ESSAYS ON SUPPLY CHAIN FINANCE

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Dedication

This work is dedicated to my mother.

Abstract

Supply chain finance is an important research area at the intersection of finance and operations management that has drawn growing attention from both academics and practitioners. This thesis contains essays addressing three different problems in supply chain finance, studying two important source of financing – trade credit and crowdfunding.

The first two essays model trade credit options in the supply chain, focusing on the interaction between operational and financial decisions. We characterize the optimal decisions of the supplier and the retailer, and show that trade credit contracts work as a risk-shifting mechanism between supply chain partners, mitigating the financial constraints, reducing market frictions, and improving the supply chain's efficiency.

In the first essay, we model a web-based retailer offering direct financing to its capital-constrained supplier who sells its products on a consignment basis. The supplier has access to bank financing and the retailer can specify debt-seniority. We show that if the retailer requires debt-seniority, then the supplier will use either bank or direct financing (but not both), and it may produce more than the first-best quantity. In contrast, if the retailer does not request seniority, the supplier may take both bank and direct financing, but it always produces less than the first-best quantity. Interestingly, the retailer does not always prefer debt-seniority.

In the second essay, we study variants of net-terms trade credit contracts in a traditional market, with market frictions of bankruptcy costs, information asymmetry and transaction costs. We provide a possible explanation for the prevalence of debt forgiveness in practice. We show that the supplier, by contracting and preannouncing forgiveness, can use one type of friction to mitigate another, and benefit both supply chain partners.

The third essay studies the social proof effect in the rapidly diffused crowdfunding market. We study how prior capital accumulation and the volume of owner-issued-referrals impact subsequent fundraising. By analyzing a campaign-level web traffic data set, we show that higher capital accumulation increases the probability of contribution for the owner-referrals, the volume of organic visitors and the total volume of organic

contributors, although it may decrease probability of contribution from the organic visitors. To study the optimal policy for entrepreneurs to use their referrals, we build a Markov-decision-process model and calibrate its parameters using our data. Our finding indicates that entrepreneurs in general should distribute their referrals over the course of the entire campaign.

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Chapter 1

Introduction

Supply chain finance is about integrating the inventory flow with the financial flow. Inventory flow has been studied extensively in the operations management literature, but the interaction between operational and financial decisions has received relatively less attention. In practice, limited working capital is a frequently-occurring constraint in corporate procurement decisions. With financial constraints, firms may not be able to execute their optimal operational decisions. Exploring the mechanism of supply chain financing is not only important for ensuring that there are adequate sources of capital, but also important for improving the supply chain efficiency and increasing the profits of supply chain partners.

In this thesis, we study two types of supply chain finance. In Chapters 2 and 3, we study Business-to-business (B2B) financing, also known as trade-credit, which is an important source of supply chain finance in the United States and other developed economies. In Chapter 4, we study Crowdfunding, which has recently emerged as an alternative to traditional sources of capital for early-stage ventures. As we explain in the next paragraph, both are economically important with significant impact on all sectors of the economy, and small businesses in particular.

It is reported that trade credit in the 1990s accounted for an average \$1.5 trillion of the book value of all assets of US corporations and represented approximately 2.5 times the combined value of all new public debt and primary equity issues during a given year (Ng et al. 2002). As of 2009, trade payables represented the second largest liability on the aggregate balance sheet of non-financial businesses in the US and they were more than triple the amounts owed to banks (Murfin and Njoroge 2012). In terms of the crowdfunding, recent industry reports indicate that crowdfunding platforms helped individuals and organizations raise approximately $2.7 billion in 2012 (Massolution 2013) globally, and estimated that the market had grown to as large as $34 billion.

We study the trade credit in Chapters 2 and 3 focusing on the interaction between operational and financial decisions. Modigliani and Miller (1958) show that in a perfect capital market, the value of a firm is independent of its financial structure. Therefore, its financial and operational decisions may be made independently. However, OR/OM papers argue that market imperfection is common in the trade-credit environment, so that the financial arrangements matter. One essential imperfection is called the agency costs. That is, in a decentralized supply chain, partners act as independent agents that maximize their own profits, giving rise to inefficiencies and extra costs. In such situations, trade credit works as an incentive that the lenders offer to the borrowers, contingent upon their operational decisions. This incentive shifts profits, as well as sales-risk, from one party to another, mitigating agency costs via different terms of trade-credit contracts. More importantly, other sources of financing (e.g. bank financing) affect such risk-shifting mechanisms and the lending and borrowing strategies change with additional market imperfections, such as bankruptcy costs, and information asymmetry. In Chapter 2 and 3, we investigate the role of trade credit as the profit-shifting and risk-shifting mechanism under different market environments, in the presence of bank financing and the above-mentioned market imperfections.

In Chapter 4, we study crowdfunding, a newly emerged source of financing for early-stage ventures. These ventures face financial constraints but may not have access to any other sources of financing. Unlike traditional financing methods, crowdfunding platforms offer entrepreneurs an opportunity to raise money from a large number of potential backers, i.e. the "crowd". Funds come from the aggregation of relatively small contributions from many individuals. Social factors have been found to play a particularly important role in the success of crowdfunding. Peer influence and observational learning are also extensively reported in the crowdfunding process. However, the joint implications of these effects and how the entrepreneurs should approach engaging with their social network in the fundraising process are not adequately explored. In Chapter 4, we investigate how the entrepreneurs may use their referrals to achieve success in the fundraising. Next, we briefly summarize the contents of each chapter and our contributions.

Retailer-Direct Financing Contracts Under Consignment

In Chapter 2, we model trade credit options in a two-sided market. The supplier sells its products via a web platform on a consignment basis. The platform owner (retailer) offers direct-financing to the supplier.

The profit-sharing and risk-shifting mechanisms are different compared to the traditional market. In a traditional market, the supplier offers a wholesale price and also determines the financial terms. The retailer buys from the supplier (incurring cost), and then sells to customers (earning revenue). The finance and product ownership flows from the supplier to the retailer, and the revenue flows in the opposite direction. Retailers in traditional markets have costs on the one side of the equation, and revenue on the other. In contrast, in the two-sided market, the retailer (platform owner) has costs and revenues on both sides, associated with transactions with customers and sellers attracted to its web site. The retailer does not own the product at any point in time as it flows from the supplier to the buyer. Both the supplier and the retailer receive the revenue only when a transaction occurs. We show that direct-financing, by shifting part of the risk to the retailer, increases the profits of both the supplier and the retailer, and improves the supply chain's efficiency.

We provide mathematical models to study the retailer's choice of debt seniority and the supplier's borrowing strategy when the supplier can also access bank financing. We find that when the retailer demands seniority, the supplier borrows from either the retailer or the bank (but never both) and the offered financial incentives may lead to the supplier overproducing. In contrast, when the retailer does not require seniority, the supplier may simultaneously borrow from both, but never overproduces. We show that in some cases, the retailer will not require debt seniority and let it pass to the bank.

We also investigate the effect of bank information asymmetry. We show that such information asymmetry makes the direct financing more attractive. The retailer is able to charge a higher interest rate, increasing its profit, and lowering the supplier's profit.

Leveraging Bankruptcy Costs to Improve Supply-Chain Performance

In Chapter 3, we study trade credit in a traditional market, with market frictions of bankruptcy costs, information asymmetry and transaction costs. We compare and contrast variants of the net-terms trade-credit contract to investigate situations in which the supplier can use one type of friction as a lever to mitigate the negative consequence of another type of friction and produce a win-win outcome for both supply chain partners. Specifically, we model two types of contracts to study the debt forgiveness in the trade credit environment – reported sales contracts and trade forgiveness (which we call TF) contracts.

In the reported sales contracts, the supplier offers trade credit to the retailer and relies on retailer-reported sales to learn the retailer's post-sales assets. If reported sales are insufficient to repay the loan, then the supplier either accepts the retailer's report or initiates bankruptcy. We show that the supplier has to trade off the potential loss from retailer's incorrect report, against the potential loss of initiating bankruptcy and paying the bankruptcy costs. The equilibrium may result in a payment less than the loan amount. That is, the supplier is often willing to forgive a part of the loan to avoid bankruptcy costs. However, the loan forgiveness is not a prior commitment. We find that in some cases the retailer can leverage private sales information to order more and receive greater forgiveness from the supplier. The supplier will charge a higher wholesale price in such cases, but still make a lower profit.

In our TF contracts, the supplier offers a certain degree of loan forgiveness up front but requires, in return, that the retailer allows a trusted third-party access to its true sales figures. The supplier bears the cost of sales verification. Our TF contract reveals the influence of debt forgiveness on both order quantities and bankruptcy costs. While eliminating information asymmetry, we find TF contract provides the supplier more flexibility of risk-shifting, and is more efficient than the two-terms contract for avoiding the bankruptcy costs and inducing adequate production. In such case, it is possible for the supplier to achieve its best possible profit.

When information asymmetry exists, our numerical examples show that the supplier can leverage the costs of bankruptcy and information asymmetry via the transaction costs of TF contract, which benefits both the supplier and the retailer as compared to the reported sales contract.

Referrals' Timing and Fundraising Success in Crowdfunding

In Chapter 4, we study how prior capital accumulation (social proof) and the volume of referrals issued by the entrepreneur impact subsequent fundraising and how the timing of referral transmission affects fundraising outcomes in the crowdfunding market.

We draw on a large-scale proprietary data set on campaign web traffic, obtained from one of the world's largest reward-based crowdfunding platforms. We observe that the probability of contribution increases with capital accumulation for those visitors who are referred by the entrepreneur, while the opposite is observed for organic visitors. We further observe a positive association between capital accumulation and the volume of organic visitors. Thus, although conversion rates decline, the total volume of impressions increases in parallel, in a countervailing fashion. Overall, we find that the combined effect of capital accumulation on fundraising outcomes is generally positive, i.e. the overall volume of organic campaign contributors increases as a campaign progresses toward its goal.

Following these descriptive analyses, we build a Markov-decision-process (MDP) model and calibrate its parameters using our data. The model considers a set of discrete time points at which the entrepreneur may decide how many referrals to use. Two types of models are studied –a concave function and a convex function. Each model reflects different assumptions about the relationship between prior capital accumulation, the entrepreneur's referral volumes, and subsequent fundraising outcomes. We find that in the convex model, an all-or-nothing policy is optimal. That is, the entrepreneur should expend either all or none of their referrals. In contrast, in the concave model, it may be optimal for the entrepreneur to issue their referrals in a distributed fashion over the course of the campaign. We evaluate the relative "fit" of these two models in our data, finding that the concave model appears to be a better representation of reality. This finding indicates that, in general, entrepreneurs should aim to distribute their referrals over the course of the entire fundraising process.

Summary of Contributions and Relevance to Practice

Our first essay contributes on the study of supply chain finance by analyzing trade credit on the consignment selling. We extend the literature by modeling the choice between bank-financing and direct-financing, and the choice of debt seniority. Our direct-financing contracts serve to inform the web-platform-retailers how to structure contract terms to improve individual and combined profits of the supply chain partners.

Our second essay extends the OM literature of trade credit by modeling debt forgiveness, and contributes on the study of supply chain contract by simultaneously considering financial constraints, bankruptcy costs and information asymmetry. Our reported-sales contracts provide a possible explanation for the prevalence of debt forgiveness in practice, and our TF contract creates an option to benefit both the supplier and the retailer by avoiding bankruptcy costs.

Our third essay contributes to both empirical and mathematical models of the crowdfunding, by studying the joint effect of prior capital accumulation and the timing of the referral transmission. Our analysis serves to guide entrepreneurs on how to use their own social network effectively to achieve funding success in the crowdfunding market.

Roadmap Comments

This thesis consists of three self-contained chapters on three different topics of supply chain finance problems. Each chapter is written as a separate paper and contains its own motivation, literature review and conclusion. Before proceeding to the three Chapters that constitute the main body of this thesis, please note that notation is unique and redefined in each chapter, and that the performance metrics of interest are also defined separately in each chapter.

Chapter 2

Retailer-Direct Financing Contracts Under Consignment

2.1 Introduction

Small businesses are widely believed to be the engines of job creation in the United States. An estimated 25 to 27 million small businesses are reported to account for 60 to 80 percent of all US jobs (Bagley 2012). Many use web platforms such as Amazon, E-bay, and Spun to sell their products. US Census Bureau (2014) reports that e-commerce has been steadily gaining momentum, increasing from about 0.8% of total retail sales in the first quarter of 2000 [$\approx$ 5.8 billion of 740 billion] to 5.9% in the third quarter of 2013 [$\approx$ 67 billion of 1.14 trillion]. Web-based retail typically occurs on a *consignment* basis. The platform owner, which we refer to as the *retailer*, does not purchase inventory. Instead, it charges sellers, which we refer to as *suppliers*, a *commission* on each sale. The commission is based on product category and covers a variety of services such as web hosting, payment collection, shipping, and handling of returns (Simchi-Levi et al. 2004).

Small business owners often find it difficult to obtain commercial loans (Rampell 2013). National Small Business Association (2014) reported that 43% of small-business owners needed funds at one point in the last four years and were unable to find any willing sources of financing. Numerous news articles track the decline in the availability of small business loans, especially since the financial crisis of 2008 (e.g. Wiersch and Shane

2013, Laderman and Gillan 2011).

Some retailers have started to offer business loans to suppliers, which we refer to as the retailer-direct financing. For example, Amazon in late 2011 launched the *Amazon Lending* program for small-business suppliers who have difficulty borrowing capital from traditional lenders. In this program, qualified suppliers are offered loans based on their business' selling performance on web platform Amazon.com. Needleman and Bensinger (2012) wrote "Merchants who spoke to The Wall Street Journal said they were offered loans ranging from $1,000 to $38,000 apiece, with interest rates ranging from less than 1% (for one of them) to 13.9% (for most who were interviewed), whereas small-business credit-card interest rates typically range from 13% to 19%." Note that in the absence of reasonably-priced business loans, small business owners often use credit cards to meet their working capital needs.

Compared to third-party lenders such as banks, the retailers are likely to have a pre-existing relationship with each small supplier, and knowledge of the market potential for their products based on web traffic and sales data. The retailers also gather the so-called "soft information", such as the character and the reliability of the firms' owners. In contrast, the banks' assessment of the loan-worthiness of a small supplier may be based on general market characteristics for that type of product, rather than supplier-specific market potential. Traditional lenders normally do not have the band width to build relationships with a large number of small borrowers (Berger and Udell 2002, Berger and Black 2011). In the United States, the USA Patriot Act has imposed KYC (know your customers) requirements on banks in an effort to prevent money laundering. When suppliers are small, the overall cost of getting to know them may be prohibitively high for banks (Taylor and Zax 2015). For these reasons, the bank and the retailer have different quality of information about the market potential of the supplier's product. Specifically, banks may underestimate the market potential of some small suppliers relative to the retailer's estimates, which we refer to as *bank information asymmetry.*

In recent years, there have been many papers that model supplier-retailer interactions under trade credit — an arrangement in which the supplier offers credit to the retailer in the conventional B2B sales environment. This chapter considers financing offered by the web platform owner who faces a two-sided market. Although it is related to the body of trade-credit literature, there are some important differences. In a traditional market, the retailer buys from the supplier (incurring cost), and then sells to customers (earning revenue). The finance and product ownership flows from the supplier to the retailer, and the revenue flows in the opposite direction. Retailers in traditional markets have costs on one side of the equation, and the revenue on the other. In contrast, in the two-sided market, the retailer (platform owner) has costs and revenues on both sides, associated with transactions with customers and sellers attracted to its web site. The retailer does not own the product at any point in time as it flows from the supplier to the buyer. Both the supplier and the retailer receive the revenue only when a transaction occurs. Many trade-credit papers assume that the wholesale price is set by the supplier who also determines the financial terms. In contrast, in the two-sided market there is no wholesale price because transactions occur on a consignment basis. Consignment rates are typically based on the product category. They are not set separately for each individual supplier.

In the sequel, we present a model in which both the bank and the retailer offer financing to the supplier, and analyze two cases. In the first case, the retailer has seniority when recouping the loan, and in the second, the bank has seniority — see, Altman and Hotchkiss (2006) for more information on the concept of debt seniority. We are interested in the following research questions. (1) What are the optimal production and financing decisions for the supplier and the retailer under different debt seniority arrangements? (2) Will direct financing improve the supply chain's efficiency? (3) Are there cases in which the retailer will not offer direct financing or prefer not to have debt seniority? (4) How does the bank information asymmetry affect the supply chain's performance?

We use mathematical models to answer the above questions. The analysis leads to the following insights.

1. The supplier's borrowing strategy depends on whether the retailer requires debt seniority or not. When the retailer demands seniority, the supplier borrows from either the retailer or the bank (but never both), whereas when the retailer does not require seniority, the supplier offers seniority to the bank, and it may simultaneously borrow from both.

2. We find that in some cases, especially when the supplier's working capital is low,

the retailer will not require debt seniority. In those cases, debt seniority will pass to the bank because that will lower the supplier's borrowing costs.

3. Although our model and its analysis are different from the standard trade-credit models, some of the insights generated by the two types of models are similar. The following is an example. The retailer, by choosing the interest rate, makes direct financing attractive for the supplier and the combined profits of the supplier and the retailer increase as a result of the direct-financing contract, thereby improving the supply chain's efficiency.

4. The availability of bank financing may cause over-production (above the *first-best* production level), lowering supply-chain efficiency, and for some parameter values it also may result in the highest possible supply-chain efficiency. The first-best production level is the amount at which the supply chain's expected total profit is maximized.

5. The bank information asymmetry implies a higher bank-loan rate, which makes bank financing less attractive to the supplier, and the retailer-direct financing more attractive. The retailer is able to charge a higher interest rate, increasing its profit, and lowering the supplier's profit.

Our models also help explain the reasons behind the above insights. For example, over-production occurs when (1) the supplier has relatively little to lose by over producing, and (2) the retailer competes with the bank by offering a low-interest loan.

We also formulate a model in which the retailer sets credit limits in addition to the interest rate. We show that credit limits will make over production less likely, and provide the retailer with another lever that it can use to influence the supplier to produce the first-best quantity. Next, we take academic license and consider a model in which the retailer chooses an optimal commission rate and an optimal direct-financing contract. We are not aware of any situation in which this happens in reality, but the model serves to satisfy academic curiosity. We find that an optimal strategy for the retailer in that case is to set interest rate to zero and recoup its financing cost by appropriately adjusting the commission. Moreover, in such cases, the supplier chooses only the retailer-direct financing, and it does not overproduce. Overall, the observations from the two model extensions mentioned above are as one would expect. By giving additional power to the retailer who now sets two parameters, the contract terms ensure that the retailer avoids sub-optimal cases that result from excessive risk shifting by the supplier.

The remainder of this chapter is organized as follows. We review the related literature in Section 2.2, and formulate models and justify assumptions in Section 2.3. We analyze the equilibrium contact for the supplier and the retailer in Sections 2.4 and 2.5 respectively, and study the effect of information asymmetry in Section 2.6. Extensions and insights are presented in Section 2.7 and we conclude in Section 2.8. All proofs are in the appendix.

2.2 Literature Review

This chapter lies at the interface of Operations Research/Management (OR/M) and finance, and therefore is related to both these bodies of literatures. In this section, we summarize, compare and contrast previous works with our efforts.

Giannetti et al. (2011) and Klapper et al. (2011) contain excellent reviews of theoretical and empirical finance papers. These papers explain the prevalence of trade credit, which typically flows from the supplier (seller) to the retailer (buyer), on the basis of certain advantages that come from direct financing. For example, (1) the supplier has informational advantage over third-party lenders, (2) the supplier is in a better position to prevent moral hazard, (3) the supplier realizes higher collateral liquidation value, (4) the supplier may use credits terms as a means of price discrimination, prevented in the US by the Robinson-Patman act, and (5) the supplier credit is tantamount to an implicit warranty assuring buyers of the product quality.

Papers in the finance literature often ignore operational decisions. One reason for this might be that Modigliani and Miller (1958) in their seminar work show that in a perfect capital market, the value of a firm is independent of its financial structure. Therefore, its financial and operational decisions may be made independently. However, OR/M papers argue that market imperfection is common. The financial arrangements also matter because the supplier and the retailer act as independent agents, giving rise to agency costs (see e.g. Barnea et al. 1981). Suppliers attempt to mitigate agency costs via the terms of the trade-credit contract. Specifically, direct financing may be viewed as an incentive that the suppliers offer to retailers, contingent upon their operational decisions.

OR/M literature on trade credit has two main streams: deterministic (EOQ) and stochastic demand models. The latter is further divided into two types of models – multiperiod and newsvendor models. This chapter belongs to the literature on the newsvendor setting. Therefore, we provide only a brief summary of works dealing with non-newsvendor settings.

In the EOQ setting, OR/M papers consider the firm's problem of minimizing the average total borrowing, inventory, and backorder costs. The firm always repays the creditor in full and there is no need to consider the risk of insolvency. Therefore, as expected, this literature shows that the terms of the trade credit do not affect optimal order quantities (Goyal 1985). Other examples of EOQ models with trade credit include Haley and Higgins (1973) and Aggarwal and Jaggi (1995).

Babich and Sobel (2004), Chao et al. (2008), and Gupta and Wang (2009) study stochastic multiperiod problems. These papers show that the retailers' inventory decisions are affected by the financial arrangements. The authors only consider decisions of one firm and parameters such as the interest rate for borrowed capital are assumed exogenous. Our work differs from this stream because it focuses on the interactions between the retailer, the supplier, and the bank. Interest rates in our models are either competitively priced or decision variables.

Turning next to those papers that model direct financing in a newsvendor setting, we find several related works in the literature. Xu and Birge (2004) study the interaction between the financial and the operational choices made by the retailer in the presence of bankruptcy costs and taxes. Kouvelis and Zhao (2011) extend this work by considering general bankruptcy costs and the retailer's wealth composition (mix of working capital and collateral) on the retailer's ordering decisions. Dada and Hu (2008) derive a nonlinear loan schedule that coordinates the channel consisting of a profit maximizing bank and a newsvendor-like retailer who faces financial constraints. Kouvelis and Zhao (2012) consider both the supplier's and the retailer's problems in a Stackelberg game setting within a perfect capital market. The authors compare trade credit with bank financing and conclude that trade credit should be preferred by the supply-chain partners. They also show that the trade-credit contract can improve the supply chain's efficiency. Caldentey and Chen (2009) model a setting similar to Kouvelis and Zhao (2012), but the supplier in their model allows the retailer to delay paying a specified fraction of the procurement cost until demand is realized. Yang and Birge (2013) study two-terms trade credit with available bank financing and characterize the optimal mix of capital including cash, trade credit and bank loan.

All of the above-mentioned papers model wholesale-price-only contracts with the supplier offering credit to the buyer. In contrast, we consider retailer-direct financing under consignment. A key feature of the consignment-selling environment is that both the supplier's and the retailer's profits are affected by realized demand at every level of demand. In contrast, in traditional contracts, the supplier's profit remains unaffected by demand after the point at which it recovers its loan in full. Also, the consignment rate is determined by the product category, independent of the financial terms. In contrast, in the standard trade-credit setting, the supplier may simultaneously choose the wholesale price as well as the financial terms. These differences lead to significant differences in analysis and conclusions, as summarized next.

Kouvelis and Zhao (2012) show that the optimal production quantity in a trade-credit contract is always strictly smaller than the first-best, assuming that the wholesale price is chosen optimally by the supplier. In contrast, because the commission rate in our model is independent of the financial terms, the supplier may over-produce and there are also cases in which the first-best is realized. These differences do not occur when the commission rate and the finance terms are simultaneously optimized by the retailer. In those cases, similar to Kouvelis and Zhao, the optimal trade-credit interest rate is zero (the risk-free rate), and the production quantity is always smaller than the first-best. Kouvelis and Zhao assume that borrowers use either the bank or the supplier financing but not both. In contrast, the supplier may borrow from both sources in our setting and an important feature of our model is the effect of debt seniority on optimal production and financing decisions. We show that the retailer's debt seniority requirement determines whether the supplier takes only one source of financing or both.

Yang and Birge (2013) extend Kouvelis and Zhao (2012) by allowing the retailer to borrow from both the bank and the supplier, but they assume that the bank always has seniority. They find that in some cases the borrower may use both bank and supplier financing and argue that this happens because of the bankruptcy costs. The phenomenon of taking both bank and direct financing can also occur in our setting, but for a different reason — because the consignment rate is not chosen optimally by the retailer. We also find that in some cases, the retailer will require debt seniority, but not in others. Its actions serve to limit the degree of risk shifting achieved via the trade financing contract.

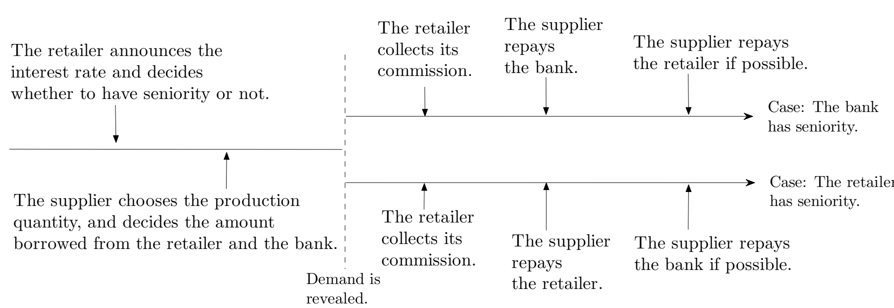
Yang and Birge (2011) study debt seniority, but they focus on net-term trade credit contracts (i.e. contracts with zero interest rate) and credit limits, whereas in our work the retailer chooses the interest rate to maximize its profit. Still, some results in theirs and our work are similar. For example, Yang and Birge conclude that when the supplier has seniority and the wholesale price is exogenous, it offers either zero credit limit or a credit limit such that the retailer only uses supplier financing. An analogous result is observed in our model when the retailer has seniority. Similarly, when the bank has seniority, the retailer may use both trade credit and bank financing in some cases. We also observe this phenomenon. The key difference is that the retailer in our model charges different interest rates to different suppliers, which is consistent with what one observes in practice. Interest rate is not modeled by Yang and Birge. We also consider the role of bank information asymmetry. This issue is particularly relevant for small-business suppliers, but not considered in previous papers.

There are some other related works that study supply-chain efficiency resulting from supplier-retailer contracts without financial arrangements. For example, Wang et al. (2004) study the consignment contract with revenue sharing and how the retail price elasticity affects the channel performance. Their consignment contracts with revenue sharing are similar to the contracts studied in this chapter. Cachon and Lariviere (2005) also study the revenue-sharing contracts under a wholesale price environment and show that these contracts can coordinate the supply chain under several different settings. Gerchak and Wang (2004) compare revenue-sharing and wholesale-price contracts in assembly systems, and develop a revenue-plus-surplus-subsidy contract to coordinate the supply chain.

2.3 Model Formulation

The small-business supplier (denoted by S) sells a single product through a retailer's web platform. The retailer (denoted by R) charges a commission rate $\alpha \in (0,1)$ per unit of sales revenue. Our setting is that of a newsvendor model with the following key parameters: unit production cost c, average retail price r (because in practice the retail price may change several times during the selling season, r is the average of anticipated prices.), demand D, the supplier's working capital κ, production quantity q, interest rate ρ, expected profit π, and superscripts "rs" and "bs" for the retailer-having-seniority and bank-having-seniority environments. The sequence of events is shown in Figure 2.1 and key notation are summarized in Table 2.1. D is non-negative with cumulative distribution function $F(\cdot)$, density $f(\cdot)$, and failure rate $z(\cdot)$. Unmet demand is lost and the leftover inventory has zero value. We use y to denote demand thresholds that cause the supplier to become insolvent.

Figure 2.1: The Sequence of Events



The expected total profit in the centralized chain equals $\pi_0(q) = rE[\min\{D,q\}] - cq$. The optimal production quantity is $q_t^* = \bar{F}^{-1}(c/r)$, which is also referred to as the *first-best* production quantity. We say that the supply chain is coordinated when the supplier chooses to produce q_t^*. Supply chain efficiency is measured by the ratio $[\pi^S(q) + \pi^R(q)]/\pi_0(q_t^*)$, where q is the amount that the supplier produces under an arbitrary contract.

The retailer offers a loan to the supplier on a take-it-or-leave-it basis, selecting the interest rate on the loan and the debt seniority. Bank financing is also available to the supplier. The supplier can choose to take direct financing or bank financing or both.

Table 2.1: Summary of Key Notation

Subscripts & Superscripts	
rs	the retailer has seniority
bs	the bank has seniority
S	the supplier
R	the retailer
opt	the decision under an optimal contract
Parameters	
α	the commission rate
c	the unit production cost
r	the unit retail price
κ	the supplier's working capital
D	the demand, with CDF $F(\cdot)$, CCDF $\bar{F}(\cdot)$, PDF $f(\cdot)$, and failure rate $z(\cdot)$
D_b	the bank's estimate of demand, with CDF $F_b(\cdot)$, CCDF $\bar{F}_b(\cdot)$
ρ_b	the competitively priced bank loan interest rate, as a function of B_d, B_b, q, ρ_d and seniority
Decision Variables	
ρ_d	the direct financing interest rate
B_d	the amount borrowed from the retailer
B_b	the amount borrowed from the bank
q	the production quantity
Variables Determined by the Decision Variables	
q_d	the supplier's optimal production quantity with direct-only financing
q_b	the supplier's optimal production quantity with bank-only financing
q_{un}	the supplier's optimal production quantity without financial constraint
q_t^*	the first-best production quantity
y_d	the insolvency threshold for the direct financing
y_b	the insolvency threshold for the bank financing

It decides the production quantity and the amounts it will borrow from the bank and the retailer. Debt seniority matters when the supplier borrows both from the bank and the retailer. In such cases, if the retailer requires seniority, then the bank loan must be junior (or the supplier must borrow only from the bank). If the retailer does not claim debt seniority, it will always be profitable for the supplier to offer seniority to the bank, which we will show later in this chapter.

We make two types of assumptions when developing mathematical models. The first set of assumptions are purely technical in nature, motivated by the need to have tractable models. Specifically, under this umbrella, we assume risk-neutral retailer and supplier, no private information except bank information asymmetry described in Section 2.1, zero risk-free interest rate, and increasing failure rate (IFR) demand distribution. These assumptions are common in newsvendor models, e.g. see Caldentey and Chen (2009) and Lariviere and Porteus (2001). Many common distributions are IFR, for example, uniform, Erlang, normal, and truncated normal distributions (see, e.g., Porteus 2002a for details). For some of our results, we also require the failure rate to be increasing convex. This property is also satisfied by many common distributions, including the uniform, the power, the exponential, the normal, and the Weibull (with shape parameter greater than 1) distributions (Zhou and Groenevelt 2008).

An important technical assumption is that there are no bankruptcy cost and tax. Similar assumption is also made in Kouvelis and Zhao (2012). We explain next why this assumption is justified in our model setting. First, web-based platform owners collect sales revenue, and return a portion of it to the supplier after collecting their commission, the loan interest (if any), the principal, and the cost of any additional services (e.g. shipping). We assume that the leftover products have no value. Thus, the retailer learns the liquidated values of assets (total revenue net of retailer costs) without incurring any assessment or court fees. (The underlying efficiency of the direct-financing mechanism, we conjecture, is a reason why the retailers may be more willing to offer direct loans.) Second, retailer-direct lending takes place in a small-business environment. The vast majority of small businesses are privately held with very small amount of equity. The retailer is unlikely to collect anything upon forcing liquidation. Its ability to collect on the loan comes from the strength of its relationship with the supplier. To round out the chapter, we discuss how the inclusion of bankruptcy costs may affect our results in Section 2.8.

Finally, for some of our results, we require that the commission rate α be no greater than $(1+\frac{c}{r})/2$. Because $c < r < \infty$, the required inequality is always satisfied when $\alpha \leq 1/2$. There is ample evidence that α does not exceed 1/2 in practice. For example, Amazon charges commission rates that vary from 6% to 45% (Amazon.com 2014). The most common commission rates are in the 8-15% range and there is only one category (accessories for Amazon devices) for which the commission rate is higher than 25%.

The second set of assumptions are motivated in part by the problem setting – i.e. the need to have a high-fidelity model of reality, and to a lesser extent by the desire for technical elegance. We list each of these assumptions next and provide appropriate justification.

1. The bank always offers a loan, and the loan is priced such that the bank's expected payoff equals the borrowed amount, i.e. the financial market is perfectly competitive. This assumption is reasonable in mature financial markets.

2. The bank knows less about the supplier than the retailer. Specifically, for the supplier that is deemed worthy of direct financing by the retailer, the bank's estimate of the demand D_b, with distribution $F_b(\cdot)$, is such that $D_b \leq_{st} D$, i.e. $F(u) \leq F_b(u)$ for all u, or equivalently $E[h(D_b)] \leq E[h(D)]$ for all increasing functions $h(\cdot)$ for which the expectations exist. Reasons why this assumption is closely related to reality are explained in Paragraph 4 of Section 2.1. We also assume that $F_b(\cdot)$ can be inferred by both the supplier and the retailer from offered bank-financing loan rates for different loan amounts and debt-seniority arrangements.

3. The consignment rate α is exogenous and does not depend on the financial terms. We justify this assumption as follows. First, retailers such as Amazon (Amazon.com 2014) routinely set the same consignment rate for groups of suppliers, based solely on their product attributes and not on other supplier characteristics. Second, the consignment rate for each supplier may remain unchanged for a long time period, whereas direct financing may be needed only in high-demand periods (e.g. Christmas season). That is, the consignment rates do not change with the direct-financing terms. Third, our analysis in Section 2.7.3 shows that if the retailer were to optimize consignment rate simultaneously with interest rate, it

would always offer direct financing with zero interest rate and the supplier would always take only direct financing. This is not true in practice. For example, Amazon charges an interest of 13.9% from most of the suppliers and many suppliers take both direct financing and bank financing, which will occur only when the consignment rate is fixed.

We present two models in the remainder of this section, which form the building blocks of our subsequent analysis. In the first model, the retailer has debt seniority, and in the second model, the bank has seniority. Before we present the two building blocks, consider the base case in which the supplier has no source of financing. With working capital κ, if the supplier produces $q \leq \kappa/c$, then its revenue equals $(1-\alpha)r \min\{D, q\}$ and the retailer's commission equals $\alpha r \min\{D, q\}$. The expected profits of the supplier and the retailer are $\pi_S^0(q) = (1-\alpha)rE[\min\{D, q\}] - cq$, subject to $q \leq \kappa/c$, and $\pi_R^0(q) = \alpha r E[\min\{D, q\}]$. The supplier's objective function is concave in q (we omit the proof in the interest of brevity) and it faces a linear constraint. Therefore, the optimal production quantity is $q_0 = \max\{q_{un}, \kappa/c\}$, where $q_{un} = \bar{F}^{-1}(c/(1-\alpha)r) \leq q_t^*$ denotes the unconstrained solution. The working capital threshold below which the supplier produces κ/c equals $\bar{\kappa}(\alpha) = c\bar{F}^{-1}(\frac{c}{(1-\alpha)r})$. Note that the supplier will not take any financing if $\kappa \geq \bar{\kappa}(\alpha)$. We focus in this chapter on a financially constrained supplier, i.e. we assume $\kappa < \bar{\kappa}(\alpha)$.

2.3.1 The Retailer Has Seniority

Suppose the retailer offers a loan with an interest rate ρ_d and requires debt seniority. The supplier accepts this offer and borrows B_d and B_b, respectively, from the retailer and the bank ("B" is mnemonic for borrowed amount). Recall that the bank is a passive lender. It does not require seniority but adjusts its interest rate according to its seniority status.

Because the supplier may borrow only the amount it needs to realize its desired production quantity, the corresponding production-quantity decision is $q = (B_b + B_d + \kappa)/c$. The supplier can fully repay $(1+\rho_d)B_d$ to the retailer if and only if $(1-\alpha)r \min\{D, q\} \geq (1+\rho_d)B_d$, i.e. if $D \geq y_d^{rs} \doteq (1+\rho_d)B_d/[(1-\alpha)r]$. Conversely, if $D < y_d^{rs}$, then the supplier becomes insolvent and the retailer collects the entire sales revenue, which equals

$r \min\{D, q\}$. Hereafter, y_d^{rs} is referred to as the insolvency threshold for the direct-financed portion of the supplier's credit. Similarly, the insolvency threshold for the bank-financed portion is $y_b^{rs} = y_d^{rs} + (1+\rho_b)B_b/[(1-\alpha)r]$. Note that $y_b^{rs} \geq y_d^{rs}$ follows from the fact that the retailer has seniority.

The supplier's and the retailer's expected profit functions are:

$$\pi_S^{rs}(q, B_b, B_d) = (1-\alpha)rE[\min\{D, q\}; D > y_b^{rs}] - [(1+\rho_d)B_d + (1+\rho_b^{rs})B_b]\bar{F}(y_b^{rs}) - \kappa. \tag{2.1}$$

$$\pi_R^{rs}(q, B_b, B_d) = \alpha r E[\min\{D, q\}] + (1-\alpha)rE[\min\{D, q\}; D \leq y_d^{rs}] + (1+\rho_d)B_d\bar{F}(y_d^{rs}) - B_d. \tag{2.2}$$

In (2.1), $E[\min\{D, q\}; D > y_b^{rs}]$ is the expected value of $\min\{D, q\}$, when demand is sufficiently high that both the retailer and the bank are repaid their respective loans in full. Similarly, in (2.2), $E[\min\{D, q\}; D \leq y_d^{rs}]$ is the expected value of $\min\{D, q\}$ when neither receives full repayment. The bank's interest rate ρ_b^{rs} satisfies the following competitive-pricing equilibrium:

$$E[(1-\alpha)r \min\{D_b, q\} - (1+\rho_d)B_d, y_d^{rs} \leq D_b \leq y_b^{rs}] + (1+\rho_b^{rs})B_b\bar{F}_b(y_b^{rs}) = B_b. \tag{2.3}$$

The competitive-pricing equilibrium in (2.3) means that the bank recovers exactly its loan amount in expectation.

The supplier chooses (q, B_b, B_d) to maximize its profit π_S^{rs} and the retailer, likewise, chooses ρ_d to maximize π_R^{rs}. In writing (2.1) and (2.2), we assume that the supplier will accept the terms offered by the retailer. In reality, the retailer needs to make an offer such that the supplier finds it attractive to do so. Details are presented in Section 2.5 that analyzes the retailer's financing decisions.

Two special cases are worth noting. If $B_b = 0$ and $B_d > 0$, then that means the supplier borrows only from the retailer. Similarly, if $B_d = 0$ and $B_b > 0$, then the supplier borrows only from the bank. The former is referred to as the *direct-only financing* scenario, and the latter as the *bank-only financing* scenario. In either case, debt seniority is irrelevant.

2.3.2 The Bank Has Seniority

If the retailer does not require debt seniority, the supplier will offer seniority to the bank. It is straightforward to argue that this is an optimal action for the supplier because by doing so, the supplier lowers the bankruptcy threshold for bank financing, which lowers the interest rate charged by the bank and increases the supplier's profit. Suppose the supplier borrows B_d and B_b from the retailer and the bank, respectively, with bank as the senior lender. Similar to Section 2.3.1, the production quantity is $q = (B_b + B_d + \kappa)/c$, and the insolvency thresholds are: $y_b^{bs} = (1 + \rho_b^{bs})B_b/[(1 - \alpha)r]$, and $y_d^{bs} = y_b^{bs} + (1 + \rho_d)B_d/[(1 - \alpha)r]$. In this case, $y_d^{bs} \geq y_b^{bs}$ follows from the fact that the bank has seniority. The supplier's and the retailer's expected profit functions are

$$\pi_S^{bs}(q, B_b, B_d) = (1 - \alpha)rE[\min\{D, q\}, D_b > y_d^{bs}] - [(1 + \rho_d)B_d + (1 + \rho_b^{bs})B_b]\bar{F}(y_d^{bs}) - \kappa. \quad (2.4)$$

$$\begin{aligned}\pi_R^{bs}(q, B_b, B_d) =& \alpha r E[\min\{D, q\}] + E[(1 - \alpha)r \min\{D, q\} - (1 + \rho_b^{bs})B_b, y_b^{bs} \leq D \leq y_d^{bs}] \\ &+ [(1 + \rho_d)B_d]\bar{F}(y_d^{bs}) - B_d. \end{aligned} \quad (2.5)$$

The interest rate ρ_b^{bs} satisfies

$$E[(1 - \alpha)r \min\{D_b, q\}, D_b \leq y_b^{bs}] + (1 + \rho_b^{bs})B_b\bar{F}_b(y_b^{bs}) = B_b. \quad (2.6)$$

The supplier chooses (q, B_b, B_d) to maximize π_S^{bs} and the retailer chooses ρ_d to maximize π_R^{bs}. Note that $\pi_S^{bs}(q, B_b = 0, B_d) = \pi_S^{rs}(q, B_b = 0, B_d)$ and $\pi_S^{bs}(q, B_b, B_d = 0) = \pi_S^{rs}(q, B_b, B_d = 0)$, i.e. seniority does not matter when the supplier takes either direct-only or bank-only financing.

2.4 Supplier's Production and Financing Decisions

In this section, we analyze the supplier's production and financing decisions when it receives a particular take-it-or-leave-it direct-financing offer from the retailer – i.e. an interest rate ρ_d and the corresponding debt-seniority requirement. Note that $(1-\alpha)r - (1 + \rho_d)c \geq 0$ must hold for the supplier to consider direct financing at rate ρ_d. This inequality is assumed to hold throughout this section. An overview of our approach is as follows.

We first investigate the supplier's production and financing decisions in Lemma 2.1 and 2.2 if it considered only one source of financing. Note that these are viable options for the supplier. We then show that borrowing from a single source is an optimal choice for the supplier when both financing sources are available, the retailer requires seniority, and the supplier accepts it. In contrast, it may be better for the supplier to simultaneously borrow from both sources when the retailer does not require seniority, in which case the seniority goes to the bank.

Lemma 2.1 (Direct-only Financing)**.** *Given an interest rate ρ_d, the supplier's optimal production quantity $q_d(\rho_d)$ can be obtained as follows:*

1. *if $c\bar{F}^{-1}(\frac{(1+\rho_d)c}{(1-\alpha)r}) \leq \kappa < c\bar{F}^{-1}(\frac{c}{(1-\alpha)r})$, then $q_d = \frac{\kappa}{c}$,*
2. *if $0 \leq \kappa \leq c\bar{F}^{-1}(\frac{(1+\rho_d)c}{(1-\alpha)r})$, then q_d satisfies*

$$\bar{F}(q_d) = \frac{(1+\rho_d)c}{(1-\alpha)r}\bar{F}(\frac{(1+\rho_d)c}{(1-\alpha)r}q_d - \frac{(1+\rho_d)\kappa}{(1-\alpha)r}). \tag{2.7}$$

Lemma 2.1 divides the supplier's production decision into two regions depending on its working capital κ and the offered rate ρ_d. In the first region, the supplier needs financing, but the interest rate is high, with the result that it uses only its own working capital. In the second region, the supplier finds the retailer's offer attractive and it borrows $(cq_d - \kappa)$ from the retailer, where q_d comes from solving Equation (2.7). The supplier's production decision is affected by the availability of direct financing only in this region. In fact, the supplier's production decision in the second region has an intuitively-appealing interpretation. The supplier chooses its production quantity such that its expected marginal revenue from selling the last unit, i.e. $(1-\alpha)r\bar{F}(q)$, equals its expected marginal borrowing cost, which is $(1+\rho_d)c\bar{F}(\frac{(1+\rho_d)c}{(1-\alpha)r}q - \frac{(1+\rho_d)\kappa}{(1-\alpha)r})$.

Lemma 2.2 (Bank-only Financing)**.** *The supplier's optimal production quantity when taking bank-only financing equals* $\max\{q_b, \kappa/c\}$, *where q_b and y_b satisfy $(1-\alpha)r\bar{F}(q_b) = c\bar{F}(y_b)/\bar{F}_b(y_b)$ and $(1-\alpha)r\int_0^{y_b}\bar{F}_b(x)dx = cq_b - \kappa$.*

Similar to Lemma 2.1, the supplier may use only its own working capital when the bank's interest rate is high. High interest rate may be a consequence of the information asymmetry, i.e. the assumption that $\bar{F}(y_b) \geq \bar{F}_b(y_b)$. If the bank and the supplier have the same information, then $q_b = q_{un}$ and Lemma 2.2 implies that the supplier's expected profit is unaffected by its financial constraint, consistent with Modigliani and Miller (1958).

Next, we study the supplier's decision when it must choose among either direct-only or bank-only or both sources of financing. In the first part of our analysis, we assume that the retailer requires seniority, i.e. if direct financing is taken, then the retailer must have seniority. The supplier's optimal production quantity within such a setting is obtained in Proposition 2.1.

Proposition 2.1 (The Retailer Has Seniority.)**.** *Given direct financing interest rate ρ_d and competitively-priced bank financing, the supplier's optimal decision is one of the following:*

1. *If $\kappa \geq \max\{cq_b, c\bar{F}^{-1}(\frac{(1+\rho_d)c}{(1-\alpha)r})\}$, then the supplier produces κ/c.*
2. *If $\kappa < \max\{cq_b, c\bar{F}^{-1}(\frac{(1+\rho_d)c}{(1-\alpha)r})\}$, then the supplier*
   (a) *either borrows $(cq_b - \kappa)$ only from the bank, and produces q_b,*
   (b) *or takes direct-only financing, and produces $q_d : \bar{F}(q_d) = \frac{(1+\rho_d)c}{(1-\alpha)r}\bar{F}(\frac{(1+\rho_d)c}{(1-\alpha)r}q_d - \frac{(1+\rho_d)\kappa}{(1-\alpha)r})$.*

The significance of Proposition 2.1 is that it proves that the supplier will choose either direct-only financing or bank-only financing. The intuition behind this result is as follows. Suppose that the supplier takes loans from both sources. If $\rho_d < \rho_b$, then the supplier will lower its finance charges by taking only direct financing. In contrast, if $\rho_d \geq \rho_b$, then for the same production quantity, ρ_b will decrease in B_b. That is, the supplier will reduce its finance charges by increasing the borrowed amount from the bank, and lowering the borrowed amount from the retailer. Because the retailer has seniority, a smaller loan from the retailer increases expected payment to the bank, and lowers the bank's interest rate. Put together, these arguments imply that *taking loans from both sources cannot be optimal.*

In Corollary 2.1, we argue that both $q_d(\rho_d)$ and $\pi_S^{rs}(q_d(\rho_d), B_b = 0, B_d = cq_d - \kappa)$ decrease in ρ_d. That is, if the retailer charges a higher interest rate, then that increases the supplier's borrowing cost. Consequently, it borrows less, produces less, and earns a smaller profit. It also follows that there must exist a maximum production quantity, which is achieved when the retailer sets $\rho_d = 0$. We denote the maximum production quantity by q_{dmax}.

Corollary 2.1. *Production quantity $q_d(\rho_d)$ and profit $\pi_S^{rs}(q_d(\rho_d), B_b = 0, B_d = cq_d - \kappa)$ decrease in ρ_d. The maximum quantity q_{dmax} under direct financing is obtained from the implicit equation $\bar{F}(q_{dmax}) = \frac{c}{(1-\alpha)r}\bar{F}(\frac{c}{(1-\alpha)r}q_{dmax} - \frac{\kappa}{(1-\alpha)r})$. Furthermore, $q_{dmax} \geq q_{un}$.*

Next, we consider the scenario in which the retailer does not demand debt seniority and consequently the supplier offers seniority to the bank if it takes both bank and direct financing. In this scenario, the supplier's optimal decisions are presented in Proposition 2.2.

Proposition 2.2 (The Bank Has Seniority.). *Given direct-financing interest rate ρ_d and that the retailer does not require seniority, the supplier's optimal production decision depends on the magnitude of ρ_d relative to a threshold interest rate $\hat{\rho}_{db} = 1/\bar{F}_b(y_b) - 1$, where y_b is obtained by solving $(1-\alpha)r\int_0^{y_b}\bar{F}_b(x)dx = cq_b - \kappa$.*

1. *If $\rho_d = 0$, then $B_b^* = 0$ and B_d^* satisfies $\bar{F}(\frac{B_d^*+\kappa}{c}) = \frac{(1+\rho_d)c}{(1-\alpha)r}\bar{F}(\frac{(1+\rho_d)}{(1-\alpha)r}B_d^*)$,*

2. *if $0 < \rho_d \leq \hat{\rho}_{db}$, then $B_b^* = (1-\alpha)r\int_{1/(1+\rho_d)}^{1}\frac{x}{f_b(x)}dx$ and B_d^* satisfies*

$$\bar{F}(\frac{B_b^* + B_d^* + \kappa}{c}) = \frac{(1+\rho_d)c}{(1-\alpha)r}\bar{F}(\frac{(1+\rho_d)}{(1-\alpha)r}B_d^* + \bar{F}_b^{-1}(\frac{1}{1+\rho_d})), \tag{2.8}$$

   with $\rho_b \leq \rho_d$ and $B_b^* \leq cq_b - \kappa$, and

3. *if $\rho_d > \hat{\rho}_{db}$, then $B_b^* = cq_b - \kappa$ and $B_d^* = 0$.*

Proposition 2.2 divides the action space for the supplier in three regions – direct-only financing (Case 1, $\rho_d = 0$), both bank and direct financing (Case 2, $\rho_d \in (0, \hat{\rho}_{db}]$), and bank-only financing (Case 3, $\rho_d > \hat{\rho}_{db}$) – depending on the offered value of ρ_d. Next, we explain the existence of Case 2 on an intuitive level. The other two cases are straightforward.

When the bank has seniority, its loan is cheaper so long as the supplier does not produce more than q_b because the supplier's marginal profit is negative when $q > q_b$. However, the retailer as the second creditor may be willing to offer a low interest rate in some cases to induce the supplier to produce more than q_b. Recall that the retailer can use the commission to offset the higher cost of its loan and make a higher profit in expectation. In such cases, the supplier takes both the bank and the direct financing offers because it also benefits from shifting risk to the retailer and producing more. This phenomenon does not happen when the retailer had seniority because the bank does not receive commission and its interest rate is never sufficiently low to increase production beyond q_d in Proposition 2.1.

2.5 Retailer's Financial Decision under Symmetric Information

In this section, we consider the retailer's choice of the terms of its loan offer to the supplier under the symmetric information ($D \doteq D_b$, i.e. D and D_b equal in distribution) scenario. The effect of information asymmetry is analyzed in Section 2.6. There are two reasons why we focus first on the symmetric-information scenario. First, we focus in this section on the effect of simultaneous availability of the bank and the retailer financing without the compounding effect of information asymmetry. Second, our model is mathematically more tractable under this scenario. Analogous to Section 2.4, we present two cases – Section 2.5.1 with retailer seniority, and Section 2.5.2 with bank seniority. However, the superscripts "rs" and "bs" are omitted in these sections to simplify notation.

2.5.1 The Retailer Has Seniority

We consider the retailer's choice of interest rate ρ_d when it has seniority. Recall that the supplier is financially constrained, i.e. $\kappa < \bar{\kappa}(\alpha)$. In Proposition 2.1, we showed that the supplier should take either bank or direct financing, but never both. In order to characterize the optimal decisions for the retailer, we rewrite the retailer's profit function (2.2) in a different form below when the supplier takes direct-only financing:

$$\pi_R^d(q_d, \rho_d)|_{\kappa \leq cq_d} = \alpha r E[\min\{D, q_d\}] + (1-\alpha) r E[\min\{D, y_d\}] - (cq_d - \kappa), \qquad (2.9)$$

where y_d, the supplier's insolvency threshold, can be obtained from the fact that $y_d \bar{F}(y_d) = (q_d - \kappa/c)\bar{F}(q_d)$ must hold for an optimal q_d — see the optimality conditions in Lemma 2.1. The retailer's optimization problem of choosing $\rho_d^* = argmax_{\rho_d} \pi_R^d(q_d(\rho_d), \rho_d)$ subject to $y_d \bar{F}(y_d) = (q_d - \kappa/c)\bar{F}(q_d)$ and $y_d \leq q_d$, can be transformed such that $q_d(\rho_d)$ is the retailer's decision variable instead of ρ_d. With a slight abuse of notation, we replace $q_d(\rho_d)$ by q_d. Also, we use $\pi_S^d(q_d)$ and $\pi_R^d(q_d)$ to denote the corresponding profit functions. The above transformation is made possible by the fact that the retailer's choice of ρ_d causes the supplier to choose production quantities described in Lemma 2.1, and from Corollary 2.1, $q_d(\rho_d)$ is monotone (decreasing) in ρ_d. Next, in Section 2.5.1, we characterize the retailer's optimal direct-financing contract, and in Section

2.5.1 we investigate the supply chain's efficiency and the size of the retailer's profit. In both cases, the retailer has seniority and the supplier does not borrow simultaneously from both lenders.

The Optimal Direct Financing Contract:

The retailer's optimal decisions are obtained in Theorem 2.1, before which we present intermediate results in two lemmas to improve expositional clarity.

Lemma 2.3. *If the failure rate $z(\cdot)$ is convexly increasing and $\alpha \leq (1 + c/r)/2$, then $\pi_R^d(q_d)$ is unimodal in $q_d \in [0, q_{dmax}]$. If the supplier takes direct-only financing, then the retailer's optimal decision q_{opt}^d is either a solution of $\frac{1-(q_d-\kappa/c)z(q_d)}{1-y_d z(y_d)} - (\frac{c}{(1-\alpha)r\bar{F}(q_d)} - \frac{\alpha}{1-\alpha}) = 0$, if the solution is smaller than q_{dmax}, or $q_{opt}^d = q_{dmax}$. Moreover, $\kappa/c < q_{opt}^d < q_t^*$.*

Lemma 2.3 shows that the retailer can make more profit by offering direct financing to induce the supplier to produce q_{opt}^d if bank financing is not available. In such cases, both the supplier and the retailer make more profit as compared to the supplier using only its available working capital.

However, the supplier will choose direct-only financing over bank-only financing only if the former is more profitable. That is, the retailer may face competition from the bank. The flip side of this argument is that the retailer will make direct financing more attractive only up to a point – the point at which its profit is equal to the profit it will make when the supplier borrows only from the bank. In other words, the availability of bank financing gives rise to reservation profits for both the supplier and the retailer. Let $\pi_S^b(q_b)$ and $\pi_R^b(q_b)$ respectively denote the supplier's and the retailer's optimal profits if the supplier takes bank-only financing. The remainder of our analysis utilizes two critical quantities — q_{ds} and q_{dr} — which are defined next. The existence of q_{ds} and q_{dr} is argued in Lemma 2.4.

Definition 2.1. Production quantity q_{ds} is a threshold at which the supplier makes the same profit under direct-only and bank-only financing options. Specifically, $q_{ds} = \{q_d : \pi_S^d(q_d) = \pi_S^b(q_b)\}$.

Definition 2.2. Production quantity q_{dr} is the maximum quantity that the retailer will induce the supplier to produce in the "rs" environment.

Lemma 2.4. *In the "rs" environment,*

1. *there exists a threshold* $q_{ds} \in [q_b, q_{dmax}]$ *such that* $\pi_S^d(q_d) \geq \pi_S^b(q_b)$ *when* $q_d \geq q_{ds}$ *and* $\pi_S^d(q_d) \leq \pi_S^b(q_b)$ *when* $q_d \leq q_{ds}$*;*

2. *there exists a threshold* $q_{dr} \in [q_{opt}^d, q_{dmax}]$ *such that* $\pi_R^d(q_d) \geq \pi_S^b(q_b)$ *when* $q_d \in [q_{opt}^d, q_{dr}]$ *and* $\pi_R^d(q_d) \leq \pi_R^b(q_b)$ *when* $q_d \in (q_{dr}, q_{dmax}]$*.*

Because $\pi_S^d(q_d)$ increases in q_d (see Corollary 2.1), the supplier makes more profit under direct financing if the retailer induces it to produce more than q_{ds}. Otherwise, direct financing is less preferable to the supplier and it will choose bank financing. Similarly, the retailer will prefer bank financing if it has to induce the supplier to produce more than q_{dr}. If $\pi_R^d(q_{dr}) = \pi_R^b(q_b)$ has no solution in $[q_{opt}^d, q_{dmax}]$, then it means that $\pi_R^d(q_d) \geq \pi_R^b(q_b)$ for every $q_d \in [q_{opt}^d, q_{dmax}]$, i.e. the retailer is more profitable under direct financing. In such cases the largest production quantity is q_{dmax}, which occurs when the interest rate is zero. These arguments lead to Theorem 2.1.

Theorem 2.1. *If the failure rate* $z(\cdot)$ *is convexly increasing and* $\alpha \leq (1 + c/r)/2$*, then there exists a threshold* $\hat{\kappa}_{bd}$ *of* κ *and the optimal direct-financing contract can be characterized as follows.*

1. *When* $\kappa \leq \hat{\kappa}_{bd}$*, the retailer will let the supplier choose bank financing.*

2. *When* $\kappa \geq \hat{\kappa}_{bd}$*, the retailer will offer an interest rate such that the corresponding order quantity is* $q_{opt}^{rs} = \max\{q_{ds}, q_{opt}^d\}$*. The optimal interest rate is obtained by solving* $\bar{F}(q_{opt}^{rs}) = \frac{(1+\rho_d)c}{(1-\alpha)r}\bar{F}(\frac{(1+\rho_d)c}{(1-\alpha)r}q_{opt}^{rs} - \frac{(1+\rho_d)\kappa}{(1-\alpha)r})$ *and the supplier borrows from the retailer.*

The intuition behind Theorem 2.1 is as follows. Because the supplier never borrows from both credit sources, the retailer either offers a low interest rate to induce the supplier to take direct-only financing, or lets the supplier borrow from the bank. When $\kappa \leq \hat{\kappa}_{bd}$, the supplier has little to lose if the demand realization is small, and much to gain if demand realization is large. If the retailer were to offer a loan, it would face a high risk of non-repayment, and the retailer's expected profit could be smaller than that under the bank financing.

At first glance, Theorem 2.1 appears to contradict programs such as Amazon Lending that are designed to provide credit to those suppliers that are severely cash constrained. This is a consequence of our assumption that the bank has the same information as the retailer. We analyze the effect of information asymmetry in Section 2.6. There, we show that as information asymmetry increases, the retailer is more likely to provide a loan to suppliers with small working capital, consistent with the observed practice.

The Supply Chain's Efficiency and the Retailer's Share:

We show in Proposition 2.3 that q_{opt}^{rs} may achieve or even exceed the first-best solution q_t^* in "rs" financing environment.

Proposition 2.3. *There exists a threshold $\hat{\kappa}_t \geq \hat{\kappa}_{bd}$, such that $q_{opt}^{rs} \geq q_t^*$ if $\hat{\kappa}_{bd} < \kappa \leq \hat{\kappa}_t$ and $q_{opt}^{rs} \leq q_t^*$ if $\kappa \geq \hat{\kappa}_t$. Moreover, if $\hat{\kappa}_t > 0$, then $q_{opt}^{rs} = q_t^*$ when $\kappa = \hat{\kappa}_t$.*

Over production happens for two reasons. First, the supplier produces more aggressively if its working capital is small, because its loss in case of insolvency is small and it has much to gain if the demand turns out to be high. Second the retailer competes with the bank and offers a low interest rate. Over production occurs when $q_{opt}^{rs} = q_{ds}$, i.e. the supplier makes the same (or slightly larger) profit by taking direct financing. In such cases, even though over-production harms the supply chain's efficiency, the retailer still makes more profit compared to bank-only financing. If the retailer increases the interest rate to reduce the supplier's production quantity, then the supplier will switch to bank-only financing.

Proposition 2.3 also shows that the highest efficiency is achieved if $\kappa = \hat{\kappa}_t > 0$. We further investigate situations in which the first-best solution may be achieved. By Theorem 2.1, the first-best solution may only occur in cases when $q_{ds} \in [q_{opt}^d, q_{dr}]$, and $q_{ds} = q_t^*$. That is, we have that $\pi_S^d(q_t^*) = \pi_S^b(q_b)$, which is equivalent to the following equalities:

$$y_d \bar{F}(y_d) = (q_t^* - \kappa/c)\bar{F}(q_t^*), \text{ and} \tag{2.10}$$

$$(1-\alpha)r \int_0^{q_t^*} \bar{F}(x)dx - (1-\alpha)r \int_0^{y_d} \bar{F}(x)dx - \hat{\kappa}_t = (1-\alpha)r \int_0^{q_b} \bar{F}(x)dx - cq_b. \tag{2.11}$$

After some algebra, we can rewrite Equation (2.11) as

$$\hat{\kappa}_t = c\Big(\int_{q_b}^{q_t^*} x dF(x) - \int_0^{y_d} x dF(x)\Big)\Big/\big(\bar{F}(q_b) - \bar{F}(q_t^*)\big). \tag{2.12}$$

If y_d and $\hat{\kappa}_t$ satisfy Equations (2.10) and (2.12), then supply chain can achieve the first-best solution and the supplier makes the same profit that it will make with bank-only financing. The retailer, therefore, makes the highest profit it can make among all possible contracts.

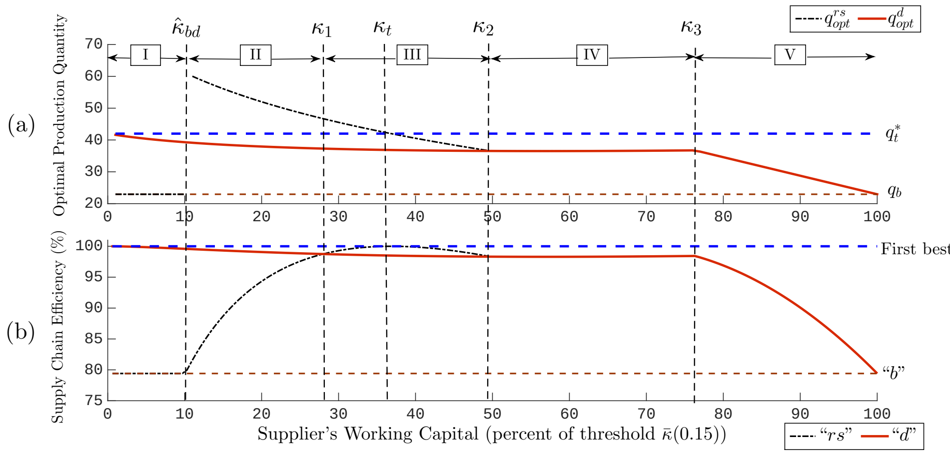
Numerical Examples:

In this section, we use numerical analysis to illustrate the optimal production quantity that results from the supplier's and the retailer's actions as a function of the supplier's working capital. We also plot the supply chain's efficiency. The problem parameters are as follows: uniform demand $U[0, 150]$, $\alpha = 0.15$, $r = 10$, and $c = 7.2$. We show the optimal production quantities in Figure 2.2(a), and the supply chain efficiency in Figure 2.2(b). The horizontal axes in both figures are the supplier's working capital, expressed as percent of $\bar{\kappa}(\alpha) = c\bar{F}^{-1}(c/[(1-\alpha)r])$. Recall that the supplier's working capital must be less than $\bar{\kappa}(\alpha)$ for it to consider debt financing. Black dash-dotted lines represent optimal "rs" contract and red solid lines represents optimal direct-only financing contract. Brown dashed lines represent bank-only financing and blue dashed lines represent the first best solution. The lines representing q_b and q_t^* are horizontal lines in Figure 2.2(a), and the corresponding supply chain efficiency is also represented by horizontal lines in Figure 2.2(b), because neither is affected by the supplier's working capital.

The supplier's production-quantity response to the optimal "rs" contract and the corresponding supply chain efficiency depends on four critical numbers: $\hat{\kappa}_{bd}$, κ_1, κ_2 and κ_3, as described below.

1. If the supplier has very little working capital ($\kappa \leq \hat{\kappa}_{bd}$, Region I), the supplier chooses bank financing (Case 1 of Theorem 2.1), so the efficiency of "rs" is lower than that of direct-only financing — the black dash-dotted line in Figure 2.2(b) lies below the red solid line in Region I.

Figure 2.2: Supply Chain Efficiency under Direct Financing



2. If $\hat{\kappa}_{bd} \leq \kappa \leq \kappa_1$ (Region II) the optimal production quantity is slightly more than q_{ds} (not shown). That is, the retailer must lower the interest rate and share more profit with the supplier until the supplier's profit is slightly larger than that under bank financing. The supplier chooses direct financing, but over-produces, which lowers the supplier chain's efficiency. In Region II, q^{rs}_{opt} is greater than q^*_t and the supply chain's efficiency is lower — compare the black dash-dotted line and the blue dashed line.

3. If $\kappa_1 \leq \kappa \leq \kappa_2$ (Region III) the optimal production quantity is still slightly greater than q_{ds}, but the supplier produces less. The supply chain's efficiency is relatively high under the "rs" contract. Moreover, if $\kappa = \hat{\kappa}_t$, we get $q^{rs}_{opt} = q^*_t$, i.e. the supply chain achieves the first-best efficiency.

4. If $\kappa \geq \kappa_2$ (Region IV & V), the efficiency of "rs" is the same as that of direct-only financing, which corresponds to the case $q^{rs}_{opt} = q^{d}_{opt}$ in Theorem 2.1. Furthermore, if $\kappa \geq \kappa_3$ (Region V), then $q^{rs}_{opt} = q^{d}_{opt} = q_{dmax}$, which means that the retailer would offer trade finance with risk-free (zero) interest rate.

2.5.2 The Bank Has Seniority

Next, we study the retailer's choice of ρ_d under the "bs" environment. The retailer chooses ρ_d to maximize its profit function (2.5) considering the supplier's response in

Proposition (2.2). The optimal ρ_d is obtained in Theorem 2.2.

Theorem 2.2. *If the failure rate $z(\cdot)$ is convexly increasing and $\alpha \leq (1 + c/r)/2$, then $\pi_R^{bs}(\rho_d)$ is unimodal in ρ_d. The optimal loan can be characterized as follows. The retailer offers $\rho_{opt}^{bs} < \hat{\rho}_{db}$, and the optimal production quantity is $q_{opt}^{bs} = \max\{q_{sol}, q_{dmax}\}$, where q_{sol} is a solution of $\frac{1-(q-\kappa/c-B_b^*/c)z(q)}{1-(y_d-y_b)z(y_d)} - (\frac{c}{(1-\alpha)r\bar{F}(q)} - \frac{\alpha}{1-\alpha}) = 0$ and the equations in Item 2 of Proposition 2.2. Moreover, $q_{un} < q_{opt}^{bs} < q_t^*$.*

The results in Theorem 2.2 are quite different from those in Theorem 2.1. Now, the retailer does not induce the supplier to choose single-source financing, instead, it may offer an interest rate such that the supplier takes both bank and direct financing. Moreover, neither over-production nor supply-chain coordination are possible in this case. Intuitively, this can be explained as follows. With the bank being the first lender, the retailer is able to offer a loan that covers only the high-risk inventory. It charges commensurately more to cover its costs. This increases the supplier's borrowing costs and thereby eliminates over-production. Similarly, the high-risk inventory increases the retailer's lending cost, which causes the retailer to not offer an interest rate that leads to the first-best solution.

Next, we provide insights into the question of seniority preference. Specifically, which type of direct financing will the retailer prefer: high-seniority, low-seniority or indifference? We find that all three cases may happen, depending on problem parameters. In some cases, the retailer may be indifferent to the order of seniority. For example, when the supplier's working capital is relatively large, the retailer offers zero interest in both scenarios and makes the same profit. In many cases, the retailer may prefer having seniority. This is mostly what one expects on an intuitive level. For example, with high-seniority the retailer may achieve its best possible profit $\pi_0(q_t^*) - \pi_S^b(q_b)$, but that level of profit is not achievable under low-seniority direct-financing contracts. More interestingly, in some cases the retailer may even prefer low-seniority.

2.6 The Effect of Information Asymmetry

Next, we study the retailer's decisions under bank information asymmetry. Recall that the bank's demand estimate is D_b, with distribution $F_b(\cdot)$, such that $F_b(x) \geq F(x)$ for all x. Similar to the symmetric information scenario, we study the two debt-seniority cases separately. In Section 2.6.1 we consider the retailer having seniority and in Section 2.6.2 we consider the bank having seniority.

2.6.1 The Retailer Has Seniority

It is clear from Proposition 2.1 that the supplier will choose either direct-only financing or bank-only financing, but never borrow from both sources. Therefore, the profits under bank-only financing still work as reservation profits to the supplier and the retailer. However, both reservation profits become smaller than those when the bank has symmetric information. In this section, we use an additional subscript 2 to specifically identify the asymmetric-information scenario. For example, q_{b2} denotes the supplier's optimal production quantity when taking bank-only financing with asymmetric information, and similar to Theorem 2.1, we define $q_{ds2} = \{q_d : \pi_S^d(q_d) = \pi_S^b(q_{b2})\}$. The optimal contracts are characterized in Theorem 2.3.

Theorem 2.3. *If the failure rate $z(\cdot)$ is convexly increasing and $\alpha \leq (1 + c/r)/2$, the optimal loan can be characterized in terms of a working capital threshold $\hat{\kappa}_{bd2}$ as follows.*

1. *When $\kappa \leq \hat{\kappa}_{bd2}$, the retailer will let the supplier choose bank financing.*
2. *When $\kappa \geq \hat{\kappa}_{bd2}$, the retailer will offer an interest rate such that the corresponding order quantity is $q_{opt2}^{rs} = \max\{q_{ds2}, q_{opt}^d\}$ and the supplier borrows from the retailer.*

Moreover, $\hat{\kappa}_{bd2} \leq \hat{\kappa}_{bd}$, $q_{ds2} \leq q_{ds}$, $\pi_R^{rs}(q_{opt}^{rs}) \leq \pi_R^{rs}(q_{opt2}^{rs})$ and $\pi_S^{rs}(q_{opt}^{rs}) \geq \pi_S^{rs}(q_{opt2}^{rs})$.

Theorem 2.3 shows that the retailer lets the supplier with small working capital choose the bank financing, and offers attractive interest rate to the supplier with relatively large working capital. However, compared to the case when the bank has symmetric information, we observe the following: (1) the retailer is more likely to make its loan offer attractive to a supplier with small capital; (2) the retailer will offer smaller incentive to the supplier to increase its production, and therefore over-production phenomenon will be less likely and smaller in magnitude; and (3) asymmetric information benefits the retailer but reduces the supplier's profit relative to the symmetric information case. The intuition behind Theorem 2.3 is as follows. The bank transfers the cost of asymmetric information to the supplier by charging a higher interest rate. Therefore, bank financing is less competitive relative to direct financing and the retailer can charge a higher interest rate, which reduces the supplier's profit and increases the retailer's profit.

2.6.2 The Bank Has Seniority

The retailer maximizes its profit function (2.5) given the supplier's response in Proposition 2.2. The optimal contract is characterized in Theorem 2.4.

Theorem 2.4. *The retailer offers* $\rho^{bs}_{opt2} < \hat{\rho}_{db}$*, and such that the optimal production quantity is* $q^{bs}_{opt2} = \max\{q_{sol2}, q_{dmax}\}$*, where* q_{sol2} *is a solution of*

$$\frac{(r\bar{F}(q) - c)c\bar{F}(y_d)[1 - (y_d - y_b)z(y_d)]}{\bar{F}(q)[(1+\rho_d)cz(y_d) - (1-\alpha)rz(q)]} + B_d\bar{F}(y_d) + \frac{(1-\alpha)r}{(1+\rho_d)z_b(y_b)}[\bar{F}_b(y_b) - \bar{F}(y_b)] = 0, \tag{2.13}$$

and Equation (2.8).

Similar to Theorem 2.4, the retailer will offer an interest rate such that the supplier takes both bank and direct financing. However, it is difficult to argue whether Equation (2.13) will have a unique solution without imposing strong conditions on the functional form of F_b. If there are multiple solutions to Equation (2.13), then the retailer needs to compare the corresponding profits and select the maximizer.

2.7 Insights and Extension

This section contains three main ideas. In the first, we numerically compare the retailer's profits in two cases – when it either requires or does not require seniority, and investigate how the information asymmetry influences the optimal contracts. In the second, we investigate whether the retailer will set credit limit to eliminate over-production, and in the third, we investigate the consequences if the retailer were to simultaneously optimize both interest and consignment rates.

2.7.1 The Attractiveness of Seniority

In our numerical examples, we assume that both D and D_b follow uniform distributions. Specifically, $D \sim U[0, 100]$ and consider three levels of information asymmetry, i.e. $D_b \sim$

$U[0, T_b]$, where $T_b = 100, 80, 60$, respectively. We set $r = 10, \alpha = 0.1, c = 6$. Figure 2.3 to 2.5 show how retailer's optimal interest rate and the corresponding profits of the supplier and the retailer change with the supplier's working capital under different seniority environments and different levels of information asymmetry.

Figure 2.3: Retailer's Optimal Profit vs. Supplier's Working Capital

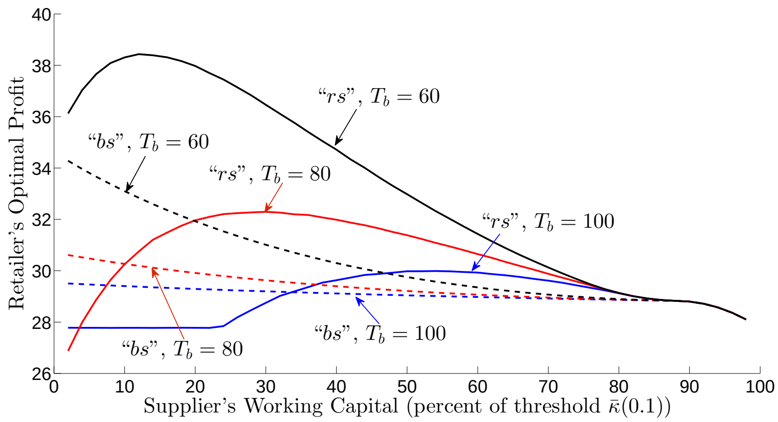
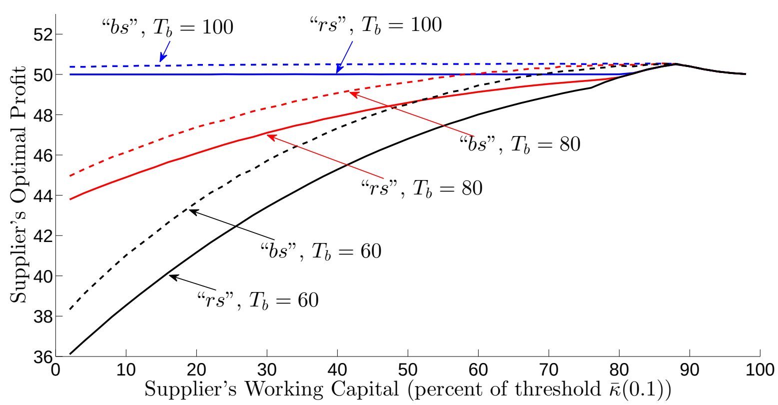
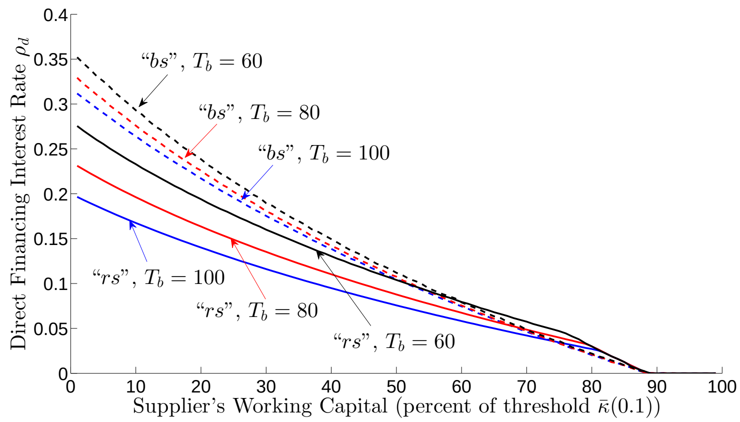


Figure 2.4: Supplier's Optimal Profit vs. Its Working Capital



The solid lines show the profits and ρ_d when the retailer is senior and the dash lines show the profits and ρ_d when the bank is senior. Blue lines show the $T_b = 100$ case, i.e. when the bank has symmetric information. The red lines show the $T_b = 80$ case, and the black lines the $T_b = 60$ case. We find that (1) there exist patterns in which the retailer prefers to not have seniority; (2) these patterns occur when the bank's

Figure 2.5: Retailer's Optimal Interest Rate vs. Supplier's Working Capital



information is close to that of the retailer (larger T_b) and the supplier's working capital is small; (3) for intermediate values of κ, the retailer prefers to be the senior lender; (4) for large κ, the loan seniority does not matter because the retailer offers a zero-interest loan in either scenario; (5) when the degree of information symmetry increases (larger T_b), the retailer charges lower interest rate and earns less profit; (6) the magnitude of the profit decrease is significant only if the supplier's working capital is small; (7) and the supplier has no incentive to voluntarily reduce (via dividends) or misreport its working capital unless it is offered zero-interest-rate loan. These results are consistent with our previous arguments that information asymmetry makes bank financing less competitive relative to the direct financing, Therefore, the retailer could charge a higher interest rate and make more profits. When the supplier's working capital is large, the retailer charges a low interest rate and the effect of bank financing is small.

2.7.2 Limited Credit Direct Financing

In this section, we study the effect of credit limit on direct financing. First, note that the retailer need not set credit limit on direct financing if the bank has seniority because in that case the supplier does not over produce. Therefore, we study credit limit in the rs-contract environment. Limited-credit rs-contract is called the lrs-contract. The retailer now specifies a maximum loan amount $\bar{B}_d$ in addition to the loan rate. We use our model to answer the following three questions. (1) Will the retailer choose credit limit such that the supplier takes both direct financing and bank financing? (2) Can credit limit eliminate over-production? (3) Will the retailer limit credit to maximize the supply chain's efficiency?

First, we argue that under an optimal "lrs" contract, the supplier never borrows from both the retailer and the bank. This can be explained as follows. Suppose the retailer chooses $(\rho_d, \bar{B}_d)$ such that the supplier's production quantity is q upon using its working capital κ and loan $\bar{B}_d$. If $q \geq q_b$, then the supplier's marginal profit is negative, therefore it will not borrow more from the bank to increase production beyond q_b. If $q \leq q_b$, then the supplier's optimal production quantity is $q^*_{lrs} \leq q_b$. The total supply-chain profit is no larger than that under bank-only financing contract. That is, the supplier and the retailer cannot simultaneously make more profits than their respective profits under bank-only financing option, which implies that the supplier will choose bank-only financing.

Next, we show that retailer may choose $(\rho_d, \bar{B}_d)$ to eliminate overproduction and coordinate the supply chain under some parameter conditions. Because bank financing works as reservation profit for the supplier, the maximum possible profit for the retailer is $\pi_0(q^*_t) - \pi^b_S(q_b)$. We then show that under some scenarios the retailer can choose $(\rho_d, \bar{B}_d)$ such that $q^{lrs}_{opt} = q^*_t$ and $\pi^{lrs}_S(q^{lrs}_{opt}) = \pi^b_S(q_b)$. In such case, the retailer achieves its best profit and the supply chain's efficiency is simultaneously maximized.

Proposition 2.4. *Assume $q^*_t z(q^*_t) \leq 1$, then there exists a threshold $\hat{\kappa}_{lt}$ such that $q^{lrs}_{opt} = q^*_t$ and $\pi^{lrs}_S(q^{lrs}_{opt}) = \pi^b_S(q_b)$ for any $\kappa \leq \hat{\kappa}_{lt}$.*

Proposition 2.4 shows that if $q^*_t z(q^*_t) \leq 1$ and $\kappa \leq \hat{\kappa}_{lt}$, then the retailer can maximize its own profit by choosing $(\rho_d, \bar{B}_d)$ such that the supplier takes direct financing, produces q^*_t and makes the same profit as that under bank-only financing. To explain why this is a possible outcome, consider the case in which the supplier over-produces and makes the same profit as that under bank-only financing. In this case, the retailer can choose to limit credit, which reduces the production quantity, and lower the interest rate commensurately to keep the supplier's profit unchanged. Consequently, the total supply chain's profit and the retailer's share will increase. We emphasize that this outcome only happens under special parameter values, i.e. in cases for which the supplier's profit increase from interest-rate decrease is exactly equal to the reduction in profit from lowering production quantity down to q_t^*. We find that overall such an outcome is more likely when the supplier's working capital is low. In such cases, it borrows more from the retailer, and if the retailer acts to lower interest rate, then that will have a significant effect on the supplier's profit, allowing the retailer a chance to achieve the first-best production level by tweaking ρ_d and $\bar{B}_d$.

2.7.3 Optimal Consignment Rate and Financial Terms

We next study the effect of an optimally chosen consignment rate on the terms of finance.

Proposition 2.5. *When the retailer jointly optimizes the consignment rate and the interest rate on the loan, the optimal direct financing contract parameters* (ρ_d^*, α^*) *are such that* $\rho_d^* = 0$.

Proposition 2.5 shows the retailer that can customize commission rates for each supplier always offers direct financing with zero interest rate, irrespective of seniority. In such cases, the supplier will take direct-only financing. The intuition behind Proposition 2.5 is as follows. When the supplier takes direct-only financing, its optimal production quantity depends on the ratio $(1+\rho_d)/(1-\alpha)$, but not on ρ_d and α individually. Therefore, the retailer can increase its profit by increasing α and decreasing ρ_d correspondingly, while keeping $(1+\rho_d)/(1-\alpha)$ unchanged until ρ_d becomes zero. Since the supplier will always take this zero-interest direct financing, $\rho_d^* = 0$ would be optimal and the terms of the competitively-priced bank financing would not be attractive.

In Proposition 2.3, we show that $\kappa/c < q_{opt}^d < q_t^*$ for every possible α. Therefore, the inequality also holds when α is chosen by the retailer. That is, the retailer's individually optimal production quantity is always smaller than that in a coordinated supply chain. The results in this section are consistent with those obtained in Kouvelis and Zhao (2012) and Yang and Birge (2013).

2.8 Concluding Remarks

In this chapter, we model trade finance options in two-sided markets. The small-business supplier sells its products via a web platform. The platform owner (retailer) charges a commission on sales. The retailer offers direct financing — setting loan interest rate and debt seniority requirements. Bank financing is also available, both when the retailer requires seniority and when it does not. Banks have asymmetric information about the market potential for the supplier's product and banks' loans are competitively priced. We show that trade finance relaxes the supplier's financial constraints and works as a risk-shifting mechanism between the supplier and the retailer. It improves both the supplier's and the retailer's profits.

From the supplier's perspective, we show that if the retailer has seniority, it takes either direct-only or bank-only financing (but never both) under an optimally-structured contract. The terms of loan may induce the supplier to produce either less than, or equal to, or more than the first-best quantity. In contrast if the bank has seniority, the supplier may borrow from both the bank and the retailer, and it never over produces.

We analyze the retailer's optimal decisions in two steps — symmetric information first, followed by asymmetric information. The former model is more tractable and therefore amenable to mathematical analysis. In that case, we show that the retailer may find it individually optimal to allow risk shifting to an extent that leads to over-production. However, this only happens when the retailer finds it attractive to require debt seniority. We then study the effect of the bank information asymmetry. The bank transfers the cost of information asymmetry to the supplier, which makes its loan less attractive to the supplier relative to the retailer's loan. This allows the retailer to charge a higher interest rate, increasing its profit and lowering the supplier's profit. Moreover, with information asymmetry, the retailer is more likely to offer terms that make its loan more attractive to the supplier, which helps explain phenomenon such as Amazon Lending. Information asymmetry also implies that the supplier produces less. Therefore the over-production phenomenon is less likely and expected to be smaller in magnitude, if it occurs.

We analyze the retailer's seniority preference and supplier's characteristics. The retailer would prefer the bank to have debt-seniority if the supplier's working capital is relatively small, and is indifferent to debt-seniority if the supplier's working capital is large. Over a large middle range, the retailer prefers to be the senior lender. With bank information asymmetry, the retailer is more likely to require debt-seniority. Moreover, if the retailer has debt-seniority and the supplier is significantly short of working capital, the retailer can avoid over-production and simultaneously induce the supplier to produce the first-best quantity by limiting credit.

In this chapter, we simplify reality to obtain tractable models, leaving room for future refinements and extensions. We describe both the limitations of our models and future opportunities next. First, we consider a particular type of bank information asymmetry, i.e. the bank's estimated demand distribution is stochastically smaller but assumed known to the supplier and the retailer. The bank's estimated demand distribution may not be ordered in this sense (weaker stochastic orders could be considered), and the asymmetry may be two-sided. That is, the retailer and the supplier may not know the bank's estimated demand distribution. At best, the supplier and the retailer may have interest rate quotes for a few requested loan amounts, from which an inaccurate estimate of the demand distribution assumed by the bank may be possible. Although we have not analyzed that case, our conjecture is that the two-sided asymmetry will harm the retailer because it will have to hedge its offer to be better than several possible offers by the bank. However, the retailer's response is difficult to predict without assuming a prior on the bank's demand distribution. Therefore, two-sided asymmetry may either benefit or harm the supplier.

Second, we assume that there are no bankruptcy costs. In addition to the reasons in Section 2.3 that justify this assumption, another reason emerges from our analysis. When the supplier considers borrowing both from the retailer and the bank, two cases arise. If the retailer has seniority, we prove that the retailer will either not offer a competitive loan, or that it will offer a loan such that the supplier borrows only from the retailer. Only when the retailer does not demand seniority, it is possible for the supplier may borrow from both sources. In that case, forced liquidation costs may arise in practice. We continue to assume no bankruptcy costs in all models in order to have consistency across all models. We believe that it would be possible to relax the zero-bankruptcy-cost assumption in future work, leading to interesting extensions of our model.

We believe that consignment selling, already a significant portion of total retail sales, is likely to be even more significant in the future. It is important therefore to consider financing models that could help the retailers and the suppliers improve their individual and combined profits. We demonstrate that retailer-direct contracts have the potential to do precisely that. Our analysis serves to inform retailers how to structure contract terms to realize the best possible outcomes. In addition to the extensions discussed above, there are other interesting avenues that could be pursued in future research dealing with retailer-direct financing. Examples include revolving credit and suppliers who produce and sell multiple products on the same web platform.

Chapter 3

Leveraging Bankruptcy Costs to Improve Supply-Chain Performance

3.1 Introduction

Modigliani and Miller (1958) show that in absence of market frictions, the value of a firm is independent of its financial structure. Because frictions such as bankruptcy costs, information asymmetry, and transaction costs are all too common in business-to-business (B2B) financing, also known as trade-credit, contracts are often designed to mitigate the negative effects of market frictions. In this chapter, we compare and contrast variants of the net-terms trade-credit contract offered by a supplier to a financially-constrained newsvending retailer. Our goal is to identify situations in which the supplier can use one type of friction as a lever to mitigate the negative consequences of another type of friction and produce a win-win outcome for both B2B partners.

In the nominal contract, which we call the *reported-sales* contract, the supplier offers trade credit to the retailer and relies on retailer-reported sales to learn the retailer's post-sales' assets. It either accepts retailer's report and payment based on reported sales, or initiates bankruptcy. The payment based on reported sales may be less than the loan amount. Thus the supplier must trade off the potential loss if it accepts the retailer's report and the retailer under-reports, against the potential loss if it initiates bankruptcy and the retailer's report is correct. In that case, the supplier will collect less after paying bankruptcy costs. Note that in this environment, there is no prior commitment by the supplier to forgive the loan. This is the most common type of loan forgiveness reported in the literature. Evans and Koch (2007) examine 498 firms that filed for either Chapter 11 or Chapter 7 proceedings during 1986-1998 and show that compared to banks, suppliers are more likely to provide both additional credit and/or forgive debt, which increases the retailers' survival rate and reduces overall bankruptcy costs. Cunat (2007) documents that when retailers experience temporary liquidity shocks that may threaten their survival, suppliers tend to forgive their debts and extend their maturity periods at no extra cost. Even without survival threats, suppliers typically forgive interest charges on payments due past the net terms date for their important customers (Loten 2012).

In the contract variant introduced in this chapter, which we call the *trade-forgiveness* (TF) contract, the supplier offers a certain degree of loan forgiveness up front but requires, in return, that the retailer allows a trusted third-party access to its true sales figures. The trusted third-party only reports the sales revenue to the supplier if the sales revenue is insufficient to cover the original loan. The supplier may incur a greater cost associated with announced forgiveness and the cost of third-party's services, but avoids the cost associated with initiating bankruptcy when that action will result in a smaller payment to the supplier. Because both the supplier and the retailer are affected by bankruptcy costs, the supplier in this contract variant may succeed in using the avoidance of bankruptcy costs as a lever to induce the retailer to cooperate.

The attractiveness of the trade-forgiveness contract to the supplier, provided it can offer terms to make the contract attractive to the retailer, depends on a variety of factors, such as the liquidated value of the retailer's assets, the cost of trusted third-party services, and the random demand faced by the retailer. It is not straightforward to identify the problem parameters under which the supplier will offer a trade-forgiveness contract, and the parameters under which it will use the nominal contract. Moreover, formal models are needed to evaluate the size of the benefit to the two parties. For these reasons, the objective of this chapter is to address the following questions. How do the reported-sales and the trade-forgiveness contracts affect the retailer's ordering decisions, the supplier's pricing decisions, and furthermore their respective profits? Note that the supplier sets wholesale prices in both contracts and the retailer is a price taker in the retail market. How does the degree of loan forgiveness, which we refer to as the *forgiveness quotient*, affect the supplier's share of risk and profits? The forgiveness quotient is the fraction of the loan that the supplier forgives if the retailer cannot repay the loan after observing its sales. When does the trade-forgiveness contract benefit both the supplier and the retailer? What is the effect of this type of contract on supply chain efficiency, where efficiency is measured with respect to the total profit in a supply chain with no financial constraints and a single decision maker.

Through mathematical analysis, we show that the supplier can use the threat of invoking bankruptcy to affect the retailer's order quantity as well as the degree to which it may misreport its sales. We prove that the supplier's optimal strategy is characterized by a threshold such that if the retailer's reported sales is below the threshold, then it will initiate bankruptcy proceedings, otherwise not. Furthermore, the threshold may be lower than the loan amount. That is, the supplier may forgive a part of the loan because bankruptcy is costly even when it does not commit to loan forgiveness up front. This result provides a possible explanation for the prevalence of loan forgiveness in B2B financing contracts. In typical wholesale-price contracts, if the supplier charges a higher wholesale price, then that results in a lower order quantity. That is not necessarily the case in reported-sales contracts. In that instance, the retailer can leverage private sales information to order more even as the supplier charges a higher wholesale price.

We also prove that TF contracts, which are two-parameters contracts like the two-terms contracts, are more efficient than two-terms contracts because they allow the supply chain to avoid certain bankruptcy costs and achieve coordination. This benefits both the supplier and the retailer. Coordination means that the total profits of the supplier and the retailer and the retailer's optimal order quantity decisions are the same as those in a centralized chain. Note that Kouvelis and Zhao (2012) and Yang and Birge (2013) have shown that the two-terms contract fails to coordinate the supply chain. We prove that once the information asymmetry is eliminated, the TF contract coordinates the supply chain if either the retailer's assets are not too low or its reservation profit is not too high. In such situations, the supplier can arbitrarily divide total profit and both the supplier and the retailer are better off as compared to the two-terms trade-credit contract. Our numerical examples show that TF contracts still benefit both the supplier and the retailer even with asymmetric information comparing to reported-sales contract as long as the fixed transaction costs are not too high.

Intuitively, by announcing the forgiveness quotient ex ante, the TF contract affects the supply chain in two ways. On the one hand, the supplier's offer of loan forgiveness encourages the retailer to order more. From this point of view, the TF contract increases the retailer's profit but costs the supplier more because its loan may not be repaid in full. On the other hand, facing a larger order quantity, the supplier has the ability to charge a higher wholesale price and collect a greater share of profits from sales. Therefore, the supplier must balance the loss from a higher forgiveness quotient and the benefit from a higher wholesale price. Essentially, the coordinating TF contract chooses a pair of forgiveness quotient and wholesale price to affect channel-optimal order quantity and maximize the supply chain profit. In addition, it prevents retailer bankruptcy, thereby lowering the supply chain's expected bankruptcy costs.

In the remainder of this chapter, we provide detailed institutional background, literature review, contribution of our chapter, mathematical models, analyses, numerical examples, and insights.

3.2 Institutional Background, Literature, and Contribution

Trade credit is an important source of supply-chain finance in the United States and other developed economies. Ng et al. (2002) reported that trade credit in the 1990s accounted for an average \$1.5 trillion of the book value of all assets of US corporations and represented approximately 2.5 times the combined value of all new public debt and primary equity issues during a given year. More recently, Murfin and Njoroge (2012) reported that as of 2009, trade payables represented the second largest liability on the aggregate balance sheet of non-financial businesses in the US and that they are more than triple the amounts owed to banks.

Two forms of trade credit contracts have been studied extensively in finance and operations management (OM) — two terms and net terms. A two-terms contract allows the retailer to either receive a discount by making an early payment or pay within a specified number of days without incurring finance charges. For example, the "2/10 net 30" contract offers a 2% discount if the amount due is paid in 10 days or less, and a permissible payment delay of up to 30 days without imposing finance charges. In the net-terms contract, early payment discount is not offered. Klapper et al. (2011) report that only about 13% of trade credit contracts offer early payment discount and that the duration of credit is largely determined by the supplier's industry characteristics. Thus, the most prevalent form of trade credit is the net terms contract, which we study in this chapter.

Various arguments have been advanced to explain the prevalence of trade credit in finance and OM papers – see for example, Giannetti et al. (2011) and Klapper et al. (2011). Such reasons include, e.g. (1) the supplier has informational advantage over third-party lenders, (2) the supplier realizes higher collateral liquidation value, (3) the supplier may use credits terms as a means of price discrimination, prevented in the US by the Robinson-Patman act, and (4) the supplier credit is tantamount to an implicit warranty assuring buyers of the product quality. We do not discuss those reasons in detail in this chapter. This chapter is concerned with loan forgiveness, which is one of the reasons. However, to the best of our knowledge, loan forgiveness has not been modeled as an explicit feature of trade-credit contracts in previous studies. The key idea behind the TF contract is that because bankruptcy is costly for both the supplier and the retailer, the supplier can offer contracts that either reduce or eliminate bankruptcy cost as an incentive in return for the retailer's cooperation in eliminating information asymmetry.

Bankruptcy costs may be both direct and indirect. Direct costs include expenses for lawyers, accountants, restructuring advisers and other professionals. Indirect costs include a wide range of opportunity costs. For example, loss of collateral's value because of the need to sell quickly, lost sales caused by customers choosing not to do business with a firm that may enter bankruptcy, and so on. In the finance literature, total bankruptcy costs are usually estimated as a proportion of the firm value (see Altman and Hotchkiss 2006). Davydenko et al. (2012) examine 175 firms that defaulted between 1997 and 2010, and report that the bankruptcy costs for an average defaulting firm were approximately 21.7% of its market value. Because by law bankruptcy costs must be recovered first, a lender receives payment only if funds remain after paying such costs – see Altman and Hotchkiss 2006 for more details. Therefore, lenders may prefer to strike a deal with the borrower and forgive a part of the loan to avoid costly bankruptcy proceedings.

Suppliers often provide net-terms' trade-credit contracts. The retailer has the option to borrow either from the supplier or the bank. The retailer may pledge assets such as existing inventories or real estate as collateral in addition to future sales revenue. Because we consider a newsvending retailer, i.e. a single-period model, net terms equal the length of the sales period.

Our work contributes on the literature by formulating models and studying the equilibrium sales report by the retailer in the trade-credit environment with bankruptcy costs. We also propose an alternative TF contract that leverages bankruptcy cost avoidance as a tool to eliminate information asymmetry in typical B2B contracts, and identify problem parameters under which the TF contract will be preferred by both B2B partners. In the remainder of this section, we position our work in relation to the literature and present its contribution.

Wilner (2002) describes a three-period sequential decision model of a financially-constrained firm borrowing either from the vendor or from the credit market. The latter option results in an early-payment discount. The firm's order-quantity decision is not modeled. It is assumed that it faces an uncertain revenue with a two-point distribution, and that it can renegotiate debt repayment after learning its revenue. Wilner argues that the money at stake (firm's current and expected future profits plus its creditor's expected future profits) is larger if debt is renegotiated, therefore trade creditors grant more concessions (as a fraction of the money at stake) in debt renegotiations. Wilner concludes that anticipating larger renegotiation concessions, not only do less financially stable firms prefer trade credit, but also all firms agree to pay a higher effective interest rate for trade credit relative to the credit market. The loan forgiveness idea in our work is similar. However, we model supplier's choice of wholesale price and the retailer's choice of order quantity, all within a framework in which demand is uncertain. These features allow us to evaluate the impact of the TF contracts on operational decisions.

The mathematical model of the TF contract we develop later in this chapter is similar to the models considered by Kouvelis and Zhao (2012), Caldentey and Chen (2009) and Yang and Birge (2013). Note that these works have different objectives and model features. In particular, they study trade-credit contracts with symmetric information. Kouvelis and Zhao (2012) considers both the supplier's and the retailer's problems in a Stackelberg game setting within a perfect capital market. The authors compare trade credit with bank financing and conclude that trade credit is the preferred option. Kouvelis and Zhao show that the two terms trade credit contracts encourages the retailer to take more risks and partially shares demand risks with the supplier. This risk sharing mechanism improves the supply chain efficiency by inducing a larger retailers order quantity but not at the level to fully coordinate the chain.

Yang and Birge (2013) study the two-terms trade-credit contract and bank financing in a similar setting with the addition of distress costs, and characterize the structure of the inventory financing portfolio including cash, trade credit and bank loan. Similar to the results in Kouvelis and Zhao (2012), they show that the two-terms trade-credit contract does not achieve coordination, since the distress costs reduce the incentive for larger order quantity.

Caldentey and Chen (2009) discuss the problem in a net terms trade credit setting, but the supplier in their model offers a contract that requires the retailer to pay a specified fraction of the procurement cost at the time of ordering, and the retailer does not have any other source of financing. Similarly, Caldentey and Chen's contract variant does not coordinate the supply chain, since their contracts shift less risks to the supplier comparing to the two-terms trade-credit contracts.

One way to compare to these works with ours is to consider a special case of our model in which the supplier can learn sales information without incurring a cost. We call that special case the TF contract with symmetric information. Viewed in that light, our model characterizes the effect of forgiveness quotient on the two parties' profits in trade-credit contracts in the presence of the bankruptcy cost. We show that an optimal TF contract with symmetric information can make both the supplier and the retailer better off as compared with the best trade-credit contract under standard terms, i.e. no loan forgiveness. Although friction mitigation, rather than supply-chain coordination, is the primary motivation behind studying the TF contract mechanism, it has the potential to coordinate the supply chain as well. The TF contracts work similarly comparing to the coordinating contracts (e.g. buyback contracts) in the spirit, but differently in the sense to avoid bankruptcy. Comparing to the two-terms trade credit contracts, TF contracts provide the supplier larger flexibility to share risks with the retailer.

Many papers study supply contracts with symmetric information and no financial constraints with the objective of achieving supply-chain coordination – see Cachon (2003) for a summary. Because we are concerned with B2B financing, we focus primarily on papers that model financial constraints. Those papers do not model information asymmetry. For example, Dada and Hu (2008) derive a nonlinear loan schedule that coordinates the channel consisting of a profit maximizing bank and a newsvending retailer who faces financial constraints. In contrast, the supplier offers trade credit to maximize its individual profit, which has different risk-shifting mechanism comparing to the bank . Contracts under financial constraints and bankruptcy costs are studied in Kouvelis and Zhao (2015). They show that buyback and revenue sharing contracts still coordinate the supply chain with only variable bankruptcy costs if the contract parameters offered properly for different order quantities. Our TF contract has the same effect of coordinating supply chain with bankruptcy costs, but differently our contract parameters do not change with the order quantity.

Asymmetric information is modeled in supply contracts without financial constraints. For example, the problem of unobservable sales is studied in Gerchak and Khmelnitsky (2003), Gerchak et al. (2007) and Heese and Kemahlioglu-Ziya (2014). The work of Gerchak and Khmelnitsky (2003) utilize a discounted dynamic framework to characterize the retailer's optimal report as a function of the supplier's delivery-response function to these reports. Gerchak et al. (2007) study the under-report under a consignment-only contract environment, where the supplier decides the level of inventory. The supplier does not know the true realized demand, as well as the true demand distribution. Gerchak et al. show that the retailer can offer the supplier an untrue demand distribution and untrue realized demand to maximize its own profit and induce the supplier to provide the system-optimal inventory quantity. Both the works discussed so far do not model audits by the supplier.

Heese and Kemahlioglu-Ziya (2014) study under-report under the contract with both wholesale-price and revenue sharing. They focus on the audits by the supplier and assume the contract terms are exogenous. They argue that it is not optimal for the supplier engaging audits to ensure truthful reporting. They show that the supplier may benefit from retailer opportunism even after accounting for auditing costs. In our model, the supplier's use of "calling bankruptcy" in reported-sales contracts has an effect similar to audit in Heese and Kemahlioglu-Ziya's paper. However, different from their model, we study equilibrium sales reporting by the retailer in the trade-credit environment. We focus on how the reported sales anticipated by the supplier affect the contract parameters (i.e. the terms offered by the supplier), which are not considered by Heese and Kemahlioglu-Ziya.

Our work is related to those that study truth-inducing mechanisms, albeit in settings quite different from ours. We summarize several such papers next. Cachon and Lariviere (2001) and Özer and Wei (2006) study information sharing in an environment in which a supplier acquires the necessary capacity before receiving an order from a manufacturer. The latter possesses private information about the end product. Cachon and Lariviere (2001) study a contract in which a high-type manufacturer may commit to paying the supplier a lump sum upon contract acceptance to convince the supplier about its true high type. Özer and Wei (2006) study the contracts that may be offered by the supplier to induce credible information sharing by the manufacturer. Other works in this setting include e.g. Gümüş (2014) study the problem considering the competition between different suppliers, and Shamir and Shin (2015) study the forecast-information sharing with the competition from other manufacturers. These models are different from ours in the following ways. First, these works do not consider financial constraints and financial options. Second, In these works, the manufacturer has the private information about the distribution of the end product demand, whereas in our work the retailer's private information is about the demand realization. Third, in their models, the information sharing decisions are made before other contract-decisions, whereas in our reported sales model the retailer reports the sales after the contracts are offered. Finally, the manufacturer has incentive to inflate the forecast, whereas in our setting the retailer may benefit from under-reporting sales.

Our work related to the work of inspection and auditing in two ways. In the reported sales contracts, the supplier's option of calling bankruptcy works as the first way of inspection. In the TF contracts, the supplier paying the transaction costs to obtain the sales information is the second way of inspection. Misreporting and auditing is studied in The economics literature, e.g. Reinganum and Wilde (1985), Border and Sobel (1987), Scotchmer (1987) and Sanchez and Sobel (1993). These studies employ the principal-agent framework and focus on auditing strategies inducing the agent to report truthfully, whereas we study the auditing in the supply chain environment and focus on the effect of auditing on demand-risk shifting and the operational decisions.

In the literature of inspection and auditing in supply chain, besides the work of Heese and Kemahlioglu-Ziya (2014) mentioned above, works of Babich and Tang (2012) and Rui and Lai (2015) compare deferred payment and inspection mechanisms for mitigating potential product adulteration by the supplier. Babich and Tang (2012) model the problem without considering the procurement decision. They find that the deferred-payment mechanism is preferable to inspection if either incentive to adulterate or the consequences of adulteration are low. Rui and Lai (2015) extends the work of Babich and Tang by considering an endogenous procurement quantity with deterministic demand function. They show that the deferred-payment mechanism can outperform the inspection mechanism when either the market size is small or the profit margin is low. Deferring payment is a form of trade credit. In this sense, the papers are related to ours, but they do not consider bankruptcy. We use bankruptcy cost avoidance as a lever in our work to mitigate the negative consequences of information asymmetry. For other works about the inspection for quality control, we refer the readers to Babich and Tang (2012) and Rui and Lai (2015) for an extensive review. In other environment, Plambeck and Taylor (2015) and Xu et al. (2015) study auditing on motivating supplier social and environmental responsibility. In all above-mentioned works about auditing in supply chain, their inspection mechanisms is not signaling, whereas in our reported sales model the retailer reports the sales as a signal to the supplier.

To summarize our contribution to the literature, we first extend the OM literature of trade credit by modeling debt forgiveness. Second, we contribute on the study of supply chain contracts by simultaneously considering financial constraints, bankruptcy costs and information asymmetry. Third, our reported-sales contracts provide a possible explanation for the prevalence of debt forgiveness in practice. Finally, our TF contract creates an option to benefit both the supplier and the retailer by avoiding bankruptcy costs.

3.3 Model Formulation

We formulate separate models for reported-sales and trade-forgiveness contracting environments. Both these models utilize some common notation and assumptions, which we list first.

Both the supplier and the retailer are risk neutral and make decisions that maximize their expected profits. The supplier chooses the wholesale price w, and the retailer chooses the order quantity q. There are other contract-specific decisions that these two players make, which we describe later in this section. Problem primitives are (1) the retail price r, (2) the unit production cost c, (3) the demand D, and (4) the retailer's assets A_R. The cumulative distribution and density functions of D are denoted by $F(\cdot)$ and $f(\cdot)$, respectively.

We assume that D has an increasing failure rate (IFR), i.e., $\frac{f(x)}{1-F(x)}$ is increasing in x. This assumption allows us to prove certain claims analytically. It is not overly restrictive because many common distributions are IFR, for example, uniform, Erlang, normal, and truncated normal distributions (see e.g. Porteus 2002a for details), and a similar assumption is also made previous OM papers on trade-credit. The unmet demand is lost with no extra penalty and the leftover inventory has zero salvage value. Without loss of generality, the risk free interest rate is set equal to zero. The complete set of notation is provided in Table 3.1. Some of this notation is explained as we develop the two model formulations in separate subsections.

In addition to the technical assumptions mentioned above, we make additional assumptions that are motivated by the problem setting. These assumptions, along with their justifications, are discussed next. The retailer has collateral assets with liquidated value A_R, which can be used to guarantee debt obligations. In order to keep the notation manageable, we assume the retailer has zero working capital.

When the retailer cannot repay its debt, it becomes insolvent. In absence of loan forgiveness, insolvency results in bankruptcy. Following the finance and OM literatures, we assume that the bankruptcy cost has two parts (Xu and Birge 2004, Lai et al. 2009, Yang and Birge 2013, Kouvelis and Zhao 2011): (1) loss of a fraction of the realized sales revenue, (2) depressed collateral value because of the need to sell quickly, represented by a fraction of the asset value. The combination of this two parts explains the bankruptcy

Table 3.1: Summary of Notation

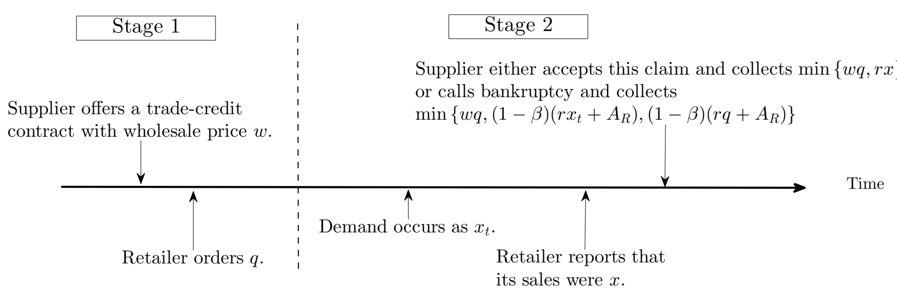
General	
w	Unit wholesale price
r	Unit retail price
c	Unit production cost
A_R	Retailer's assets
D	Random demand
$F(\cdot)$	The cumulative distribution function(CDF) of the demand
$\bar{F}(\cdot)$	$1 - F(\cdot)$
$f(\cdot)$	The probability density function (PDF) of the demand
$z(\cdot)$	Failure rate of the demand distribution, i.e., $z(\cdot) = f(\cdot)/\bar{F}(\cdot)$.
q	Order quantity.
β	Bankruptcy cost parameter
π_S	The supplier's expected profit function
π_R	The retailer's expected profit function
Reported Sales Trade Credit Contracts	
x	Arbitrary reported sales by the retailer
x_t	Actual realized demand
x_R	Retailer's optimal reported sales
B	Supplier's decision on calling bankruptcy or not: $B = 1$ means calling bankruptcy
Trade-Forgiveness Contracts	
λ	Forgiveness quotient
q_c^*	The traditional newsvendor optimal order quantity, i.e., $q_c^* = \bar{F}^{-1}(\frac{c}{r})$
$q^*(w, \lambda)$	The retailer's optimal order quantity given TF contract (w, λ)
y_1	The forgiveness threshold for the retailer under TF contract
y_2	The insolvency threshold for the retailer under TF contract
κ	Fixed transition costs
π_R^0	The retailer's reservation profit

costs as a proportion of the firm's market value. The bankruptcy costs parameter is denoted by β. Bankruptcy costs are recovered first before lenders receive payment.

3.3.1 Reported-Sales Contract

This is the default contract. The supplier receives a report of sales from the retailer but does not observe the actual sales. It must then decide whether to accept the retailer's report, and the concomitant payment, or initiate bankruptcy proceedings. We model the two players' decisions in the setting of a two-stage game. The sequence of events is shown in Figure 3.1.

Figure 3.1: Sequence of Events of Reported Sales Contracts



In Stage-1, the supplier sets the wholesale price w and the retailer chooses order quantity q. The retailer obtains these items on credit, owing the supplier wq. In Stage-2, the demand x_t is realized. It is observed by the retailer, but not the supplier. The retailer decides the reported sales amount x. The supplier's response B is binary – whether to call bankruptcy or not. We say that $B = 0$ means the supplier accepts the retailer's reported sales and $B = 1$ means that the supplier calls bankruptcy. If the supplier chooses $B = 0$, then it collects $wq \wedge rx$. [In the previous expression, the symbol $\wedge$ is used to denote minimum, i.e. $a \wedge b = \min\{a, b\}$ for any a and b for which this operation is defined.] In that case, the retailer's net profit is $r(x_t \wedge q - x)$. If the supplier chooses $B = 1$, then it collects $wq \wedge (1-\beta)(r(x_t \wedge q) + A_R)$. The retailer also pays bankruptcy costs and its net profit is $((1-\beta)(r(x_t \wedge q) + A_R) - wq)^+ - A_R$.

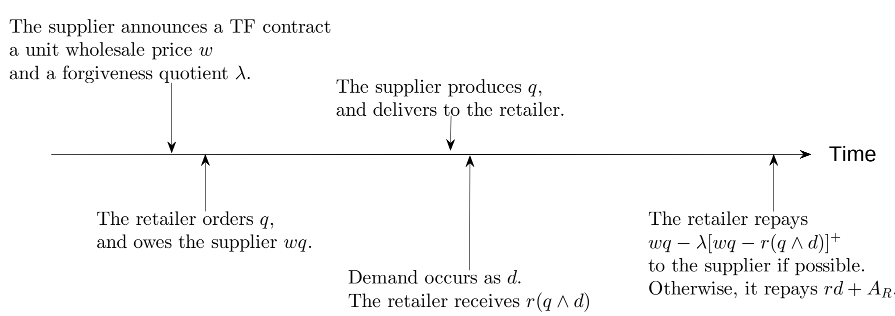
We consider a randomized strategy for the supplier, i.e. it chooses the probability p of $B = 1$. In summary, the Stage-1 decisions are w and q, and the Stage-2 decisions are x and p.

3.3.2 Trade-Forgiveness (TF) contracts

The TF contract parameters are (w, λ), where w is the wholesale price and $\lambda \in [0, 1)$ is the forgiveness quotient. Specifically, the supplier offers to forgive a proportion of the debt that the retailer is unable to pay upon realization of sales, and in return the retailer allows the supplier to observe the true demand via a trusted third-party. That is, if the debt amount is wq and the selling revenue is $r(q \wedge D)$, then the forgiven amount is $\lambda(wq - r(q \wedge D))^+$. Forgiveness and sales verification are relevant only if $wq > r(q \wedge D)$.

In the TF contract, the forgiveness serves as an incentive for the retailer to cooperate and allows the supplier to observe the true demand. Also, a fixed transaction cost may occur for the information verification, which we assume is κ. In order to compare the profits in TF contracts and those in other contracts, we assume that the retailer has a reservation profit, denoted as π_R^0. We show the sequence of events in Figure 3.2.

Figure 3.2: Sequence of Events of TF Contracts



Before closing this section, we present an observation from the analysis of the standard newsvendor model, i.e. when there are no financial constraints. We utilize this observation in proving certain claims later in our analysis.

Observation 3.1. *In the financially-unconstrained newsvendor model, the optimal order quantity* $q_c^* = \bar{F}^{-1}(\frac{c}{r})$. *Given any* $F(\cdot)$ *and* r, q_c^* *can be viewed as a function of the production cost* c, *and that it defines a supply curve* $\frac{c}{r} = \bar{F}(q_c^*(c))$. *Taking derivative with respect to* c *on both sides of this equation gives* $-f(q_c^*)\frac{\partial q_0^*}{\partial c} = \frac{1}{r}$. *Then,* $q_c^* z(q_c^*) = q_c^* f(q_c^*)/\bar{F}(q_c^*) = -1/[\frac{c}{q_c^*}\frac{\partial q_c^*}{\partial c}]$, *where* $\frac{c}{q_c^*}\frac{\partial q_c^*}{\partial c}$ *defines the production-cost elasticity of the amount of supply that the firm brings to the market.*

In this paper, we assume the firm's optimal supply is either elastic or unitary elastic, which means the elasticity of the supply is less than or equal to -1 (see, e.g. Kreps 1990 for details). Put differently, the optimal supply is sensitive to the unit production cost. Our assumption is equivalent to the inequality $q_c^* z(q_c^*) \leq 1$. Note that this assumption is a constraint only on the exogenous parameters r, c and $F(\cdot)$, but does not depend on any of the decision variables. In the OM literature, Lariviere and Porteus (2001) shows that this condition holds if the firm is served by a manufacturer in a price-only contract. In our model the supplier is not necessarily a manufacturer, but the exogenous market parameters are assumed to satisfy this condition. Similar assumption is also made in Yang and Birge (2013).

3.4 Analysis: The Reported-Sales Contract

We analyze Stage-2 decisions first, followed by Stage-1. For Stage-1 decisions, we devote a separate subsection to each player's problem.

3.4.1 Analysis of Stage-2 Decisions

The retailer moves first, but we consider the supplier's decision first because the retailer must anticipate what the supplier will do in response to its choices. That is, we fix x as the reported sales and x_t as the actual realized demand. It should be clear that if $rx \geq wq$, then there will be no bankruptcy and the supplier has no decision to make. Therefore, the analysis below focuses on the case when $rx < wq$. In such case, the supplier collects rx if it chooses $B = 0$. Otherwise, it calls bankruptcy and collects

$wq \wedge (1-\beta)(A_R + r(x_t \wedge q))$. The supplier's expected profit is

$$\pi_S(B|x,q) = \begin{cases} rx & \text{if } B = 0 \text{ and} \\ E[wq \wedge (1-\beta)(A_R + r(q \wedge X_t(x)))] & \text{otherwise.} \end{cases} \tag{3.1}$$

In the above expression, $X_t(x)$ is the supplier's belief about realized demand, given reported sales x. Although the random demand D does not depend on x, the supplier's belief about realized demand $X_t(x)$ is affected by the reported sales. The supplier's optimal strategy is

$$p^*(x) = \arg\max_{0 \le p \le 1} (1-p)rx + pE[wq \wedge (1-\beta)(A_R + rX_t(x)) \wedge (1-\beta)(A_R + rq)]. \tag{3.2}$$

Because $rx < wq$ implies $x < q$ and the retailer will not report x such that $A_R + r(x_t \wedge q) - rx < 0$, the supplier's belief about the realized demand $X_t(x)$ conditional upon reported x, follows the conditional distribution $F(x_t | x_t \ge (x - \frac{A_R}{r})^+)$, where $F(\cdot)$ is the distribution function of the random demand D. (Note that if the retailer reports $x < A_R/r$ then the supplier receives no information update about the realized demand.) Therefore, the supplier's optimization problem becomes

$$p^*(x) = \arg\max_{0 \le p \le 1} (1-p)rx + pE[wq \wedge (1-\beta)(A_R + r(D \wedge q)) | D \ge (x - \frac{A_R}{r})^+]. \tag{3.3}$$

Therefore, optimal p is either 0, or 1, or the supplier is indifferent to its choice of p, and we get

$$p^*(x) = \begin{cases} 0 & \text{if } rx > E[wq \wedge (1-\beta)(A_R + r(D \wedge q)) | D \ge (x - \frac{A_R}{r})^+], \\ \text{any } p \in [0,1] & \text{if } rx = E[wq \wedge (1-\beta)(A_R + r(D \wedge q)) | D \ge (x - \frac{A_R}{r})^+], \\ 1 & \text{otherwise.} \end{cases} \tag{3.4}$$

Next, we consider the retailer's objective, given x_t. We assume that $rx \le wq$ because the retailer has no reason to report sales amount that is greater than wq.

$$\pi_R(x|x_t,q) = \begin{cases} r(x_t \wedge q - x) & \text{if } B = 0 \\ ((1-\beta)(r(x_t \wedge q) + A_R) - wq)^+ - A_R & \text{otherwise.} \end{cases} \tag{3.5}$$

The retailer's expected profit is

$$\pi_R(x) = (1 - p^*(x))(r(x_t \wedge q - x)) + p^*(x)(((1-\beta)(r(x_t \wedge q) + A_R) - wq)^+ - A_R). \tag{3.6}$$

Note that $A_R + r(x_t \wedge q - x) \geq (1-\beta)(r(x_t \wedge q) + A_R) - wq$ always holds if $(1-\beta)(r(x_t \wedge q) + A_R) - wq \geq 0$. Then, the retailer clearly does not want the supplier to choose $p^*(x) = 1$ because in that case it makes less profit. It is reasonable to expect that the retailer will choose the smallest value of x that allows it to make some money, i.e. smallest x such that $rx > E[wq \wedge (1-\beta)(A_R + r(D \wedge q))|D \geq (x - \frac{A_R}{r})^+]$. For ease of exposition, we assume that the supplier will not enforce bankruptcy if it is indifferent to its choice of p. Then the retailer would choose the **smallest** value of x such that

$$rx = E[wq \wedge (1-\beta)(A_R + rD) \wedge (1-\beta)(A_R + rq)|D \geq (x - \frac{A_R}{r})^+]. \tag{3.7}$$

We prove in Proposition 3.1 that Equation (3.7) always has a solution. This solution depends on all of the problem parameters. We use notation x_R to denote the solution of (3.7). If Equation (3.7) has more than one solution, then x_R is chosen as the smallest one. In this sense, the retailer's reported sales amount is unique. That is, if the retailer reports x_R, then the supplier will not enforce bankruptcy and the retailer makes net profit $r(x_t \wedge q - x_R)$. However, it may be not possible for the retailer to report x_R in the case that $A_R + r(x_t \wedge q - x_R) < 0$, and in that case the supplier will call bankruptcy and the retailer will make zero profit. Since the supplier does not observe the actual realized demand, it cannot tell if the retailer is lying.

When the retailer reports x_R and that happens to be less than wq, the supplier believes that the actual demand is low, and therefore willing to forgive the retailer to avoid paying the bankruptcy costs. When the retailer reports sales amount less than x_R, the supplier cannot tell if the retailer's report is accurate or not and calls bankruptcy in part to maintain a credible threat. That is, to enforce truth revelation in those cases when the retailer benefits from doing so. That is, x_R is minimum reported sales that the supplier is indifferent between calling bankruptcy or not, which works as a bankruptcy-threat threshold for the retailer.

Proposition 3.1. *There exists an x_R solving Equation (3.7). Specifically,*

1. *if $(1-\beta)A_R \geq wq$, then $x_R = wq/r$,*
2. *if $E[wq \wedge (1-\beta)(A_R + rq) \wedge (1-\beta)(A_R + rD)] \leq A_R < wq/(1-\beta)$, then $x_R = E[wq/r \wedge (1-\beta)(A_R/r + q) \wedge (1-\beta)(A_R/r + D)]$ and $x_R \leq A_R/r$,*

3. *if* $E[wq \wedge (1-\beta)(A_R + rq) \wedge (1-\beta)(A_R + rD)] > A_R$*, then* $x_R > A_R/r$ *satisfies* $\beta x_R z(x_R - A_R/r) \leq 1$ *and*

$$rx_R\bar{F}(x_R - A_R/r) - \int_{x_R - A_R/r}^{\infty} wq \wedge (1-\beta)(A_R + r(q \wedge \xi))dF(\xi) = 0. \tag{3.8}$$

If $x_t \geq x_R - A_R/r$*, then the retailer will report* x_R *and the supplier will not call bankruptcy. Otherwise, the retailer will report some* $x < x_t + A_R/r$*, and the supplier will enforce bankruptcy.*

In Case 1 and 2 of Proposition 3.1, we have $x_R - A_R/r \leq 0$, and therefore $x_t \geq x_R - A_R/r$ always holds. That is, if the retailer's assets are large enough, then the retailer is able to choose the reported sales amount such that the supplier never calls bankruptcy, regardless of the realized demand. In Case 1, the retailer's assets are very high, then the retailer willingly chooses to fully repay the supplier to avoid paying the bankruptcy costs on its assets. In Case 2, the retailer has medium-high assets. Instead of paying the full debt, it reports an acceptable amount such that the supplier will not call bankruptcy. Case 3 of Proposition 3.1 corresponds to the situation in which the retailer's assets are relatively low. In that case, $x_R - A_R/r > 0$, and therefore the supplier will call bankruptcy if $x_t \leq x_R - A_R/r$, i.e. the realized demand is too low for the retailer to report x_R. In those cases, even if the retailer reports the actual demand, the supplier will enforce bankruptcy. This decision by the supplier serves to enforce appropriate revelation of sales in the other two cases.

Corollary 3.1. *The reported sales revenue is equal to the amount owed only if* $\beta = 0$ *or* $(1-\beta)A_R \geq wq$*.*

Corollary 3.1 shows that the retailer is willing to pay the full debt only if either there are no bankruptcy costs or its assets are relatively large. That is, for the former case the supplier can always call bankruptcy without any costs, and for the latter case the retailer would avoid paying the bankruptcy costs on its assets.

3.4.2 The Retailer's Ordering Decisions in Stage-1

In Stage 1, the retailer's expected profit function is

$$\pi_R(q|w) = E[(A_R + r(D \wedge q) - rx_R)^+] - A_R. \tag{3.9}$$

In the above expression, x_R is determined by Proposition 3.1. Because in Proposition 3.1 the value of x_R depends on both wq and $(1-\beta)(A_R+rq)$, we consider several cases, starting first with the case in which $w \leq (1-\beta)r$. In that case $wq \leq (1-\beta)(A_R+rq)$ is always true, and Equation(3.7) becomes

$$rx = E[wq \wedge (1-\beta)(A_R + rD)|D \geq (x - \frac{A_R}{r})^+]. \tag{3.10}$$

Definition 3.1. Let q_α denote the supremum of q such that $E[wq \wedge (1-\beta)(A_R+rD)] \leq A_R$. That is, q_α denotes the order quantity such that the retailer reports the sales as A_R/r.

Then, the retailer's optimal decisions are shown in Proposition 3.2.

Proposition 3.2. *Given a trade credit contract with wholesale price $w \leq (1-\beta)r$, the retailer's expected profit function is quasiconcave and its optimal ordering decisions are as follows.*

1. *If $r\bar{F}(\frac{(1-\beta)A_R}{w}) \leq w$, then $\bar{F}(q^*) = \frac{w}{r}$ and $x_R^* = wq^*$.*
2. *If $r\bar{F}(\frac{(1-\beta)A_R}{w}) > w$ and $r\bar{F}(q_\alpha) \leq w\bar{F}(\frac{wq_\alpha}{(1-\beta)r} - \frac{A_R}{r})$, then q^* satisfies $r\bar{F}(q^*) = w\bar{F}(\frac{wq^*}{(1-\beta)r} - \frac{A_R}{r})$ and $x_R^* = \int_0^\infty \frac{w}{r}q^* \wedge (1-\beta)(\frac{A_R}{r} + \xi)dF(\xi) \leq \frac{A_R}{r}$.*
3. *If $r\bar{F}(q_\alpha) > w\bar{F}(\frac{wq_\alpha}{(1-\beta)r} - \frac{A_R}{r})$ and $r\bar{F}(q_\alpha)(1-\beta z(0)\frac{A_R}{r}) \leq w\bar{F}(\frac{wq_\alpha}{(1-\beta)r} - \frac{A_R}{r})$, then $q^* = q_\alpha$ and $x_R^* = \frac{A_R}{r}$.*
4. *Otherwise, q^* satisfies $r\bar{F}(q^*)(1-\beta x_R z(x_R - \frac{A_R}{r})) = w\bar{F}(\frac{wq^*}{(1-\beta)r} - \frac{A_R}{r})$ and $x_R^* > \frac{A_R}{r}$ satisfies Equation (3.8).*

In the first case of Proposition 3.2, the retailer's assets are high, and therefore it will fully repay the supplier. Its order quantity will be the same as that of a financially-unconstrained newsvendor. In the remaining three cases, the retailer will report a sales amount that the supplier will find acceptable, without initiating bankruptcy. Note that the retailer will not repay the loan in full in all three cases.

From Proposition 3.1, the retailer's marginal profit as a function of q depends on whether $x_R \leq A_R/r$ or not. Therefore, in Case 2 of Proposition 3.2, the retailer selects x_R to avoid bankruptcy and chooses its q such that its margin equals zero. In Case 3, the bankruptcy cost is still high, and the retailer's profit increases with q in the range of no bankruptcy. Therefore, it is profitable for the retailer to order up to a point until which it faces no risk of bankruptcy. In Case 4, the retailer's assets are relatively low and it is willing to take the risk of bankruptcy. Therefore, it chooses q such that its marginal profit equals zero with positive probability of bankruptcy.

Next, we investigate the retailer's profits when $w > (1-\beta)r$. In that case, it is possible that $wq > (1-\beta)(A_R + rq)$. If the supplier calls bankruptcy, the maximum loan repayment will be $(1-\beta)(A_R+rq)$ instead of wq, which may encourage the retailer to order aggressively. If $wq > (1-\beta)(A_R+rq)$, i.e. $(w-(1-\beta)r)q > (1-\beta)A_R$ (Note that this case happens only if $w > (1-\beta)r$), then Equation (3.7) becomes

$$rx_R = E[(1-\beta)(A_R + rD) \wedge (1-\beta)(A_R + rq)|D \geq (x_R - \frac{A_R}{r})^+]. \tag{3.11}$$

Interestingly, x_R does not depend on w, and therefore both π_R and π_S do not depend on w. The retailer's problem becomes

$$\begin{aligned} \max_q \quad & \pi_R(q|w) = E[(A_R + r(D \wedge q) - rx_R)^+] - A_R \qquad (3.12) \\ s.t. \quad & rx_R = E[(1-\beta)(A_R + rD) \wedge (1-\beta)(A_R + rq)|D \geq (x_R - \frac{A_R}{r})^+], \\ & (w-(1-\beta)r)q \geq (1-\beta)A_R. \end{aligned}$$

Definition 3.2. Let $x_{Rm} = \max\{A_R/r, \hat{x}\}$, where $\hat{x}$ solves equation $\hat{x}z((\hat{x}-A_R/r)^+) = 1$. Let q_m solve $rx_{Rm}\bar{F}((x_{Rm}-A_R/r)^+) - \int_{(x_{Rm}-A_R/r)^+}^{\infty}(1-\beta)(A_R+r(q_m \wedge \xi))dF(\xi) = 0$.

Lemma 3.1. *In the range of q such that $(w-(1-\beta)r)q \geq (1-\beta)A_R$, the retailer's optimal order quantity is $\tilde{q}^* = q_m$ and $\tilde{x}_R^* = x_{Rm}$ if $q_m \geq \frac{(1-\beta)A_R}{(w-(1-\beta)r)}$. Otherwise, $\tilde{q}^* = \frac{(1-\beta)A_R}{(w-(1-\beta)r)}$ and $\tilde{x}_R^*$ satisfies Equation (3.11).*

The solution described in Lemma 3.1 solves the subproblem (3.12). We still need to compare its profit value with the value of $\pi_R(q|w)$ under additional constraints: the constraints (3.10) and the constraint implied by the inequality $(w-(1-\beta)r)q \leq (1-\beta)A_R$. We next show that there exists a wholesale price threshold $\bar{w}$ such that $(\tilde{q}^*, \tilde{x}_R^*)$ is global optimal to the retailer if $w \geq \bar{w}$. Otherwise, the optimal order quantity is chosen such that $(w-(1-\beta)r)q \leq (1-\beta)A_R$.

Definition 3.3. Let $\bar{w}$ denote the wholesale price such that

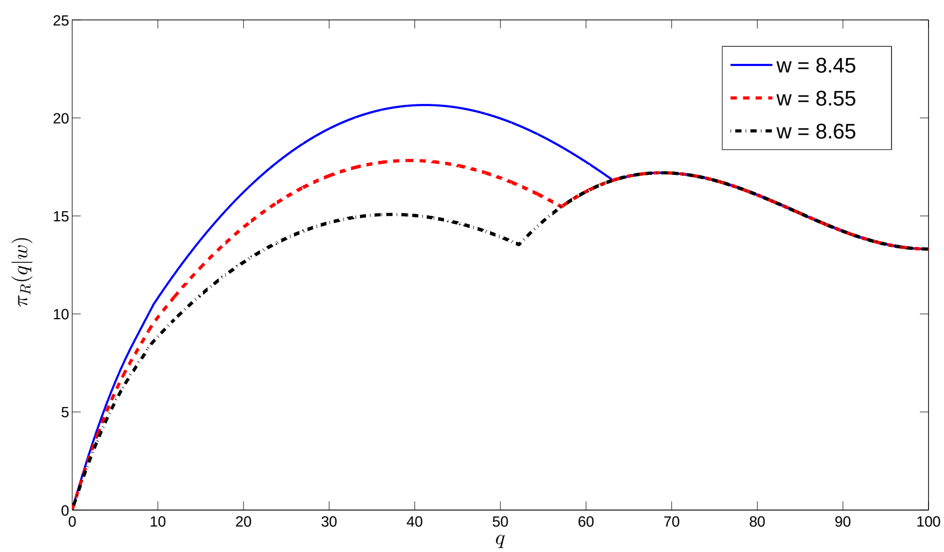
$$\pi_R(\tilde{q}^*, \tilde{x}_R^*|\bar{w}) = \max\{\pi_R(q|\bar{w})|q \leq \frac{1-\beta)A_R}{w-(1-\beta)r}\}$$

.

Lemma 3.2. *If $w \leq \bar{w}$, then $(w-(1-\beta)r)q^* \leq (1-\beta)A_R$. Otherwise, $q^* = \tilde{q}^*$ and $x^* = \tilde{x}_R^* = x_{Rm}$, where $(\tilde{q}^*, \tilde{x}_R^*)$ is obtained from Lemma 3.1.*

It is clear from Equation (3.10) that x_R increases with w. Therefore, the retailer's optimal profit decreases with w if $(w-(1-\beta)r)q \leq (1-\beta)A_R$. At $w = \bar{w}$, the retailer is indifferent between choosing $(w-(1-\beta)r)q \leq (1-\beta)A_R$ or $(w-(1-\beta)r)q \geq (1-\beta)A_R$. If $w < \bar{w}$, the retailer prefers smaller order quantity such that $(w-(1-\beta)r)q \leq (1-\beta)A_R$. If $w > \bar{w}$, the wholesale price is high and increasing w will not affect the retailer's decision since the loan repayment amount no longer depends on w. Figure 3.3 illustrates how the retailer's profit function changes with q when $w > (1-\beta)r$.

Figure 3.3: Retailer's Profits when $w > (1-\beta)r$



In Figure 3.3, the non-smooth points are at $q = \frac{(1-\beta)A_R}{w-(1-\beta)r}$ at different levels of w. Given each w, the corresponding profit function has two peaks. The left peak indicates the optimal q when $q < \frac{(1-\beta)A_R}{w-(1-\beta)r}$ and the right peak indicates the optimal q when $q > \frac{(1-\beta)A_R}{w-(1-\beta)r}$. As w increases, the left peak becomes lower, i.e. both the profit and the order quantity get smaller, whereas the right peak remains the same. When w is sufficiently high, (i.e. above $\bar{w}$ in Lemma 3.2), the global optimal q jumps from the left peak to the right peak. This means, when the supplier charges a high wholesale price, anticipating that its loan repayment will be at least as much as the value of the retailer's assets, the threat of bankruptcy is less important to the retailer. The retailer orders aggressively and reports low sales. This increases the retailer's profit. We will explore the supplier's decision in the next section.

Next, we investigate the retailer's decision if the wholesale price is larger than $(1-\beta)r$. First of all, in such cases, the retailer's profit function may not be quasiconcave, e.g. see Figure 3.3. Therefore, it is difficult to analytically identify the global optimal solution. We provide all local optima, which are candidates for the global optimum, in Proposition 3.3. Given these candidate solutions, the global optimal solution may be chosen by selecting the local optimum that maximizes the profit of the retailer among all local optima.

Proposition 3.3. *If $w > (1-\beta)r$, then the retailer's optimal order quantity is chosen from one of the following.*

1. $\bar{F}(q_1^*) = \frac{w}{r}$ *such that* $q_1^* \leq (1-\beta)A_R/w$, *and* $x_R^* = wq^*$.
2. q_2^* *satisfies* $r\bar{F}(q_2^*) = w\bar{F}(\frac{wq_2^*}{(1-\beta)r} - \frac{A_R}{r})$ *such that* $(1-\beta)A_R/w \leq q_2^* \leq q_\alpha$, *and* $x_R^* = \int_0^\infty \frac{w}{r}q_2^* \wedge (1-\beta)(\frac{A_R}{r}+\xi)dF(\xi) \leq \frac{A_R}{r}$.
3. $q_3^* = q_\alpha$ *and* $x_R^* = \frac{A_R}{r}$.
4. q_4^* *satisfies* $r\bar{F}(q_4^*)(1-\beta x_R z(x_R - \frac{A_R}{r})) = w\bar{F}(\frac{wq_4^*}{(1-\beta)r} - \frac{A_R}{r})$ *such that* $q_\alpha \leq q_4^* \leq \frac{(1-\beta)}{w-(1-\beta)r}A_R$, *and* $x_R^* > \frac{A_R}{r}$ *satisfies Equation (3.8).*
5. $q_5^* = q_m$ *if* $q_m \geq \frac{(1-\beta)}{w-(1-\beta)r}A_R$, *and* $x_R^* = x_{Rm}$.

The retailer chooses $q^* = \arg\max_{i=1,2,\cdots,5} \pi_R(q_i^*|w)$.

Proposition 3.3 provides a necessary condition for the retailer's optimal order quantity. These candidates are either points at which the retailer's marginal profit of changing q equals zero, or the non-smooth points of the retailer's profit function. Candidate 3 occurs at a non-smooth point because of the bankruptcy costs. That is, q_3^* is the largest order quantity such that the retailer avoids the risk of bankruptcy. The solution corresponding to Candidates 2 and 4 may not be unique. The retailer must compare the profits from all candidates to choose the global optimal solution.

3.4.3 The Supplier's Decisions in Stage-1

In this section, we investigate the supplier's choice of the wholesale price w. The supplier maximizes its expected profit function shown in Equation (3.13).

$$\pi_S(w, q(w)) = (1-\beta)E[(A_R + rD); D \leq (x_R - A_R/r)^+] + rx_R\bar{F}((x_R - A_R/r)^+) - cq, \tag{3.13}$$

where x_R is determined by Proposition 3.1. Substituting x_R into the above equation, we obtain that

$$\pi_S(w, q(w)) = E[wq \wedge (1-\beta)(A_R + rD \wedge rq)] - cq. \tag{3.14}$$

By Lemma 3.2, we have that

$$\pi_S(w, q(w)) = \begin{cases} E[wq \wedge (1-\beta)(A_R + rD)] - cq & \text{if } w \leq \bar{w}, \\ E[(1-\beta)(A_R + rD \wedge rq_m)] - cq. & \text{otherwise.} \end{cases} \tag{3.15}$$

Similar to the retailer's optimal decision when $w > (1-\beta)r$, we cannot obtain sufficient conditions for the supplier's global optimal decisions. We next provide all candidate solutions, one of which maximizes the supplier's profit.

Definition 3.4.

1. Let (w_{s1}, q_{s1}) denote a solution to $\bar{F}(q)(1 - qz(q)) = \frac{c}{r}$ and $\bar{F}(q) = \frac{w}{r}$.
2. Let (w_{s2}, q_{s2}) denote a solution to $\bar{F}(q) = \frac{w}{r}$ and $q = (1-\beta)A_R/w$.
3. Let (w_{s3}, q_{s3}) denote a solution to $r\bar{F}(q) = w\bar{F}(\frac{wq}{(1-\beta)r} - \frac{A_R}{r})$ and $r\bar{F}(q)(qz(q) - 1) + c(1 - \frac{wq}{(1-\beta)r}z(\frac{wq-(1-\beta)A_R}{(1-\beta)r})) = 0$.
4. Let (w_{s4}, q_{s4}) denote a solution to $r\bar{F}(q) = w\bar{F}(\frac{wq}{(1-\beta)r} - \frac{A_R}{r})$ and $q = q_\alpha(w)$.
5. Let (w_{s5}, q_{s5}) denote a solution to $r\bar{F}(q)(1 - \beta x_R z(x_R - \frac{A_R}{r})) = w\bar{F}(\frac{wq}{(1-\beta)r} - \frac{A_R}{r})$ and $q = q_\alpha(w)$.
6. Let (w_{s6}, q_{s6}) denote a solution to $\frac{\partial \pi_S}{\partial w} = 0$ and $r\bar{F}(q)(1 - \beta x_R z(x_R - \frac{A_R}{r})) = w\bar{F}(\frac{wq}{(1-\beta)r} - \frac{A_R}{r})$.
7. Denote $w_{s7} = \bar{w} - \epsilon_1$, where $\epsilon_1 > 0$ is an arbitrary small constant.

8. Denote $w_{s8} = \bar{w} + \epsilon_2$, where ϵ_2 is a positive constant.

Proposition 3.4. *The supplier's optimal decisions on w are as*

$$w^* = \arg \max_{i=1,2,\cdots,8} \pi_S(w_{s,i}, q^*(w_{s,i})),$$

where $q^*(w_{s,i})$ is obtained from Proposition 3.2 and 3.3.

3.5 Analysis: The Trade-Forgiveness (TF) Contract

3.5.1 The Retailer's Decisions

Suppose the retailer accepts a TF contract with parameters (w, λ). That is, the retailer pays wholesale price w, knows that it will be allowed a forgiveness quotient λ, and agrees to let a trusted third-party verify sales. The supplier pays a fixed sales verification cost κ, which does not enter into the retailer's decisions. In order to determine the retailer's corresponding order quantity, we define two threshold demand levels, y_1 and y_2, as follows. Given q, y_1 is the minimum demand level at which the retailer can fully repay supplier's loan without requiring loan forgiveness, i.e. $y_1 = \frac{1}{r}(wq)$. Similarly, y_2 is a threshold such that if demand is less than y_2, then the retailer becomes insolvent, i.e. $y_2 = \frac{1}{r}[wq - \frac{A_R}{1-\lambda}]^+$. It should be clear that $y_2 \leq y_1$ and that if $D < y_2$, then the retailer loses all its assets. If $D \in [y_2, y_1]$, then the retailer takes advantage of loan forgiveness offered by the supplier. Its payment to the supplier in that case equals $wq - \lambda(wq - r\min\{q, D\})$. Finally, if $D \geq y_1$, the retailer pays the supplier in full. Combining these case, the retailer's expected profit is:

$$\pi_R(q) = \int_{y_1}^{\infty} [r\min\{q,\xi\} - wq]dF(\xi) + \int_{y2}^{y_1} [r\xi - wq + \lambda(wq - r\min\{q,\xi\})]dF(\xi) \\ - A_R F(y_2) \tag{3.16}$$

$$= \begin{cases} \int_0^q r\bar{F}(\xi)d\xi - \lambda\int_0^{y_1} r\bar{F}(\xi)d\xi - (1-\lambda)wq, & \text{if } y_2 = 0, \\ \int_0^q r\bar{F}(\xi)d\xi - \lambda\int_0^{y_1} r\bar{F}(\xi)d\xi - (1-\lambda)\int_0^{y_2} r\bar{F}(\xi)d\xi - A_R, & \text{if } y_2 > 0 \end{cases} \tag{3.17}$$

In Equation (3.16), the first term is the retailer's profit when it pays the loan in full, the second term captures the scenario when loan forgiveness is exercised and finally the third term corresponds to the event in which insolvency occurs. The following proposition provides the retailer's optimal order quantity.

Proposition 3.5. *Given a contract (w, λ), the retailer's optimal order quantity $q^*(w, \lambda)$ is obtained by solving the following implicit equation.*

$$r\bar{F}(q) = \lambda w\bar{F}(y_1) + (1-\lambda)w\bar{F}(y_2), \tag{3.18}$$

where $y_1 = \frac{1}{r}(wq)$ *and* $y_2 = \frac{1}{r}[wq - \frac{A_R}{1-\lambda}]^+$.

In Proposition 3.5, $r\bar{F}(q)$ is the retailer's marginal revenue from sales, $w\bar{F}(y_1)$ is its borrowing cost if no forgiveness incurs, and $(1-\lambda)w(\bar{F}(y_2) - \bar{F}(y_1))$ is its marginal cost if the debt is partially forgiven. Thus, Proposition 3.5 shows that the retailer chooses its order quantity such that its marginal revenue equals the marginal cost.

Before proceeding further, we present a lemma that is needed for the proofs of our subsequent results. Note that in Lemma 3.3, $q_c^* = \bar{F}^{-1}(\frac{c}{r})$, which was defined earlier as the optimal order quantity in the financially-unconstrained newsvendor model.

Lemma 3.3. *If* $q_c^* z(q_c^*) \leq 1$, *then* $q\bar{F}([q-M]^+)$ *is concave and strictly increasing in* $q \in [0, q_c^*]$, *where* $M \geq 0$ *is an arbitrary constant.*

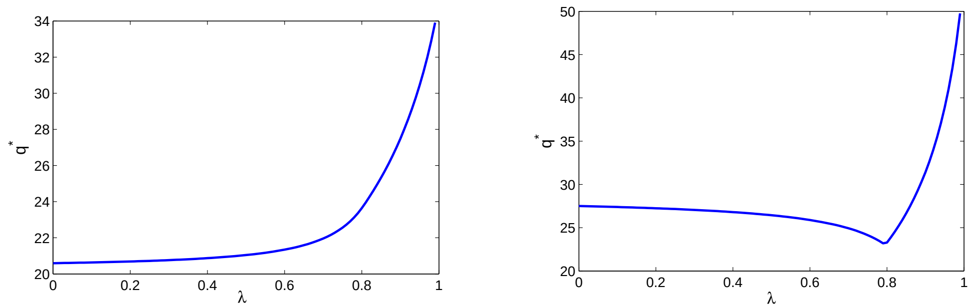
Next, we examine how the retailer's optimal decision changes in contract parameters w and λ.

Corollary 3.2. *For a fixed λ, the retailer's optimal order quantity q^* decreases in w.*

This result is consistent with intuition that when offered a higher wholesale price, the retailer's ordering cost increases, so the retailer will order less. In term of the forgiveness quotient, our first intuition might be that when the supplier forgives more, the retailer will order more, however, this intuition is not always true. It can be easily checked that when the retailer is sufficiently rich (i.e. $y_2 \equiv 0$), the optimal q increases with λ. This is however not true when the retailer faces the risk of insolvency. The following examples show that in such instances, the retailer may order less when the forgiveness quotient is higher.

In the two examples, we fix $r = 10$, $c = 5$, $w = 8.5$, $A_R = 40$. The demand distribution is Gamma with parameters (2, 20) in Figure 3.4(a), and is exponential with parameter 0.02 in Figure 3.4(b). We see in the first example, the retailer's optimal order quantity increases in λ, which is consistent with our intuition that the more the supplier forgives the more the retailer orders. However, the outcome in the second example is

Figure 3.4: Relationship between q^* and λ.



(a) D $\sim$ Gamma(2,20)

(b) D$\sim$ Exponential(0.02)

not expected. In this case the retailer's optimal order quantity first decreases and then increases in λ. This observation can be explained as follows. Consider a retailer with few assets. If λ is small (e.g. $\lambda = 0.2$ in Figure 3.4(b)), the retailer has high risk of insolvency and it has little to lose when it becomes insolvent. The retailer does not go insolvent only if demand is high. Therefore, the retailer orders aggressively such that it makes much profit if the demand is realized high. However, as λ increases (e.g. $\lambda = 0.5$ in Figure 3.4(b)), the retailer's chance of insolvency goes down. That is, given the same order quantity and realized demand, the retailer may lose everything with a small λ but may have some assets left with a high λ. Therefore, the expected marginal costs with the high λ could be larger than that with the small λ upon increasing q. For this reason, in the high λ scenario the retailer may consider lowering its order quantity to avoid a greater loss if the demand is realized low. Next, with even higher λ (e.g. $\lambda = 0.9$ in Figure 3.4(b)), the retailer gambles using the supplier's money, so it may increase its order quantity as expected.

3.5.2 The Supplier's Decisions

Suppose the realized demand is x. If $x \geq y_1$, then the supplier receives wq. If $y_2 \leq x \leq y_1$, the supplier receives $wq - \lambda(wq - rx)$. If $x \leq y_2$, then $rx + A_R \leq wq - \lambda(wq - rx)$. It is always profitable for the supplier to receive $rx + A_R$ and not to call bankruptcy. Putting these components together, the supplier's expected profit is

$$\pi_S(w, \lambda, q) = wq\bar{F}(y_1) + \int_{y_2}^{y_1} [wq - \lambda(wq - rx)]dF(x) + \int_0^{y_2} [rx + A_R]dF(x) - cq - \kappa. \quad (3.19)$$

Next, we present one of our key results, which explains that the supplier can choose w and λ to induce the retailer to pick any order quantity that the supplier desires. Recall from Equation (3.18) that the retailer's optimal order quantity satisfies

$$q\bar{F}(q) = \lambda\frac{w}{r}q\bar{F}(\frac{w}{r}q) + (1-\lambda)\frac{w}{r}q\bar{F}([\frac{w}{r}q - \frac{A_R}{(1-\lambda)r}]^+), \tag{3.20}$$

which means that $q\bar{F}(q)$ is a convex combination of $\frac{w}{r}q\bar{F}(\frac{w}{r}q)$ and $\frac{w}{r}q\bar{F}([\frac{w}{r}q - \frac{A_R}{(1-\lambda)r}]^+)$. Clearly, we have $w < r$ and $\frac{w}{r}q\bar{F}(\frac{w}{r}q) \leq \frac{w}{r}q\bar{F}([\frac{w}{r}q - \frac{A_R}{(1-\lambda)r}]^+)$. Therefore for every desired q and w, if we have $\frac{w}{r}q\bar{F}(\frac{w}{r}q) \leq q\bar{F}(q) \leq \frac{w}{r}q\bar{F}([\frac{w}{r}q - \frac{A_R}{(1-\lambda)r}]^+)$, then there must exist some $\lambda \in [0, 1)$ such that the above equation holds. Lemma 3.3 shows that $\frac{w}{r}q\bar{F}(\frac{w}{r}q) \leq q\bar{F}(q)$ always holds as long as $q \in [0, q_c^*]$. In Proposition 3.6 we identify conditions under which $\bar{F}(q) \leq \frac{w}{r}\bar{F}([\frac{w}{r}q - \frac{A_R}{(1-\lambda)r}]^+)$ holds.

Definition 3.5. For each q, let $\hat{w}(q)$ denote the maximum of $\{c, w_1\}$, where the w_1 solves the equation $\frac{w}{r}\bar{F}(\frac{wq-A_R}{r}) = \bar{F}(q)$.

That is, $\hat{w}(q)$ denotes the wholesale price that induces the retailer tp order q if the supplier offers no forgiveness. Because for a fixed q the function $\frac{wq}{r}\bar{F}(\frac{wq-A_R}{r})$ is strictly increasing in w (by Lemma 3.3), it follows that $\hat{w}(q)$ is unique. Now, we are ready to present our key result.

Proposition 3.6. *Let $q^* \in (0, q_c^*]$ denote a supplier-selected order quantity. Then, for every $w \in [\hat{w}(q^*), r)$, the supplier can choose a $\lambda \in [0, 1)$ such that the tuple (q^*, w, λ) satisfies Equation (3.18).*

Proposition 3.6 shows that by selecting parameters (w, λ), the supplier can arbitrarily decide the retailer' optimal order quantity, but the wholesale price must not be too small. Otherwise, the desired forgiveness quotient may not exist. For example in Figure 3.5(a), the order quantity increases with λ. In that case, if the retailer orders more than q^* under the TF contract with $(w, 0)$, then the forgiveness quotient cannot reduce the order quantity, and therefore the retailer will always order more than what the supplier desires.

Since the supplier can arbitrarily decide the retailer's optimal order quantity, in particular the supplier-selected order quantity can be q_c^*, the optimal order quantity in a centralized supply chain. Therefore, the choice of q_c^* will result in maximizing both total profits in the centralized and decentralized settings. The difference between these two profits equals the value of the fixed transition cost, i.e. κ. The total profit of the decentralized chains is the same as the profit of the centralized chain. That is,

$$\pi_R(q, w, \lambda) + \pi_S(q, w, \lambda) = \int_0^\infty r \min\{q, x\} dF(x) - cq - \kappa = \pi_c(q) - \kappa \tag{3.21}$$

Let $\pi_c^* := \pi_c(q_c^*)$ denote the optimal centralized supply chain profit. Also, let $\hat{\pi}_S$ denote the supplier's profit given $q = q_c^*$, $w = \hat{w}(q_c^*)$ and $\lambda = 0$ (i.e. substitute $q = q_c^*$, $w = \hat{w}(q_c^*)$ and $\lambda = 0$ in Equation (3.19). We describe next the supplier's optimal decisions if $\pi_R^0 \le \pi_c^* - \hat{\pi}_S - \kappa$. We show that the resulting TF contract coordinates the supply chain.

Theorem 3.1. *If $\pi_R^0 \le \pi_c^* - \hat{\pi}_S - \kappa$, and the supplier offers a TF contract, then it will offer a TF contract with (w, λ) such that the equilibrium order quantity is q_c^*, and the retailer and the supplier make profits π_R^0 and $\pi_c^* - \pi_R^0 - \kappa$, respectively.*

Note that we cannot as yet say whether the supplier will prefer the TF contract over alternatives, e.g. the reported-sales contract or the two-terms contract. That comparison is the focus of the next section.

In Theorem 3.1, the optimal TF contract coordinates the supply chain and the retailer makes profit π_R^0. Therefore, the optimal parameters (w, λ) are obtained by solving the following simultaneous equations.

$$\int_{wq_c^*/r}^\infty [r \min\{q_c^*, x\} - wq] dF(x) + \int_0^{wq_c^*/r} [(1-\lambda)(rx - wq_c^*)] dF(x) = \pi_R^0, \tag{3.22}$$

$$r\bar{F}(q_c^*) = \lambda w \bar{F}(wq_c^*/r) + (1-\lambda)w. \tag{3.23}$$

From Theorem 3.1, we find that whether the TF contracts coordinates the supply chain or not depends on π_R^0 and the threshold $\hat{\pi}_S$. Next, in Proposition 3.7 we explore how these thresholds change with the supplier's and the retailer's assets.

Proposition 3.7. *For any π_R^0, there exists a threshold $\hat{A}_R \le cq_c^*$ such that the retailer makes profit π_R^0, and the supplier makes profit as high as $\pi_c^* - \pi_R^0 - \kappa$ under the optimal TF contract if $A_R \ge \hat{A}_R$.*

In Proposition 3.7, we show that as the retailer's assets increase, it acts less aggressively because it has more to lose if bankruptcy occurs. So the supplier is able to offer a low forgiveness quotient and a correspondingly low wholesale price. In this situation, the retailer collects a greater share of the total supply chain profit. This explains the fact that when the retailer's assets increase, its share of the coordinated supply chain's profit also increases. In addition, when its assets are sufficiently large, the retailer no longer needs loan forgiveness and the wholesale price could be as low as the production cost, thus in that case the total profit could be arbitrarily allocated between the supplier and the retailer.

3.6 Insights & Numerical Examples

In this section, we compare the TF contract with the two-terms and the reported-sales contracts via numerical examples to provide additional insights.

3.6.1 TF vs. Two-terms Contracts

In this section, we compare the TF contract with the two-terms trade credit contract. The TF contract essentially removes information asymmetry upon paying a fee κ to the trusted third-party and offering forgiveness quotient λ.

In the two-terms trade credit contract, the supplier offers two wholesale prices w_1 and w_2. The retailer can pay the supplier either the unit price w_1 as soon as it makes the purchase, or w_2 (where $w_1 \leq w_2$) at the end of the season. Recall that the length of the selling season equals the net terms period. When $w_1 = w_2$, the two-terms contract reduces to a net-terms contract, which is also the same as a TF contract with $\lambda = 0$. We denote the profit of the supplier and the retailer under an optimal two-terms contract as π_S^{tc} and π_R^{tc} respectively.

In Theorem 3.1 we show that with TF contract, the supplier can make any profit between $\hat{\pi}_S$ and π_c^*, where $\hat{\pi}_S$ is the supplier's profit under a net terms contract such that $q = q_c^*$. Because π_S^{tc} is the supplier's profit under an optimal two-terms contract, and $\hat{\pi}_S$ is the supplier's profit under a particular net-terms contract, it is straightforward to argue that $\pi_S^{tc} \geq \hat{\pi}_S$. Because the value of κ does not affect the choice of contract parameters, we first compare the two contracts upon setting $\kappa = 0$. Let $\epsilon > 0$ be a quantity such that $\pi_c^* \geq \pi_R^{tc} + \pi_S^{tc} + \epsilon$. This inequality holds because two-terms contracts cannot achieve supply chain coordination. Then, the supplier can choose the

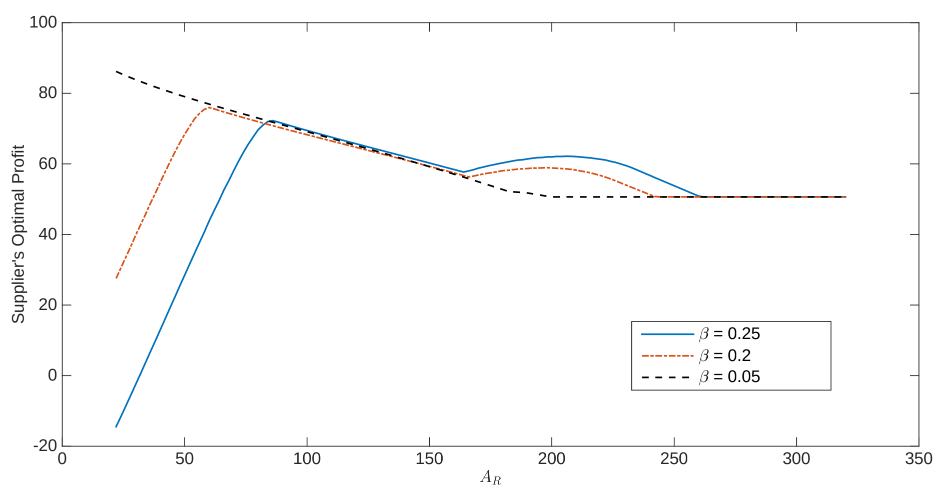
TF contract such that it makes profit $\pi_S^{tc} + \epsilon/2$ and in this case the retailer's profit is $\pi_c^* - (\pi_S^{tc} + \epsilon/2) > \pi_R^{tc}$. This establishes that the supplier can choose the TF contract such that both itself and the retailer are better off on account of debt forgiveness.

The arguments above may not work well when κ is large. That is, if the cost of the third-party's sales-verification services is large, the TF contract may not be superior to the two-terms contract in terms of the supplier's profit, although it can still achieve supply-chain coordination.

3.6.2 TF vs. Reported-Sales Contracts

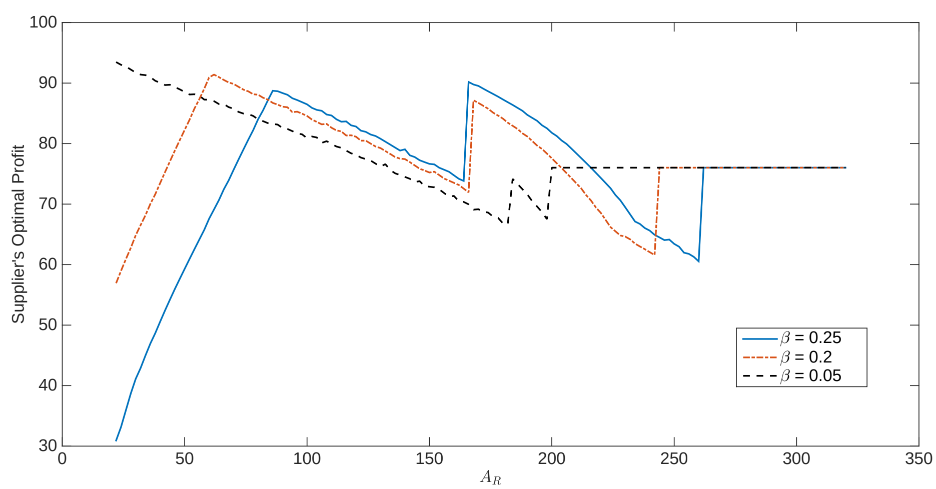
In this section, we use numerical examples to compare the reported-sales contract with the TF contract under different values of the retailer's assets A_R and different bankruptcy costs parameter β. In these examples, D follows the uniform distribution $U[0, 100]$, $r = 10$, and $c = 5.5$. The supplier's optimal profits under the reported-sales contract for different values of the bankruptcy costs are shown in Figure 3.5. The supplier's optimal profits under the TF contracts (without κ) are shown in Figure 3.6. The difference between these two profits (the profit under the TF contracts minus the profit under the reported-sales contract) is shown in Figure 3.7.

Figure 3.5: Effect of A_R and β on Supplier's Profit under Reported-Sales Contract



In Figure 3.5, we see that when A_R is small ($A_R < 60$), the supplier's profit is smaller when β is larger, whereas when A_R is between 170 and 240, the supplier's profit is larger when β is larger. An explanation is as follows. When the retailer's assets are small, it is mostly the supplier who pays the bankruptcy costs, therefore its profit is smaller in β. When the retailer's assets are relatively large, the retailer also experiences bankruptcy costs and it is therefore willing to report higher sales to avoid bankruptcy threat from the supplier. For medium and large assets, the supplier is less sensitive to β because it recovers most, if not all, of its loan. Also, we see that for each β, larger values of the retailer's assets do not uniformly increase the supplier's profit. The reason is that when retailer's assets are large, it acts less aggressively (orders less) to avoid expected bankruptcy costs, which decreases the supplier's profit.

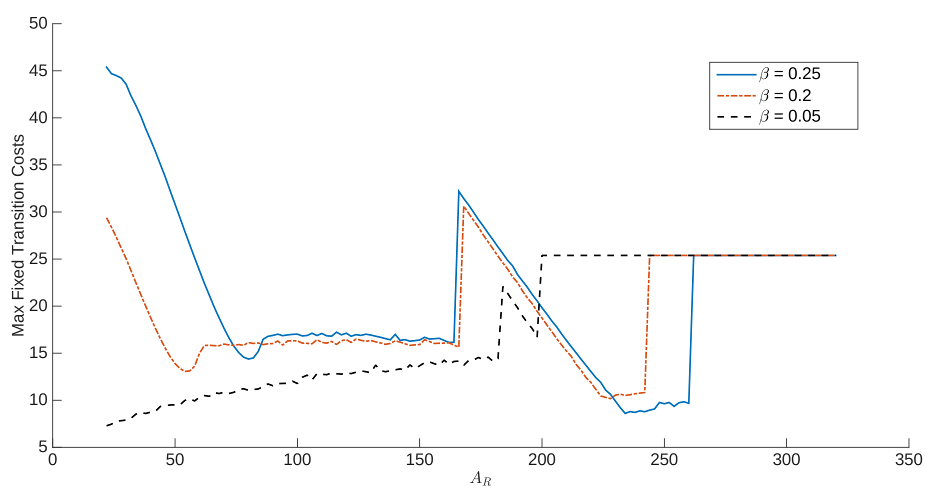
Figure 3.6: Supplier's Optimal Profit under TF contracts (without fixed costs)



In Figure 3.6 we examine the supplier's optimal profit with TF contracts, under the constraint that the retailer makes at least the same profit as it does under the reported-sales contracts. We find that the supplier's profits under TF contracts have similar patterns with its profits under reported-sales contracts in Figure 3.5. The reasons are explained as follows. As shown in our analysis of TF contracts, the supplier can induce the retailer to order as the chain-coordinating quantity, therefore the supplier's profits mainly depend on the retailer's reservation profit, i.e. its optimal profit under reported-sales contracts. In this sense, the explanation follows that of Figure 3.5, and also the patterns of supplier's profit are similar.

The differences of the supplier's profits under these two contracts are shown in Figure 3.7. That is, Figure 3.7 shows the maximum fixed transition costs that the supplier should be willing to pay to learn the retailer's true sales. We find that all lines are above zero, which means that if there is no fixed transition cost, TF contract can benefit the supplier and make the retailer earn at least the same as that under the reported-sales contract. These comparisons show that by announcing the forgiveness quotient ex ante, the supplier gains earns greater profits by increasing the total supply-chain profit. Moreover, the maximum accepted fixed cost is generally smaller in β when the retailer's asset A_R is small, and generally larger in β when A_R is large. The reason is that for relatively small A_R, it is mainly the supplier that pays the bankruptcy costs, and therefore the TF contracts deliver a greater benefit for the supplier by avoiding the bankruptcy costs. In contrast, for relatively large A_R, the retailer mainly pays the bankruptcy cost. The larger bankruptcy costs alleviate the supplier's loss from the retailer's misreport, and therefore reduce the gap of supplier's profits between the reported sales contracts and TF contracts.

Figure 3.7: Maximum Fixed Transition Costs



3.7 Concluding Remarks

In this chapter, we study the interaction between the financial and operational decisions of the supply chain with financial constraints, bankruptcy costs, transaction costs and information asymmetry. Two type of contracts are modeled to study the debt forgiveness in the trade credit environment – reported sales contracts and TF contracts.

In the reported sales contracts, we show that the supplier has to trade off the potential loss of retailer's misreport, against the potential loss of initiating bankruptcy and paying the bankruptcy costs. The equilibrium may result in a payment less than the loan amount. That is, the supplier is willing to forgive part of the loan even though the forgiveness is not prior-committed. We find that in some cases the retailer can leverage private sales information to order more and anticipate the supplier to forgive more, even as the supplier charges a higher wholesale price, which may reduce the supplier's profit.

Our TF contract model reveals the influence of debt forgiveness on both order quantities and bankruptcy costs. When eliminating information asymmetry, we find TF contract is more efficient than the two-terms contract by avoiding the bankruptcy costs and achieving coordination. We show that under some conditions of the retailer's assets and its reservation profit, it is possible for the supplier to achieve its best possible profit by offering a TF contract. When information asymmetry exists, our numerical examples show that TF contract benefits both the supplier and the retailer comparing to the reported sales contract, as long as the fixed transaction costs are not too large.

There are several opportunities for future work along the lines presented in this work. The first extension of this approach is to problems considering the financially-constrained supplier, and the supplier itself may take a loan from the financial market. In such case, the supplier needs to consider its own risk of bankruptcy as well, and we conjecture that this will weaken the power of debt forgiveness.

The other extension is to study the debt forgiveness in a multi-period model. We study the problem in the newsvendor setting, which is more suitable for a short-term lending relationship. For a long-term loan environment, the multi-period model may be more plausible. In such case, the challenge arises not only from both parties dynamically maximizing their own profits, but also from the additional decision dimension of the supplier – to call bankruptcy or to extend the loan to next period. The model complexity may require us to give up some features captured in this chapter.

A third extension of this problem may involve multiple suppliers who sell to multiple retailers. Two special cases may be of particular interest – a supply chain with one supplier and multiple retailers or a supply chain with one retailer and multiple suppliers. In these instances, the interaction between multiple players is expected to be a key challenge.

Chapter 4

Referrals' Timing and Fundraising Success in Crowdfunding

4.1 Introduction

Crowdfunding has recently emerged as an alternative to traditional sources of capital for early-stage ventures. Unlike traditional financing methods, crowdfunding platforms offer entrepreneurs an opportunity to raise money from a large number of potential backers, i.e. the "crowd." Funds come from the aggregation of relatively small contributions from many individuals. Crowdfunding has diffused rapidly over the last few years. Recent industry reports indicate that crowdfunding platforms helped individuals and organizations raise approximately $2.7 billion in 2012 (Massolution 2013), globally, and estimated that the market had grown to as large as $34 billion[1] in 2015 (Massolution 2015). Concurrent with the growth of crowdfunding markets, many academic papers have begun to explore the antecedents and characteristics of successful fundraising campaigns.

Social factors have been found to play a particularly important role. For example,

[1] A breakdown of this estimate into the four types of crowdfunding models is as follows: Peer-to-peer Lending $25B, Donation Crowdfunding $2.85B, Reward Crowdfunding $2.68B, and Equity Crowdfunding $2.5B.

Agrawal et al. (2015) have demonstrated that family and friends play a critical role in successful fundraising, and Mollick (2014) has found that campaign success is highly correlated with the size of the organizer's social network. Research has also reported extensive evidence of peer influence and observational learning in the crowdfunding process (Agrawal et al. 2015, Zhang and Liu 2012, Kuppuswamy and Bayus 2015, Burtch et al. 2013). Despite these observations, few papers have explored the joint implications of these findings for entrepreneurs' and how they should approach engaging with their social network in the fundraising process.

Over the course of the crowdfunding campaign, an entrepreneur's primary vehicle for soliciting support from social connections is the campaign referral. On most crowdfunding platforms, these referrals are implemented via website widgets that integrate with various communication channels, e.g., Facebook, Twitter, Email and so on. When clicked, a referral message is automatically generated and then issued on behalf of the user. These messages typically contain a URL link that directs a recipient back to the campaign URL. We consider this referral action, and aim to address the following research questions: First, how do prior capital accumulation (social proof) and the volume of referrals issued by the entrepreneur impact subsequent fundraising? Second, how does the timing of referral transmission affect fundraising outcomes? Third, in turn, what are the optimal referral strategies for a campaign organizer?

We draw on a large-scale proprietary data set on campaign web traffic, obtained from one of the world's largest reward-based crowdfunding platforms. This data reflects inbound campaign traffic pertaining to 1,147 reward-based crowdfunding campaigns that were conducted between August 2012 and March 2013. We study the probability that a given visitor contributes funds toward the target project, and the effects of prior capital accumulation (social proof) on subsequent fundraising. We find that visitors who are referred by the entrepreneur are, on average, more likely to contribute than organic visitors who arrive of their own accord. We further observe that the probability of contribution increases with capital accumulation for those visitors who are referred by the entrepreneur, while the opposite is observed for organic visitors. The latter result confirms the findings of past work in reward-based crowdfunding about the presence of a "bystander" or "crowding out" effect (Kuppuswamy and Bayus 2015, Burtch et al. 2013). Despite the fact that, conditional on their arrival, organic visitors' become less likely to contribute as a campaign progresses toward its goal, we observe a positive association between capital accumulation and the volume of organic visitors. Thus, although conversion rates decline, the total volume of impressions increases in parallel, in a countervailing fashion. Overall, we find that the combined effect of capital accumulation on fundraising outcomes is generally positive, i.e. the overall volume of organic campaign contributors increases as a campaign progresses toward its goal.

Following these descriptive analyses, we build a Markov-decision-process (MDP) model and calibrate its parameters using our data. The model considers a set of discrete time points at which the entrepreneur may decide how many referrals to use. The entrepreneur is assumed to have a finite number of referrals, i.e. his or her social network. Our objective it to gain some insights into how the amount of capital accumulated affects the optimal timing of referrals. For reasons of analytical tractability, we consider two types of models, each reflecting different assumptions about the relationship between prior capital accumulation, an entrepreneur's referral volumes, and subsequent fundraising outcomes. Specifically, these relationships are described either by a concave function or a convex function. We evaluate the relative "fit" of these two models in our data, finding that the concave model appears to be a better representation of reality. However, the fit of a given model may vary by campaign; that is, the convex model may be a more reasonable representation in some situations.

The two models can be traced back to different assumptions about the effect of capital accumulation on the chances that a visitor to the web site will contribute. The convex model assumes increasing marginal returns to prior capital accumulation and referral volumes, and the concave model assumes decreasing marginal returns. Both types of models are possible, depending on the type of the project and the characteristics of the population of visitors. Other functional forms of the relationship are also possible. We study two possibilities based on behavioral assumptions similar to those made in other papers, e.g. Monahan (1983) and Prastacos (1983).

Interestingly, the optimal referral policies could be quite different under different market environments. In the convex model, an all-or-nothing policy is optimal. That is, entrepreneurs should expend either all of their referrals or none of their referrals. In contrast, in the concave model, i.e., decreasing marginal returns, it may be optimal for the entrepreneurs to issue their referrals in a distributed fashion over the course of the campaign. Because the concave model is found to fit the data better, this findings indicate that, in general, entrepreneurs should aim to distribute their referrals over the course of the entire fundraising process.

4.2 Study Context

On most crowdfunding platforms, a campaign organizer (entrepreneur) must specify a few common characteristics before they can launch the campaign. Specifically, he or she must specify the funding goal or target (in dollars), a funding duration (in days), a campaign category, e.g., technology, design, health, and the menu of contributor rewards. Additionally, some crowdfunding platforms, such as the one we study, provide the entrepreneur with a choice about whether or not to institute a provision-point mechanism, i.e., a fixed, or all-or-nothing fundraising. Under this funding model, all contributions are returned to the crowd if the funding target is not reached by the end of the funding duration. In contrast, under a thresholdless or flexible funding model, the entrepreneur retains all contributions, even if the funding goal is not reached. Each of the above factors are observable to any campaign website visitor, along with some other notable pieces of information, including a real-time indication of the number of days remaining in the fundraising, what percentage of the funding goal has already been achieved, and how many individuals have pledged funds toward the campaign.

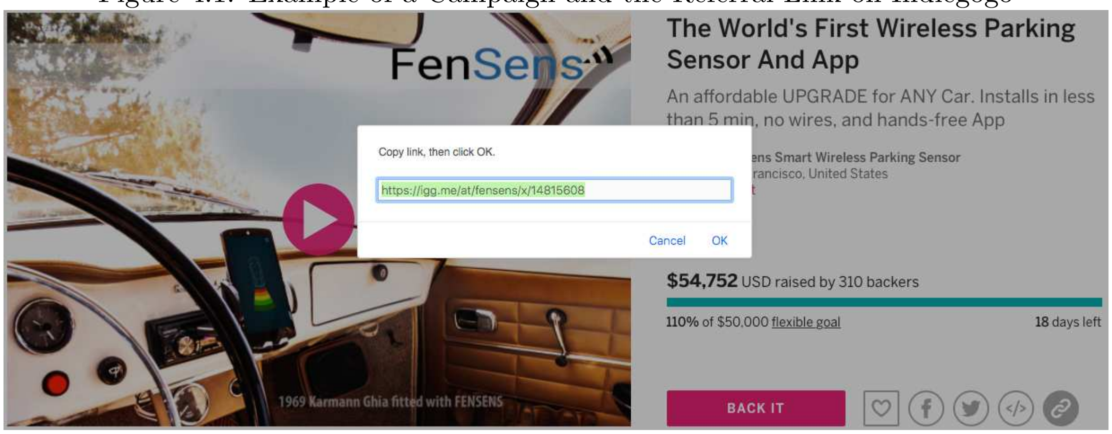
As noted previously, on most crowdfunding sites, including the one we study here, the basic unit of contribution solicitation is the campaign *referral*. In each campaign's Share Bar, e.g see Figure 4.1, users who are logged in can grab a unique link to the campaign, share it with others, and keep track of the number of visitors they bring to the page. Nunnelly (2013) has the following to say about the role of the Share Bar on the Indiegogo website:

> *"Our Insights team dug deeper into our data, and perhaps unsurprisingly, there is a strong positive relationship between the number of referrals a campaign has and the amount of funds raised (as well as percentage of goal raised)... Still, referrals are not just numbers... You want to be pursuing quality leads through referrals by connecting with like-minded and similarly*

passionate audiences, rather than simply spamming the highest number of eyeballs possible...”

Clearly, referrals play an important role in crowdfunding campaign success. Entrepreneurs need to use their referrals strategically if they wish to convert visitors into campaign contributors.

Figure 4.1: Example of a Campaign and the Referral Link on Indiegogo



4.3 Literature Review

Our work contributes to both empirical and mathematical models of crowdfunding. Among empirical papers, several works study the role and behavior of different types of potential backers. Agrawal et al. (2015) and Kuppuswamy and Bayus (2015) both study the role of family and friends. In both these two works, one-time backers who only fund a single project or first-time backers who pledge to a project before contributing to any other are considered as friends. Agrawal et al. (2015) compare local and distant funders by examining the data from crowdfunding platform Sellaband for new musical artists. They find different patterns between local and distant funders and show that the existence of distance-related frictions in online lending is largely explained by the likelihood that some funders have an offline social relationship with the artist. Kuppuswamy and Bayus (2015) consider friend-backers as well as the family-backers who have the same last name as the fundraiser. They examine data from Kickstarter and argue that most of the contributors are one-time backers (join and pledge in the same day) that likely come from the entrepreneurs own social circle. They show that bakers are more likely to contribute in the first and last week of the funding cycle. We also investigate the role of the funderaiser's social network but by analyzing the referrals by the project-owner with the goal of providing guidance on the optimal timing for the use of referrals.

Several works study how the level of funds raised affects the contribution decisions of later visitors to the website. Both Agrawal et al. (2015) and Kuppuswamy and Bayus (2015) show that contribution propensity increases with how much of the goal has been pledged (i.e. "herd effect"). However, these two works do not consider the influence of timing, i.e. how much time it takes for a project to reach a certain percent of goal. We study how investors' contribution decisions are affected by current funding status in different level of elapsed time of the campaign.

Similar to our work, Li and Duan (2014) also study how investors' contribution decisions are affected by both current funding status and elapsed time of the campaign. They shows that investors are more likely to back a project that has already attracted a critical mass of funding goal. For the same level of funding achieved, the contribution propensity declines over time. We also control for the type of visitors (referrals or not). Our results are different and we find that while the non-referrals may be less likely to contribute as the funding level increases, the same conclusion does not apply to referrals. This effect is also consistent with the results in Burtch et al. (2013). They study how the contribution amount and reading time of others affect the behavior of crowd funders for online journalism projects. They find evidence in support of a substitution model, in which the marginal utility contributors gain is diminished by the contribution of others. "Herd effect" is also studied in the micro-loan market Prosper.com by Zhang and Liu (2012). They find lenders engage in active observational learning (rational herding). The herd effect is amplified by exhibiting signals of low quality of the project, but weakened by high project quality.

Mollick (2014) examines how the crowdfunding success is affected by the number of Facebook friends of the founder, the project quality (with a video or not, spelling error, quick update), and different cities of the campaigns. The author also studies whether the entrepreneurs will deliver the rewards in time or not, and finds evidence that overfunded campaigns are more likely to delay delivery of the rewards. Burtch et al. (2015) study the impact of the information (privacy) control mechanisms on crowdfunder behavior via a randomized field experiment.

Among papers containing mathematical models, Belleflamme et al. (2014) compare pre-order and profit-sharing mechanism without considering uncertainty, except in an extension of their model with uncertain product quality. Hu et al. (2015) study the product and pricing decisions for a fixed funding (all-or-nothing) model. They propose a two-period game model in which two cohorts of buyers with different valuations of a perk of the campaign arrive at a crowdfunding project and make sign-up decision sequentially. The entrepreneur set prices (not necessarily the same) for both periods at the beginning of the funding. They suggest different perk-pricing strategies depending on different market characteristics and argue that crowdfunding leads to a smaller reward quality gap between two different perks. Alaei et al. (2016) consider a dynamic model in which consumers arrive sequentially and make decisions about whether to pledge or not. The consumers' decisions depend on their anticipation of the success probability of the project, i.e. how many pledges are needed in the future. The authors suggest that the entrepreneurs should keep the price at a high level to generate as much funding from as few consumers as possible. These works focus on the pricing decisions of the fundraisers, who make one-time decisions. We contribute to the literature by building a dynamic model, studying the timing of the fundraiser referring others to maximizes the total raised amount.

4.4 Data Set

Our data set contains web-traffic records of 1,147 US-based campaigns that were hosted by a leading reward-based crowdfunding platform. All of these campaigns began after August 22^{nd} 2012 and ended before March 18^{th} 2013. Because the crowdfunding platform does not examine the quality of the campaigns and the effort spent by the campaign-owner, the quality of the campaigns are unknown. We limit our analysis to campaigns that achieved at least 10% of their funding target by the end of their respective funding durations, and consider those campaigns as having met a minimum quality standard.

All campaigns in our sample have a fundraising duration that is no longer than 120 days, which was required by the platform at the time the data were achieved. Within these 1,147 campaigns, 1,104 employed a flexible, thresholdless funding model, while only 43 employed a provision-point mechanism, or a fixed-funding model. Given the rarity of fixed-funding campaigns in the sample, our analyses focus on flexible, or thresholdless fundraisers. We further exclude 8 outlier campaigns, namely those that raised more than 350% of their funding targets, as well as 4 campaigns that indicate a very high number of refunds were issued by the platform to campaign contributors. Our data did not contain reasons why refunds were issued.

We define the key variables used in our analysis in Table 4.1 and present summary of statistics in Table 4.2.

Table 4.1: Definition of Key Variables

Variable	**Definition**
Goal	Funding target of campaigns.
Funding Duration	Days from the start to the end of the campaign
Balance	Percentage of goal raised at the end of the campaign.
Visits	Number of visits on the website of the campaign. We eliminate self-visits (i.e. visits by the campaign owner and his/her partners).
Owner-referrals	Visits referred by the campaign owner or his/her partners.
Other-visits	Visits include: non-referrals (visits that are not referred); and other-referred (referred visits, but not by the owner or his/her partners).
Backers	Visitors who contribute (or pledge) to the campaign. The same person may back a campaign more than once.
Num. of Perks	Total number of different perks offered by a campaign.
Min Deliver Perk Price	Minimum perk price with delivery date of a campaign.
Period	Funding period of the campaign. Period $= i$ denotes the time interval of $(\frac{i-1}{5}, \frac{i}{5})$ of the funding duration, where $i = 1, 2, \cdots, 5$.
Balance at visit	Percentage of the goal raised at the time of visit to the campaign.
Current balance	Percentage of the goal raised at the beginning of each period.
Pd_Contri.	Percentage of goal pledged in a period of a campaign. (not accumulated over periods).

Table 4.2: Campaign Descriptive Statistics

Variable	**Mean**	**Std. dev**	**Min**	**Max**
Goal ($)	9,927	21,404	500	400,000
Funding Duration (Days)	47.7	21.65	2	120
Num. of Visits	1,028	3,356	4	64,874
Owner-referrals (% of Visits)	27.47	19.43	0	88.24
Num. of Backers	59.1	157.5	1	2,470
Owner-referral Backers (% of Backers)	34.56	29.47	0	100
Balance (% of Goal)	58.06	47.05	10	318
Owner-referral Funding (% of Balance)	34.31	31.5	0	100
Num. of Perks	7.282	3.405	1	23
Min Deliver Perk Price ($)	33.56	214.5	1	6,000

Note. n=1,092.

4.5 Data Analysis

4.5.1 Contribution Probability from Owner-referrals

In this section, we investigate how a campaign's capital accumulation, or progression toward its fundraising target, at the time of a visit, may affect the owner-referrals' probability of contribution. In addition to fundraising progress, the visitor's contribution decision is likely to also be affected by the campaign's target funding goal, as well as the funding duration and the number of days remaining. We therefore fit the logistic regression model expressed by Equation (1). Let $Y_{(i,t)}$ denote the visitor's binary contribution decision, where i denotes the i^{th} campaigns and t denotes the t^{th} period. $Y_{(i,t)} = 1$ indicates that a contribution was made. Then, we write the model in a linear form, as follows.

$$\begin{aligned} Y_{(i,t)} =& \beta_{0,t} + \beta_{1,t} \cdot \log(Balance\_at\_Visit_{(i,t)}) + \beta_2 \cdot \log(Goal_i) + \beta_3 \cdot Funding\_Duration_i \\ &+ \beta_4 \cdot Num\_of\_Perks_i + \beta_5 \cdot Min\_Deliver\_Perk\_Price_i + \epsilon_{(i,t)}. \end{aligned} \tag{1}$$

We evaluate how well Equation (1) fits our data on all owner-referrals. Descriptive statistics for these sub-samples are provided in Tables 4.3 and the regression results are presented in Table 4.4. In Table 4.4, the interaction terms of Period and log(Balance at Visit) interpret how the visitors' pledging decisions change with the percentage of goal reached that they see in each period.

Table 4.3: Descriptive Statistics of Owner-referrals

		Balance at Visit (% of Goal)				Pledge or Not (0 or 1)	
Period	**Freq.**	**Mean**	**Std. dev.**	**Min**	**Max**	**Mean**	**Std. dev.**
1	68,329	15	23.4	0	270	.0626	.242
2	30,520	28.9	30.1	0	267	.0629	.243
3	27,424	39.3	36.2	0	288	.064	.245
4	23,437	48.3	39.8	0	318	.0768	.266
5	39,708	72.5	54.3	0	367	.0931	.291
Total	189,418	36.9	42.6	0	367	.071	.257

We find that the coefficients of log(Balance at Visit) are all positive and significant ($p < 0.001$) for owner-referrals. This suggests that individuals who are referred by the campaign organizer respond positively to capital accumulation, exhibiting a higher probability of contribution when the campaign balance is larger. Our results are robust under probit regression models and when replacing log(Balance at Visit) with variables Balance at Visit (The results are shown in Table A.2 in the appendix).

4.5.2 Contribution of Other Visitors

In this section, we investigate the contribution of Other-visits. Different from the Owner-referrals, the visit volume of Other-visits are not controllable by the campaign owner. Therefore, both the contribution probability and the visit volume of Other-visits may be affected by the campaign's capital accumulation. We first investigate the contribution probability from Other-visits. Similar to Section 4.5.1, we evaluate how well Equation (1) fits our data on Other-visits. Descriptive statistics for these sub-samples are provided in Tables 4.5 and the regression results are presented in Table 4.6.

We find that the coefficients of log(Balance at Visit) are all negative and significant ($p < 0.001$) for organic visitors. That is, it appears that individuals who arrive at a campaign of their own accord respond negatively to capital accumulation, consistent with prior work that has reported evidence of a bystander effect Kuppuswamy and Bayus (2015) and crowding out Burtch et al. (2013).

Although the contribution probability of organic visitors is clearly decreasing in campaign capital accumulation, this does not necessarily mean that a higher balance

Table 4.4: Regression Result of Model 1 for Owner-referrals

	(1) Logistic Model	(2) Probit Model
Pledge or Not		
Period=2	-0.0435	-0.0231
	(-0.55)	(-0.60)
Period=3	-0.293**	-0.135**
	(-2.72)	(-2.65)
Period=4	-0.0788	-0.0416
	(-0.65)	(-0.71)
Period=5	0.499***	0.244***
	(5.34)	(5.19)
Period=1 × log(Balance at Visit)	0.217***	0.108***
	(19.10)	(19.14)
Period=2 × log(Balance at Visit)	0.159***	0.0788***
	(7.07)	(7.12)
Period=3 × log(Balance at Visit)	0.218***	0.105***
	(7.51)	(7.55)
Period=4 × log(Balance at Visit)	0.199***	0.0985***
	(6.39)	(6.48)
Period=5 × log(Balance at Visit)	0.0900***	0.0451***
	(4.17)	(4.12)
log(Goal)	0.0124	0.00549
	(1.53)	(1.37)
Funding Duration	-0.00136**	-0.000688***
	(-3.21)	(-3.30)
Num_of_Perks	-0.0293***	-0.0143***
	(-11.29)	(-11.19)
Min_Deliver_Perk_Price	-0.000254***	-0.000112***
	(-4.35)	(-4.57)
Constant	-2.808***	-1.579***
	(-37.22)	(-42.68)
Chi2	989.90	1000.14
$p-value$	2.6e-203	1.6e-205
N	195907	195907

t statistics in parentheses
* $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

Table 4.5: Descriptive Statistics of Other-visits

		Balance at Visit (% of Goal)				Pledge or Not (0 or 1)	
Period	**Freq.**	**Mean**	**Std. dev.**	**Min**	**Max**	**Mean**	**Std. dev.**
1	226,372	31.5	33.3	0	282	.0458	.209
2	140,889	42.4	39.5	0	267	.0353	.185
3	139,922	52.4	44.5	0	288	.0382	.192
4	128,666	64.3	50.3	0	318	.0481	.214
5	187,069	83.3	61.3	0	367	.0723	.259
Total	822,918	53.8	50.3	0	367	.0491	.216

will cause an aggregate reduction in contributions from organic visitors. This is because capital accumulation may also affect the total number of organic visitors arriving at the campaign page, if *popular* campaigns achieve greater awareness and receive greater attention.

In the following analysis, we investigate how a project's accumulation of capital and owner-referral volumes affect organic visits and contribution, as well as visits and contributions by other-referrals, i.e., referrals initiated by other platform users. Here, *Other-visits* refers collectively to all visits that are not initiated by an owner referral, and *Other-backers* refers to all backers that are not initiated by an owner referral. For each period, t, we first investigate the number of Other-visits via fitting OLS regression Model 2, and then investigate the number of Other-backers via fitting OLS regression Model 3. The key variables in Model 2 and Model 3 are explained in Table 4.7. We include this variable because that the owner-referrals may share the campaign with others, not only via the referral links, but also via the word-of-mouth. It is also possible that the owner-referrals may pledge on the campaign without using the referral links. From all these aspects, the number of owner-referrals may potentially increase the visits of others.

Table 4.6: Regression Result of Model 1 for Other-visits

	(1) Logistic Model	(2) Probit Model
Pledge or Not		
Period=2	-0.120*	-0.0634**
	(-2.33)	(-2.70)
Period=3	0.166**	0.0747**
	(2.92)	(2.83)
Period=4	0.212***	0.0998***
	(3.49)	(3.48)
Period=5	0.418***	0.202***
	(8.40)	(8.18)
Period=1 × log(Balance at Visit)	-0.0415***	-0.0214***
	(-5.91)	(-6.43)
Period=2 × log(Balance at Visit)	-0.0952***	-0.0435***
	(-6.92)	(-7.05)
Period=3 × log(Balance at Visit)	-0.145***	-0.0678***
	(-9.95)	(-10.21)
Period=4 × log(Balance at Visit)	-0.0976***	-0.0472***
	(-6.68)	(-6.90)
Period=5 × log(Balance at Visit)	-0.0400***	-0.0203***
	(-3.77)	(-3.82)
log(Goal)	-0.0977***	-0.0481***
	(-26.21)	(-27.27)
Funding Duration	0.00657***	0.00324***
	(26.85)	(27.68)
Num_of_Perks	-0.0185***	-0.00892***
	(-12.99)	(-13.27)
Min_Deliver_Perk_Price	-0.0000216	-0.0000114
	(-0.94)	(-1.02)
Constant	-1.901***	-1.132***
	(-49.39)	(-62.32)
Chi2	6,285.39	6,196.68
$p-value$	0	0
N	851,214	851,214

t statistics in parentheses
* $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

Table 4.7: Definition of Additional Variables

Variable	**Definition**
NOV	Number of Other-visits of a campaign in each period.
NOB	Number of backers from Other-visits of a campaign in each period.
Current Balance	Percentage of the goal raised at the beginning of each period.
Owner_Ref	Number of visits of Owner-referrals.

$$\begin{aligned}\log(NOV_{(i,t)}) = &\beta_{0,t} + \beta_{1,t} \cdot \log(Current\_Balance_{(i,t)}) + \beta_{2,t} \cdot \log(Owner\_Ref_{(i,t)}) \\ &+ \beta_3 \cdot \log(Goal_i) + \beta_4 \cdot Funding\_Duration_i \\ &+ \beta_5 \cdot Num\_of\_Perks_i + \beta_6 \cdot Min\_Deliver\_Perk\_Price_i + \epsilon_{(i,t)}\end{aligned} \tag{Model 2}$$

$$\begin{aligned}\log(NOB_{(i,t)}) = &\beta_{0,t} + \beta_{1,t} \cdot \log(Current\_Balance_{(i,t)}) + \beta_{2,t} \cdot \log(Owner\_Ref_{(i,t)}) \\ &+ \beta_3 \cdot \log(Goal_i) + \beta_4 \cdot Funding\_Duration_i \\ &+ \beta_5 \cdot Num\_of\_Perks_i + \beta_6 \cdot Min\_Deliver\_Perk\_Price_i + \epsilon_{(i,t)}\end{aligned} \tag{Model 3}$$

The descriptive statistics are shown in Table A.3 in the appendix. The regression results are shown in Table 4.8. We first look at the second column of Table 4.8. The coefficients of log(Current_Balance) and log(Owner_Referrals) are all positive and significant ($p < 0.001$) except for the coefficient of log(Current_Balance) in period 1. This exception may be because that the majority of the value of the current balance for period 1 is zero, i.e. most entrepreneurs do not fund themselves at their funding starts. In all other periods, the number of Other-visits increases with both the current balance and the number of owner-referrals. Therefore, high balance and more referrals attract larger group of visits to the campaign.

We next look at the third column of Table 4.8. The coefficients of log(Current_Balance) are positive and significant ($p < 0.001$) for period 2 to 5, and the coefficients of log(Current_Balance) in period 1 is positive with $0.01 < p < 0.05$. That is, although high current balance may decrease the contribution probability of Other-visitors, it may attract more people to visit the project website, and therefore the combined effects may result in more backers overall. In terms of the owner-referrals, the coefficients of log(Owner_Referrals) are positive and significant ($p < 0.001$) in period 4 and 5. This implies the owner-referrals will attract more backers when they occur close to the end of the funding cycle.

4.6 Optimal Referral Strategy

On the one hand, the contribution probability of owner-referrals is affected by the project's current balance. On the other hand, contributions from these referrals will add to the current balance, and thereby affect the pledging decisions of other visitors. That is, the essential trade-off that and entrepreneur faces can be summed up as follows. If the entrepreneur uses referrals in early funding periods, then referrals may see low current balance and may not contribute. If the entrepreneur refers late, then the current balance may be too low to attract other visitors in earlier periods. The cumulative effect of lower balance may also affect the conversion rate of the owner referrals. That is, the effectiveness of late referrals may be low because they will exhibit lower propensity to contribute if the balance is low. The referral strategy is important to the entrepreneurs and an optimal recourse is not obvious. We next build a dynamic decision model to study this problem.

4.6.1 Model Formulation

We consider a finite-horizon Markov Decision Process (MDP) model. The campaign is divided into n periods. In period t, the state of the project is described by two quantities. The first is the percent of goal reached at the beginning of this period, denoted by x_t. The second is the available number of referrals at the beginning of this period, denoted by c_t. The entrepreneur decides the number of referrals he/she will use in each period, denoted by k_t. In each period, $k_t \leq c_t$ is satisfied by the entrepreneur's choice of k_t. We assume that the total number of referrals is limited, and the limit is denoted by $\bar{c}$.

Given x_t and c_t, we propose the following model to specify the expected amount of money raised in period t. That is, we assume that the expected amount of money raised in period t depends on the balance at the beginning of this period, and not on how the campaign achieves that percentage of its goal. In Section 4.6.4, we will fit the model in

Table 4.8: Regression Results for NOV and NOB

	log(NOV)	log(NOB)
Period=2	-1.311***	-0.745***
	(-9.83)	(-6.81)
Period=3	-1.807***	-1.087***
	(-11.97)	(-8.73)
Period=4	-2.166***	-1.307***
	(-13.58)	(-9.11)
Period=5	-2.063***	-1.233***
	(-11.27)	(-7.66)
Period=1 × log(Current_Bal)	0.0341	0.138*
	(0.45)	(1.99)
Period=2 × log(Current_Bal)	0.478***	0.219***
	(15.43)	(8.47)
Period=3 × log(Current_Bal)	0.514***	0.252***
	(13.93)	(8.18)
Period=4 × log(Current_Bal)	0.577***	0.290***
	(15.33)	(8.27)
Period=5 × log(Current_Bal)	0.542***	0.316***
	(12.38)	(8.10)
Period=1 × log(Owner_Ref)	0.170***	0.0472*
	(6.88)	(2.11)
Period=2 × log(Owner_Ref)	0.131***	-0.00913
	(6.07)	(-0.47)
Period=3 × log(Owner_Ref)	0.163***	0.0436*
	(7.31)	(2.31)
Period=4 × log(Owner_Ref)	0.188***	0.0874***
	(8.99)	(4.30)
Period=5 × log(Owner_Ref)	0.249***	0.177***
	(12.78)	(9.16)
log(Goal)	0.610***	0.388***
	(40.52)	(27.49)
Funding Duration	0.00160*	0.000763
	(2.40)	(1.21)
Num_of_Perks	0.0612***	0.0418***
	(12.87)	(9.57)
Min_Deliver_Perk_Price	0.0000626	-0.00000108
	(1.03)	(-0.03)
Constant	-2.230***	-2.272***
	(-16.16)	(-18.25)
R^2	0.51	0.33
$p-value$	0	0
N	5,388	5,388

t statistics in parentheses
* $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

Equation (4.1) to the data via an OLS regression model.

$$r_t(x_t, c_t, k_t) = h_t(x_t, k_t)E[D_t], \text{ where } k_t \leq c_t. \tag{4.1}$$

In Equation (4.1), D_t is a random variable. We assume that $D_1, D_2, \cdots, D_n$ are independent, and that without loss of generality $E[D_t] = 1$ for every t. The function $h_t(x, k)$ is a continuous function of (x, k) and increasing in both x and k. Suppose in period t the states are (x_t, c_t), the chosen number of referrals is k_t and the realization of D_t is d_t, then in period $t + 1$, the states are

$$x_{t+1} = x_t + h_t(x_t, k_t)d_t, \tag{4.2}$$

$$c_{t+1} = c_t - k_t. \tag{4.3}$$

Let $u_t(x_t, c_t)$ denote the optimal expected contributed amount from period t through period n. Then for $t = 1, 2, \cdots, n-1$,

$$u_t(x_t, c_t) = \max_{0 \leq k_t \leq c_t} h_t(x_t, k_t) + E_{D_t}\big[u_{t+1}\big(x_t + h_t(x_t, k_t)D_t, c_t - k_t\big)\big], \tag{4.4}$$

$$u_n(x_n, c_n) = h_n(x_n, c_n). \tag{4.5}$$

Note that the entrepreneur will use all remaining referrals in period n. Let

$$v_t(x_t, c_t, k_t) = h_t(x_t, k_t) + E_{D_t}\big[u_{t+1}\big(x_t + h_t(x_t, k_t)D_t, c_t - k_t\big)\big], \tag{4.6}$$

which means that $u_t(x_t, c_t) = \max_{0 \leq k_t \leq c_t} v_t(x_t, c_t, k_t)$.

Next, we characterize the optimal policies under two different assumptions about the function $h(x, k)$: the concave model and the convex model.

4.6.2 The Concave Models

Suppose $h_t(x_t, k_t)$ is jointly concave in (x_t, k_t) for each t. Then we can prove that $v_t(x_t, c_t, k_t)$ is concave in (x_t, c_t, k_t). This helps us identify an optimal referral strategy. Proofs of all propositions from this chapter are presented in Appendix B.3.

Proposition 4.1. *If $h_t(x_t, k_t)$ is concave in (x_t, k_t), then $v_t(x_t, c_t, k_t)$ is concave in (x_t, c_t, k_t) for every t.*

Let

$$h_t^{[1]}(\tilde{x},\tilde{k}) := \frac{\partial h_t(x_t,k_t)}{\partial x_t}|_{x_t=\tilde{x},k_t=\tilde{k}}, \qquad h_t^{[2]}(\tilde{x},\tilde{k}) := \frac{\partial h_t(x_t,k_t)}{\partial k_t}|_{x_t=\tilde{x},k_t=\tilde{k}}, \tag{4.7}$$

$$u_t^{[1]}(\tilde{x},\tilde{c}) := \frac{\partial u_t(x_t,c_t)}{\partial x_t}|_{x_t=\tilde{x},c_t=\tilde{c}}, \text{and} \qquad u_t^{[2]}(\tilde{x},\tilde{c}) := \frac{\partial u_t(x_t,c_t)}{\partial c_t}|_{x_t=\tilde{x},c_t=\tilde{c}}. \tag{4.8}$$

Then taking derivative of $v_t(x_t,c_t,k_t)$ w.r.t k_t, we have

$$\begin{aligned}\frac{\partial v_t}{\partial k_t}(x_t,c_t,k_t) =& h_t^{[2]}(x_t,k_t) + E[u_{t+1}^{[1]}(x_t+h_t(x_t,k_t)D_t, c_t-k_t)h_t^{[2]}(x_t,k_t)D_t \\ &- u_{t+1}^{[2]}(x_t+h_t(x_t,k_t)D_t, c_t-k_t)]\end{aligned} \tag{4.9}$$

Therefore, the optimal policy is characterized as follows.

$$k_t^* = \begin{cases} 0, & \text{if } \frac{\partial v_t(x,c,k)}{\partial k}|_{k=0} \le 0 \\ c, & \text{if } \frac{\partial v_t(x,c,k)}{\partial k}|_{k=c} \ge 0 \\ k : \frac{\partial v_t(x,c,k)}{\partial k} = 0, & \text{otherwise.} \end{cases}$$

That is, if $\frac{\partial v_t(x,c,k)}{\partial k}|_{k=0} \le 0$, then the optimal decision is to refer no one in period t. If $\frac{\partial v_t(x,c,k)}{\partial k}|_{k=c} \ge 0$, then the optimal decision is to refer everyone in period t. Otherwise, the optimal decision is to use a portion of the referrals such that the marginal contribution of referring in period t equals the total marginal contribution of referring in all subsequent periods.

In a continuous-state DP model, the state space is usually discretized so that the DP functional equation that characterizes the solution need only be solved for a finite number of values of the state vector (Johnson et al. 1993). In our case, we discretized both the space of number of referrals c and the space of the current balance x. In particular, we assume that the value set for c is $\mathcal{C} = \{0,1,2,\cdots,\bar{c}\}$. Also, we discretize the funding amount in each period. That is, by adjusting the distribution of D_t, we assume that the upper bound of $h_t(x_t,k_t)D_t$ is M (For example, see Table A.3, the maximum percentage of goal raised in a period is 228%), and its value set is $\{0,1,2,\cdots,M\}$. Therefore, the value set of x_t in period t is $\mathcal{X}_t = \{0,1,2,\cdots,tM\}$. Then an algorithm for finding the optimal referral policy may be constructed as follows.

1. Set $t = n$ and $u_n(x,c) = h_n(x,c)$ for all $x \in \mathcal{X}_n$ and $c \in \mathcal{C}$.

2. Substitute $t-1$ for t, and set $x = 0$.

(a) Set $c = 0$.

i. Set

$$k_t^*(x,c) = \begin{cases} c & \text{if } c = 0 \text{ or } v_t(x,c,c) \geq v_t(x,c,c-1), \\ \min\{k|v_t(x,c,k+1) - v_t(x,c,k) \leq 0\} & \text{otherwise.} \end{cases} \tag{4.10}$$

ii. Set $u_t(x,c) = v_t(x,c,k_t^*(x,c))$.

iii. If $c = \bar{c}$ go to Step 2(b), otherwise substitute $c+1$ for c, and return to (i).

(b) If $x = tM$, go to Step 3, otherwise substitute $x+1$ for x, and return to (a).

3. If $t = 1$, stop, otherwise return to Step 2.

4.6.3 The Convex Models

Suppose $h_t(x,k)$ is convex in (x,k) for every t. Then, we can prove that $v_t(x,c,k)$ is convex in (x,c,k). Therefore, the optimal policy is all or nothing, i.e. either refer everyone or no one.

Proposition 4.2. *If $h_t(x,k)$ is convex in (x,k), then $v_t(x,c,k)$ is convex in (x,c,k) for every t. The optimal policy is to refer either everyone or no one.*

From Proposition 4.2, the optimal policy is referring either everyone or no one. That is,

$$u_t(x,c) = \max\{v_t(x,c,0), v_t(x,c,c)\} \tag{4.11}$$

The optimal policy can be implemented in the help of the following algorithm. Note that we use the discrete approximation introduced in Section 4.6.2.

1. Set $t = n$ and $u_n(x,c) = h_n(x,c)$ for all $x \in \mathcal{X}_n$ and $c \in \{0, \bar{c}\}$.

2. Substitute $t-1$ for t, and set $x = 0$.

   (a) Evaluate $v_t(x,0,0)$.

   (b) Set

$$k_t^*(x,\bar{c}) = \begin{cases} 0 & \text{if } v_t(x,\bar{c},0) \geq v_t(x,\bar{c},\bar{c}), \\ \bar{c} & \text{otherwise.} \end{cases} \tag{4.12}$$

(c) Set $u_t(x, \bar{c}) = v_t(x, \bar{c}, k_t^*(x, \bar{c}))$.

(d) If $x = tM$, go to Step 3, otherwise substitute $x + 1$ for x, and return to (b).

3. If $t = 1$, stop, otherwise return to Step 2.

4.6.4 Model Comparison

In this section, we fit two models to our data. One of the model (Model 4) has a concave structure and the other (Model 5) has a convex structure. The model fit is evaluated with the help of OLS regression models. Recall that the variable Pd_Contri is defined in Table 4.1 as the percentage of goal pledged in a period of a campaign, variables $Currnet\_Balance$ and $Owner\_Ref$ are defined in Table 4.7. The descriptive statistics are shown in Table A.3. The results are shown in Table 4.9 and Table 4.10.

$$\begin{aligned}\log(Pd\_Contri_{(i,t)}) =& \beta_{0,t} + \beta_{1,t} \cdot \log(Current\_Balance_{(i,t)}) + \beta_{2,t} \cdot \log(Owner\_Ref_{(i,t)}) \\ &+ \beta_3 \cdot \log(Goal_i) + \beta_4 \cdot Funding\_Duration_i \\ &+ \beta_5 \cdot Num\_of\_Perks_i + \beta_6 \cdot Min\_Deliver\_Perk\_Price_i + \epsilon_{(i,t)}\end{aligned} \tag{Model 4}$$

$$\begin{aligned}\log(Pd\_Contri_{(i,t)}) =& \beta_{0,t} + \beta_{1,t} \cdot Current\_Balance_{(i,t)} + \beta_{2,t} \cdot Owner\_Ref_{(i,t)} \\ &+ \beta_3 \cdot \log(Goal) + \beta_4 \cdot Funding\_Duration_i \\ &+ \beta_5 \cdot Num\_of\_Perks_i + \beta_6 \cdot Min\_Deliver\_Perk\_Price_i + \epsilon_{(i,t)}\end{aligned} \tag{Model 5}$$

The OLS regression Model 4 fits our concave model in the following way. Taking the exponential transform on both sides of Model 4, we have

$$\begin{aligned}Pd\_Contri_{(i,t)} =& (Current\_Balance_{(i,t)})^{\beta_{1,t}} \cdot (Owner\_Ref_{(i,t)})^{\beta_{2,t}} \\ &\cdot \exp\{\beta_{0,t} + \beta_3 \cdot \log(Goal_i) + \beta_4 \cdot Funding\_Duration_i\} \\ &\cdot \exp\{\beta_5 \cdot Num\_of\_Perks_i + \beta_6 \cdot Min\_Deliver\_Perk\_Price_i\} \\ &\cdot \exp\{\epsilon_{(i,t)}\}\end{aligned} \tag{4.13}$$

Comparing Equation (4.1) and (4.13), the OLS regression evaluates the value function $r_t(x_t, c_t, k_t)$ by assuming that D_t follows a log-normal distribution with the mean equals

1, and

$$h_t(x_t, k_t) = \alpha_{0,t}(x_t + 1)^{\beta_{1,t}}(k_t + 1)^{\beta_{2,t}}. \tag{4.14}$$

where $\alpha_{0,t} = e^{\beta_{0,t}+\beta_3\cdot\log(Goal)+\beta_4\cdot Funding\_Duration+\beta_5\cdot Num\_of\_Perks+\beta_6\cdot Min\_Deliver\_Perk\_Price}$. Note that $\alpha_{0,t}$ is a constant for a given campaign. Moreover, the Hessian matrix of the function $h_t(x_t, k_t)$ is

$$H_t = \begin{bmatrix} \alpha_{0,t}\beta_{1,t}(\beta_{1,t}-1)(x_t+1)^{\beta_{1,t}-2}(k_t+1)^{\beta_{2,t}} & \alpha_{0,t}\beta_{1,t}\beta_{2,t}(x_t+1)^{\beta_{1,t}-1}(k_t+1)^{\beta_{2,t}-1} \\ \alpha_{0,t}\beta_{1,t}\beta_{2,t}(x_t+1)^{\beta_{1,t}-1}(k_t+1)^{\beta_{2,t}-1} & \alpha_{0,t}\beta_{2,t}(\beta_{2,t}-1)(x_t+1)^{\beta_{1,t}}(k_t+1)^{\beta_{2,t}-2} \end{bmatrix}. \tag{4.15}$$

H_t is negative definite if $0 < \beta_{1,t} < 1$, $0 < \beta_{2,t} < 1$, and $\beta_{1,t}(\beta_{1,t} - 1)\beta_{2,t}(\beta_{2,t} - 1) - \beta_{1,t}^2\beta_{2,t}^2 > 0$, which is equivalent to $\beta_{1,t} + \beta_{2,t} < 1$, $\beta_{1,t} > 0$ and $\beta_{2,t} > 0$. That is, if $\beta_{1,t} + \beta_{2,t} < 1$, $\beta_{1,t} > 0$ and $\beta_{2,t} > 0$, then $h_t(x_t, k_t)$ is jointly concave in (x_t, k_t). Examining the coefficients of log(Current_Bal) and log(Owner_Ref) in Table 4.9, we have $\beta_{1,t} + \beta_{2,t} < 1$, $\beta_{1,t} > 0$ and $\beta_{2,t} > 0$ for every t. Therefore, OLS regression Model 4 provides an evaluation of our concave model.

Next, we check that OLS regression Model 5 provides an evaluation of our convex model. Similar to the concave model, we have that the OLS regression evaluates the function $r_t(x_t, c_t, k_t)$ by assuming that D_t follows a log-normal distribution with the mean equals 1, and

$$h_t(x_t, k_t) = \exp\{\alpha_{0,t} + \beta_{1,t}x + \beta_{2,t}k\}, \tag{4.16}$$

where $\alpha_{0,t} = e^{\beta_{0,t}+\beta_3\cdot\log(Goal)+\beta_4\cdot Funding\_Duration+\beta_5\cdot Num\_of\_Perks+\beta_6\cdot Min\_Deliver\_Perk\_Price}$. It is clear that the function (4.16) is jointly convex in (x_t, k_t). Therefore, the OLS regression Model 5 provide an evaluation of our convex model.

Moreover, the concave and convex models are compared based AIC and BIC in Table 4.9 and Table 4.10. The results show that the concave model has lower AIC and BIC, and therefore fits our data better. We further investigate our model fit by considering random effect and fixed effect over project categories, respectively. The regression results are shown in Table A.4 and A.5 in the appendix. These tables confirm that our results in Table 4.9 and Table 4.10 are robust when considering the random effect and fixed effect over project categories.

Table 4.9: Concave Model Regression

	log(Pd_Contri)
Period=2	-0.785***
	(-5.96)
Period=3	-1.254***
	(-8.70)
Period=4	-1.747***
	(-11.25)
Period=5	-1.559***
	(-8.48)
Period=1 × log(Current_Bal)	0.473***
	(8.73)
Period=2 × log(Current_Bal)	0.164***
	(4.71)
Period=3 × log(Current_Bal)	0.247***
	(6.57)
Period=4 × log(Current_Bal)	0.353***
	(8.81)
Period=5 × log(Current_Bal)	0.345***
	(7.33)
Period=1 × log(Owner_Ref)	0.248***
	(11.43)
Period=2 × log(Owner_Ref)	0.196***
	(8.82)
Period=3 × log(Owner_Ref)	0.227***
	(10.63)
Period=4 × log(Owner_Ref)	0.284***
	(13.81)
Period=5 × log(Owner_Ref)	0.373***
	(17.68)
log(Goal)	-0.0768***
	(-5.11)
Funding Duration (Days)	-0.00142*
	(-2.03)
Num_of_Perks	0.0174***
	(3.69)
Min_Deliver_Perk_Price	-0.0000613
	(-1.66)
Constant	2.187***
	(15.67)
R^2	0.25
$p-value$	0
N	5,388
AIC	16,070
BIC	16,196

t statistics in parentheses
* $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

Table 4.10: Convex Model Regression

	log(Pd_Contri)
Period=2	-0.936***
	(-14.76)
Period=3	-1.069***
	(-17.33)
Period=4	-1.116***
	(-17.29)
Period=5	-0.662***
	(-8.68)
Period=1 × Current_Bal	6.003***
	(5.05)
Period=2 × Current_Bal	1.000***
	(5.05)
Period=3 × Current_Bal	0.824***
	(6.20)
Period=4 × Current_Bal	0.708***
	(6.00)
Period=5 × Current_Bal	0.654***
	(5.30)
Period=1 × Owner_Ref	0.00179***
	(5.21)
Period=2 × Owner_Ref	0.00326***
	(6.45)
Period=3 × Owner_Ref	0.00259***
	(4.03)
Period=4 × Owner_Ref	0.00457***
	(6.57)
Period=5 × Owner_Ref	0.00264***
	(3.44)
log(Goal)	-0.0283
	(-1.82)
Funding Duration (Days)	-0.00240**
	(-3.25)
Num_of_Perks	0.0271***
	(5.49)
Min_Deliver_Perk_Price	-0.0000436
	(-1.04)
Constant	2.483***
	(19.67)
R^2	0.17
$p-value$	9.4e-185
N	5,388
AIC	16,614
BIC	16,740

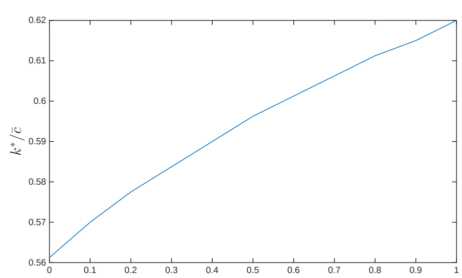
t statistics in parentheses
* $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

4.7 Insights & Discussions

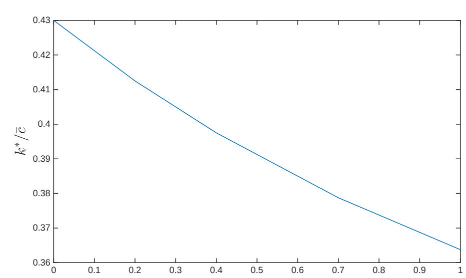
One may intuitively expect that the optimal policy is always increasing or always decreasing in the current balance x_t. However, our analysis shows that this is not true. We show two two-period examples in Figure 4.2. In both cases, $h_t(x_t, k_t)$ are defined as in Equation (4.14). In Figure 4.2(a), the parameters are $\alpha_{0,1} = 0.25, \alpha_{0,2} = 0.15, \alpha_{1,1} = 0.6, \alpha_{1,2} = 0.3, \alpha_{2,1} = 0.1, \alpha_{2,2} = 0.12$. In Figure 4.2(b), the parameters are $\alpha_{0,1} = 0.15, \alpha_{0,2} = 0.15, \alpha_{1,1} = 0.4, \alpha_{1,2} = 0.8, \alpha_{2,1} = 0.1, \alpha_{2,2} = 0.12$. In both examples, we set $\mu_1 = \mu_2 = 1$, $D_1 = 1$ and $\bar{c} = 80$. In each figure, the horizontal axis is the percent of goal reached at the beginning of period 1, and the vertical axis is the optimal number of referrals in period 1. We see that given the first sets of parameters k^* increases with x. However, with the second sets of parameters, k^* decreases with x.

Figure 4.2 explanations are as follows. Suppose $h_t(x_t, k_t)$ are supermodular. Increasing x_t will increase the marginal contribution of k in period t, and the total marginal contribution of k from period $t + 1$ to period n. However, because the total number of referrals is fixed, increasing k in period t will cause a loss of potential contributions in the following periods. Although, by increasing x_t in the concave model, the increment of the marginal contribution of k in period t is larger than the increment in period $t+1$, it may be less than the total increment from period $t+1$ to period n. Therefore, how k_t^* change with x_t depends on the difference of the magnitude of the marginal contribution of k in each period.

Figure 4.2: Optimal Number of Referrals vs. Current Percent of Goal Reached



(a) Increase with x



(b) Decrease with x

Next, we consider two special cases in two-period models. In the two-period model, the value function of the entrepreneur is

$$v(x, c, k) = h_1(x, k) + E[h_2(x + h_1(x, k)D, c - k)], \tag{4.17}$$

and the entrepreneur choose k to maximize $v(x, c, k)$.

Proposition 4.3. *Suppose* $h_t(x, k) = \exp\{\alpha_{0,t} + \alpha_{1,t}x + \alpha_{2,t}k\}$*,* $t = 1$ *or* 2*.*

1. *If* $\alpha_{i,1} \geq \alpha_{i,2}$*, for* $i = 1, 2$*, then for every* c *there exists a threshold* $\hat{x}$ *such that if* $x < \hat{x}$*, then* $k^* = 0$*, and if* $x \geq \hat{x}$ *then* $k^* = c$*.*

2. *If* $\alpha_{i,1} \geq \alpha_{i,2}$*, for* $i = 0, 1, 2$*, then* $\hat{x} = 0$*. That is,* $k^* = c$ *for every* $x \geq 0$ *and* $c \geq 0$*.*

Proposition 4.4. *Suppose* $h_t(x_t, k_t) = f(x_t)g(k_t)$*,* $t = 1, 2$*, when* $f(\cdot)$ *and* $g(\cdot)$ *are positive, increasing and concave. Then* $k^* \leq c/2$*.*

Proposition 4.3 provides a threshold policy that if the current balance is larger than the threshold, then it is optimal to refer all, else refer none. The second statement specifies a situation in which referring all at the very beginning is always optimal. However, in Proposition 4.4 we provide a case in which referring all contacts at the beginning is not optimal.

4.8 Concluding Remarks

In this Chapter, we explore how the referrals initiated by the entrepreneurs are related to the success of fundraising in the crowdfunding market. Our work contributes to both empirical and mathematical models of the crowdfunding study.

By analyzing a large-scale data set on campaign web traffic, we find that the prior capital accumulation is positively associated with the probability of contribution for the visitors who are referred by the entrepreneur, but negatively associated with the probability for the organic visitors. However, we observe that the volume of organic visitors increases with the capital accumulation, countervailing the decline of the conversion rate, which leads to a generally positive combined effect.

Following the descriptive analysis, we build a Markov-decision-process to study how the amount of capital accumulated affects the optimal timing of referrals. Depending on the relationship between prior capital accumulation, the entrepreneur's referrals volumes, and the subsequent fundraising outcomes, we study a convex model and a concave model, and characterize the optimal referring policies. In the convex model, we show that an all-or-nothing policy is optimal. In contrast, in the concave model, it may be optimal for the entrepreneur to issue their referrals in a distributed fashion over the course of the fundraising process. We further evaluate and compare these two models in our data, and find that the concave model is a better representation of reality. That is, if the entrepreneur uses all referrals in early funding period, then referrals may observe low current balance and not contribute. If the entrepreneur refers late, then the current balance may be too low to attract other visitors in earlier periods. Therefore, the entrepreneur trades off between refer-early and refer-late, and should optimally use referrals in a distributed fashion.

We believe that crowdfunding, already a significant source of financing for early-stage ventures, is likely to grow even more significant in the future. Therefore, it is important to study the mechanisms available to entrepreneurs to obtain successful crowdfunding campaigns, and help entrepreneurs raise more money. Our analysis serves to inform entrepreneurs how to use their own social network effectively to achieve funding success in the crowdfunding market. We focus on the timing and volume of the referral. In addition to these factors, the structure of the entrepreneurs' social network may also affect the funding success, which could be pursued in future research.

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Appendix A

Additional Tables of Statistical Analysis for Chapter 4

Table A.1: Regression with Variable Balance at Visit for Owner-referrals

	(1) Logistic Model	(2) Probit Model
Pledge or Not		
Period=2	0.00891	0.00567
	(0.26)	(0.34)
Period=4	0.0807*	0.0401*
	(2.00)	(2.00)
Period=5	0.377***	0.190***
	(11.50)	(11.45)
Period=1 × Balance at Visit	0.701***	0.368***
	(15.11)	(14.58)
Period=2 × Balance at Visit	0.256***	0.130***
	(4.09)	(4.06)
Period=3 × Balance at Visit	0.497***	0.249***
	(8.81)	(8.68)
Period=4 × Balance at Visit	0.415***	0.210***
	(7.87)	(7.74)
Period=5 × Balance at Visit	0.177***	0.0889***
	(5.82)	(5.71)
log(Goal)	0.0113	0.00465
	(1.39)	(1.16)
Funding Duration (Days)	-0.00146***	-0.000767***
	(-3.45)	(-3.68)
Num_of_Perks	-0.0307***	-0.0149***
	(-11.63)	(-11.54)
Min_Deliver_Perk_Price	-0.000246***	-0.000109***
	(-4.36)	(-4.60)
Constant	-2.427***	-1.391***
	(-34.69)	(-40.31)
Chi2	897.50	877.36
$p-value$	1.7e-183	3.6e-179
N	195907	195907

t statistics in parentheses
* $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

Table A.2: Regression with Variable Balance at Visit for Other-visits

	(1) Logistic Model	(2) Probit Model
Pledge or Not		
Period=2	-0.333***	-0.156 ***
	(-14.07)	(-14.66)
Period=3	-0.176***	-0.0830***
	(-7.42)	(-7.66)
Period=4	-0.0445	-0.0209
	(-1.85)	(-1.88)
Period=5	0.363***	0.176***
	(19.13)	(19.17)
Period=1 × Balance at Visit	-0.224***	-0.110***
	(-7.38)	(-7.92)
Period=2 × Balance at Visit	-0.126***	-0.0575***
	(-3.36)	(-3.56)
Period=3 × Balance at Visit	-0.249***	-0.114***
	(-7.71)	(-8.10)
Period=4 × Balance at Visit	-0.109***	-0.0531***
	(-4.08)	(-4.42)
Period=5 × Balance at Visit	-0.0754***	-0.0375***
	(-5.31)	(-5.40)
log(Goal)	-0.0982***	-0.0483***
	(-26.27)	(-27.36)
Funding Duration (Days)	0.00665***	0.00330***
	(27.20)	(28.19)
Num_of_Perks	-0.0184***	-0.00894***
	(-12.91)	(-13.31)
Min_Deliver_Perk_Price	-0.0000155	-0.00000816
	(-0.67)	(-0.73)
Constant	-1.950***	-1.160***
	(-54.45)	(-69.05)
Chi2	6250.54	6145.70
$p - value$	0	0
N	851214	851214

t statistics in parentheses
* $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

Table A.3: Decriptive Statistics of Projects by Period

Variable	**Mean**	**Std. dev.**	**Min**	**Max**
Period 1				
Pd_Contri (% of Goal)	19.2	24	0	228
Current Balance (% of Goal)	.43	2.73	0	58.3
NOV	218	1,490	0	44,114
NOB	11.7	53.8	0	1,177
Owner Ref	64.7	135	0	2498
Period 2				
Pd_Contri (% of Goal)	8.76	12.2	0	114
Current Balance (% of Goal)	19.6	24.2	0	228
NOV	127	457	0	8,681
NOB	5.14	14.5	0	234
Owner Ref	27.7	62.2	0	615
Period 3				
Pd_Contri (% of Goal)	7.73	11.1	0	150
Current Balance (% of Goal)	28.4	29.8	0	278
NOV	129	770	0	18,336
NOB	5.55	30.6	0	882
Owner Ref	24.2	66	0	1,099
Period 4				
Pd_Contri (% of Goal)	8.28	13.3	0	194
Current Balance (% of Goal)	36.4	34.3	0	298
NOV	121	442	0	8,398
NOB	6.35	20.8	0	307
Owner Ref	21.9	57	0	701
Period 5				
Pd_Contri (% of Goal)	16.2	21.9	0	160
Current Balance (% of Goal)	44.8	39.5	0	318
NOV	171	616	0	12,729
NOB	13.2	53.5	0	1,141
Owner Ref	35.8	118	0	2,426

Table A.4: Concave Model with Random and Fixed Effect

	(1) OLS log(Pd_Contri)	(2) Random Effect log(Pd_Contri)	(3) Fixed Effect log(Pd_Contri)
Period=2	-0.785***	-0.773***	-0.767***
	(-5.96)	(-6.46)	(-6.40)
Period=3	-1.254***	-1.231***	-1.219***
	(-8.70)	(-8.89)	(-8.78)
Period=4	-1.747***	-1.698***	-1.672***
	(-11.25)	(-10.77)	(-10.58)
Period=5	-1.559***	-1.497***	-1.468***
	(-8.48)	(-9.15)	(-8.95)
Period=1 × log(Current_Bal)	0.473***	0.476***	0.475***
	(8.73)	(6.40)	(6.37)
Period=2 × log(Current_Bal)	0.164***	0.163***	0.162***
	(4.71)	(5.40)	(5.35)
Period=3 × log(Current_Bal)	0.247***	0.243***	0.240***
	(6.57)	(6.96)	(6.86)
Period=4 × log(Current_Bal)	0.353***	0.344***	0.339***
	(8.81)	(8.80)	(8.64)
Period=5 × log(Current_Bal)	0.345***	0.332***	0.325***
	(7.33)	(8.44)	(8.23)
Period=1 × log(Owner_Ref)	0.248***	0.252***	0.254***
	(11.43)	(11.92)	(11.99)
Period=2 × log(Owner_Ref)	0.196***	0.198***	0.199***
	(8.82)	(8.90)	(8.93)
Period=3 × log(Owner_Ref)	0.227***	0.226***	0.228***
	(10.63)	(10.15)	(10.19)
Period=4 × log(Owner_Ref)	0.284***	0.281***	0.280***
	(13.81)	(12.72)	(12.64)
Period=5 × log(Owner_Ref)	0.373***	0.372***	0.373***
	(17.68)	(18.71)	(18.70)
log(Goal)	-0.0768***	-0.0779***	-0.0849***
	(-5.11)	(-5.36)	(-5.74)
Funding Duration (Days)	-0.00142*	-0.00165*	-0.00170*
	(-2.03)	(-2.38)	(-2.44)
Num_of_Perks	0.0174***	0.0162**	0.0165**
	(3.69)	(3.21)	(3.20)
Min_Deliver_Perk_Price	-0.0000613	-0.0000347	-0.00000342
	(-1.66)	(-0.51)	(-0.05)
Constant	2.187***	2.178***	2.255***
	(15.67)	(16.54)	(17.32)
Chi2		1780.62	
R^2	0.25		0.24
$p-value$	0	0	0
N	5388	5388	5388
AIC	16070	16051	15993
BIC	16196	16190	16118

t statistics in parentheses
* $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

Table A.5: Convex Model with Random and Fixed Effect

	(1) OLS log(Pd_Contri)	(2) Random Effect log(Pd_Contri)	(3) Fixed Effect log(Pd_Contri)
Period=2	-0.936***	-0.932***	-0.930***
	(-14.76)	(-15.61)	(-15.55)
Period=3	-1.069***	-1.063***	-1.060***
	(-17.33)	(-17.27)	(-17.19)
Period=4	-1.116***	-1.105***	-1.099***
	(-17.29)	(-17.22)	(-17.09)
Period=5	-0.662***	-0.647***	-0.640***
	(-8.68)	(-9.94)	(-9.81)
Period=1 × Current_Bal	6.003***	6.052***	6.051***
	(5.05)	(4.85)	(4.84)
Period=2 × Current_Bal	1.000***	0.994***	0.992***
	(5.05)	(6.99)	(6.96)
Period=3 × Current_Bal	0.824***	0.812***	0.805***
	(6.20)	(7.04)	(6.96)
Period=4 × Current_Bal	0.708***	0.685***	0.674***
	(6.00)	(6.78)	(6.65)
Period=5 × Current_Bal	0.654***	0.625***	0.611***
	(5.30)	(7.09)	(6.91)
Period=1 × Owner_Ref	0.00179***	0.00186***	0.00189***
	(5.21)	(7.33)	(7.45)
Period=2 × Owner_Ref	0.00326***	0.00331***	0.00335***
	(6.45)	(5.98)	(6.03)
Period=3 × Owner_Ref	0.00259***	0.00262***	0.00265***
	(4.03)	(5.03)	(5.08)
Period=4 × Owner_Ref	0.00457***	0.00459***	0.00462***
	(6.57)	(7.61)	(7.64)
Period=5 × Owner_Ref	0.00264***	0.00268***	0.00271***
	(3.44)	(9.19)	(9.27)
log(Goal)	-0.0283	-0.0299*	-0.0369*
	(-1.82)	(-1.98)	(-2.40)
Funding Duration (Days)	-0.00240**	-0.00274***	-0.00283***
	(-3.25)	(-3.77)	(-3.87)
Num_of_Perks	0.0271***	0.0274***	0.0284***
	(5.49)	(5.16)	(5.27)
Min_Deliver_Perk_Price	-0.0000436	-0.0000138	0.0000132
	(-1.04)	(-0.19)	(0.18)
Constant	2.483***	2.469***	2.557***
	(19.67)	(19.97)	(21.13)
Chi2		1104.09	
R^2	0.17		0.16
$p-value$	9.4e-185	3.9e-223	0
N	5388	5388	5388
AIC	16614	16586	16524
BIC	16740	16724	16649

t statistics in parentheses
* $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

Appendix B

Proofs of Theorems, Propositions, and Lemmas

B.1 Proofs of Theorems, Propositions, and Lemmas of Chapter 2

Proof of Lemma 2.1. Only a financially constrained supplier uses direct financing and in such cases, we must have $cq_d \geq \kappa$. Upon substituting $B_b = 0$ and $(1+\rho_d)(cq-\kappa) = (1-\alpha)ry_d$ in the relevant part of the supplier's profit function in Equation (2.1), we obtain:

$$\pi_S^d(q) = (1-\alpha)r\int_0^\infty \min\{x,q\}dF(x) - (1-\alpha)r\int_0^{y_d} xdF(x) - (1-\alpha)ry_d\bar{F}(y_d) - \kappa. \quad \text{(B.1)}$$

Because $\frac{\partial y_d}{\partial q} = \frac{(1+\rho_d)c}{(1-\alpha)r}$, we next obtain

$$\frac{\partial \pi_S^d(q)}{\partial q} = (1-\alpha)r\bar{F}(q) - (1-\alpha)r\bar{F}(y_d)\frac{\partial y_d}{\partial q} = (1-\alpha)r\bar{F}(q) - (1+\rho_d)c\bar{F}(y_d), \quad \text{(B.2)}$$

$$\frac{\partial^2 \pi_S^d(q)}{\partial q^2} = -(1-\alpha)rf(q) + \frac{[(1+\rho_d)c]^2}{(1-\alpha)r}f(y_d). \quad \text{(B.3)}$$

In Equation (B.2), if $(1+\rho_d)c \geq (1-\alpha)r$, then $\frac{\partial \pi_S^d(q)}{\partial q} \leq 0$ because $y_d \leq q$ implies $\bar{F}(y_d) \geq \bar{F}(q)$. In such cases, the optimal solution is $q_d = \kappa/c$. Alternatively, if $(1+\rho_d)c \leq (1-\alpha)r$, then for q such that $\frac{\partial \pi_S^d(q)}{\partial q} \geq 0$, we have $\frac{\bar{F}(q)}{\bar{F}(y_d)} \geq \frac{(1+\rho_d)c}{(1-\alpha)r}$. These arguments lead to the following inequalities:

$$\frac{\partial^2 \pi_S^d(q)}{\partial q^2}\Big|_{\frac{\partial \pi_S^d(q)}{\partial q}\geq 0} \leq -(1-\alpha)rf(q) + (1+\rho)c\frac{\bar{F}(q)}{\bar{F}(y_d)}f(y_d) \tag{B.4}$$

$$= (1-\alpha)r\bar{F}(q)[-z(q) + \frac{(1+\rho_d)c}{(1-\alpha)r}z(y_d)] \leq 0. \tag{B.5}$$

The inequality in (B.5) comes from $\frac{(1+\rho_d)c}{(1-\alpha)r} \leq 1$ and the IFR assumption, which means $z(q) \geq z(y_d)$. To summarize what we have shown so far, either $\pi_S^d(q)$ decreases in q or concavely increases for $q \geq \kappa/c$, which establishes that the differentiable function $\pi_S^d(q)$ is unimodal.

If $c\bar{F}^{-1}(\frac{(1+\rho)c}{(1-\alpha)r}) \leq \kappa$, we have $\frac{\partial \pi_S^d(q)}{\partial q}\big|_{q=\kappa/c} = (1-\alpha)r\bar{F}(\kappa/c) - (1+\rho_d)c \leq 0$, which means $\frac{\partial \pi_S^d(q)}{\partial q} \leq 0$ for every q. Thus $q_d = \kappa/c$. Otherwise, if $c\bar{F}^{-1}(\frac{(1+\rho_d)c}{(1-\alpha)r}) \geq \kappa$, then optimal q is a solution to the first-order optimality equation $(1-\alpha)r\bar{F}(q) - (1+\rho_d)c\bar{F}(y_d) = 0$. If $c\bar{F}^{-1}(\frac{(1+\rho_d)c}{(1-\alpha)r}) \geq \kappa$ and $(1-\alpha)r\bar{F}(q) - (1+\rho_d)c\bar{F}(y_d) = 0$ does not have a solution, then it means that $\frac{\partial \pi_S^d(q)}{\partial q} \geq 0$ for all q. However, we note that $\pi_S^d(\kappa/c) > \pi_S^d(+\infty) = -\kappa$, which is a contradiction. This completes the proof . □

Proof of Lemma 2.2. Similar to the proof of Lemma 2.1, upon substituting $B_d = 0$ and $(1+\rho_b)(cq-\kappa) = (1-\alpha)ry_b$ in the relevant part of the supplier's profit function in Equation (2.1), we obtain:

$$\pi_S^b(q) = (1-\alpha)r\int_0^\infty \min\{x,q\}dF(x) - (1-\alpha)r\int_0^{y_b} xdF(x) - (1-\alpha)ry_b\bar{F}(y_b) - \kappa. \tag{B.6}$$

We also substitute $B_d = 0$ and rewrite Equation (2.3) as $(1-\alpha)r\int_0^{y_b}\bar{F}_b(x)dx = cq-\kappa$. Taking derivative w.r.t. q on both sides of this equation, we obtain $\partial y_b/\partial q = c/[(1-\alpha)r\bar{F}_b(y_b)]$. Next, we take derivative w.r.t. q on $\pi_S^b(q)$ and obtain

$$\frac{\partial \pi_S^d(q)}{\partial q} = (1-\alpha)r\bar{F}(q) - (1-\alpha)r\bar{F}(y_b)\frac{\partial y_b}{\partial q} = (1-\alpha)r\bar{F}(q) - c\frac{\bar{F}(y_b)}{\bar{F}_b(y_b)}. \tag{B.7}$$

$$\frac{\partial^2 \pi_S^d(q)}{\partial q^2} = -(1-\alpha)rf(q) + \frac{cf(y_b)}{\bar{F}_b(y_b)}\frac{\partial y_b}{\partial q} - \frac{cf(y_b)\bar{F}(y_b)}{\bar{F}_b(y_b)^2}\frac{\partial y_b}{\partial q}. \tag{B.8}$$

For all q such that $\frac{\partial \pi_S^d(q)}{\partial q} \geq 0$, we have $\bar{F}(q) \geq \bar{F}(y_b)\frac{\partial y_b}{\partial q}$ and $(1-\alpha)r \geq \frac{\bar{F}(y_b)}{\bar{F}(q)}\frac{c}{\bar{F}_b(y_b)}$.

Therefore,

$$\frac{\partial^2 \pi_S^b(q)}{\partial q^2}\Big|_{\frac{\partial \pi_S^b(q)}{\partial q}\geq 0} \leq -(1-\alpha)rf(q) + \frac{cf(y_b)}{\bar{F}_b(y_b)}\frac{\partial y_b}{\partial q} \tag{B.9}$$

$$= -(1-\alpha)rz(q)\bar{F}(q) + \frac{c}{\bar{F}_b(y_b)}z(y_b)\bar{F}(y_b)\frac{\partial y_b}{\partial q} \leq 0. \tag{B.10}$$

This means that $\pi_S^b(q)$ either decreases in q or concavely increases for $q \geq \kappa/c$. Let q_b satisfies $(1-\alpha)r\bar{F}(q_b) = c\bar{F}(y_b)/\bar{F}_b(y_b)$ and $(1-\alpha)r\int_0^{y_b}\bar{F}_b(x)dx = cq_b - \kappa$. Then $\pi_S^b(q)$ is maximized by $\max\{q_b, \kappa/c\}$, which completes the proof. □

Proof of Proposition 2.1. We will prove by contradiction that the supplier never borrows from both the bank and the retailer. Given ρ_d, suppose the supplier's optimal decision is (q_d^*, B_b^*, B_d^*) such that $B_b^* > 0$, $B_d^* > 0$ and $cq_d^* = B_b^* + B_d^* + \kappa$.

Case 1: if $\rho_d < \rho_b$, then

$$\begin{aligned}
\pi_S^{rs}(q_d^*, B_b^*, B_d^*) &= E\big[\big((1-\alpha)r\min\{D, q_d^*\} - (1+\rho_d)B_d^* - (1+\rho_b)B_b^*\big)^+\big] - \kappa \\
&= E\big[\big((1-\alpha)r\min\{D, q_d^*\} - (1+\rho_d)(cq_d^* - \kappa) - (\rho_b - \rho_d)B_b^*\big)^+\big] - \kappa \\
&< E\big[\big((1-\alpha)r\min\{D, q_d^*\} - (1+\rho_d)(cq_d^* - \kappa)\big)^+\big] - \kappa \\
&= \pi_S^{rs}(q_d^*, B_b = 0, B_d = cq_d^* - \kappa).
\end{aligned}$$

Therefore (q_d^*, B_b^*, B_d^*) is not an optimal solution to the supplier, which is the contradiction.

Case 2: if $\rho_d \geq \rho_b$, we will argue that for any fixed q and (ρ_b, B_b, B_d) satisfying Equation (2.3) and $cq = B_b + B_d + \kappa$, we have $\partial\rho_b/\partial B_b < 0$. Then, similar to Case 1, we have

$$\begin{aligned}
\pi_S^{rs}(q_d^*, B_b^*, B_d^*) &= E\big[\big((1-\alpha)r\min\{D, q_d^*\} - (1+\rho_b)(cq_d^* - \kappa) - (\rho_d - \rho_b)B_d^*\big)^+\big] - \kappa \\
&< \pi_S^{rs}(q_d^*, B_b = cq_d^* - \kappa, B_d = 0).
\end{aligned}$$

This also contradicts the assumption that (q_d^*, B_b^*, B_d^*) is optimal.

Next, we prove that $\partial\rho_b/\partial B_b < 0$. Using $\frac{\partial y_d}{\partial B_b} = -\frac{1+\rho_d}{(1-\alpha)r}$, $\frac{\partial y_b}{\partial B_b} = \frac{\rho_b - \rho_d}{(1-\alpha)r} + \frac{B_b}{(1-\alpha)r}\frac{\partial \rho_b}{\partial B_b}$ and taking derivative w.r.t. B_b on both side of Equation (2.3), after some algebra we obtain $B_b\bar{F}_b(y_b)\frac{\partial\rho_b}{\partial B_b} = 1 - (1+\rho_d)\bar{F}_b(y_d) - (\rho_b - \rho_d)\bar{F}_b(y_b)$. Rewriting Equation (2.3), we have $\int_{y_d}^{y_b}(1-\alpha)rx - (1+\rho_d)(cq-\kappa)dF_b(x) = B_b\big(1 - (1+\rho_d)\bar{F}_b(y_d) - (\rho_b - \rho_d)\bar{F}_b(y_b)\big)$. Because $(1-\alpha)ry_b = (1+\rho_d)(cq-\kappa) + (\rho_b - \rho_d)B_b \leq (1+\rho_d)(cq-\kappa)$, we get that

$\int_{y_d}^{y_b}(1-\alpha)rx-(1+\rho_d)(cq-\kappa)dF_b(x) < 0$, and thus $1-(1+\rho_d)\bar{F}_b(y_d)-(\rho_b-\rho_d)\bar{F}_b(y_b) < 0$. Therefore, $\frac{\partial \rho_b}{\partial B_b} < 0$, which completes the proof. $\square$

Proof of Corollary 2.1. To prove Corollary 2.1, we first need to show that $\partial q_d(\rho_d)/\partial \rho_d \leq 0$ and $\partial \pi_S^d(\rho_d)/\partial \rho_d \leq 0$ when q and ρ_d satisfy $\bar{F}(q) = \frac{(1+\rho_d)c}{(1-\alpha)r}\bar{F}(\frac{(1+\rho_d)c}{(1-\alpha)r}q - \frac{(1+\rho_d)\kappa}{(1-\alpha)r})$. Taking derivative with respect to ρ_d on both sides of the above equation, we have

$$\begin{aligned}-(1-\alpha)rf(q)\frac{\partial q_d}{\partial \rho_d} =& c\bar{F}(\frac{(1+\rho_d)c}{(1-\alpha)r}q_d - \frac{(1+\rho_d)\kappa}{(1-\alpha)r}) - (1+\rho_d)cf(\frac{(1+\rho_d)c}{(1-\alpha)r}q_d \\ &- \frac{(1+\rho_d)\kappa}{(1-\alpha)r})[\frac{cq-\kappa}{(1-\alpha)r} + \frac{(1+\rho_d)c}{(1-\alpha)r}\frac{\partial q_d}{\partial \rho_d}].\end{aligned}$$

Simplifying the above equation, we get

$$\frac{\partial q_d}{\partial \rho_d} = \frac{\frac{\bar{F}(y_d)}{1+\rho_d}[1-y_d z(y_d)]}{\frac{\bar{F}(q_d)}{\bar{F}(y_d)}f(y_d) - \frac{\bar{F}(y_d)}{\bar{F}(q_d)}f(q_d)} = \frac{\frac{\bar{F}(y_d)}{1+\rho_d}[1-y_d z(y_d)]}{\bar{F}(q_d)z(y_d) - \bar{F}(y_d)z(q_d)} \leq 0.$$

where the inequality is a consequence of the following facts: (1) $1-y_d z(y_d) \geq 0$ (Lemma B.1), (2) $z(y_d) \leq z(q_d)$ (IFR) and (3) $\bar{F}(q_d) \leq \bar{F}(y_d)$. This completes a proof of the claim that q_d decreases in ρ_d.

Next, we show π_S^d decreases in ρ_d. We rewrite Equation (2.1) as follows.

$$\begin{aligned}\pi_S^d(q,\rho_d) =& (1-\alpha)r\int_0^\infty \min\{x,q\}dF(x) - (1-\alpha)r\int_0^{y_d} xdF(x) \\ &- (1-\alpha)ry_d\bar{F}(y_d) - \kappa \\ =& (1-\alpha)r\int_0^q \bar{F}(x)dx - (1-\alpha)r\int_0^{y_d}\bar{F}(x)dx - \kappa. \\ \frac{\partial \pi_S^d(q_d(\rho_d),\rho_d)}{\partial \rho_d} =& (1-\alpha)r\bar{F}(q_d)\frac{\partial q_d}{\partial \rho_d} - (1-\alpha)r\bar{F}(y_d)\frac{\partial y_d}{\partial \rho_d} \\ =& (1-\alpha)r\bar{F}(q_d)\frac{\partial q_d}{\partial \rho_d} - (1-\alpha)r\bar{F}(y_d)\frac{cq_d-\kappa}{(1-\alpha)r} \\ &- (1-\alpha)r\bar{F}(y_d)\frac{(1+\rho_d)c}{(1-\alpha)r}\frac{\partial q_d}{\partial \rho_d} \\ =& -(cq_d-\kappa)\bar{F}(y_d) \leq 0.\end{aligned}$$

It follows that there must exist a maximum production quantity, which is achieved when the retailer sets $\rho_d = 0$. The maximum quantity q_{dmax} under direct financing is then obtained from the implicit equation $\bar{F}(q_{dmax}) = \frac{c}{(1-\alpha)r}\bar{F}(\frac{c}{(1-\alpha)r}q_{dmax} - \frac{\kappa}{(1-\alpha)r})$. It is clear that $q_{dmax} \geq q_{un}$. $\square$

Proof of Proposition 2.2. In this proof, we omit the superscript "bs" of y_d^{bs} and y_b^{bs} for notational simplicity. We rewrite Equation (2.4) as

$$\pi_S^{rs}(B_b, B_d) = (1-\alpha)r\int_0^q \bar{F}(x)dx - (1-\alpha)r\int_0^{y_d} \bar{F}(x)dx - \kappa. \tag{B.11}$$

Because $cq = B_b + B_d + \kappa$, we have $\frac{\partial q}{\partial B_b} = \frac{\partial q}{\partial B_d} = 1/c$. Similarly, because $y_d = y_b + (1+\rho_d)B_d/[(1-\alpha)r]$ and $(1-\alpha)r\int_0^{y_b}\bar{F}_b(x)dx = B_b$, we have $\frac{\partial y_d}{\partial B_d} = \frac{(1+\rho_d)}{(1-\alpha)r}$ and $\frac{\partial y_d}{\partial B_b} = \frac{\partial y_b}{\partial B_b} = \frac{1}{(1-\alpha)r\bar{F}_b(y_b)}$. Then

$$\frac{\partial \pi_S^{rs}}{\partial B_b} = (1-\alpha)r\bar{F}(q)\frac{\partial q}{\partial B_b} - (1-\alpha)r\bar{F}(y_d)\frac{\partial y_d}{\partial B_b} = \frac{(1-\alpha)r}{c}\bar{F}(q) - \frac{\bar{F}(y_d)}{\bar{F}_b(y_b)}. \tag{B.12}$$

$$\frac{\partial \pi_S^{rs}}{\partial B_d} = (1-\alpha)r\bar{F}(q)\frac{\partial q}{\partial B_d} - (1-\alpha)r\bar{F}(y_d)\frac{\partial y_d}{\partial B_d} = \frac{(1-\alpha)r}{c}\bar{F}(q) - (1+\rho_d)\bar{F}(y_d). \tag{B.13}$$

It is clear that $B_b = B_d = 0$ is not optimal, therefore the optimal candidates are (1) $B_b > 0$ and $B_d > 0$, in which case $\frac{\partial \pi_S^{rs}}{\partial B_b} = \frac{\partial \pi_S^{rs}}{\partial B_d} = 0$; (2) $B_b = 0$ and $B_d > 0$, in which case $\frac{\partial \pi_S^{rs}}{\partial B_b} \le 0$ and $\frac{\partial \pi_S^{rs}}{\partial B_d} = 0$; and (3) $B_b > 0$ and $B_d = 0$, in which case $\frac{\partial \pi_S^{rs}}{\partial B_b} = 0$ and $\frac{\partial \pi_S^{rs}}{\partial B_d} \le 0$.

We first argue that $B_b^* = 0$ if and only if $\rho_d = 0$. If $\rho_d = 0$, we have $\frac{\partial \pi_S^{rs}}{\partial B_b} \le \frac{\partial \pi_S^{rs}}{\partial B_d}$ by comparing (B.12) and (B.13). If $\bar{F}_b(y_b) < 1$, then we must have $\frac{\partial \pi_S^{rs}}{\partial B_b}(B_b^*, B_d^*) < 0$ and $\frac{\partial \pi_S^{rs}}{\partial B_d}(B_b^*, B_d^*) = 0$. Otherwise $\bar{F}_b(y_b) = 1$. In both cases, we obtain $B_b^* = 0$. For the "only if" part of our claim, we note that $B_b^* = 0$ only occurs if $\bar{F}_b(y_b) = 1$ and the optimality conditions are $\frac{\partial \pi_S^{rs}}{\partial B_b} \le \frac{\partial \pi_S^{rs}}{\partial B_d} = 0$. In such case, we have $1 \ge 1 + \rho_d$, i.e. $\rho_d = 0$. Therefore, if $\rho_d = 0$ we have $B_b^* = 0$ and $\frac{\partial \pi_S^{rs}}{\partial B_d} = 0$, i.e. $\bar{F}(\frac{B_d^*+\kappa}{c}) = \frac{c}{(1-\alpha)r}\bar{F}(\frac{1}{(1-\alpha)r}B_d^*)$. If $\rho_d > 0$, then we have $B_b^* > 0$.

Next, we obtain the optimality condition when $\rho_d > 0$, which depends on the threshold $\hat{\rho}_{db} = 1/\bar{F}_b(y_b) - 1$, where y_b satisfies $(1-\alpha)r\int_0^{y_b}\bar{F}_b(x)dx = cq_b - \kappa$. If $\rho_d > 0$, then $\frac{\partial \pi_S^{rs}}{\partial B_b}(B_b^*, B_d^*) = 0$, i.e. $\frac{(1-\alpha)r}{c}\bar{F}(q) = \frac{\bar{F}(y_d)}{\bar{F}_b(y_b)}$.

If $B_d^* = 0$, then we have $y_b = y_d$ and $\frac{(1-\alpha)r}{c}\bar{F}(q) = \frac{\bar{F}(y_d)}{\bar{F}_b(y_b)}$, thus $B_b^* = cq_b - \kappa$. Then $\frac{\partial \pi_S^{rs}}{\partial B_d}|_{B_d=0} = \frac{\bar{F}(y_d)}{\bar{F}_b(y_b)} - (1+\rho_d)\bar{F}(y_d) = \bar{F}(y_d)(\hat{\rho}_{db} - \rho_d)$.

If $\rho_d < \hat{\rho}_{db}$, then $\frac{\partial \pi_S^{rs}}{\partial B_d}\Big|_{B_d=0} > 0$, so $B_d = 0$ is not optimal. Therefore the optimality condition is $\frac{\partial \pi_S^{rs}}{\partial B_b} = \frac{\partial \pi_S^{rs}}{\partial B_d} = 0$. That is, $(1+\rho_d)\bar{F}_b(y_b) = 1$ and $\bar{F}(q) = \frac{(1+\rho_d)c}{(1-\alpha)r}\bar{F}(y_d)$.

If $\rho_d \ge \hat{\rho}_{db}$, we will show that $B_d^* > 0$ is not optimal. The proof is by contradiction. Suppose $(B_d^*, B_b^*) > 0$ is an optimal solution, then $\frac{\partial \pi_S^{rs}}{\partial B_b} = \frac{\partial \pi_S^{rs}}{\partial B_d} = 0$. In that case we have $\bar{F}_b(y_b) = 1/(1+\rho_d)$, which implies y_b, and thus B_b, increase in ρ_d. If $\rho_d \geq \hat{\rho}_{db}$, then $\frac{\partial \pi_S^{rs}}{\partial B_d}\big|_{B_d=0} \leq 0$ when $B_b = cq_b - \kappa$, and in that case $\bar{F}_b(y_b) = 1/(1+\hat{\rho}_{db})$ which also implies $B_b^* \geq cq_b - \kappa$. Next,

$$\begin{aligned}\frac{\partial^2 \pi_S^{rs}}{\partial B_d \partial B_b} &= -\frac{(1-\alpha)r}{c^2} f(q) + \frac{(1+\rho_d)}{(1-\alpha)r\bar{F}(y_b)} f(y_d) \\ &= \frac{(1-\alpha)r}{c^2}[-\bar{F}(q)z(q) + \frac{(1+\rho_d)^2 c^2}{(1-\alpha)^2 r^2}\frac{1}{(1+\rho_d)\bar{F}(y_b)}\bar{F}(y_d)z(y_d)].\end{aligned}$$

We have $\frac{\partial^2 \pi_S^{rs}}{\partial B_d \partial B_b} \leq 0$ if $\frac{\partial \pi_S^{rs}}{\partial B_d} \geq 0$, because $z(q) \geq z(y_d)$ and $(1+\rho_d)\bar{F}_b(y_b) \geq 1$ when $B_b \leq B_b^*$. Therefore, we have $\frac{\partial \pi_S^{rs}}{\partial B_d}\big|_{B_d=0}$ is either negative or decreasing in B_b when $cq_b - \kappa \leq B_b \leq B_b^*$, and thus $\frac{\partial \pi_S^{rs}}{\partial B_d}(B_d = 0, B_b = B_b^*) \leq 0$. Also,

$$\frac{\partial^2 \pi_S^{rs}}{\partial B_d^2} = -\frac{(1-\alpha)r}{c^2}\bar{F}(q)z(q) + \frac{(1+\rho_d)^2}{(1-\alpha)r}\bar{F}(y_d)z(y_d).$$

It is easy to see $\frac{\partial^2 \pi_S^{rs}}{\partial B_d^2} \leq 0$ if $\frac{\partial \pi_S^{rs}}{\partial B_d} \geq 0$. Therefore, $\frac{\partial \pi_S^{rs}}{\partial B_d} \geq 0$ is either negative or decreasing in B_d, and thus we have either $\frac{\partial \pi_S^{rs}}{\partial B_d}(B_d = B_d^*, B_b = B_b^*) \leq 0$ or $\frac{\partial \pi_S^{rs}}{\partial B_d}(B_d = B_d^*, B_b = B_b^*) \leq \frac{\partial \pi_S^{rs}}{\partial B_d}(B_d = 0, B_b = B_b^*) \leq 0$. Hence, $B_d^* > 0$ is not optimal, and the optimal solution is $B_d^* = 0$. $\square$

Proof of Lemma 2.3. We first show that $q_{opt}^d > \kappa/c$. If $\kappa < c\bar{F}^{-1}(c/[(1-\alpha)r])$, it is clear that

$$\alpha r E[\min\{D, q_d\}] > \pi_R^d(\kappa/c) = \alpha r E[\min\{D, \kappa/c\}], \text{ if } \kappa/c < q_d \leq q_{un}.$$

In what follows, we show that $\pi_R^d(q_d) - \pi_R^d(\kappa/c) > 0$. Note that y_d is chosen such that (q_d, y_d) satisfy $(q_d - \kappa/c)\bar{F}(q_d) = y_d\bar{F}(y_d)$.

$$\begin{aligned}\pi_R^d(q_d) - \pi_R^d(\kappa/c) &> (1-\alpha)r\int_0^{y_d} x dF(x) + (1-\alpha)r y_d \bar{F}(y_d) - (cq_d - \kappa) \\ &\geq (1-\alpha)r\int_0^{y_d} x dF(x) + (1-\alpha)r y_d \bar{F}(y_d) - (1-\alpha)r(q_d - \kappa/c)\bar{F}(q_d) \\ &= (1-\alpha)r\int_0^{y_d} x dF(x) \geq 0. \qquad \text{(B.14)}\end{aligned}$$

Therefore, $\pi_R^d(q_d) > \pi_R^d(\kappa/c)$ for every $q_d \in (\kappa, q_{un}]$, and it follows immediately that $\pi_R^d(q_{un}) > \pi_R^d(\kappa/c)$, so κ/c is not optimal. We have thus proved that the retailer will choose ρ_d such that $q_d > \kappa/c$.

Next, to prove Proposition 2.3, we show that $\pi_R^d(q_d)$ either decreases or concavely increases in q_d. We also show that there exists a q_{opt}^d obtained by solving $\frac{\partial \pi_R^d}{\partial q_d} = 0$ such that $\pi_R^d(q_d)$ is decreasing when $q_d \geq q_{opt}^d$, and increasing when $q_d \in [0, q_{opt}^d]$. Therefore, $\pi_R^d(q_d)$ is maximized either by q_{opt}^d or by the production quantity upper limit q_{dmax}. The latter occurs when $q_{opt}^d > q_{dmax}$. Details are provided below.

From $y_d\bar{F}(y_d) = (q_d - \kappa/c)\bar{F}(q_d)$, we have $\frac{\partial y_d}{\partial q_d} = \frac{\bar{F}(q_d)[1-(q_d-\frac{\kappa}{c})z(q_d)]}{\bar{F}(y_d)[1-y_d z(y_d)]}$. Then

$$\frac{\partial \pi_R^d}{\partial q_d} = \alpha r\bar{F}(q_d) + (1-\alpha)r\bar{F}(y_d)\frac{\partial y_d}{\partial q_d} - c \tag{B.15}$$

$$= \bar{F}(q_d)[\alpha r + (1-\alpha)r\frac{1-(q_d-\frac{\kappa}{c})z(q_d)}{1-y_d z(y_d)}] - c \tag{B.16}$$

$$= (1-\alpha)r\bar{F}(q_d)[\frac{1-(q_d-\frac{\kappa}{c})z(q_d)}{1-y_d z(y_d)} + \frac{\alpha}{1-\alpha} - \frac{c}{(1-\alpha)r\bar{F}(q_d)}]. \tag{B.17}$$

Define $h(q_d) := \frac{1-(q_d-\frac{\kappa}{c})z(q_d)}{1-y_d z(y_d)}$. Then

$$\frac{\partial h(q_d)}{\partial q_d} = -\frac{z(q_d)+(q_d-\frac{\kappa}{c})z'(q_d)}{1-y_d z(y_d)} + (\frac{1-(q_d-\kappa/c)z(q_d)}{1-y_d z(y_d)})^2\frac{\bar{F}(q_d)[z(y_d)+y_d z'(y_d)]}{\bar{F}(y_d)[1-y_d z(y_d)]}.$$

If $\frac{\partial \pi_R^d}{\partial q_d} \geq 0$ and $\alpha \leq (1+c/r)/2$, then we have

$$\frac{1-(q_d-\frac{\kappa}{c})z(q_d)}{1-y_d z(y_d)} \geq -\frac{\alpha}{1-\alpha} + \frac{c}{(1-\alpha)r\bar{F}(q_d)} \geq -\frac{\alpha}{1-\alpha} + \frac{2\alpha-1}{(1-\alpha)} = -1.$$

Because $1 - y_d z(y_d) \geq 1 - (q_d - \frac{\kappa}{c})z(q_d)$ (IFR) and $1 - y_d z(y_d) \geq 0$ (Lemma B.1 at the end of this Online Supplement), we then have $(\frac{1-(q_d-\kappa/c)z(q_d)}{1-y_d z(y_d)})^2 \leq 1$. Also because $\bar{F}(y_d) \geq \bar{F}(q_d)$, $q_d - \frac{\kappa}{c} \geq y_d$, $z(q_d) \geq z(y_d)$ (IFR), and $z'(q_d) \geq z'(y_d)$ ($z(\cdot)$ is convex), we have $\frac{\partial h(q_d)}{\partial q_d} \leq 0$.

Then

$$\frac{\partial^2 \pi_R^d}{\partial q_d^2} = -(1-\alpha)rf(q_d)[\frac{\alpha}{1-\alpha} + h(q_d)] + (1-\alpha)r\bar{F}(q_d)\frac{\partial h(q_d)}{\partial q_d}.$$

If $\frac{\partial \pi_R^d}{\partial q_d} \geq 0$, we have $h(q_d) + \frac{\alpha}{1-\alpha} \geq 0$ and $\frac{\partial h(q_d)}{\partial q_d} \leq 0$, so $\frac{\partial^2 \pi_R^d}{\partial q_d^2} \leq 0$. From the above analysis, it follows that $\pi_R^d(q_d)$ either decreases or concavely increases in q_d. Therefore, there exists a q_{opt}^d such that $\pi_R^d(q_d)$ is decreasing when $q_d \geq q_{opt}^d$, and increasing when $q_d \in [0, q_{opt}^d]$, where q_{opt}^d is solved by $\frac{\partial \pi_R^d}{\partial q_d} = 0$, i.e. $\frac{1-(q_d-\kappa/c)z(q_d)}{1-y_d z(y_d)} - (\frac{c}{(1-\alpha)r\bar{F}(q_d)} - \frac{\alpha}{1-\alpha}) = 0$. If the solution is larger than q_{dmax}, then $\pi_R^d(q_d)$ increases for all $q_d \in [0, q_{dmax}]$, so we set $q_{opt}^d = q_{dmax}$. $\square$

Proof of Lemma 2.4. We start from the first statement. In Corollary 2.1 we showed that $\pi_S^d(q_d)$ increases in q_d. Recall that q_{dmax} is the optimal order quantity under zero interest rate and $\pi_S^b(q_b)$ is the supplier's optimal profit under competitively priced interest rate, which implies $\pi_S^d(q_{ds}) = \pi_S^b(q_b) \leq \pi_S^d(q_{dmax})$. Therefore, $q_{ds} \leq q_{dmax}$.

Next, we argue that $q_{ds} \geq q_b$. Similar to (B.14) we have that

$$\begin{aligned}
\pi_R^d(q_b) - \pi_R^b(q_b) &= (1-\alpha)r\int_0^{y_d} xdF(x) + (1-\alpha)ry_d\bar{F}(y_d) - (cq_b - \kappa) \\
&\geq (1-\alpha)r\int_0^{y_d} xdF(x) + (1-\alpha)ry_d\bar{F}(y_d) - (1-\alpha)r(q_b - \kappa/c)\bar{F}(q_b) \\
&= (1-\alpha)r\int_0^{y_d} xdF(x) \geq 0,
\end{aligned}$$

where y_d satisfies $(q_b - \kappa/c)\bar{F}(q_b) = y_d\bar{F}(y_d)$. Therefore, $\pi_R^d(q_b) \geq \alpha r E[\min\{D, q_b\}] = \pi_R^b(q_b)$. The total profit $\pi_0(q_b)$ does not depend on trade financing. An immediate consequence of proving $\pi_R^d(q_d = q_b) \geq \pi_R^b(q_b)$ is that $\pi_S^d(q_b) \leq \pi_S^b(q_b)$. Therefore, $q_{ds} \geq q_b$.

Next, we prove the second statement. Again, we have that $\pi_R^d(q_b) \geq \alpha r E[\min\{D, q_b\}] = \pi_R^b(q_b)$. Also we show in Lemma 2.3 that $\pi_R^d(q_d)$ decreases in $q_d \in [q_{opt}^d, q_{dmax}]$ and $\pi_R^d(q_{opt}^d) \geq \pi_R^d(q_b) \geq \pi_R^b(q_b)$. Therefore if there exists a $q_{dr} \in [q_{opt}^d, q_{dmax}]$ such that $\pi_R^d(q_{dr}) = \pi_R^b(q_b)$, then $\pi_R^d(q_d) \leq \pi_R^b(q_b)$ for every $q_d > q_{dr}$. If $\pi_R^d(q_{dr}) = \pi_R^b(q_b)$ has no solution in $[q_{opt}^d, q_{dmax}]$, it means $\pi_R^d(q_d) \geq \pi_R^b(q_b)$ for every $q_d \in [q_{opt}^d, q_{dmax}]$. In that case, we set $q_{dr} = q_{dmax}$. □

Proof of Theorem 2.1. The proof of this theorem is directly from Lemma 2.3 and Lemma 2.4.

Lemma 2.4 shows that $\pi_S^d(q_d) \geq \pi_S^b(q_b)$ only if $q_d \geq q_{ds}$. Lemma 2.4 also shows that $\pi_R^d(q_d) \geq \pi_R^b(q_b)$ only if $q_d \leq q_{dr}$. This leads to the following three cases:

1. When $q_{ds} \leq q_{opt}^d$, the retailer's best choice q_{opt}^d is achievable.

2. When $q_{opt}^d \leq q_{ds} \leq q_{dr}$, the optimal order quantity can only be chosen between q_{ds} and q_{dr}, and in this case $\pi_R^d(q_d)$ decreases in q_d, so the retailer will induce the supplier to order q_{ds}.

3. When $q_{dr} \leq q_{ds}$, there is no feasible solution in the interval (q_{ds}, q_{dr}), therefore bank financing is optimal.

Next, we prove that $\frac{\partial(q_{dr}-q_{ds})}{\partial\kappa} \geq 0$ at $q_{dr} = q_{ds}$. Therefore, $q_{dr} - q_{ds}$ crosses zero at most once (from negative to positive). Thus there exists a threshold $\hat{\kappa}_{bd}$ such that $q_{dr} \leq q_{ds}$ if $\kappa \leq \hat{\kappa}_{bd}$, and $q_{dr} \geq q_{ds}$ if $\kappa \geq \hat{\kappa}_{bd}$.

Taking derivative w.r.t. κ on both sides of $\pi_S^d(q_{ds}) = \pi_S^b(q_b)$ and using Equation $y_d\bar{F}(y_d) = (q - \kappa/c)\bar{F}(q)$, we obtain

$$\begin{aligned}
&(1-\alpha)r\bar{F}(q_{ds})\frac{\partial q_{ds}}{\partial\kappa} \\
&-(1-\alpha)r\frac{(\partial q_{ds}/\partial\kappa - 1/c)\bar{F}(q_{ds}) - (q_{ds}-\kappa/c)f(q_{ds})(\partial q_{ds}/\partial\kappa)}{1-y_{ds}z(y_{ds})} - 1 = 0, \qquad \text{(B.18)}\\
\Rightarrow &(1-\alpha)r\bar{F}(q_{ds})(1 - \frac{1-(q_{ds}-\kappa/c)z(q_{ds})}{1-y_{ds}z(y_{ds})})\frac{\partial q_{ds}}{\partial\kappa} \\
&= 1 - \frac{(1-\alpha)r\bar{F}(q_{ds})/c}{1-y_{ds}z(y_{ds})} = 1 - \frac{(1+\rho_{ds})\bar{F}(y_{ds})}{1-y_{ds}z(y_{ds})}. \qquad \text{(B.19)}
\end{aligned}$$

Because $\frac{\bar{F}(y_{ds})}{1-y_{ds}z(y_{ds})} \geq 1$ (Lemma B.1) and $\frac{1-(q_{ds}-\kappa/c)z(q_{ds})}{1-y_{ds}z(y_{ds})} \leq 1$, we then have $\frac{\partial q_{ds}}{\partial\kappa} \leq 0$. Similarly, in terms of q_{dr} we have

$$\big(c - r\bar{F}(q_{dr}) + (1-\alpha)r\bar{F}(q_{dr})(1 - \frac{1-(q_{dr}-\kappa/c)z(q_{dr})}{1-y_{dr}z(y_{dr})})\big)\frac{\partial q_{dr}}{\partial\kappa} = 1 - \frac{(1+\rho_{dr})\bar{F}(y_{dr})}{1-y_{dr}z(y_{dr})}. \qquad \text{(B.20)}$$

If $q_{dr} = q_{ds}$ (denoted as q_{rs}), then by definition $\pi_R^d(q_{rs}) + \pi_S^d(q_{rs}) = \pi_R^b(q_b) + \pi_S^b(q_b)$. Lemma 2.4 shows $q_{ds} \geq q_b$, so we conclude that $q_{rs} \geq q_t^*$ (recall q_t^* is the first best). Therefore, $c - r\bar{F}(q_{rs}) \geq 0$. Comparing Equation (B.19) and (B.20), we note that if $q_{dr} = q_{ds}$ then $\frac{\partial q_{dr}}{\partial\kappa} \leq 0$ and $\frac{\partial q_{dr}}{\partial\kappa}/\frac{\partial q_{ds}}{\partial\kappa} \leq 1$. Therefore, $\frac{\partial(q_{dr}-q_{ds})}{\partial\kappa} \geq 0$ (since $\frac{\partial q_{ds}}{\partial\kappa} \leq 0$), which completes the proof. □ □

Proof of Proposition 2.3. From the proof of Theorem 2.1, we have $\frac{\partial q_{ds}}{\partial\kappa} \leq 0$. Also, we have $q_{ds} = q_{dr} \geq q_t^*$ if $\kappa = \hat{\kappa}_{bd}$. Therefore, $\hat{\kappa}_t$ exists. □

Proof of Theorem 2.2. From Proposition 2.2, we only need to consider the case $0 \leq \rho_d \leq \hat{\rho}_{db}$.

Take derivative on $\pi_R^{bs} = \pi_0 - \pi_S^{bs}$, then we have

$$\frac{\partial \pi_R^{bs}(B_b^*, B_d^*, \rho_d)}{\partial \rho_d} = (r\bar{F}(q) - c)\frac{\partial q}{\partial \rho_d} - \frac{\partial \pi_S^{bs}}{\partial B_b}\frac{\partial B_b}{\partial \rho_d} + \frac{\partial \pi_S^{bs}}{\partial B_d}\frac{\partial B_d}{\partial \rho_d} + \frac{\partial \pi_S^{bs}}{\partial \rho_d} \tag{B.21}$$

$$= (r\bar{F}(q) - c)\frac{\partial q}{\partial \rho_d} + (1-\alpha)r\bar{F}(y_d)\frac{\partial y_d}{\partial \rho_d} = (r\bar{F}(q) - c)\frac{\partial q}{\partial \rho_d} + B_b^*\bar{F}(y_d). \tag{B.22}$$

$$\frac{\partial q}{\partial \rho_d} = \frac{\bar{F}(y_d)[1 - (y_d - y_b)z(y_d)]}{(1+\rho_d)\bar{F}(q)[z(y_d) - \frac{(1-\alpha)r}{(1+\rho_d)c}z(q)]} \leq 0. \tag{B.23}$$

If $\rho_d = \hat{\rho}_{db}$, then $B_d^* = 0$ and $q = q_{un}$, therefore $\frac{\partial \pi_R^{bs}(B_b^*, B_d^*, \rho_d)}{\partial \rho_d} = \frac{\alpha c}{1-\alpha}\frac{\partial q}{\partial \rho_d} < 0$. Thus, $\rho_d = \hat{\rho}_{db}$ is not optimal.

If $q_d \geq q_t^*$, then $r\bar{F}(q) - c \leq 0$ and thus $\frac{\partial \pi_R^{bs}(B_b^*, B_d^*, \rho_d)}{\partial \rho_d} > 0$. Therefore, $q_d \geq q_t^*$ is not optimal.

After some algebra, we get

$$\frac{\partial \pi_R^{bs}(B_b^*, B_d^*, \rho_d)}{\partial \rho_d} = (1-\alpha)r\bar{F}(q)\frac{\partial q}{\partial \rho_d}\left(\frac{1 - (q - \kappa/c - B_b^*/c)z(q)}{1 - (y_d - y_b)z(y_d)} - \frac{c}{(1-\alpha)r\bar{F}(q)} + \frac{\alpha}{1-\alpha}\right). \tag{B.24}$$

Similar to the proof in Lemma 2.3, we can argue that $\frac{1-(q-\kappa/c-B_b^*/c)z(q)}{1-(y_d-y_b)z(y_d)} - \frac{c}{(1-\alpha)r\bar{F}(q)} + \frac{\alpha}{1-\alpha}$ is either increasing or concavely decreasing in ρ_d. Therefore, the optimal ρ_{opt}^{bs} let q_{opt}^{bs} be either a solution of $\frac{1-(q-\kappa/c-B_b^*/c)z(q)}{1-(y_d-y_b)z(y_d)} - \frac{c}{(1-\alpha)r\bar{F}(q)} + \frac{\alpha}{1-\alpha} = 0$ or $q_{opt}^{bs} = q_{dmax}$. □

Proof of Theorem 2.3. The proof is analogous to that of Theorem 2.1. Using a series of arguments similar to the proof of Theorem 2.1, we can show that the existence of $\hat{\kappa}_{bd2}$. If $\kappa \leq \hat{\kappa}_{bd2}$, the retailer will let the supplier choose the bank financing. If $\kappa \geq \hat{\kappa}_{bd2}$, we have the optimal production quantity $q_{opt2}^{rs} = \max\{q_{ds2}, q_{opt}^d\}$.

Next, we prove that $q_{ds2} \leq q_{ds}$ and $\hat{\kappa}_{bd2} \leq \hat{\kappa}_{bd}$.

From Lemma 2.2 we have $q_{b2} \leq q_b$, since $\bar{F}(q_{b2}) = \frac{c}{(1-\alpha)r}\frac{\bar{F}(y_{b2})}{\bar{F}_b(y_{b2})} \geq \frac{c}{(1-\alpha)r} = \bar{F}(q_b)$. Therefore, we have $\pi_S^b(q_{b2}) \leq \pi_S^b(q_b)$ and $\pi_R^b(q_{b2}) \leq \pi_R^b(q_b)$. Then by Definition 1, we know $\pi_S^d(q_{ds2}) \leq \pi_S^d(q_{ds})$, which leads to $q_{ds2} \leq q_{ds}$, since π_S^d increases in q_d (Corollary 2.1). Next by Definition 2, we know $\pi_R^d(q_{dr2}) \leq \pi_R^d(q_{dr})$, which leads to $q_{dr2} \geq q_{dr}$, since π_R^d decreases in $q_d \in [q_{opt}^d, q_{max}]$ (Lemma 2.3). Therefore, it follows that $q_{dr2} - q_{ds2} \geq q_{dr} - q_{ds}$. In the proof of Theorem 2.1, we show that $q_{dr} - q_{ds}$ crosses zero at most once (from negative side to positive side). Similarly, we can argue that $q_{dr2} - q_{ds2}$ also crosses zero at most once (from negative side to positive side). Because $q_{dr2} - q_{ds2} \geq q_{dr} - q_{ds}$,

$q_{dr2} - q_{ds2}$ reaches zero before $q_{dr} - q_{ds}$ as κ increases, and thus $\hat{\kappa}_{bd2} \leq \hat{\kappa}_{bd}$. (Recall that $\hat{\kappa}_{bd2}$ and $\hat{\kappa}_{bd}$ are the values of κ such that $q_{dr2} = q_{ds2}$ and $q_{dr} = q_{ds}$.)

Next, we show that $\pi_R^{rs}(q_{opt}^{rs}) \leq \pi_R^{rs}(q_{opt2}^{rs})$ and $\pi_S^{rs}(q_{opt}^{rs}) \geq \pi_S^{rs}(q_{opt2}^{rs})$. This is true because that $q_{opt2}^{rs} = \max\{q_{ds2}, q_{opt}^d\}$, $q_{opt}^{rs} = \max\{q_{ds}, q_{opt}^d\}$, and $q_{ds2} \leq q_{ds}$. Therefore $q_{opt}^{rs} \geq q_{opt2}^{rs} \geq q_{opt}^d$. By Lemma 2.3, π_R^d decreases in $q_d \in [q_{opt}^d, q_{max}]$, so $\pi_R^{rs}(q_{opt}^{rs}) \leq \pi_R^{rs}(q_{opt2}^{rs})$. By Corollary 2.1, π_S^d increases in q_d, so $\pi_S^{rs}(q_{opt}^{rs}) \geq \pi_S^{rs}(q_{opt2}^{rs})$. $\square$

Proof of Theorem 2.4. Take derivative of (2.5) w.r.t ρ_d. After some algebra, we obtain

$$\begin{aligned}\frac{\partial \pi_R^{bs}(B_b^*, B_d^*, \rho_d)}{\partial \rho_d} &= \frac{(r\bar{F}(q) - c)c\bar{F}(y_d)[1 - (y_d - y_b)z(y_d)]}{\bar{F}(q)[(1+\rho_d)cz(y_d) - (1-\alpha)rz(q)]} + B_d\bar{F}(y_d) \\ &+ \frac{(1-\alpha)r}{(1+\rho_d)z_b(y_b)}[\bar{F}_b(y_b) - \bar{F}(y_b)]. \end{aligned} \tag{B.25}$$

Similar to the proof of Theorem 2.2, we have that $\frac{\partial \pi_R^{bs}(B_b^*, B_d^*, \rho_d)}{\partial \rho_d} < 0$ if $\rho_d = \hat{\rho}_{db}$. Therefore, the optimality condition is $\frac{\partial \pi_R^{bs}(B_b^*, B_d^*, \rho_d)}{\partial \rho_d} = 0$ or the optimal production quantity is q_{dmax}. $\square$

Proof of Proposition 2.4. We only need to show that for every $\kappa \leq \hat{\kappa}_{lt}$, we have $\pi_{ldwb}^S(\rho_{ldwb} = 0, q_{ldwb} = q_t^*) \geq \pi_b^S(q_b)$. Then the retailer can increase ρ_{ldwb} and changes $\bar{B}_d$ correspondingly such that $q_{ldwb}^{**} = q_t^*$ and $\pi_S^{ldwb}(q_{ldwb}^{**}) = \pi_S^b(q_b)$. Then if $\rho_{ldwb} = 0, q_{ldwb} = q_t^*$, we have that

$$\begin{aligned}\pi_S^{ldwb}|_{(\rho_{ldwb}=0, q_{ldwb}=q_t^*)} - \pi_S^b(q_b) =& (1-\alpha)r\int_0^{q_t^*} \bar{F}(x)dx - (1-\alpha)r\int_0^{\frac{cq_t^*-\kappa}{(1-\alpha)r}} \bar{F}(x)dx \\ &- (1-\alpha)r\int_0^{q_b} \bar{F}(x)dx + (1-\alpha)r\int_0^{y_b} \bar{F}(x)dx. \end{aligned} \tag{B.26}$$

$$\begin{aligned}&\frac{\partial(\pi_S^{ldwb}(\rho_{ldwb} = 0, q_{ldwb} = q_t^*) - \pi_S^b(q_b))}{\partial \kappa} \\ &= \bar{F}\left(\frac{cq_t^* - \kappa}{(1-\alpha)r}\right) - (1-\alpha)r[\bar{F}(q_b)\frac{\partial q_b}{\partial \kappa} - \bar{F}(y_b)\frac{\partial y_b}{\partial \kappa}]. \end{aligned} \tag{B.27}$$

Taking derivative w.r.t κ on both sides of $(1-\alpha)r\int_0^{y_b} \bar{F}_b(x)dx = cq_b - \kappa$, we get $(1-\alpha)r\bar{F}_b(y_b)\frac{\partial y_b}{\partial \kappa} = c\frac{\partial q_b}{\partial \kappa} - 1$. From Lemma 2.2, we have that $(1-\alpha)r\bar{F}_b(y_b) = c\bar{F}(y_b)/\bar{F}(q_b)$, and therefore $\bar{F}(y_b)\frac{\partial y_b}{\partial \kappa} = \bar{F}(q_b)\frac{\partial q_b}{\partial \kappa} - \bar{F}(q_b)/c$. Thus, we have

$$\frac{\partial(\pi_S^{ldwb}(\rho_{ldwb}=0, q_{ldwb}=q_t^*) - \pi_S^b(q_b))}{\partial \kappa} = \bar{F}\big(\frac{cq_t^* - \kappa}{(1-\alpha)r}\big) - \frac{(1-\alpha)r}{c}\bar{F}(q_b) \le 0,$$

where the last inequality follows from the fact that $q_b \le \bar{F}^{-1}(\frac{c}{(1-\alpha)r})$.

Next, we show that if $q_t^* z(q_t^*) \le 1$, then we have $\pi_S^{ldwb}(\rho_{ldwb} = 0, q_{ldwb} = q_t^*) - \pi_S^b(q_b^*) \ge 0$ at $\kappa = 0$. Therefore, there exists a threshold $\hat{\kappa}_{lt}$ such that $\pi_S^{ldwb}(\rho_{ldwb} = 0, q_{ldwb} = q_t^*) - \pi_S^b(q_b^*) \ge 0$ for any $\kappa \le \hat{\kappa}_{lt}$, which will complete the proof.

If $\kappa = 0$, we rewrite Equation (B.26) as

$$\begin{aligned}
\pi_S^{ldwb} - \pi_S^b &\ge (1-\alpha)r\big(\int_{q_b}^{q_t^*} \bar{F}(x)dx - \int_0^{\frac{cq_t^*}{(1-\alpha)r}} \bar{F}(x)dx + \frac{cq_b}{(1-\alpha)r}\big) \\
&\ge (1-\alpha)r\big(\int_{q_b}^{q_t^*} \bar{F}(x)dx - \int_{\frac{cq_b}{(1-\alpha)r}}^{\frac{cq_t^*}{(1-\alpha)r}} \bar{F}(x)dx\big) \\
&= (1-\alpha)r\big(\int_{q_b}^{q_t^*} \bar{F}(x)dx - \int_{q_b}^{q_t^*} \frac{c}{(1-\alpha)r})\bar{F}(\frac{cx}{(1-\alpha)r})dx\big) \ge 0.
\end{aligned}$$

The last inequality comes from the fact that $x\bar{F}(x)$ increase in x for all x such that $xz(x) \le 1$, and also from the fact that $xz(x) \le q_t^* z(q_t^*) \le 1$. □

Proof of Proposition 2.5. First, we consider the case that only direct financing is available. In this case, the retailer's profit is

$$\pi_R^d = \alpha r \int_0^q \bar{F}(x)dx + (1-\alpha)\int_0^{\frac{(1+\rho_d)c}{(1-\alpha)r}(q-\kappa/c)} \bar{F}(x)dx - (cq-\kappa), \tag{B.28}$$

with the constraint that $\bar{F}(q) = \frac{(1+\rho_d)c}{(1-\alpha)r}\bar{F}(\frac{(1+\rho_d)c}{(1-\alpha)r}(q-\kappa/c))$.

Note that when $\frac{(1+\rho_d)c}{(1-\alpha)r}$ is fixed, q is also fixed, and therefore both $\int_0^q \bar{F}(x)dx$ and $\int_0^{\frac{(1+\rho_d)c}{(1-\alpha)r}(q-\kappa/c)} \bar{F}(x)dx$ are fixed. Because $\int_0^q \bar{F}(x)dx \ge \int_0^{\frac{(1+\rho_d)c}{(1-\alpha)r}(q-\kappa/c)} \bar{F}(x)dx$, π_R^d increases with α for every fixed $\frac{(1+\rho_d)c}{(1-\alpha)r}$. That is, we can increase α and decrease ρ_d correspondingly to keep $\frac{(1+\rho_d)c}{(1-\alpha)r}$ unchanged, until ρ_d reaches zero. Hence, at optimality we must have $\rho_d^* = 0$.

Second, at $\rho_d^* = 0$ the supplier will only take direct financing, so the availability of bank financing does not change the retailer's decision. □

Lemma B.1. *For every q and ρ_d that satisfy Equation (2.7), we must have $0 \le 1 - y_d z(y_d) \le \bar{F}(y_d)$ whenever $y_d = (1+\rho_d)(cq-\kappa)/[(1-\alpha)r]$.*

Proof of Lemma B.1. We can rewrite Equation (2.7) as $y_d\bar{F}(y_d) = (q-\kappa/c)\bar{F}(q)$. Let $g(x) = x\bar{F}(x)$. Then $\frac{dg(x)}{dx} = \bar{F}(x) - xf(x) = \bar{F}(x)[1-xz(x)]$. Thus, $g(x)$ decreases in x if $xz(x) \ge 1$. In our setting, if we assume $y_d z(y_d) > 1$, then we must have $y_d\bar{F}(y_d) \ge q\bar{F}(q) > (q-\kappa/c)\bar{F}(q)$, which contradicts with the fact that q and y_d satisfy $y_d\bar{F}(y_d) = (q-\kappa/c)\bar{F}(q)$. Therefore, we must have $y_d z(y_d) \le 1$.

Next, we prove $1 - y_d z(y_d) \le \bar{F}(y_d)$, i.e. $\bar{F}(y_d) + y_d z(y_d) \ge 1$. Note $\frac{\partial(\bar{F}(y_d)+y_d z(y_d))}{\partial y_d} = -z(y_d)\bar{F}(y_d) + z(y_d) + yz'(y_d) \ge 0$. Also, $\bar{F}(y_d) + y_d z(y_d)\big|_{y_d=0} = 1$. Therefore, we have $\bar{F}(y_d) + y_d z(y_d) \ge 1$. □

B.2 Proofs of Theorems, Propositions, and Lemmas of Chapter 3

Proof of Proposition 3.1. We first show that Equation (3.7) always has a solution. We have that

$$\begin{aligned}&rx - E[wq \wedge (1-\beta)(A_R + rD \wedge rq)|D \ge x - \frac{A_R}{r}]\\ =&\frac{rx\bar{F}((x-A_R/r)^+) - \int_{(x-A_R/r)^+}^{\infty}[wq \wedge (1-\beta)(A_R+rq) \wedge (1-\beta)(A_R+r\xi)]dF(\xi)}{\bar{F}((x-A_R/r)^+)}.\end{aligned} \tag{B.29}$$

Let $\Phi(x)$ denote the numerator of the above expression, i.e. $\Phi(x) := rx\bar{F}((x-A_R/r)^+) - \int_{(x-A_R/r)^+}^{\infty} wq \wedge (1-\beta)(A_R+rq) \wedge (1-\beta)(A_R+r\xi)dF(\xi)$. Because $\Phi(x)$ is positive at $x = q$ and negative at $x = 0$, this implies that there always exists at least one x_R such that $\Phi(x_R) = 0$.

Next, we explain three different conditions to find the minimal solution of (3.7).

In the first case, if $(1-\beta)A_R \ge wq$, then $(1-\beta)(A_R + r\xi) \ge wq$ for every $\xi \ge 0$. Therefore, $\Phi(x) := rx\bar{F}((x-A_R/r)^+) - wq\bar{F}((x-A_R/r)^+)$. Thus, $x_R = wq/r$ solves the equation.

In the second case, $E(wq \wedge (1-\beta)(A_R+rq) \wedge (1-\beta)(A_R+rD)] \le A_R < wq/(1-\beta)$. We have $\Phi(x = A_R/r) = A_R - E[wq \wedge (1-\beta)(A_R+rq) \wedge (1-\beta)(A_R+rD)] \ge 0$, and $\Phi(x=0) < 0$. Also, $\Phi(x)$ is clearly increasing in x if $x \le A_R/r$, therefore the minimal solution of (3.7) is achieved at $x_R \leq A_R/r$. Thus, $x_R = E[wq/r \wedge (1-\beta)(A_R/r+q) \wedge (1-\beta)(A_R/r+D)]$ and $x_R \leq A_R/r$.

In the third case, $E(wq \wedge (1-\beta)(A_R+rq) \wedge (1-\beta)(A_R+rD)] > A_R$. As shown in the second case, we have $\Phi(x) < 0$ for every $x \leq A_R/r$. Therefore, the minimal solution of (3.7) is achieved at $x_R > A_R/r$. Because $(1-\beta)(A_R+r\xi) \leq (1-\beta)x \leq [wq \wedge (1-\beta)(A_R+rq)]$ if $\xi \leq x - A_R/r$, which implies $\int_0^{x-A_R/r}[wq \wedge (1-\beta)(A_R+rq) \wedge (1-\beta)(A_R+r\xi)]dF(\xi) = \int_0^{x-A_R/r}(1-\beta)(A_R+r\xi)dF(\xi)$. Then we rewrite $\Phi(x)$ as

$$\begin{aligned}\Phi(x) =& rx\bar{F}((x-A_R/r)) + \int_0^{x-A_R/r}(1-\beta)(A_R+r\xi)dF(\xi)\\ &- \int_0^{\infty}[wq \wedge (1-\beta)(A_R+rq) \wedge (1-\beta)(A_R+r\xi)]dF(\xi).\end{aligned} \tag{B.30}$$

Then

$$\frac{\partial\Phi(x)}{\partial x} = r\bar{F}(x-A_R/r)(1-\beta xz(x-A_R/r)). \tag{B.31}$$

Therefore, $\Phi(x)$ increases in x if $(1-\beta xz(x-A_R/r)) \geq 0$ and decreases in x if $(1-\beta xz(x-A_R/r)) < 0$. Therefore, the minimal solution of (3.7) is achieved at $x_R > A_R/r$ and $1-\beta xz(x-A_R/r) \geq 0$ such that $\Phi(x) = 0$.

If $x_t \geq x_R - A_R/r$, then the retailer will report x_R and the supplier will not call bankruptcy. If $x_t < x_R - A_R/r$, then x_R is no longer feasible to the retailer, and therefore the supplier will enforce bankruptcy. $\square$

Proof of Corollary 3.1. Clearly, we have $rx_R \leq wq$. We want to prove that $rx_R < wq$ if $\beta > 0$ and $(1-\beta)A_R < wq$, and that $rx_R = wq$ if $\beta = 0$ or $(1-\beta)A_R \geq wq$.

We first prove that $rx_R < wq$ if $\beta > 0$ and $(1-\beta)A_R < wq$. In Equation (3.7), we see that $(1-\beta)(A_R+r(x-A_R/r)) = (1-\beta)rx < wq$ because clearly $rx \leq wq$. That is, at $D = x - A_R/r$, we have $wq > (1-\beta)(A_R+rD)$. Therefore, $wq > (1-\beta)(A_R+rD)$ with positive probability. Thus, the right-hand-side of Equation (3.7) is less than wq, so $rx_R < wq$.

Next, we show that if $\beta = 0$ or $(1-\beta)A_R \geq wq$, then $rx_R = wq$. From Proposition 3.1, we know that $rx_R = wq$ if $(1-\beta)A_R \geq wq$. In the remaining, we show that $rx_R = wq$ if $\beta = 0$ and $A_R < wq$. It is easy to check this situation suits the third case of Proposition 3.1, and $x = wq/r$ satisfies the optimality condition. Thus, $x_R = wq/r$ if $\beta = 0$ or $(1-\beta)A_R \geq wq$. $\square$

Proof of Proposition 3.2. From Proposition 3.1, x_R is chosen differently in three cases. When $w \le (1-\beta)r$, we have that

$$\begin{cases} x_R = wq/r & \text{if } q \le \frac{(1-\beta)A_R}{w}, \\ x_R = \int_0^\infty \frac{w}{r} q \wedge (1-\beta)(A_R/r+\xi)dF(\xi) & \text{if } \frac{(1-\beta)A_R}{w} < q \le q_\alpha, \\ x_R\text{: } rx\bar{F}(x-A_R/r) - \int_{x-A_R/r}^\infty wq \wedge (1-\beta)(A_R+r\xi)dF(\xi) = 0 & \text{otherwise.} \end{cases} \tag{B.32}$$

Therefore, substituting x_R in Equation (3.9), we obtain

$$\pi_R(q|w) = \begin{cases} r\int_0^q \bar{F}(\xi)d\xi - wq + A_R & \text{if } q \le \frac{(1-\beta)A_R}{w}, \\ r\int_0^q \bar{F}(\xi)d\xi - r(1-\beta)\int_0^{\frac{wq}{r(1-\beta)}-\frac{A_R}{r}} \bar{F}(\xi)d\xi + \beta A_R & \text{if } \frac{(1-\beta)A_R}{w} < q \le q_\alpha, \\ r\int_0^q \bar{F}(\xi)d\xi - r\int_0^{x_R-A_R/r} \bar{F}(\xi)d\xi. & \text{otherwise.} \end{cases} \tag{B.33}$$

Taking the derivative w.r.t. q, we obtain

$$\frac{\partial \pi_R(q|w)}{\partial q} = \begin{cases} r\bar{F}(q) - w, & \text{if } q \le \frac{(1-\beta)A_R}{w}, \\ r\bar{F}(q) - w\bar{F}(\frac{wq}{(1-\beta)r} - \frac{A_R}{r}), & \text{if } \frac{(1-\beta)A_R}{w} < q \le q_\alpha, \\ r\bar{F}(q) - \frac{w\bar{F}(\frac{wq}{(1-\beta)r}-\frac{A_R}{r})}{1-\beta x_R z(x_R-A_R/r)}, & \text{otherwise,} \end{cases} \tag{B.34}$$

where the last equation comes from $\frac{\partial y_R}{\partial q} = \frac{w\bar{F}(\frac{wq}{(1-\beta)r}-\frac{A_R}{r})}{r\bar{F}(x_R-A_R/r)\left(1-\beta x_R z(x_R-A_R/r)\right)}$.

Note that $\frac{\partial \pi_R(q|w)}{\partial q}$ is continuous except at $q = q_\alpha$ and $\frac{\partial \pi_R(q|w)}{\partial q}$ is decreasing at $q = q_\alpha$. In the following, we show that $\frac{\partial \pi_R(q|w)}{\partial q}$ is either negative or decreasing in q for all q. Therefore, $\pi_R(q|w)$ is quasiconcave in q.

Case 1: $q \le (1-\beta)A_R/w$.

In such case, we have $\frac{\partial \pi_R(q|w)}{\partial q} = r\bar{F}(q) - w$, which decreases in q.

Case 2: $\frac{(1-\beta)A_R}{w} < q \le q_\alpha$.

In such case, we have that

$$\frac{\partial \pi_R(q|w)}{\partial q} = r\bar{F}(q) - w\bar{F}(\frac{wq}{(1-\beta)r} - \frac{A_R}{r}). \tag{B.35}$$

If $\frac{\partial \pi_R(q|w)}{\partial q} \geq 0$, then

$$\frac{\partial^2 \pi_R(q|w)}{\partial q^2} = -rf(q) + \frac{w^2}{(1-\beta)r} f(\frac{wq}{(1-\beta)r} - \frac{A_R}{r}) \tag{B.36}$$

$$\leq -rf(q) + \frac{w}{(1-\beta)r} \frac{r\bar{F}(q)}{\bar{F}(\frac{wq}{(1-\beta)r} - \frac{A_R}{r})} f(\frac{wq}{(1-\beta)r} - \frac{A_R}{r}) \tag{B.37}$$

$$= r\bar{F}(q)\big(-z(q) + \frac{w}{(1-\beta)r} z(\frac{wq}{(1-\beta)r} - \frac{A_R}{r})\big) \tag{B.38}$$

$$\leq 0. \tag{B.39}$$

The first inequality holds because $w \leq \frac{r\bar{F}(q)}{\bar{F}(\frac{wq}{(1-\beta)r} - \frac{A_R}{r})}$ if $\frac{\partial \pi_R(q|w)}{\partial q} \geq 0$. The second inequality holds because $q \geq \frac{wq}{(1-\beta)r} - \frac{A_R}{r}$ and IFR imply $z(q) \geq z(\frac{wq}{(1-\beta)r} - \frac{A_R}{r})$. Therefore, in this case $\frac{\partial \pi_R(q|w)}{\partial q}$ is either negative or decreasing in q.

Case 3: $q > q_\alpha$

In such case, we have that

$$\frac{\partial x_R}{\partial q} = \frac{w\bar{F}(\frac{wq}{(1-\beta)r} - \frac{A_R}{r})}{r\bar{F}(x_R - A_R/r)\big(1 - \beta x_R z(x_R - A_R/r)\big)} \geq 0. \tag{B.40}$$

$$\frac{\partial \pi_R(q|w)}{\partial q} = r\bar{F}(q) - \frac{w\bar{F}(\frac{wq}{(1-\beta)r} - \frac{A_R}{r})}{1 - \beta x_R z(x_R - A_R/r)}. \tag{B.41}$$

If $\frac{\partial \pi_R(q|w)}{\partial q} \geq 0$, we have that

$$\frac{\partial^2 \pi_R(q|w)}{\partial q^2} = -rf(q) + \frac{w^2}{(1-\beta)r} \frac{f(\frac{wq}{(1-\beta)r} - \frac{A_R}{r})}{1 - \beta x_R z(x_R - A_R/r)} \tag{B.42}$$

$$- \frac{\beta z(x_R - A_R/r) + \beta x_R z'(x_R - A_R/r)}{(1 - \beta x_R z(x_R - A_R/r))^2} \frac{\partial x_R}{\partial q} \tag{B.43}$$

$$\leq -rf(q) + \frac{w^2}{(1-\beta)r} \frac{f(\frac{wq}{(1-\beta)r} - \frac{A_R}{r})}{1 - \beta x_R z(x_R - A_R/r)} \tag{B.44}$$

$$\leq r\bar{F}(q)\big(-z(q) + \frac{w}{(1-\beta)r} z(\frac{wq}{(1-\beta)r} - \frac{A_R}{r})\big) \tag{B.45}$$

$$\leq 0. \tag{B.46}$$

Inequality B.44 holds because $z(x_R - A_R/r) \geq 0$, $z'(x_R - A_R/r) \geq 0$, and $\frac{\partial x_R}{\partial q} \geq 0$. Inequality B.45 holds because $0 \leq 1 - \beta x_R z(x_R - A_R/r) \leq 1$. The last inequality holds because $z(q) \geq z(\frac{wq}{(1-\beta)r} - \frac{A_R}{r})$. Therefore, in this region $\frac{\partial \pi_R(q|w)}{\partial q}$ is either negative or decreasing in q.

Therefore, $\frac{\partial \pi_R(q|w)}{\partial q}$ is either negative or decreasing in q in all three cases. Therefore, examining (B.34), we have

1. If $r\bar{F}(\frac{(1-\beta)A_R}{w}) \leq w$, then $\frac{\partial \pi_R(q|w)}{\partial q}\Big|_{q\geq \frac{(1-\beta)A_R}{w}} \leq 0$. Therefore the optimality condition is $\bar{F}(q^*) = \frac{w}{r}$ and $x_R^* = wq^*$.

2. If $r\bar{F}(\frac{(1-\beta)A_R}{w}) > w$ and $r\bar{F}(q_\alpha) \leq w\bar{F}(\frac{wq_\alpha}{(1-\beta)r} - \frac{A_R}{r})$, then $\frac{\partial \pi_R(q|w)}{\partial q}\Big|_{q\leq \frac{(1-\beta)A_R}{w}} \geq 0$ and $\frac{\partial \pi_R(q|w)}{\partial q}\Big|_{q\geq q_\alpha} \leq 0$. Therefore, q^* satisfies $r\bar{F}(q^*) = w\bar{F}(\frac{wq^*}{(1-\beta)r} - \frac{A_R}{r})$ and $x_R^* = \int_0^\infty \frac{w}{r}q^* \wedge (1-\beta)(\frac{A_R}{r} + \xi)dF(\xi) \leq \frac{A_R}{r}$.

3. If $r\bar{F}(q_\alpha) > w\bar{F}(\frac{wq_\alpha}{(1-\beta)r} - \frac{A_R}{r})$ and $r\bar{F}(q_\alpha)(1-\beta z(0)\frac{A_R}{r}) \leq w\bar{F}(\frac{wq_\alpha}{(1-\beta)r} - \frac{A_R}{r})$, then $\frac{\partial \pi_R(q|w)}{\partial q}\Big|_{q\leq q_\alpha} \geq 0$ and $\frac{\partial \pi_R(q|w)}{\partial q}\Big|_{q\geq q_\alpha} \leq 0$. Therefore, $q^* = q_\alpha$ and $x_R^* = \frac{A_R}{r}$.

4. Otherwise, q^* satisfies $r\bar{F}(q^*)(1-\beta x_R z(x_R - \frac{A_R}{r})) = w\bar{F}(\frac{wq^*}{(1-\beta)r} - \frac{A_R}{r})$ and $x_R^* > \frac{A_R}{r}$ satisfies Equation (3.8).

This completes the proof. $\square$

Proof of Lemma 3.1. We show below that q_m and x_{Rm} maximize $\pi_R(q) = E[(A_R + r(D\wedge q) - rx_R)^+]$ with no constraint on q. If $q > q_m$, then the above $\pi_R(q)$ decreases in q..Therefore, in the range of q such that $(w-(1-\beta)r)q \geq (1-\beta)A_R$, the retailer's optimal order quantity is $\tilde{q}^* = q_m$ and $\tilde{x}_R^* = x_{Rm}$ if $q_m \geq \frac{(1-\beta)A_R}{(w-(1-\beta)r)}$. Otherwise, $\tilde{q}^* = \frac{(1-\beta)A_R}{(w-(1-\beta)r)}$ and $\tilde{x}_R^*$ satisfies Equation (3.11).

If $E[(1-\beta)(A_R + rD)\wedge(1-\beta)(A_R + rq)] \leq A_R$, i.e. $(1-\beta)rE[D\wedge q] \leq \beta A_R$, then $x_R = (1-\beta)E[D\wedge q] + (1-\beta)A_R/r$. In such case $x_R \leq A_R/r$, and

$$\pi_R(q|w) = rE[(D\wedge q)] - rx_R + A_R = \beta rE[(D\wedge q)] + \beta A_R. \tag{B.47}$$

In such case, $\pi_R(q|w)$ always increases in q, up to the point such that $x_R = A_R/r$.

If $E[(1-\beta)(A_R + rD)\wedge(1-\beta)(A_R + rq)] \geq A_R$, i.e. $(1-\beta)rE[D\wedge q] \geq \beta A_R$, then x_R satisfies that $rx_R\bar{F}(x_R - A_R/r) - \int_{x_R - A_R/r}^\infty (1-\beta)(A_R + r(q\wedge\xi))dF(\xi) = 0$, and

$$\pi_R(q|w) = E[(A_R + r(D\wedge q) - rx_R)^+] \tag{B.48}$$

$$= \int_{x_R - A_R/r}^\infty A_R + r(q\wedge\xi)dF(\xi) - rx_R\bar{F}(x_R - A_R/r) \tag{B.49}$$

$$= \frac{\beta}{(1-\beta)} rx_R\bar{F}(x_R - A_R/r). \tag{B.50}$$

Then $\frac{\partial \pi_R}{\partial x_R} = \frac{\beta}{(1-\beta)} r\bar{F}(x_R - A_R/r)(1 - x_R z(x_R - A_R/r))$. If $A_R z(0)/r > 1$, then the above expression is negative for all x_R, and therefore the retailer's profit is maximized by choosing $x_R = A_R/r$. Otherwise, $\frac{\partial \pi_R}{\partial x_R}$ crosses zero only once (from positive to negative) since $(1 - x_R z(x_R - A_R/r)$ decreases in x_R. Therefore, to maximize $\pi_R(q|w)$, the retailer will choose x_R such that $x_R z(x_R - A_R/r) = 1$ and choose q such that $r x_R \bar{F}(x_R - A_R/r) - \int_{x_R - A_R/r}^{\infty} (1-\beta)(A_R + r(q \wedge \xi))dF(\xi) = 0$.

Next, if $q > q_m$, then $x_R \geq x_{Rm}$ and therefore $1 - x_R z(x_R - A_R/r) \leq 0$, which means π_R decreases in x_R and further decreases in q. This completes the proof. □

Proof of Lemma 3.2. It is clear from Equation (3.10) that x_R increases with w. Therefore, the retailer's profit decreases with w for every fixed q, so the retailer's optimal profit decreases with w if $(w - (1-\beta)r)q \leq (1-\beta)A_R$. From the definition of $\bar{w}$, at $w = \bar{w}$, the retailer is indifferent between choosing $(w - (1-\beta)r)q \leq (1-\beta)A_R$ or $(w - (1-\beta)r)q \geq (1-\beta)A_R$. If $w < \bar{w}$, the retailer's local optimal profit under the constraint $(w - (1-\beta)r)q \leq (1-\beta)A_R$ is larger than the local optimal profit under the constraint $(w - (1-\beta)r)q \geq (1-\beta)A_R$. Similarly, if $w > \bar{w}$, the retailer's local optimal profit under the constraint $(w - (1-\beta)r)q \leq (1-\beta)A_R$ is smaller than the local optimal profit under the constraint $(w - (1-\beta)r)q \geq (1-\beta)A_R$. This completes the proof. □

Proof of Proposition 3.3. The optimal solution is obtained at the point that the retailer's marginal profit equals zero, or at the non-smooth point of the retailer's profit function (3.9). The non-smoothness arises from the retailer's choice of x_R in Proposition 3.1.

The proof of the candidates 1 to 4 are the same to the proof of Proposition 3.2. The conditions of candidates 5 are from Lemma 3.1. The only remaining non-smooth point is $q = \frac{(1-\beta)A_R}{w-(1-\beta)r}$. Because $w > (1-\beta)r$, from Equation (3.7) we know that $\frac{\partial x_R}{\partial q}$ is larger when q approaching $\frac{(1-\beta)A_R}{w-(1-\beta)r}$ from the left than that when q approaching $\frac{(1-\beta)A_R}{w-(1-\beta)r}$ from the right. Therefore, from Equation (3.9), we know that $\frac{\partial \pi_R}{\partial q}$ is smaller when q approaching $\frac{(1-\beta)A_R}{w-(1-\beta)r}$ from the left than that when q approaching $\frac{(1-\beta)A_R}{w-(1-\beta)r}$ from the right, and therefore $\frac{(1-\beta)A_R}{w-(1-\beta)r}$ is a local minimal point and thus not optimal. □

Proof of Proposition 3.4. The optimal solution is obtained at the point that the supplier's marginal profit equals zero, or at the non-smooth point of the supplier's profit function (3.15). The non-smoothness arises from the retailer's choice of q in Proposition 3.3. Depending on different choice of the optimal q, we consider the five cases below.

Case 1: If $\bar{F}(q) = \frac{w}{r}$ and $q \le (1-\beta)A_R/w$. Then $\pi_S(w,q) = (w-c)q$, and

$$\frac{\partial q}{\partial w} = -\frac{1}{rf(q)}. \tag{B.51}$$

$$\frac{\partial \pi_S}{\partial w} = q + (w-c)\frac{\partial q}{\partial w} = -\frac{1}{rf(q)}(\bar{F}(q)(1-qz(q)) - \frac{c}{r}). \tag{B.52}$$

In such case, the first order condition is $\bar{F}(q)(1-qz(q)) - \frac{c}{r} = 0$ and $\bar{F}(q) = \frac{w}{r}$. Also, we need to satisfy the constraint that $q \le (1-\beta)A_R/w$. Therefore, in this case the optimal solution is (w_{s1}, q_{s1}) if $q_{s1} \le q_{s2}$, and (w_{s2}, q_{s2}) otherwise.

Case 2: If $r\bar{F}(q) = w\bar{F}(\frac{wq}{(1-\beta)r} - \frac{A_R}{r})$ and $(1-\beta)A_R/w \le q \le q_\alpha(w)$, then $\pi_S(w,q) = E[wq \wedge (1-\beta)(A_R + rD)] - cq$, and

$$\frac{\partial q}{\partial w} = -\frac{1 - \frac{wq}{(1-\beta)r}z(\frac{wq-(1-\beta)A_R}{(1-\beta)r})}{wz(q) - \frac{w^2}{(1-\beta)r}z(\frac{wq-(1-\beta)A_R}{(1-\beta)r})}, \tag{B.53}$$

$$\frac{\partial \pi_S}{\partial w} = q\bar{F}(\frac{wq-(1-\beta)A_R}{(1-\beta)r}) + (r\bar{F}(q) - c)\frac{\partial q}{\partial w} \tag{B.54}$$

$$= \frac{r}{w}q\bar{F}(q) - (r\bar{F}(q) - c)\frac{1 - \frac{wq}{(1-\beta)r}z(\frac{wq-(1-\beta)A_R}{(1-\beta)r})}{wz(q) - \frac{w^2}{(1-\beta)r}z(\frac{wq-(1-\beta)A_R}{(1-\beta)r})} \tag{B.55}$$

$$= \frac{r\bar{F}(q)(qz(q)-1) + c(1 - \frac{wq}{(1-\beta)r}z(\frac{wq-(1-\beta)A_R}{(1-\beta)r})}{wz(q) - \frac{w^2}{(1-\beta)r}z(\frac{wq-(1-\beta)A_R}{(1-\beta)r})}. \tag{B.56}$$

The first order condition is $r\bar{F}(q) = w\bar{F}(\frac{wq}{(1-\beta)r} - \frac{A_R}{r})$ and $r\bar{F}(q)(qz(q)-1) + c(1 - \frac{wq}{(1-\beta)r}z(\frac{wq-(1-\beta)A_R}{(1-\beta)r}) = 0$. Since we cannot argue quasi-concavity in this case, the candidates of the optimal solution are (w_{s2}, q_{s2}), (w_{s4}, q_{s4}), and (w_{s3}, q_{s3}) if $q_{s2} \le q_{s3} \le q_{s4}$.

Case 3: If $q = q_\alpha(w)$ and $x_R = A_R/r$, then $\pi_S(w,q) = A_R - cq$.

In such case $\frac{\partial q}{\partial w} = -q/w < 0$. Therefore, $\frac{\partial \pi_S}{\partial w} = -c\frac{\partial q}{\partial w} = cq/w > 0$. That is, the supplier's profit increases with w.

Case 4: If $r\bar{F}(q)(1 - \beta x_R z(x_R - \frac{A_R}{r})) = w\bar{F}(\frac{wq}{(1-\beta)r} - \frac{A_R}{r})$ and $q \ge q_\alpha(w)$, then $\pi_S(w,q) = E[wq \wedge (1-\beta)(A_R + rD)] - cq$.

The first order condition is $\frac{\partial \pi_S}{\partial w} = 0$ and $r\bar{F}(q)(1 - \beta x_R z(x_R - \frac{A_R}{r})) = w\bar{F}(\frac{wq}{(1-\beta)r} - \frac{A_R}{r})$. Therefore, the optimal solution in this case is (w_{s5}, q_{s5}) if $q_{s5} \le q_{s6}$, or (w_{s6}, q_{s6}) if $w_{s6} \le \bar{w}$. Moreover, if $w_{s6} > \bar{w}$, then π_S increases in w until $\bar{w}$.

<u>Case 5</u>: We know that q_m does not depend on w. However, as shown in Lemma 3.2 and Figure 3.3, the supplier's profit may be discontinuous at $w = \bar{w}$. Therefore, w_{s7} can be optimal if $\pi_S(w)$ increases in w up to $\bar{w}$. Also, w_{s8} can also be optimal if the jump at $\bar{w}$ is positive and large enough.

$\square$

Proof of Proposition 3.5. If $wq \le \frac{A_R}{1-\lambda}$, then $y_2 = 0$. In this case, taking derivative of the first equation of (3.17) with respect to q, we have

$$\frac{\partial \pi_R(q)}{\partial q} = r\bar{F}(q) - \lambda w \bar{F}(\frac{wq}{r}) - (1-\lambda)w,$$
$$\frac{\partial^2 \pi_R(q)}{\partial q^2} = -rf(q) + \lambda \frac{w^2}{r} f(\frac{wq}{r})$$

Note that for every q such that $\frac{\partial \pi_R(q)}{\partial q} \ge 0$, we have $r\bar{F}(q) \ge w[1 - \lambda F(\frac{wq}{r})]$. Upon assuming $\frac{\partial \pi_R(q)}{\partial q} \ge 0$ and substituting the previous inequality into $\frac{\partial^2 \pi_R(q)}{\partial q^2}$, we have

$$\begin{aligned}\frac{\partial^2 \pi_R(q)}{\partial q^2} &\le -rf(q) + \lambda w \frac{f(\frac{wq}{r})}{1-\lambda F(\frac{wq}{r})} \bar{F}(q) = \bar{F}(q)[-rz(q) + \lambda w \frac{f(\frac{wq}{r})}{1-\lambda F(\frac{wq}{r})}] \\ &\le \bar{F}(q)[-rz(q) + \lambda w z(\frac{wq}{r})] \le 0.\end{aligned}$$

The last inequality follows from the fact that $z(\cdot)$ is assumed to be increasing. This shows that when $wq \le \frac{A_R}{1-\lambda}$, $\pi_R(q)$ is either decreasing or concave in q.

Next, when $wq \ge \frac{A_R}{1-\lambda}$, $y_2 = \frac{1}{r}[wq - \frac{A_R}{1-\lambda}]$, then by the second equation in (3.17),

$$\frac{\partial \pi_R(q)}{\partial q} = r\bar{F}(q) - \lambda w \bar{F}(y_1) - (1-\lambda)w\bar{F}(y_2),$$
$$\frac{\partial^2 \pi_R(q)}{\partial q^2} = -rf(q) + \lambda \frac{w^2}{r} f(y_1) + (1-\lambda)\frac{w^2}{r} f(y_2).$$

Again, upon assuming $\frac{\partial \pi_R(q)}{\partial q} \ge 0$, we have

$$\begin{aligned}\frac{\partial^2 \pi_R(q)}{\partial q^2} &\le -rf(q) + \lambda w \frac{\lambda f(y_1) + (1-\lambda)f(y_2)}{\lambda \bar{F}(y_1) + (1-\lambda)\bar{F}(y_2)} \bar{F}(q) \\ &= w\bar{F}(q)[-\frac{r}{w}z(q) + \frac{\lambda f(y_1) + (1-\lambda)f(y_2)}{\lambda \bar{F}(y_1) + (1-\lambda)\bar{F}(y_2)}] \\ &= w\bar{F}(q)[-\frac{r}{w}z(q) + \frac{\lambda \bar{F}(y_1)}{\lambda \bar{F}(y_1) + (1-\lambda)\bar{F}(y_2)} z(y_1) + \frac{(1-\lambda)\bar{F}(y_2)}{\lambda \bar{F}(y_1) + (1-\lambda)\bar{F}(y_2)} z(y_2)] \\ &\le 0.\end{aligned}$$

The last inequality results from the fact that $z(q) \geq z(y_1) \geq z(y_2)$.

Therefore, when $wq \geq \frac{A_R}{1-\lambda}$, $\pi_R(q)$ is also either decreasing or concave in q. Moreover, note that $\frac{\partial \pi_R}{\partial q}$ is continuous at $q = \frac{A_R}{(1-\lambda)w}$ (because in the two regions defined by $wq \leq \frac{A_R}{1-\lambda}$ and $wq \geq \frac{A_R}{1-\lambda}$, the first order derivatives of $\pi_R(q)$ have the same form), and that $\frac{\partial \pi_R}{\partial q}(0) = r - w > 0$.

Thus, $\pi_R(q)$ is unimodal in q and an unique optimal order quantity is obtained by solving the following equation:

$$r\bar{F}(q) = \lambda w \bar{F}(y_1) + (1-\lambda) w \bar{F}(y_2). \tag{B.57}$$

Hence proved. $\square$

Proof of Lemma 3.3. Note that by IFR property, $z(q)$ is increasing, so $qz(q)$ is strictly increasing in q, which means that $qz(q) < 1$ for all $q \in (0, q_c^*)$. Therefore, if $q \leq M$, then $q\bar{F}([q-M]^+) = q$, which is concave and strictly increasing in q. Otherwise, if $q \geq M$, then

$$\begin{aligned}
\frac{\partial}{\partial q} q\bar{F}([q-M]^+) &= \bar{F}(q-M) - qf(q-M) \\
&= \bar{F}(q-M)[1 - qz(q-M)] \\
&\geq \bar{F}(q-M)[1 - qz(q)] > 0.
\end{aligned}$$

The last strict inequality holds because $\bar{F}(q-M) \geq \bar{F}(q) \geq \bar{F}(q_c^*) > 0$. Therefore $q\bar{F}([q-M]^+)$ is strictly increasing in q. Also from the IFR property and the fact that $1 - qz(q) \geq 0$, we have that $\bar{F}(q-M)[1 - qz(q-M)]$ decreases in q. Therefore $q\bar{F}([q-M]^+)$ is concave. Hence proved. $\square$

Proof of Corollary 3.2. First, we show that when q satisfies the Equation (3.18), we must have $y_1 z(y_1) \leq 1$. This is true since $\bar{F}(q)$ is a convex combination of $\frac{w}{r}\bar{F}(y_1)$ and $\frac{w}{r}\bar{F}(y_2)$, also $\bar{F}(y_1) \leq \bar{F}(y_2)$, we must have $\bar{F}(q) \geq \frac{w}{r}\bar{F}(y_1)$, i.e., $q\bar{F}(q) \geq y_1\bar{F}(y_1)$. Then by Lemma 3.3, we must have $y_1 z(y_1) \leq 1$.

When $y_2 > 0$, taking derivative on both sides of the Equation (3.18) with respect to w, we have

$$-rf(q)\frac{\partial q}{\partial w} = \lambda\bar{F}(y_1) - \lambda w f(y_1)[\frac{q}{r} + \frac{w}{r}\frac{\partial q}{\partial w}] + (1-\lambda)\bar{F}(y_2) - (1-\lambda) w f(y_2)[\frac{q}{r} + \frac{w}{r}\frac{\partial q}{\partial w}].$$

After simplifying the above equation, we have

$$\frac{\partial q}{\partial w} = \frac{\lambda[\bar{F}(y_1) - y_1 f(y_1)] + (1-\lambda)[\bar{F}(y_2) - y_1 f(y_2)]}{-rf(q) + \lambda \frac{w^2}{r} f(y_1) + (1-\lambda)\frac{w^2}{r} f(y_2)}. \tag{B.58}$$

In the proof of Proposition 3.5, we showed that $\frac{\partial^2 \pi_R(q)}{\partial q^2} = -rf(q) + \lambda \frac{w^2}{r} f(y_1) + (1 - \lambda)\frac{w^2}{r} f(y_2) \leq 0$ for any q that satisfies Equation (3.18). Therefore, the denominator of Equation (B.58) is negative. The numerator of (B.58) can be simplified as follows.

$$\begin{aligned}
&\lambda[\bar{F}(y_1) - y_1 f(y_1)] + (1-\lambda)[\bar{F}(y_2) - y_1 f(y_2)] \\
&= \lambda \bar{F}(y_1)[1 - y_1 z(y_1)] + (1-\lambda)\bar{F}(y_2)[1 - y_1 z(y_2)] \\
&\geq \lambda \bar{F}(y_1)[1 - y_1 z(y_1)] + (1-\lambda)\bar{F}(y_2)[1 - y_1 z(y_1)] \\
&\geq 0.
\end{aligned}$$

where the first inequality comes from $z(y_2) \leq z(y_1)$ and the second inequality from $1 - y_1 z(y_1) \geq 0$. Therefore, when $y_2 > 0$, $\frac{\partial q}{\partial w} \leq 0$. Similarly, it is easy to show when $y_2 = 0$, we also have $\frac{\partial q}{\partial w} \leq 0$, which establishes that q^* decreases in w. Hence proved. $\square$

Proof of Proposition 3.6. We multiply by q on both sides of Equation (3.18) to obtain

$$q\bar{F}(q) = \lambda(\frac{wq}{r})\bar{F}(\frac{wq}{r}) + (1-\lambda)(\frac{wq}{r})\bar{F}([\frac{wq}{r} - \frac{A_R}{(1-\lambda)r}]^+) \tag{B.59}$$

We fix q and denote $g(w,\lambda) = \lambda(\frac{wq}{r})\bar{F}(\frac{wq}{r}) + (1-\lambda)(\frac{wq}{r})\bar{F}([\frac{wq}{r} - \frac{A_R}{(1-\lambda)r}]^+)$. For every $w \leq r$, we have $\frac{wq}{r} \leq q \Rightarrow (\frac{wq}{r})\bar{F}(\frac{wq}{r}) \leq q\bar{F}(q)$ (by Lemma 3.3). It means that for every w, we have $g(w,1) \leq q\bar{F}(q)$.

Next, it is clear that $g(r,0) = q\bar{F}([q - \frac{A_R}{r}]^+) \geq q\bar{F}(q)$. By Lemma 3.3, $g(w,0)$ increases in w. Therefore there exist w_1 such that $g(w_1,0) = q\bar{F}(q)$, i.e., $\frac{w_1}{r}\bar{F}(\frac{w_l q - A_R}{r}) = \bar{F}(q)$. Let $\hat{w} = \max\{c, w_1\}$, then for all $w \geq \hat{w}$, we have $g(w,0) \geq q\bar{F}(q)$. Therefore, by continuity, for any $w \in [\hat{w}, r)$, there exists $\lambda \in [0,1)$ such that $g(w,\lambda) = q\bar{F}(q)$. Hence proved. $\square$

Proof of Theorem 3.1. By Proposition 3.6, for every $w \in [\hat{w}(q_c^*), r)$ we can choose $\lambda \in [0,1)$ such that the order quantity is q_c^*. Therefore, we have $\pi_R + \pi_S = \pi_c^* - \kappa$. Moreover, note that when $w \to r$, we have $\lambda \to 1$, $y_2 \to 0$, $y_1 \to q_c^*$, and therefore

$\pi_S \to \pi_c^* - \kappa$. Hence, by continuity, the supplier can make any profit between $\hat{\pi}_S$ and $\pi_c^* - \kappa$. Therefore, the supplier makes the profit $\pi_c^* - \pi_R^0 - \kappa$ for the retailer accepting the offer. Hence proved. $\square$

Proof of Proposition 3.7. Following Theorem 3.1, we only need to prove that $\hat{\pi}_S$ decreases in A_R when $A_R \geq cq_c^*$. Let $\hat{\pi}_R := \pi_c(q_c^*) - \hat{\pi}_S - \kappa$ and note that $\pi_c(q_c^*)$ does not depend on A_R. We then prove that $\hat{\pi}_R$ increases in A_R.

$\hat{\pi}_R$ is achieved by choosing $\lambda = 0$ and $w = w_l$ such that $\frac{w_l}{r}\bar{F}(\frac{w_l q_c^* - A_R}{r}) = \bar{F}(q_c^*)$. In this case $y_2 = \frac{wq_c^* - A_R}{r}$ and

$$\hat{\pi}_R(A_R) = \int_0^{q_c^*} r\bar{F}(x)dx - \int_0^{y_2} r\bar{F}(x)dx - A_R. \tag{B.60}$$

Rewriting $\frac{w}{r}\bar{F}(\frac{wq_c^* - A_R}{r}) = \bar{F}(q_c^*)$, we have $(y_2 + \frac{A_R}{r})\bar{F}(y_2) = q_c^*\bar{F}(q_c^*)$. Thus, $\frac{\partial y_2}{\partial A_R} = \frac{\bar{F}(y_2)/r}{(y_2 + \frac{A_R}{r})f(y_2) - \bar{F}(y_2)}$, and

$$\frac{\partial \hat{\pi}_R(A_R)}{\partial A_R} = -r\bar{F}(y_2)\frac{\partial y_2}{\partial A_R} - 1 = \frac{\bar{F}(y_2)}{1 - (y_2 + \frac{A_R}{r})z(y_2)} - 1 \geq \frac{\bar{F}(y_2)}{1 - y_2 z(y_2)} - 1 \tag{B.61}$$

Denote $H(y_2) = \frac{\bar{F}(y_2)}{1 - y_2 z(y_2)} - 1$, then $\frac{\partial H(y_2)}{\partial y_2} = \frac{y_2 f(y_2) z(y_2) + y_2 \bar{F}(y_2) z'(y_2)}{(1 - y_2 z(y_2))^2} \geq 0$. It follows that

$$\frac{\partial \hat{\pi}_R(A_R)}{\partial A_R} \geq H(y_2) \geq H(0) = 0 \tag{B.62}$$

Hence $\hat{\pi}_R(A_R)$ increases in A_R.

$\square$

B.3 Proofs of Propositions and Lemmas of Chapter 4

Proof of Proposition 4.1. We prove the proposition by induction. First note that $u_n(x_n, c_n)$ is concave in (x_n, c_n). Assume that $u_{t+1}(x_{t+1}, c_{t+1})$ is concave in (x_{t+1}, c_{t+1}), and let $\Phi_{t+1}(x_t, c_t, k_t, d_t) = u_{t+1}\big(x_t + h_t(x_t, k_t)d_t, c_t - k_t\big)$ for every realization d_t, then we prove below that $\Phi_{t+1}(x_t, c_t, k_t, d_t)$ is concave in (x_t, c_t, k_t) for every d_t. For any two points $(x_t^{(1)}, c_t^{(1)}, k_t^{(1)})$, $(x_t^{(2)}, k_t^{(2)}, c_t^{(2)})$, and any $0 < \theta < 1$, let $\bar{\theta} := 1 - \theta$, and we have

$$\Phi_{t+1}(\theta x_t^{(1)} + \bar{\theta} x_t^{(2)}, \theta c_t^{(1)} + \bar{\theta} c_t^{(2)}, \theta k_t^{(1)} + \bar{\theta} k_t^{(2)}, d_t)$$
$$= u_{t+1}\big(\theta x_t^{(1)} + \bar{\theta} x_t^{(2)} + h_t(\theta x_t^{(1)} + \bar{\theta} x_t^{(2)}, \theta k_t^{(1)} + \bar{\theta} k_t^{(2)}) d_t, \theta c_t^{(1)} + \bar{\theta} c_t^{(2)} - (\theta k_t^{(1)} + \bar{\theta} k_t^{(2)})\big) \quad \text{(B.63)}$$
$$\geq u_{t+1}\big(\theta x_t^{(1)} + \bar{\theta} x_t^{(2)} + \theta h_t(x_t^{(1)}, k_t^{(1)}) d_t + \bar{\theta} h_t(x_t^{(2)}, k_t^{(2)}) d_t, \theta(c_t^{(1)} - k_t^{(1)}) + \bar{\theta}(c_t^{(2)} - k_t^{(2)}))\big) \quad \text{(B.64)}$$
$$= u_{t+1}\big(\theta(x_t^{(1)} + h_t(x_t^{(1)}, k_t^{(1)}) d_t) + \bar{\theta}(x_t^{(2)} + h_t(x_t^{(2)}, k_t^{(2)}) d_t), \theta(c_t^{(1)} - k_t^{(1)}) + \bar{\theta}(c_t^{(2)} - k_t^{(2)}))\big) \quad \text{(B.65)}$$
$$\geq \theta u_{t+1}\big(x_t^{(1)} + h_t(x_t^{(1)}, k_t^{(1)}) d_t, c_t^{(1)} - k_t^{(1)}\big) + \bar{\theta} u_{t+1}\big(x_t^{(2)} + h_t(x_t^{(2)}, k_t^{(2)}) d_t, c_t^{(2)} - k_t^{(2)}\big) \quad \text{(B.66)}$$
$$= \theta \Phi(x_t^{(1)}, c_t^{(1)}, k_t^{(1)}, d_t) + \bar{\theta} \Phi(x_t^{(2)}, c_t^{(2)}, k_t^{(2)}, d_t). \quad \text{(B.67)}$$

Inequality B.64 holds because $h_t(x_t, k_t)$ is jointly concave, and Inequality B.66 holds because $u_{t+1}(x, c)$ is jointly concave in (x, c). Therefore, $\Phi_{t+1}(x_t, c_t, k_t, d_t)$ is concave in (x_t, c_t, k_t) for every d_t.

Also, $v_t(x_t, c_t, k_t)$ is concave in (x_t, c_t, k_t) because $v_t(x_t, c_t, k_t)$ is defined by taking expectation of D_t on $\Phi_{t+1}(x_t, c_t, k_t, D_t)$. Further, $u_t(x_t, c_t) = \max_{k_t \leq c_t} v_t(x_t, c_t, k_t)$ is concave in (x_t, c_t) (see e.g. Theorem A.4 in Porteus 2002b), which completes the proof. □

Proof of Proposition 4.2. The proof of Proposition 4.2 is similar to that of Proposition 4.1 by induction. First note that $u_n(x, c)$ is convex in (x, c). Assume that $u_{t+1}(x, c)$ is convex in (x, c), and let $\Phi_{t+1}(x, c, k, d_t) = u_{t+1}\big(x_t + h_t(x_t, k) d_t, c_t - k\big)$ for every realization d_t. Similar to the proof of Proposition 4.1, it is easy to show that $\Phi_{t+1}(x, c, k, d_t)$ is convex in (x, c, k) for every d_t (by only changing the direction of the inequality). Therefore, $v_t(x, c, k)$ is convex in (x, c, k) (see Equation 4.6). Therefore, $u_t(x, c) = \max_{k \leq c} v_t(x, c, k)$ is convex in (x, c) (see e.g. Theorem A.3 in Porteus 2002b), which completes the proof. □

Proof of Proposition 4.3. If $h_t(x, k) = \exp\{\alpha_{0,t} + \alpha_{1,t} x + \alpha_{2,t} k\}$, $t = 1$ or 2, then

$$v(x, c, k) = \exp\{\alpha_{0,1} + \alpha_{1,1} x + \alpha_{2,1} k\} + \exp\{\alpha_{0,2} + \alpha_{1,2} x + \alpha_{2,2}(c - k)\} E[\exp\{\alpha_{1,2} h_1(x, k) D\}] \quad \text{(B.68)}$$

From Proposition 4.2, we know that k^* equals either c or 0. Therefore, we only need to compare $v(x,c,c)$ with $v(x,c,0)$.

$$\begin{aligned} v(x,c,c) &= \exp\{\alpha_{0,1} + \alpha_{1,1}x + \alpha_{2,1}c\} \\ &\quad + \exp\{\alpha_{0,2} + \alpha_{1,2}x\} E[\exp\{\alpha_{1,2} e^{\alpha_{0,1}+\alpha_{1,1}x+\alpha_{2,1}c} D\}] \qquad \text{(B.69)} \\ v(x,c,0) &= \exp\{\alpha_{0,1} + \alpha_{1,1}x\} \\ &\quad + \exp\{\alpha_{0,2} + \alpha_{1,2}x + \alpha_{2,2}c\} E[\exp\{\alpha_{1,2} e^{\alpha_{0,1}+\alpha_{1,1}x} D\}] \qquad \text{(B.70)} \end{aligned}$$

$$\begin{aligned} v(x,c,c) - v(x,c,0) =& e^{\alpha_{0,2}+\alpha_{1,2}x} \big(e^{\alpha_{0,1}-\alpha_{0,2}+(\alpha_{1,1}-\alpha_{1,2})x} (e^{\alpha_{2,1}c} - 1) \big) \\ & + e^{\alpha_{0,2}+\alpha_{1,2}x} \big(E[\exp\{\alpha_{1,2} e^{\alpha_{0,1}+\alpha_{1,1}x} e^{\alpha_{2,1}c} D\}] \big) \\ & - e^{\alpha_{0,2}+\alpha_{1,2}x} \big(E[\exp\{\alpha_{1,2} e^{\alpha_{0,1}+\alpha_{1,1}x} D + \alpha_{2,2}c\}] \big) \qquad \text{(B.71)} \end{aligned}$$

If $\alpha_{i,1} \geq \alpha_{i,2}$, for $i = 1,2$, then it is clear that $e^{\alpha_{0,1}-\alpha_{0,2}+(\alpha_{1,1}-\alpha_{1,2})x}(e^{\alpha_{2,1}c} - 1)$ increases in x. Next, we define

$$\phi(x) = E[\exp\{\alpha_{1,2} e^{\alpha_{0,1}+\alpha_{1,1}x} e^{\alpha_{2,1}c} D\}] - E[\exp\{\alpha_{1,2} e^{\alpha_{0,1}+\alpha_{1,1}x} D + \alpha_{2,2}c\}],$$

and we show that $\phi(x)$ increases in x.

$$\begin{aligned} \frac{\partial \phi(x)}{\partial x} =& E[\exp\{\alpha_{1,2} e^{\alpha_{0,1}+\alpha_{1,1}x} e^{\alpha_{2,1}c} D\} \alpha_{1,2}\alpha_{1,1} e^{\alpha_{0,1}+\alpha_{1,1}x} e^{\alpha_{2,1}c} D] \\ & - E[\exp\{\alpha_{1,2} e^{\alpha_{0,1}+\alpha_{1,1}x} D\} e^{\alpha_{2,2}c} \alpha_{1,2}\alpha_{1,1} e^{\alpha_{0,1}+\alpha_{1,1}x} D] \qquad \text{(B.72)} \\ \geq& 0. \qquad \text{(B.73)} \end{aligned}$$

Therefore, if $\alpha_{i,1} \geq \alpha_{i,2}$, for $i = 1,2$, then $v(x,c,c) - v(x,c,0)$ increases in x. Thus, there exists a threshold $\hat{x}$. If $x < \hat{x}$, then $v(x,c,c) - v(x,c,0) \leq 0$, thus $k^* = 0$, and if $x \geq \hat{x}$ then $v(x,c,c) - v(x,c,0) \geq 0$, thus $k^* = c$.

Next, we show that If $\alpha_{i,1} \geq \alpha_{i,2}$, for $i = 0,1,2$, then $v(x,c,c) - v(x,c,0) \geq 0$ at $x = 0$. Therefore, $k^* = c$ for every $x \geq 0$ and $c \geq 0$. Substitute $x = 0$ in Equation (B.71), we have

$$\begin{aligned} v(0,c,c) - v(0,c,0) =& e^{\alpha_{0,2}} \big(e^{\alpha_{0,1}-\alpha_{0,2}} (e^{\alpha_{2,1}c} - 1) \big) + e^{\alpha_{0,2}} \big(E[\exp\{\alpha_{1,2} e^{\alpha_{0,1}} e^{\alpha_{2,1}c} D\}] \big) \\ & - e^{\alpha_{0,2}} \big(E[\exp\{\alpha_{1,2} e^{\alpha_{0,1}} D + \alpha_{2,2}c\}] \big) \qquad \text{(B.74)} \end{aligned}$$

Denote $\theta = e^{\alpha_{2,1}c}$, and $y = E[\exp\{\alpha_{1,2} e^{\alpha_{0,1}} D\}]$. Then $\theta \geq 1$ and $y > 0$. Note $E[\exp\{\alpha_{1,2} e^{\alpha_{0,1}} \theta D\}] \geq (E[\exp\{\alpha_{1,2} e^{\alpha_{0,1}} D\}])^{\theta}$, and $e^{\alpha_{0,1}-\alpha_{0,2}} \geq 1$. Therefore,

$$v(0,c,c) - v(0,c,0) \geq e^{\alpha_{0,2}} \big(\theta - 1 + y^{\theta} - \theta y \big) \qquad \text{(B.75)}$$

The derivative of the right-hand-side of Expression (B.75) is $\theta y^{\theta-1} - \theta$. It implies that $\theta - 1 + y^{\theta} - \theta y$ is minimized by $y = 1$ if $\theta \geq 1$. Therefore, $v(0, c, c) - v(0, c, 0) \geq e^{\alpha_{0,2}}(\theta - 1 + 1 - \theta) = 0$. Therefore, $v(x, c, c) - v(x, c, 0) \geq 0$ at $x = 0$, and thus $k^* = c$. □

Lemma B.2. *If $h_2(x, k)$ is component-wise concave and supermodular on (x, k), and $h_1(x, k)$ is concave in k for every x, then $v(x, c, k)$ is concave in k.*

Proof of Lemma B.2. We have that

$$\frac{\partial v}{\partial k}(x, c, k) = h_1^{[2]}(x, k) + E[h_2^{[1]}(x + h_1(x, k)D, c - k)h_1^{[2]}(x, k)D - h_2^{[2]}(x + h_1(x, k)D, c - k)]. \tag{B.76}$$

We need to show that $\frac{\partial v}{\partial k}(x, c, k)$ decreases in k in Equation (B.76). We have that $h_1^{[2]}(x, k)$ decreases in k because $h_1(x, k)$ is concave in k. $h_2^{[1]}(x + h_1(x, k)D, \cdot)$ decreases in k because $h_2(x, k)$ is concave in x, and $h_2^{[1]}(\cdot, c - k)$ decreases in k because $h_2(x, k)$ is supermodular on (x, k). Therefore, $E[h_2^{[1]}(x + h_1(x, k)D, c - k)h_1^{[2]}(x, k)D]$ decreases in k. Similarly, $E[h_2^{[2]}(x + h_1(x, k)D, c - k)]$ increases in k because $h_2(x, k)$ is concave in k and supermodular on (x, k). Hence, $\frac{\partial v}{\partial k}(x, c, k)$ decreases in k. □

Proof of Proposition 4.4. From lemma B.2, we note that the $v(x, c, k)$ is concave in k if $h(x, k) = f(x)g(k)$, then we want to show that $\frac{\partial v}{\partial k}(x, c, k) \leq 0$ at $k = c/2$, and then we will have $k^* \leq c/2$. Denote $h^{[1]}(x, k) = f'(x)g(k)$ and $h^{[2]}(x, k) = f(x)g'(k)$. Substitute $h^{[1]}(x, k)$ and $h^{[2]}(x, k)$ in Equation (B.76), we obtain

$$\frac{\partial v}{\partial k}(x, c, k) = f(x)g'(k) + E[f'(x + h_1(x, k)D)g(c - k)f(x)g'(k)D - f(x + h_1(x, k)D)g'(c - k)]. \tag{B.77}$$

From lemma B.2, $v(x, c, k)$ is concave in k. Therefore,to prove $k^* \leq c/2$, we only need to show that $\frac{\partial v}{\partial k}(x, c, k) \leq 0$ at $k = c/2$. Then,

$$\frac{\partial v}{\partial k}(x, c, \frac{c}{2}) = f(x)g'(\frac{c}{2}) + E[f'(x + f(x)g(\frac{c}{2})D)g(\frac{c}{2})f(x)g'(\frac{c}{2})D - f(x + f(x)g(\frac{c}{2})D)g'(\frac{c}{2})].$$

Because $f(x)$ is concave, we have $f(x+f(x)g(\frac{c}{2})D) - f(x) \geq f'(x+f(x)g(\frac{c}{2})D)f(x)g(\frac{c}{2})D$ for every D. Therefore,

$$\frac{\partial v}{\partial k}(x, c, \frac{c}{2}) \leq f(x)g'(\frac{c}{2}) + E[(f(x + f(x)g(\frac{c}{2})D) - f(x))g'(\frac{c}{2}) - f(x + f(x)g(\frac{c}{2})D)g'(\frac{c}{2})] = 0. \tag{B.78}$$

Thus, $\frac{\partial v}{\partial k}(x, c, k) \leq 0$ at $k = c/2$ and therefore, $k^* \leq c/2$. □