machinery, such as "the logical product of two or more propositions," as showing that a knowledge of number must be presupposed. In general,

> It is impossible to give a definition without enunciating a phrase, and difficult to enunciate a phrase without putting in it a name of a number,

or at least the world "several," or at least a word in the plural. And then the slope is slippery, and at each instant one risks falling into a *petitio principi* (1905, 821).

Here Poincaré is making what is, by our lights, an elementary logical mistake. As Frege warned, there can be an appearance of circularity; but this appearance is dispelled when one distinguishes uses of numerical expressions that can be replaced by purely quantificational devices from the full-blooded uses of such expressions that the formal definition is meant to underwrite. The former involve no arithmetic, and no mathematical induction; hence there is no *petitio*. In a reply to Poincaré, Couturat makes this point reasonably well, if rather untechnically (Couturat 1906). He argues, for example, that what is presupposed is not the *number* one but rather unity (or, as Frege might prefer to say, objecthood). Poincaré never cedes the point. In (1906), he abandons it for the sake of the discussion, to avoid, as he puts it "the spectacle of an interminable guerrilla war." He adds, "I continue to think that M. Courturat defines the clear by the obscure, and one cannot speak of x and y without thinking *two*" (1906, 294).

Thus Poincaré does not accept the logical distinctions that would rebut the charge of *petitio*. The distinction he does invoke, between "clear" and "obscure," is a psychological one. Indeed, in this context clarity and obscurity amount to no more than familiarity and unfamiliarity. (I take it that *we* do not find the quantificational notions to which Poincaré is objecting particularly obscure.) In this, Poincaré is going beyond his point that mathematical induction is psychologically convincing enough that no further grounding of it is needed (all we need do is cite "intuition"); he is claiming that any attempt to ground it further *must* fail, because it will have to reduce familiar notions to ones less familiar. Now that is not the logicists' criterion of success; they were under no illusion that the notions to which arithmetic is to be reduced would be more familiar to their audience. The contrast between Poincaré and Frege here is striking (there is, however, no evidence that Poincaré was acquainted with Frege's work). Frege explicitly mentions that familiarity poses a danger, since a familiar inferential step will seem self-evidently correct to us, "without our ever being conscious of the subordinate steps condensed within it" (1884, 102). As a result, such a step will be too readily deemed intuitive, rather than composed of smaller steps each of which is purely logical. Thus familiarity can obscure the need for analysis of our reasoning.

Poincaré's example "one cannot speak of *x* and *y* without thinking *two*" highlights the psychologistic nature of his argument. Despite Couturat's urging that *logically* the number two is not presupposed in the use of two variables, Poincaré rests on the alleged psychological fact that in this case we do always think of two. Again the contrast with Frege is stark. Frege argues emphatically that the "ideas and changes of ideas which occur during the course of mathematical thinking" are irrelevant to the foundations of arithmetic (1884, vi); and that, more generally, a sharp distinction must be made between the mental or physical conditions under which a person comes to understand, appreciate, or believe a proposition and the ultimate rational basis of the proposition. In the absence of such a distinction, Frege adeptly urges, we would have to take into account in mathematics the phosphorus content of the human brain (ibid.); this, for Frege, is a reductio. Indeed, this distinction is required even to make proper sense of the claim that mathematics is a priori. It may well be true that a person cannot do mathematics without having had experiences; but such a fact is irrelevant to the grounds for the propositions of mathematics, which is what the claim of a priori status concerns (Frege 1884, 3 and 12).

The contrast with Frege shows how Poincaré is—despite his disclaimer—construing the project of the foundations of mathematics as being concerned with matters of the psychology of mathematics and faulting logicism for getting it wrong. Now there is a subtle aspect to this difference between Poincaré and the logicists. I claimed, noncontroversially, that Poincaré's objection is logically in error: the notions used in the definition of number do not presuppose number. But to accept this is to accept the notion of presupposition that is first made available through modern logic. We all agree that the notions "in no case," "a class with one object," and so on, do not presuppose any number theory, because of the way they can be rendered using (first-order) quantificational logic. In our acceptance of this rendering as being the criterion of presupposition, we have already accepted at least part of the picture that the logicists were urging. If we take Poincaré to reject even such a use of logic, and to claim that if certain thoughts always occur to us in such cases then the content of those thoughts is "presupposed," we can make some sense of his objection. It is, of course, questionable whether such a position is ultimately coherent; but, in any case, the distance between Poincaré's stance and that of any logicist is at this point unbridgeable.

More generally, the Fregean distinctions I've been stressing—between habitually accepted inferences and fully analyzed inferences, and between empirical conditions a person must satisfy in order to arrive at certain propositions and the ultimate rational basis for the propositions—must be given content by establishing a means for saying what analysis and justification look like: that is, for saying what counts as showing that one proposition is the basis for another. Only thus can the opposition be exhibited between a Fregean project and one of describing how people actually come to hold various propositions. It is the new logic itself that provides those means. Thus the questions that logicism is asking are themselves first given sense by the framework the logicists develop for answering them. And that the new logic does this, by being explicatory of "analysis" and of "justification," is part of the reason Frege takes it to deserve the honorific "logic."

In this there is a circle of a peculiarly philosophical sort. If you don't buy the picture, you needn't buy the picture. But, in fact, entrance into the circle is hard to resist: for the notion of nonpsychologistic justification can be seen as, and was urged to be, just the fully thought-through version of the sort of rigor at which proofs in mathematics, in the ordinary, unreconstructed, sense, already aim. (This point had particular force given the achievements of the arithmetiziers of analysis.) Poincaré can then be painted, or caricatured, as holding that in pursuing mathematics we seek such justifications, but with regard to matters he wants to think of as basic (because they are familiar), the proper story is a psychological one. For Frege, to be content with this is to make the rigor of *all* proofs as illusion.

In any case, the objection I've discussed is, by our lights, simply mistaken, and Poincaré does not return to it after 1905 (although he apparently thinks well enough of it to reprint the relevant paragraphs in [1908]). He does, however, give a more sophisticated *petitio* argument in (1909b), in criticizing Russell's theory of types. Descendants of this argument are still discussed today, in large measure due to Parsons (1965). The clearest version focuses on the fact that to formulate the (formal) system of logic, to which arithmetic is to be reduced, we inductively define the formulas and the notion of derivation. The very foundations of logic, it is alleged, are thereby shown to require induction.

This version of the argument may appear to have more force than Poincaré's original one; but, I believe, the appearance is misleading. Its seeming force stems from the fact that, in contemporary logic, formal systems are automatically treated from a metatheoretical standpoint. Consider, for example, a Hilbertian view: here the justification of a mathematical proposition starts with the assertion that the proposition, as a formal object, is a theorem of a certain formal system. That assertion is, of course, a metamathematical one, requiring notions defined by recursion. Thus, even before the question of a consistency proof for the system enters, arithmetic must be invoked. (As before, this causes the Hilbertian no problem, since he does not claim to eliminate induction.) Contemporary foundational discussions, although ordinarily not committed to Hilbert's formalism, share this starting point; investigation of the status of a proposition begins with a metamathematical assertion about its provability in some system.

With respect to logicism, though, the formal system plays a markedly different role. The logical system Frege or Russell proposes is meant to be the universal language, inside of which all reasoning takes place. There is no metatheoretical stance either available or needed. Justification is just what is done inside the system. To give the ultimate basis for a proposition is to give the actual proof inside the system, starting from first principles; that is, it is to assert the proposition with its ground, not to assert the metaproposition "this sentence is a theorem." The role of formality in logicism is limited: it enables us to be sure that all the principles needed for the justification of the proposition have been made explicit, but it is neither essential to the nature of the justification nor constitutive of its being a justification. To a logicist, the charge the number theory is being invoked is based, in the end, on thinking that not judgments but the manifestations of judgments—that is, the formal symbols—are the objects of study. Frege steadfastly argued against this construal. Logic is not *about* manipulations of signs on paper, even though it may be a psychological necessity for us, in order to be sure that we are proceeding logically, to verify proofs by syntactic means.

Frege explicitly mentions a similar issue:

> A delightful example of the way in which even mathematicians can confuse the grounds of proof with the mental or physical conditions to be satisfied if the proof is to be given is to be found in E. Schröder. Under the heading "Special Axiom" he produces the following: "The principle I have in mind might well be called the Axiom of Symbolic Stability. It guarantees us that throughout all our arguments and deductions the symbols remain constant in our memory—or preferably on paper." (1884, viii)

Of course, if we are to be able to do mathematics, the signs will have to remain constant before our eyes; but that hardly says that mathematics presupposes the physics of inkblots. Similarly, if we are to be able to do logic we probably will have to be able to count, so that, for example, we can count the number of parentheses in a formula. On the logicist view, that does not show that logic presupposes counting.[5]

In sum, the logicists' conception of logic as the universal framework of rational discourse reduces the sophisticated form of Poincaré's objection to the original, naive form. That conception yields a distinction between the justification of a proposition and the means for expressing and checking putative justifications. Knowledge about the formal system of logic pertains only to the latter; possession of such knowledge is among the conditions under which a person, in fact, will be able to give correct justifications. As we have seen, it is a central tenet of antipsychologism that such conditions are irrelevant to the rational grounds for a proposition. Thus the objection is defeated.

Of course, Poincaré does not accept any such distinction; if the formal system is "comprehensible" only to one who already knows arithmetic, then the theorems of the formal system presuppose arithmetic (see 1909b, 469). Here again, "presupposition" is for Poincaré a notion based in psychology, not logic; the dispute between Poincaré and the logicists amounts, at bottom, to a difference in the conception of the foundational enterprise.

Thus in the end none of the forms of the *petitio* argument has force against logicism. Poincaré's ultimate reliance in them on a psychologistic view of the questions in the foundations of mathematics underlines what is involved in the logicists' establishment of their radically antipsychologistic view. One can reject that view, but, as Poincaré himself exhibits, such a rejection leaves little foundational role for logic at all. The rejection pervasively affects the basic notions of justification, presupposition, and proof; as a result, the general questions one asks about mathematics are considerably transformed. On this score, some current approaches that seek naturalistic accounts in the philosophy of mathematics are in the same position as psychologism.[6] Consequently, the extent to which such approaches are capable of addressing the problems that gave rise to the subject of foundations of mathematics is highly unclear.

II

Poincaré first enunciates the vicious circle principle in (1906), and his subsequent foundational writings focus almost exclusively on it. By 1906, Poincaré had gained a far more extensive acquaintance with the new logic. In particular, he had read Whitehead (1902), in which the logicist construction of the numbers is outlined, and Russell (1905), in which Russell discusses various paradoxes and gives the famous delineation of routes to avoid them—namely, the zigzag theory, the theory of limitation of size, and the no-classes theory.

Poincaré introduces the vicious circle principle in a discussion of Richard's paradox, the paradox of the set of definable decimals. This set is denumerable, and from an enumeration of it a Cantor-diagonal decimal can be formed. That decimal is not among the enumerated ones but is, it seems, definable and hence is in the set. Poincaré draws the vicious circle principle from Richard's own solution of the paradox (Richard 1905). The set E of definable decimals, Poincaré says, must be construed as composed just of the decimals that can be defined "without introducing the notion of the set E itself. Failing that, the definition of E would contain a vicious circle" (1906, 307). He then proposes as a general principle that only those definitions that do not contain a vicious circle should be taken to determine sets.

An analysis of the Richard paradox as resting on some type of circularity is extremely plausible, since the paradox exploits a notion of definability that allows definitions containing reference to the notion of definability itself. Most contemporary solutions of the paradox recognize this and proceed by stratifying notions of definability. On such views, the circle that would otherwise exist is peculiar to semantic or intensional notions like "definability." Poincaré's analysis, however, has a different cast. He focuses his challenge not on the notion of definability but on the set of decimals specified by invoking the notion.[7] This leads Poincaré to apply the analysis more widely; he descries the same circularity in the set-theoretic paradoxes as well.

Thus Poincaré immediately goes on to claim that the vicious circle principle solves the Burali-Forti paradox of the order type of the set of ordinals—a paradox that Poincaré, like Russell but unlike Burali-Forti himself, recognized from the first to be a true paradox. He says, "One introduces there the set E of all ordinal numbers; that means all ordinal numbers that one can define without introducing the notion of the set *E* itself" (1906, 307). Thus paradox is blocked. The extension of the vicious circle analysis to this paradox is a bold step, for there is a real distinction between this case and that of the Richard paradox. An item qualifies for membership in the Richard set only if there is a sequence of words that defines it. Since the truth or falsity of the latter condition may depend on the composition of the Richard set, a circle is apparent. In contrast, an item qualifies for membership in the Burali-Forti set if and only if it is an ordinal, and this condition in no way depends on the Burali-Forti set. Nonetheless, Poincaré alleges a vicious circle: one that arises if the condition is applied universally. Therefore, he takes the vicious circle principle to restrict the range of candidates for membership in the set, namely, to items that can be given independently of the set.

Poincaré's treatment of the Burali-Forti paradox may have been influenced by a lesson Russell drew from it: "There are some properties such that, given any class of terms all having such a property, we can always define a new term also having the property in question. Hence we can never collect all the terms having the said property into a whole" (Russell 1905, 144). For Russell, this lesson does not constitute an analysis or solution of the paradox; it is only an indication of what is to be analyzed. Poincaré can be seen as taking it in a more definitive manner: a specification of a set must be interpreted as simply not applicable to any "new" term producible from that set.

So far, then, Poincaré is using the vicious circle principle to bar from membership in a set anything that in some sense presupposes that set. In this form, the principle can also be used to block the Cantor and Russell paradoxes. This is not yet, however, the full strength of the principle that Poincaré urges. (Indeed, under a suitable interpretation of "presupposes," both the simple theory of types and the so-called iterative conception of set wind up abiding by a restriction of this sort.)

The full force of Poincaré's principle first emerges in his use of it to criticize the logicist definition of number. A finite number is defined as a number that belongs to every inductive set, that is, every set that contains 0 and contains $n + 1$ whenever it contains n. Poincaré claims that to avoid a vicious circle, the inductive sets invoked in the definition cannot include those specified by reference to the set of finite numbers. If such a restriction is adopted, then mathematical induction cannot be deduced for such sets; many basic laws of arithmetic become unprovable, and the logicist claim is defeated. Now, the vicious circle principle, so construed, restricts not just the candidates for membership in the set being defined, but also the range of the quantifiers in the definition. This yields the notion of predicativity in its modern sense: quantifiers occurring in specifications of sets must be construed as not including in their range of variation the set that is being defined, or anything defined by reference to that set.[8]

Poincaré makes one further application of the vicious circle principle in (1906). Earlier (1905, 25-29), he had pointed out that the then extant proof of the Schröder-Bernstein theorem invoked mathematical induction by utilizing an inductive construction of a sequence of sets. This belied the claim that cardinal number theory can be developed without special mention of the finite numbers. In 1906 Zermelo sends Poincaré a different proof: instead of an inductive construction, the intersection *U* of all sets with a certain property *P* is formed; the theorem is obtained by showing that a particular set defined from U has property *P*, and thus is among the sets intersected. Poincaré invokes the vicious circle principle to block the argument, for it rules out from the sets intersected any set in whose definition the intersection *U* itself figures. Here again, the principle is taken to restrict the range of quantifiers in the definition of a set.

The restriction Poincaré urges can be put, in the formal mode, thus: the quantifiers in a specification of a set cannot legitimately be instantiated by names that contain reference to that set. In (1906) he gives a general reason for his restriction, namely, that a purely logical proof must start from identities and definitions, and proceed in such a way that when definienda are replaced by definientia, one obtains an "immense tautology" (1906, 316). Impredicative definitions block such a replacement; hence they cannot be allowed in logic. In the polemical setting of 1906, this remark cuts no ice: logicists and set theorists did not think that their proofs must reduce to tautologies. On the technical level, however, it is a tremendously prescient remark. Although adherence to a predicativity constraint does not ensure reducibility to tautology in a literal sense, in many settings it does yield conservative extension results, which fail once impredicative definitions are allowed.

Poincaré's proposal met much criticism. In particular, Russell and Zermelo both found the same flaw in it: the very formulation of the vicious circle principle, they charge, violates the principle.

It is precisely the form of definition said to be predicative that con-

> tains something circular; for unless we already have the notion, we cannot know at all what objects might at some time be determined by it and would therefore have to be excluded. (Zermelo 1908, 191)

That is, if the specification "class of ϕs" is taken as meaning "class of things that have ϕ but do not presuppose the class of ϕs," then the specification involves the notion of the class of ϕs, and so a vicious circle has been introduced. Russell repeats the point often, and likens it to an attempt to avoid insulting a person with a long nose by remarking "When I speak of noses, I except such as are inordinately long" (Russell 1908, 155).

Now, as was pointed out above, for Poincaré the vicious circle principle amounts to a restriction on legitimate instantiations for the quantified variables in a specification of a set. In their criticism, Zermelo and Russell are assuming that such a restriction must arise from explicit limitations on the quantifiers, limitations that are part of the content of the specification. It is those explicit limitations that introduce the circle. Poincaré never answered this objection. For him, one presumes, a specification of a set stands as is, without explicit reference to the cases that are ruled out. The effect of the vicious circle principle comes in the *application* of the specification. The restriction on instantiations need not be grounded in an explicit anterior fixing of the quantifiers' ranges.

The difference between Poincaré and the logicians here is fundamental. For Russell and Zermelo, inference is a matter of universal and subject-neutral logical laws. An inference seemingly licensed by such a general law can be blocked only by so construing the content of the statement concerned as to make the law inapplicable. In this sense, logic is the arbiter of content. But, as we may surmise from section I, Poincaré does not share this conception of universal logical laws; hence he does not recognize any need for a notion of fixed content to ground restrictions on inference.

Such a denial is unintelligible from Russell's and Zermelo's point of view and is unsatisfactory from most contemporary ones. Contemporary predicative systems arrange their entities in a hierarchy; a definition with quantifiers ranging over entities of level at most n in the hierarchy automatically defines something at level $n + 1$. As a result, the defined entity need not be invoked to restrict the ranges of the quantifiers in its definition.

Zermelo presses another criticism as well, which has become a standard "classical" argument against the predicativity constraint.

> After all, an object is not created through...a "determination"; rather, every object can be determined in a wide variety of ways.... A definition may very well rely upon notions that are equivalent to [i.e., have the same extension as] the one to be defined. (1908, 191).

Zermelo thus totally rejects the vicious circle principle.

Russell, in contrast, adopts the vicious circle analysis and frames his logical system accordingly. The result is the ramified theory of types. Russell's disagreement with Zermelo can be focused on Zermelo's second objection. That objection requires a principle of extensionality about the entities under consideration, and here Russell parts company from the set theorists (and from Frege as well). Russell's logical theory concerns not extensional entities like sets, but intensional entities, namely, properties and propositions. Zermelo's point about equivalent notions is thereby rendered irrelevant; different definitions do determine different entities. Moreover, since the ranges of quantifiers in a definition are essential to the content of the defined proposition or property, support can be obtained for a hierarchy of propositions and properties that abides by the predicativity constraint. Vicious circles are thereby avoided, but not by ad hoc fiats; rather, the restrictions "result naturally and inevitably from our positive doctrines" (Russell 1908, 155). Thus, although Russell accepts Poincaré's proposal, philosophically his system is quite differently based.[9] Indeed, Poincaré is unsympathetic to Russell's theory (see 1909b, §§3-4); he continues to be hostile to formal systems, and to any call for more fundamental laws that would issue in the predicativity constraint.

Zermelo presses one further objection to Poincaré. He notes that many standard mathematical proofs involve impredicative reasoning, citing in particular the "fundamental theorem of algebra" (every algebraic equation has a root in the complex numbers), and, more generally, the panoply of theorems on maxima and minima (Zermelo 1908, 191). Thus the predicativity constraint would require the rejection of reasoning and results universally accepted by mathematicians. This criticism has particular force against Poincaré, who explicitly denies that foundational studies could affect mathematics: "[Mathematics] will pursue step by step its accustomed conquests, which are definitive and which it will never have to abandon" (1906, 307).

Poincaré is quick to reply (1909a, 199). He agrees that the standard proof of the fundamental theorem of algebra uses impredicative reasoning, and he sketches a maneuver that avoids it. (The maneuver relies on the replacement of a set of real numbers with a certain greatest lower bound by a sequence of rational numbers with the same bound.) Indeed, the same sort of maneuver is later used by Weyl (1918) to show that all the basic theorems of the theory of continuous functions, including those on maxima and minima, are predicatively provable. However, Poincaré ends his reply with a curious paragraph.

> More generally, if we envisage a set E of positive real numbers. . . one can prove that this set possesses a lower limit e; this lower limit is defined *after* the set E; and there is not petitio principii since e is not in general part of E. In certain particular cases, it can happen that e is part of E. In these cases, there is no more a petitio principii, since e is not part of *E in virtue of its definition*, but as a result of a proof posterior to both the definition of E and that of e. (1909a, 199)

The paragraph is odd in that it seems to have nothing to do with predicativity. In it, Poincaré notes the distinction between the greatest lower bound ("lower limit") of a set of reals and a minimum attained by a member of the set; and he warns that the assertion that a greatest lower bound is a minimum requires a further proof. But the impredicative reasoning to which Poincaré objects does not involve any claim that an object, defined by reference to a certain set, is a member of the set purely in virtue of its definition; a further proof is, in fact, supplied.

Zermelo is quick to exploit this oddity. In a reply, he shows that Poincaré's remark, altered by replacing the words "real numbers" and "lower limit" by "sets" and "intersection," would vindicate his logicist-style definition of finite number. He concludes, "I am able to evidence the legitimacy of my proof by thus appealing to the authority of M. Poincaré himself" (Zermelo 1909, 193).

The oddity of the passage can be dispelled, however, by reflection on what Poincaré means by "a proof posterior to both the definition of E and that e." The oddity arises if "posterior" is read as meaning, simply, "subsequent." Yet Poincaré probably intends "posterior" to be understood in light of the vicious circle principle, in which case a stronger stricture emerges. For a proof that is posterior in such a sense cannot contain certain instantiations; in particular, no quantifier in the specification of E can be instantiated by a term that makes reference to e. The original proof of the fundamental theorem does not abide by this stricture. Moreover, when so understood, the passage cannot be used to legitimate Zermelo's definition of finite number. In short, the passage should be read not as expounding the vicious circle principle, but as relying on it. As a result, Poincaré's charge that illicit conclusions are being drawn about an object "in virtue of its definition" is subtler than it appears. Indeed, in the passage Poincaré is using the vicious circle principle to push together the obvious fallacy of assuming a greatest lower bound to be a minimum and the controversial step of applying a definition impredicatively.[10] At bottom, he is urging, the same fallacy lurks in both.

This is further evidenced in Poincaré's characterization, in his last paper on foundations, of impredicative definitions as being composed of two *postulates*: "The object X to be defined has such-and-such relation to all individuals of genre G; X is of genre G" (1912, 154). Now if the genre G is taken to be a given set of reals and the relation is taken to be that of greatest lower bound, then the two postulates yield the conflation of greatest lower bound and minimum, an illegitimate conflation, to be sure, but not because of any impredicativity. However, if the genre represents not the set of reals but the range of quantifiers used in specifying the set, then the two postulates exhibit the form of an impredicative definition.

Among Poincaré's other late writings on foundations, (1909b) is particularly revealing. He starts with an extended discussion of the Richard paradox, taking it to show that whether a given sentence defines a real number, and if so which one, may depend on whether the Richard set has already been defined. Hence "definability" means something different before the Richard set is defined and after. As he puts it, the classification of real numbers into definable or not is "mutable." He then infers generally that impredicative definitions are fallacious, because after such a definition is enunciated, the entity taken to be defined by it may cause a change in meaning in that very definition. Impredicative definition thus engenders fallacies of equivocation. Thus, as before, Poincaré analyzes the Richard paradox as arising not from anything peculiar in the notion of definability, but from difficulties caused by an object introduced by a definition; the emphasis here, though, is on the general lesson learned from the paradox.

For Poincaré, the Richard paradox is not just one antinomy among many to be solved, as Russell viewed it. Rather, its solution exposes the mutability in mathematical definitions generally; hence the solution must apply to all of mathematics. Underlying this step is the idea that all entities that mathematics legitimately treats must be definable. Indeed, Poincaré urges, "the only objects about which it is permissible to reason are those which can be defined in a finite number of words (1909b, 464). Yet since, for Poincaré, definability has no formal analysis, no prior limit can be put on the objects that may, one day, be subject to our reasoning. Poincaré's rejection of the notion of a fixed range of quantifiers is explicit here. A universal theorem does not relate to all objects, imaginable or not, in such a range; rather, it asserts only that each particular case of the theorem—each case defined in a finite number of words that will be considered by the mathematicians or by succeeding generations of mathematicians—can be verified (1909b, 480).

This construal of generality, and the insistence on definability, reflect Poincaré's repeated denial of the existence of the actual infinite. "Every proposition about the infinite should be the translation, the abbreviated statement, of propositions about the finite" (1909b, 482). Poincaré is not here calling for some kind of finitist reduction, however. As a result of the lack of predetermined limits to "definability," the array of propositions about the finite that are abbreviated by a given proposition about the infinite is open-ended rather than fixed.

From all this, the view of mathematics that moves Poincaré may be surmised. Mathematics is a matter of our thinking and verbalizing mathematically. There is nothing beyond our words and thoughts that can anchor or ground the discipline. The idea that our classifications are what we reason about, is a direct rejection of the tenet, central to logicism (but not limited to it), that logic applies to a realm of fixed content. Once that tenet is rejected, logic cannot be thought of as the basic rationality-framing subject that Frege and Russell took it to be; indeed, logic loses all claim to priority. It is no wonder, then, that Poincaré is blind to the advances of the new logic and resists the purely logical sense of presupposition. Similarly, Poincaré's view of the vicious circle principle as governing applications of definitions, rather than as constraining the content of those definitions, becomes explicable. The immutability of our classifications is to be secured by our subsequent behavior: we must not do anything that would cause mutation. Immutability is *not* secured by reference to a logical structure present from the start.

The priority of the finite in Poincaré also stems from his psychologism. The finitude of our minds is, presumably, the basis for his insistence that everything about which we can reason must be definable in a finite number of words, where definability is left unanalyzed, its extent a matter of the conduct of future generations of mathematicians. Moreover, Poincaré pervasively assumes intuitive knowledge of the realm of the finite and the principles governing it. From this ground, the question of the justification of such principles applied to the infinite can first be launched. For logicists, in contrast, the distinction between finite and infinite is first made out after logic is in place; there might be questions of whether infinite structures exist, but there can be no question of whether logic is also "true of" the infinite. Poincaré does, in the end, recognize this contrast. Characteristically, he supports his own stance on priorities by reference to the order in which students can most effectively be instructed in mathematics. This is, as he puts it, "how the human spirit naturally proceeds" (1909b, 482). He concludes,

> M. Russell will doubtless tell me that these are not matters of psychology, but of logic and epistemology. I shall be driven to respond that there is not logic and epistemology independent of psychology. This profession of faith will probably close the discussion, since it will show an irremediable divergence of views.

Poincaré is here being far more perspicacious than many subsequent writers on the foundations of mathematics. The divergent views are not divergent answers to the same questions; they are deeply opposed construals of those questions. What counts as giving "foundations" for mathematics is entirely different for Poincaré than for Russell, Frege, or Zermelo. This difference is not readily adjudicable through the arguments found in the explicit controversies of the literature.

That is not to say that the differences are not susceptible to argument at all; the arguments, though, may have to operate at a far more general philosophical level. It is puzzling that Poincaré's apparently naturalistic view issues in prohibitions of forms of mathematical reasoning that are widely accepted and that have led, by themselves, to no mathematical crises. Thus, it can be suspected that Poincaré's strictures ultimately rest on more than straightforward psychological grounds. In that case, a foothold will be provided for Frege's powerful arguments against psychologism; if it is not to succumb to these arguments, Poincaré's view may well have to be elaborated into a full-blown metaphysical psychologism such as Brouwer's.

For all that, the notion of predicativity is undeniably illuminating and important. One can only admire Poincaré's genius in arriving at a notion that would prove to be technically so fecund. There remain, however, many questions as to its philosophical bases and the proper place of predicativity considerations in the philosophy of logic and mathematics; these questions require, and deserve, much further investigation.[11]

Notes

1. This marks a distinction between Poincaré's writings on geometry and those on foundations. (I use the latter term throughout this paper to mean foundations of analysis and number theory.)

2. Poincaré's psychological rendering of Kantian terms is explicit in an earlier article, where he says that mathematical induction is "an affirmation of a power of the mind" and that "the mind has a direct intuition of this power" (Poincaré 1894, 382).

3. For more on this theme, see Goldfarb (1982). In that article, I also note that the question of whether mathematics is analytic or synthetic in Kant's sense is not an issue for the logicists, since the new logic renders the distinction as Kant drew it philosophically insignificant. (Frege in [1884] redefines "analytic"; Russell in [1902] baldly asserts that logic is synthetic in Kant's sense.) Poincaré is unaware of this. He represents the logicists as claiming that mathematics is analytic in Kant's sense, and he often speaks of his own aim, against the logicists, as being that of showing this claim false.

4. In an article on the teaching of mathematics (1899), Poincaré stresses the importance of imparting intuition, in just this sense.

5. A similar point is made by Russell in explicit reply to Poincaré (Russell 1910, 252).

6. I have in mind here particularly the approach urged by Kitcher (1984). I do not mean to include Quine's very different sort of naturalism.

7. On this point Poincaré was quickly criticized by Giuseppe Peano (1906). The distinction Peano suggests, between semantic paradoxes and set-theoretic paradoxes, was given canonical form in Ramsey (1925).

8. A terminological note: Originally Russell used "predicative" for properties that define classes, i.e., those ϕ such that $[\chi \mid \phi\chi]$ exists. Poincaré proposes in (1906) that properties are predicative only if they contain no vicious circles. In later writings, Poincaré simply *defines* "predicative" to mean containing no vicious circles. Russell follows this later usage, which is the one that has come down to us.

9. The justifications for Russell's ramified theory of types are discussed in detail in Goldfarb (forthcoming).

10. The distinction between greatest lower bound and minimum became a well-known issue in late-nineteenth-century analysis, through critical scrutiny of Dirichlet's fallacious argument for the Dirichlet Principle, which relied on conflation of the two. The problem of finding a correct proof for the principle received much attention, and Poincaré himself contributed to its solution. Monna (1975) gives an excellent historical account of this issue.

11. I am grateful to Burton Dreben, Peter Hylton, and Thomas Ricketts for helpful suggestions and comments.

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