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Kronecker's Place in History

At a conference on the history and philosophy of mathematics, it seems especially appropriate to talk about Kronecker's place in history. My objective is to show that the prevalence of one philosophical viewpoint in contemporary mathematics has had a distorting influence on the way that that place—and with it several major issues in the history and philosophy of mathematics—have been viewed in our time.

Consider the story of Georg Cantor (1845-1918) and Leopold Kronecker (1823-91) as it is told by present-day writers:

Cantor was the founder of set theory—he even gave it is name. As such, he was the first to speak the language of modern mathematics and the first to deal rigorously with the transfinite entities that are the central notions of most of mathematics. His creation took great courage and had to overcome the prejudices of many generations of mathematicians who had regarded the notion of a completed infinite as a symptom of unrigorous thinking at best and, more likely, of complete nonsense. Chief among Cantor's opponents was Kronecker, "the uncrowned king of the German mathematical world" of those days. Kronecker was small in stature, but large in power. He was a wealthy man who could indulge in mathematics as a hobby, and he was the sort of difficult and prejudiced man who could make life unpleasant for those around him. He held extreme views on the foundations of mathematics, insisting that all of mathematics be based on the natural numbers, a limitation that would obviously nullify not only a large part of the mathematics of his own time but also almost all that has been done since. His combativeness and his fanaticism poisoned his relations with his great contemporary in Berlin, Karl Weierstrass, and caused Weierstrass great unhappiness in his last years. But the primary victim of Kronecker's attacks was Cantor, a former student in Berlin, who, from his outpost in Halle, was trying to win that recognition for his theories. Cantor suffered a severe depression in 1884, verging on a nervous breakdown. One cannot determine a precise cause for such a breakdown, but he was under the dual strain of trying to solve the problem of the continuum hypothesis and of trying to withstand the opposition of Kronecker to his entire work. In the end, Cantor prevailed, although the effort nearly cost him his sanity. Kronecker died in 1891 and the torch of mathematics passed to a new generation, headed by David Hilbert, which recognized the true value of Cantor's theory and brought about the dawning of the modern age of mathematics, based on Cantor's firm foundation.

I'm sure you recognize the characters and the story. I would like you to consider, however, another interpretation of that controversy of a hundred years ago. Even Kronecker's enemies admit he was a superb mathematician, and he has many impressive friends. In the twentieth century, such outstanding mathematicians as Erich Hecke, Carl Ludwig Siegel, and Andre Weil have studied Kronecker's works intensively and have built on them in their own work. As far as Kronecker's personality is concerned, I have been able to find little evidence of his alleged hostility and aggressiveness. Although he was clearly a man who had strong opinions, there is every indication that his manners and his morals were thoroughly gentlemanly. I would rather accept the genial picture of him given by Frobenius in his *Gedaechtnisrede* than the ones given by E. T. Bell and Constance Reid. In the philosophy of mathematics, Kronecker's ideas have undeniably been part of a minority view, but it is a view that has continued to have its proponents, including Henri Poincaré, L. E. J. Brouwer, Hermann Weyl, and Errett Bishop. To have a name for the views that, roughly speaking, these men had in common I will use Brouwer's term "intuitionism," meaning the notion that mathematics must ultimately be based on the irreducible intuition of *counting*, of the natural numbers as a potentially infinite sequence.

As I understand it, the majority view rejects the intuitionistic views primarily on pragmatic grounds. It is thought that intuitionism would disallow too much of modern mathematics and would make that which remained too fussy and complicated. Cantor said, "The essence of mathematics lies in its freedom," and intuitionism is seen as a straitjacket.[1] But it is hard to imagine that anyone, even Cantor, believed that the essence of mathematics lay in its freedom. The essence of mathematics lies, rather, in its *truth*, by which I mean its power to convince us of its correctness.

If the use of transfinite logic diminishes its power to convince, and if, as the intuitionists maintain, virtually all of classical mathematics can be given an intuitionist foundation, then the use of transfinite logic in mathematics is both unwarranted and undesirable.

Kronecker believed that "someday people will succeed in 'arithmetizing' all of mathematics, that is, in founding it on the single foundation of the number-concept in its narrowest sense."[2] He said that this was his goal and that if he did not succeed then surely others who came after him would. This view inevitably put him in opposition not only to Cantor, whose "sets" could only appear to Kronecker as figments, but also to his colleague Weierstrass, who likewise had undertaken the "arithmetization of analysis." We have seen in our own time in the example of Errett Bishop how an outstanding mathematician can arrive at views of mathematics that place him, like it or not, in complete opposition to his colleagues, however much he might respect them. Quite simply, there is no way for a Bishop or a Kronecker to say politely, "I'm sorry, but your proofs do not prove what you think they do, and for you to convince me that your mathematics is worth my time you have to rethink in a fundamental way all you have done and show me that you have understood and can answer my objections. Meanwhile, I will pursue what seems to me a much more worthwhile project, that of establishing all of mathematics on a firm foundation, something I am confident I can do."

Kronecker, unlike Bishop, published nothing on his constructive program. He was engrossed in the last years of his life with his work on the theory of algebraic numbers and algebraic functions, and his intention to give an intuitionist development of the foundations of mathematics was never, as far as we know, seriously begun. Nonetheless, his thinking on foundational questions is evident in much of his work on mathematics proper, and, in my opinion, it is worthy of the highest respect. Therefore, I think it would give a truer picture of what happened between him and Cantor if the story were told in something like the following way.

The Weierstrass school, of which Cantor was a member, found that, in developing the theory of functions and in following up Cauchy's efforts to put the calculus on a firm foundation, they needed to use arguments dealing with infinity such as those that had been introduced by Bolzano earlier in the nineteenth century. Cantor went even further, completely disregarding the taboo against competed infinities and dealing with infinite collections that he called *Menge* or sets. Kronecker believed that none of these ways of dealing with infinity was acceptable or indeed necessary. He had a grand conception that all of mathematics would be based on the intuition of the natural numbers, but he never carried this conception to fruition. Naturally, Kronecker's denial of the validity of Weierstrass's arguments deeply wounded Weierstrass, and their relations went from great friendship in the early years to almost total alienation in Kronecker's last years, even though Kronecker insisted, perhaps insensitively, that a disagreement over mathematical questions should not affect their personal relations. Cantor's reaction to Kronecker's opposition was even stronger, in the first place because he was in a much weaker position vis-à-vis Kronecker because of his youth and his position at a provincial university, and in the second place because Cantor's personality contained a strain of paranoia that deeply disturbed at one time or another his relations with many contemporaries other than Kronecker, including such ostensible allies as Weierstrass, H. A. Schwarz, and G. Mittag-Leffler. Still, Cantor's ideas were taken up by Dedekind (who had in fact anticipated many of them) and Hilbert, and they became the basis of a new mode of mathematics that proved very fruitful and has dominated mathematics ever since, with only an occasional Brouwer or Weyl to oppose it.

I believe that the pendulum is beginning to swing back Kronecker's way, not least because of the appearance of computers on the scene, which has fostered a great upsurge of algorithmic thinking. Mathematicians are more and more interested in making computational sense of their abstractions. If I am right in thinking that Kronecker has been undervalued and caricatured by historians because they have been following the lead of the philosophers of mathematics and of mathematicians themselves, Kronecker's place in history may be about to improve considerably.

I will do what I can to bring this about. Surely one of the principal reasons Kronecker is so little studied today (except, it seems to me, by the best mathematicians) is that his works are so very difficult to read. Jordan very aptly called them in 1870 "l'envie et le désespoir des géometres."[3] This shows that the difficulty we experience today in reading Kronecker's works is not merely the difficulty of reading an old text written in an outdated terminology. Kronecker's difficult style was difficult for his contemporaries, too. I have been working at it for many years, especially his Kummer Festschrift entitled "Grundzuege einer arithmetischen Theorie der algebraischen Groessen."[4] I feel my efforts have been rewarded and that I will soon be able to publish papers that will convey something of what Kronecker was doing and that will, I hope, awaken the interest of others in studying Kronecker.

Let me conclude by giving a few indications of what it is that I find so valuable in Kronecker's works and how it relates to the history and philosophy of mathematics. It is said of Kronecker, as it is of Brouwer, that he succeeded in his mathematics by ignoring his intuitionist principles, but I think this allegation is completely untrue in both cases. Kronecker's principles permeate his mathematics—the problems he studies, the goals he sets for himself, the way he structures his theories. One of the parts of his Kummer Festschrift that I have studied most is his theory of what he calls "divisors," which is a first cousin of Dedekind's theory of "ideals." Two ways in which Kronecker's theory differs from Dedekind's show the difference in philosophy between the two men.

First, Dedekind regarded as the principal task of the theory the definition of "ideals" in such a way that the theorem on unique factorization into primes, which is false for algebraic numbers, becomes true for ideals. Kronecker took an altogether different view. He noted that the notion of "prime" was *relative to the field of numbers under consideration* and that if the field was extended things that had been prime might no longer be. For this reason, the theorem on unique factorization into primes properly belongs to a later part of the theory, after the basic concepts have been defined in a way that is independent of the field under consideration. Those of you who have studied algebraic number theory will undoubtedly have studied it using Dedekind's ideals and will remember that when you go from one field to another you have to manipulate the ideals—intersecting them with the lower field if the new field is smaller and taking the ideal in the larger field they generate if the new field is larger. In Kronecker's theory there is none of this. Divisors are defined and handled in such a way that nothing changes if the field is extended or if it is contracted.

Second, Kronecker defined his divisors, in essence, by telling how to *compute* with them. In Dedekind's terms, this amounts to giving an algorithm for determining, given a set of generators of an ideal, whether a given element of the field is or is not in the ideal. Nothing of the kind enters in Dedekind's theory because for Dedekind the definition was complete and satisfactory once the ideal was defined as a set, albeit an infinite set with no criterion for membership. For Kronecker, such a definition was by no means unthinkable, but it was certainly undesirable, and you can be sure he would settle for it only as a last resort. These are just two ways in which Kronecker's theory was affected—and improved—by his philosophy.

I have said above that, as far as we know, Kronecker never made a serious beginning in writing out his proposals for an intuitionist foundation of mathematics. This is not to say, however, that his hope that either he or a successor of his would do this was mere talk. I believe he had a unified view of all branches of mathematics and had, in many instances, fully thought-out ideas on how to base them on intuitionist principles. Had he devoted more time to this task and, I must add, had he been a better expositor, intuitionist ideas might not have had to endure a century of ostracism. It is my hope that this ostracism is now drawing to a close and that if not I then one of my successors will open the eyes of the mathematical world to the wealth of ideas hidden between the covers of Kronecker's collected works.

Notes

1. G. Cantor, *Gesammelte Abhandlugen* (Berlin: Lokay, 1932), p. 182.
2. L. Kronecker, *Werke*, vol. 3 (Leipzig: Teubner, 1899), p. 253.
3. C. Jordan, Introduction to "Traité des substitutions et des équations algébriques," Paris, 1870.
4. Kronecker's *Werke* was published in five volumes by Teubner, Leipzig, between 1895 and 1930. This paper is in vol. 2.