19. See note 15.
20. I am indebted to Professor A. Taub for this point.
21. H. Müller zum Hagen, P. Yodzis, H.-J. Seifert, "On the Occurrence of Naked Singularities in General Relativity," *Communications in Mathematical Physics* 34 (1973): 135–148; 37 (1974): 29–40.
22. By the "boundary of space-time", I mean the *b*-boundary (B. G. Schmidt, "A new Definition of Singular Points in General Relativity," *General Relativity and Gravitation* 1 (1971): 269–280). Data which can be expressed as scalars on the frame bundle sometimes have limiting values on this boundary (C. J. S. Clarke, "The Classification of Singularities," *General Relativity and Gravitation* 6 (1975): 35–40), and it might be hoped that this would generalise to genuinely singular situations.
23. See, for example, J. C. Graves, *The Conceptual Foundations of Contemporary Relativity Theory* (Cambridge, Mass.: M. I. T. Press, 1971).

JOHN EARMAN

Till the End of Time

1. Introduction

What could it mean to say that time has a beginning or an end? Is it possible that time has a beginning or an end? In this paper I shall not be concerned with these questions in their full generality, for I shall be concerned only with physically interesting possibilities. I cannot specify at the outset what is to count as a physically interesting possibility in the present context—substantial discussion will be needed to uncover the factors relevant to such a specification. In the sense in which I am using it, the notion of a physically interesting possibility is broader than that of a physical possibility; any actual physical possibility is a physically interesting possibility, but not conversely, although every physically interesting possibility must be intimately related to actual physical possibilities. It would seem good strategy to discuss physical possibilities first, before proceeding to the murkier concept of physically interesting possibilities. This would indeed be sound strategy, except for the fact that we do not know what counts as a physical possibility in the present context. Thus it is necessary to plunge right into murkier waters.

The particular approach that I shall explore is certainly not the only one, nor do I claim it is the best. However, it does have a virtue, albeit a negative one: it reveals that we are not now in a position to give meaningful answers to the questions posed above, and that in order to arrive at such a position it is necessary to settle a number of other questions first, some of which belong to mathematics, some to physics, and some to metaphysics.[1] Since the recognition of ignorance is often the first step toward wisdom, it is to be hoped that the way will be paved for more positive results.

2. Aristotle and Leibniz on the Beginning and End of Time

Initially, Aristotle's theory of time seems to allow for the possibility of a beginning or an end for time. According to Aristotle, time is the measure or numerable aspect of motion. Hence, if motion has a beginning or an end, time would have a beginning or an end. But Aristotle forecloses this possibility, for he says that time cannot have a beginning or an end, and that, therefore, motion must also be eternal. His argument here is that a first or a last instant of time is impossible since in conceiving of an instant of time we must conceive of it as being preceded and succeeded by other instants.[2]

Today this argument does not seem very compelling. We have learned to be suspicious of any argument that purports to prove the impossibility of Y by seeking to demonstrate the inconceivability of Y and by utilizing the premise that if Y is inconceivable, Y is impossible. I shall not stop to give a detailed analysis of Aristotle's argument, for my main concern is to examine the implications of modern science for the questions at issue, and from the point of view of modern science, Aristotle's theory of time has only a curiosity value. For from the modern point of view, time must be seen as the temporal aspect of the more fundamental entity, space-time, and modern science countenances space-time structures which, in a precise sense, do not harbor any physical change and whose temporal aspects are, in a precise sense, infinite in both past and future.[3]

However, Aristotle's views do raise some points that help to focus the issues under discussion. Aristotle seems to take the statement, "Time comes to an end," to mean, "There is a last instant for time." He does not seem to have conceived of the possibility that time could *come to an end* without coming to *an* end—that time could, so to speak, "run out" in the future direction without there being a last instant. Suppose, for example, that time can be represented by the metric space (I, d_e) where I is the open interval $(-\infty, +1)$ of the real line $\mathbb{R}$ and d_e is the usual Euclidean metric (it is understood that the positive direction of $\mathbb{R}$ corresponds to the future direction of time). In this case there is no last instant for time, but time is finite in the future in the sense that there is a finite upper bound on how far one can go in the future direction from any given point of time, i.e., for any $x \in I$, there is a finite N_x such that for any $y \in I$ where $y > x$, $d_e(x, y) < N_x$.[4] I suppose Aristotle might have responded that such a possibility is not a real one, and that to conceive of an instant of time, we must conceive of other instants which precede and succeed the first by as great an interval as we like. But such a response should be taken, I think, as indicating that its author has limited powers of conception.

Conversely, the existence of a future end point for time is not sufficient

by itself to guarantee that time comes to an end in the sense intended, at least not if we are willing to extend the concept of a metric space (X, d) so that d may be mapping from $X \times X$ to $[0, +\infty]$. For example, let X be the extended real line $\widetilde{\mathbb{R}}$ which is obtained from $\mathbb{R}$ by adjoining the points $\pm\infty$ to $\mathbb{R}$, defining $-\infty < x < +\infty$ for any $x \in \mathbb{R}$. Define $d(x, y) = d_e(x, y)$ and $d(\pm\infty, x) = d(x, \pm\infty) = +\infty$ for $x \in \mathbb{R}$; $d(+\infty, +\infty) = d(-\infty, -\infty) = 0$, and $d(+\infty, -\infty) = d(-\infty, +\infty) = +\infty$. Time as represented by (X, d) has first and last instants, but they are infinitely far in the past and future so that time never runs out in the past or future directions.

Thus in what I shall refer to as the Aristotelian conception of the beginning (end) of time, two elements are involved: (1) time is finite in the past (future) in some appropriate sense, and (2) there is a past (future) end point for time. I shall begin by investigating condition (1); only in the latter part of the paper shall we be in a position to deal with (2) in a meaningful way. It might be thought that if we have a model which satisfies condition (1), we can always make it into a model which illustrates the Aristotelian conception of the beginning or the end of time by adjoining end points. We shall see, however, that things are not so simple; the illusion of simplicity is fostered by two pernicious tendencies: first, the tendency to think of time as an autonomous entity rather than as an aspect of space-time, and second, the tendency to think of time as being represented by an interval of the real line. In these respects, the discussion of this section and, indeed, most of the discussion of these matters in the philosophical literature has been misleading.

I turn now to Leibniz's views. Leibniz believed that there are possible worlds which have a beginning or an end. But when combined with the following argument, his Principle of Sufficient Reason poses a problem for the actuality of any such world: "Time can be continued to infinity. For since a whole of time is similar to a part, it will be related to another whole of time as its part is to it. Thus it must always be understood as being capable of being continued into another greater time."[5] So for any possible world W which has, say, a beginning, there is another world W' which extends W to past infinity. What sufficient reason could God have for actualizing W rather than W'? And does not the Principle of Plenitude suggest that He would choose W' over W?

Leibniz addresses himself to these questions in a letter to Bourguet.[6] If nature is "always equally perfect, though in variable ways, it is more probable that it had no beginning" (because, presumably, W' would then be more perfect than W). On the other hand, if nature decreases in perfection as we go backward in time, and if the perfection decreases at such a rate that, within a finite period, we reach zero and negative perfection, then it is more probable that time has a beginning (since, presumably, W would then be more perfect than W'). Leibniz simply left the matter hanging since he was unwilling to commit himself on whether the perfection of nature changes at such a rate as to make a beginning for time probable, and even on whether it is changing at all.

As we shall see below, relativity theorists have had to confront issues similar to those with which Leibniz struggled. But we shall also see that certain relativistic space-time models circumvent the main problem.

3. The Meaning of Temporal Finiteness within a Relativistic Space-time Framework

If we take seriously the notion that time must be thought of as the temporal aspect of space-time, we are led to ask what sort of space-time structure would illustrate the notions of the beginning and the end of time. In what follows, I shall work with relativistic space-times, and this for two reasons. First, and most obviously, current evidence indicates that actual space-time is relativistic. The fact that the possibilities to be discussed below can be constructed within this framework makes for some initial confidence that they will lead to physically interesting possibilities. Secondly, and less obviously, certain of the possibilities to be discussed cannot be realized within the orthodox Newtonian framework.

Definition 1. A *relativistic space-time* $\mathcal{S}$ is a triple $\langle M, g, \nabla \rangle$ where M is a connected, four-dimensional differentiable manifold, g is a Lorentzian metric for M, and ∇ is the unique symmetric linear connection compatible with g.[7]

According to general relativity theory, which will be taken as our guide to physically interesting possibilities, space-time structure and the distribution of matter-energy are not independent. Thus we must consider cosmological models.

Definition 2. A *cosmological model* $\mathfrak{M}$ is a pair $\langle \mathcal{S}, \mathcal{E} \rangle$ where $\mathcal{S}$ is a space-time and $\mathcal{E}$ is an energy-momentum tensor.[8]

Conditions for a physically interesting example of how time can have a beginning or an end will be conditions on cosmological models. However,

I shall concentrate primarily on conditions which the space-time $\mathcal{S}$ must satisfy, for once we have settled on these conditions, it will prove easy to find a cosmological model which incorporates $\mathcal{S}$ and which satisfies all the conditions that one might plausibly impose on $\mathcal{S}$ and on $\mathcal{S}$ and $\mathcal{E}$ together.

It will be assumed that $\mathcal{S}$ is temporally orientable so that it can be assigned a consistent time sense or direction, and that one of the two possible time orientations has been singled out so that one can meaningfully speak of the future direction of time.[9] The restriction imposed by this requirement is not a strong one; for if $\mathcal{S}$ is not temporally orientable, there always exists a covering space-time that is. And in any case, the questions at issue do not even arise unless $\mathcal{S}$ possesses the assumed feature. We shall shortly see that other features as well need to be assumed for the same reason.

It is easy to display relativistic space-times that are finite in their temporal aspects. For example, start with Minkowski space-time $\mathcal{S}_{Min}$ and obtain the hypertorodial space-time $\mathcal{S}_{roll}$ by rolling up M_{Min} in the spatial sense and then rolling up the resulting space in the temporal sense.[10] The temporal aspect of $\mathcal{S}_{roll}$ is finite in a straightforward sense, but this sense of finiteness is not appropriate for our present concerns; for time in $\mathcal{S}_{roll}$ is not bounded in the past or future in any appropriate way, and, indeed, there is no past or future in the usual sense, since every point of M_{roll} lies to the past and future of itself and every other point.

The moral is that for $\mathcal{S}$ to illustrate how time can be finite in the past or future, something like time in the usual sense must be present to start with. Exactly which temporal features $\mathcal{S}$ must display is open to debate, but certainly $\mathcal{S}_{roll}$ does not qualify. Some of the features that might plausibly be required are given in the following definitions.

Definition 3. The (time-oriented) space-time $\mathcal{S}$ has *global time order* if and only if there do not exist any nontrivial, closed, future-directed, timelike curves.[11]

If $\mathcal{S}$ has a temporal order according to definition 3, then the relation $<$ on $M \times M$ where $x < y$ (read "x is chronologically earlier than y") is defined to hold just in case there is a differentiable map σ: $[0, 1] \rightarrow M$ such that $\sigma(0) = x$, $\sigma(1) = y$, and $\dot{\sigma}(\lambda)$ is future-pointing for all $\lambda \in [0, 1]$, is transitive and asymmetric. Hence M is partially ordered by $\leqslant$, where $x \leqslant y$ holds just in case $x < y$ or $x = y$.

Definition 4. $\mathcal{S}$ possesses a *global time function* if and only if there is a smooth map t: $M \to \mathbb{R}$ such that $t(x) < t(y)$ whenever $x < y$.

The existence of a time order does not necessarily imply the existence of a global time function. If $\mathcal{S}$ does have a global time function, we can project out a one-dimensional time T: an "instant" (a point of T) corresponds to a time slice of M, i.e., a set $S_\lambda \underset{\text{def}}{=} \{x\colon x \in M \text{ and } t(x) = \lambda,\ \lambda = \text{constant}\}$. The points of T are totally ordered by the relation of temporal precedence induced on T by the natural order of the slices S_λ.

Definition 5. Suppose now that $\mathcal{S}$ has a global time function t. Consider the field of future-pointing unit normals to the S_λ. If this field is sufficiently smooth, its integral curves will form a timelike congruence. t is *metric* if and only if each pair of the S_λ are a constant space-time distance apart as measured along the curves of the congruence.

If t is metric by definition 5 and T is the one-dimensional time associated with t, we can find a metric space (X, d), where X is an interval of $\mathbb{R}$, and a one-one correspondence f: $T \to X$ such that for a, $b \in T$, $d(f(a), f(b))$ is the space-time distance between the slices corresponding to a and b. However, this representation of T as a metric space can be misleading for our purposes. *If* there is a timelike curve of maximal proper length from some point $x \in M$ to S_λ, then this curve must be a geodesic normal to S_λ; and the curves of the congruence in definition 5 are geodesics which are normal to S_λ. But it is *not* always the case that there is a timelike curve of maximal length from some arbitrary $x \in M$ to S_λ. If the S_λ have the Cauchy property defined below, then such a maximal curve will always exist.

Definition 6. A time-slice S of the space-time $\mathcal{S}$ (i.e., a properly embedded spacelike submanifold without boundaries) is a *Cauchy surface* of $\mathcal{S}$ if and only if every future-directed timelike curve without end point intersects S once and only once.

Suppose that $\mathcal{S}$ possesses a global time function. No future-directed timelike curve can intersect any of the corresponding S_λ more than once. We ordinarily assume that if a process "goes on forever" (has no past or future end point), then there must be for each instant of T a stage of the process dated by that instant. This assumption will not be satisfied unless the S_λ have the Cauchy property.

Since the existence of a global time function and the existence of Cauchy surfaces are strong requirements, it would seem desirable to have

a definition of temporal finiteness that does not rely on these requirements. Towards this end, one might try

Definition 7. Let $\mathcal{S}$ be a time ordered space-time. A *half-curve* of $\mathcal{S}$ is a curve which has one end point and which has been extended as far as possible in some direction from that point. A future- (past-) directed timelike half curve is *complete in the future (past)* if and only if proper length as measured along the curve from its end point assumes arbitrarily large values; otherwise the curve is *incomplete in the future (past)*. $\mathcal{S}$ is *finite in the future (past)* or *future- (past-) bounded* if and only if every future- (past-) directed timelike half-curve is incomplete in the future (past).

Note that *any* relativistic space-time contains some incomplete timelike half-curves. However, in the more commonly known cases, all timelike half-curves of bounded acceleration are complete; in particular, all timelike half-geodesics are complete, geodesics being curves of zero acceleration.

Unfortunately, definition 7 is unsatisfactory. Start with Minkowski space-time $\mathcal{S}_{Min}$ (which for sake of illustration is taken to be two-dimensional), and remove all those points on or above some chosen null hypersurface. (Note: if $\mathcal{S} = \langle M, g, \nabla\rangle$ is a space-time, then the result of removing a closed set of points from M and restricting g and ∇ to the remainder is again a space-time.) The resulting space-time $\mathcal{S}_{null}$ (see Figure 1) will be finite in the future according to definition 7. But should $\mathcal{S}_{null}$ be counted as finite in the future in the sense being sought? It is

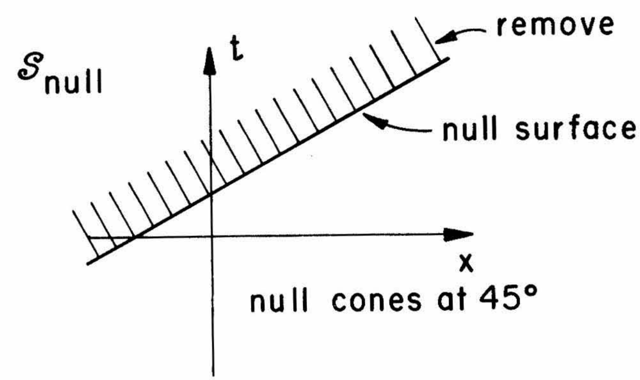

Figure 1

true that in $\mathcal{S}_{null}$ every possible observer will "run out of time." But time itself can be said to go on forever, since for every metric time function t, the metric of the associated one-dimensional time T is not future-bounded.

A somewhat more complicated example gives an even clearer indication of the shortcomings of definition 7. Start again with $\mathcal{S}_{Min}$ and remove spacelike hypersurfaces at regular intervals as indicated in Figure 2. The resulting space-time $\mathcal{S}_{strip}$ is finite in both the past and future according to definition 7; but clearly its temporal aspect is not finite in the intended sense.

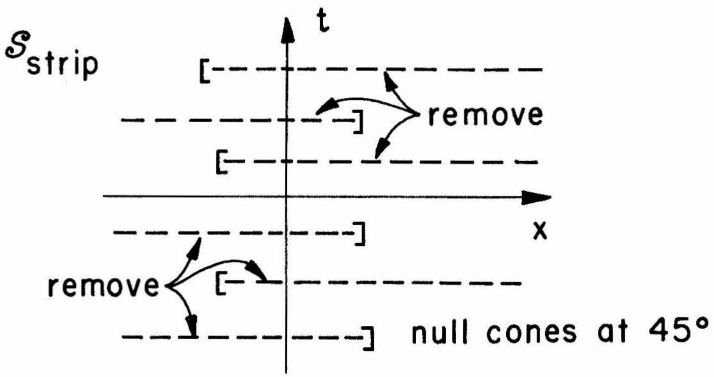

Figure 2

The moral is that the collection of individual timelike curves may give a misimpression of temporal finitude; global time functions may have to be consulted in order to gain an accurate impression. But the exact form the consultation should take is not easy to specify. Start yet again with $\mathcal{S}_{Min}$, and let (x, t) be a pseudo-Cartesian coordinate system for M_{Min}. Obtain the truncated space-time $\mathcal{S}_{trun}$ by deleting all those points whose temporal coordinates satisfy $t \geqslant 1$ (see Figure 3). As restricted to M_{trun}, t is a global metric time function for $\mathcal{S}_{trun}$, and the associated temporal metric is future-bounded. Consider, however, another pseudo-Cartesian coordinate system (x', t') for $\mathcal{S}_{Min}$. As restricted to M_{trun}, t' is also a metric time function for $\mathcal{S}_{trun}$, but the metric of the associated time is *not* in general future-bounded.

My own reaction is that, in the sense intended, $\mathcal{S}_{trun}$ should be regarded as finite in the future. The existence of a time function t' that

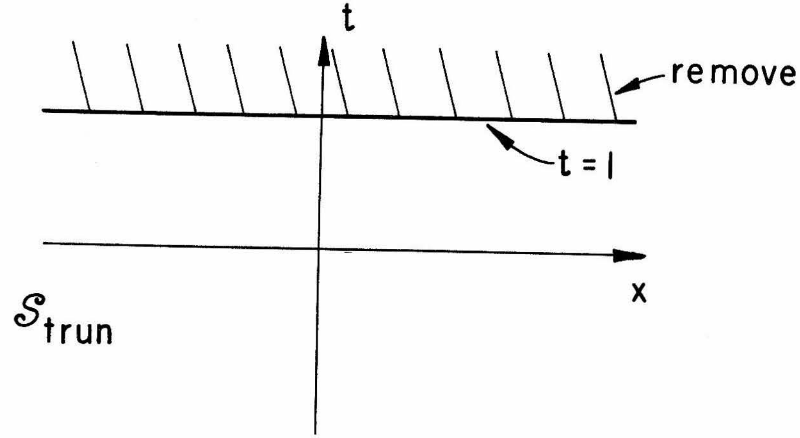

Figure 3

makes $\mathcal{S}_{trun}$ look infinite in the future does not detract from this conclusion since the original time function t has a preferred status with respect to the present problem: the t = constant time slices are Cauchy surfaces for $\mathcal{S}_{trun}$ whereas the t' = constant hypersurfaces are not. If my view is correct, then definition 7 is satisfactory when applied to space-times with a Cauchy surface. In such a case there exists a global time function all of whose time slices are Cauchy surfaces. And past (future) boundedness in the sense of definition 7 is equivalent to the past (future) boundedness of the metric associated with such a time function.

However, as the examples of $\mathcal{S}_{null}$ and $\mathcal{S}_{strip}$ reveal, definition 7 cannot be applied with confidence beyond this restricted class of cases. It is difficult to think of a general criterion that can be so applied. Suppose, for example, that $\mathcal{S}$ has a global metric time function but that no global metric time function for $\mathcal{S}$ has time slices with the Cauchy property. Shall we then decree a democracy among metric time functions and say that $\mathcal{S}$ is temporally finite if and only if all the temporal metrics of the one-dimensional times associated with the metric time functions are appropriately bounded? This would be too strong a requirement. If we remove one additional point from $\mathcal{S}_{trun}$ then there are no Cauchy surfaces in the resulting space-time, and, consequently, it would not be counted as temporally finite in the future according to the criterion under consideration. This is an unacceptable result.

At this juncture it must be strongly emphasized that the above examples are of more than mere academic interest; for physical space-time singularities, e.g., "curvature singularities," can play the role of the deleted regions. And such singularities have been shown to be a pervasive feature of the solutions to Einstein's field equations (see below).

In summary, unless a space-time satisfies some very strong causality conditions, it may not make sense to ask, "Is time finite or infinite?" Even when such conditions are satisfied, it is difficult to formulate an adequate general criterion for temporal finiteness, if indeed one exists. That the finite-infinite distinction becomes blurred in the relativistic context is an interesting point, and it deserves more attention in the philosophical literature. But having made it, I propose to avoid it in what follows by concentrating on cases in which we can agree that an appropriate type of finiteness obtains.

Before closing this section I want to comment on the meaning of the Aristotelian conceptions of the beginning and the end of time. It might be thought that the way to illustrate these conceptions within the relativistic context is to modify the picture of space-time so that a space-time can be a manifold with boundary, the boundary being the disjoint union of two spacelike three-manifolds, one of which can be interpreted as the "first instant" and the other as the "last instant." Prima facie, the existence of such boundaries is problematic. Their existence prompts one to ask such questions as "What explains the appearance of these boundaries?" and "What is beyond these boundaries?" Perhaps such questions can be dismissed as not legitimate. Or perhaps there are quite straightforward answers, e.g., the answer to the second question may be simply "Nothing!" What is problematic is how in general to attach a "future (past) boundary" to a space-time that is future- (past-) bounded. In some cases there is an obvious and natural procedure, but such cases turn out not to provide physically interesting illustrations of how time can have an end or a beginning, or so I shall argue below. For the moment, let us set aside the questions of boundaries and of how to implement condition (2) of the Aristotelian conceptions of the beginning and the end of time.

4. Truncated Space-times

Although the example of the truncated Minkowski space-time $\mathcal{S}_{trun}$ presented above is a trivial example, it poses a nontrivial problem for the philosophy of time: are there good reasons for believing that actual space-time cannot be truncated in the way $\mathcal{S}_{trun}$ is?

If one holds that there are no good reasons, then one would be wise to receive extreme unction as soon as possible, since time may run out any second now. Moreover, such a position implies a profound skepticism with respect to our knowledge of the past and future; it implies that we do not in fact know the great bulk of the things we ordinarily claim to know about the past and future. For if a person has no reason to believe p, then he does not know p; and if p is a presupposition of q and he does not know that p, then he does not know that q.[12] And, needless to say, most of our ordinary knowledge claims about the past and future presuppose that there is a past and future. Thus my claim to know that I existed five seconds ago (as measured along my world-line), or that barring certain catastrophes, the universe will still exist five seconds from now, presupposes respectively that actual space-time is not truncated in the past (future) in such a way that proper length along my world-line as measured backwards (forwards) in time from now never reaches five seconds; hence, if I have no good reason to believe that space-time is not truncated in the manner described, I do not know that I existed five seconds ago or that the universe will exist five seconds from now.

One's initial reaction to such skepticism is apt to be that it is too absurd to be taken seriously. Whether or not this reaction can be sustained remains to be seen.

On the other hand, if one holds that there are good reasons to reject $\mathcal{S}_{trun}$ and its like as a model for actual space-time, then these reasons must be supplied. We shall see that this order is less simple to fill than one might think at first glance. But before going into details, it will be helpful to contrast the above examples with another sort.

5. Another Example—Temporally Finite but Untruncated Space-time

Definition 8. A space-time $\mathcal{S}' = \langle M', g', \nabla' \rangle$ *properly extends* the space-time $\mathcal{S} = \langle M, g, \nabla \rangle$, if and only if M is isometrically embeddable as a proper subset of M. $\mathcal{S}$ is *inextendible* if and only if there is no $\mathcal{S}'$ that properly extends $\mathcal{S}$.

Can there be a space-time that is inextendible and yet is finite in the past or future? If so, the finiteness cannot be the result of truncation surgery.[13] The answer is affirmative; specific examples will be studied below (see section 8).

We can now see one of the virtues of the four-dimensional point of view, from which time is seen as an aspect of space-time. For from the one-dimensional point of view, there is no distinction between the temporal finiteness of $\mathcal{S}_{trun}$ and the inextendible space-times referred to above. And from the one-dimensional point of view, it is hard to fault Leibniz's assertion that time is always capable of being extended into a greater time. But not so in the four-dimensional view! Another virtue of the four-dimensional view will emerge below; namely, that general relativity provides a mechanism for realizing temporal finiteness in space-time.

6. Arguments against Truncation

There are physical considerations, both observational and theoretical, that can be brought to bear against certain kinds of geodesic incompleteness. For example, we never observe particles in inertial motion simply to pop out of existence, and conservation principles weigh against our ever observing this. But such considerations do not seem to operate against $\mathcal{S}_{trun}$; for here it is the whole matrix of existence that, so to speak, pops out.

(It is worth noting that in the present context, the problems attending the notion of space coming to an end are not wholly different from those attending the notion of time coming to an end. If we truncate Minkowski space-time in the spatial sense (e.g., delete all those points whose pseudo-Cartesian coordinates (x, t) satisfy $|x| \geq A$, where A is a positive constant), all the ancient problems arise about what happens when we poke a spear at the "edges" of space; but these problems arise here in a form not contemplated by the ancients, and, in fact, in the present context they are part and parcel of the problems involved in interpreting incomplete timelike curves. On the other hand, there are important differences between space and time. For example, adding the time slice consisting of all those points of M_{Min} whose temporal coordinates satisfy $t = 1$ as a future boundary for M_{trun} does not help to resolve any problems. But it is possible to deal with the timelike incompleteness arising from truncating in the spatial sense by adding a spatial boundary (in the above example, the set of all points satisfying $|x| = A$) and by treating this boundary as a rigid reflecting barrier; now, of course, the problems raised by the ancients do arise in the form they contemplated.)

Nor do the basic laws of relativistic physics help to rule out truncated

space-times. Thus, for example, it follows from the local nature of Einstein's field equations that if the cosmological model $\langle \mathcal{S}, \mathcal{E} \rangle$ is a solution, so is $\langle \mathcal{S}_{trun}, \mathcal{E}_{trun} \rangle$.

At this point, common sense cries out: "If my world line cannot be extended backwards in time from now more than five seconds, then most of my memory impressions are mistaken. But this is absurd." But exactly where does the absurdity lie? There is no logical contradiction, nor—apparently—any inconsistency with the basic laws of physics. Of course, it does seem that the likelihood that the universe switched itself on, so to speak, a few seconds ago in such a way that people are endowed with the memory impressions they in fact have, is very low. But this estimate of likelihood is surely not an estimate of probability based on either observed relative frequencies or on the implications of the laws of physics. The point becomes clearer when we contrast the present case with a case in which we assume that space-time is not truncated in the past, and ask for the probability of the spontaneous appearance of fossillike objects. Here laws and observed relative frequencies can be brought to bear, e.g., statistical mechanics tells us that the probability of a spontaneous fluctuation producing the 'fossils' is vanishingly small. No such argument is available in the case in which we ask for the likelihood of the universe being truncated in the past. The estimate of low likelihood in this case seems simply an expression of our unwillingness to accept a certain kind of explanation. But whether or not this unwillingness can be backed by philosophically respectable reasons is the question at issue.

Moreover, even if memory impressions, 'fossils' and the like did provide reasons for rejecting truncation in the past, there still remains the case of truncation in the future. Here I am inclined to think that the best we can hope for is a pragmatic vindication of the rejection of future-truncated space-times as models of the actual universe. Long and frustrating experience has revealed the fruitlessness of attempting to refute radical skepticism with respect to our knowledge of the external world. However, one can hope to show that such skepticism is inquiry-limiting and in this way justify proceeding as if the skepticism were false. And so I believe it is with future truncation.

Intuition would have it otherwise—the actual space-time world cannot be truncated in the future. Nor is this intuition raw and untutored. Indeed, it is not much of an exaggeration to say that this intuition is the starting point of recent research on gravitational collapse. Consider the original Schwarzschild vacuum solution to the Einstein field equations. (The reader need not be familiar with the details in order to follow the argument.) Focus on an observer whose world-line approaches the Schwarzschild radius. It is found that the observer would have to use up an infinite amount of Schwarzschild coordinate-time in order to reach this radius. But another calculation shows that only a finite amount of proper time elapses. Therefore, either the observer encounters some sort of violent agency at the Schwarzschild radius and is snuffed out of existence, or he simply runs out of time and ceases to exist, or he crosses over the Schwarzschild radius into a region of space-time not covered by the Schwarzschild coordinates. The first possibility can be ruled out, since it can be shown that the space-time metric is perfectly regular as one approaches the Schwarzschild radius (the so-called Schwarzschild singularity is a coordinate singularity, not a real singularity). The second possibility is ruled out on the grounds that "it would be unreasonable to suppose that the observer's experience could simply cease after some finite time, without his encountering some violent agency." [14] We are left then with the last alternative. If we admit this, we are driven to ask what happens to a massive object with exterior Schwarzschild field once it collapses within its Schwarzschild radius. This is the beginning of the story of gravitational collapse, a story that cannot be told here. What should be told here is that the argument just considered is essentially an application of Leibniz's Principle of Sufficient Reason. Had Leibniz lived to read it, he would no doubt have claimed that it provides an illustration of his doctrine that physics rests on metaphysical principles.

Without taking a stand on this doctrine, I am simply going to assume in what follows that certain kinds of truncated space-time do not provide physically interesting examples of how time can be finite. The final judgment about the justifiability of this assumption must await further investigation.

7. Conditions for Physically Interesting Finite Pasts and Futures

The above discussion leads to the following conditions (or more properly, to condition schema, since the details are left open) on a cosmological model $\langle \mathcal{S}, \mathcal{E} \rangle$ if it is to serve as a physically interesting example of how time can be finite in the past (future):

(1) $\mathcal{S}$ possesses certain properties, among which are the property of

having a global time order and, possibly, the properties of definitions 4–6.

(2) $\mathcal{S}$ is past- (future-) bounded.

(3) $\mathcal{E}$ has certain features, among which is, perhaps, the feature that there are no negative energy densities (see below).

(4) $\mathcal{S}$ and $\mathcal{E}$ together satisfy certain conditions, among which is, perhaps, that of being a solution of Einstein's field equations.

(5) $\langle \mathcal{S}, \mathcal{E} \rangle$ is maximal with respect to (1), (3), and (4), i.e., there is no $\langle \mathcal{S}', \mathcal{E}' \rangle$ which properly extends $\langle \mathcal{S}, \mathcal{E} \rangle$ and which satisfies (1), (3), and (4).

The condition most open to question is (5). Finiteness in the past or future might seem so objectionable that we would expect nature to go on building even if she could do so only by employing building blocks that do not satisfy (1), (3), and (4). This suggests that we strengthen (5) to

(5′) $\mathcal{S}$ is inextendible.

On the other hand, (5) and (5′) may prove to be too strong. If nature starts by building a cosmological model $\langle \mathcal{S}, \mathcal{E} \rangle$, we cannot require her to go on building until a model that is maximal with respect to properties P is reached if there is no such maximal space-time. And unless the set of properties P is chosen with care, it may not be provable that every space-time having properties P is contained in a space-time that is maximal with respect to P. If $\mathcal{S} = \langle M, g, \nabla \rangle$ has a Cauchy surface C and if we consider only those extensions $\mathcal{S}' = \langle M', g', \nabla' \rangle$ of $\mathcal{S}$ such that the image $C' = \phi(C)$ of C under the embedding map is a Cauchy surface of $\mathcal{S}'$, then it can be shown that there is a unique maximal extension $\tilde{\mathcal{S}}$ of $\mathcal{S}$. [15] But although $\tilde{\mathcal{S}}$ may be maximal in this sense, it may not be a good model for illustrating how time can be finite. For by deleting portions of Minkowski space-time $\mathcal{S}_{Min}$, we can obtain a space-time $\mathcal{S}_{del}$ which is maximal in this sense and which is past and future-bounded; but $\mathcal{S}_{del}$ is just as objectionable as the original truncated space-time $\mathcal{S}_{trun}$ discussed in section 4. [16]

Another approach is to use the concept of a framed space-time, a space-time $\mathcal{S} = \langle M, g, \nabla \rangle$ together with an orthonormal tetrad at some point of M. If we restrict attention to framed space-times and extensions in which the embedding map carries the preferred frame of the one onto the preferred frame of the other, it can be proven that every space-time is contained in a maximal space-time.[17] However, the use of preferred frames is somewhat artificial. And it is not obvious that any framed space-time with properties P can be extended to a framed space-time which is maximal with respect to P.

All this suggests that (5) may have to be weakened to

(5″) $\langle \mathcal{S}, \mathcal{E} \rangle$ is not extendible to an $\langle \mathcal{S}', \mathcal{E}' \rangle$ which satisfies (1), (3), and (4) and which itself is not past- (future-) bounded.

However, (5″) may be too weak. For there are truncated cosmological models that cannot be extended as a solution to Einstein's field equations to a model that is not past- or future-bounded (consider, for example, truncated versions of the Friedmann models—see the following section); such truncated models will again be as objectionable as the original one.

Fortunately we can carry forward the discussion without having to decide precisely what form of the inextendibility condition to impose; for we shall see that there are cosmological models which satisfy the strongest form of the inextendibility condition and which have the other properties we desire.

8. Some Physically Interesting Examples of Temporally Finite Space-times

The considerations of the preceding section are closely related to the definition of "singular space-time" adopted by a number of physicists: $\mathcal{S}$ is said to be singular if no extension of $\mathcal{S}$ is timelike geodesically complete (i.e., no extension is such that every timelike half-geodesic is complete). Several theorems have been proved about the existence of singularities in the cosmological models of the general theory of relativity. These theorems will not in general be relevant to our present concerns; for we are not interested in timelike incompleteness in general—we are interested only in cases in which the incompleteness is of such a global nature that time runs out for all possible observers and in which $\mathcal{S}$ satisfies the other conditions discussed in section 7. There is one general class of cases in which the singularity theorems will be of general interest; namely, those in which $\mathcal{S}$ contains a Cauchy surface. For, in the first place, such an $\mathcal{S}$ will possess a global time function. Second, if $\mathcal{S}$ is timelike geodesically incomplete, we may be able to prove that the incompleteness is of a global nature. Moreover, the existence of a Cauchy surface adds a bonus—it makes possible the implementation of Laplacian

Determinism; thus it may be possible to predict (retrodict) from the state of the universe at a given instant, the end (beginning) of time.

In fact, general relativity predicts that in many seemingly physically interesting cases, time is finite in the past or future. More precisely, let $\langle \mathcal{S}, \mathcal{E} \rangle$ be a cosmological model such that Einstein field equations (without cosmological constant)[18] are satisfied and $\mathcal{S}$ possesses a Cauchy surface C. Then if the convergence of the future-pointing unit normals to C is everywhere greater than some positive constant C_0 (this means that at the instant corresponding to C, the universe is everywhere contracting at a rate at least as great as that given by C_0), and if a condition on the energy-momentum tensor is satisfied (in typical cases, this condition can be violated only by having negative energy densities or large negative pressures), then as measured from C, no future-directed timelike curve of $\mathcal{S}$ has a proper length greater than $3/C_0$.[19]

The Friedmann cosmological models satisfy the hypotheses of this theorem and/or the temporal converse of the convergence condition; thus they are finite in the past and/or in the future. Moreover, the space-times involved are inextendible—none of them can be embedded as a proper subset of any space-time, much less a solution to Einstein's field equations, and they are therefore maximal in the strongest sense. Thus they would seem to qualify as physically interesting examples of how time can be infinite in the past or future.[20]

9. Examples Reconsidered

How physically realistic are such models? Two opposing views on this question in particular and on the singularity theorems in general have emerged. According to one view, no space-time can be regarded as being physically realistic unless it is timelike geodesically complete. The fact that timelike geodesic incompleteness is a pervasive feature of the cosmological models of general relativity is taken as an indication that something is drastically wrong with the theory.[21] The opposing attitude is that nothing is wrong with the theory and that, therefore, we must learn to live with singularities. Thus C. Misner[22] argues that since observational evidence together with Einstein's theory suggests that time in our universe has a beginning, we had better get used to the notion that time has an "absolute zero."

However, Misner goes on to say that "*the universe is meaningfully* *infinitely old because infinitely many things have happened since its beginning.*"[23] Viewed as a way of making the notion of the beginning of time more palatable, this statement is unexceptionable; but viewed in another light, it threatens to undermine the analysis given above. For in support of this statement, Misner introduces a time scale Ω which is related logarithmically to the proper time scale and which is "attractive as a primary standard" in that "significant epochs (e.g., galaxy formation, nucleo-synthesis hardon era, etc.) are spaced at reasonable intervals of Ω."[24] But on the Ω scale, the universe is infinitely old. Therefore, if the Ω scale were accepted as the "primary standard," time would have no beginning on the above analysis since it would not be past-bounded on the Ω scale. And, it might be asked, what can justify the use of the proper time scale to the exclusion of all other time scales? The answer is that it may be useful to employ the Ω scale or some other scale in discussing some phenomenon, e.g., galaxy formation; but if one is to believe the theory in the context of which the discussion is taking place, space-time is equipped with a metric that gives the measure of space-time distances, and it is this metric that we must use in answering the question, "Does time have a beginning or an end?"

10. Singularities and the End Points of Time

Misner uses the term "singularity" to cover not only the case of timelike geodesic incompleteness but also the case in which there is some "infinity," e.g., infinite curvature or infinite mass density. In suggesting that a singularity may well have occurred in our universe at some finite proper time in the past, he presumes that the singularity may well involve some such infinity. But this sort of talk is not consistent with the point of view we have adopted so far, for such talk assumes there are "singular points" at which the curvature or mass density becomes infinite; but we have been assuming that space-time is a Lorentzian manifold and that, in particular, the metric of M is everywhere non-singular (i.e., defined and differentiable at every point of M).[25] Thus, in speaking of a Friedmann universe, we mean Friedmann space-time $\mathcal{S}_F$ with singular points omitted, and in the statement that $\mathcal{S}_F$ is not extendible to a larger space-time $\mathcal{S}_F'$, $\mathcal{S}_F'$ must be taken to be without singular points. (For example, the proof of the inextendibility takes the following form for the spatially closed Friedmann space-time which is finite in the past and future. Assume for purposes of contradiction that M_F can be isometrically embedded as a

proper subset of M_F' and let ϕ (M_F) denote the image of M_F under the embedding map. Then we can find a point $p \in (M_F' - \phi(M_F))$ and a sequence of points $x_i \in \phi(M_F)$ such that the sequence converges to p and the scalar curvature $\mathcal{R}(x_i)$ diverges as $x_i \rightarrow$ p. But this yields a contradiction, since the curvature invariants are differentiable functions of space-time position.)

There is good reason not to admit singular points as part of the space-time manifold M: unless some limitation is put on the type of singularity we admit, we shall not have any theory at all, and there does not seem to be any good motivation for picking out the admissible singularities or for prescribing how many and in what configuration they will be placed on the manifold.[26]

Still, it is possible that singular points can be treated by joining them to M so as to form a "boundary" for M. Some such move must be made if our picture of space-time is to accommodate the notion that the scalar curvature "becomes infinite." (It is at this point that I part company with R. Swinburne,[27] who argues that if the best confirmed cosmological theory implies that there is a time t_0 in the past at which matter was infinitely dense, then we can conclude that the universe (which Swinburne takes to mean the collection of all physical objects that are spatially related to the earth) came into existence after t_0, since an infinitely dense state of matter is physically impossible. But if it really is physically impossible, then the conclusion we must draw from Swinburne's argument is that the theory in question is false. Moreover, on my interpretation of relativity theory, models like those of Friedmann do not imply that there was or will be a time at which matter was or will be infinitely dense; rather, they entail the more radical consequence that time itself is finite in the past or future. On the other hand, such models do imply that singular states do "occur"—not at any point of time, but at ideal points which are attached to space-time by some procedure yet to be described.)

Also we would like to have a means of illustrating the Aristotelian conception of the beginning or end of time, i.e., a means of representing first or last instants of time.

Mathematically, what we want is a prescription for associating with any future- (past-) bounded space-time $\mathcal{S}$ a "future (past) boundary" for $\mathcal{S}$. One immediately runs into a problem if this task is interpreted in the following literal way: for any future- or past-bounded space-time $\mathcal{S} = \langle M, g, \nabla \rangle$, find a manifold with boundary $\overline{M}$ such that $\overline{M} = M \cup \partial(M)$

and the boundary $\partial(M)$ is the disjoint union of the "future boundary" and the "past boundary." In the case of the truncated Minkowski space-time $\mathcal{S}_{trun}$, the obvious and natural way to obtain $\overline{M}_{trun}$ is to take the "future boundary" to be the set of all points whose temporal coordinates satisfy $t = 1$. But the very reason why there is an obvious procedure in this case—the fact that there is an extension $\mathcal{S}_{Min}$ of $\mathcal{S}_{trun}$ so that the future boundary of $\mathcal{S}_{trun}$ can be derived by taking the closure of M_{trun} in M_{Min}—means that condition (5′) of section 7 is violated. And it can be shown that for any time-orientable space-time $\mathcal{S} = \langle M, g, \nabla \rangle$, if there exists a manifold with boundary $\overline{M} = M \cup \partial(M)$ such that the boundary $\partial(M)$ is a spacelike three-manifold, then $\mathcal{S}$ is extendible to an $\mathcal{S}' = \langle M', g', \nabla' \rangle$ so that $\partial(M) = Cl(M) \cup Cl(M' - M)$ where Cl denotes the operation of taking the closure in M'. Thus we have a dilemma. If $\mathcal{S}$ is, say, a future-bounded and inextendible space-time, then the "future boundary" for $\mathcal{S}$ cannot be represented as a boundary of the space-time manifold of $\mathcal{S}$, at least not in the way the time slice $t = 1$ of Minkowski space-time does for $\mathcal{S}_{trun}$. On the other hand, if $\mathcal{S}$ is extendible to a larger space-time, it violates condition (5′) and thus may fail to qualify as an interesting example of how time can come to an end. To sum up, the straightforward way of trying to illustrate the Aristotelian conceptions of the beginning and end of time seems to be blocked.

A more sophisticated approach is needed. In order to illustrate the possibilities, I shall briefly describe the g-boundary approach.[28] For any given space-time $\mathcal{S} = \langle M, g, \nabla \rangle$, this approach associates with each incomplete timelike geodesic an ideal end point. An equivalence relation is defined on the timelike geodesics, and two ideal end points are identified just in case their corresponding geodesics are equivalent modulo this relation. The resulting set of ideal points forms the g-boundary $\partial_g(M)$; a topology is defined for $\partial_g(M)$, and a prescription is given for attaching $\partial_g(M)$ to M to form a manifold $\overline{M}_g$ with g-boundary. In some cases a differential and metric structure can also be defined for $\partial_g(M)$. $\overline{M}_g$ will *not* in general be a manifold with a boundary. However, if $\overline{M}$ is a manifold with boundary $\partial(M)$ such that every incomplete timelike geodesic of M strikes $\partial(M)$, then given the space-time $M \equiv \overline{M} - \partial(M)$, the g-boundary approach can be used to extend M to $\overline{M}$.

In the case of the spatially closed Friedmann universe, the g-boundary does not consist, as one might expect, of two points corresponding to the initial and final singular states; rather, it consists of two three-spheres S^3,

and topologically, Friedmann space-time-plus-g-boundary is the product of S^3 and a closed and bounded interval of $\mathbb{R}$. Similarly, if the g-boundary approach is applied to a conical space-time with vertex removed, it does not give back the vertex point and only the vertex point. However, a modification of the equivalence relation on the timelike geodesics will lead to this result.[29] In addition to alternate schemes for equating timelike geodesics, there are alternative topologies and metric structures for $\partial_g(M)$. There doesn't seem to be any sense in choosing one alternative once and for all to the exclusion of all others; different alternatives may be better for illuminating different aspects of "singularities." Consequently there is no one right way to represent the "first" or "last moment of time."

11. Before the Beginning and After the End

The truncated Minkowski space-time of section 4 illustrates how time can be finite in the future; but the success of this example is the success of stipulation. We stipulate in effect that time is finite in the future by erasing an infinite portion of Minkowski space-time; there seems to be no reason to believe—and some reason not to believe—that this particular stipulation could be physically realized. In contrast, the Friedmann cosmological models provide a more interesting illustration in that they provide us with a mechanism for realizing temporal finiteness. Still, it could be claimed that these examples succeed only by virtue of stipulation; this time the stipulation specifies what we are to count as a space-time. Why, it might be asked, could we not change our picture of space-time so that other regular regions of space-time are joined onto the "initial" and "final" singularities of the Friedmann models, making them passing episodes in a longer history?

I shall approach this question somewhat obliquely by considering first another question. We have assumed that space-time is a connected manifold. Why, it might be asked, could we not change this picture to allow for the existence of other regions of space-time totally disconnected from ours? Well, clearly we could. But the question remains as to what the new picture of space-time amounts to, and in particular, what is meant by the "existence" of these other regions. Does every possible connected space-time "exist" (in this new sense of existence) as a region (of the new extended space-time) that is disconnected from ours? If so, the new picture does not really contain any innovation; the "existence" of a disconnected space-time region such that __________ means no more than it is possible that ________. Disconnected regions of space-time are only a device for picturing other possible worlds. On the other hand, if not every possible space-time "exists" as a region disconnected from ours, how do we tell which of them "exists"? For the "existence" of the other regions to make any empirical difference, these other regions must interact in some fashion with our region. This interaction cannot be described in anything like the usual spatio-temporal terms we use to describe causal interactions since, by hypothesis, these other regions do not enjoy any of the usual spatio-temporal relations with our region. What sort of interaction, then, can it be?

Does talk about what happens "before" and "after" the "initial" and "final" singularities in the Friedmann universes make any more sense than talk about events that happen "out there," where "out there" indicates a region of space-time disconnected from ours? If the singularities are true space-time singularities and not just regions where matter is very dense—and we have assumed that they are true space-time singularities—then they seem to separate us from the "other" regions that "join on" "before" and "after" just as effectively as disconnectedness separates us from the other regions "out there." The "before" and "after" regions might just as well be other possible worlds.

This interpretation is opposed to the more usual picture of an "oscillating universe." In the case of the spatially closed Friedmann universe, there are formal solutions to the differential equations governing the temporal behavior of the radius R of the universe which, if taken literally, would allow one to picture the universe as oscillating between the singular points where $R = 0$. (There are other solutions in which r does not oscillate.) But here as elsewhere, one can be misled by taking a picture too literally, for from the point of view presented above, these mathematical solutions are purely formal. As we have seen above, the "singular points" at which $R = 0$ can be represented only by sophisticated mathematical techniques, and on some representations they are not even points but rather three-spheres. Second, although continuity considerations can be used to help characterize certain aspects of singularities, they do not seem to provide a means of carrying us "through" the singularities and into "other" space-time regions on the "other side" of the singularities. Compare the situation here with that in Newtonian gravitational theory of point mass particles. When the particles collide, Newton's $1/r^2$ law blows up. Under certain conditions, however, solutions to the equations of

motion can be extended through the singularities by means of analytic continuation. This procedure is possible only because of the constant background of Newtonian space-time, which gives the means of defining relevant senses of continuity of solutions. But in the relativistic case, the space-time background itself becomes singular, and hence no means of defining a "continuous" extension through the singularity is at hand.

I am not claiming that we could never come to possess any empirically well-grounded principle that would "carry us through" space-time singularities. What I do claim is that we do not now possess such a principle and that it is difficult to see how such a principle could be constructed and confirmed within present relativity physics.[30]

Two final points. First, the arguments given in this section take for granted certain elements of the currently accepted picture of space-time. This picture may well be dropped in the future in favor of some radically different picture; but this is a matter which at the present time must be left to writers of science fiction. Second, if the sentiments of this section are rejected, then one must conclude that no physically interesting example of how time can be finite in the future or past can be constructed within the current framework of physics. Such a negative conclusion would be interesting in its own right.

12. Concluding Remarks

Many of the above considerations rely heavily on the space-time metric. This is no accident, for most of the crucial distinctions I have drawn cannot be made in terms of topological, or affine, or even conformal structure. Some philosophers hold that the metric element is "nonintrinsic" and "conventional" in a way that, say, the topological structure is not. I do not share this view, but I wish to point out that if it is correct, then the answers to many of the questions which philosophers have asked about the beginning and end of time are matters of convention.

Unless the line of analysis I have pursued in this paper is very misleading, the answers to the questions posed at the outset lie somewhere in a thicket of problems growing out of the intersection of mathematics, physics, and metaphysics. This paper has only located the thicket and engaged in a little initial bush-beating. This is not much progress, but knowing which bushes to beat is a necessary first step.

Some philosophers will be disappointed that the thicket is populated by so many problems of a technical and scientific nature. On the contrary, I am encouraged by this result because it shows that a long-standing philosophical problem has a nontrivial and, indeed, a surprisingly large content. Moreover, this result is a good illustration of the artificiality and danger of trying to separate philosophy from science. If I am right, some of the best philosophy of time is being done today by physicists or, as I would prefer to say, by natural philosophers. Conversely, this work inevitably brings a confrontation with traditional philosophical problems.

Notes

1. The history of philosophy seems to me to be littered with disputes which could not possibly have been resolved in a satisfactory manner by the disputants, simply because they were not in a position to settle the relevant questions.

2. See *Physics*, Bk. VIII, 251b.

3. See my paper "Space-Time, or How to Solve Philosophical Problems and Dissolve Philosophical Muddles Without Really Trying," *Journal of Philosophy*, 67 (1970): 259–277. See also S. Shoemaker, "Time Without Change," *ibid.*, 66 (1969): 363–382.

4. It might be objected here that although time as represented by the metric space $((-1, +1), d_e)$ is finite in both the past and the future according to this criterion, time is not "really" finite since in the topology induced by the metric, $(-1, +1)$ is noncompact. But all this shows is that compactness does not always capture the relevant sense of finiteness.

5. L. E. Loemker, ed., *Leibniz: Philosophical Papers and Letters* (New York: D. Reidel, 1970), p. 669.

6. *Ibid.*, p. 664.

7. Technically g is a symmetric tensor field of type (0, 2) which is defined on all of M, which is nondegenerate, and which has index 1 or 3. The differentiability classes of M and g are left open since nothing important in what follows turns on this question. The technical definition of ∇ is not needed here; for present purposes, the important fact about ∇ is that it allows us to define a notion of parallel transport on M and hence a notion of space-time geodesic (i.e., auto-parallel curve).

8. Technically, $\mathscr{G}$ is a symmetric tensor field of type (2, 0).

9. For more details, see the first reference of note 3 and my paper, "An Attempt to Add Some Direction to 'The Problem of Director of Time,' " *Philosophy of Science*, 41 (1974): 15–47.

10. Let (x, t) be a pseudo-Euclidean coordinate system for M_{Min}. First identify those points with coordinates (x_1, t_1) and (x_2, t_2) just in case $t_1 = t_2$ and $x_1 = x_2$ modulo m; then identify two points of the resulting space just in case their spatial coordinates are the same and their temporal coordinates are equal modulo n (m, n positive integers).

11. I will use "curve" ambiguously to denote a map σ: $[0, 1] \rightarrow M$ and the image $\sigma([0, 1])$.

12. Although the notion of presupposition may be problematical in general, it seems transparent enough in the following examples.

13. Actually, things are more complicated than this, for a space-time $\langle M, g, \nabla\rangle$ may be inextendible but still locally extendible in the sense that there is an open subset $U \subset M$ such that: (1) the closure of U in M is noncompact, but (2) $\langle U, g|_U, \nabla|_U\rangle$ can be isometrically embedded in a larger space-time so that the closure of the image of U is compact. Since there are enough complications already, I shall not discuss this one here. For some implications of local extendibility for determinism, see my paper "Laplacian Determinism in Classical and Relativistic Physics," forthcoming in the University of Pittsburgh series in the Philosophy of Science.

14. R. Penrose, "Gravitational Collapse," *Revisita del Nuovo Cimento*, 1 (1969): 252–275.

15. For a precise statement and proof of this result, see Y. Choquet-Bruhat and R.

Geroch, "Global Aspects of the Cauchy Problem in General Relativity," *Communications in Mathematical Physics*, 14 (1969): 329–335.

16. For example, let M_{del} consist of all those points of M_{Min} in the intersection of the future lobe of the null cone at (0, 0, 0, 0) and the past lobe of the null cone at (0, 0, 0, 1), (x, y, z, t) being a pseudo-Euclidean coordinate system. $C = \{(x, y, z, t)$: $t = 1/2$ and $x^2 + y^2 + z^2 < 1/2\}$ is a Cauchy surface for $\mathscr{S}_{del}$. $\mathscr{S}_{del}$ cannot be extended in such a way that C remains a Cauchy surface. This example is due to Choquet-Bruhat, *ibid.*

17. See R. Geroch, "Singularities," in M. Carmeli, ed., *Relativity* (New York: Plenum Press, 1970).

18. In many cases the addition of a cosmological term will not block the singularity results.

19. See S. W. Hawking, "The occurence of singularities in cosmology," *Proceedings of the Royal Society of London A*, 294 (1966): 511–521.

20. The Friedmann models were chosen as illustrations because of their simplicity; but precisely because of their simplicity—in particular, their high degree of symmetry and the absence of any pressure term in their equation of state—they are somewhat unrealistic. The singularity theorem cited above applies to models that are much more realistic in these respects.

21. See R. Penrose, "Structure of Space-Time," in C. M. DeWitt and J. A. Wheeler, eds., *Battelle Rencontres* (New York: W. A. Benjamin, 1968).

22. "Absolute Zero of Time," *Physical Review*, 186 (1969): 1328–1333.

23. *Ibid.*, p. 1331.

24. *Ibid.*

25. When I speak of curvature, I am referring to the rational scalar invariants formed from the Riemann Tensor and its derivatives. The so-called irrational scalar curvature invariants can become infinite in regions where the metric is nonsingular.

26. See J. A. Wheeler, "Geometrodynamics and the Problem of Motion," *Reviews of Modern Physics*, 33 (1961): 63–78.

27. *Space and Time* (New York: St. Martin's Press, 1968), chapter 15.

28. See R. Geroch, "Local Characterization of Singularities in General Relativity," *Journal of Mathematical Physics*, 9 (1968): 450–465.

29. See D. A. Feinblum, "Global Singularities and the Taub-NUT Metric," *Journal of Mathematical Physics*, 11 (1970): 2713–2720. Recently, a seemingly more satisfactory definition of space-time singularities has been given by B. G. Schmidt, "A New Definition of Singular Points in General Relativity," *General Relativity and Gravitation*, 1 (1971): 269–280. But this definition is too complicated to be reviewed here.

30. One possible exception has been mentioned in C. J. S. Clarke's contribution to this volume; namely, that the laws of gravitation might be put in a distributional form such that singularities could be countenanced as part of the space-time manifold. However, one would guess that there are solutions to Einstein's field equations which (1) are temporally finite and (2) cannot be extended in even a distributional sense.