Higher Picard groupoids and Dijkgraaf-Witten theory

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Dedication

To Milli, Mayya and Tina

Abstract

In the first part of this thesis we propose a model for additive ∞-categories based on Γ-spaces and construct the archetype example of an additive ∞-category, namely the (quasi)-category of higher Picard groupoids, $\mathcal{P}ic$. The goal of the second part of this thesis is to categorify Dijkgraaf-Witten (DW) theory, aiming at providing foundation for a direct construction of DW theory as an Extended Topological Quantum Field Theory. The main tool is cohomology with coefficients in a Picard groupoid, namely the Picard groupoid of hermitian lines.

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Chapter 1

Introduction

In principle, a (higher [1]) Picard groupoid is a *coherently commutative group object* in a suitable (higher) category of (pointed) groupoids. In this thesis, following [Seg74], we model our (higher) Picard groupoids as fibrant objects in the *stable Q-model category*. This model category structure is one of the main constructions of [Sch99] and it is reviewed in Section 2.5.2. These fibrant objects are also known as *very special* Γ*-spaces*. We explicitly address the question of sufficiency of very special Γ-spaces as models for (higher) Picard groupoids and implicitly recommend the suitability of the chosen model for the purpose of Homological Algebra. Another approach to higher Picard groupoids was introduced by Ando, Blumberg and Gepner in [ABG]. This approach is a categorification of the notion of a Picard group of modules over a commutative ring spectrum. The paper defines a higher Picard groupoid of an $\mathcal{O}$-monoidal stable presentable ∞-category $\mathcal{R}$ (see definition [Lur, 2.1.2.13]) which is just the *space* of invertible objects of $\mathcal{R}$ and where $\mathcal{O}^{\otimes}$ is a suitable ∞-operad (see definition [Lur, 2.1.1.10]) whose underlying ∞-category is isomorphic to $\mathcal{O}$ and which satisfies conditions defined in definition [ABG, 1.4]. This means that these higher Picard groupoids may not be algebras over

[1] We will take the liberty of skipping *higher* when referring to higher Picard groupoids in relation to Part I of the thesis, whenever it does not create confusion.

the ∞-operad $Comm^{\otimes}$ (see definition [Lur, 2.1.1.18]). The Picard groupoids proposed in this thesis are necessarily *grouplike* E_{∞} *spaces.* Every Picard groupoid in the sense of definition 2.25 determines a *Comm*-monoidal space in which every object is invertible with respect to the monoidal structure. A fundamental difference between the two approaches is that the approach taken by Ando, Blumberg, and Gepner is based on geometric objects, namely cocartesian fibrations $E \to N(\Gamma^{op})$ (of ∞-operads) whereas our approach is based on functors $\Gamma^{op} \to \mathbf{Kan_{\bullet}}$ (of ordinary categories), which are algebraic objects. This is analogous to treating principal G-bundles over X as maps $X \to BG$. A third approach to higher Picard groupoids is proposed in [May01]. This paper defines a Picard groupoid as a dualizable object in a suitably defined symmetric monoidal ∞-category of (pointed) simplicial sets. In this approach, the invertibility of each object is established by the dualizability data. The Picard groupoids defined in this thesis can be given a dualizability characterization as proposed in [May01].

The main objective of part I of this thesis is to propose a model for *additive* ∞*-categories* which is based on Γ-spaces. A significant part of this thesis is concerned with the construction of the archetype example of an additive ∞-category in this proposed model, namely the ∞-category of Picard groupoids, denoted by $\mathcal{P}ic$. The motivation for our model comes from (ordinary) additive categories. Every (ordinary) additive category has an associated *direct sum* bifunctor which endows the additive category with a symmetric monoidal structure. An additive ∞-category can be, similarly, endowed with a (weak) symmetric monoidal structure, which is presented as a Γ-space, see Section 5.1.

Γ-spaces were introduced by Segal in [Seg74]. Bousfield and Friedlander considered a larger category of Γ-spaces. They constructed two model category structures on their category of Γ-spaces, $\Gamma\mathcal{S}$, in [BF78] with the purpose of comparing Γ-spaces to spectra.

In [Sch99], Schwede describes a simplicial, monoidal model category structure on the category $\Gamma\mathcal{S}$, which he called the *strict Q-model category structure.* The weak equivalences and fibrations of this model structure are degreewise weak equivalences and fibrations in the model category $(\mathbf{sSets}_\bullet, \mathbf{Kan})$. The cofibrations of this model structure are called *Q-cofibrations.* The strict Q-model category structure is a *Bousfield localization* of another model category structure on Γ-spaces which was also constructed in [Sch99] and which was called the *stable Q-model category structure.* In Chapter 3, we construct another model category structure on $\Gamma\mathcal{S}$, which we call the *strict JQ-model category structure.* The cofibrations of this model structure are the same as Q-cofibrations. The weak equivalences and fibrations of this model structure are, respectively, degreewise weak equivalences and fibrations in the "Joyal" model category structure on (pointed) simplicial sets, $(\mathbf{sSets}_\bullet, \mathbf{Q})$. More precisely, the strict Q-model category structure is a *Bousfield localization* of the strict JQ-model category structure described in Chapter 3. Unlike Schwede's strict Q-model structure, which is simplicial in the traditional sense, the strict JQ-model category is a $(\mathbf{sSets}, \mathbf{Q})$-enriched model category, see Theorem 3.15. The fibrant objects of the strict categorical Q-model category structure should be viewed as internal (quasi)categories in the category of Γ-spaces. Our proposed additive ∞-categories are fibrant objects in the strict categorical Q-model category structure.

In Chapter 4 we construct a (simplicial) *derived category of Picard groupoids* and then define the ∞-*category of Picard groupoids* to be its coherent nerve. The notion of a coherently commutative group in the ∞-category $\mathcal{K}_\bullet$ of Kan complexes is weaker than the notion of a higher Picard groupoid, which is a fibrant object in the stable Q-model category. The former need not be presented by an honest functor $\Gamma^{op} \to \mathcal{K}_\bullet$ but rather by a functor which preserves composition only up to coherent homotopy. In other words, a coherently commutative group in $\mathcal{K}_\bullet$ may be presented by a simplicial map $F : N(\Gamma^{op}) \to \mathcal{K}_\bullet$ which satisfies additional conditions given in Definition 4.15. We call this presentation of a coherently commutative group a *weak Picard groupoid*. Another main result of this Chapter is that, up to equivalence, every weak Picard groupoid can be *strictified* to an honest functor defining a fibrant object in the stable Q-model category, see Theorem 4.20.

The main result of part I of this thesis is presented in Chapter 5. We begin the Chapter by constructing a simplicial morphism $\underline{\mathcal{P}ic}^{\oplus}_{w}$, see (5.1), which encodes a *coherently commutative monoid* structure on $\mathcal{P}ic$. The "multiplication" in this coherently commutative monoid is just the direct sum of Picard groupoids. Most of the Chapter is devoted to *strictifying* the simplicial morphism $\underline{\mathcal{P}ic}^{\oplus}_{w}$ into a Γ-space. The main result of part I, namely Theorem 5.16, establishes the existence of a JQ-fibrant Γ-space, denoted by $\underline{\mathcal{P}ic}^{\oplus}$, which is weakly equivalent to a functor of simplicial categories representing the simplicial morphism $\underline{\mathcal{P}ic}^{\oplus}_{w}$, namely

$$\mathbf{Lan}_i(\psi(F)) : \mathbf{F}\mathfrak{C}(N(\Gamma^{op})) \to (\mathbf{sSets}_\bullet, \mathbf{Q})^{cf},$$

see 5.3, in the functor model category $(\mathbf{sSets}_\bullet, \mathbf{Q})^{\mathbf{F}\mathfrak{C}(N(\Gamma^{op}))}$. In other words, for all $m^+ \in \Gamma^{op}$, there is a categorical equivalence between the (pointed) quasi-categories $\underline{\mathcal{P}ic}^{\oplus}(m^+)$ and $\prod\limits_{1}^{m}\mathcal{P}ic$. The above mentioned functor $\mathbf{Lan}_i(\psi(F))$ is an extension, along a cofibration of simplicial categories, see 5.2, of the functor of simplicial categories

$$\psi(F) : \mathfrak{C}(N(\Gamma^{op})) \to (\mathbf{sSets}_\bullet, \mathbf{Q})^{cf},$$

which is uniquely determined by adjointness, by the simplicial morphism

$$\underline{\mathcal{P}ic}^{\oplus}_{w} : N(\Gamma^{op}) \to N((\mathbf{sSets}_\bullet, \mathbf{Q})^{cf}).$$

This extension is carried out in Theorem 5.20.

The *strictification* carried out in Theorem 5.16 is stronger that the *strictification* of weak Picard groupoids carried out in Theorem 4.20. More precisely, a *strictification* of the simplicial diagram $\underline{\mathcal{P}ic}^{\oplus}_{w}$ is possible using the techniques of Theorem 4.20. However, this would result in a Γ-space which would only be (degreewise) weakly homotopy equivalent to a product of (finitely many) copies of $\mathcal{P}ic$. Such a strictification would be insufficient for our purpose of modelling *additive functors* by morphisms of Γ-spaces which is described in the next paragraph.

We observe that every additive quasi-category X, see Definition 5.3, determines a coherently commutative monoid whose "multiplication" is defined as a direct sum in X. One may similarly *strtictify* this monoid to obtain a JQ-fibrant Γ-space $\underline{X}^{\oplus}$. This allows us to propose a model for *additive functors* between additive quasi-categories. We define an additive functor from X to another additive quasi-category Y to be a morphism of Γ-spaces $\underline{F}^{\oplus} : \underline{X}^{\oplus} \to \underline{Y}^{\oplus}$. Further, we propose that the quasi-category $\mathcal{M}ap_{\Gamma\mathcal{S}}(Q\underline{X}^{\oplus}, \underline{Y}^{\oplus})$ models a *(derived)* ∞*-category of additive functors* from X to Y, where the Γ-space $Q\underline{X}^{\oplus}$ is a cofibrant replacement of $\underline{X}^{\oplus}$ in the JQ-model category. This proposed model is itself an additive quasi-category. More precisely, the quasi-category $\mathcal{M}ap_{\Gamma\mathcal{S}}(Q\underline{X}^{\oplus}, \underline{Y}^{\oplus})$, very naturally extends to a JQ-fibrant Γ-space, namely $Hom(Q\underline{X}^{\oplus}, \underline{Y}^{\oplus})$.

The second part of this thesis is devoted to a construction of a TQFT called the *Dijkgraaf-Witten* theory. There has been recurring interest in mathematical physics to Dijkgraaf-Witten theory, see for example [Mon15], [BR96], [SV] and [FPSV15]. An (ordinary) *Picard groupoid* $\mathcal{A}$ is a symmetric monoidal groupoid such that for each object $a \in \mathcal{A}$, the following two functors $a + - : \mathcal{A} \to \mathcal{A}$ and $- + a : \mathcal{A} \to \mathcal{A}$, which are obtained by the bifunctor $- + - : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ giving the symmetric monoidal structure on $\mathcal{A}$, are equivalences in the sense of categories. We observe that the nerve of an (ordinary) Picard groupoid determines a (higher) Picard groupoid uniquely up to equivalence, see example 2.2. Thus one may regard an (ordinary) Picard groupoid as a (higher) Picard groipoid. The category of Picard groupoids, denoted $\mathbf{2}\mathcal{P}ic$, is enriched over itself as per [Sch08, Lemma 2.1]. $\mathbf{2}\mathcal{P}ic$ should be considered as the analog of the category of abelian groups **Ab** in the world of bicategories, whereas $\mathcal{P}ic$ plays a similar role in the world of ∞-categories. The Picard groupoid of primary interest in our work is the Picard groupoid of *hermitian lines* denoted by $\mathcal{L}$. The objects of $\mathcal{L}$ are one dimensional complex vector spaces having a hermitian inner product structure on them. The morphisms of $\mathcal{L}$ are linear isometries. The main tool used in our work is simplicial cohomology with coefficients in the Picard groupoid $\mathcal{L}$. This cohomology should be understood as a derived functor to the category $\mathbf{2}\mathcal{P}ic$. More precisely, given a simplicial set $X_\bullet$, $H^n(X_\bullet; \mathcal{L})$ is itself a Picard groupoid. This cohomolgy was defined in [CMM04] and further developed in [dRMMV05]. D. Freed and F. Quinn in [FQ93] constructed a pairing between cocycles $Z^{n+1}(Y; U(1))$ and cycles $Z_n(Y; \mathbb{Z})$, where Y is a smooth n-dimensional closed manifold, and between $Z^{n+1}(X; U(1))$ and $Z_{n+1}(X, \partial X; \mathbb{Z})$, where X is a compact $n+1$-dimensional smooth manifold with boundary $i : \partial X = \partial X_- \coprod \partial X_+ \subset X$. Both these pairing resemble a *cap product* but are obviously different. Categorifying the coefficents resulted in lowering the cohomological degree thus allowing us to redefine these pairings as the cohomology operation of cap product. More precisely, $\pi_0(H^n(Y; \mathcal{L})) \cong H^{n+1}(Y; U(1))$ and $\pi_0(H^n(X; \mathcal{L})) \cong H^{n+1}(X; U(1))$. Using this cap product, we were able to write two maps on cohomology with coefficients in $\mathcal{L}$ which we call "integration along fibers" of $\pi : Y \times Map(Y, BG) \to Map(Y, BG)$ and $\pi : X \times Map(X, BG) \to Map(X, BG)$ respectively, where G is a finite group. The first of them is the following functor of Picard groupoids

$$\pi_* : H^n(Y \times Map(Y, BG); \mathcal{L}) \to H^0(Map(Y, BG); \mathcal{L}).$$

The existence of the second maps was claimed by J. Lurie and G. Heuts in [HL14] but it was only the categorification of coefficients which allowed us to construct the following functor

$$\pi_* : H^n(X \times Map(X, BG); \mathcal{L}) \to H^0(Map(X, BG); \mathcal{L}^I),$$

where $\mathcal{L}^I$ is the Picard groupoid of arrows of $\mathcal{L}$ with a symmetric monoidal structure is obtained "pointwise" from that of $\mathcal{L}$. Using these two maps we give a direct construction of a TQFT. R. Dijkgraaf and E. Witten in [DW90] constructed a gauge theory with a finite gauge group G as a "toy model," a tool for studying more general gauge theories with compact gauge groups. Their goal was to describe this theory, known as *Dijkgraaf-Witten theory*, as a Topological Quantum Field Theory (TQFT), *i.e.*, a functor on the category of 3-dimensional (3d) cobordisms to that of vector spaces, starting with an action given by a cocycle $\alpha \in Z^3(G; U(1))$. The primary goal of our work is to present a construction of the Dijkgraaf-Witten theory as a TQFT by categorifying the coefficient group $U(1)$. Starting from a cocycle $\alpha \in ObH^n(BG; \mathcal{L})$ we construct a TQFT functor

$$Z : \mathbf{Cob}(n+1) \to \mathbf{Vect},$$

where the domain is the category of $(n+1)$d cobordisms and the codomain is the category of complex vector spaces. Consider the following diagram

$$\begin{array}{ccc} Y \times Map(Y, BG) & \xrightarrow{ev} & BG \\ \pi \downarrow & & \\ Map(Y, BG). & & \end{array}$$

Passing to cohomology, we get the following composite diagram

$$H^n(BG; \mathcal{L}) \xrightarrow{ev^*} H^n(Y \times Map(Y, BG); \mathcal{L}) \xrightarrow{\pi_*} H^0(Map(Y, BG); \mathcal{L}).$$

We observe that an object of the Picard groupoid $H^0(Map(Y, BG); \mathcal{L})$ is equivalent to a *local system* over $Map(Y, BG)$ with values in $\mathcal{L}$ which is a functor from the first fundamental groupoid of the space $Map(Y, BG)$ to the Picard groupoid $\mathcal{L}$. We define

$$\pi_* ev^* \alpha := \mathcal{L}_Y : \Pi_1(Map(Y, BG)) \to \mathcal{L}$$

For any $Y \in Ob\ \mathbf{Cob}(n+1)$, we define the value $Z(Y)$ of the *TQFT functor* to be the space of global sections of the local system $\mathcal{L}_Y$ over $Map(Y, BG)$ constructed above:

$$Z(Y) := H^0(Map(Y, BG); \mathcal{L}_Y) := \lim \mathcal{L}_Y \in \mathbf{Vect},$$

where the limit is taken for a natural extension $\Pi_1 Map(Y, BG) \xrightarrow{\mathcal{L}_Y} \mathcal{L} \to \mathbf{Vect}$ of the functor $\mathcal{L}_Y$, denoted by the same symbol. The limit exists, because the category **Vect** is complete.

Now consider the following diagram

$$\begin{array}{ccc} X \times Map(X, BG) & \xrightarrow{ev} & BG \\ {\scriptstyle\pi}\big\downarrow & & \\ Map(X, BG). & & \end{array}$$

Passing to cohomology, we get the following composite diagram

$$H^n(BG; \mathcal{L}) \xrightarrow{ev^*} H^n(X \times Map(X, BG); \mathcal{L}) \xrightarrow{\pi_*} H^0(Map(X, BG); \mathcal{L}^I).$$

We define

$$\pi_* ev^* \alpha := \mathcal{L}_X : \Pi_1(Map(X, BG)) \to \mathcal{L}^I.$$

There are two obvious functors $s, t : \mathcal{L}^I \to \mathcal{L}$. We observe that $s \circ \mathcal{L}_X = p_-^* \mathcal{L}_{\partial_- X}$ and $t \circ \mathcal{L}_X = p_+^* \mathcal{L}_{\partial_+ X}$. This observation means that the local system $\mathcal{L}_X$ is just a natural transformation $\mathcal{L}_X : p_-^* \mathcal{L}_{\partial_- X} \Rightarrow p_-^* \mathcal{L}_{\partial_+ X}$. Now I describe the construction of a linear map

$$Z(X) : Z(\partial_- X) \to Z(\partial_+ X)$$

or

$$Z(X) : H^0(Map(\partial_- X, BG); \mathcal{L}_{\partial_- X}) \to H^0(Map(\partial_+ X, BG); \mathcal{L}_{\partial_+ X}).$$

We construct a push-pull mechanism along the diagram of spaces:

$$Map(\partial_- X, BG) \xleftarrow{p_-} Map(X, BG) \xrightarrow{p_+} Map(\partial_+ X, BG).$$

The pullback

$$p_-^* : H^0(Map(\partial_- X, BG); \mathcal{L}_{\partial_- X}) \to H^0(Map(X, BG); p_-^* \mathcal{L}_{\partial_- X})$$

is easy. So is an intermediate map:

$$H^0(\mathcal{L}_X) : H^0(Map(X, BG); p_-^*\mathcal{L}_{\partial_- X}) \to H^0(Map(X, BG); p_+^*\mathcal{L}_{\partial_+ X}).$$

The pushforward

$$(p_+)_* : H^0(Map(X, BG); p_+^*\mathcal{L}_{\partial_+ X}) \to H^0(Map(\partial_+ X, BG); \mathcal{L}_{\partial_+ X})$$

is another novelty of this thesis. We construct this a as *transfer*.

Part I

Higher Picard groupoids as a model of a higher additive category

Chapter 2

A review of Γ-spaces

The category of Γ-spaces was introduced by Segal [Seg74], who showed that it has a homotopy category which is equivalent to the homotopy category of connective spectra. Bousfield and Friedlander [BF78] considered a larger category of Γ-spaces in which the ones introduced by Segal appeared as *special* Γ*-spaces*, see definition 2.18. Their category admits a closed model category structure with a notion of *stable weak equivalences*. The homotopy category of this closed model category, obtained by inverting stable weak equivalences is also equivalent to the homotopy category of connective spectra. One advantage of the approach of Bousfield and Friedlander, [BF78], in relating Γ-spaces and spectra is that their closed simplicial model category structure is related by a Quillen-pair of functors to a simplicial model category structure on the category of spectra, see [BF78, Section 5.7]. Lydakis constructs a symmetric monoidal smash product of Γ-spaces in [Lyd99]. The main result of [Lyd99] is that the smash product is compatible with the model category structures of [BF78] and corresponds to the smash product of spectra under the equivalence of homotopy theories of [BF78, Section 5.7]. Then Schwede introduced two (Quillen) closed model category structures on the category of Γ-spaces in [Sch99]. The notion of cofibrations, called *Q-cofibrations*, in these two model category structures is the same and it is a slight weakening of the notion of cofibrations of Bousfield and Friedlander [BF78]. In particular every Q-cofibration is a cofibration in the sense of [BF78]. The two model category structures are called the *strict Q-model* and the *stable Q-model* category structure following the notions of *strict Q-equivalences* and *stable Q-equivalences*. We will briefly describe each of the model category structures mentioned above later in this chapter.

2.1 The notion of Γ-spaces and examples

We start by explaining the original idea of a Γ-space. Any simplicial abelian group A has a classifying space which we denote BA. The classifying space BA is itself a simplicial abelian group with a classifying space B^2A. The sequence $A, BA, B^2A\ldots$ forms an (omega) spectrum. A Γ-space is a generalization of the notion of a simplicial abelian group. Instead of giving a multiplication law on a space A, one specifies for each $n \geq 0$, a space A_n and a homotopy equivalence

$$p_n : A_n \to \prod_{1\leq i\leq n} A_1,$$

and an *n-fold multiplication law* $m_n : A_n \to A_1$. The maps p_n and m_n are required to satisfy certain conditions associated to associativity and commutativity of the multiplication law. A Γ-space for which all maps p_n are isomorphisms is simply a simplicial abelian monoid. A Γ-space has a notion of a classifying space which is again a Γ-space, so it defines a spectrum. This idea was generalized by Bousfield and Friedlander in [BF78], where the requirement of homotopy equivalence on the maps p_n was dropped from the definition of a Γ-space.

For any non-negative integer n, let n^+ denote the pointed set $\{0, 1, \ldots, n\}$ with 0 as the basepoint. The category Γ^{op} is the full subcategory of pointed sets, with objects n^+, for all $n \in \mathbb{Z}^+$. The category Γ^{op} is equivalent to the opposite of the category Γ used by Segal in his definition of Γ-spaces, see [Seg74, Definition 1.1]. It is easy to see that for all $k^+, l^+ \in \Gamma^{op}$, $k^+ \vee l^+ \cong (k+l)^+$. At this stage we would like to mention a few morphisms in Γ^{op} which will be used repeatedly in this text. For all $k^+, l^+ \in \Gamma^{op}$, we have *projection morphisms* $\delta^{k+l}_k : (k+l)^+ \to k^+$ and $\delta^{k+l}_l : (k+l)^+ \to l^+$. Another set of morphisms which will be mentioned repeatedly are called the *multiplication maps*. For each $n^+ \in \Gamma^{op}$, we have a map $m_n : n^+ \to 1^+$, in Γ^{op}, defined by $m_n(i) = 1$, for all non-zero $i \in n^+$.

Definition 2.1. The *category of* Γ*-spaces*, denoted $\Gamma\mathcal{S}$, is the full subcategory of the category of functors from Γ^{op} to the category of pointed simplicial sets $\mathbf{sSets}_\bullet$, with objects all functors F such that $F(0^+) \cong *$. Morphisms in $\Gamma\mathcal{S}$ are natural transformations. An object of the category $\Gamma\mathcal{S}$ will be called a Γ*-space*.

The following example shows how ordinary Picard groupoids fit into our definition of a Γ-space and is thus a bridge between the first and the second part of this thesis:

Example 2.2. Let A be an abelian group. It is a well known fact that the Eilenberg-Maclane complex $K(A, n)$, $n \geq 0$, has as q simplices the *normalized* n-cocycles of the representable simplicial set $\Delta[q] := \mathbf{sSets}(-, [q])$ with coefficients in A, see [EA53]. Let $\mathcal{A}$ be an ordinary Picard groupoid, see definition in section 8.1. For each $n \geq 0$, we will define a (pointed) Kan complex $K(\mathcal{A}, n)$. A q-simplex in $K(\mathcal{A}, n)$ is a pair $(P : \Delta[q]_n \to Ob(\mathcal{A}), g : \Delta[q]_{n+1} \to Mor(\mathcal{A}))$ such that for any $(a_0, a_1, \ldots, a_{n+1}) \in \Delta[q]_{n+1}$, $g\,(a_0, a_1, \ldots, a_n, a_{n+1})$ is a morphism

$$g\,(a_0, \ldots, a_{n+1}) : \sum_{i=0}^{n+1} (-1)^i P\,(d_i\,(a_0, \ldots, a_{n+1})) \to 0$$

in $\mathcal{A}$. Besides (P, g) has to satisfy the following cocycle condition. For any $(a_0, \ldots, a_{n+2}) \in \Delta[q]_{n+2}$

$$\sum_{j=0}^{n+2} (-1)^j \sum_{i=0}^{n+1} (-1)^i d_i d_j\,(a_0, \ldots, a_{n+2}) \overset{\sum_{j=0}^{n+2}(-1)^j g d_j (a_0, \ldots, a_{n+2})}{\longrightarrow} 0.$$

Moreover, since we want *normalized cocycles*, the conditions $Ps_j(a_0, \ldots, a_{n-1}) = 0$ and

$$\sum_{i=0}^{n+1} (-1)^i P d_i s_k(a_0, \ldots, a_n) \overset{gs_k(a_0,\ldots,a_n)}{\longrightarrow} 0,$$

are also satisfied for all $(a_0, \ldots, a_{n-1}) \in \Delta[q]_{n-1}$, $(a_0, \ldots, a_n) \in \Delta[q]_n$, $0 \leq j \leq n-1$ and $0 \leq k \leq n$. The collection of Kan complexes $\{K(\mathcal{A}, n)\}_{n \geq 0}$ forms an Ω-spectrum, see [CMM04, Prop. 3.2], which we also denote by $K(\mathcal{A}, n)$, by abuse of notation. This Ω-spectrum determines a Picard groupoid $\Phi(\mathbb{S}, K(\mathcal{A}, n))$ which is defined in degree n as follows:

$$\Phi(\mathbb{S}, K(\mathcal{A}, n))(n^+) = \mathcal{M}ap_{Sp}(\underbrace{\mathbb{S} \times \cdots \times \mathbb{S}}_{n-copies}, K(\mathcal{A}, n)).$$

2.2 Function objects and smash product

A Γ-space is a functor from the category Γ^{op} into the category of pointed simplicial sets. Further, the category $\Gamma\mathcal{S}$ is a full subcategory of the functor category $\mathbf{sSets}_\bullet^{\Gamma^{op}}$, whose object set consists of all functors with domain category Γ^{op} and codomain category $\mathbf{sSets}_\bullet$. The simplicial enrichment of $\Gamma\mathcal{S}$ is inherited from the simplicial enrichment of $\mathbf{sSets}_\bullet^{\Gamma^{op}}$. The following two definitions put a simplicial enrichment on the category of functors from a small category $\mathcal{C}$ into a simplicial model category $\mathcal{M}$, denoted by $\mathcal{M}^{\mathcal{C}}$. See definition D.3 for a definition of a simplicial model category.

Definition 2.3. Let $F : \mathcal{C} \to \mathcal{M}$ be an object of $\mathcal{M}^{\mathcal{C}}$ and K be a simplicial set. Then

1. A functor $F \otimes K : \mathcal{C} \to \mathcal{M}$ is defined on objects by

   $$(F \otimes K)(c) := F(c) \otimes K,$$

   for every object $c \in \mathcal{C}$, and on morphisms by

   $$(F \otimes K)(f) := F(f) \otimes K,$$

   where f is a morphism in $\mathcal{C}$.

2. A functor $\mathbf{hom}_{\mathcal{M}^{\mathcal{C}}}(K, F) : \mathcal{C} \to \mathcal{M}$ is defined on objects by

$$\mathbf{hom}_{\mathcal{M}^{\mathcal{C}}}(K, F)(c) := \mathbf{hom}_{\mathcal{M}}(K, F(c)),$$

for all objects $c \in \mathcal{C}$ and on morphisms by

$$\mathbf{hom}_{\mathcal{M}^{\mathcal{C}}}(K, F)(f) := \mathbf{hom}_{\mathcal{M}}(K, F(f)),$$

for all morphisms $f : c \to c'$ in $\mathcal{M}$.

Definition 2.4. Let $F' : \mathcal{C} \to \mathcal{M}$ and $F : \mathcal{C} \to \mathcal{M}$ be two objects of $\mathcal{M}^{\mathcal{C}}$, then $Map_{\mathcal{M}^{\mathcal{C}}}(F', F)$ is defined to be the simplicial set whose set of n-simplices is the set of all maps $F' \otimes \Delta[n] \to F$ and whose face and degeneracy maps are induced by those between the representable simplicial sets $\Delta[n]$.

There are three different kinds of function objects for any pair of Γ-spaces, $(F', F) \in Ob(\Gamma\mathcal{S}) \times Ob(\Gamma\mathcal{S})$. The first one is the pointed set of morphisms of Γ-spaces (natural transformations), $\Gamma\mathcal{S}(F', F)$. The second is the simplicial set $Map_{\Gamma\mathcal{S}}(F', F)$ which provides the simplicial enrichment to the model category $\Gamma\mathcal{S}$ and is inherited from the simplicial enrichment of the functor category $\mathbf{sSets}_{\bullet}^{\Gamma^{op}}$, as described in definition 2.4. $Map_{\Gamma\mathcal{S}}(F', F)(n^+)$ is the pointed set $\Gamma\mathcal{S}(F' \wedge (\Delta[q])^+, F)$ as its pointed set of q-simplices. The pointed simplicial set $(\Delta[q])^+$ above is the simplicial set $\Delta[q]$ given an external basepoint and $F' \wedge (\Delta[q])^+$ is the Γ-space which takes n^+ to the pointed simplicial set $F'(n^+) \wedge (\Delta[q])^+$. Finally, Lydakis defines a third function object, in [Lyd99, definition 2.1], which is an internal hom Γ-space, denoted $Hom(F', F)$, and is defined in degree m as follows:

$$Hom(F', F)(m^+) := Map_{\Gamma\mathcal{S}}(F', F(m^+ \wedge -)),$$

where the Γ-space $F(m^+ \wedge -)$ takes n^+ to $F(m^+ \wedge n^+)$. One can check that the definition of $Hom(F', F)$ is functorial in both variables i.e. there is a bi-functor

$$Hom : (\Gamma\mathcal{S})^{op} \times \Gamma\mathcal{S} \to \Gamma\mathcal{S} \tag{2.1}$$

whose object function is defined by $(F', F) \mapsto Hom(F', F)$. A pointed simplicial set is a pair (K, a), where K is a simplicial set and $a : 1 \to K$ is a morphism of simplicial sets whose domain is the terminal object in **sSets**. We will supress the mention of the basepoint a as long as this does not lead to any confusion. The simplicial model category structure on the stable Q-model category specifies a tensor product with simplicial sets and a mapping space consisting of maps from a simplicial set to a Γ-space. Both these notions are bi-functorial and we describe them next. We first describe the tensor product bi-functor:

$$- \wedge - : \Gamma\mathcal{S} \times \mathbf{sSets}_{\bullet} \to \Gamma\mathcal{S}$$

For a Γ-space F and a (pointed) simplicial set K, the Γ-space $F \wedge K$ is defined, for each $k^+ \in \Gamma^{op}$, as follows:

$$(F \wedge K)(k^+) := F(k^+) \wedge K.$$

We construct another bi-functor

$$\mathbf{hom}_{\Gamma\mathcal{S}}(-,-) : (\mathbf{sSets}_{\bullet})^{op} \times \Gamma\mathcal{S} \to \Gamma\mathcal{S},$$

For a pointed simplicial set K and any Γ-space F, we define a Γ-space $\mathbf{hom}(K, F)$ as follows:

$$\mathbf{hom}_{\Gamma\mathcal{S}}(K, F)(k^+) := (F(k^+))^K_{\bullet}$$

for all $k^+ \in \Gamma^{op}$ and where the Kan complex $(F(k^+))^K_{\bullet}$ is the *simplicial set of basepoint preserving maps* from the pointed simplicial set $e : 1 \to K$ to $a : 1 \to F(k^+)$ which is defined by the following equilizer diagram

$$\begin{tikzcd}
(F(k^+))^K_{\bullet} \arrow[r] & (F(k^+))^K \arrow[rr, "(F(k^+))^e"] \arrow[dr] & & (F(k^+))^1 \\
& & 1 \arrow[ur, "a"'] &
\end{tikzcd}$$

The simplicial set $(F(k^+))_\bullet^K$ is pointed by considering the map $0 : K \to F(k^+)$, which maps every vertex of K to the basepoint of $F(k^+)$, as the basepoint. The structure morphisms of the Γ-space $\mathbf{hom}_{\Gamma\mathcal{S}}(K, F)$ are obtained by composition with those of F.

Definition 2.5. A $\Gamma \times \Gamma$ space is a functor from the product category $\Gamma^{op} \times \Gamma^{op}$ to the category $\mathbf{sSets}_\bullet$ which takes the object $(0^+, 0^+) \in Ob(\Gamma \times \Gamma)$ to the terminal pointed simplicial set. The category of all $\Gamma \times \Gamma$-spaces and natural transformations between them will be denoted $\Gamma\Gamma\mathcal{S}$.

Definition 2.6. The external smash product of two Γ-spaces, F' and F is the $\Gamma \times \Gamma$-space, denoted $F' \widetilde{\wedge} F$, which takes the object (m^+, n^+) to $F(m^+) \wedge F(n^+)$.

Every Γ-space F determines a $\Gamma \times \Gamma$-space which we denote by RF and which is defined by

$$RF(m^+, n^+) := F(m^+ \wedge n^+).$$

This construction is functorial.

Definition 2.7. The functor

$$R : \Gamma\mathcal{S} \to \Gamma\Gamma\mathcal{S}$$

defined above will be called the *smash product inclusion* functor.

2.3 Representable Γ-spaces

for every $n \geq 0$ we have a *representable* Γ-space which we denote by Γ^n. This Γ-space is defined as follows:

$$\Gamma^n(m^+) := \Gamma^{op}(n^+, m^+),$$

for all $m^+ \in \Gamma^{op}$. We would like to review some properties of representable Γ-spaces which would be repeatedly used in the later Chapters of this thesis.

Lemma 2.8. *([Lyd99, 2.6]) For all $n \geq 0$ and $F \in Ob(\Gamma\mathcal{S})$, the (pointed) simplicial set $\mathcal{M}ap_{\Gamma\mathcal{S}}(\Gamma^n, F)$ is isomorphic to $F(n^+)$.*

The inclusion of the category Γ^{op} into the category of pointed simplicial sets determines a Γ-space, namely

$$\mathbb{S} : \Gamma^{op} \hookrightarrow \mathbf{sSets_\bullet}.$$

This Γ-space represents the *Sphere spectrum* in the world of Γ-spaces. The Γ-space Γ^1 is isomorphic to $\mathbb{S}$.

Lemma 2.9. *For all $n, m \geq 0$ and $F \in Ob(\Gamma\mathcal{S})$, the map*

$$\Gamma^m \vee \Gamma^n \to \Gamma^{m+n},$$

which is induced by the two projection maps δ^{m+n}_m and δ^{m+n}_n in Γ^{op}, see (A.1)*, is a stable Q-equivalence.*

We will leave the proof of this lemma as an exercise for the interested reader.

2.4 Relation with Spectra

In this section we want to review the relation between spectra and Γ-spaces. In particular, we want to recall a pair of functors between the category of Spectra **Sp** and the category $\Gamma\mathcal{S}$. We will further explain how these functors form Quillen pairs on both the *stable* and *strict* model category structures on **Sp** and $\Gamma\mathcal{S}$. These four model category structures were constructed in [BF78] and are reviewed below. We begin by recalling the definition of a spectrum in the sense of [BF78]

Definition 2.10. ([BF78, Definition 2.1]) A spectrum X consists of a sequence of pointed simplicial sets X^n, for all $n \geq 0$, together with maps $\sigma^n : S^1 \wedge X^n \to X^{n+1}$, where $S^1 = \Delta[1]/\partial\Delta[1] \in \mathbf{sSets_\bullet}$.

A map of spectra consists of a sequence of simplicial maps $f^n : X^n \to Y^n$, which strictly commuting with the suspension maps i.e. for all $n \in \mathbb{Z}^+$, $\sigma^n(1 \wedge f^n) = f^{n+1}\sigma^n$.

Definition 2.11. A map of spectra $f : X \to Y$ will be called

1. A *strict weak equivalence* of spectra if $f^n : X^n \to Y^n$ is a weak homotopy equivalence for all $n \geq 0$.

2. A *strict fibration* of spectra if $f^n : X^n \to Y^n$ is a Kan fibration for all $n \geq 0$.

3. A *strict cofibration* of spectra if the simplicial map $f^0 : X^0 \to Y^0$ and the following induced maps
$$X^{n+1} \coprod_{S^1 \wedge X^n} S^1 \wedge Y^n \to Y^{n+1}$$
are cofibrations of simplicial sets for all $n \geq 0$.

Proposition 2.12. *([BF78, Prop 2.2]) Strict weak equivalences, fibrations and cofibrations of spectra induce a proper simplicial model category structure on* **Sp***.*

Notation 2.13. The model category described in proposition 2.12 will be denoted by $\mathbf{Sp}^{strict}$

Definition 2.14. A map of spectra $f : X \to Y$ will be called

1. A *stable weak equivalence* of spectra if $f_\bullet : \pi_\bullet X \to \pi_\bullet Y$ is an is a group isomorphism and where
$$\pi_\bullet X := colim_k \pi_{\bullet+k} X^k.$$

2. A *stable cofibration* of spectra if it is a strict cofibration of spectra.

3. A *stable fibration* of spectra if it has the right lifting property with respect to every acyclic fibrations of spectra.

Proposition 2.15. *([BF78, Theorem 2.3]) Stable weak equivalences, fibrations and cofibrations of spectra induce a proper simplicial model category structure on* $\mathbf{Sp}$.

Notation 2.16. The model category described in proposition 2.15 will be denoted by $\mathbf{Sp}^{stable}$

Next we will describe a pair of adjoint functors which encode the transition from Γ-spaces to spectra and vice-versa. We first describe the right adjoint functor

$$\Phi(\mathbb{S}, -) : \mathbf{Sp} \to \Gamma\mathcal{S}$$

To any spectrum X, the above functor associates a Γ-space which is defined, in degree n, as follows:

$$\Phi(\mathbb{S}, F)(n^+) := \mathcal{M}ap_{\mathbf{Sp}}(\underbrace{\mathbb{S} \times \cdots \times \mathbb{S}}_{n-copies}, X).$$

In order to define a left adjoint to $\Phi(\mathbb{S}, -)$, we need to understand that a Γ-space F can be *prolonged* to a functor from $\mathbf{sSets}_\bullet$ to $\mathbf{sSets}_\bullet$. This prolonged functor is denoted by

$$\underset{\sim}{F} : \mathbf{sSets}_\bullet \to \mathbf{sSets}_\bullet$$

and can be described as a *coend* construction, see [Mac71, IX.6]. If K is a pointed simplicial set, then we define

$$\underset{\sim}{F}(K) := \int^{n^+ \in \Gamma^{op}} K^n \wedge F(n^+).$$

The prolonged functor preserves weak equivalences of simplicial sets [BF78, Proposition 4.9] and comes with a coherent natural map $K \wedge \underset{\sim}{F}(L) \to \underset{\sim}{F}(K \wedge L)$ which makes it a simplicial functor. This prolonged functor defines a spectrum whose n-th term is the (pointed) simplicial set $\underset{\sim}{F}(S^n)$, where $S^n = \underbrace{S^1 \wedge \cdots \wedge S^1}_{n-copies}$. We denote this spectrum by $\underset{\sim}{F}\mathbb{S}$. This construction defines a functor

$$\underset{\sim}{-}\mathbb{S} : \Gamma\mathcal{S} \to \mathbf{Sp},$$

which is a left adjoint to $\Phi(\mathbb{S}, -)$.

A pair of model category structures, called the *strict* and *stable* model category structures, was constructed on the category $\Gamma\mathcal{S}$ in [BF78]. These model categories are Quillen equivalent to the strict Q-model category and the stable Q-model category respectively. We will not review these model categories here but the interested reader can see [BF78, Theorem 3.5] and [BF78, Theorem 5.2]. The pair of functors $\left(\underset{\sim}{-}\mathbb{S}, \Phi(\mathbb{S}, -)\right)$ forms a Quillen pair between the stable model category (in the sense of [BF78]) and the model category $\mathbf{Sp}^{stable}$.

2.5 The Q-model category structures

There are multiple model category structures on the category of Γ-spaces $\Gamma\mathcal{S}$. The most important of these model category structures is the *stable Q-model category structure* described in [Sch99], which is a *localization* of the *strict Q-model category structure* also described in [Sch99]. In this section we review both of these model category structures. Both of these model category structures are derived from a model category structure of the functor category $\mathbf{sSets}_\bullet^{\Gamma^{op}}$ which we recall before we begin our review. This model category structure is called the *projective model category structure* on $\mathcal{M}^{\mathcal{C}}$.

Theorem 2.17. *([Hir02, Theorem 11.6.1]) Let $\mathcal{C}$ be a small category and $\mathcal{M}$ be a cofibrantly generated model category with generating cofibrations $\mathcal{I}$ and generating trivial fibrations $\mathcal{J}$, then the functor category $\mathcal{M}^{\mathcal{C}}$ inherits a cofibrantly generated model category structure. In this structure, am map $f : F' \to F$ is*

1. *a weak equivalence if $f(c) : F'(c) \to F(c)$ is a weak equivalence in $\mathcal{M}$ for every object $c \in \mathcal{M}$,*

2. *A fibration if $f(c) : F'(c) \to F(c)$ is a fibration in $\mathcal{M}$ for every object $c \in \mathcal{M}$, and*

3. *A cofibration is a natural transformation which has the left lifting property with respect to all natural transformations which are both weak equivalences and fibrations.*

Further, the model category $\mathcal{M}^{\mathcal{C}}$ is simplicial if the model category $\mathcal{M}$ is simplicial.

The homotopy groups of a Γ-space X are those of the associated spectrum

$$\pi_n X := colim_i \pi_{n+i}|X(S^i)|.$$

These groups are always trivial in negative dimensions. Before we recall the stable Q-model category structure, we recall a couple of definitions

Definition 2.18. A Γ-space F is called *special* if the morphism of (pointed) simplicial sets $F((k+l)^+) \to F(k^+) \times F(l^+)$ induced by the projections $(k+l)^+ \cong k^+ \vee l^+ \to k^+$ and $(k+l)^+ \cong k^+ \vee l^+ \to l^+$ is a weak equivalence for all k and l.

For a special Γ-space F, the following zig-zag of maps of simplicial sets

$$F(1^+) \times F(1^+) \xleftarrow{\sim} F(2^+) \to F(1^+)$$

induces an abelian monoid structure on $\pi_0(F(1^+))$. The map on the right is induced by the map $m : 2^+ \to 1^+$, in Γ^{op} which is defined by $m(1) = 1 = m(2)$.

Definition 2.19. A Γ-space F is called *very special* if it is special and the monoid $\pi_0(F(1^+))$ is a group.

2.5.1 The strict Q-model category structure

A map of Γ-spaces is called a *strict Q-fibration* (resp. *strict Q-equivalence*) if it is a Kan fibration (resp. weak equivalence) of simplicial sets when evaluated at every object of the category Γ. *Strict Q-cofibrations* are maps of Γ-spaces that have a left lifing property with respect to all strict acyclic Q-fibrations. The following *strict model category structure* is due to [Qui67].

Lemma 2.20. *The strict Q-equivalences, strict Q-fibrations and strict Q-cofibrations make the category of* Γ*-space into a closed simplicial model category.*

Now we describe the Q-stable model category structure on $\Gamma\mathcal{S}$.

2.5.2 The stable Q-model category structure

A map of Γ-spaces is called a *stable Q-equivalence* if it induces isomorphisms on all homotopy groups. The stable Q-model category structure is obtained by localizing the strict model category structure on the category of Γ-spaces with respect to stable equivalences. The *stable Q-cofibrations* are exactly the strict Q-cofibrations and the *stable Q-fibrations* are maps of Γ-spaces. which have right lifting property with respect to all stable acyclic cofibrations.

Definition 2.21. A Γ-space F will be called countable if the disjoint union of all simplices in all the simplicial sets $F(n^+)$, $n^+ \in \Gamma^{op}$, is a countable set.

Countable Γ-spaces characterize stable Q-fibrations as described by the following lemma:

Lemma 2.22. *([Sch99, Lemma A5]) Let* $f : F \to G$ *be a morphism of* Γ*-spaces which has the right lifting property with respect to all stable acyclic Q-cofibrations between countable* Γ*-spaces. Then the map has the right lifting property with respect to all stable acyclic Q-cofibrations.*

Definition 2.23. A closed model category $\mathcal{C}$, see [Qui67], is *cofibrantly generated* if it is complete and cocomplete and there exist a set of cofibrations $\mathcal{I}$ and a set of acyclic cofibrations $\mathcal{J}$ such that fibrations are precisely the $\mathcal{J}$-injectives; the acyclic fibrations are precisely the $\mathcal{I}$-injectives; the domain of every map in $\mathcal{I}$ and $\mathcal{J}$ is small relative to $\mathcal{I} - cof_{reg}$ and $\mathcal{J} - cof_{reg}$, see B, respectively. The maps in $\mathcal{I}$ and $\mathcal{J}$ will be referred to as *generating cofibrations* and *generating acyclic cofibrations* respectively.

Theorem 2.24. *([Sch99, Theorem 1.5]) The stable Q-fibrations, stable Q-cofibarions and stable Q-equivalences provide the category of Γ-spaces, $\Gamma\mathcal{S}$, with a cofibrantly generated closed simplicial model category. A Γ-space F is fibrant in this model category structure if and only if it is very special and $F(n^+)$ is fibrant simplicial set i.e. a Kan complex for all $n^+ \in \Gamma^{op}$. A strict Q-fibration between stably Q-fibrant Γ-spaces is a stable Q-fibration.*

A map in $\Gamma\mathcal{S}$ is a stable acyclic fibration if and only if it is pointwise an acyclic fibration of simplicial sets. This is equivalent to having the right lifting property with respect to the Q-cofibrations

$$\Gamma^n \wedge (\partial\Delta[i])^+ \longrightarrow \Gamma^n \wedge (\Delta[i])^+,$$

for all $n \in \mathbb{Z}^+$ and $i \in \mathbb{Z}^+ \bigcup \{0\}$. These maps thus form the set of generating cofibrations $\mathcal{I}$. As generating acyclic cofibrations, one may choose any set $\mathcal{J}$ of representatives of isomorphism classes of Q-cofibrations between countable Γ-spaces which are also stable Q-equivalences.

Definition 2.25. A *Picard groupoid* is a fibrant object of the Q-stable model category. In other words it is a very special Γ-space in which the simplicial set in any degree is a (pointed) Kan complex.

Chapter 3

The strict JQ-model category structure on Γ-spaces

Schwede introduced two model category structures on Γ-spaces called the *strict Q-model category* structure and the *stable Q-model category* structure in [Sch99]. The strict Q-model category structure is obtained by restricting the projective model category structure, see definition 2.17, on the functor category $\mathbf{sSets}_\bullet^{\Gamma^{op}}$ to the full subcategory of Γ-spaces. In this section we obtain a new model category structures on Γ-spaces which is a variation of Schwede's strict Q-model category structure. Unlike existing model category structures on the category of Γ-spaces which focus on commutative monoidal structures on spaces, the intent of the model category structure proposed here is to study weakly commutative monoidal structures on (quasi) categories. We begin by recalling the notion of a *categorical equivalence* of (pointed) simplicial sets which is essential for defininig weak equivalences of the desired model category structure.

Definition 3.1. A morphism of simplicial sets $f : A \to B$ is called a categorical equivalence if for any quasi-category X, the induced morphism on the homotopy categories of mapping spaces

$$ho(\mathcal{M}ap_{\mathbf{sSets}}(f,X)) : ho(\mathcal{M}ap_{\mathbf{sSets}}(B,X)) \to ho(\mathcal{M}ap_{\mathbf{sSets}}(A,X)),$$

is an equivalence of (ordinary) categories.

Categorical equivalences are weak equivalences of a cofibrantly generated model category structure on simplicial sets called the "Joyal" model category structure and we will denote the "Joyal" model category by $(\mathbf{sSets}, \mathbf{Q})$, see [Joy08b, Theorem 6.12] for the definition of the "Joyal" model category structure.

Definition 3.2. We call a map of Γ-spaces

1. A *strict JQ-fibration* if it is degreewise a *pseudo-fibration* i.e. a fibration of simplicial sets in the Joyal model category structure on simplicial sets.

2. A *strict JQ-equivalence* if it is degreewise a *categorical equivalence* i.e. a weak equivalence of simplicial sets in the Joyal model category structure on simplicial sets, see [Lur09b].

Two immediate consequence of the above definitions are the following: Strict acyclic JQ-fibrations are the same as the strict acyclic Q-fibrations and hence the cofibrations in the JQ-model category structure, namely are the morphisms having left lifting pproperty with respect to acyclic JQ-fibrations are the same as Q-cofibrations.

Theorem 3.3. *Strict JQ-equivalences, strict JQ-fibrations and Q-cofibrations provide the category of* Γ*-spaces with a closed model category structure.*

Proof. The closed model category axiom CM1 is clear from the existing model category structures. The axioms CM2 and CM3 follows from the fact that strict categorical Q-cofibrations are the same as Q-cofibrations and degreewise definition of strict JQ-weak equivalences and strict JQ-fibrations and CM2 in the "Joyal" model category structure on simplicial sets. One half of CM4 follows by definition of strict JQ-fibrations. The second half of CM4, i.e. the left lifting property of cofibrations with respect to strict acyclic JQ-fibrations follows from the following two facts: Firstly, an acyclic fibration of simplicial sets in the Joyal model category structure is the same as a trivial fibration of simplicial sets, i.e. an acyclic fibration in the classical model category structure on simplicial sets. This means that strict acyclic Q-fibrations are the same as the strict acyclic JQ-fibrations. Secondly, the strict Q-model category structure tells us that the class of maps in $\Gamma\mathcal{S}$, having left lifting property with respect to strict acyclic Q-fibrations is exactly that of Q-cofibrations. Now we move to CM5. Since cofibrations and acyclic fibrations of the strict Q-model category structure and the proposed strict JQ-model category structure are the same, therefore the "cofibration/acyclic fibration" part of the factorization axiom CM5 follows from the corresponding factorization in the strict Q-structure. We use the small object argument, see appendix B, to prove the other half of the factorization axiom CM5. Let $\mathcal{J}$ be a set of maps of Γ-spaces which are both Q-cofibrations and also strict JQ-equivalences. The set $\mathcal{J}$ permits the small object argument because the category of Γ-spaces is locally presentable. By the small object argument, every map f in the category of Γ-spaces can be factored as $f = qi$, where i is a regular $\mathcal{J}$-cofibration, see Appendix B, and q is a map having right lifting property with respect to all maps in $\mathcal{J}$. Every map in the set $\mathcal{J}$ is an injective strict JQ-equivalence. We claim that the class of injective strict JQ-equivalences is closed under cobase change and transfinite induction. An injective strict JQ-equivalence is degreewise an an acyclic cofibration of simplicial sets, in the Joyal model category structure. Further a pushout of Γ-spaces is degreewise a pushout of simplicial sets. The above two facts along with [Hir02, Prop. 10.3.4], which states that the class of trivial cofibrations in a model category is closed under pushouts and transfinite compositions, prove that the cobase change of an injective, therefore the map i is a strict JQ-equivalence. The map i is also a Q-cofibration because the class of Q-cofibrations is closed under cobase change and transfinite composition. □

We want to provide a characterization of fibrant objects in the JQ-model category structure but in order to do so, we will need a characterization of fibrant objects in the model category $(\mathbf{sSets}, \mathbf{Q})$.

Proposition 3.4. *The following conditions on a simplicial set X are equivalent*

1. *X is a quasi-category.*

2. *the projection map $p_2 : X^{\Delta[2]} \to X^{\Lambda^1[2]}$, induced by the horn inclusion $h_2^1 : \Lambda^1[2] \to \Delta[2]$. is a trivial fibration.*

The following proposition provides the desired characterization of the fibrant objects in the strict JQ-model category structure:

Proposition 3.5. *A Γ-space F is a fibrant object of the strict JQ-model category structure if and only if the following morphism of Γ-spaces*

$$\mathbf{hom}_{\Gamma\mathcal{S}}(h_2^1, F) : \mathbf{hom}_{\Gamma\mathcal{S}}(\Delta[2], X) \to \mathbf{hom}_{\Gamma\mathcal{S}}(\Lambda^1[2], X),$$

h_2^1 is the (inner) horn inclusion $h_2^1 : \Lambda^1[2] \to \Delta[2]$, is a strict acyclic JQ-fibration.

Proof. For any $k^+ \in Ob(\Gamma^{op})$, the (pointed) simplicial set $F(k^+)$ is a quasicategory if and only if, the simplicial map $\mathbf{hom}_{\Gamma\mathcal{S}}(h_2^1, F)(k^+)$ is a trivial fibration of simplicial sets. □

Definition 3.6. A *strict* Γ*-space quasicategory* is a fibrant object of the strict JQ-model category structure.

Lemma 3.7. *Let F', F and G be a triple of Γ-spaces such that the Γ-space F' is Q-cofibrant and F and G are strict JQ-fibrant Γ-space. Then for any strict acyclic JQ-fibration $p : F \to G$, the induced morphism*

$$Hom(F', p) : Hom(F', F) \to Hom(F', G)$$

is a strict acyclic JQ-fibration.

Proof. By adjointness, it would be sufficient to establish the existence of the dotted arrow whenever we have a (outer) commutative diagram

$$\begin{tikzcd} M \wedge F' \arrow[r] \arrow[d, "i \wedge id"'] & F \arrow[d, "p"] \\ N \wedge F' \arrow[r] \arrow[ur, dotted] & G \end{tikzcd}$$

where $i : M \to N$ is a strict acyclic Q-cofibration. In this case, the *pushout product axiom* of the stable Q-model category structure, [Sch99, Lemma 1.7], implies that the map $i \wedge id : M \wedge F' \to N \wedge F'$ is a Q-cofibration. Now the strict JQ-model category structure, Theorem 3.3, garuntees the existence of the dotted arrow.

□

Corollary 3.8. *Let F', F be a pair of Γ-spaces such that F' is Q-cofibrant and F is a strict JQ-fibrant Γ-space. Then the function object $Hom(F', F)$ is a strict JQ-fibrant Γ-space.*

Proof. If F is a strict JQ-fibrant Γ-space then the following map

$$\mathbf{hom}_{\Gamma\mathcal{S}}(h^1_2, F) : \mathbf{hom}_{\Gamma\mathcal{S}}(\Delta[2], F) \to \mathbf{hom}_{\Gamma\mathcal{S}}(\Lambda^1[2], F),$$

which is induced by the (inner) horn inclusion $h^1_2 : \Lambda^1[2] \to \Delta[2]$, is a strict acyclic JQ-fibrations. Now, according to Lemma 3.7, the induced maps

$$Hom(F', \mathbf{hom}_{\Gamma\mathcal{S}}(h^1_2, F)) : Hom(F', \mathbf{hom}_{\Gamma\mathcal{S}}(\Delta[2], F)) \to Hom(F', \mathbf{hom}_{\Gamma\mathcal{S}}(\Lambda^1[2], F))$$

are strict acyclic JQ-fibration. In light of adjointness discussed in appendix E, we observe that the morphisms

$$\mathbf{hom}_{\Gamma\mathcal{S}}(h^1_2, Hom(F', F)) : \mathbf{hom}_{\Gamma\mathcal{S}}(\Delta[2], Hom(F', F)) \to \mathbf{hom}_{\Gamma\mathcal{S}}(\Lambda^1[2], Hom(F', F))$$

are strict acyclic JQ-fibrations, for all n and i as specified above. Thus we conclude that $Hom(F', F)$ is a strict JQ-fibrant Γ-space.

□

The rest of this section is dedicated to obtaining a characterization of strict JQ-equivalences between fibrant objects. Specifying a pair of vertex $(x, y) \in F(n^+)_0 \times F(n^+)_0$ is equivalent to specifying a morphism $(x, y)^n_F : \Gamma^n \to F \times F$.

Notation 3.9. The morphism of Γ-spaces $(x, y)^n_F : \Gamma^n \to F \times F$ should be viewed as a pair of morphisms $x : \Gamma^n \to F$ and $y : \Gamma^n \to F$. Since the Γ-space Γ^n is discrete, the simplicial maps $x(m^+)$ and $y(n^+)$ are competely determined by two maps of (pointed) sets $x(m^+) : \Gamma^n(m^+) \to F(m^+)_0$ and $y(m^+) : \Gamma^n(m^+) \to F(m^+)_0$. The image of any $s \in \Gamma^n(m^+)$, under the morphism of (pointed) sets $(x, y)^n_F(m^+) : \Gamma^n(m^+) \to F(m^+) \times F(m^+)$ will be denoted by $(x^{n,m}_s, y^{n,m}_s)$.

For every pair $m, n \in \mathbb{Z}^+$, we define a (pointed) simplicial set, which we denote $Hom^n_F(x, y)(m^+)$, as follows:

$$Hom^n_F(x, y)(m^+) := \bigvee_{(x^{n,m}_s, y^{n,m}_s) \in (x,y)^n_F(m^+)} Hom_{F(m^+)}(x^{n,m}_s, y^{n,m}_s),$$

where the simplicial set $Hom_{F(m^+)}(x^{n,m}_s, y^{n,m}_s)$ is defined as the fiber, over the pair of vertices $(x^{n,m}_s, y^{n,m}_s) \in F(m^+)_0 \times F(m^+)_0$, of the fibration of simplicial sets

$$(s, t) : F(m^+)_\bullet^{\Delta[1]^+} \to F(m^+) \times F(m^+),$$

which is induced by the pair simplicial inclusion maps $i_0 : \Delta^0 \hookrightarrow \Delta^1$ and $i_1 : \Delta^0 \hookrightarrow \Delta^1$, see appendix C for more details.

Lemma 3.10. *Let F be a strict JQ-fibrant Γ-space and let n be a fixed nonnegative integer. Then the following collection of simplicial sets $\{Hom_F^n(x,y)(m^+)\}_{m^+ \in \Gamma^{op}}$, defined above, glue together to form a Γ-space which we denote $Hom_F^n(x,y)$.*

Proof. The Γ-space, F, assigns to any morphism, $f_m^k : m^+ \to k^+$, in Γ^{op}, a simplicial map $F(f_m^k)(m^+) : F(m^+) \to F(k^+)$. Similarly, there is an assignment of a map of (pointed) sets, $\Gamma^n(f_m^k)$. Since (x,y) is a pair of morphisms of Γ-spaces, the map of (pointed) sets

$$(x,y)_F^n(f_m^k) : (x,y)_F^n(m^+) \to (x,y)_F^n(m^+),$$

can be described as follows:

$$(x,y)_F^n(f_m^k)(x_s^{n,m}, y_s^{n,m}) = (x_{s'}^{n,k}, y_{s'}^{n,k}),$$

where $s' = \Gamma^n(f_m^k)(s)$. Now, one can check that each map f_m^k defines a simplicial morphism

$$Hom_F^n(x_s^{n,m}, y_s^{n,m})(f_m^k) : Hom_{F(m^+)}(x_s^{n,m}, y_s^{n,m}) \to Hom_{F(k^+)}(x_{s'}^{n,k}, y_{s'}^{n,k}).$$

Therefore for each $(x_s^{n,m}, y_s^{n,m}) \in (x,y)_F^n(m^+)$, we get a simplicial map

$$i_{s'} \circ Hom_F^n(x_s^{n,m}, y_s^{n,m})(f_m^k) : Hom_{F(m^+)}(x_s^{n,m}, y_s^{n,m}) \to Hom_F(x,y)(k^+),$$

where $i_{s'}$ is the following inclusion map:

$$i_{s'} : Hom_{F(k^+)}(x_{s'}^{n,k}, y_{s'}^{n,k}) \hookrightarrow \bigvee_{(x_s^{n,k}, y_s^{n,k}) \in (x,y)_F^n(k^+)} Hom_{F(k^+)}(x_s^{n,k}, y_s^{n,k}).$$

Thus we get a map from the coproduct of all domain spaces to $Hom_F(x,y)(k^+)$ which is the desired simplicial map

$$Hom_F^n(x,y)(f_m^k) : Hom_F(x,y)(m^+) \to Hom_F(x,y)(k^+).$$

□

Proposition 3.11. *A morphism between two strict JQ-fibrant* Γ*-spaces,* $f : F \to G$*, is a strict JQ-equivalence if and only if for each* $(x, y) \in F(n^+)_0 \times F(n^+)_0$ *and all* $n \in \mathbb{Z}^+$*, the induced morphism on the mapping* Γ*-spaces,*

$$f^n(x, y) : Hom^n_F(x, y) \to Hom^n_G(f(x), f(y))$$

is a strict Q-equivalence and the induced maps on the homotopy categories

$$Ho(f(n^+)) : Ho(F(n^+)) \to Ho(G(n^+))$$

are equivalences of (ordinary) categories.

Proof. If f is a strict JQ-equivalence between fibrant objects, then for all $n \in \mathbb{Z}^+$ and all pairs $(x, y) \in F(n^+)_0 \times F(n^+)_0$, there is an equivalence of Kan complexes

$$f(n^+)_{x,y} : Hom_{F(n^+)}(x, y) \to Hom_{G(n^+)}(f(x), f(y)).$$

Since coproducts in the category of (pointed) simplicial sets are homotopy copruducts, the above implies that

$$f^n(x, y)(n^+) : Hom^n_F(x, y)(n^+) \to Hom^n_G(f(x), f(y))(n^+)$$

is an equivalence of simplicial sets. The second condition about the equivalence of homotopy categories is obvious. Converseley, if, for all $n \in \mathbb{Z}^+$ $f^n(x, y)(n^+)$ is a weak equivalence then for each $s \in \Gamma^n(n^+)$, the inclusion maps

$$i_s : Hom_{F(n^+)}(x^{n,n}_s, y^{n,n}_s) \hookrightarrow Hom^n_F(x, y)(n^+)$$

and

$$i_s : Hom_{G(n^+)}(f(x^{n,n}_s), f(y^{n,n}_s)) \hookrightarrow Hom^n_G(f(x), f(y))(n^+)$$

induce, for any Kan complex X, a weak equivalences

$$\prod_{s \in \Gamma^n n^+} X^{Hom_{G(n^+)}(f(x^{n,n}_s), f(y^{n,n}_s))} \to \prod_{s \in \Gamma^n n^+} X^{Hom_{F(n^+)}(x^{n,n}_s, y^{n,n}_s)}.$$

Since the geometric realization functor and the homotopy groups preserve finite products, each component map

$$X^{Hom_{G(n^+)}(f(x_s^{n,n}),f(y_s^{n,n}))} \to X^{Hom_{F(n^+)}(x_s^{n,n},y_s^{n,n})}$$

is a weak equivalence. Now the simplicial model category structure on the category of simplicial sets implies that the morphism

$$Hom_{F(n^+)}(x_s^{n,n}, y_s^{n,n}) \to Hom_{G(n^+)}(f(x_s^{n,n}), f(y_s^{n,n}))$$

is a weak equivalence of simplicial sets.

□

3.1 Enrichment of the JQ-model category

In this subsection, we describe an enrichment of the JQ-model category. We begin by reviewing the notion of enrichment of a model category over a *monoidal model category*.

Definition 3.12. A *monoidal model category* is a closed monoidal category $\mathcal{C}$ with a model category structure, such that $\mathcal{C}$ satisfies the following conditions:

1. The monoidal structure $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a Quillen bifunctor.

2. Let $QS \overset{q}{\to} S$ be the cofibrant replacement for the unit object S, obtained by using the functorial factorization system to factorize $0 \to S$ into a cofibration followed by a trivial fibration. Then the natural map

$$QS \otimes X \overset{q \otimes 1}{\to} S \otimes X$$

is a weak equivalence for all cofibrant X. Similarly, the natural map $X \otimes QS \overset{1 \otimes q}{\to} X \otimes S$ is a weak equivalence for all cofibrant X.

Example 3.13. The model category of simplicial sets with the Joyal model category structure, $(\mathbf{sSets}, \mathbf{Q})$ is a monoidal model category.

Definition 3.14. Let $\mathbf{S}$ be a monoidal model category. An $\mathbf{S}$-*enriched model category* is an $\mathbf{S}$ enriched category $\mathbf{A}$ equipped with a model category structure (on its underlying category) such that there is a Quillen adjunction of two variables, see definition E.2, $(\otimes, \mathbf{hom}_{\mathbf{A}}, \mathcal{M}ap_{\mathbf{A}}, \phi, \psi) : \mathbf{A} \times \mathbf{S} \to \mathbf{A}$.

Both strict and stable Q-model category structures on the category $\Gamma\mathcal{S}$ are simplicial, see definition D.3. The strict JQ-model category structure is NOT simplicial. The following proposition tells us that the strict JQ-model category is an $(\mathbf{sSets}, \mathbf{Q})$-enriched model category in the sense of definition 3.14.

Proposition 3.15. *The category $\Gamma\mathcal{S}$ endowded with the strict JQ-model category is an $(\mathbf{sSets}, \mathbf{Q})$-enriched model category.*

Proof. The simplicial stable Q-model category structure provides a two variable adjunction

$$(\wedge, \mathbf{hom}_{\Gamma\mathcal{S}}, \mathcal{M}ap_{\Gamma\mathcal{S}}, \phi, \psi) : \Gamma\mathcal{S} \times \mathbf{sSets} \to \Gamma\mathcal{S}.$$

We just need to verify that this adjunction of two variables is a Quillen adjunction for the relavent model category structures on $\Gamma\mathcal{S}$ and $\mathbf{sSets}$. In order to do so, we will verify condition (2) of Lemma E.3. Let $g : W \to X$ be a cofibration in $(\mathbf{sSets}, \mathbf{Q})$ and let $p : Y \to Z$ be a strict JQ-fibration, then we have to verify that the induced map

$$\mathbf{hom}^{\square}_{\Gamma\mathcal{S}}(g, p) : \mathbf{hom}_{\Gamma\mathcal{S}}(X, Y) \to \mathbf{hom}_{\Gamma\mathcal{S}}(X, Z) \underset{\mathbf{hom}_{\Gamma\mathcal{S}}(W,Z)}{\times} \mathbf{hom}_{\Gamma\mathcal{S}}(W, Y)$$

is a fibration in $(\mathbf{sSets}, \mathbf{Q})$ which is acyclic if either of g or p is acyclic. It would be sufficiant to check that the above morphism is degreewise a fibration in $(\mathbf{sSets}, \mathbf{Q})$, i.e. for all $n^+ \in \Gamma^{op}$, the morphism

$$\mathbf{hom}^{\square}_{\Gamma\mathcal{S}}(g, p)(n^+) : \mathbf{hom}_{\Gamma\mathcal{S}}(X, Y)(n^+) \to \mathbf{hom}(X, Z)_{\Gamma\mathcal{S}}(n^+) \underset{\mathbf{hom}_{\Gamma\mathcal{S}}(W,Z)(n^+)}{\times} \mathbf{hom}_{\Gamma\mathcal{S}}(W, Y)(n^+),$$

is a fibration in $(\mathbf{sSets}, \mathbf{Q})$. This follows from the definition of the functor $\mathbf{hom}(-,-)$ and the monoidal model category structure on $(\mathbf{sSets}, \mathbf{Q})$.

$\square$

Chapter 4

The higher category of Picard groupoids

In this chapter we construct a higher category of Picard groupoids. We do so by first constructing a *derived category of Picard groupoids*, which is a (fibrant) simplicial category and then the desired higher category is obtained by applying the simplicial nerve functor. Before we begin this construction, we want to describe some properties of Picard groupoids which would be useful in the construction and also in subsequent chapters.

Proposition 4.1. *If F is a fibrant object of the model category $\Gamma\mathcal{S}$, then the Γ-space $F(n^+ \wedge -)$ is also fibrant.*

Proof. $F(n^+ \wedge -)(1^+) = F(n^+)$ and since F is fibrant, the simplicial set $F(n^+)$ is equivalent to $\prod\limits_1^n F(1^+)$ therefore $\pi_0(F(n^+ \wedge -)(1^+)) \cong \pi_0(\prod\limits_1^n F(1^+))$ which is a group. Notice the isomorphisms $(n^+ \wedge (k+l)^+) \cong \bigvee\limits_1^n (k+l)^+ \cong (\bigvee\limits_1^n k^+) \vee (\bigvee\limits_1^n l^+) \cong ((\bigvee\limits_1^n k^+) + (\bigvee\limits_1^n l^+))$. The two projection maps $(k+l)^+ \to k^+$ and $(k+l)^+ \to l^+$ induce an equivalence of pointed simplicial sets $F((\bigvee\limits_1^n k^+) + (\bigvee\limits_1^n l^+)) \to F(\bigvee\limits_1^n k^+) \times F(\bigvee\limits_1^n l^+)$. Composing with the isomorphisms above, we get the following equivalence of pointed simplicial sets $F(n^+ \wedge -)((k+l)^+) \to F(n^+ \wedge -)(k^+) \times F(n^+ \wedge -)(l^+)$. □

Proposition 4.2. *Let F be a stable Q-fibrant Γ-space, then the Γ-space $F(n^+ \wedge -)$ is stable Q-equivalent to $\prod\limits_1^n F$.*

Proof. We begin by observing that the symmetric monoidal structure defined in [Lyd99] is based on a choice of a bi-functor $- \wedge - : \Gamma^{op} \times \Gamma^{op} \to \Gamma^{op}$. For $n^+ \in Ob\Gamma^{op}$, there are n maps $p_i : n^+ \to 1^+$ defined by $p_i(j) = 1$, if $i = j$ and $p_i(j) = 0$ otherwise. Each p_i determines a natural transformation $p_i \wedge - : n^+ \wedge - \stackrel{\cdot}{\to} 1^+ \wedge -$. Composition of each $p_i \wedge -$ with F determines the following morphism of Γ-spaces

$$F(p_i \wedge -) : F(n^+ \wedge -) \to F.$$

These n morphisms determine the following morphism of Γ-spaces

$$(F(p_1 \wedge -), \dots, F(p_n \wedge -)) : F(n^+ \wedge -) \to \prod_1^n F.$$

In degree one, this morphism specifies a morphism of (pointed) simplicial sets

$$(F(p_1), \dots, F(p_n)) : F(n^+) \to \prod_1^n F(1^+).$$

Since F is a fibrant Γ-space, the above map is a weak equivalence of Kan complexes. Further, both $F(n^+ \wedge -)$ and $\prod\limits_1^n F$ are fibrant Γ-spaces, therefore a morphism between them which specifies a weak equivalence of their degree one simplicial sets is a stable Q-equivalence. Thus $(F(p_1), \dots, F(p_n)) : F(n^+) \to \prod\limits_1^n F(1^+)$ is a stable Q-equivalence. □

An essential ingredient in the construction of the desired bi-functor is that for any stable Q-fibrant Γ-space F and a Γ-space F', the mapping Γ-space $Hom(QF', F)$ is a stable Q-fibrant Γ-space

Theorem 4.3. *Let F and F' be two objects of the stable Q-model category and F is also fibrant. Then the Γ-space $Hom(QF', F)$, where QF' is the cofibrant replacement of F', is a fibrant object of the stable Q-model category.*

Proof. By definition, $Hom(QF', F)((k+l)^+) = \mathcal{M}ap_{\Gamma\mathcal{S}}(QF', F((k+l)^+ \wedge -))$. The projection maps $\delta_k^{k+l} : (k+l)^+ \to k^+$ and $\delta_l^{k+l} : (k+l)^+ \to l^+$ induce maps of Γ-spaces $F(\delta_k^{k+l} \wedge -) : F((k+l)^+ \wedge -) \to F(k^+ \wedge -))$ and $F(\delta_l^{k+l} \wedge -) : F((k+l)^+ \wedge -) \to F(l^+ \wedge -))$. These two maps induce the following map of Γ-spaces

$$(F(\delta_k^{k+l} \wedge -), F(\delta_l^{k+l} \wedge -)) : F((k+l)^+ \wedge -) \to F(k^+ \wedge -) \times F(l^+ \wedge -).$$

In degree one, the above map specifies the following morphism of pointed simplicial sets

$$(F(\delta_k^{k+l}), F(\delta_l^{k+l})) : F((k+l)^+) \to F(k^+) \times F(l^+).$$

This morphism is an equivalence of (pointed) simplicial sets because, by assumption, F is a stable Q-fibrant Γ-space. Hence the morphism of Γ-spaces $(F(\delta_k^{k+l} \wedge -), F(\delta_l^{k+l} \wedge -))$ is a stable Q equivalence. This implies that

$$\begin{aligned}\mathcal{M}ap_{\Gamma\mathcal{S}}(QF', (F(\delta_k^{k+l} \wedge -), F(\delta_l^{k+l} \wedge -))) : \mathcal{M}ap_{\Gamma\mathcal{S}}(QF', F((k+l)^+ \wedge -)) \\ \to \mathcal{M}ap_{\Gamma\mathcal{S}}(QF', F(k^+ \wedge -) \times F(l^+ \wedge -))\end{aligned}$$

is an equivalence of (pointed) Kan complexes. Composition with the obvious projection maps gives us the following stable Q equivalence

$$\mathcal{M}ap_{\Gamma\mathcal{S}}(QF', F((k+l)^+ \wedge -)) \to \mathcal{M}ap_{\Gamma\mathcal{S}}(QF', F(k^+ \wedge -)) \times \mathcal{M}ap_{\Gamma\mathcal{S}}(QF', F(l^+ \wedge -)).$$

This shows that the projection maps $(k+l)^+ \to k^+$ and $(k+l)^+ \to l^+$ induce the following weak equivalence of (pointed) Kan complexes

$$Hom(QF', F)((k+l)^+) \to Hom(QF', F)(k^+) \times Hom(QF', F)(l^+).$$

It remains to show that $\pi_0(Hom(QF', F)(1^+))$ is a group. In order to do so, it will be sufficient to show that the following diagram is a pullback square in the ∞-category $\mathcal{K}_\bullet$

$$\begin{tikzcd}
Hom(QF', F)(2^+) \arrow[r, "Hom(QF', F)(m)"] \arrow[d, "Hom(QF', F)(p_1)"'] & Hom(QF', F)(1^+) \arrow[d] \\
Hom(QF', F)(1^+) \arrow[r] & 0
\end{tikzcd} \tag{4.1}$$

This follows from the assumption that the Γ-space F is very special. Under this assumption, proposition 6.1 implies that the following diagram is a pullback square in $\mathcal{L}_\infty$

$$\begin{tikzcd}
F(2^+ \wedge -) \arrow[r, "F(m\wedge -)"] \arrow[d, "F(p_1\wedge -)"'] & F \arrow[d] \\
F \arrow[r] & 0
\end{tikzcd}$$

therefore the following diagram is a pullback square in $\mathcal{K}_\bullet$

$$\begin{tikzcd}
\mathcal{M}ap_{\Gamma\mathcal{S}}(QF', F(2^+ \wedge -)) \arrow[r, "\mathcal{M}ap_{\Gamma\mathcal{S}}(QF', F(m\wedge -))"] \arrow[d, "\mathcal{M}ap_{\Gamma\mathcal{S}}(QF', F(p_1\wedge -))"'] & \mathcal{M}ap_{\Gamma\mathcal{S}}(QF', F) \arrow[d] \\
\mathcal{M}ap_{\Gamma\mathcal{S}}(QF', F) \arrow[r] & 0
\end{tikzcd}$$

This shows that the diagram 4.1 is a pullback square in $\mathcal{K}_\bullet$ and this completes the proof the theorem.

$\square$

Let $(\Gamma\mathcal{S})^f$ denote the full (simplicial) subcategory of fibrant objects of the stable Q-model category. The projection from a fibrant Γ-space onto its degree one Kan complex defines a functor of simplicial categories

$$\Omega_\infty : (\Gamma\mathcal{S})^f \to \mathcal{K}_\bullet.$$

We want to show that $(\Gamma\mathcal{S})^f$ is a fibrant simplicial category. In order to do so we prove the following lemma

Lemma 4.4. *Let $F', F \in Ob\Gamma\mathcal{S}$ and let F also be fibrant object of $\Gamma\mathcal{S}$, then the simplicial set $\mathcal{M}ap_{\Gamma\mathcal{S}}(QF', F)$ is a Kan complex.*

Proof. The argument is based on the properties of a simplicial model categories. Since $F \to *$ is a stable Q-fibration and QF' is a Q-cofibrant object, therefore the induced map

$$\mathcal{M}ap_{\Gamma\mathcal{S}}(QF', F) : \mathcal{M}ap_{\Gamma\mathcal{S}}(QF', F) \to \mathcal{M}ap_{\Gamma\mathcal{S}}(QF, *) \cong *,$$

is a Kan fibration. Thus the simplicial set $\mathcal{M}ap_{\Gamma\mathcal{S}}(QF', F)$ is a Kan complex.

□

Remark. The above lemma is valid for any simplicial model category.

Now we define another simplicial category

Definition 4.5. We denote by $\mathbf{D}(\Gamma\mathcal{S}^f)$, the simplicial category whose objects are the same as those of $\Gamma\mathcal{S}^f$ and the simplicial mapping space of Γ-space maps between any two objects, $F', F \in \mathbf{D}(\Gamma\mathcal{S}^f)$, is defined as follows:

$$Map_{\mathbf{D}(\Gamma\mathcal{S}^f)}(F', F) := \mathcal{M}ap_{\Gamma\mathcal{S}}(QF', QF).$$

The simplicial category $\mathbf{D}(\Gamma\mathcal{S}^f)$ will be called the *derived category of Picard groupoids.*

Proposition 4.6. *The derived category of Picard groupoids, $\mathbf{D}(\Gamma\mathcal{S}^f)$, is a fibrant simplicial category.*

Proof. We begin by proving that $\mathbf{D}(\Gamma\mathcal{S}^f)$ is a simplicial category. The underlying (ordinary) category has the same set of objects as $\mathbf{D}(\Gamma\mathcal{S}^f)$ and the Hom set $Hom_{\mathbf{D}(\Gamma\mathcal{S}^f)}(F', F); = Map_{\mathbf{D}(\Gamma\mathcal{S}^f)}(F', F)$. For any triple of objects $F, G, H \in Ob(\mathbf{D}(\Gamma\mathcal{S}^f))$, the composition law is defined as follows:

$$c^{\mathbf{D}(\Gamma\mathcal{S}^f)}_{F,G,H} := c^{\Gamma\mathcal{S}}_{QF,QG,QH}.$$

The identity map id_F is defined as the map id_{QF}. The fibrant part of the proposition statement follows from proposition 4.4 and the fact that in every model category, a cofibrant replacement of a fibrant object is both cofibrant and fibrant. □

We let $\Gamma\mathcal{S}^{cf}$ denote the full (simplicial) subcategory of the stable Q-model category whose objects are both fibrant and cofibrant.

Lemma 4.7. *There is a DK-equivalence*

$$Q^{ob} : \mathbf{D}(\Gamma\mathcal{S}^f) \to \Gamma\mathcal{S}^{cf},$$

which maps an object of $\Gamma\mathcal{S}$ *to a cofibrant replacement in the Q-stable model category.*

Proof. The simplicial functor Q^{ob} is defined on objects by $Q^{ob}(F) := QF$, for all $F \in \mathbf{D}(\Gamma\mathcal{S}^f)$, where Q is the cofibrant replacement functor specified by the chosen functorial factorization system on the stable Q-model category, see appendix B. For any other Picard groupoid G, we define the simplicial map

$$Q^{ob}_{F,G} : \mathcal{M}ap_{\mathbf{D}(\Gamma\mathcal{S}^f)}(F, G) \to \mathcal{M}ap_{\Gamma\mathcal{S}^{cf}}(F, G)$$

to be the identity function on the (pointed) Kan complex $\mathcal{M}ap_{\Gamma\mathcal{S}}(QF, QF)$.

In order to prove that $\pi_0(Q^{ob})$ is an equivalence of (ordinary) categories, it is sufficient to observe that for each object $H \in \Gamma\mathcal{S}^{cf}$, the cofibrant replacement functor Q provides an acyclic fibration $QH \to H$ and every stable Q-equivalence in $\Gamma\mathcal{S}^{cf}$ induces an isomorphism in $\pi_0(\Gamma\mathcal{S}^{cf})$. □

In order to understand the (weak) homotopy type of the derived category of Picard groupoids, we need the following definition:

Definition 4.8. We say that two simplicial categories are *weakly equivalent* if they can be joined by a finite string of weak equivalences, see [DK80c, sec 2.4] or [Ber07].

The following proposition indicates that the derived category of Picard groupoids $\mathbf{D}(\Gamma\mathcal{S}^f)$ is a *simplicial homotopy category* of the model category of Γ-spaces $\Gamma\mathcal{S}$ because it is equivalent to a *hammock localization* of $\Gamma\mathcal{S}$ see [DK80a] and [DK80c].

Proposition 4.9. *The derived category of Picard groupoids $\mathbf{D}(\Gamma\mathcal{S}^f)$ is weakly equivalent, as a simplicial category to the simplicial category $L^H\Gamma\mathcal{S}$.*

An easy consequence of the above proposition is the following corollary:

Corollary 4.10. *The derived category of Picard groupoids $\mathbf{D}(\Gamma\mathcal{S}^f)$ is equivalent, as a simplicial category to the simplicial category $L^H\Gamma\mathcal{S}^f$.*

Definition 4.11. The ∞-category $N(\mathbf{D}(\Gamma\mathcal{S}^f))$, where N is the *coherent nerve* functor, will be called the ∞-category *of Picard groupoids* and will be denoted by $\mathcal{P}ic$.

Remark. The ∞-category $\mathcal{P}ic$ is a pointed quasicategory with the basepoint being the trivial Γ-space.

We define the *homotopy category* of $\mathcal{P}ic$, $ho(\mathcal{P}ic)$, to be the homotopy category of the simplicial category $\mathbf{D}(\Gamma\mathcal{S}^f)$ i.e. $Ob(ho(\mathcal{P}ic)) = Ob(\mathbf{D}(\Gamma\mathcal{S}^f))$ and $Hom_{ho(\mathcal{P}ic)}(F', F) := \pi_0 Hom_{\mathbf{D}(\Gamma\mathcal{S}^f)}(F', F)$. Further, $Hom_{ho(\mathcal{P}ic)}(F', F) = \pi_0 Hom(F', F)(1^+)$ and $Hom(F', F)$ is a Picard groupoid, this implies that the homotopy category $ho(\mathcal{P}ic)$ is enriched over the category of Abelian groups.

For every $n^+ \in Ob(\Gamma^{op})$ and for any pair of objects $(F', F) \in \mathbf{D}(\Gamma\mathcal{S}^f) \times \mathbf{D}(\Gamma\mathcal{S}^f)$, we define the simplicial Hom set as follows:

$$Hom_{\mathbf{D}(\Gamma\mathcal{S}^f)[n^+]}(F', F) := \mathcal{M}ap_{\Gamma\mathcal{S}}(QF', QF(n^+ \wedge -)).$$

Proposition 4.12. *The simplicial sets in the collection $\{Hom_{\mathbf{D}(\Gamma\mathcal{S}^f)[m^+]}(F', F)\}_{m \geq 0}$ glue together to form a Picard groupoid which we denote $\underline{Hom}_{\mathbf{D}(\Gamma\mathcal{S}^f)}(F', F)$.*

Proof. We observe that the simplicial set $Hom_{\mathbf{D}(\Gamma\mathcal{S}^f)[m^+]}(F', F)$ is the same as $Hom(QF', QF)(m^+)$. Thus the simplicial sets in the collections glue into the Γ-space $Hom(QF', QF)$. Now the result follows from proposition 4.3. □

Let us recall the definition of an *equivalence* in an ∞-category

Definition 4.13. A morphism in an ∞-category X is an *equivalence* if it induces an isomorphism in the homotopy category $ho(X)$.

We claim that a morphism in $\mathcal{P}ic$ is an equivalence iff it is a stable Q-equivalence between fibrant Γ-spaces. The following proposition verifies this claim

Proposition 4.14. *A morphism $f \in (\mathcal{P}ic)_1$ is an equivalence in $\mathcal{P}ic$ iff it is a stable Q-equivalence between fibrant Γ-spaces $d_1(f)$ and $d_0(f)$.*

4.1 Strictification of Picard groupoids

The rest of this chapter is devoted to proving an equivalence between the ∞-category $\mathcal{P}ic$ and the quasicategory $\mathcal{L}_\infty$, see A.6. This equivalence encodes the idea that every *weak Picard groupoid*, see definition 4.15, can be *strictified*. We begin by recalling, [Lur09b, Prop. 4.2.4.4], that the evaluation map

$$ev : (\mathbf{sSets}_\bullet^{\Gamma^{op}})^{cf} \times \Gamma^{op} \to (\mathbf{sSets}_\bullet)^{cf}$$

induces a categorical equivalence of quasicategories

$$\mathcal{U} : N((\mathbf{sSets}_\bullet^{\Gamma^{op}})^{cf}) \to \mathcal{K}_\bullet^{N(\Gamma^{op})}. \tag{4.2}$$

The functor (simplicial) category $\mathbf{sSets}_\bullet^{\Gamma^{op}}$ above has been endowed with the (simplicial) projective model category structure as described in 2.17. The quasi-category $\mathcal{K}_\bullet$ above is the simplicial nerve of the fibrant simplicial category $\mathbf{Kan}_\bullet$.

Definition 4.15. A *weak Picard groupoid*, F, is a simplicial map

$$F : N(\Gamma^{op}) \to \mathcal{K}_\bullet,$$

which satisfies the following properties:

1. For every $n^+ \in \Gamma^{op}$, the following morphism which is induced by the n morphisms of (pointed) Kan complexes $F(\delta_i^n)$, where $1 \leq i \leq n$ and δ_i^n are the projection maps in the category Γ^{op} as described in A.1,

$$(F(\delta_1^n), \ldots, F(\delta_n^n)) : F(n^+) \to \prod_1^n F(1^+)$$

is an equivalence in the quasicategory $\mathcal{K}_\bullet$.

2. The (pointed) Kan complex $F(0^+)$ is contractible.

3. The following two diagrams are pullback squares in $\mathcal{K}_\bullet$

$$\begin{tikzcd} F(2^+) \arrow[r, "F(m_2)"] \arrow[d, "F(\delta_1^2)"'] & F(1^+) \arrow[d] \\ F(1^+) \arrow[r] & * \end{tikzcd} \qquad \begin{tikzcd} F(2^+) \arrow[r, "F(m_2)"] \arrow[d, "F(\delta_1^2)"'] & F(1^+) \arrow[d] \\ F(1^+) \arrow[r] & * \end{tikzcd} \tag{4.3}$$

Remark. A *weak Picard groupoid* is an object of the ∞-category $\mathcal{L}_\infty$.

Definition 4.16. The ∞-category $\mathcal{L}_\infty$, see example A.7 in appendix A will be refered to as the ∞*-category of weak Picard groupoids.*

Proposition 4.17. *If an object $F \in N((\mathbf{sSets}_\bullet^{\Gamma^{op}})^{cf})_0$ is a Picard groupoid, then $\mathcal{U}(F)$ is a weak Picard groupoid. If $\mathcal{U}(F)$ is a weak Picard groupoid, then there exists a Picard groupoid, G, and an equivalence in the quasi-category $\mathcal{K}_\bullet^{N(\Gamma^{op})}$*

$$f : N(\Gamma^{op}) \times \Delta[1] \to \mathcal{K}_\bullet,$$

such that $d_1(f) = N(G)$ and $d_0(f) = N(F)$.

Proof. Any $F \in N((\mathbf{sSets}_\bullet^{\Gamma^{op}})^{cf})_0$ factors through $\mathbf{Kan}_\bullet$ as follows

$$\begin{tikzcd} N(\Gamma^{op}) \arrow[rr, "F"] \arrow[dr] & & \mathbf{sSets}_\bullet \\ & \mathbf{Kan}_\bullet \arrow[ur, hook] & \end{tikzcd}$$

We denote the simplicial map obtained by restriction of codomain, also by F, *i.e.* $F : \Gamma^{op} \to \mathbf{Kan}_\bullet$. We recall that in the present situation,

$$\mathcal{U}(F) = N(F) : N(\Gamma^{op}) \to \mathcal{K}_\bullet.$$

Let us first assume that F is a Picard groupoid, then the following two observations prove that $\mathcal{U}(F)$ is a weak Picard groupoid:

1. The degree zero function $\mathcal{U}(F)_0$ is the same as the object function of the Picard groupoid F.

2. For all $n^+ \in \Gamma^{op}$ and $1 \leq i \leq n$, the simplicial morphisms $F(\delta_i^n) = \mathcal{U}(F)(\delta_i^n)$ and $F(m_2) = \mathcal{U}(F)(m_2)$.

Conversely, let us assume that $\mathcal{U}(F)$ is a weak Picard groupoid. Now the Picard groupoid G is constructed as follows: We define $G(m^+) = F(m^+)$, if $m^+ \neq 0^+$ and $G(0^+) = *$. Let $f_m^k : m^+ \to k^+$ be a morphism in Γ^{op}, then $G(f_m^k) = F(f_m^k)$, if $k^+ \neq 0^+$ and $m^+ \neq 0^+$. We define $G(f_m^k)$ to be the unique terminal morphism if $k^+ = 0^+$ and $m^+ \neq 0^+$. We define $G(f_m^k)$ to be the unique initial morphism if $k^+ \neq 0^+$ and $m^+ = 0^+$. $\square$

Proposition 4.17 implies that the (simplicial) functor $\mathcal{U}$ maps every object of the category $(\Gamma\mathcal{S})^{cf}$, regarded as a full subcategory of $(\mathbf{sSets}_\bullet^{\Gamma^{op}})^{cf}$, to a weak Picard groupoid. Further, regarding $\mathcal{L}_\infty$ as a full subcategory of $\mathcal{K}_\bullet^{N(\Gamma^{op})}$, there is an obvious inclusion (simplicial) mapping

$$i_{\mathcal{U}} : N((\Gamma\mathcal{S})^{cf}) \hookrightarrow \mathcal{L}_\infty.$$

Proposition 4.18. *The inclusion map*

$$i_{\mathcal{U}} : N((\Gamma\mathcal{S})^{cf}) \hookrightarrow \mathcal{L}_\infty.$$

is fully-faithful.

Proof. We begin the proof by observing that the simplicial category $(\Gamma\mathcal{S})^{cf}$ is a full simplicial subcategory of $(\mathbf{sSets}_\bullet^{\Gamma^{op}})^{cf}$. Further, the ∞-category $\mathcal{L}_\infty$ is a full ∞-subcategory of $\mathcal{K}_\bullet^{N(\Gamma^{op})}$. We want to show that the following map

$$Hom_{N((\Gamma\mathcal{S})^{cf})}(F, G) \to Hom_{\mathcal{L}_\infty}(i_{\mathcal{U}}(F), i_{\mathcal{U}}(G))$$

is an equivalence of Kan complexes. But in light of the above two statements, this is equivalent to showing that the following map

$$Hom_{N(\mathbf{sSets}_\bullet^{\Gamma^{op}})^{cf}}(F, G) \to Hom_{\mathcal{K}_\bullet^{N(\Gamma^{op})}}(\mathcal{U}(F), \mathcal{U}(G))$$

is an equivalence of Kan complexes. This follows from the fact that the simplicial map $\mathcal{U}$ is a categorical equivalence of quasicategories.

$\square$

If a simplicial set X is a quasi-category then the homotopy category of X, $ho(X)$ has a simple construction. This construction is described in [Lur09b, 1.2.3]. We will refer to this description in the formulation and proof of the next proposition.

A morphism $f : F \to G$ in the quasi-category $\mathcal{K}_\bullet^{N(\Gamma^{op})}$ is an equivalence (see definition 4.13) iff there exists another morphism $f^{-1} : G \to F$ such that

$$[f] \circ [f^{-1}] = [id_F] \quad and \quad [f^{-1}] \circ [f] = [id_G]$$

in the homotopy category $ho(\mathcal{K}_\bullet^{N(\Gamma^{op})})$. In other words there exsits a pair of 2-simplicies $H_L, H_R \in (\mathcal{K}_\bullet^{N(\Gamma^{op})})_2$ satisfying

$$d_0(H_L) = f^{-1} \;\; and \;\; d_2(H_L) = f \;\; and \;\; [d_1(H_L)] = [id_F] \tag{4.4}$$

and

$$d_0(H_R) = f \;\; and \;\; d_2(H_R) = f^{-1} \;\; and \;\; [d_1(H_R)] = [id_G] \tag{4.5}$$

Proposition 4.19. *Let $F : N(\Gamma^{op}) \to \mathcal{K}_\bullet$ be an object of the quasi-category $\mathcal{K}_\bullet^{N(\Gamma^{op})}$ and let G be a weak Picard groupoid. Further, let $f : F \to G$ be an equivalence in $\mathcal{K}_\bullet^{N(\Gamma^{op})}$. Then F is a weak Picard groupoid.*

Proof. For every $m^+ \in N(\Gamma^{op})_0$, we fix a pair of 2-simplices H_L and H_R in $\mathcal{K}_\bullet^{N(\Gamma^{op})}$ which satify (4.4) and (4.5). This pair provides us with a pair of 2-simplices in $\mathcal{K}_\bullet$

$$H_L(m^+, -) : \Delta[2] \to \mathcal{K}_\bullet \;\text{ and }\; H_R(m^+, -) : \Delta[2] \to \mathcal{K}_\bullet.$$

Further $d_2(H_L(m^+, -)) = f(m^+)$, $d_0(H_L(m^+, -)) = f^{-1}(m^+)$ and $[d_1(H_L(m^+, -))] = [id_{F(m^+)}]$. Similarly $d_2(H_R(m^+, -)) = f^{-1}(m^+)$, $d_0(H_R(m^+, -)) = f(m^+)$ and $d_0(H_L(m^+, -)) = id_{G(m^+)}$. This proves that, for each $m^+ \in N(\Gamma^{op})_0$, $f(m^+) : F(m^+) \to G(m^+)$ is an equivalence in $\mathcal{K}_\bullet$.

For every morphism $f^m_k : m^+ \to k^+$ in $N(\Gamma^{op})_1$, the 2-simplex H_L provides a diagram $H(f^m_k, -) : \Delta[1] \times \Delta[2] \to \mathcal{K}_\bullet$ which we depict as follows:

$$\begin{tikzcd}
F(m^+) \arrow[r, "f(m^+)"] \arrow[d, "F(f^m_k)"'] & G(m^+) \arrow[r, "f^{-1}(m^+)"] \arrow[d, "G(f^m_k)"] & F(m^+) \arrow[d, "F(f^m_k)"] \\
F(k^+) \arrow[r, "f(k^+)"'] & G(k^+) \arrow[r, "f^{-1}(k^+)"'] & F(k^+)
\end{tikzcd} \tag{4.6}$$

In particular, the m morphisms $\delta^m_i : m^+ \to 1^+$, each provide us with a diagram

$$d_2(H_L(\delta^m_i, -)) : \Delta[1] \times \Delta[1] \to \mathcal{K}_\bullet$$

and therefore together provide a diagram

$$f(\delta^m_i) := (d_2(H_L(\delta^m_1, -)), \ldots, d_2(H_L(\delta^m_m, -)) : \Delta[1] \times \Delta[1] \to \mathcal{K}_\bullet,$$

which we depict by

$$\begin{tikzcd} F(m^+) \arrow[r, "\big(F(\delta_1^m),\dots F(\delta_1^m)\big)"] \arrow[d, "f(m^+)"'] & F(1^+) \times \cdots \times F(1^+) \arrow[d, "\prod_1^m f(1^+)"] \\ G(m^+) \arrow[r, "\big(G(\delta_1^m),\dots G(\delta_1^m)\big)"'] & G(1^+) \times \cdots \times G(1^+) \end{tikzcd} \tag{4.7}$$

We observe that the morphisms $f(m^+)$, $\prod\limits_1^m f(1^+)$ and $(G(\delta_1^m), \dots G(\delta_1^m))$ are equivalences in $\mathcal{K}_\bullet$. In this situation the morphism

$$(F(\delta_1^m), \dots F(\delta_1^m)) : F(m^+) \to F(1^+) \times \cdots \times F(1^+)$$

is also an equivalence in $\mathcal{K}_\bullet$.

Similarly the morphisms $F(m_2)$, $F(\delta_1^2)$ and $F(\delta_2^2)$ provide us with a pair of diagrams

$$f(m_2, \delta_1^2) := (d_2(H_L(m_2, -)), d_2(H_L(\delta_1^2, -)) : \Delta[1] \times \Delta[1] \to \mathcal{K}_\bullet,$$

and

$$f(m_2, \delta_2^2) := (d_2(H_L(m_2, -)), d_2(H_L(\delta_2^2, -)) : \Delta[1] \times \Delta[1] \to \mathcal{K}_\bullet,$$

which we depict by

$$\begin{tikzcd} F(2^+) \arrow[r, "\big(F(m_2),F(\delta_1^2)\big)"] \arrow[d, "f(2^+)"'] & F(1^+) \times F(1^+) \arrow[d, "f(1^+)\times f(1^+)"] \\ G(2^+) \arrow[r, "\big(G(m_2),G(\delta_1^2)\big)"'] & G(1^+) \times G(1^+) \end{tikzcd} \qquad \begin{tikzcd} F(2^+) \arrow[r, "\big(F(m_2),F(\delta_2^2)\big)"] \arrow[d, "f(2^+)"'] & F(1^+) \times F(1^+) \arrow[d, "f(1^+)\times f(1^+)"] \\ G(2^+) \arrow[r, "\big(G(m_2),G(\delta_2^2)\big)"'] & G(1^+) \times G(1^+) \end{tikzcd}$$

An argument similar to the one presented above proves that the top horizontal morphisms

$$\big(F(m_2), F(\delta_1^2)\big) : F(2^+) \to F(1^+) \times F(1^+)$$

and

$$\big(F(m_2), F(\delta_2^2)\big) : F(2^+) \to F(1^+) \times F(1^+)$$

are both equivalences in $\mathcal{K}_\bullet$. This is equivalent to proving that the two diagrams (6.1) are both pullback squares. Thus we have proved that F is a weak Picard groupoid in the sense of definition 4.15.

$\square$

Using the above propositions, we want to now show that the simplicial map $i_{\mathcal{U}}$ is a categorical equivalence of quasicategories.

Theorem 4.20. *The inclusion simplicial map*

$$i_{\mathcal{U}} : N((\Gamma\mathcal{S})^{cf}) \hookrightarrow \mathcal{L}_\infty$$

is a categorical equivalence.

Proof. A simplicial map $f : \mathcal{K} \to \mathcal{L}$ is a categorical equivalence iff the simplicial functor $\mathfrak{C}(f) : \mathfrak{C}(\mathcal{K}) \to \mathfrak{C}(\mathcal{L})$ is a DK-equivalence, see definition (D.4). We will prove the theorem by showing that

$$\mathfrak{C}(i_{\mathcal{U}}) : \mathfrak{C}(N((\Gamma\mathcal{S})^{cf})) \to \mathfrak{C}(\mathcal{L}_\infty)$$

is a DK-equivalence. For every pair of objects $F, G \in Ob(\mathfrak{C}(N((\Gamma\mathcal{S})^{cf})))$, the proposition 4.18 implies that the simplicial map

$$\mathfrak{C}(i_{\mathcal{U}})_{F,G} : Map_{\mathfrak{C}(N((\Gamma\mathcal{S})^{cf}))}(F, G) \to Map_{\mathfrak{C}(\mathcal{L}_\infty)}(i_{\mathcal{U}}(F), i_{\mathcal{U}}(G)) \tag{4.8}$$

is a weak homotopy equivalence of simplicial sets for every pair (F, G) of Picard groupoids. It remains to show that

$$\pi_0(\mathfrak{C}(i_{\mathcal{U}})) : \pi_0(\mathfrak{C}(N((\Gamma\mathcal{S})^{cf}))) \to \pi_0(\mathfrak{C}(\mathcal{L}_\infty))$$

is an equivalence of (ordinary) categories.

For every pair (F, G) of Picard groupoids, the weak homotopy equivalence (4.8) shows that

$$\pi_0(\mathfrak{C}(i_{\mathcal{U}}))_{F,G} : \pi_0(\mathfrak{C}(N((\Gamma\mathcal{S})^{cf})))(F, G) \to \pi_0(\mathfrak{C}(\mathcal{L}_\infty))(i_{\mathcal{U}}(F), i_{\mathcal{U}}(G))$$

is a bijection. It remains to show that $\pi_0(\mathfrak{C}(i_{\mathcal{U}}))$ is essentially surjective. The simplicial category $\mathfrak{C}(\mathcal{L}_\infty)$ is a full (simplicial) subcategory of $\mathfrak{C}(\mathcal{K}_\bullet^{N(\Gamma^{op})})$. The categorical equivalence (4.2) implies that for any $H \in Ob(\mathfrak{C}(\mathcal{L}_\infty))$, there exists an $F \in Ob(\mathfrak{C}(N((\mathbf{sSets}_\bullet^{\Gamma^{op}}))^{cf}))$ and an isomorphism $[f] : \mathcal{U}(F) \to H$ in $\pi_0(\mathfrak{C}(\mathcal{K}_\bullet^{N(\Gamma^{op})}))$. Since $\pi_0(\mathfrak{C}(\mathcal{K}_\bullet^{N(\Gamma^{op})}))$ is isomorphic to $ho(\mathcal{K}_\bullet^{N(\Gamma^{op})})$, therefore F and H are isomorphic in $ho(\mathcal{K}_\bullet^{N(\Gamma^{op})})$. Hence there is an equivalence $f : F \to G$ in $\mathcal{K}_\bullet^{N(\Gamma^{op})}$. In this situation, proposition 4.19 implies that $\mathcal{U}(F)$ is a weak Picard groupoid. Proposition 4.17 garuntees the existence of a Picard groupoid $F' \in Ob(\mathfrak{C}(N((\Gamma\mathcal{S})^{cf})))$ such that there is an equivalence $f' : \mathcal{U}(F') \to \mathcal{U}(F)$ in $\mathcal{K}_\bullet^{N(\Gamma^{op})}$. Thus we get an isomorphism $[f']$ in $\pi_0(\mathfrak{C}(\mathcal{L}_\infty))$. Now the composite isomorphism

$$[f] \circ [f'] : \mathcal{U}(F') \to H$$

is in $\pi_0(\mathfrak{C}(\mathcal{L}_\infty))$ which proves that the functor $\pi_0(\mathfrak{C}(i_{\mathcal{U}}))$ is an equivalence of (ordinary) categories. □

Chapter 5

A higher additive category structure on $\mathcal{P}ic$

The main result of this thesis will be presented in this chapter. We begin by constructing a *coherently commutative monoid* structure on the higher category of Picard groupoids, $\mathcal{P}ic$. This structure is presented by the simplicial diagram 5.1. The main result, 5.16, says that this coherently commutative monoid structure can be "strictified" into a Γ-space, uniquely upto equivalence. This "strictified" Γ-space is denoted by $\underline{\mathcal{P}ic}^{\oplus}$. Further, there is a categorical equivalence between the infinite loop space $\Omega_\infty(\underline{\mathcal{P}ic}^{\oplus})$ and the (pointed) quasicategory $\mathcal{P}ic$. The Γ-space $\underline{\mathcal{P}ic}^{\oplus}$ is a fibrant object in the strict JQ-model category structure. The "multiplication" in this coherently commutative monoid is just the direct sum of Picard groupoids. The motivation of this Γ-space structure comes from (ordinary) additive categories which are provided with a symmetric monoidal category structure by the direct sum functor.

One may similarly associate a coherently commutative monoid structure, denoted

$\underline{X}_w^{\oplus}$, any *additive quasi-category* X, see definition 5.3. We claim that the process described to *strictify* $\underline{\mathcal{P}ic}_w^{\oplus}$ can be applied to $\underline{X}_w^{\oplus}$ to obtain a Γ-space, denoted $\underline{X}^{\oplus}$, representing the additive quasi-category X. This allows us to propose a model for an additive functor between two additive quasi-categories X and Y. We define an additive functor from X to Y to be a morphism of Γ-spaces

$$\underline{F}^{\oplus} : \underline{X}^{\oplus} \to \underline{Y}^{\oplus}.$$

In the first part of this Chapter, we construct a simplicial map

$$\underline{\mathcal{P}ic}_w^{\oplus} : N(\Gamma^{op}) \to N((\mathbf{sSets}_{\bullet}, \mathbf{Q})^{cf}). \tag{5.1}$$

We define the degree zero function of the simplicial map 5.1 as follows:

$$(\underline{\mathcal{P}ic}_w^{\oplus})_0 : n^+ \mapsto \prod_1^n \mathcal{P}ic.$$

We observe that a simplicial map

$$f : \prod_1^n \mathcal{P}ic \to \prod_1^m \mathcal{P}ic$$

is completely determined by m maps

$$f_i : \prod_1^n \mathcal{P}ic \to \mathcal{P}ic,$$

where $1 \leq i \leq n$. These maps will be called the *component maps* of f. The degree one function of our simplicial map 5.1 will be completely determined by a sequence of "product functors" each of which is a right adjoint to the diagonal simplicial map

$$\delta_n : \mathcal{P}ic \to \prod_1^n \mathcal{P}ic.$$

The next proposition proves the existence of these "product functors"

Proposition 5.1. *For every $n \in \mathbb{N}$, the functor $\delta_n : \mathcal{P}ic \to \prod_1^n \mathcal{P}ic$ has a right adjoint.*

Proof. Since $\mathcal{P}ic$ is a presentable ∞-category, therefore any finite product of copies of $\mathcal{P}ic$, namely $\prod\limits_{1}^{n}\mathcal{P}ic$, is also presentable ∞-category. Further, the simplicial map δ preserves colimits, therefore by the *Adjoint functor theorem*, see [Lur09b, Theorem], it has a right adjoint. $\square$

Remark. Any right adjoint of the functor δ_n is also a left adjoint.

Notation 5.2. A right adjoint of the diagonal map δ_n will be denoted by $\overset{h}{\underset{n}{\oplus}}$ and $\overset{h}{\underset{n}{\oplus}}(p_1, p_2, \ldots, p_n)$ will be denoted by $p_1 \overset{h}{\oplus} p_2 \ldots \overset{h}{\oplus} p_n$.

Let $f : m^+ \to k^+$ be a morphism in the category Γ^{op}. If $k = 0$, then we define $\underline{\mathcal{P}ic}_w^{\oplus}(f)$ to be the zero morphism, namely the morphism $0_m : \prod\limits_{1}^{m}\mathcal{P}ic \to \mathcal{P}ic$, which maps every n-simplex to the n-fold degeneracy of the the zero object of $\mathcal{P}ic$, namely $\epsilon_n(*)$. Let us assume that $k \neq 0$. We will define m component maps which will completely specify $\underline{\mathcal{P}ic}_w^{\oplus}(f)$.

If the preimage of $s \in k^+$ is the emptyset, then we define the sth component of $(\underline{\mathcal{P}ic}_w^{\oplus}(f))$ to be the zero morphism, otherwise let the preimage of $s \in k^+$ be the set

$$\{k_1^s, k_2^s, \ldots, k_l^s\} \subset m^+$$

For each $f : m^+ \to k^+$, we define a simplicial map $q_f^s : \prod\limits_{1}^{m}\mathcal{P}ic \to \prod\limits_{1}^{m}\mathcal{P}ic$ as follows

$$q_f^s(p_1, p_2, \ldots, p_m) = ((q_f^s)_1, (q_f^s)_2, \ldots, (q_f^s)_m),$$

where $(q_f^s)_j = p_j$ if $j \in \{k_1^s, k_2^s, \ldots, k_l^s\}$ and $(q_f^s)_j = *$ otherwise, where $1 \leq j \leq m$.

In this case, we define the sth component, $(s \leq k)$, or the simplicial map $(\underline{\mathcal{P}ic}_w^{\oplus}(f))_s : \prod\limits_{1}^{m}\mathcal{P}ic \to \mathcal{P}ic$ to be the simplicial map, which in degree n is defined by the following function

$$(p_1, p_2, \ldots, p_m) \mapsto \overset{h}{\underset{m}{\oplus}}(q_f^s(p_1, p_2, \ldots, p_m)) = * \overset{h}{\oplus} * \overset{h}{\oplus} * \ldots \overset{h}{\oplus} p_{i_1} \overset{h}{\oplus} \ldots p_{i_{l_s}} \overset{h}{\oplus} \ldots \overset{h}{\oplus} * .$$

Therefore $\underline{\mathcal{P}ic}^{\oplus}_{w}(f)$ is the simplicial map whose k components are given by (simplicial) maps $\overset{h}{\underset{m}{\oplus}}(q^s_f(-))$, where $1 \leq s \leq k$. Thus we have fully defined the function $\left(\underline{\mathcal{P}ic}^{\oplus}_{w}\right)_1$.

Let $b = g \circ f$ be a composite morphism in Γ^{op}, then $g \circ f$ is a nondegenerate 2-simplex of the simplicial set $N(\Gamma^{op})$. The desired function $(\underline{\mathcal{P}ic}^{\oplus}_{w})_2$ should map a composite arrow $g \circ f$ to a homotopy between the maps $(\underline{\mathcal{P}ic}^{\oplus}_{w})_1(g \circ f)$ and $(\underline{\mathcal{P}ic}^{\oplus}_{w})_1(h)$. This homotopy is constructed below using the counit of adjunction (of ∞-categories) whose adjoint functors are $\left(\delta_m, \overset{h}{\underset{m}{\oplus}}\right)$,

$$\epsilon_m : \delta_m \circ \overset{h}{\underset{m}{\oplus}} \Rightarrow id.$$

This counit is presented by a simplicial homotopy

$$\Upsilon_m : \prod_1^m \mathcal{P}ic \times \Delta[1] \to \prod_1^m \mathcal{P}ic,$$

such that $\Upsilon_m(-,0) = \delta_m \circ \overset{h}{\underset{m}{\oplus}}$ and $\Upsilon_m(-,1) = id$.

Consider the following 2-simplex in the simplicial set $N(\Gamma^{op})$

$$m^+ \xrightarrow{f} k^+ \xrightarrow{g} v^+.$$

The simplicial map $\underline{\mathcal{P}ic}^{\oplus}_{w}(f) : \prod_1^n \mathcal{P}ic \to \prod_1^m \mathcal{P}ic$ has m components. The sth component is described in (5). The simplicial homotopy Υ_m defined above determines another simplicial homotopy

$$\Upsilon^s_f := \Upsilon_m(q^s_f \times id) : \prod_1^m \mathcal{P}ic \times \Delta[1] \to \prod_1^m \mathcal{P}ic,$$

such that $\Upsilon^s_f(-,0) = \delta_m \circ (\underline{\mathcal{P}ic}^{\oplus}_{w}(f))_s$ and $\Upsilon^s_f(-,0) = q^s_f$. This provides us with l simplicial homotopies

$$(\Upsilon^s_f)_{k^s_j} : \prod_1^m \mathcal{P}ic \times \Delta[1] \to \mathcal{P}ic,$$

where $1 \leq j \leq l$. We observe that $(\Upsilon^s_f)_{k^s_j}(-,0) = (\underline{\mathcal{P}ic}^{\oplus}_{w}(f))_s$ and $(\Upsilon^s_f)_{k^s_j}(-,1) = (q^s_f)_{k^s_j}$. Similarly for every t, where $1 \leq t \leq v$, we get r simplicial homotopies

$$(\Upsilon^t_g)_{v^t_j} : \prod_1^k \mathcal{P}ic \times \Delta[1] \to \mathcal{P}ic,$$

where $g(v_j^t) = t$, for $1 \leq j \leq r$. Further, $(\Upsilon_g^t)_{v_j^t}(-,0) = (\underline{\mathcal{P}ic}_w^{\oplus}(g))_t$ and $(\Upsilon_g^t)_{v_j^t}(-,1) = (q_g^t)_{v_j^t}$.

Now we are ready to define the tth $(1 \leq t \leq v)$ component of the homotopy

$$(H_{g\circ f}^b)_t : \prod_1^m \mathcal{P}ic \times \Delta[1] \to \mathcal{P}ic,$$

such that $(H_{g\circ f}^b)_t(-,0) = (\underline{\mathcal{P}ic}_w^{\oplus}(g\circ f))_t$ and $(H_{g\circ f}^b)_t(-,1) = (\underline{\mathcal{P}ic}_w^{\oplus}(b))_t$. The component map $(H_{g\circ f}^b)_t$ is defined as follows:

$$(H_{g\circ f}^b)_t := \overset{h}{\underset{r}{\oplus}}(((\Upsilon_g^t)_{v_1^t})\circ(\overset{h}{\underset{l_1}{\oplus}}((\Upsilon_f^{v_1^t})_{k_1^{v_1^t}},\ldots,(\Upsilon_f^{v_1^t})_{k_{l_1}^{v_1^t}})),\ldots,((\Upsilon_g^t)_{v_r^t})\circ(\overset{h}{\underset{l_r}{\oplus}}((\Upsilon_f^{v_r^t})_{k_1^{v_r^t}},\ldots,(\Upsilon_f^{v_r^t})_{k_{l_r}^{v_r^t}})))$$

Thus we have completely specified the homotopy $H_{g\circ f}^b)_t$ and therefore specified the function $(\underline{\mathcal{P}ic}_w^{\oplus})_2$. Proposition **??** implies that our construction can extend be extended to higher degrees and therefore the desired simplicial map can be defined.

Remark. The simplicial diagram $\underline{\mathcal{P}ic}_w^{\oplus}$ should be viewed as a (weak) coherently commutative monoid struture on $\mathcal{P}ic$.

The above construction of a coherently commutative monoid is based entirely on the existence of a sequence of adjoint functors $(\delta_n, \overset{h}{\underset{n}{\oplus}})$.

Definition 5.3. A null-pointed quasi-category X will be called an *additive* quasi-category if, for all $n \in \mathbb{Z}^+$, there exists a functor (of quasi-categories) $\overset{h}{\underset{n}{\oplus}} : \prod_1^n X \to X$ which is both right and left adjoint to the diagonal functor (of quasi-categories) $\delta_n : X \to \prod_1^n X$.

Remark. The above construction can be applied to any additive quasi-category. Therefore, to every additive quasi-category X, one can associate a coherently commutative monoid $\underline{X}_w^{\oplus}$ which is presented by a simplicial morphism

$$\underline{X}_w^{\oplus} : N(\Gamma^{op}) \to N((\mathbf{sSets_\bullet}, \mathbf{Q})^{cf}).$$

5.1 Strictification of the symmetric monoidal structure

This section of is devoted to "strictifying" the simplicial diagram $\underline{\mathcal{P}ic}_w^{\oplus}$ into an honest functor which defines a Γ-space. In order to do so, we would need to review the theory of categories enriched over monoidal model categories.

Let $\mathbf{S}$ be a monoidal model category and let $\mathbf{Cat}_{\mathbf{S}}$ denote the category of (small) $\mathbf{S}$-enriched categories in which morphisms are given by $\mathbf{S}$-enriched functors.

Definition 5.4. Let $\mathbf{S}$ be a monoidal model category. We say that a functor $F : \mathcal{C} \to \mathcal{C}'$ in $Cat_{\mathbf{S}}$ is a *weak equivalence* if the induced functor $hF : h\mathcal{C} \to h\mathcal{C}'$ is a functor of $h\mathbf{S}$ enriched categories. In other words, F is a weak equivalence if and only if

1. For every pair of objects $X, Y \in Ob(\mathcal{C})$, the induced map

$$Map_{\mathcal{C}}(X, Y) \to Map_{\mathcal{C}'}(F(X), F(Y))$$

   is a weak equivalence in S.

2. Every object $Y \in \mathcal{C}'$ is equivalent to $F(X)$ in the homotopy category $h\mathcal{C}'$ for some $X \in \mathcal{C}$.

Notation 5.5. We let $[0]_{\mathbf{S}}$ denote the $\mathbf{S}$-enriched category which has only one object X and $Map_{[0]_{\mathbf{S}}}(X, X) := \mathbf{1}_{\mathbf{S}}$, where $\mathbf{1}_{\mathbf{S}}$ is the unit of the monoidal structure on $\mathbf{S}$.

Definition 5.6. We let C_0 denote the collection of all morphism in $\mathbf{S}$ which satisfy the following two conditions

1. The inclusion $\emptyset \hookrightarrow [0]_{\mathbf{S}}$.

2. The induced maps $[1]_S \to [1]'_S$, where the maps $S \to S'$ range over a set of generators for a weakly saturated set of class of cofibrations in $\mathbf{S}$.

We recall the following proposition from [Lur09b] which is generalization of the main result of [Ber07].

Proposition 5.7. *[Lur09, A.3.2.4] Let* $\mathbf{S}$ *be a combinatorial monoidal model category. Assume that every object of* $\mathbf{S}$ *is cofibrant and the collection of weak equivalences in* $\mathbf{S}$ *is stable under filtered colimits. Then there exists a left proper combinatorial model category structure on* $\mathbf{Cat_S}$ *chracterized by the following conditions:*

(C) The class of cofibrations in $\mathbf{Cat_S}$ *is the smallest weakly saturated class of morphisms containing the set of morphisms* C_0 *appearing above.*

(W) The class of weak equivalences is the class of maps defined in definition 5.4.

Remark. The monoidal model categories $(\mathbf{sSets}, \mathbf{Kan})$ and $(\mathbf{sSets}, \mathbf{Q})$, satisfy the hypothesis of the above proposition.

Remark. If the monoidal model category $\mathbf{S} = (\mathbf{sSets}, \mathbf{Kan})$, then the above proposition recovers the model category structure on simplicial categories, which is the main result of [Ber07].

Proposition 5.8. *The cofibrations in the model category* $\mathbf{Cat}_{(\mathbf{sSets},\mathbf{Q})}$ *are the same as those in the model category* $\mathbf{Cat}_{(\mathbf{sSets},\mathbf{Kan})}$.

Proof. The cofibrations are the same because the collection of morphisms generating cofibrations of the two model category structures is the same. □

An obvious consequence of the above proposition is the following corollary:

Corollary 5.9. *The acyclic fibrations in the model category* $\mathbf{Cat}_{(\mathbf{sSets},\mathbf{Q})}$ *are the same as those in the model category* $\mathbf{Cat}_{(\mathbf{sSets},\mathbf{Kan})}$

Another easy consequence of the above proposition is the following:

Corollary 5.10. *Let* X *be a small category enriched over simplicial sets, then another small simplicial category* QX *is a cofibrant replacement of* X *in the model category* $\mathbf{Cat}_{(\mathbf{sSets},\mathbf{Q})}$ *if and only if it is a cofibrant replacement of* X *in* $\mathbf{Cat}_{(\mathbf{sSets},\mathbf{Kan})}$.

The following proposition will be useful in proving the main result.

Proposition 5.11. *The simplicially enriched functor category* $((\mathbf{sSets}_\bullet, \mathbf{Q})^{\Gamma^{op}})^{cf})$ *is a fibrant object in the model category* $\mathbf{Cat}_{(\mathbf{sSets}, \mathbf{Q})}$.

A simplicial map $F : N(\Gamma^{op}) \to N((\mathbf{sSets}, \mathbf{Q})^{cf})$ uniquely determines, by adjointness, a map of simplicial categories

$$\psi(F) : \mathfrak{C}(N(\Gamma^{op})) \to (\mathbf{sSets}, \mathbf{Q})^{cf}.$$

The simplicially enriched category $\mathfrak{C}(N(\Gamma^{op}))$ is a cofibrant object of the model category $(\mathbf{sSets}, \mathbf{Kan})$ and therefore, by proposition 5.8, a cofibrant object of $(\mathbf{sSets}, \mathbf{Q})$. The counit of the (Quillen) equivalence

$$(\mathfrak{C}, N, \phi, \psi) : \mathbf{Cat}_{(\mathbf{sSets}, \mathbf{Kan})} \to (\mathbf{sSets}, \mathbf{Q}),$$

provides a DK-equivalence $\epsilon : \mathfrak{C}(N(\Gamma^{op})) \to \Gamma^{op}$. We observe that the objects of the cofibrant simplicial category $\mathfrak{C}(N(\Gamma^{op}))$ are the same as those of Γ^{op}. We want to replace this DK-equivalence by a functor (of simplicial categories) which is a weak equivalence in the model category $\mathbf{Cat}_{(\mathbf{sSets}, \mathbf{Q})}$. In order to do so, we introduce, for each set $\mathcal{O}$, a category $\mathbf{Cat}_{\mathcal{O}}$ whose objects are small categories with object set $\mathcal{O}$ and whose morphisms are functors which are identity on objects. We also get, for every set $\mathcal{O}$, a model category whose objects are all simplicial categories having object set $\mathcal{O}$. These simplicial categories should be viewed as simplicial objects in the category $\mathbf{Cat}_{\mathcal{O}}$. The model category is denoted by $\mathcal{S}Cat_{\mathcal{O}}$ and the model category structure is described by the following theorem

Theorem 5.12 ([DK80b], Prop. 7.2)**.** *The category* $\mathcal{S}Cat_{\mathcal{O}}$*, with the following three classes of maps*

1. *A class of weak equivalences consisting of* DK*-equivalences.*

2. *A class of fibrations consisting of functors of simplicial categories $f : \mathcal{C} \to \mathcal{D}$, which for every pair of objects $X, Y \in \mathcal{O}$, induce a fibration of simplicial sets $Map_{\mathcal{C}}(X,Y) \to Map_{\mathcal{D}}(X,Y)$.*

3. *A class of cofibrations which consists of functors (of simplicial categories) which have the left lifting property with respect to all functors which lie in both classes above.*

is a simplicial, proper model category.

This model category structure allows us to factorize the DK-equivalence ϵ, defined above, as follows:

$$\begin{tikzcd} & \mathbf{F}\mathfrak{C}(N(\Gamma^{op})) \arrow[dr, "p"] & \\ \mathfrak{C}(N(\Gamma^{op})) \arrow[ur, "i"] \arrow[rr, "\epsilon"'] & & \Gamma^{op} \end{tikzcd} \tag{5.2}$$

where i is a cofibration and p is an acyclic fibration in the model category $\mathcal{S}Cat_{\mathcal{O}}$. We notice that the functor (of simplicial categories) p is also a fibration of simplicial categories in the sense of [Ber07], see definition **??**. Thus, corollary 5.9 proves that p is an acyclic fibration in the model category $\mathbf{Cat}_{(\mathbf{sSets},\mathbf{Q})}$. We claim that the functor of simplicial categories $\psi(F)$, see (5.1), extends to a functor of simplicial categories along the cofibration i described in (5.2). This extended functor is denoted as follows:

$$\mathbf{Lan}_i(\psi(F)) : \mathbf{F}\mathfrak{C}(N(\Gamma^{op})) \to (\mathbf{sSets}_{\bullet}, \mathbf{Q})^{cf}. \tag{5.3}$$

Further, this extended functor has the same object function as that of $\psi(F)$. We prove this claim later in this Chapter in Theorem 5.20. The following lemma is an easy consequence of the discussion above

Lemma 5.13. *The simplicial category $\mathbf{F}\mathfrak{C}(N(\Gamma^{op}))$ is a cofibrant replacement of the (discrete simplicial) category Γ^{op} in the model category $\mathbf{Cat}_{(\mathbf{sSets},\mathbf{Kan})}$.*

Thus the simplicial map $\psi(F)$ uniquely determines a homotopy class of maps (*i.e.* a morphism) in the Hom set

$$Hom_{hCat_{(\mathbf{sSets},\mathbf{Q})}}(\Gamma^{op}, (\mathbf{sSets}_\bullet, \mathbf{Q})^{cf}) \cong \mathbf{Cat}_{(\mathbf{sSets},\mathbf{Q})}(\mathbf{F}\mathfrak{C}(N(\Gamma^{op})), (\mathbf{sSets}_\bullet, \mathbf{Q})^{cf})/\sim .$$

The following two propositions will play an important part in proving the main result of this thesis. The first proposition helps us understand the composition of the homotopy classes of functors of simplicial categories.

Proposition 5.14. *[Lur09b, A.3.4.10] Let $\mathbf{A}$ be a combinatorial, $(\mathbf{sSets}, \mathbf{Q})$-enriched model category, and let $\mathcal{C}$ be a small, cofibrant $(\mathbf{sSets}, \mathbf{Q})$-enriched category. Let $f, f' : \mathcal{C} \to \mathbf{A}$ be two functors of simplicial categories, Then the following conditions are equivalent:*

1. *The homotopy classes $[f]$ and $[f']$ coincide in $Hom_{\mathbf{Cat}_{(\mathbf{sSets},\mathbf{Q})}}(\mathcal{C}, \mathbf{A}^{cf})$.*
2. *The maps f and f' are weakly equivalent when regarded as objects of the category $\mathbf{A}^{\mathcal{C}}$ endowed with the projective model category structure, see definition 2.17.*

The second proposition is a special case of [Lur09b, A.3.4.13.]. It tells us that the simplicially enriched functor category $(\mathbf{A}^{\mathcal{C}})^{cf}$ behaves as an *internal mapping object* in the homotopy category of the model cateegory $\mathbf{Cat}_{(\mathbf{sSets},\mathbf{Q})}$.

Proposition 5.15. *Let $\mathbf{A}$ be a combinatorial, $(\mathbf{sSets}, \mathbf{Q})$-enriched model category, and let $\mathcal{C}$ be a small $(\mathbf{sSets}, \mathbf{Q})$-enriched category. Then the evaluation map*

$$ev : (\mathbf{A}^{\mathcal{C}})^{cf} \times \mathcal{C} \to \mathbf{A}^{cf}$$

has the following property: For every small $(\mathbf{sSets}, \mathbf{Q})$-enriched category $\mathcal{D}$, composition with the map ev induces a bijection

$$Hom_{h\mathbf{Cat}_{(\mathbf{sSets},\mathbf{Q})}}(\mathcal{D}, (\mathbf{A}^{\mathcal{C}})^{cf}) \cong Hom_{h\mathbf{Cat}_{(\mathbf{sSets},\mathbf{Q})}}(\mathcal{D} \times \mathcal{C}, \mathbf{A}^{cf}).$$

The functor category $\mathbf{A}^{\mathcal{C}}$ above is endowed with the projective *model category structure.*

It is easy to see that $[0]_{(\mathbf{sSets},\mathbf{Q})} \times \Gamma^{op} \cong \Gamma^{op}$. The proposition above gives us the following bijection:

$$Hom_{h\mathbf{Cat}_{(\mathbf{sSets},\mathbf{Q})}}([0]_{(\mathbf{sSets},\mathbf{Q})}, (\mathbf{sSets}_\bullet, \mathbf{Q})^{\Gamma^{op}cf}) \cong Hom_{h\mathbf{Cat}_{(\mathbf{sSets},\mathbf{Q})}}(\Gamma^{op}, (\mathbf{sSets}_\bullet, \mathbf{Q})^{cf}). \tag{5.4}$$

Propositions 5.14 and 5.15 along with the fact that $[0]_{(\mathbf{sSets},\mathbf{Q})}$ is a cofibrant object of the model category $\mathbf{Cat}_{(\mathbf{sSets},\mathbf{Q})}$, together imply that the evaluation map, ev, induces a bijection between the set $Hom_{h\mathbf{Cat}_{(\mathbf{sSets},\mathbf{Q})}}(\Gamma^{op}, (\mathbf{sSets}_\bullet, \mathbf{Q})^{cf}$ and the set of equivalence classes of weakly equivalent functors in the model category $(\mathbf{sSets}_\bullet, \mathbf{Q})^{\Gamma^{op}}$. This leads us to the main result of part I of this thesis. The following theorem, which is the main result of part I of this thesis, should be understood as a strictification of the simplicial diagram $\underline{\mathcal{P}ic}_w^{\oplus} : N(\Gamma^{op}) \to N((\mathbf{sSets}_\bullet, \mathbf{Q})^{cf})$ because the functor of simplicial categories

$$\mathbf{Lan}_i(\psi(F)) : \mathbf{F}\mathfrak{C}(N(\Gamma^{op})) \to (\mathbf{sSets}_\bullet, \mathbf{Q})^{cf},$$

see (5.3), represents the simplicial morphism $\underline{\mathcal{P}ic}_w^{\oplus}$ in the category of simplicial categories. The functor category $(\mathbf{sSets}_\bullet, \mathbf{Q})^{\mathbf{F}\mathfrak{C}(N(\Gamma^{op}))}$ is assumed to have the projective model category structure as described in definition 2.17.

Theorem 5.16. *There exists a JQ-fibrant* Γ*-space* $\underline{\mathcal{P}ic}^{\oplus} : \Gamma^{op} \to (\mathbf{sSets}_\bullet, \mathbf{Q})$ *such that the functor*

$$\underline{\mathcal{P}ic}^{\oplus} \circ p : \mathbf{F}\mathfrak{C}(N(\Gamma^{op})) \to ((\mathbf{sSets}_\bullet, \mathbf{Q}))^{cf},$$

where $p : \mathbf{F}\mathfrak{C}(N(\Gamma^{op})) \to \Gamma^{op}$ *is the acyclic fibration establishing* $\mathbf{F}\mathfrak{C}(N(\Gamma^{op}))$ *as a cofibrant replacement of* Γ^{op}*, see* (5.2) *is weakly equivalent in the (functor) model category* $(\mathbf{sSets}_\bullet, \mathbf{Q})^{\mathbf{F}\mathfrak{C}(N(\Gamma^{op}))}$*, to the extended functor*

$$\mathbf{Lan}_i(\psi(\underline{\mathcal{P}ic}_w^{\oplus})) : \mathbf{F}\mathfrak{C}(N(\Gamma^{op})) \to (\mathbf{sSets}_\bullet, \mathbf{Q})^{cf},$$

Proof. The two propositions 5.14 and 5.15 together imply that there exists a functor of simplicial categories $\underline{\mathcal{P}ic}^{\oplus} : [0]_{(\mathbf{sSets},\mathbf{Q})} \to ((\mathbf{sSets}_\bullet, \mathbf{Q})^{\Gamma^{op}})^{cf}$ such that the following composite functor

$$\mathbf{F}\mathfrak{C}(N(\Gamma^{op})) \to \Gamma^{op} \cong [0]_{(\mathbf{sSets},\mathbf{Q})} \times \Gamma^{op} \overset{\underline{\mathcal{P}ic}^{\oplus} \times id}{\to} ((\mathbf{sSets}_{\bullet}, \mathbf{Q})^{\Gamma^{op}})^{cf} \times \Gamma^{op} \overset{ev}{\to} (\mathbf{sSets}_{\bullet}, \mathbf{Q})^{cf},$$

which is the same as $\underline{\mathcal{P}ic}^{\oplus} \circ p$, is weakly equivlent to $\mathbf{Lan}_i(\psi(\underline{\mathcal{P}ic}^{\oplus}_{w}))$, in the model category $(\mathbf{sSets}_{\bullet}, \mathbf{Q})^{\mathbf{F}\mathfrak{C}(N(\Gamma^{op}))}$. We identify the functor (of simplicial categories) $\underline{\mathcal{P}ic}^{\oplus}$ with an object of $((\mathbf{sSets}_{\bullet}, \mathbf{Q})^{\Gamma^{op}})^{cf}$, which we also denote by $\underline{\mathcal{P}ic}^{\oplus}$. The above two statements imply that the (pointed) quasicategory $\underline{\mathcal{P}ic}^{\oplus}(0^+)$ is (categorically) equivalent to the terminal simplicial set $*$. We can assume that $\underline{\mathcal{P}ic}^{\oplus}(0^+) = *$. Thus $\underline{\mathcal{P}ic}^{\oplus}$ is a Γ-space such that there is a categorical equivalence between the (pointed quasicategories) $\Omega_{\infty}(\underline{\mathcal{P}ic}^{\oplus})$ and $\mathcal{P}ic$. □

Finally, we present a proof of the claim made earlier regarding the extension of the functor $\psi(F)$ along the cofibration i which was defined in (5.2). We will view an object of the category $\mathcal{S}Cat_{\mathcal{O}}$ as a simplicial object in the category of categories having object set $\mathcal{O}$. In order to prove the desired result, we would need a characterization of cofibrations in $\mathcal{S}Cat_{\mathcal{O}}$. To formulate it we need the following two definitions:

Definition 5.17. A morphism $f : \mathcal{C} \to \mathcal{D}$ in $\mathcal{S}Cat_{\mathcal{O}}$ will be called a *free map* if it satisfies the following conditions:

1. The morphism f is injective.

2. In each dimension k, $\mathcal{D}_k$ admits a unique (free) factorization $d_k = f(\mathcal{C}_k) * \mathbf{F}_k$, in which $\mathbf{F}_k$ is a free category, and finally

3. For each $k \geq 0$, all degeneracies of generators of $\mathbf{F}_k$ are generators of $\mathbf{F}_{k+1}$.

Definition 5.18. A morphism $f : \mathcal{C} \to \mathcal{D}$ is called a *strong retract* of a map $f' : \mathcal{C} \to \mathcal{D}'$

if there exists a commutative diagram

$$\begin{tikzcd}
 & \mathcal{C} \arrow[ddl, "f"'] \arrow[d, "f'"] \arrow[ddr, "f"] & \\
 & \mathcal{D}' \arrow[dr] & \\
\mathcal{D} \arrow[ur] \arrow[rr, "id"'] & & \mathcal{D}
\end{tikzcd}$$

Now we are ready to present a characterization of cofibrations in $\mathcal{S}Cat_{\mathcal{O}}$

Theorem 5.19. *A morphism in $\mathcal{S}Cat_{\mathcal{O}}$ is a cofibration if and only if it is a strong retract of a free map.*

Now we have all the tools needed to prove our extension claim made earlier.

Theorem 5.20. *A functor of simplicial categories $F : \mathfrak{C}(N(\Gamma^{op})) \to (\mathbf{sSets}, \mathbf{Q})$ extends, along the cofibration $i : \mathfrak{C}(N(\Gamma^{op})) \to \mathbf{F}\mathfrak{C}(N(\Gamma^{op}))$, see* (5.2)*, to a functor of simplicial categories $\mathbf{Lan}_i(F) : \mathbf{F}\mathfrak{C}(N(\Gamma^{op})) \to (\mathbf{sSets}, \mathbf{Q})$ in such a way that the object function of the extended functor, $\mathbf{Lan}_i(F)$ is the same as the object function of the F.*

Proof. The commutative diagram (5.2) tells us that the object functor of the cofibration $i : \mathfrak{C}(N(\Gamma^{op})) \to \mathbf{F}\mathfrak{C}(N(\Gamma^{op}))$ is the identity map. The characterization of cofibrations in $\mathcal{S}Cat_{\mathcal{O}}$, see Theorem 5.19, tells us that i is a strong retract of a free map $f : \mathfrak{C}(N(\Gamma^{op})) \to \mathbf{F}(\Gamma^{op})$ in $\mathcal{S}Cat_{\mathcal{O}}$, therefore it is sufficient to extend the functor F along the free map f. Further, the object function of f is also the identity. Definition 5.17 tells us that the codomain category $\mathbf{F}(\Gamma^{op})$ is a coproduct of $\mathfrak{C}(N(\Gamma^{op}))$ with a (degreewise) free (simplicial) category which we denote by $\mathbf{F} \in \mathcal{S}Cat_{\mathcal{O}}$. Thus the functor (of simplicial categories) $F : \mathfrak{C}(N(\Gamma^{op})) \to (\mathbf{sSets}, \mathbf{Q})$ can be extended to a functor (of simplicial categories) $\mathbf{Lan}_i(F) : \mathbf{F}\mathfrak{C}(N(\Gamma^{op})) \to (\mathbf{sSets}, \mathbf{Q})$.

$\square$

Remark. Let X be an additive quasi-category, then the above strictification process can be applied to the coherently commutative monoid associated to X, namely $\underline{X}^{\oplus}_{w}$, see remark (5), to produce a Γ-space $\underline{X}^{\oplus}$.

The above remark allows us to propose a model for additive functors between additive quasi-categories

Definition 5.21. Let X and Y be two additive quasi-categories. A morphism of Γ-spaces $F : \underline{X}^{\oplus} \to \underline{Y}^{\oplus}$ will be called an *additive functor* from X to Y.

All additive functors from X to Y form a simplicial set, namely $\mathcal{M}ap_{\Gamma\mathcal{S}}(\underline{X}^{\oplus}, \underline{Y}^{\oplus})$. If X is a cofibrant object of the JQ-model category, then $\mathcal{M}ap_{\Gamma\mathcal{S}}(\underline{X}^{\oplus}, \underline{Y}^{\oplus})$ is a quasi-category.

Definition 5.22. Let X and Y be additive quasi-categories. The quasi-category $\mathcal{M}ap_{\Gamma\mathcal{S}}(Q\underline{X}^{\oplus}, \underline{Y}^{\oplus})$ will be called the *derived* ∞*-category of additive functors* from X to Y, where the Γ-space $Q\underline{X}^{\oplus}$ is a cofibrant replacement of $\underline{X}^{\oplus}$ in the JQ-model category.

Remark. There is a natural inclusion of simplicial sets

$$\mathcal{M}ap_{\Gamma\mathcal{S}}(\underline{X}^{\oplus}, \underline{Y}^{\oplus}) \hookrightarrow \mathcal{M}ap_{\Gamma\mathcal{S}}(Q\underline{X}^{\oplus}, \underline{Y}^{\oplus}),$$

which is induced by the acyclic fibration $Q\underline{X}^{\oplus} \to \underline{X}^{\oplus}$.

5.2 The mapping spaces of $\mathcal{P}ic$

For any pair of objects $(x, y) \in \mathcal{P}ic_0 \times \mathcal{P}ic_0$, we can associate with the Γ-space $\mathcal{P}ic$, another function object which is a Γ-space. We denote this function object by $Hom^{R}_{\underline{\mathcal{P}ic}^{\oplus}}(x, y)$. In degree m, this Γ-space is defined as follows:

$$Hom^{R}_{\underline{\mathcal{P}ic}^{\oplus}}(x, y)(m^+) := Hom^{R}_{\mathcal{P}ic}(x, y(m^+ \wedge -)),$$

see appendix C for the definition of the simplicial set on the right. Any morphism $f^k_m : m^+ \to k^+$, in Γ^{op} induces the following morphism

$$Hom^R_{\underline{\mathcal{P}ic}^{\oplus}}(x, f^k_m) : Hom^R_{\mathcal{P}ic}(x, y(m^+ \wedge -)) \to Hom^R_{\mathcal{P}ic}(x, y(k^+ \wedge -)).$$

Proposition 5.23. *For any pair of objects $(x, y) \in \mathcal{P}ic_0 \times \mathcal{P}ic_0$, the function object $Hom^R_{\underline{\mathcal{P}ic}^{\oplus}}(x, y)$ is a Picard groupoid.*

Proof. We begin the proof by defining another Γ-space which we denote $Sing_{Q^\bullet}\underline{Hom}_{\mathbf{D}(\Gamma\mathcal{S}^f)}(x, y)$. This Γ-space is obtained by composing the Picard groupoid $\underline{Hom}_{\mathbf{D}(\Gamma\mathcal{S}^f)}(x, y)$ with the singular simplicial set functor,

$$Sing_{Q^\bullet} : \mathbf{sSets} \to \mathbf{sSets},$$

described in [Lur09b, section 2.2.2]. Since $Sing_{Q^\bullet}$ is a right Quillen functor and $\underline{Hom}_{\mathbf{D}(\Gamma\mathcal{S}^f)}(x, y)$ is a Picard groupoid, it follows that the composite Γ-space $Sing_{Q^\bullet}\underline{Hom}_{\mathbf{D}(\Gamma\mathcal{S}^f)}(x, y)$ is a Picard groupoid. Now [Lur09b, Proposition 2.2.2.13] implies that for all $m^+ \in Ob(\Gamma^{op})$, we have the following natural isomorphism of simplicial sets

$$Sing_{Q^\bullet}\underline{Hom}_{\mathbf{D}(\Gamma\mathcal{S}^f)}(x, y)(m^+) \cong Hom^R_{\underline{\mathcal{P}ic}^{\oplus}}(x, y)(m^+)$$

Thus the Γ-space $Hom^R_{\underline{\mathcal{P}ic}^{\oplus}}(x, y)$ is isomorphic to the Γ-space $Sing_{Q^\bullet}\underline{Hom}_{\mathbf{D}(\Gamma\mathcal{S}^f)}(x, y)$ and hence it is a Picard groupoid.

$\square$

Now we define another function object for $\mathcal{P}ic$. For any pair of objects $(x, y) \in \mathcal{P}ic_0 \times \mathcal{P}ic_0$, we define

$$Hom^{\oplus}_{\mathcal{P}ic}(x, y) := Hom^R_{\mathcal{P}ic}(x, y(m^+ \wedge -)).$$

For any morphism $f^k_m : m^+ \to k^+$, in Γ^{op}, we define

$$Hom^{\oplus}_{\mathcal{P}ic}(x, f^k_m) : Hom^{\oplus}_{\mathcal{P}ic}(x, y(m^+ \wedge -)) \to Hom^{\oplus}_{\mathcal{P}ic}(x, y(k^+ \wedge -)).$$

Theorem 5.24. *The function object $Hom^{\oplus}_{\mathcal{P}ic}(x, y)$ is a Picard groupoid for any pair of objects $(x, y) \in \mathcal{P}ic_0 \times \mathcal{P}ic_0$.*

Proof. We begin by showing that $Hom^{\oplus}_{\mathcal{P}ic}(x, y)$ is a special Γ-space for any pair of objects $(x, y) \in \mathcal{P}ic_0 \times \mathcal{P}ic_0$ i.e. For any $(k + l)^+ \in Ob(\Gamma^{op})$, we want to show that the following simplicial morphism, which is induced by the projection maps $\delta^{(k+l)^+}_{k^+}$ and $\delta^{(k+l)^+}_{l^+}$,

$$Hom^{\oplus}_{\mathcal{P}ic}(x, y)((k + l)^+) \to Hom^{\oplus}_{\mathcal{P}ic}(x, y)(k^+) \times Hom^{\oplus}_{\mathcal{P}ic}(x, y)(l^+)$$

is a homotopy equivalence of (pointed) Kan complexes. We observe that for all $m^+ \in Ob(\Gamma^{op})$, the mapping space $Hom^{\oplus}_{\mathcal{P}ic}(x, y)(m^+)$ is the same as $Hom_{\mathcal{P}ic}(x, y(m^+ \wedge -))$. Now, the naturality in the definition of mapping spaces of quasicategories in appendix C provides us, for every pair of objects $k^+, l^+ \in Ob(\Gamma^{op})$, the following commutative diagram

$$\begin{tikzcd}
Hom^{\oplus}_{\mathcal{P}ic}(x, y)((k + l)^+) \arrow[r, "f"] \arrow[d] & Hom^{\oplus}_{\mathcal{P}ic}(x, y)(k^+) \times Hom^{\oplus}_{\mathcal{P}ic}(x, y)(l^+) \arrow[d] \\
Hom^{R}_{\underline{\mathcal{P}ic}^{\oplus}}(x, y)((k + l)^+) \arrow[r, "f^R"'] & Hom^{R}_{\underline{\mathcal{P}ic}^{\oplus}}(x, y)(k^+) \times Hom^{R}_{\underline{\mathcal{P}ic}^{\oplus}}(x, y)(l^+)
\end{tikzcd}$$

where the maps f and f^R are the simplicial maps

$$f = Hom^{\oplus}_{\mathcal{P}ic}(x, y)(\delta^{(k+l)^+}_{k^+}) \times Hom^{\oplus}_{\mathcal{P}ic}(x, y)(\delta^{(k+l)^+}_{l^+})$$

and

$$f^R = Hom^{R}_{\underline{\mathcal{P}ic}^{\oplus}}(x, y)(\delta^{(k+l)^+}_{k^+}) \times Hom^{R}_{\underline{\mathcal{P}ic}^{\oplus}}(x, y)(\delta^{(k+l)^+}_{l^+})$$

respectively. The vertical maps, in the above commutative diagram are the natural weak equivalences described in appendix C. Now according to the 2 out of 3 property of weak equivalences, it is sufficient to show that the map f^R is a simplicial weak equivalence. But this is an immediate consequence of the Picard groupoid structure on the function object $Hom^{R}_{\mathcal{P}ic}(x,y)$ described in proposition 5.23 above. Thus we have proved that the Γ-space $Hom^{\oplus}_{\mathcal{P}ic}(x,y)$ is special.

The second part of the proof is to show that the Γ-space $Hom^{\oplus}_{\mathcal{P}ic}(x,y)$ is very special. In order to do so, we will show that the following two commutative diagrams

$$\begin{tikzcd} Hom^{\oplus}_{\mathcal{P}ic}(x,y)(2^+) \arrow[r, "Hom^{\oplus}_{\mathcal{P}ic}(x,y)(m_2)"] \arrow[d, "Hom^{\oplus}_{\mathcal{P}ic}(x,y)(p_1)"'] & Hom^{\oplus}_{\mathcal{P}ic}(x,y)(1^+) \arrow[d] \\ Hom^{\oplus}_{\mathcal{P}ic}(x,y)(1^+) \arrow[r] & 0 \end{tikzcd} \tag{5.5}$$

and

$$\begin{tikzcd} Hom^{\oplus}_{\mathcal{P}ic}(x,y)(2^+) \arrow[r, "Hom^{\oplus}_{\mathcal{P}ic}(x,y)(m_2)"] \arrow[d, "Hom^{\oplus}_{\mathcal{P}ic}(x,y)(p_2)"'] & Hom^{\oplus}_{\mathcal{P}ic}(x,y)(1^+) \arrow[d] \\ Hom^{\oplus}_{\mathcal{P}ic}(x,y)(1^+) \arrow[r] & 0 \end{tikzcd} \tag{5.6}$$

are homotopy pullback squares in the category of simplicial sets.

□

Chapter 6

Properties of $\mathcal{P}ic$.

The category of Abelian groups, **Ab**, is an additive category. Every abelian group is an *commutative group object* in **Ab**. The next lemma describes a similar situation in the ∞-category

Lemma 6.1. *Every object of the ∞-category $\mathcal{P}ic$ defines an infinite loop object in the ∞-category $\mathcal{P}ic$.*

Proof. For each $F \in \mathcal{P}ic$, we define a morphism of ∞-categories, $\mathcal{L}_{\infty}(F) : N(\Gamma^{op}) \to \mathcal{P}ic$ as follows: For each $n^+ \in \Gamma^{op}$, we define $\mathcal{L}_{\infty}(F)(n^+) := F(n^+ \wedge -)$. It is easy to see that $\mathcal{L}_{\infty}(F)(0^+) \simeq 0$, where 0 is a terminal (hence a zero) object in $\mathcal{L}_{\infty}$. For each arrow $i : m^+ \to n^+$ in Γ^{op}, $\mathcal{L}_{\infty}(F)(i)$ is the natural transformation $F(m^+ \wedge -) \dot{\to} F(n^+ \wedge -)$, induced by i. In higher degrees, the morphism $\mathcal{L}_{\infty}(F)$ is defined by vertical composition of such natural transformations.

By the previous proposition,

$$\mathcal{L}_{\infty}(F)(n^+) \simeq \prod_{1}^{n} \mathcal{L}_{\infty}(F)(1^+).$$

It remains to show that the following two diagrams are pullback squares in $\mathcal{P}ic$

$$\begin{tikzcd} F(2^+ \wedge -) \arrow[r, "F(m\wedge -)"] \arrow[d, "F(p_1\wedge -)"'] & F \arrow[d] \\ F \arrow[r] & 0 \end{tikzcd} \qquad \begin{tikzcd} F(2^+ \wedge -) \arrow[r, "F(m\wedge -)"] \arrow[d, "F(p_2\wedge -)"'] & F \arrow[d] \\ F \arrow[r] & 0 \end{tikzcd} \tag{6.1}$$

By remark A, it is sufficient to establish that the first diagram is a pullback square. The morphism $F(m)$ in the diagram above is the natural transformation induced by the morphism $m : 2^+ \to 1^+$ in Γ^{op} which defined by $m(1) = m(2) = 1$. The morphisms $F(m)$ and $F(p_1)$ induce the following morphism of Γ-spaces

$$F(2^+ \wedge -) \overset{(F(m\wedge -),F(p_1\wedge -))}{\longrightarrow} F \times F.$$

In degree one, the above morphism specifies an equivalence of Kan complexes therefore $(F(m), F(p_1))$ is a stable Q-equivalence. This implies that the above diagram of Γ-spaces, 6.1, is a pullback square in $\mathcal{P}ic$. $\square$

It is not too difficult to check that the construction of the infinite loop object in the proof of the proposition 6.1 is functorial i.e. we have the following morphism of ∞-categories

$$\mathcal{L}_\infty(-) : \mathcal{P}ic \to \mathcal{L}_\infty(\mathcal{P}ic) \tag{6.2}$$

which is defined in degree zero by $F \mapsto \mathcal{L}_\infty(F)$. The following theorem is an open problem and will NOT be proved in this thesis.

Theorem 6.2. *The morphism 6.2 induces an equivalence of $\mathcal{P}ic$ with the ∞-category of infinite loop objects in $\mathcal{P}ic$.*

Part II

Categorification of Dijkgraaf-Witten Theory

Chapter 7

Classical Dijkgraaf-Witten Theory

R. Dijkgraaf and E. Witten in [DW90] constructed a gauge theory with a finite gauge group G as a "toy model," a tool for studying more general gauge theories with compact gauge groups. Their goal was to describe this theory, known as *DW theory*, as a Topological Quantum Field Theory (TQFT), *i.e.*, a functor on the category of 3-dimensional (3d) cobordisms to that of vector spaces, starting with an action given by a cocycle $\alpha \in Z^3(G; U(1))$. Dijkgraaf and Witten indicated that the vector space $\Phi(Y)$ corresponding to a closed oriented 2d manifold Y was closely related to the set $Hom(\pi_1(Y), G)/G$ of equivalence classes of principal G-bundles over Y and that it could be constructed by cutting the surface Y into pairs of pants, as Φ was expected to be a functor. The linear map $\Phi(X) : \partial_- X \to \partial_+ X$ corresponding to a 3d oriented cobordism X between closed manifolds $\partial_- X$ and $\partial_+ X$ depended on such choices as the choice of a map $Hom(\pi_1(X, x_0), G) \to Map(X, BG)$, the choice of a basepoint x_0, the choice of a chain, via triangulation, representing the relative fundamental cycle $[X] \in H_3(X, \partial X; \mathbb{Z})$, which was interpreted as "lattice gauge theory." One can say that, from the categorical point of view, Dijkgraaf and Witten constructed a TQFT functor on a certain subcategory of cobordisms decorated with appropriate extra structure utilized in their constructions. They used an orbifold approach to taking the homotopy quotient by G, that is to say, worked with the G-set $Hom(\pi_1(Y), G)$.

D. Freed and F. Quinn in [FQ93, Fre94] streamlined the construction of the TQFT functor Φ, so that $\Phi(X)$ would no longer depend on the choice of a representative of the fundamental cycle $[X]$ and thereby would produce a TQFT functor on the category of cobordisms. They also generalized the construction to n-dimensional cobordisms. Their main tool was to define pairings between cocycles in $Z^{n+1}(Y, U(1))$ and cycles $Z_n(Y, \mathbb{Z})$ and between $Z^{n+1}(X, U(1))$ and cycles $Z_{n+1}(X, \partial X; \mathbb{Z})$, resembling but certainly different from cap product, which would not even be defined because of dimension considerations. Freed and Quinn introduced the idea of an *invariant section* of a flat hermitian line bundle over a groupoid. This is a particular case of the idea of the limit of a functor, and in this context, is akin to taking a global section.

J. Lurie in [HL14] sketched a different construction of Dijkgraaf-Witten theory. Rather than using the orbifold $Hom(\pi_1(Y), G)/G$, he modeled the set of equivalence classes of principal G-bundles on the mapping space $Map(Y, BG)$. Given a cohomology class $\alpha \in H^{n+1}(BG; U(1))$ and a closed oriented n-manifold Y, he used a "push-pull" construction $\pi_* ev^* \alpha \in H^1(Map(Y, BG); U(1))$ for the diagram

$$\begin{array}{ccc} Y \times Map(Y, BG) & \xrightarrow{ev} & BG \\ {\scriptstyle\pi}\big\downarrow & & \\ Map(Y, BG) & & \end{array}$$

to obtain a hermitian line bundle $\mathcal{L}_Y$ over Y. Then he defined the TQFT functor Φ on objects by taking the space

$$\Phi(Y) := H^0(Map(Y, BG), \mathcal{L}_Y)$$

of global sections. He used *ambidexterity*, a natural isomorphism

$$H^0(Map(Y,BG),\mathcal{L}_Y) \xrightarrow{\sim} H_0(Map(Y,BG),\mathcal{L}_Y),$$

to produce a linear map

$$\Phi(X) : \Phi(\partial_- X) \to \Phi(\partial_+ X),$$

using push-pull again, now along the diagram

$$Map(\partial_- X, BG) \xleftarrow{p_-} Map(X,BG) \xrightarrow{p_+} Map(\partial_+ X, BG).$$

Lurie's construction deliberately avoided the following subtlety. The hermitian line bundle $\mathcal{L}_Y$ is determined by the cohomology class α only up to isomorphism. Starting with a cocycle $\alpha \in Z^{n+1}(BG;U(1))$ would partially fix the problem, because the resulting cocycle $\pi_* ev^* \alpha \in Z^1(Map(Y,BG);U(1))$ is not quite the same as a hermitian line bundle: isomorphic, but different hermitian line bundles may correspond to the same cocycle, whereas the cocycle is determined by a hermitian line bundle only up to condoundary. Moreover, the push-pull cocycle $\pi_* ev^* \alpha$ will depend on the choice of a cycle representing the fundamental class $[Y] \in H_n(Y;\mathbb{Z})$.

In the current paper, we replace the coefficient group $U(1)$ with an equivalent Picard groupoid, namely the Picard groupoid $\mathcal{L}$ of hermitian lines, and notice that an object of $H^0(M,\mathcal{L})$ is exactly a flat hermitian line bundle over M, see Section 9.

The paper [**?**] attempted the construction of an Extended Topological Quantum Field Theory (ETQFT), which is defined on cobordisms with corners, rather than boundary, and a generalization of the DW theory to the case of a compact group G. The construction utilizes the Cobordism Hypothesis, which asserts that an ETQFT is determined by its value on zero-dimensional manifolds. The two-dimensional case of the cobordism hypothesis was proved by C. J. Schommer-Pries in [SP09], and the full version was proven by Lurie in [Lur09a]. However, Freed, Hopkins, Lurie, and Teleman emphasize the importance of a direct construction, which has not been been done yet.

This paper arose from the authors' trying to find an approach to this hypothetical direct construction of an ETQFT. In the process we have realized that Freed and Quinn's pairing makes sense as a cohomological operation, cap product, if the group $H^{n+1}(Y;U(1))$ is replaced with cohomology $H^n(Y;\mathcal{L})$ with coefficients in the Picard groupoid $\mathcal{L}$ of hermitian line bundles. Categorifying the coefficients goes along with lowering the cohomological degree, thus opening a way to defining cap products as well as extending the TQFT to an ETQFT by further categorification to higher Picard groupoids and higher gerbes.

Another novel feature of our approach is that we do not use ambidexterity, but rather a transfer map in the context of cohomology with coefficients in Picard groupoids. In principle, one can view the transfer map as an avatar of ambidexterity, but it might be argued that using an avatar is less demanding than engaging the full power of a deity.

Chapter 8

The setup of DW Theory

We will consider (flat) hermitian line gerbes over simplicial sets. To deal with gerbes over manifolds and topological spaces, we will associate simplicial sets to them in a standard way: by taking singular simplices or the nerve of an open cover. Flat hermitian line gerbes are analogous to more traditional gerbes over topological spaces with the constant sheaf $U(1)$ as the band, whether given as stacks of groupoids, via gluing (descent) data, or as higher bundles, [BM05, Bry93, Moe02, Mur94]. We will take the liberty of omitting the adjective "flat" when referring to flat hermitian line bundles and gerbes.

We will describe cohomology with coefficients in Picard groupoids over simplicial sets and later apply this construction to cobordisms, which are manifolds, rather than simplicial sets. This may be done by working with the simplicial set of singular chains associated to the cobordism or by using the nerve of a sufficiently fine open covering, see examples in Section 9.

8.1 Cohomology with coefficients in Picard groupoids

A *Picard groupoid* is a symmetric monoidal groupoid in which every object is invertible, up to isomorphism, with respect to the tensor product, which, by a slight abuse of notation, we denote $+$. More precisely, for each object s of a Picard groupoid $\mathcal{A}$, the functors $t \mapsto s+t$, and $t \mapsto t+s$ define autoequivalences of $\mathcal{A}$ as a category. In this case, one can define a functor $\mathcal{A} \to \mathcal{A}$, $s \mapsto -s$, and natural isomorphisms

$$m = m_s : s + (-s) \to 0, \qquad n = n_s : (-s) + s \to 0$$

such that $l_s(m_s + id_s) = r_s(id_s + n_s)\alpha_{s,-s,s}$ for all objects s of $\mathcal{A}$, where 0 is the zero (also known as unit) object of $\mathcal{A}$ and

$$\alpha_{s,t,u} : (s+t)+u \to s+(t+u) \qquad \text{and} \tag{8.1}$$

$$l_s : 0 + s \to s, \quad r_s : s + 0 \to s \tag{8.2}$$

are the natural transformations of the monoidal structure on $\mathcal{A}$. We will assume that $-0 = 0$, $m_0 = r_0$, and $n_0 = l_0$. Another structure natural transformation is a symmetry:

$$\beta_{s,t} : s + t \to t + s,$$

making $\mathcal{A}$ to be a symmetric monoidal category. Given a Picard groupoid $\mathcal{A}$, let $\pi_0(\mathcal{A})$ denote the abelian group of its connected components and $\pi_1(\mathcal{A})$ denote the abelian group of automorphisms of the zero object.

A *homomorphism* between two Picard groupoids $\mathcal{A}$ and $\mathcal{B}$ is a functor $F : \mathcal{A} \to \mathcal{B}$ and an assignment of a *coherence morphism* which is an arrow of $\mathcal{B}$, $\phi^F_{s,t} : F(s)+F(t) \to F(s+t)$, to every pair of objects $s, t \in \mathcal{A}$ which is natural in both variables s and t such that the assignment respects the symmetry natural transformations β of $\mathcal{A}$ and $\mathcal{B}$ in the following sense:

$$F(\beta_{s,t}) \circ \phi^F_{s,t} = \phi^F_{t,s} \circ \beta_{F(s),F(t)}$$

and also respects the associativity in the following sense:

$$\phi^F_{s,t+u} \circ (id_{F(s)} + \phi^F_{t,u}) \circ \alpha^{\mathcal{B}}_{F(s),F(t),F(u)} = F(\alpha^{\mathcal{A}}_{s,t,u}) \circ \phi^F_{s+t,u} \circ (\phi^F_{s,t} + id_u),$$

for each triple of objects $s, t, u \in \mathcal{A}$ and where $\alpha^{\mathcal{A}}$ and $\alpha^{\mathcal{B}}$ are the associativity natural transformations of $\mathcal{A}$ and $\mathcal{B}$ respectively.

A homomorphism between two Picard groupoids $F : \mathcal{A} \to \mathcal{B}$ will be called a *strict homomorphism* if the coherence morphisms $\phi^F_{s,t}$ are identities for all pairs $s, t \in \mathcal{A}$ and $F(0) = 0$.

Given two homomorphisms F and $F' : \mathcal{A} \to \mathcal{B}$, a *monoidal natural transformation from* F *to* F' is a natural transformation $\theta : F \Rightarrow F'$ which is compatible with the coherence morphisms of both homomorphisms F and F' in the following sense:

$$\phi^{F'}_{s,t} \circ (\theta_s + \theta_t) = \theta_{s+t} \circ \phi^F_{s,t},$$

for all pairs $s, t \in \mathcal{A}$.

Given any two Picard groupoids $\mathcal{A}$ and $\mathcal{B}$, the category whose objects are all homomorphisms from $\mathcal{A}$ to $\mathcal{B}$ and whose morphisms are monoidal natural transformations between these homomorphisms has the structure of a Picard groupoid which we denote by $[\mathcal{A}, \mathcal{B}]$, see [**?**] for a detailed proof of this assertion. One can associate another Picard groupoid with $\mathcal{A}$ and $\mathcal{B}$ which we denote by $\mathcal{A} \otimes \mathcal{B}$, and which will be called the *tensor product*. We will not recall its construction, which is rather elaborate, see [**?**], but mention that the tensor product 2-functor is determined by an adjunction

$$[\mathcal{A}, [\mathcal{B}, \mathcal{C}]] \xrightarrow{\sim} [\mathcal{A} \otimes \mathcal{B}, \mathcal{C}]$$

in the bicategory of Picard groupoids. This bicategory also has a unit object I for the monoidal structure. The bicategory of Picard groupoids, not only has an internal hom as indicated above, but it has the structure of a $\mathbf{2}\mathcal{P}ic$-category, see appendix F for a definition of a $\mathbf{2}\mathcal{P}ic$-category. More precisely, Picard groupoids, homomorphisms between

Picard groupoids and monoidal natural transformations between homomorphisms form a $\mathbf{2}\mathcal{P}ic$-category which we denote by $\mathbf{2}\mathcal{P}ic$. Further, $\mathbf{2}\mathcal{P}ic$ is the archtype example of a $\mathbf{2}\mathcal{P}ic$-category. Our point of view on $\mathbf{2}\mathcal{P}ic$ is that it is the analog of the category of Abelian groups, **Ab**, in the world of bicategories.

The groupoid of lines, i.e., one-dimensional vector spaces, and G-torsors for a given abelian group G have natural structures of Picard groupoids with respect to tensor products and the product of torsors over G, respectively. We will later focus our attention on the Picard groupoid $\mathcal{L}$ of hermitian lines, where the hermitian form on the tensor product of hermitian lines is the tensor product of the hermitian forms one each line.

Let $X_\bullet$ be a simplicial set and $\mathcal{A}$ be a Picard groupoid. We will define cohomology $H^\bullet(X_\bullet, \mathcal{A})$ of $X_\bullet$ with values in $\mathcal{A}$, following [CMM04] and [dRMMV05]. Similar cohomology may be defined for topological spaces and, more generally, with coefficients in sheaves of Picard groupoids.

Let us associate with $X_\bullet$ and $\mathcal{A}$ a *cosimplicial Picard groupoid*, that is to say, a cosimplicial object in the category of Picard groupoids, defined as the "mapping space" $\mathcal{A}^{X_\bullet} := Map(X_\bullet, \mathcal{A})$: for each $n \geq 0$, we define the Picard groupoid $\mathcal{A}^{X_n}$ whose objects are maps $X_n \to Ob\mathcal{A}$, morphisms are maps $X_n \to Mor\mathcal{A}$, and the tensor product and morphism composition are defined "point-wise." The cosimplicial structure is comprised of homomorphisms of Picard groupoids:

$$\begin{tikzcd}[column sep=large]
\mathcal{A}^{X_0} \arrow[r, shift left=2, "d_0^*"] \arrow[r, shift right=2, "d_1^*"'] & \mathcal{A}^{X_1} \arrow[l, "s_0^*" description] \arrow[r, shift left=3, "d_0^*"] \arrow[r] \arrow[r, shift right=3, "d_2^*"'] & \mathcal{A}^{X_2} \arrow[l, shift left=1.5, "s_0^*" description] \arrow[l, shift right=1.5, "s_1^*"'] \arrow[r, shift left=4, "d_0^*"] \arrow[r, shift left=1.5] \arrow[r, shift right=1.5] \arrow[r, shift right=4, "d_3^*"'] & \cdots, \arrow[l, shift left=3, "s_0^*" description] \arrow[l] \arrow[l, shift right=3, "s_2^*"']
\end{tikzcd}$$

where the coface and codegeneracy homomorphisms $d_i^* : \mathcal{A}^{X_n} \to \mathcal{A}^{X_{n+1}}$ and $s_j^* : \mathcal{A}^{X_{n+1}} \to \mathcal{A}^{X_n}$ are obtained by composition with the face maps $d_i : X_{n+1} \to X_n$ and degeneracy maps $s_j : X_n \to X_{n+1}$ of the simplicial set $X_\bullet$, respectively.

Now, by taking alternating sums, we obtain a (*cochain*) *complex of Picard groupoids*:

$$
\begin{tikzcd}
C^\bullet(X_\bullet,\mathcal{A}):\ 0 \arrow[r] & \mathcal{A}^{X_0} \arrow[r, "d"] \arrow[rr, bend left, "0"] & \mathcal{A}^{X_1} \arrow[r, "d"] \arrow[rr, bend right, "0"'] \arrow[d, phantom, "\chi\Uparrow" near start, shift left=0] & \mathcal{A}^{X_2} \arrow[r, "d"] \arrow[rr, bend left, "0"] \arrow[d, phantom, "\Downarrow\chi"] & \mathcal{A}^{X_3} \arrow[r, "d"] & \dots
\end{tikzcd}
$$

with $d = \sum_{i=0}^{n+1}(-1)^i d_i^* : \mathcal{A}^{X_n} \to \mathcal{A}^{X_{n+1}}$ and a monoidal transformation $\chi : d^2 \Rightarrow 0$, obtained in a unique way from the structure isomorphisms α, m and n. This system of coboundary homomorphisms d and monoidal transformations χ is *coherent*, i.e., $\chi d = d\chi$ as 2-cells $d^3 \Rightarrow 0$. Let Kom($\mathbf{2}\mathcal{P}ic$) denote the $\mathbf{2}\mathcal{P}ic$-category of complexes of Picard groupoids. The objects of Kom($\mathbf{2}\mathcal{P}ic$) are complexes of Picard groupoids. A 1-morphism between $\mathcal{A}^\bullet, \mathcal{B}^\bullet \in Ob(\text{Kom}(\mathbf{2}\mathcal{P}ic))$, pictured below:

$$
\begin{tikzcd}
\dots\mathcal{A}^{n-1} \arrow[r, "d_\mathcal{A}"] \arrow[d, "f^{n-1}"'] \arrow[rr, bend left, "0"] \arrow[dr, phantom, "\phi^n\swarrow"] & \mathcal{A}^n \arrow[r, "d_\mathcal{A}"] \arrow[d, "f^n"] \arrow[dr, phantom, "\swarrow\phi^{n+1}"] & \mathcal{A}^{n+1}\dots \arrow[d, "f^{n+1}"] \\
\dots\mathcal{B}^{n-1} \arrow[r, "d_\mathcal{B}"'] \arrow[rr, bend right, "0"'] & \mathcal{B}^n \arrow[r, "d_\mathcal{B}"'] & \mathcal{B}^{n+1}\dots
\end{tikzcd},
$$

is a pair $F = (f, \phi)$, where f is a sequence of homomorphisms $f^n : \mathcal{A}^n \to \mathcal{B}^n$ and ϕ is a sequence of monoidal natural transformations $\phi^n : f^n d_\mathcal{A} \Rightarrow d_\mathcal{B} f^{n-1}$ in $\mathbf{2}\mathcal{P}ic$, satisfying the following coherence conditions $\phi^{n+1}d_\mathcal{A} = d_\mathcal{B}\phi^n$ and $(f^{n+1}\chi_\mathcal{A}) \circ (\phi^{n+1}d_\mathcal{A}) \circ (d_\mathcal{B}\phi^n) = \chi_\mathcal{B} f^{n-1}$. A 2-morphism $(f,\phi) \Rightarrow (f',\phi')$ is a sequence $\{\gamma_n\}_{n\in\mathbb{Z}}$, where $\gamma_n : f^n \Rightarrow f'^n$ is a monoidal natural transformation, for all $n \in \mathbb{Z}$, and the following coherence condition is satisfied: $(\gamma_{n+1}d_\mathcal{A}) \circ \phi_n = \phi'_n \circ (d_\mathcal{B}\gamma_n)$. It would be useful to describe an alternative, equivalent, notion of a 2-morphism in Kom($\mathbf{2}\mathcal{P}ic$) which is a generalization of *cochain homotopy* to the Picard groupoid context. In this notion, a 2-morphism is also a pair $H = (h, \psi)$, where $h^n : \mathcal{A}^n \to \mathcal{B}^{n-1}$ and ψ is a sequence of monoidal natural transformations $\psi^n : d_{\mathcal{B}} h^n + f^n \Rightarrow f'^n + h^{n+1} d_{\mathcal{A}}$ satisfying an obvious coherence condition. We leave the establishment of an equivalence between the two notions of 2-morphisms in $\mathrm{Kom}(\mathbf{2}\mathcal{P}ic)$ as an excercise for an interested reader.

The cohomology $H^\bullet(X_\bullet, \mathcal{A})$ of $X_\bullet$ with coefficients in a Picard groupoid $\mathcal{A}$ is defined as the cohomology of the complex $(\mathcal{A}^{X_\bullet}, d, \chi)$ of Picard groupoids. The cohomology of a complex of Picard groupoids may be defined as follows. In principle, to define the nth cohomology $H^n(X_\bullet, \mathcal{A})$, we want to take the kernel $Kerd$ of the homomorphism $d : \mathcal{A}^{X_n} \to \mathcal{A}^{X_{n+1}}$ and then the cokernel of the homomorphism $d' : \mathcal{A}^{X_{n-1}} \to Kerd$ induced by $d : \mathcal{A}^{X_{n-1}} \to \mathcal{A}^{X_n}$, but these need to be defined in a suitable categorified sense. In particular, the kernels, cokernels, and cohomology will depend on two subsequent coboundary homomorphisms d as well as χ and be Picard groupoids. The objects of the category $Ker(d, \chi)$ (of *n-cocycles*) are pairs (a, ϕ) in which a is an object of $\mathcal{A}^{X_n}$ and $\phi : da \to 0$ is a morphism in $\mathcal{A}^{X_{n+1}}$ satisfying a *cocycle condition*:

$$d(\phi) = \chi_a : d^2(a) \to 0.$$

A morphism $(a, \phi) \to (a', \phi')$ in $Ker(d, \chi)$ is given by a morphism $f : a \to a'$ in $\mathcal{A}^{X_n}$ such that $\phi' \circ d(f) = \phi$. The monoidal structure on $Ker(d, \chi)$ is inherited from that of $\mathcal{A}$. The kernel $Ker(d, \chi)$ naturally participates in a complex of Picard groupoids, as follows:

$$\begin{tikzcd} \mathcal{A}^{X_{n-2}} \arrow[r, "d"] \arrow[rr, bend left, "0"] & \mathcal{A}^{X_{n-1}} \arrow[r, "d'"] \arrow[u, phantom, "\chi'\Uparrow"] & Ker(d, \chi) \end{tikzcd} .$$

The cohomology $H^n(X_\bullet, \mathcal{A})$ is defined as the cokernel $Coker(d', \chi')$ in this complex. The cokernel $Coker(d', \chi')$ is a Picard category whose objects are the same as those of $Ker(d, \chi)$, i.e., of the type (a, ϕ), where a is an object of $\mathcal{A}^{X_n}$ and $\phi : da \to 0$ is a morphism in $\mathcal{A}^{X_{n+1}}$ satisfying the cocycle condition above. A morphism $(a,\phi) \to (a',\phi')$ in $Coker(d',\chi')$ is given by an equivalence class of pairs (b,f), where b is an object of $\mathcal{A}^{X_{n-1}}$ and $f : (a,\phi) \to (d'b + a', \chi_b + \phi')$ is a morphism in $Ker(d,\chi)$. Two morphisms (b,f) and $(b',f') : (a,\phi) \to (a',\phi')$ are *equivalent*, if there is a pair (c,g) with c being an object of $\mathcal{A}^{X_{n-2}}$ and $g : b \to dc + b'$ a morphism in $\mathcal{A}^{X_{n-1}}$ such that the following diagram commutes:

$$\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, "f'"'] & d'b + a' \arrow[r, "d'(g)+id"] & (d'dc + d'b') + a' \arrow[d, "\alpha"] \\
d'b' + a' & 0 + (d'b' + a') \arrow[l, "l"] & dd'c + (d'b' + a'). \arrow[l, "\chi'_c+id+id"]
\end{tikzcd}$$

One can check that $\pi_0(H^n(X_\bullet,\mathcal{A})) \cong \pi_1(H^{n+1}(X_\bullet,\mathcal{A}))$.

The *simplicial homology of a simplicial set* $X_\bullet$ *with coefficients in a Picard groupoid* $\mathcal{A}$ may be defined similarly by looking at the *simplicial Picard groupoid* $\mathcal{A}X_\bullet$ whose n-simplices are formal "linear combinations" $a_1s_1 + \cdots + a_ks_k$ of pairwise distinct elements $s_1,\dots,s_k$ in X_n with coefficients $a_1,\dots,a_k$ in $\mathcal{A}$. Perhaps, a better way of looking at $\mathcal{A}X_\bullet$ is to view it as $\mathcal{A}$-valued functions on $X_\bullet$ with finite support and apply the same treatment to it as that for $\mathcal{A}^{X_\bullet}$. In particular, summing up the face homomorphisms gives rise to a *chain complex of Picard groupoids* $C_\bullet(X_\bullet,\mathcal{A})$, which determines the homology Picard groupoids $H_n(X_\bullet,\mathcal{A})$ for $n \geq 0$.

When A is an abelian group, we will think of it as a *discrete* Picard groupoid, denoted $A[0]$, with A being the set of objects and identities being the only morphisms, so as $\pi_0(A[0]) = A$ and $\pi_1(A[0]) = 0$. Then the (co)homology with coefficients in the Picard groupoid $A[0]$ will be related to the usual simplicial (co)homology with coefficients in the group A as follows:

$$\begin{aligned}
\pi_0 H^\bullet(X_\bullet; A[0]) &= H^\bullet(X_\bullet; A), \\
\pi_0 H_\bullet(X_\bullet; A[0]) &= H_\bullet(X_\bullet; A).
\end{aligned}$$

8.2 Relative cohomology

Let $\mathcal{A} \in \mathbf{2}\mathcal{P}ic$, let $X_\bullet$ be a simplicial set, let $Y_\bullet \subset X_\bullet$ be a simplicial subset. There is an inclusion map $Y_\bullet \hookrightarrow X_\bullet$ in that category of simplicial sets. This inclusion induces a 1-morphism

$$i_\bullet : C_\bullet(Y_\bullet, \mathcal{A}) \hookrightarrow C_\bullet(X_\bullet, \mathcal{A})$$

in $\mathrm{Kom}(\mathbf{2}\mathcal{P}ic)$. We define relative homology $H_\bullet(X_\bullet, Y_\bullet, \mathcal{A})$ to be the homology of the 2-chain complex given by the cokernel of $i_\bullet$ in $\mathrm{Kom}(\mathbf{2}\mathcal{P}ic)$. We call this 2-chain complex, given by the cokernel, a *relative 2-chain complex* so $H_\bullet(X_\bullet, Y_\bullet, \mathcal{A})$ is the homology of the relative 2-chain complex $C_\bullet(X_\bullet, Y_\bullet, \mathcal{A})$. The *nth.* degree of the relative 2-chain complex is the Picard groupoid given by the cokernel, in the category of Picard groupoids, of the map $i_n : C_n(Y_\bullet, \mathcal{A}) \hookrightarrow C_n(X_\bullet, \mathcal{A})$. Relative cohomology is defined similarly, $H^\bullet(X_\bullet, Y_\bullet, \mathcal{A})$ is the cohomology of the relative 2-cochain complex given by the cokernel of the following map, induced by the inclusion $Y_\bullet \hookrightarrow X_\bullet$

$$i^\bullet : C^\bullet(Y_\bullet, \mathcal{A}) \to C^\bullet(X_\bullet, \mathcal{A}).$$

The objects of $i^\bullet(C^n(Y_\bullet, \mathcal{A}))$ are those functions, $X_n \to Ob(\mathcal{A})$, which vanish outside of Y_n. $C^n(X_\bullet, Y_\bullet, \mathcal{A})$ is a Picard subgroupoid of $C^n(X_\bullet, \mathcal{A})$ whose objects are the same as those of $C^n(X_\bullet, \mathcal{A})$. A morphisms in $C^n(X_\bullet, Y_\bullet, \mathcal{A})$ is a certain equivalence class of morphisms in $C^n(X_\bullet, \mathcal{A})$. The cokernel also gives a 1-morphism, in $\mathrm{Kom}(\mathbf{2}\mathcal{P}ic)$, $p^\bullet : C^\bullet(X_\bullet, \mathcal{A}) \to C^\bullet(X_\bullet, Y_\bullet, \mathcal{A})$ and a 2-morphism $\phi^\bullet : p^\bullet \circ i^\bullet \Rightarrow 0 : C^\bullet(Y_\bullet, \mathcal{A}) \to C^\bullet(X_\bullet, Y_\bullet, \mathcal{A})$, where 0 is the zero homomorphism. If $\alpha \in Ob(C^n(Y_\bullet, \mathcal{A}))$ then the natural transformation ϕ^n assigns to α, a morphism $i(\alpha) \to 0$ in $C^n(X_\bullet, Y_\bullet, \mathcal{A})$. In other words those objects of $C^n(Y_\bullet, \mathcal{A})$ are isomorphic to the zero object in $C^n(X_\bullet, Y_\bullet, \mathcal{A})$.

8.3 Functoriality

The nth cohomology (and nth homology) defined above is a functor of $\mathbf{2}\mathcal{P}ic$-categories, see appendix F, $H^n : \mathrm{Kom}(\mathbf{2}\mathcal{P}ic) \to \mathbf{2}\mathcal{P}ic$. Moreover, every $F \in Mor_{\mathrm{Kom}(\mathbf{2}\mathcal{P}ic)}(\mathcal{A}^\bullet, \mathcal{B}^\bullet)$ determines a morphism $H^\bullet(F) \in Mor_{\mathrm{Kom}(\mathbf{2}\mathcal{P}ic)}(H^\bullet(\mathcal{A}^\bullet),$
$H^\bullet(\mathcal{B}^\bullet))$ on cohomology. This fact follows from properties of relative kernels and cokernels; for a direct proof of this fact see [dRMMV05].

Note that the described cohomology and homology are (strictly) functorial with respect to simplicial maps. If $\mathcal{A}$ is Picard groupoind and $f : X_\bullet \to Y_\bullet$ is a simplicial map, then we get a strict morphism between the corresponding cochain complexes of Picard groupoids $f^* : C^\bullet(Y_\bullet, \mathcal{A}) \to C^\bullet(X_\bullet, \mathcal{A})$, which yields a strict morphism on cohomology $f^* : H^n(Y_\bullet, \mathcal{A}) \to H^n(X_\bullet, \mathcal{A})$ for $n \geq 0$. Moreover, a simplicial homotopy between two simplicial maps induces a monoidal natural transformation on cohomology, *cf.* [BCC93, Proposition 2.1] and [CMM04, Proposition 2.3(i)] and the discussion of 2-morphisms in $\mathrm{Kom}(\mathbf{2}\mathcal{P}ic)$ in Section 8.1. The same statements are true for homology.

8.4 The long 2-exact sequence

We begin this subsection by recalling the notion of a short 2-exact sequence of Picard groupoids. Here we will only recall this notion in a subcategory of $\mathbf{2}\mathcal{P}ic$ which has the same objects as $\mathbf{2}\mathcal{P}ic$ and whose morphisms are homomorphisms which preserve the unit of addition. For the general case see [BV02, **?**]. A complex

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{A} \arrow[r, "F"'] \arrow[rr, bend left, "0"] & \mathcal{C} \arrow[r, "G"'] \arrow[phantom, u, "\phi\Uparrow"] & \mathcal{B} \arrow[r] & 0 \end{tikzcd}$$

is called a *short 2-exact sequence of Picard groupoids* if the unique morphism $\overline{G}$: $Coker(F, id_0) \to \mathcal{B}$ is full and faithful and further, $\pi_1(Ker(F, \phi)) = 0$ and $\pi_0(Coker(G, \phi)) =$

0.

Definition 8.1. A 2-*exact sequence of complexes of Picard groupoids* is a diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{A}^\bullet \arrow[r, "F^\bullet"'] \arrow[rr, bend left, "0"] & \mathcal{B}^\bullet \arrow[u, phantom, "\phi^\bullet \Uparrow"] \arrow[r, "G^\bullet"'] & \mathcal{C}^\bullet \arrow[r] & 0, \end{tikzcd} \tag{8.3}$$

where $F^\bullet$ and $G^\bullet$ are 1-morphisms and $\phi^\bullet$ is a 2-morphism in Kom($\mathbf{2}\mathcal{P}ic$), such that in every degree, the above diagram in Kom($\mathbf{2}\mathcal{P}ic$), reduces to a short 2-exact sequence of Picard groupoids.

The following example of a short 2-exact sequence is of particular interest and would be referenced frequently.

Example 8.2. Let $X_\bullet$ be a simplicial set and let $Y_\bullet \subset X_\bullet$ be a simplicial subset. Then for any Picard groupoid $\mathcal{A}$, there is a morphism $i : \mathcal{C}^\bullet(Y_\bullet; \mathcal{A}) \to \mathcal{C}^\bullet(X_\bullet; \mathcal{A})$ of (cochain) complexes of Picard groupoids. This morphism determines a short 2-exact sequence of complexes of Picard groupoids:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{C}^\bullet(X_\bullet, Y_\bullet; \mathcal{A}) \arrow[r] \arrow[rr, bend left, "0"] & \mathcal{C}^\bullet(X_\bullet; \mathcal{A}) \arrow[u, phantom, "\pi \Uparrow"] \arrow[r, "i"'] & \mathcal{C}^\bullet(Y_\bullet; \mathcal{A}) \arrow[r] & 0. \end{tikzcd} \tag{8.4}$$

The inclusion of simplicial sets induces another morphism of (chain) complexes of Picard groupoids, $i : \mathcal{C}_\bullet(Y_\bullet; \mathcal{A}) \to \mathcal{C}_\bullet(X_\bullet; \mathcal{A})$. This morphism determines a short 2-exact sequence of (chain) complexes of Picard groupoids:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{C}_\bullet(Y_\bullet; \mathcal{A}) \arrow[r, "i"'] \arrow[rr, bend left, "0"] & \mathcal{C}_\bullet(X_\bullet; \mathcal{A}) \arrow[u, phantom, "\pi \Uparrow"] \arrow[r] & \mathcal{C}_\bullet(X_\bullet, Y_\bullet; \mathcal{A}) \arrow[r] & 0. \end{tikzcd} \tag{8.5}$$

A short 2-exact sequence of complexes (8.3) has an associated *long* 2-*exact sequence* of cohomology

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[r] & H^n(\mathcal{A}^\bullet) \arrow[r, "H^n(F)"'] \arrow[rr, bend left=40, "0"] \arrow[rr, phantom, bend left=20, "H^n(\phi)\Uparrow"] & H^n(\mathcal{B}^\bullet) \arrow[r, "H^n(G)"] \arrow[rr, bend right=40, "0"'] \arrow[rr, phantom, bend right=20, "\Downarrow\Sigma^n"] & H^n(\mathcal{C}^\bullet) \arrow[r, "\partial^n"] \arrow[rr, bend left=40, "0"] \arrow[rr, phantom, bend left=20, "\Psi^n\Uparrow"] & H^{n+1}(\mathcal{A}^\bullet) \arrow[r, "H^{n+1}(F)"'] & H^{n+1}(\mathcal{B}^\bullet) \arrow[r] & \cdots
\end{tikzcd} \tag{8.6}$$

We will briefly outline the construction of the 1-morphism ∂^n and the 2-morphism Ψ^n here. For a more elaborate description of the various components of this long exact sequence, we refer the interested reader to Section 4 of [dRMMV05]. For an outline we will refer to the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{A}^n \arrow[r, "f^n"] \arrow[d, "d^n_{\mathcal{A}}"'] \arrow[rr, bend left=30, "0"] \arrow[rr, phantom, bend left=15, "\phi^n\Uparrow"] \arrow[dr, phantom, "\lambda^n\,\Nearrow"] & \mathcal{B}^n \arrow[r, "g^n"] \arrow[d, "d^n_{\mathcal{B}}"'] \arrow[dr, phantom, "\Nearrow\,\mu^n"] & \mathcal{C}^n \arrow[r] \arrow[d, "d^n_{\mathcal{C}}"] & \cdots \\
\cdots \arrow[r] & \mathcal{A}^{n+1} \arrow[r, "f^{n+1}"'] \arrow[rr, bend right=30, "0"'] \arrow[rr, phantom, bend right=15, "\phi^{n+1}\Downarrow"] & \mathcal{B}^{n+1} \arrow[r, "g^{n+1}"'] & \mathcal{C}^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

Let $(C_n, c_n : d^n_{\mathcal{C}}(C_n) \to 0)$ be an object in $Ker(d^n_{\mathcal{C}})$; since $\pi_1(Coker(g_n, \phi^n)) = 0$, there is a $B_n \in \mathcal{B}^n$ and $i : g^n(B_n) \to C_n$. Since the following pair

$$(d^n_{\mathcal{B}}(B_n), c_n \circ d^n_{\mathcal{C}}(i) \circ \mu_n(B_n) : g^{n+1}(d^n_{\mathcal{B}}(B_n)) \to d^n_{\mathcal{C}}(g^n(B_n)) \to d^n_{\mathcal{C}}(C_n) \to 0)$$

is an object of $Ker(g^{n+1})$ and the factorization of $f^{n+1} : \mathcal{A}^{n+1} \to \mathcal{B}^{n+1}$ through $Ker(g^{n+1})$ is an equivalence, there are $A_{n+1} \in \mathcal{A}^{n+1}$ and $j : f^{n+1}(A_{n+1}) \to d^n_{\mathcal{B}}(B_n)$ such that

$$g^{n+1}(j) \circ c_n \circ d^n_{\mathcal{C}}(i) \circ \mu_n(B_n) = \phi^{n+1}(A_{n+1}).$$

We now need an arrow $a_{n+1} : d_{\mathcal{A}}^{n+1} \to 0$. Since the factorization f'^{n+2} of f^{n+2} through $Ker(g^{n+2})$ is an equivalence of categories, it is enough to find an arrow $f'^{n+2}(d_{\mathcal{A}}^{n+1}(A_{n+1})) \to f'^{n+2}(0)$. This is given by the following:

$$\chi_{\mathcal{B}}(B_n) \circ d_{\mathcal{B}}^{n+1}(j) \circ \lambda^{n+1}(A_{n+1}) : f^{n+2}(d_{\mathcal{A}}^{n+1}(A_{n+1})) \to 0 \cong f^{n+2}(0).$$

We put $\partial^n(C_n, c_n) := (A_{n+1}, a_{n+1})$. This is an object of $H^{n+1}\mathcal{A}^\bullet$: the condition $d_{\mathcal{A}}^{n+1}(a_{n+1}) = \chi_{\mathcal{A}}(A_{n+1})$ can be easily checked by applying the faithful functor f^{n+3}. The arrow function of the functor ∂^n has a much more elaborate description and moreover it is not used in the construction of our theory. We will refer the interested reader to [dRMMV05].

Before we can describe a construction of the 2-morphism Ψ^n, we need another description of $H^n(\mathcal{C}^\bullet)$. Since (f^n, ϕ^n, g^n) is a 2-short exact sequence, $\mathcal{C}^n$ is equivalent to the cokernel of f^n, we get the following alternative description of $H^n(\mathcal{C}^\bullet)$. An object is a pair

$$(B_n \in \mathcal{B}^n, [A_{n+1} \in \mathcal{A}_{n+1}, a_{n+1} : d_{\mathcal{B}}^n(B_n) \to f^{n+1}(A_{n+1})]),$$

where $[A_{n+1}, a_{n+1}] \in Mor_{Coker(f^{n+1}, id_0)}(d_{\mathcal{B}}^n(B_n), 0)$, such that there exists an arrow $t^{n+2} : d_{\mathcal{A}}^n(A_{n+1}) \to 0$ making the following diagram commutative

$$\begin{array}{ccc} d_{\mathcal{B}}^{n+1}(d_{\mathcal{B}}^n(B_n)) & \xrightarrow{d_{\mathcal{B}}^{n+1}(a_{n+1})} & d_{\mathcal{B}}^{n+1}(f^{n+1}(A_{n+1})) \\ {\scriptstyle \chi_{\mathcal{B}}(B_n)}\Big\downarrow & & \Big\downarrow{\scriptstyle (\lambda^{n+1})^{-1}(A_{n+1})} \\ 0 & \xleftarrow[f^{n+2}(t^{n+2})]{} & f^{n+2}(d_{\mathcal{A}}^{n+1}(A_{n+1})). \end{array}$$

Note that t^{n+2} is necessarily unique because f^{n+2} is faithful. Now we begin the construction of Ψ^n, given an object

$$(B_n \in \mathcal{B}^n, [A_{n+1} \in \mathcal{A}^{n+1}, a_{n+1} : d_{\mathcal{B}}^n(B_n) \to f^{n+1}(A_{n+1})]),$$

in $H^n(C^\bullet)$, we apply ∂^n and $H^{n+1}(f)$ and obtain the following object of $H^{n+1}(\mathcal{B}^\bullet)$:

$$(f^{n+1}(A_{n+1}), f^{n+2}(t^{n+2}) \circ \lambda_{n+1}^{-1}(A_{n+1}) : d_{\mathcal{B}}^{n+1}(f^{n+1}(A_{n+1})) \to f^{n+2}(d_{\mathcal{A}}^{n+1}(A_{n+1})) \to 0).$$

This object is naturally isomorphic to the unit of the addition $0 \in H^{n+1}(\mathcal{B}^\bullet)$ via the following morphism which we take as the definition of Ψ^n on the object $(B_n, [A_{n+1}, a_{n+1}])$

$$\Psi^n(B_n, [A_{n+1}, a_{n+1}]) := [B_n \in \mathcal{B}^n, a_{n+1}^{-1} : f^{n+1}(A_{n+1}) \to d_{\mathcal{B}}^n(B_n)].$$

The following example describes the images under the morphism ∂^n and the natural transformation Ψ^n of an object in degree n of the cohomology sequence associated to the 2-short exact sequence (8.5).

Example 8.3. Let X be a compact finite-dimensional manifold with boundary ∂X. We denote by $H_n(X, \partial X; \mathcal{L})$ the nth homology Picard groupoid of the chain complex $C_\bullet(X_\bullet, \partial X_\bullet; \mathcal{L})$. Let $(X_n, [X'_{n-1}, x'_{n-1}]) \in Ob\ H_n(X, \partial X; \mathcal{L})$, where $X_n \in Ob\ C_n(X_\bullet; \mathcal{L})$ and the morphism $[X'_{n-1}, x'_{n-1}] : d(X_n) \to 0$ in $Mor\ C_{n-1}(X_\bullet; \partial X_\bullet; \mathcal{L})$ consists of an object $X'_{n-1} \in C_{n-1}(\partial X_\bullet; \mathcal{L})$ and a morphism $x'_{n-1} : d^n(X_n) \to X'_n \in Mor\ C_{n-1}(X_\bullet; \mathcal{L})$. The coboundary $\partial_n(X_n, [X'_{n-1}, x'_{n-1}])$ is the pair $(X'_{n-1}, x'_{n-1}) \in Ob\ H_{n-1}(\partial X; \mathcal{L})$. The natural transformation $\Psi_n(X_n, [X'_{n-1}, x'_{n-1}])$ is a morphism in $Hom_{H_{n-1}(X;\mathcal{L})}((X'_{n-1}, x'_{n-1}), 0)$ given by the equivalence class $[X_n, (x'_{n-1})^{-1}]$. Thus every object in $H_n(X, \partial X; \mathcal{L})$ produces a morphism in $H_{n-1}(X; \mathcal{L})$.

8.5 The Cap Product

In this section we develop a cap product between cohomology with coefficients in a Picard groupoid and homology with coefficients in the Picard groupoid $\mathbb{Z}[0]$:

$$\cap : H_\bullet(X_\bullet, \mathbb{Z}[0]) \otimes H^\bullet(X_\bullet, \mathcal{A}) \to H_\bullet(X_\bullet, \mathcal{A}).$$

In order to do that, we will define a chain map i.e a morphism in Kom($\mathbf{2}\mathcal{P}ic$)

$$H_\bullet(X_\bullet, \mathbb{Z}[0]) \to [H^\bullet(X_\bullet, \mathcal{A}), H_\bullet(X_\bullet, \mathcal{A})].$$

We start by defining the following chain map

$$\cap^{ch} : C_\bullet(X_\bullet, \mathbb{Z}[0]) \to [C^\bullet(X_\bullet, \mathcal{A}), C_\bullet(X_\bullet, \mathcal{A})] \tag{8.7}$$

where the right hand side is the chain complex $[C^\bullet(X_\bullet, \mathcal{A}), C_\bullet(X_\bullet, \mathcal{A})]_\bullet$ defined in Appendix 2. We define the map in degree p as follows: On objects the chain map is given by

$$\sigma_q \mapsto \prod_{q \geq p} F_q,$$

where $F_q \in Mor(\mathbf{2}\mathcal{P}ic)$ is defined on objects by

$$F_q : \alpha \mapsto \alpha(d_f^p(\sigma_q))d_l^{q-p}(\sigma_q),$$

where d_f^p and d_l^{q-p} are the restrictions of $d : C_{q+1}(X_\bullet, \mathbb{Z}[0]) \to C_q(X_\bullet, \mathbb{Z}[0])$ to the simplex determined by the first $p+1$ vertices and the last $q-p+1$ vertices of σ_q respectively. On morphisms, F_q given by

$$F_q : \{\{\alpha \to \beta\} \mapsto \{\alpha(d_f^p(\sigma_q))d_l^{q-p}(\sigma_q) \to \beta(d_f^p(\sigma_q))d_l^{q-p}(\sigma_q)\}$$

The map on the right side is determined by the natural transformation $\alpha \to \beta$. This chain map induces a map on homology

$$H_\bullet(X_\bullet, \mathbb{Z}[0]) \to H_\bullet([C^\bullet(X_\bullet, \mathcal{A}), C_\bullet(X_\bullet, \mathcal{A})]). \tag{8.8}$$

Composition with the following obvious morphism gives us the desired chain map

$$H_\bullet([C^\bullet(X_\bullet, \mathcal{A}), C_\bullet(X_\bullet, \mathcal{A})]) \to [H^\bullet(X_\bullet, \mathcal{A}), H_\bullet(X_\bullet, \mathcal{A})]. \tag{8.9}$$

8.6 Relative cap product

We now construct a relative version of the cap product. The 2-functor $[C^\bullet(X_\bullet; \mathcal{A}), -] : \mathrm{Kom}(\mathbf{2}\mathcal{P}ic) \to \mathrm{Kom}(\mathbf{2}\mathcal{P}ic)$ and the chain map (8.7), determine a composite 1-morphism and a 2-morphism $\phi_\bullet$ in Kom($\mathbf{2}\mathcal{P}ic$)

$$\begin{tikzcd}
C_\bullet(Y_\bullet;\mathbb{Z}[0]) \arrow[r, "i_\bullet"] \arrow[rr, bend left, "0"] & C_\bullet(X_\bullet;\mathbb{Z}[0]) \arrow[r, "\cap^{ch}_{rel}"] \arrow[u, phantom, "\phi_\bullet \Uparrow"] & {[C^\bullet(X_\bullet;\mathcal{A}), C_\bullet(X_\bullet, Y_\bullet;\mathcal{A})]}
\end{tikzcd}$$

In order to define the 1-morphism $\cap^{ch}_{rel}$ and the 2-morphism $\phi_\bullet$ in the above diagram, we need to define a restriction of the chain map 8.7. The image of the restriction of this chain map to $i_\bullet(C_\bullet(Y_\bullet;\mathbb{Z}[0]))$ is contained in the 2-(chain) complex $[C^\bullet(X_\bullet;\mathcal{A}), C_\bullet(Y_\bullet;\mathcal{A})]$, this determines the following commutative diagram

$$\begin{tikzcd}
i_\bullet(C_\bullet(Y_\bullet;\mathbb{Z}[0])) \arrow[r, "\cap^{ch}|_{i_\bullet(C_\bullet(Y_\bullet;\mathbb{Z}[0]))}"] \arrow[dr, "\cap^{ch}_{Y_\bullet}"'] & {[C^\bullet(X_\bullet;\mathcal{A}), C_\bullet(X_\bullet;\mathcal{A})]} \\
& {[C^\bullet(X_\bullet;\mathcal{A}), C_\bullet(Y_\bullet;\mathcal{A})]} \arrow[u, "{[C^\bullet(X_\bullet;\mathcal{A}), i_\bullet]}"']
\end{tikzcd}$$

The following composite chain map will be called the *restricted relative cap product chain map*.

$$\cap^{ch}_{res} : C_\bullet(Y_\bullet;\mathbb{Z}[0]) \xrightarrow{i_\bullet} i_\bullet(C_\bullet(Y_\bullet;\mathbb{Z}[0])) \xrightarrow{\cap^{ch}_{Y_\bullet}} [C^\bullet(X_\bullet;\mathcal{A}), C_\bullet(Y_\bullet;\mathcal{A})] \tag{8.10}$$

The 2-morphism $\phi_\bullet$ is the composition $\cap^{ch}_{res} \circ ([C^\bullet(X_\bullet;\mathcal{A}), \pi^{\mathcal{A}}_\bullet)])$ as described in the following diagram

$$\begin{tikzcd}
C_\bullet(Y_\bullet;\mathbb{Z}[0]) \arrow[r, "i_\bullet"] \arrow[rr, bend left, "0"] \arrow[d, "\cap^{ch}_{res}"'] & C_\bullet(X_\bullet;\mathbb{Z}[0]) \arrow[r, "p_\bullet"] \arrow[d, "\cap^{ch}"'] \arrow[dr, "\cap^{ch}_{rel}"'] \arrow[u, phantom, "\pi_\bullet \Uparrow"] & C_\bullet(X_\bullet, Y_\bullet;\mathbb{Z}[0]) \arrow[d, dashed, "u"] \arrow[dl, phantom, "\Leftarrow \lambda_\bullet"] \\
{[C^\bullet(X_\bullet;\mathcal{A}), C_\bullet(Y_\bullet;\mathcal{A}]} \arrow[r] \arrow[rr, bend right, "0"'] & {[C^\bullet(X_\bullet;\mathcal{A}), C_\bullet(X_\bullet;\mathcal{A})]} \arrow[r] \arrow[d, phantom, "\Downarrow {[C^\bullet(X_\bullet;\mathcal{A}), \pi^{\mathcal{A}}_\bullet)]}"] & {[C^\bullet(X_\bullet;\mathcal{A}), C_\bullet(X_\bullet, Y_\bullet;\mathcal{A})]} \\
& {} &
\end{tikzcd}$$

where $\pi_\bullet^{\mathcal{A}}$ is the following 2-morphism

$$\begin{tikzcd}
C_\bullet(Y_\bullet;\mathcal{A}) \arrow[r, "i_\bullet"] \arrow[rr, bend left, "0"] & C_\bullet(X_\bullet;\mathcal{A}) \arrow[u, phantom, "\pi_\bullet^{\mathcal{A}}\Uparrow"] \arrow[r, "p_\bullet^{\mathcal{A}}"'] & C_\bullet(X_\bullet,Y_\bullet;\mathcal{A})
\end{tikzcd}$$

The universality of the cokernel determines a unique pair consisting of a 1-morphism in $\mathrm{Kom}(\mathbf{2}\mathcal{P}ic)$

$$u : C_\bullet(X_\bullet, Y_\bullet; \mathbb{Z}[0]) \to [C^\bullet(X_\bullet;\mathcal{A}), C_\bullet(X_\bullet, Y_\bullet;\mathcal{A})],$$

and a 2-morphism in $\mathrm{Kom}(\mathbf{2}\mathcal{P}ic)$, $\lambda_\bullet : u \circ p_\bullet \Rightarrow \cap^{ch}_{rel}$ such that the following diagram commutes

$$\begin{tikzcd}
u \circ p_\bullet \circ i_\bullet \arrow[r, Rightarrow, "u\cdot\pi_\bullet"] \arrow[d, Rightarrow, "\lambda_\bullet\cdot i_\bullet"'] & u \circ 0 \arrow[d, Rightarrow] \\
\cap^{ch}_{rel} \circ i_\bullet \arrow[r, Rightarrow, "\phi_\bullet"'] & 0
\end{tikzcd}$$

For more details on the universality of this cokernel we refer the interested reader to [**?**]. This unique 1-morphism, u, induces the following 1-morphism on passing to homology

$$H_\bullet(X_\bullet, Y_\bullet, \mathbb{Z}[0]) \to H_\bullet([C^\bullet(X_\bullet,\mathcal{A}), C_\bullet(X_\bullet, Y_\bullet, \mathcal{A})]). \tag{8.11}$$

Composition with the following chain map

$$H_\bullet([C^\bullet(X_\bullet,\mathcal{A}), C_\bullet(X_\bullet, Y_\bullet, \mathcal{A})]) \to [H^\bullet(X_\bullet,\mathcal{A}), H_\bullet(X_\bullet, Y_\bullet, \mathcal{A})]. \tag{8.12}$$

and the adjointness of the tensor product gives us the desired chain map

$$\cap : H_\bullet(X_\bullet, Y_\bullet, \mathbb{Z}[0]) \otimes H^\bullet(X_\bullet,\mathcal{A}) \to H_\bullet(X_\bullet, Y_\bullet, \mathcal{A}). \tag{8.13}$$

which we will call the relative cap product.

As in the classical case, the boundary map in homology is natural with respect to relative cap product, in a sense made precise below.

Proposition 8.4. *The following diagram of Picard groupoids is commutative up to natural isomorphism for $p-1 \geq q \geq 0$:*

$$\begin{tikzcd}
H_p(X_\bullet, Y_\bullet; \mathbb{Z}[0]) \otimes H^q(X_\bullet, \mathcal{A}) \arrow[r, "\cap"] \arrow[d, "\partial \otimes i^*"'] \arrow[dr, phantom, "\swarrow"] & H_{p-q}(X_\bullet, Y_\bullet; \mathcal{L}) \arrow[d, "\partial"] \\
H_{p-1}(Y_\bullet; \mathbb{Z}[0]) \otimes H^q(Y_\bullet, \mathcal{A}) \arrow[r, "\cap"] & H_{p-q-1}(Y_\bullet; \mathcal{L})
\end{tikzcd} .$$

Chapter 9

Hermitian Line Gerbes

In this section we describe geometric objects which we call (*flat*) *hermitian line n-gerbes*. Then we give an example describing a flat hermitian line 2-gerbe over the simplicial set BG, where G is a (discrete) group. We move on to describe certain geometic objects over a topological space X which are classified by the Čech cohomology of X with coefficients in $U(1)$ and which we call *flat hermitian line 1-gerbes* over X. We describe these in two ways. For the first description, we define a category $C(X;\mathfrak{U}_I)$, associated to an open cover of X and show that hermitian line 0-cocycles on the simplicial set $N(C(X;\mathfrak{U}_I))$ represent (flat) hermitian line 0-gerbes. Our second description is that a flat hermitian line 0-gerbe can be represented by a functor from the first fundamental groupoid of X into the Picard groupoid of hermitian lines $\mathcal{L}$. Finally, we move on to describe higher hermitian line gerbes over X. A classification of the gerbes described in this section, along the lines of [Sha] is highly desirable.

Definition 9.1. A *hermitian line n-gerbe on* a simplicial set $X_\bullet$ is an an n-cocycle on the simplicial set $X_\bullet$ with values in $\mathcal{L}$ i.e. an object $K \in ObH^n(X_\bullet, \mathcal{L})$ of degree n cohomology Picard groupoid of $X_\bullet$ with coefficients in the Picard groupoid $\mathcal{L}$ of hermitian lines.

Remark. A n-gerbe should be properly defined as a 0-cocycle with coefficients in an appropriate Picard $(n+1)$-groupoid but this would be out of scope of this paper.

The following example shows that a 2-gerbe on BG, where G is a finite group, is exactly the same as a 2-cocycle with values in hermitian lines as defined in [**?**].

Example 9.2. Any group G can be viewed as a category with a single object. The simplicial set BG is the nerve of this category. An object of $H^2(BG;\mathcal{L})$ consists of a pair $(\beta, \phi : d\beta \to 0)$, where $\beta \in ObC^2(BG;\mathcal{L})$ and ϕ is an arrow in $C^3(BG;\mathcal{L})$ and d is the differential of the 2-complex $C^\bullet(BG;\mathcal{L})$. β is a set function whose domain is the underlying set of $G\times G$ and codomain is $Ob\mathcal{L}$ i.e. it assigns to each pair $(g_1, g_2) \in G\times G$ a hermitian line $l_{g_2,g_1} \in Ob\mathcal{L}$. The arrow ϕ gives, for every triple $(g_1, g_2, g_3) \in G\times G\times G$, a following isomorphism in $\mathcal{L}$

$$t_{g_3,g_2,g_1} : l_{g_3,g_2} - l_{g_3,g_2g_1} + l_{g_3g_2,g_1} - l_{g_2,g_1} \to \mathbb{C}.$$

The morphism $d(\phi) : d^2(\beta) \to 0$ gives, for every quadruple $(g_1, g_2, g_3, g_4) \in G\times G\times G\times G$, the following isomorphism

$$t_{g_4,g_3,g_2} - t_{g_4,g_3,g_{21}} + t_{g_4,g_{32},g_1} - t_{g_{43},g_2,g_1} + t_{g_3,g_2,g_1},$$

which is the canonical isomorphism $\chi_\beta((g_1, g_2, g_3, g_4)) : d^2\beta((g_1, g_2, g_3, g_4)) \to \mathbb{C}$.

By the definition above, a flat hermitian line 0-gerbe on a simplicial set $X_\bullet$ is just an object of the Picard groupoid $H^0(X_\bullet;\mathcal{L})$. Thus, given a topological space X, we may look at hermitian line 0-gerbes over X in two ways: associating simplicial sets $Sing_\bullet X$ and $N(C(X;\mathfrak{U}_I))$ to X, where $N(C(X;\mathfrak{U}_I))$ is the simplicial sets obtained by taking the nerve of a category associated to the cover of X, $C(X;\mathfrak{U}_I)$, which we now define:

Definition 9.3. We define $C(X;\mathfrak{U}_I)$ to be a category whose object set is the collection $\mathfrak{U}_I = \{U_i : i \in I\}$, which is a chosen open cover of the topological space X. If the set

$U_{i,j} \neq \emptyset$, then $Hom_{C(X;\mathfrak{U}_I)}(U_i, U_j) = \{U_{i,j}\}$, otherwise the set $Hom_{C(X;\mathfrak{U}_I)}(U_i, U_j) = \emptyset$. Composition in $C(X; \mathfrak{U}_I)$ is defined as follows: $U_{i,j} \circ U_{j,k} := U_{i,j,k} := U_{i,j} \cap U_{j,k}$. $id_{U_i} := U_{ii}$. The source of an arrow $U_{i,j}$ is U_i and its target is U_j.

This leads to two interpretations of flat hermitian line 0-gerbes: Definitions 9.4 and 9.5 below.

Definition 9.4. A *flat hermitian line* 0*-gerbe* over X, $\mathcal{G}^0(\Lambda, \theta)$, is defined by the following data

1. A function $\Lambda_0 : (Sing_\bullet X)_0 \to Ob(\mathcal{L})$, i.e. an assignment of a hermitian line to each point of X.

2. A function $\Lambda_1 : (Sing_\bullet X)_1 \to Mor(\mathcal{L})$ which assigns to each $f \in (Sing_\bullet X)_1$, a linear isometry $\Lambda_1(f) : \Lambda_0\partial_1(f) \to \Lambda_0\partial_0(f)$ in $\mathcal{L}$ such that for all $f, g \in (Sing_\bullet X)_1$ satisfying $\partial_1(f) = \partial_0(g)$, $\Lambda_1(g \circ f) = \Lambda(g) \circ \Lambda(f)$.

This data is subject to the following condition. For each $n \geq 2$, there exists a function $\Lambda_n : (Sing_\bullet X)_n \to Mor(\mathcal{L})$ such that for all $\sigma_n \in (Sing_\bullet X)_n$, $\Lambda_n(\sigma_n) = \Lambda_{n-1}(\partial_0\sigma_n) \circ \Lambda_{n-1}(\partial_1\sigma_n) \circ \cdots \circ \Lambda_{n-1}(\partial_{n-1}\sigma_n)$.

Remark. The above definition assigns a hermitian line to each point of X. Further, two homotopic paths in X, relative to endpoints, are assigned the same linear isometry. In other words the above data is equivalent to defining a functor from the first fundamental groupoid of the space X, $\Pi_1(X)$, to $\mathcal{L}$.

Definition 9.5. A *flat hermitian line* 0*-gerbe* over X, $\mathcal{G}^0(\Lambda, \theta)$, is defined by the following data

1. A constant hermitian line bundle Λ_i over every open set U_i for all $i \in I$.

2. For each ordered pair of distinct indices $(i,j) \in I \times I$, a constant, non-zero section

$$\theta_{i,j} \in \Gamma(U_{i,j}; \Lambda_i \otimes \Lambda_j)$$

This data is subject to a cocycle condition, on $U_{i,j,k}$ which we denote by $\delta\theta \Rightarrow 0$. The cocycle condition is that over any three fold intersections $U_{i,j,k}$, we can tensor the three sections of the coboundary to give a trivialization of the following hermitian line bundle

$$(\Lambda_i \otimes \Lambda_j) \bigotimes (\Lambda_i \otimes \Lambda_k)^{-1} \bigotimes (\Lambda_j \otimes \Lambda_k).$$

over $U_{i,j,k}$. Notice that the above hermitian line bundle is canonically trivial, so the cocycle condition is the requirement that the following

$$\theta_{i,j} - \theta_{i,k} + \theta_{j,k}$$

be the canonical section of this trivial hermitian line bundle over $U_{i,j,k}$.

Remark. Each point $x \in X$ has a neighborhood U_i such that the hermitian line bundle Λ_i is isomorphic to the trivial hermitian line bundle $U_i \times \mathbb{C}$. Further, the specification of constant, non-zero section $\theta_{i,j}$ is the same as specifying a hermitian line bundle isomorphism $g_{i,j} : \Lambda_i|_{U_{i,j}} \to \Lambda_j|_{U_{i,j}}$, which restricts to the same linear isometry on every fiber. These two observations along with the data in the definition above are sufficient to construct a (flat) hermitian line bundle over the space X.

Now we move on to define higher hermitian line gerbes. Our definition of a flat hermitian line 1-gerbe closely follows the definition of a "1-gerb" developed in [Cha98].

Definition 9.6. A *flat hermitian line* 1-*gerbe* over X, $\mathcal{G}^1(\Lambda, \theta)$, is defined by the following data

1. A constant hermitian line bundle Λ_i^j over the intersection $U_{i,j}$ for every ordered pair $(i,j) \in I \times I$ and $i \neq j$, such that Λ_i^j and Λ_j^i are dual to each other.

2. For each ordered triple of distinct indices $(i,j,k) \in I \times I \times I$, a nowhere zero section

$$\theta_{i,j,k} \in \Gamma(U_{i,j,k}; \Lambda_i^j \otimes \Lambda_j^k \otimes \Lambda_k^i)$$

such that the sections of reorderings of triples (i,j,k) are related in the natural way.

This data is subject to a cocycle condition, on $U_{i,j,k,l}$ which we denote by $\delta\theta \Rightarrow 0$. The cocycle condition is that over any four fold intersections $U_{i,j,k,l}$, we can tensor the four sections of the coboundary to give a trivialization of the following hermitian line bundle

$$(\Lambda_i^j \otimes \Lambda_j^k \otimes \Lambda_k^i) \bigotimes (\Lambda_i^j \otimes \Lambda_j^l \otimes \Lambda_l^i)^{-1} \bigotimes (\Lambda_i^k \otimes \Lambda_k^l \otimes \Lambda_l^i) \bigotimes (\Lambda_j^k \otimes \Lambda_k^l \otimes \Lambda_l^j)^{-1}. \quad (9.1)$$

over $U_{i,j,k,l}$. Notice that the above hermitian line bundle is canonically trivial, so the cocycle condition is the requirement that the following

$$\theta_{i,j,k} - \theta_{i,j,l} + \theta_{i,k,l} - \theta_{j,k,l}$$

be the canonical section of this trivial hermitian line bundle over $U_{i,j,k,l}$.

The tensor product of two flat hermitian line 1-gerbes is obtained by tensoring line bundles and sections in an obvious way.

Let $(\alpha, \phi) \in ObH^1(N(C(X; \mathfrak{U}_I); \mathcal{L})$. To each $U_{i,j}$, the cochain α assigns a hermitian line l_i^j and the morphism ϕ specifies a linear isometry for each $U_{i,j,k}$

$$\phi(U_{i,j,k}) : l_i^j - l_k^i + l_j^k \to \mathbb{C}.$$

Equivalently, the specification of this linear isometry is the specification of a constant function $t_{i,j,k} : U_{i,j,k} \to U(1)$. In other words

$$t_{i,j,k}(x) = \phi(U_{i,j,k}),$$

$\forall x \in U_{i,j,k}$. Put constant hermitian line bundles $\Lambda_i^j = U_{i,j} \times l_i^j$ over each $U_{i,j}$. Then $t_{i,j,k}$ gives a trivialization of the coboundary line bundle $\Lambda_i^j \otimes \Lambda_j^k \otimes \Lambda_k^i$. We define the section

$$\theta_{i,j,k}(x) = t_{i,j,k}^{-1}(x, e_1).$$

The morphism

$$dt_{i,j,k} = d\phi(U_{i,j,k,l}) = t_{i,j,k} - t_{i,j,l} + t_{i,k,l} - t_{j,k,l}$$

gives a trivialization of the line bundle (9.1) over $U_{i,j,k,l}$. The following section corresponds to the above trivialization of the the hermitian line bundle (9.1)

$$\theta_{i,j,k} - \theta_{i,j,l} + \theta_{i,k,l} - \theta_{j,k,l}.$$

Clearly this is the canonical section.

Conversely, a flat hermitian line 1-gerbe over X defines a 1-cocycle in $H^1(N(C(X;\mathfrak{U}_I);\mathcal{L}))$. We leave the easy verification of this fact as an excercise for the reader.

Definition 9.7. Let $\mathcal{G}^1(\Lambda,\theta)$ and $\mathcal{H}^1(\Upsilon,\eta)$ be two flat hermitian line 1-gerbes over X and let (g,ϕ) and (h,ψ) the two 1-cocycles in $H^1(N(C(X;\mathfrak{U}_I);\mathcal{L}))$ determined by them, then the the two gerbes $\mathcal{G}^1(\Lambda,\theta)$ and $\mathcal{H}^1(\Upsilon,\eta)$ are equivalent if there exists a morphism $(g,\phi) \to (h,\psi)$ in $H^1(N(C(X;\mathfrak{U}_I));\mathcal{L})$.

If $\mathcal{G}^1(\Lambda,\theta)$ and $\mathcal{H}^1(\Upsilon,\eta)$ are equivalent, then there are hermitian line bundle isomorphisms

$$\Lambda_i^j \cong \Upsilon_i^j,$$

over each $U_{i,j}$, such that the isomorphisms induce a mapping

$$\theta_{i,j,k} \mapsto \eta_{i,j,k}.$$

Definition 9.8. A flat hermitian line 1-gerbe $\mathcal{G}^1(\Lambda, \theta)$ is *globally trivialized* by displaying a basis λ_i^j for each line bundle Λ_i^j such that on each $U_{i,j,k}$, we can express the sections on three fold intersections, in terms of coordinates specifed specified by the data and the ring $C^\infty(U_{i,j,k}; U(1))$, as follows:

$$\theta_{i,j,k} = 1(x)\lambda_i^j \otimes \lambda_j^k \otimes \lambda_k^i,$$

where $1(x) \in C^\infty(U_{i,j,k}; U(1))$ is the constant function which assigns to each point $x \in U_{i,j,k}$, the identity of the group $U(1)$.

Remark. Let $\mathcal{G}$ be a globally trivial flat hermitian line 1-gerbe and (α, ϕ) be the hermitian line 1-cocycle determined by $\mathcal{G}$. Then for $U_{i,j,k}$

$$\phi(U_{i,j,k}) : l_i^j - l_j^k + l_k^i \to \mathbb{C}$$

is the canonical isomorphism.

Definition 9.9. A flat hermitian line 1-gerbe is *trivial* if it is equivalent to the *zero* 1-*gerbe over* X, which is the flat hermitian line 1-gerbe determined by the cocycle $(0, id_0) \in H^1(N(C(X; \mathfrak{U}_I)); \mathcal{L})$.

The notion of a trivial Čech hermitian line 1-gerbe can equivalently be defined by a geometric entity called an *object*, which we define next.

Definition 9.10. Given a flat hermitian line 1-gerbe $\mathcal{G}^1(\Lambda, \theta)$, an *object compatible with* $\mathcal{G}^1$, denoted $\mathcal{O}(L, m)$ is specified by the following data

1. Constant hermitian line bundles L_i over each U_i;

2. Hermitian line bundle isomorphisms over each intersection $U_{i,j}$

$$m_i^j : L_i \cong \Lambda_i^j \otimes L_j;$$

such that the composition on three fold intersection

$$L_i \longrightarrow (\Lambda_i^j \otimes \Lambda_j^k \otimes \Lambda_k^i) \otimes L_i$$

is exactly

$$(id \otimes m_k^i) \circ (id \otimes m_j^k) \circ m_i^j \equiv \theta_{i,j,k} \otimes id.$$

Here we are abusing notation by denoting the trivialization determined by the section $\theta_{i,j,k}$ also by $\theta_{i,j,k}$.

Proposition 9.11. *Let $\mathcal{G}^1$ be a flat hermitian line 1-gerbe over X and let (α, ϕ) be the Čech hermitian line 1-cocycle determined by $\mathcal{G}^1$. Then $\mathcal{G}^1$ has an object, $\mathcal{O}(L, m)$, compatible with it iff there is a Čech hermitian line 0-chain $\beta \in ObC^0(N(C(X; \mathfrak{U}_I)); \mathcal{L})$ and a morphism $f : (\alpha, \phi) \to (d\beta, \chi_\beta)$, in $Ker(d, \chi)$, such that (β, f) is a representative of a morphism $[(\beta, f)] : (\alpha, \phi) \to (0, id_0)$ in $H^1(N(C(X; \mathfrak{U}_I)); \mathcal{L})$.*

Proof. Let $\mathcal{G}^1$ be a trivial flat hermitian line 1-gerbe over X as above. Then there exists a morphism $[(\beta, f)] : (\alpha, \phi) \to (0, id_0)$ in $H^1(N(C(X; \mathfrak{U}_I)); \mathcal{L})$. Choose a representative (β, f) of this morphism. Now we define the constant line bundle, L_i, over each U_i as follows: $L_i := U_i \times \beta(U_i)$. The linear isometry $f(U_{i,j}) : \alpha(U_{i,j}) \to (\beta(U_i) - \beta(U_j))$ determines a morphism of hermitian line bundles

$$m_i^j : L_i \to \Lambda_i^j \otimes L_j$$

over each $U_{i,j}$. The condition over three fold intersections, in definition 9.10, follows from the equation $\chi_\beta \circ d(f) = \phi$. Conversely, given an object compatible with a trivial flat hermitian line 1-gerbe $\mathcal{G}^1$, one can define the isomorphism $[(\beta, f)] : (\alpha, \phi) \to (0, id_0)$ in $H^1(N(C(X; \mathfrak{U}_I)); \mathcal{L})$. $\square$

Finally, we are ready to define a *flat hermitian line* 2*-gerbe* over X.

Definition 9.12. A *flat hermitian line 2-gerbe over X*, $\mathcal{G}^2(\mathcal{G},\mathcal{O},\theta)$, is defined by the following data

1. A flat hermitian line 1-gerbe $\mathcal{G}_i^j$ over the intersection $U_{i,j}$ for every ordered pair $(i,j) \in I \times I$ and $i \neq j$ such that $\mathcal{G}_i^j$ and $\mathcal{G}_j^i$ are dual to each other.

2. For each ordered triple of distinct indices $(i,j,k) \in I \times I \times I$, an object $\mathcal{O}_{i,j,k}$ compatible with the coboundary gerbe

$$\mathcal{G}_i^j \otimes \mathcal{G}_j^k \otimes \mathcal{G}_k^i$$

such that the sections of reorderings of triples (i,j,k) are related in the natural way.

3. For each ordered quadruple of distinct indices $(i,j,k,l) \in I \times I \times I \times I$, trivializations $\theta_{i,j,k,l}$ of coboundaries of objects

$$\mathcal{O}_{i,j,k} \otimes \mathcal{O}_{i,j,l}^{-1} \otimes \mathcal{O}_{i,k,l} \otimes \mathcal{O}_{j,k,l}^{-1}$$

on $U_{i,j,k,l}$. Notice that each pair $(\mathcal{O}_{i,j,k} \otimes \mathcal{O}_{i,j,l}^{-1})$ is a line bundle over $U_{i,j,k,l}$ so asking for a *trivialization* of the object is ligitimate.

This data is subject to a cocycle condition, on $U_{i,j,k,l,m}$ which we denote by $\delta\theta \Rightarrow 0$. The cocycle condition is that over any five fold intersections $U_{i,j,k,l,m}$, we can tensor the five sections of the coboundary objects to give a trivialization of the following hermitian object

$$(\mathcal{O}_{i,j,k} \otimes \mathcal{O}_{i,j,l}^{-1} \otimes \mathcal{O}_{i,k,l} \otimes \mathcal{O}_{j,k,l}^{-1}) \bigotimes (\mathcal{O}_{i,j,k} \otimes \mathcal{O}_{i,j,m}^{-1} \otimes \mathcal{O}_{i,k,m} \otimes \mathcal{O}_{j,k,m}^{-1})^{-1}$$

$$\bigotimes (\mathcal{O}_{i,j,l} \otimes \mathcal{O}_{i,j,m}^{-1} \otimes \mathcal{O}_{i,l,m} \otimes \mathcal{O}_{j,l,m}^{-1}) \bigotimes (\mathcal{O}_{i,k,l} \otimes \mathcal{O}_{i,k,m}^{-1} \otimes \mathcal{O}_{i,l,m} \otimes \mathcal{O}_{k,l,m}^{-1})^{-1}$$

$$\bigotimes (\mathcal{O}_{j,k,l} \otimes \mathcal{O}_{j,k,m}^{-1} \otimes \mathcal{O}_{j,l,m} \otimes \mathcal{O}_{k,l,m}^{-1})$$

Notice that the above object is canonically trivial, so the cocycle condition is that the following

$$\theta_{i,j,k,l} - \theta_{i,j,k,m} + \theta_{i,j,l,m} - \theta_{i,k,l,m} + \theta_{j,k,l,m}$$

is the canonical section.

A hermitian line 2-cocycle (α, ϕ) represents a flat hermitian line 2-gerbe over X. We outline a construction of a flat hermitian line 2-gerbe starting from the 2-cocycle (α, ϕ). A flat hermitian line 1-gerbe, $\mathcal{G}^i_j(\Lambda, \theta)$ over $U_{i,j}$ for every pair $(i, j) \in I \times I$, is determined by the 2-cocycle (α, ϕ) as follows: Over each three-fold intersection, $U_{i,j,k}$, a constant hermitian line bundle $\Lambda_{i,j,k}$ is defined by $\Lambda_{i,j,k} := U_{i,j,k} \times \alpha(U_{i,j,k})$. On every four-fold intersection $U_{i,j,k,l}$, the section $\theta_{i,j,k,l}$ is determined by the linear isometry

$$\phi(U_{i,j,k,l}) : \alpha(U_{i,j,k}) - \alpha(U_{i,k,l}) + \alpha(U_{i,j,l}) - \alpha(U_{j,k,l}) \to \mathbb{C}.$$

This section satisfies the cocycle condition $\delta\theta \Rightarrow 0$ over five-fold intersections, thus defining a flat hermitian line 1-gerbe over $U_{i,j}$. Notice that the coboundary flat hermitian line 1-gerbe $\mathcal{G}^i_j \otimes \mathcal{G}^j_k \otimes \mathcal{G}^i_k$ over $U_{i,j,k}$, is trivial, therefore there exists an object $\mathcal{O}_{i,j,k}$ compatible with this trivial coboundary gerbe. This object $\mathcal{O}_{i,j,k}$ is specified by the 2-chain $\alpha \in C^2(N(C(X; \mathfrak{U}_I)); \mathcal{L})$. Over each four-fold intersection $U_{i,j,k,l}$, a section $\theta_{i,j,k,l}$ of the coboundary Object $\mathcal{O}_{i,j,k} \otimes \mathcal{O}^{-1}_{i,j,l} \otimes \mathcal{O}_{i,k,l} \otimes \mathcal{O}^{-1}_{j,k,l}$ is specified by the linear isometry $\phi(U_{i,j,k,l})$.

Chapter 10

Dijkgraaf-Witten Theory

We would like to recover Dijkgraaf-Witten's construction [DW90] of a TQFT. In principle, we follow their construction, using Freed-Quinn's hermitian-line incarnation [FQ93], and placing it further within the framework of cohomology with coefficients in the Picard groupoid of hermitian lines.

10.1 Hermitian line corresponding to a closed n-manifold

We start with an n-cocycle α which is an object of the Picard groupoid $H^n(BG; \mathcal{L})$. For each map $f : Y \to BG$ from a closed n-manifold Y, we take the pullback $f^*\alpha$. Consider the cap product

$$\cap : H^n(Y; \mathcal{L}) \otimes H_n(Y; \mathbb{Z}[0]) \to H_0(Y; \mathcal{L}),$$

which is a morphism of Picard groupoids. If we substitute the given cocycle α in the first factor, we will get a morphism

$$f^*\alpha \cap - : H_n(Y; \mathbb{Z}[0]) \to H_0(Y; \mathcal{L}). \tag{10.1}$$

What we would like to do is to apply this morphism to the fundamental cycle of Y. However, in the homology with coefficients in a Picard groupoid, be it a discrete one, such as $\mathbb{Z}[0]$, no single object represents the fundamental cycle canonically. It is rather a full subgroupoid (not monoidal) C_Y formed by all possible cycles representing the fundamental cycle and connected by equivalence classes of morphisms given by n-boundaries modulo $(n+1)$-boundaries: a morphism $y \to y'$ is given by an $(n+1)$-chain x such that $y' = y + dx$; two morphisms $x : y \to y'$ and $x' : y \to y'$ are equivalent if there is an $(n+2)$-chain w such that $x = dw + x'$. Thus, we can restrict the above morphism (10.1) to this *fundamental-cycle groupoid* C_Y and get a functor

$$f^*\alpha \cap - : C_Y \to H_0(Y; \mathcal{L}).$$

If we compose this functor with the *degree map*

$$H_0(Y; \mathcal{L}) \to \mathcal{L}$$

which takes each linear combination $a_1 y_1 + \cdots + a_k y_k$ of points $y_1, \ldots, y_k$ in Y with coefficients $a_1, \ldots, a_k$ in $\mathcal{L}$ to the sum $a_1 + \cdots + a_k$, which is an object in $\mathcal{L}$, we obtain a functor

$$F : C_Y \to \mathcal{L} \tag{10.2}$$

from the fundamental-cycle groupoid to the groupoid of hermitian lines. Now we take the limit of this functor. The existence of the limit is guaranteed by the following fact.

Proposition 10.1. *The functor*

$$F : C_Y \to \mathcal{L},$$

which represents the cap product of the cocycle α with the fundamental-cycle groupoid C_Y, has a limit,

$$\lim_{C_Y} F,$$

in the category $\mathcal{L}$ of hermitian lines.

Proof. The limit of the functor F may be realized by Freed-Quinn's *invariant-section construction*: an invariant section is a collection of elements in $\{s(y) \in F(y) \mid y \in ObC_Y\}$ such that for each morphism $x : y \to y'$ in C_Y, we have $F(x)s(y) = s(y')$. The space of invariant sections is a hermitian line, in other words, the limit of F exists, if the functor has *no holonomy*, *i.e.*, $F(x) = id$ for each automorphism $x : y \to y$. This is indeed the case, due to the following argument.

Being an object of $H^n(Y; \mathcal{L})$, the cocycle α is represented by a pair (a, ϕ), where a is an object of $C^n(Y; \mathcal{L})$, *i.e.*, a function $a : S^n(Y) \to Ob\mathcal{L}$, and $\phi : da \to 0$ is a morphism in $C^{n+1}(Y; \mathcal{L})$, *i.e.*, a function $S^{n+1}(Y) \to Mor\mathcal{L}$. The functor $F : C_Y \to \mathcal{L}$ acts in the following way on objects and morphisms of the groupoid C_Y:

$$F(y) = a(y) \qquad \text{for } y \in ObC_Y,$$

and

$$F(x) : a(y) \to a(y') \qquad \text{for } x \in MorC_Y,\ y' = y + dx,$$

is defined by $\phi(x) : a(y') - a(y) = a(dx) = da(x) \to 0$ as a composition of it with the structure natural transformations (8.1)-(8.2) and their inverses.

Now suppose we have an automorphism $x : y \to y$, which in particular means that we have a chain $x \in ObC_{n+1}(Y; \mathbb{Z}[0])$, such that $dx = 0$. Since $H_{n+1}(Y; \mathbb{Z}[0])$ is trivial whenever $\dim Y = n$, the cycle x must be a boundary: $x = dw$ for some w. This renders the equivalence class of the morphism x to be trivial. $\square$

10.2 Linear isometry corresponding to an $(n+1)$-cobordism

Now let X be a compact $n+1$-manifold with boundary $i : \partial X = \partial X_- \coprod \partial X_+ \subset X$. As a starting point, we use the same n-cocycle α, which is an object of the Picard groupoid $H^n(BG; \mathcal{L})$. For any continuous function $f : X \to BG$, a pullback of α along f gives an n-cocycle $f^*\alpha$, which is an object of the Picard groupoid $H^n(X;\mathcal{L})$. Consider the relative cap product

$$\cap : H^n(X;\mathcal{L}) \otimes H_{n+1}(X,\partial X;\mathbb{Z}[0]) \to H_1(X,\partial X;\mathcal{L}),$$

which is a morphism of Picard groupoids. If we substitute $f^*\alpha$ in the first factor, we will get a functor

$$f^*\alpha \cap - : H_{n+1}(X,\partial X;\mathbb{Z}[0]) \to H_1(X,\partial X;\mathcal{L}). \tag{10.3}$$

As above, we restrict this functor to the *relative fundamental-cycle groupoid* $C_{X,\partial X}$ which is the full subgroupoid of $H_{n+1}(X,\partial X;\mathbb{Z}[0])$ whose objects are all possible relative cycles representing the relative fundamental class of X. The restriction gives us a functor

$$f^*\alpha \cap - : C_{X,\partial X} \to H_1(X,\partial X;\mathcal{L}).$$

We compose this functor first with the 2-morphism Ψ_1 from (the chain version of) the long 2-exact sequence (8.6) and then the degree map

$$\begin{tikzcd} C_{X,\partial X} \arrow[r, "f^*\alpha\cap -"] & H_1(X,\partial X;\mathcal{L}) \arrow[r, "\partial_1"] \arrow[rr, bend left, "0"] & H_0(\partial X;\mathcal{L}) \arrow[r, "H_0(i)"] \arrow[phantom, u, "\Psi_1 \Uparrow"] & H_0(X;\mathcal{L}) \arrow[r, "deg"] & \mathcal{L}\,. \end{tikzcd} \tag{10.4}$$

This diagram gives us a 2-morphism $t : F \Rightarrow 0$, where $F : C_{X,\partial X} \to \mathcal{L}$ is the composite functor in the lower row. Consider the following diagram:

$$\begin{tikzcd} H_{n+1}(X,\partial X;\mathbb{Z}[0]) \arrow[r, "f^*\alpha\cap -"] \arrow[d, "\partial"] & H_1(X,\partial X;\mathcal{L}) \arrow[r, "\partial_1"] \arrow[rr, bend left, "0"] & H_0(\partial X;\mathcal{L}) \arrow[r, "H_0(i)"] \arrow[phantom, u, "\Psi_1 \Uparrow"] & H_0(X;\mathcal{L}) \arrow[r, "\mathrm{deg}"] & \mathcal{L}\,, \\ H_n(\partial X;\mathbb{Z}[0]) \arrow[urr, "f|_{\partial X}^*\alpha\cap -"'] \arrow[ur, phantom, "\Nearrow"] \end{tikzcd} \tag{10.5}$$

where the bottom 2-morphism is comes from Proposition 8.4. When we restrict the boundary 1-morphism $\partial : H_{n+1}(X, \partial X; \mathbb{Z}[0]) \to H_n(\partial X; \mathbb{Z}[0])$ to the full subcategory $C_{X,\partial X}$, we get the following commutative diagram:

$$\begin{tikzcd} C_{X,\partial X} \arrow[r] \arrow[d, "\partial"'] \arrow[dr, phantom, "\circlearrowright"] & H_{n+1}(X, \partial X; \mathbb{Z}[0]) \arrow[d, "\partial"] , \\ -C_{\partial_- X} \times C_{\partial_+ X} \arrow[r] & H_n(\partial X; \mathbb{Z}[0]) \end{tikzcd}$$

where $-C_{\partial_- X}$ is the negative fundamental-cycle groupoid of $\partial_- X$, the full subcategory in $H_n(\partial X; \mathbb{Z}[0])$ made up by representatives of the negative fundamental class of $\partial_- X$ in $H_n(\partial_- X; \mathbb{Z}) \subset H_n(\partial X; \mathbb{Z})$. By stacking together the last two diagrams, we obtain the following diagram:

$$\begin{tikzcd} C_{X,\partial X} \arrow[r, "F"] \arrow[r, bend left=50, "0"] \arrow[r, phantom, bend left=25, "\Psi_F \Uparrow"] \arrow[d, "\partial"] \arrow[dr, phantom, "\nearrow\!\!\!\nearrow X_F"] & \mathcal{L} , \\ -C_{\partial_- X} \times C_{\partial_+ X} \arrow[ur, "F_- + F_+"'] & \end{tikzcd} \tag{10.6}$$

where $F_- := f|^*_{\partial_- X}\alpha \cap -$ and $F_+ := f|^*_{\partial_+ X}\alpha \cap -$ appended by $H_0(i)$ and deg as in (10.5).

Applying the limit functor, we get canonical morphisms

$$-\lim_{C_{\partial_- X}} F_- + \lim_{C_{\partial_+ X}} F_+ \to \lim_{-C_{\partial_- X} \times C_{\partial_+ X}} (F_- + F_+) \to \lim_{C_{X,\partial X}} F \to 0$$

in $\mathcal{L}$, whence a morphism

$$l_f : \lim_{C_{\partial_- X}} F_- \to \lim_{C_{\partial_+ X}} F_+,$$

which translates into a canonical linear isometry between hermitian lines.

10.3 The Dijkgraaf-Witten theory TQFT functor

Given a finite group G, for each $\alpha \in H^n(BG;\mathcal{L})$, we construct the Dijkgraaf-Witten theory TQFT functor,

$$Z^\alpha : \mathbf{Cob}(n+1) \to \mathbf{Vect},$$

from the category $\mathbf{Cob}(n+1)$ of cobordisms to the category $\mathbf{Vect}$ of complex vector spaces, using the ingredients developed in the preceding sections. We first construct the values of the functor on objects. Observe that for every $Y \in Ob\ \mathbf{Cob}(n+1)$, Proposition 10.1 delivers a canonical hermitian line for each $f \in Map(Y, BG)$. We claim that these lines glue into a flat hermitian line bundle over $Map(Y, BG)$, or a *local system* with values in $\mathcal{L}$, *i.e.*, a functor

$$\mathcal{L}_Y : \Pi_1 Map(Y, BG) \to \mathcal{L}$$

from the fundamental groupoid of the mapping space $Map(Y, BG)$ to $\mathcal{L}$.

A morphism in $\Pi_1 Map(Y, BG)$ is a homotopy class $[f]$ of a map $f : Y \times I \to BG$. We can think of $Y \times I$ as the identity cobordism between two copies of Y. Applying the construction of Section 10.2, we get a morphism in $\mathcal{L}$,

$$l_f : \lim_{C_Y} F_0 \to \lim_{C_Y} F_1.$$

We define $\mathcal{L}_Y([f]) := l_f$. The cocycle $f^*\alpha$ does depend on the representative f of the homotopy class $[f]$, see Section 8.3, however the difference disappears at the homology level after applying the cap product with $f^*\alpha$ and the "boundary homomorphism" $\partial_1 : H_1(Y \times I, \partial(Y \times I); \mathcal{L}) \to H_0(\partial(Y \times I); \mathcal{L})$ in (10.5). Note that $f|^*_{\partial(Y\times I)}\alpha$ does not depend on the representative of the homotopy class $[f]$, because the homotopy is supposed to be relative to the boundary. Thus, the diagram (10.6) does not depend of the choice of a representative of the homotopy class $[f]$, and the local system $\mathcal{L}_Y$ is well defined.

One can view the construction of a local system $\mathcal{L}_y$ as "integration of $ev^*\alpha$ along fibers" of π or a construction of the push-pull in cohomology with values in Picard groupoids along the following diagram:

$$\begin{array}{ccc} Y \times Map(Y, BG) & \xrightarrow{ev} & BG \\ {\scriptstyle\pi}\big\downarrow & & \\ Map(Y, BG), & & \end{array}$$

$$H^n(BG;\mathcal{L}) \xrightarrow{ev^*} H^n(Y \times Map(Y, BG);\mathcal{L}) \xrightarrow{\pi_*} H^0(Map(Y, BG);\mathcal{L}),$$

where $\pi_* ev^*\alpha := \mathcal{L}_Y$, by definition, and we recall that objects of $H^0(Map(Y, BG);$ $\mathcal{L})$ are identified with local systems or 0-gerbes, see Section 9.4.

For any $Y \in Ob\, \mathbf{Cob}(n+1)$, we define the value $Z^\alpha(Y)$ of the *TQFT functor* to be the space of global sections of the local system $\mathcal{L}_Y$ over $Map(Y, BG)$ constructed above:

$$Z^\alpha(Y) := H^0(Map(Y, BG); \mathcal{L}_Y) := \lim \mathcal{L}_Y \in \mathbf{Vect},$$

where the limit is taken for a natural extension $\Pi_1 Map(Y, BG) \xrightarrow{\mathcal{L}_Y} \mathcal{L} \to \mathbf{Vect}$ of the functor $\mathcal{L}_Y$, denoted by the same symbol. The limit exists, because the category $\mathbf{Vect}$ is complete.

Now we construct the arrow function of the TQFT functor. This can also be viewed as a construction of "fiberwise integral." Let X be an $(n+1)$-dimensional cobordism from $\partial_- X$ to $\partial_+ X$. We get two local systems $\mathcal{L}_{\partial_- X}$ and $\mathcal{L}_{\partial_+ X}$ over the mapping spaces $Map(\partial_- X, BG)$ and $Map(\partial_+ X, BG)$, respectively. Let $p_\pm : Map(X, BG) \to Map(\partial_\pm X, BG)$ denote the natural restriction morphisms. We start with constructing a morphism $\mathcal{L}_X : p_-^*\mathcal{L}_{\partial_- X} \to p_+^*\mathcal{L}_{\partial_+ X}$ of local systems on $Map(X, BG)$. *i.e.*, a natural transformation between functors $p_-^*\mathcal{L}_{\partial_- X}$ and $p_+^*\mathcal{L}_{\partial_+ X} : \Pi_1(Map(X, BG)) \to \mathcal{L}$. For each $f \in Map(X, BG)$, by invoking the construction of Section 10.2 once again, we get two functors $F_\pm : C_{\partial_\pm X} \to \mathcal{L}$ and the following morphism

$$l_f : \lim_{C_{\partial_- X}} F_- \to \lim_{C_{\partial_+ X}} F_+$$

in $\mathcal{L}$. Note that the fiber of each pull-back local system $p_\pm^*\mathcal{L}_{\partial_\pm X}$ over $f \in Map(X, BG)$ is by definition the fiber of $\mathcal{L}_{\partial_\pm X}$ over $p_\pm(f)$, and that fiber is $\lim_{C_{\partial_\pm X}} F_\pm$ by the construction of Section 10.1. We define $\mathcal{L}_X(f)$ to be $l_f : p_-^*\mathcal{L}_{\partial_- X}|_f \to p_+^*\mathcal{L}_{\partial_+ X}|_f$ on objects $f \in Map(X, BG)$ of $\Pi_1(Map(X, BG))$. A morphism $f \to g$ in the fundamental groupoid $\Pi_1(Map(X, BG))$ is represented by a homotopy $h \in Map(X \times I, BG)$ between maps f and $g \in Map(X, BG)$. To see that $\mathcal{L}_X$ consitutes a natural transformation, we need to see that the diagram

$$\begin{tikzcd} p_-^*\mathcal{L}_{\partial_- X}|_f \arrow[r, "l_f"] \arrow[d, "p_-^* l_{h|_{\partial_- X\times I}}"'] & p_+^*\mathcal{L}_{\partial_+ X}|_f \arrow[d, "p_+^* l_{h|_{\partial_- X\times I}}"] \\ p_-^*\mathcal{L}_{\partial_- X}|_g \arrow[r, "l_g"] & p_+^*\mathcal{L}_{\partial_+ X}|_g \end{tikzcd} \tag{10.7}$$

commutes. Indeed, the homotopy gives a morphism $H : f^*\alpha \to g^*\alpha$ in the Picard groupoid $H^n(X; \mathcal{L})$. Using the bifunctoriality of the cap product, we get a 2-morphism $f^*\alpha \cap - \Rightarrow g^*\alpha \cap -$ added to Diagram (10.4), resulting in a commutative triangle

$$\begin{tikzcd} F \arrow[rr, Rightarrow, "\Psi_H"] \arrow[dr, Rightarrow, "\Psi_F"'] & & G \arrow[dl, Rightarrow, "\Psi_G"] \\ & 0 & \end{tikzcd}$$

on top of the upper part of Diagram (10.6) and, similarly, a commutative square

$$\begin{tikzcd} (F_- + F_+) \circ \partial \arrow[r, Rightarrow, "\Psi_{\partial H}"] \arrow[d, Rightarrow, "X_F"'] & (G_- + G_+) \circ \partial \arrow[d, Rightarrow, "X_G"] \\ F \arrow[r, Rightarrow, "\Psi_H"] & G \end{tikzcd}$$

on top of the lower part of Diagram (10.6), with $\Psi_{\partial H}$ coming from the 2-morphism $f|_{\partial X}^*\alpha \cap - \Rightarrow g|_{\partial X}^*\alpha \cap -$ added to the bottom triangle in (10.4). Passing to the limits, we see that (10.7) is commutative.

Now, after the morphism $\mathcal{L}_X : p_-^*\mathcal{L}_{\partial_- X} \to p_+^*\mathcal{L}_{\partial_+ X}$ of local systems on $Map(X, BG)$ is constructed, we are ready to construct a linear map

$$Z^\alpha(X) : Z^\alpha(\partial_- X) \to Z^\alpha(\partial_+ X)$$

or

$$Z^\alpha(X) : H^0(Map(\partial_- X, BG); \mathcal{L}_{\partial_- X}) \to H^0(Map(\partial_+ X, BG); \mathcal{L}_{\partial_+ X}).$$

The plan is to describe a push-pull along the diagram of spaces:

$$Map(\partial_- X, BG) \xleftarrow{p_-} Map(X, BG) \xrightarrow{p_+} Map(\partial_+ X, BG).$$

The pullback

$$p_-^* : H^0(Map(\partial_- X, BG); \mathcal{L}_{\partial_- X}) \to H^0(Map(X, BG); p_-^* \mathcal{L}_{\partial_- X})$$

is easy. So is an intermediate map:

$$H^0(\mathcal{L}_X) : H^0(Map(X, BG); p_-^* \mathcal{L}_{\partial_- X}) \to H^0(Map(X, BG); p_+^* \mathcal{L}_{\partial_+ X}).$$

The pushforward

$$(p_+)_* : H^0(Map(X, BG); p_+^* \mathcal{L}_{\partial_+ X}) \to H^0(Map(\partial_+ X, BG); \mathcal{L}_{\partial_+ X})$$

is not straightforward, and its existence relies on the specifics of the topology of mapping spaces to BG for a finite group G.

Recall that the space $Map(X, BG)$ may naturally be realized at the classifying space for principal G-bundles over X. This leads to a natural homotopy equivalence

$$Map(X, BG) \sim \coprod_{[P \to X]} BAut(P),$$

where the disjoint union is taken over isomorphism classes $[P \to X] \simeq \pi_0 Map(X, BG)$ of principal G-bundles $P \to X$. The map $p_+ : Map(X, BG) \to Map(\partial_+ X, BG)$ is homotopy equivalent to the natural restriction map

$$p'_+ : \coprod_{[P \to X]} BAut(P) \to \coprod_{[P_+ \to \partial_+ X]} BAut(P_+),$$

which is a finite covering map over each connected component $BAut(P_+)$, sometimes with empty fiber.

We will define the pushforward

$$(p_+)_* : H^0(Map(X,BG); p_+^*\mathcal{L}_{\partial_+ X}) \to H^0(Map(\partial_+ X, BG); \mathcal{L}_{\partial_+ X})$$

as a transfer map

$$(p'_+)_* : H^0\left(\coprod_{[P\to X]} BAut(P); (p'_+)^*\mathcal{L}_{\partial_+ X}\right) \to H^0\left(\coprod_{[P_+\to \partial_+ X]} BAut(P_+); \mathcal{L}_{\partial_+ X}\right),$$

which will be constructed using the definition of H^0 as a limit over the fundamental groupoid. Indeed, for every path γ_+ in $BAut(P_+)$, we take all its lifts to the component $BAut(P)$ over $BAut(P_+)$, which is a finite, possibly zero, number. For each such path γ, we have a linear isometry $(p'_+)^*\mathcal{L}_{\partial_+ X}(\gamma) : (p'_+)^*\mathcal{L}_{\partial_+ X}(\gamma(0)) \to (p'_+)^*\mathcal{L}_{\partial_+ X}(\gamma(1))$, which, by definition of $(p'_+)^*$, is equal to the isometry $\mathcal{L}_{\partial_+ X}(\gamma_+) : \mathcal{L}_{\partial_+ X}(\gamma_+(0)) \to \mathcal{L}_{\partial_+ X}(\gamma_+(1))$. Since $H^0\left(\coprod_{[P\to X]} BAut(P); (p'_+)^*\mathcal{L}_{\partial_+ X}\right)$ is a limit of the functor $(p'_+)^*\mathcal{L}_{\partial_+ X}$, we have a canonical commutative diagram of linear maps:

$$\begin{tikzcd}
 & H^0\left(\coprod_{[P\to X]} BAut(P); (p'_+)^*\mathcal{L}_{\partial_+ X}\right) \arrow[dl] \arrow[dr] & \\
(p'_+)^*\mathcal{L}_{\partial_+ X}(\gamma(0)) \arrow[rr] \arrow[d, equal] & & (p'_+)^*\mathcal{L}_{\partial_+ X}(\gamma(1)) \arrow[d, equal] \\
\mathcal{L}_{\partial_+ X}(\gamma_+(0)) \arrow[rr] & & \mathcal{L}_{\partial_+ X}(\gamma_+(1)).
\end{tikzcd}$$

If, given γ_+, we add the linear maps $H^0\left(\coprod_{[P\to X]} BAut(P); (p'_+)^*\mathcal{L}_{\partial_+ X}\right) \to \mathcal{L}_{\partial_+ X}(\gamma_+(0))$ over all possible γ's covering γ_+ and do the same for maps to $\mathcal{L}_{\partial_+ X}(\gamma_+(1))$, we will get a commutative diagram

$$\begin{tikzcd}
 & H^0\left(\coprod_{[P\to X]} BAut(P); (p'_+)^*\mathcal{L}_{\partial_+ X}\right) \arrow[dl] \arrow[dr] & \\
\mathcal{L}_{\partial_+ X}(\gamma_+(0)) \arrow[rr] & & \mathcal{L}_{\partial_+ X}(\gamma_+(1)).
\end{tikzcd}$$

Since $H^0\left(\coprod_{[P_+\to\partial_+ X]} BAut(P_+);\mathcal{L}_{\partial_+ X}\right)$ is a limit of the functor $\mathcal{L}_{\partial_+ X}$, we get a canonical linear map

$$H^0\left(\coprod_{[P\to X]} BAut(P);(p'_+)^*\mathcal{L}_{\partial_+ X}\right)\to H^0\left(\coprod_{[P_+\to\partial_+ X]} BAut(P_+);\mathcal{L}_{\partial_+ X}\right),$$

which we declare to be the *transfer* $(p'_+)_*$.

Finally, the TQFT functor

$$Z^\alpha(X): H^0(Map(\partial_- X, BG);\mathcal{L}_{\partial_- X})\to H^0(Map(\partial_+ X, BG);\mathcal{L}_{\partial_+ X})$$

is defined as the composition of $(p_+)_*$, $H^0(\mathcal{L}_X)$, and p_-^*.

The invariance of Z^α under diffeomorphisms $X'\to X''$ of cobordisms is obvious, as a diffeomorphism induces an isomorphism of simplicial sets $Sing(X')$ and $Sing(X'')$ representing the cobordisms and leads to isomorphic diagrams (10.5) and (10.6) in a strict sense, thus giving the same isometry l_f of hermitian lines in Section 10.2.

References

[ABG] M. Ando, A. J. Blumberg, and D. Gepner, *Parametrerized spectra, multiplicative thom spectra, and the twisted umkehr map*, Preprint.

[BCC93] M. Bullejos, P. Carrasco, and A. M. Cegarra, *Cohomology with coefficients in symmetric cat-groups. An extension of Eilenberg-Mac Lane's classification theorem*, Math. Proc. Cambridge Philos. Soc. **114** (1993), no. 1, 163–189.

[Ber07] J.E. Bergner, *A model category structure on the category of simplicial categories*, Trans. Amer. Math. Soc. **359** (2007), 2043–2058.

[BF78] A. K. Bousfield and E. M. Friedlander, *Homotopy theory of* Γ*-spaces, spectra and bisimplicial sets.*, Geometric applications of homotopy theory II, Lecture Notes in Math. (1978), no. 658.

[BM05] L. Breen and W. Messing, *Differential geometry of gerbes*, Adv. Math. **198** (2005), no. 2, 732–846.

[BR96] D. Birmingham and M. Rakowski, *On Dijkgraaf-Witten type invariants*, Lett. Math. Phys. **37** (1996), no. 4, 363374.

[Bry93] J.-L. Brylinski, *Loop spaces, characteristic classes and geometric quantization*, Birkhauser, 1993.

[BV02] D. Bourn and E. M. Vitale, *Extensions of symmetric cat-groups*, Homol. Homotopy Appl **4** (2002), 103–162.

[Cha98] D. S. Chatterjee, *On gerbs*, Ph.D. thesis, Trinity college, Cambridge, 1998.

[CMM04] P. Carrasco and J. Martínez-Moreno, *Simplicial cohomology with coeffiecients in symmetric categorical groups*, Appl. Categ. Structures **12** (2004), no. 3, 257–285.

[DK80a] W. G. Dwyer and D. M. Kan, *Calculating simplicial localizations*, Journal of Pure and Appl. Algebra **18** (1980), 17–35.

[DK80b] ______, *Simplicial localizations of categories*, Journal of Pure and Appl. Algebra **17** (1980), 267–284.

[DK80c] W.G. Dwyer and D. M. Kan, *Function complexes in homotopical algebra*, Topology **19** (1980), 427–440.

[dRMMV05] A. del Río, J. Martínez-Moreno, and E. M. Vitale, *Chain complexes of symmetric categorical groups*, J. Pure and Appl. Algebra **196** (2005), 279–312.

[DS11] D. Dugger and D. I. Spivak, *Mapping spaces in quasi-categories*, Algebr. Geom. Topol. **11** (2011), no. 1, 263–325.

[DW90] R. Dijkgraaf and E. Witten, *Topological guage theories and group cohomology*, Commun. Math. Phys. **129** (1990), no. 2, 393–429.

[EA53] S Eilenberg and MacLane A., *On groups $H(\pi, n)$. i*, Ann. of Math **58** (1953), 55–106.

[FPSV15] J. Fuchs, J. Priel, C. Schweigert, and A Valentino, *On the brauer groups of symmetries of abelian Dijkgraaf-Witten theories*, Comm. Math. Phys. **339** (2015), no. 2, 385405.

[FQ93] D. Freed and F. Quinn, *Chern-Simons theory with finite gauge group*, Commun. Math. Phys. **156** (1993), no. 3, 435–472.

[Fre94] D. Freed, *Higher algebraic structures and quantization*, Commn. Math. Phys. **159** (1994), no. 2, 343–398.

[Hir02] Phillip S. Hirchhorn, *Model categories and their localizations*, Mathematical Surveys and Monographs, vol. 99, Amer. Math. Soc., Providence, RI, 2002.

[HL14] G. Heuts and J. Lurie, *Ambidexterity*, Contemporary Math. **613** (2014), 79–110.

[Hov99] M. Hovey, *Model categories*, Mathematical Surveys and Monographs, vol. 63, Amer. Math. Soc., Providence, RI, 1999.

[Joy08a] A. Joyal, *Notes on quasi-categories*, `http://www.math.uchicago.edu/~may/IMA/Joyal.pdf`, 2008.

[Joy08b] ———, *Theory of quasi-categories and applications*, `http://mat.uab.cat/~kock/crm/hocat/advanced-course/Quadern45-2.pdf`, 2008.

[Lur] Jacob Lurie, *Higher algebra*, `http://www.math.harvard.edu/~lurie`, Preprint.

[Lur09a] J. Lurie, *On the classification of topological field theories*, Current developments in mathematics, 2008, Int. Press, Somerville, MA, 2009, pp. 129–280.

[Lur09b] Jacob Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009.

[Lyd99] M. Lydakis, *Smash products and* Γ- *spaces*, Math. Proc. Camb. Soc. **126** (1999).

[Mac71] S. MacLane, *Categories for the working mathematician*, Springer-Verlag, 1971.

[May01] J. P. May, *Picard groups, grothendieck rings, and burnside rings of categories*, Advances in Math. **163** (2001), no. 1, 1–16.

[Moe02] I. Moerdijk, *Introduction to the language of stacks and gerbes*, arxiv:Math/0212226 (2002).

[Mon15] Samuel Monnier, *Higher abelian Dijkgraaf-Witten theory*, Lett. Math. Phys. **105** (2015), no. 9, 13211331.

[Mur94] M. K. Murray, *Bundle gerbs*, arXiv:dg-ga/9407015 (1994).

[Qui67] D. G. Quillen, *Homotopical algebra*, Lecture notes in Mathematics, Springer-Verlag, 1967.

[Rie14] E. Riehl, *Categorical homotopy theory*, New Mathematical Monographs, Cambridge University Press, New York, NY, 2014.

[Sch99] S. Schwede, *Stable homotopical algebra and* Γ- *spaces*, Math. Proc. Camb. Soc. **126** (1999), 329.

[Sch08] V. Schmitt, *Enrichments over symmetric picard categories*, arXiv/0812.0150 (2008).

[Seg74] G. Segal, *Categories and cohomology theories*, Topology **13** (1974), 293–312.

[Sha] A. Sharma, *On the homotopy theory of G - spaces*, `https://arxiv.org/abs/1512.03698`, Preprint.

[SP09] C. J. Schommer-Pries, *The classification of two-dimensional extended topological field theories*, ProQuest LLC, Ann Arbor, MI, 2009, Thesis (Ph.D.)–University of California, Berkeley.

[SV] A. Sharma and A. A. Voronov, *Categorification of Dijkgraaf-Witten theory*, `https://arxiv.org/abs/1511.00295`, Preprint IPMU15-0216.

Appendix A

Infinite loop objects

We begin this section by reviewing the theory of infinite loop spaces. There are many approaches to study infinite loop spaces but we will only review Segal's approach. Following [Joy08b] and [Joy08a], we denote the ∞-category of Kan complexes by $\mathcal{K}$ and will be refered to as the ∞-category of spaces. $\mathcal{K}$ is the coherent nerve of the simplicial category of Kan complexes. We begin by describing the category of finite sets Γ^{op} whose opposite was introduced in [Seg74]. The objects of Γ^{op} are pointed finite sets and morphisms are the morphisms of pointed sets. Every object of Γ^{op} is isomorphic to the pointed set $n^+ = \{0, 1, \ldots, n\}$, pointed by $0 \in n^+$. For each $1 \leq k \leq n$, let δ_k be the pointed map $n^+ \to 1^+$ defined by putting

$$\delta^n_k = \begin{cases} 1 & \text{if} \quad x = k \\ 0 & \text{if} \quad x \neq k \end{cases} \tag{A.1}$$

Now we briefly review the theory of a limit sketches in ∞-categories. A *projective cone* in a simplicial set A is a map of simplicial sets $c : 1 \star K \to A$, where $1 \star K$ denotes the *join* of the terminal simplicial set 1 with K. For details on the join of simplicial sets see [Joy08b].

Definition A.1. A *limit sketch* is a pair (A, P), where A is a simplicial set and P is a set of projective cones in A.

If X is an ∞-category and (A, P) a limit sketch then

Definition A.2. A map $f : A \to X$ is called a *model* of a limit sketch (A, P) in X if it takes every limit cone $c : 1 \star K \to A$ in L to an exact cone $fc : 1 \star K \to X$.

We shall write $f : A/P \to X$ to indicate that the map $f : A \to X$ is a model of the limit sketch (A, P). The subcategory of the ∞-category X^A spanned by all models of the limit sketch (A, P) will be called the ∞-category of models of (A, P) in X and will be denoted by $\mathcal{M}odel(A/P, X)$. If X is the ∞-category $\mathcal{K}$, then it will be denoted just by $\mathcal{M}odel(A/P)$.

Example A.3. Let X be an ∞-category which has finite limits. Consider the limit sketch $(N(\Delta^{op}), P)$, where P is the set of all projective cones $c : 1 \star \Lambda^2[2] \to N(\Delta^{op})$ such that the two, non-identity, arrows of the 2-horn are mapped to the following two composite maps

$$[n-m] \overset{\delta_0}{\to} [n-m+1] \overset{\delta_0}{\to} \dots \overset{\delta_0}{\to} [n-1] \overset{\delta_0}{\to} [n]$$

and

$$[m] \overset{\delta_{m+1}}{\to} [m+1] \overset{\delta_{m+2}}{\to} \dots \overset{\delta_{n-1}}{\to} [n-1] \overset{\delta_n}{\to} [n].$$

and the remaining two, non-identity, arrows of the square $1 \star \Lambda^2[2]$ are mapped to $\epsilon_0 : [0] \to [n-m]$ and $\epsilon_m : [0] \to [m]$ respectively. A *category object* in X is a model of the limit sketch $(N(\Delta^{op}), P)$ in X. In other words, a simplicial object is a category object if it satisfies the *Segal condition*. The ∞-category of category objects in X is denoted $Cat(X)$ and is defined as $Cat(X) := \mathcal{M}odel((N(\Delta^{op})/P), X)$. A monoid in X is a model of the limit sketch $(N(\Delta^{op}), P \cup \{1\})$ in X. In other words a monoid in X is a category object $C : N(\Delta^{op}) \to X$ such that $C_0 \simeq 1$, where 1 is a final object of X. The ∞-category of monoids in X is denoted $Mon(X)$ and is defined as $Mon(X) := \mathcal{M}odel((N(\Delta^{op})/P \cup \{1\}), X)$.

Example A.4. The idea of a *groupoid* object in X is that it is a category object $C : N(\Delta^{op}) \to X$ which takes the square

$$\begin{tikzcd} {[0]} \arrow[r, "\epsilon_0"] \arrow[d, "\epsilon_0"'] & {[1]} \arrow[d, "\delta_0"] \\ {[1]} \arrow[r, "\delta_1"'] & {[2]} \end{tikzcd}$$

to a pullback square

$$\begin{tikzcd} C_2 \arrow[r, "m"] \arrow[d, "\partial_0"'] & C_1 \arrow[d, "t"] \\ C_1 \arrow[r, "t"'] & C_0 \end{tikzcd}$$

in X. The morphisms δ_0 and δ_1 above can be replaced by δ_1 and δ_2. A groupoid object is defined as a model of a certain limit sketch. This limit sketch is obtained by adding the following pair of cones to the set P above. Let $g_1 : 1 \star \Lambda^2[2] \to N(\Delta^{op})$ be a projective cone which maps the two non-identity morphisms of the 2-horn to the maps $\delta_0 : [1] \to [2]$ and $\delta_1 : [1] \to [2]$ and the remaining two non-identity morphisms to $\epsilon_0 : [0] \to [1]$ and $g_2 : 2 \star \Lambda^2[2] \to N(\Delta^{op})$ be a projective cone which maps the two non-identity morphisms of the 2-horn to the maps $\delta_2 : [1] \to [2]$ and $\delta_1 : [1] \to [2]$ and the remaining two non-identity morphisms to $\epsilon_0 : [0] \to [1]$. Let $(N(\Delta^{op}), P^G)$ denote the limit sketch $(N(\Delta^{op}), P \cup \{g_1, g_2\})$. Now we can define a *groupoid object* in X to a model of the limit sketch $(N(\Delta^{op}), P^G)$. The ∞-category of groupoid objects in X is denoted $\mathcal{G}(X)$ and is defined as $\mathcal{G}(X) := \mathcal{M}odel(N(\Delta^{op})/P^G, X)$. A *group* in X is a model of the limit sketch $(N(\Delta^{op}), P^G \cup \{1\})$ in X. In other words a group in X is a groupoid object $G : N(\Delta^{op} \to X$ such that $G_0 \simeq 1$, where 1 is a final object of X. The ∞-category of groups in X is denoted $\mathcal{A}_\infty(X)$ and is defined as $\mathcal{A}_\infty(X) := \mathcal{M}odel((N(\Delta^{op})/P \cup \{1\}), X)$.

Remark. If X is the ∞-category underlying the model category of topological spaces $\mathbf{Top}_\bullet$, then a group will be called an A_∞-space. We will sometimes refer to a group as an A_∞-space.

The ∞-category of groups in $\mathcal{K}_\bullet$ will be denoted by $\mathcal{A}_\infty$.

A Γ-*object* in X is defined to be a map of simplicial sets $E : N(\Gamma^{op}) \to X$ such that $E(0^+) \simeq 1$, where 1 is a terminal object in X. From the morphisms $E(\delta_k^n) : E(n^+) \to E(1+)$, we obtain a morphism

$$(E(\delta_1^n), \dots, E(\delta_n^n)) : E(n^+) \to \prod_1^n E(1^+)$$

We shall say that E is an $\mathcal{E}_\infty$-*object* if $(E(\delta_1^n), \dots, E(\delta_n^n))$ is invertible for every $n \geq 0$. An $\mathcal{E}_\infty$ object in the ∞-category $\mathcal{K}_\bullet$ will be called an $\mathcal{E}_\infty$-space. The ∞-category of $\mathcal{E}_\infty$-spaces will be denoted by $\mathcal{E}_\infty$.

Example A.5. The notion of an $\mathcal{E}_\infty$-object can be defined as a model of a limit sketch. The n projection maps A.1 define a discrete cone $c_n : 1 \star n \to N(\Gamma^{op})$, where n is the unordered set $\{1, \dots, n\}$ considered as a discrete simplicial set. The *limit sketch of* $\mathcal{E}_\infty$ *objects* $(N(\Gamma^{op}), E)$ is defined by putting $E = \{c_n; n \geq 0\}$. The ∞-category of $\mathcal{E}_\infty$ objects in an ∞-category X is defined as follows

$$\mathcal{E}_\infty(X) := Model(\Gamma^{op}/E, X).$$

In particular, the ∞-category of $\mathcal{E}_\infty$ spaces is defined by $\mathcal{E}_\infty := Model(\Gamma^{op}/E, \mathcal{K}_\bullet)$.

There is a functor $i : \Delta^{op} \to \Gamma^{op}$ obtained by putting $i([n]) = Hom(\Delta[n], S^1)$, where $S^1 = \Delta[1]/\partial\Delta[1]$ is the pointed simplicial circle. The map $\mathcal{K}_\bullet{}^i : \mathcal{K}_\bullet{}^{N(\Gamma^{op})} \to \mathcal{K}_\bullet^{N(\Delta^{op})}$ takes an $\mathcal{E}_\infty$ space E to the monoid $i^\bullet : \Delta^\circ \to \mathcal{K}_\bullet$ which will be referred to as the *monoid underlying the* $\mathcal{E}_\infty$ *space* E.

Definition A.6. An $\mathcal{E}_\infty$-object E will be called an *infinite loop object* in X if its underlying monoid $i^\bullet(E)$ is a group in X. An infinite loop object in $\mathcal{K}_\bullet$ will be called an *infinite loop space*.

In topology, the term *infinite loop space* is used for the degree one pointed topological space of an infinite loop object in the ∞-category of pointed topological spaces (the ∞-category underlying the model category $\mathbf{Top}_\bullet$).The ∞-category of infinite loop objects in an ∞-category X will be denoted by $\mathcal{L}_\infty(X)$. The ∞-category of infinite loop spaces will be denoted just by $\mathcal{L}_\infty$.

The following example defines an infinite loop space as a model of a limit sketch

Example A.7. The limit sketch whose models define infinite loop objects in an ∞-category X is obtained by adding another cones to the limit sketch of $\mathcal{E}_\infty$ objects described in example A.5. There is a morphism $m : 2^+ \to 1^+$ in Γ^{op} which is defined by $m(1) = m(2) = 1$. We want to obtain two projective cones $m_1 : 1 \star \Lambda^2[2] \to N(\Gamma^{op})$ and $m_1 : 1 \star \Lambda^2[2] \to N(\Gamma^{op})$ by extending a morphism of simplicial sets $t : \Lambda^2[2] \to N(\Gamma^{op})$ which we define below. The simplicial set $\Lambda^2[2]$ is generated by by *elementary face operators* δ^2_1 and δ^2_0. The simplicial map t is defined by assigning to both generators the unique map $1^+ \to 0^+$ in Γ^{op} i.e. $t(\delta^2_1) = t(\delta^2_1) = 1^+ \to 0^+$. One extension of t to the join $1 \star \Lambda^2[2]$ is determined by the arrows m and $\delta^2_2 : 2^+ \to 1^+$ in Γ^{op} this extension defines the cone m_1. Another extension of the morphism t to the join $1 \star \Lambda^2[2]$ is determined by the arrows m and $\delta^2_1 : 2^+ \to 1^+$ in Γ^{op} this extension defines the cone m_2.

The *limit sketch of infinite loop spaces* $(N(\Gamma^{op}), L)$ is defined by defining the set of cones $L := E \cup \{m_1, m_2\}$. The ∞-category of $\mathcal{L}_\infty$ objects in an ∞-category X is defined as follows

$$\mathcal{L}_\infty(X) := Model(\Gamma^{op}/L, X).$$

In particular, the ∞-category of $\mathcal{L}_\infty$ spaces is defined by $\mathcal{L}_\infty := Model(\Gamma^{op}/E, \mathcal{K}_\bullet)$.

Remark. If a morphism of simplicial sets $f : N(\Gamma^{op}) \to X$ takes either one of the cones m_1 or m_2 to an exact cone in an ∞-category X, then f takes both cones to exact cones in X

We now give some examples of infinite loop objects

Example A.8. Let A be an abelian group, then the constant simplicial set $A : \Delta^o \to$ **Set** such that $A_n = A$, for all $n \geq 0$ is an infinite loop space. Further, $\pi_0 A = A$ and $A = \Omega(BA)$, where BA is the classifying space $K(A, 1)$.

Example A.9. Let A be a symmetric monoidal groupoid such that for every object $a \in A$ the following two functors are equivalences of ordinary categories

$$a \otimes - : A \to A;$$

$$- \otimes a : A \to A.$$

The nerve of A, $N(A)$, is an infinite loop space.

Appendix B

The small object argument

Given any category $\mathcal{C}$, one may construct a *morphism category of* $\mathcal{C}$ whose objects are arrows of $\mathcal{C}$ and whose morphisms are commutative diagrams. We denote this category by $Mor(\mathcal{C})$.

Definition B.1. A *functorial factorization system* on a category $\mathcal{C}$ is an ordered pair (α, β) of functors, whose domain and codomain category is $Mor(\mathcal{C})$, such that $f = \beta(f) \circ \alpha(f)$ for all $f \in Mor(\mathcal{C})$. In particular, the domain of $\alpha(f)$ is the same as the domain of f, the codomain of $\alpha(f)$ is the domain of $\beta(f)$ and the codomain of $\beta(f)$ is the codomain of f.

If $\mathcal{C}$ is a cocomplete category, then by applying a functorial factorization to the unique arrow from the initial object of $\mathcal{C}$ to any object of $\mathcal{C}$, one gets a functor which we call the *cofibrant replacement functor* determined by the functorial factorization system.

Proposition B.2. *A functorial factorization system,* (α, β)*, on a cocomplete category* $\mathcal{C}$ *uniquely determines a cofibrant replacement functor* $Q : \mathcal{C} \to \mathcal{C}$ *and also a natural transformation* $Q \Rightarrow id_{\mathcal{C}}$.

Given a cocomplete category $\mathcal{C}$ and a set $\mathcal{J}$ of maps, we denote

1. by $\mathcal{J} - inj$ the subcategory of $\mathcal{C}$ consisting of maps which have the right lifting property with respect to the maps in $\mathcal{J}$. Maps in $\mathcal{J} - inj$ will be refered to as *$\mathcal{J}$-injectives.*

2. by $\mathcal{J} - cof$ the subcategory of $\mathcal{C}$ consisting of maps which have the left lifting property with respect to the maps in $\mathcal{J} - inj$. Maps in $\mathcal{J} - cof$ will be refered to as *$\mathcal{J}$-cofibrations.*

3. by $\mathcal{J} - cof_{reg}$ the subcategory of (possibly transfinite) compositions of maps that can be obtained by cobase change from maps in $\mathcal{J}$. Maps in $\mathcal{J} - cof_{reg}$ will be referred to as *regular $\mathcal{J}$-cofibrations.*

Quillen's small object argument [Qui67, II p.3.4] has the following transfinite analog

Lemma B.3. *Let $\mathcal{C}$ be a cocomplete category and $\mathcal{J}$ be a set of maps in $\mathcal{C}$ whose domains are small with respect to $\mathcal{J} - cof_{reg}$. Then*

1. *there is a functorial factorization of maps $f \in Mor(\mathcal{C})$ as $f = qi$ with $q \in \mathcal{J} - inj$ and $i \in \mathcal{J} - cof_{reg}$.*

2. *every $\mathcal{J}$-cofibration is a retract of a regular $\mathcal{J}$-cofibration.*

Corollary B.4. *Every cofibrantly generated model category has a functorial factorization system.*

B.1 A Functorial Factorization system on $\Gamma\mathcal{S}$

Corollary B.4 garuntees the existence of a functorial factorization system and therefore also the existence of a cofibrant replacement functor on $\Gamma\mathcal{S}$. In this section we provide a construction for both on the category $\Gamma\mathcal{S}$.

Appendix C

Mapping spaces of quasicategories

Any justification of quasicategories as a model for $(\infty, 1)$-categories would have to be based on the existence of hom-spaces between objects (vertices) of the quasicategory. Further, these hom-spaces should represent some well defined homotopy types. These homotopy types, as objects of the enriched homotopy category of spaces $\mathcal{H}$, should have the property that their underlying set, which is obtained by applying the functor $\pi_0 : \mathcal{H} \to \mathbf{Set}$, coincides with the hon-sets of the homotopy category associated to the quasicategory. In this chapter we discuss three possible models for hom-spaces for a quasicategory and show that all three are homotopically equivalent Kan-complexes.

Exploiting the cartesian closure of the category of simplicial sets, for any quasicategory X, we have another quasicategory $X^{\Delta[1]}$, whose vertices are the 1-simplices in X and whose n-simplices are (simplicial) maps $\Delta[n] \times \Delta[1] \to X$. For any pair of vertices $(x, y) \in X_0 \times X_0$, we define our first model for a hom-space, denoted $Hom_X(x, y)$, by the following pullback

$$\begin{tikzcd}
Hom_X(x,y) \arrow[rr] \arrow[d] & & X^{\Delta[1]} \arrow[d] \\
* \arrow[r, "{(x,y)}"'] & X \times X \cong X^{\partial\Delta[1]} &
\end{tikzcd}$$

The map $X^{\Delta[1]} \to X^{\partial\Delta[1]}$ is a trivial fibration of simplicial sets and the class of fibrations is closed under pullbacks, therefore the simplicial set $Hom_X(x,y)$ is a Kan complex. An n-simplex in $Hom_X(x,y)$ is a map $\Delta[n] \times \Delta[1] \to X$ such that the image of $\Delta[n] \times \{0\}$ is degenerate at x and the image of $\Delta[n] \times \{1\}$ is degenerate at y. In particular, a 1-simplex looks like

$$\begin{tikzcd}
x \arrow[r, "f"] \arrow[d, equal] \arrow[dr] & y \arrow[d, equal] \\
x \arrow[r, "g"'] & y
\end{tikzcd}$$

Now it is easy to see that

$$\pi_0 Hom_X(x,y) = Hom_{hX}(x,y).$$

A less symmetric construction is also possible. Let $Hom_X^R(x,y)$ denote the simplicial set whose 0-simplices are all $f \in X_1$ such that $d_1(f) = x$ and $d_0(f) = y$, whose 1-simplices are all $F \in X_2$ such that $d_2(F) = s_0(x)$ and $\epsilon_0^{n+1}(F) = y$ i.e. a 1-simplex looks like this:

$$\begin{tikzcd}
& x \arrow[dr] \arrow[d, phantom, "\sim"] & \\
x \arrow[ur, equal] \arrow[rr, "g"'] & \, & y
\end{tikzcd}$$

An n-simplex in $Hom_X^R(x,y)$ is an $(n+1)$-simplex of X, whose $(n+1)th$ vertex is degenerate at y and whose $(n+1)th$ face is degenerate at x. Dually, $Hom_X^L(x,y)$ is the simplicial set whose n-simplices are $(n+1)$-simplices of X whose zeroth vertex is x and whose zeroth face is degenerate at y. Once again it is easy to see that

$$\pi_0 Hom^L_X(x,y) = \pi_0 Hom^R_X(x,y) = Hom_{hX}(x,y).$$

The main objective of this chapter is to exibit natural categorical equivalences between the three mapping spaces described above. Our exposition follows [DS11] and [Rie14]. Each of the three mapping spaces have the same zero simplices. We want to describe n-simplex of $Hom^L_X(x,y)$ and $Hom^R_X(x,y)$ with the help of the following two pushout diagrams:

$$\begin{tikzcd} \Delta[n] \arrow[r] \arrow[d, "d_0"'] & \Delta[0] \arrow[d] \\ \Delta[n+1] \arrow[r] & \Delta[n+1]_{0|1} \end{tikzcd} \qquad \begin{tikzcd} \Delta[n] \arrow[r] \arrow[d, "d_{n+1}"'] & \Delta[0] \arrow[d] \\ \Delta[n+1] \arrow[r] & \Delta[n+1]_{n|n+1} \end{tikzcd}$$

The notation above requires explaination which we provide now. Surjection $\Delta[n] \to \Delta[1]$ correspond to integers $0 \leq i < n$, which partition the vertices of $\Delta[n]$ into $[0, 1, \ldots, i]$, which is the fiber over the vertex 0 of $\Delta[1]$, and $[i, i+1, \ldots, n]$, which is the fiber over the vertex 1 of $\Delta[1]$. We denote by $\Delta[n]_{i|i+1}$, the quotient space obtained by collapsing the face of $\Delta[n]$ which is spanned by vertices $[0, 1, \ldots, i]$ to a point and the face spanned by the vertices $[i, i+1, \ldots, n]$ to another point. This quotient space has two vertices and one nondegenerate k-simplex for each nondegenerate k-simplex of $\Delta[n]$ whose image surjects onto $\Delta[1]$.

Similarly, we want to describe the simplicial set $Hom_X(x,y)$ with the help of a pushout diagram, which is the following:

$$\begin{tikzcd} \Delta[n] \times \partial\Delta[1] \arrow[r] \arrow[d, "id \times i_1"'] & \partial\Delta[1] \arrow[d] \\ \Delta[n] \times \Delta[1] \arrow[r] & C^n_{cyl} \end{tikzcd}$$

We write C^n_L and C^n_R for $\Delta[n+1]_{0|1}$ and $\Delta[n+1]_{n|n+1}$ respectively. This notation emphasizes that the above constructions define three cosimplicial objects $C^\bullet_L, C^\bullet_R, C^\bullet_{cyl}$, taking values in the category of simplicial sets and maps preserving two chosen basepoints. The target of these cosimplicial objects is the slice category $\partial\Delta[1]/\mathbf{sSets}$, which we denote by $\mathbf{sSets}_{\bullet,\bullet}$. Now the quasicategory X with two chosen vertices, x and y, is an object of the category $\mathbf{sSets}_{\bullet,\bullet}$. Now we are ready to define the three mapping spaces using the pushout diagrams above. These definitions are as follows:

$$Hom_X^L(x,y) = \mathbf{sSets}_{\bullet,\bullet}(C_L^\bullet, X);$$

$$Hom_X^R(x,y) = \mathbf{sSets}_{\bullet,\bullet}(C_R^\bullet, X);$$

and

$$Hom_X(x,y) = \mathbf{sSets}_{\bullet,\bullet}(C_{cyl}^\bullet, X);$$

Appendix D

Simplicial Model categories

Before we define a *simplicial model category*, we recall the definition of a *simplicial category*

Definition D.1. A *simplicial category* (also called a *simplicially enriched category*) $\mathcal{M}$ is a category together with the following data:

1. For every pair of objects X, Y of $\mathcal{M}$, a simplicial set $Map_{\mathcal{M}}(X, Y)$ which is called the *function complex* from X to Y or the *simplicial mapping space* from X to Y.

2. For every triple of objects X, Y and Z of $\mathcal{M}$, a map of simplicial sets

$$c^{\mathcal{M}}_{X,Y,Z} : Map_{\mathcal{M}}(X, Y) \times Map_{\mathcal{M}}(Y, Z) \to Map_{\mathcal{M}}(X, Z),$$

which is called the *composition law.*

3. For every object X of $\mathcal{M}$, a map of simplicial sets $i_X : \Delta[0] \to Map_{\mathcal{M}}(X, X)$, and finally

4. For every pair of objects X, Y of $\mathcal{M}$, a bijection $Map_{\mathcal{M}}(X, Y)_0 \cong \mathcal{M}(X, Y)$ which commutes with the composition law.

This data is subject to the following three axioms for all quadruple of objects W, X, Y and Z of $\mathcal{M}$:

(Associativity) The following diagram commutes

$$\begin{tikzcd}
(Map(Y,Z) \times Map(X,Y)) \times Map(W,X) \arrow[rr] \arrow[d, "\cong"'] & & Map(X,Z) \times Map(W,X) \arrow[dd, "c^{\mathcal{M}}_{W,X,Z}"] \\
Map(Y,Z) \times (Map(X,Y) \times Map(W,X)) \arrow[d, "id_{Map(Y,Z)} \times c^{\mathcal{M}}_{W,Y,Z}"'] & & \\
Map(Y,Z) \times Map(W,Y) \arrow[rr, "c^{\mathcal{M}}_{W,Y,Z}"'] & & Map(W,Z).
\end{tikzcd}$$

(Left unit) The following diagram commutes

$$\begin{tikzcd}
* \times Map(X,Y) \arrow[rr, "i_Y \times id_{Map(X,Y)}"] \arrow[dr] & & Map(X,Y) \times Map(X,X) \arrow[dl] \\
& Map(X,Y). &
\end{tikzcd}$$

(Right unit) The following diagram commutes

$$\begin{tikzcd}
Map(X,Y) \times * \arrow[rr, "id_{Map(X,Y)} \times i_X"] \arrow[dr] & & Map(X,Y) \times Map(X,X) \arrow[dl] \\
& Map(X,Y). &
\end{tikzcd}$$

Definition D.2. A simplicial category $\mathcal{M}$ will be called a *fibrant* simplicial category if, for each pair of objects X, Y of $\mathcal{M}$, the simplicial mapping space $Map_{\mathcal{M}}(X,Y)$ is a Kan complex.

Remark. A simplicial category if fibrant if and only if it is a fibrant object of the model category structure described in [Ber07].

After discussing the notion of a simplicial category, now we want to define the notion of a *simplicial model category*

Definition D.3. A simplicial model category is a model category $\mathcal{M}$ which is also a simplicial category in the sense of definition D.1 such that the following two axioms are satisfied:

(M6) For any two objects $X, Y \in \mathcal{M}$ and every simplicial set K there are objects $X \otimes K$ and $\mathbf{hom}_{\mathcal{M}}(K, \mathcal{M})$ of $\mathcal{M}$ such that we have isomorphisms of simplicial sets

$$Map(X \otimes K, Y) \cong Map_{\mathbf{sSets}}(K, Map_{\mathcal{M}}(X, Y)) \cong Map_{\mathcal{M}}(X, \mathbf{hom}_{\mathcal{M}}(K, \mathcal{M})),$$

which are natural in X, Y and K.

(M7) Let $i : A \to B$ be a cofibration in $\mathcal{M}$ and $p : X \to Y$ is a fibration in $\mathcal{M}$, that the map of simplicial sets

$$Map_{\mathcal{M}}(B, X) \overset{i^* \times p_*}{\to} Map_{\mathcal{M}}(A, X) \times_{Map_{\mathcal{M}}(A,Y)} Map_{\mathcal{M}}(B, Y)$$

is a fibration, which is a trivial fibration if either i or p is a weak equivalence.

Remark. A simplicial model category is a $(\mathbf{sSets}, \mathbf{Kan})$-enriched model category in the sense of definition 3.14.

A model category structure was constructed on the category of all (small) simplicial categories in [Ber07]. We now review some relevent notions from that model category structure.

Definition D.4. A functor of simplicial categories, $F : \mathcal{C} \to \mathcal{D}$, is called a *DK-equivalence* if it satisfies the following two conditions:

1. For any pair of objects $c_1, c_2 \in \mathcal{C}$, the following morphism

$$F_{c_1,c_2} : \mathcal{M}ap_{\mathcal{C}}(c_1, c_2) \to \mathcal{M}ap_{\mathcal{C}}(F(c_1), F(c_2)).$$

is a weak homotopy equivalence of simplicial sets.

2. The induced functor $\pi_0(F) : \pi_0(\mathcal{C}) \to \pi_0(\mathcal{D})$ is an equivalence of (ordinary) categories.

Appendix E

Enrichment over Γ-spaces

The stable Q-model category structure on $\Gamma\mathcal{S}$ is a *monoidal* model category structure, see [Sch99]. The objective of this chapter is to show that $\Gamma\mathcal{S}$ is a $\Gamma\mathcal{S}$ enriched model category. We begin with a few definitions:

Definition E.1. Suppose $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ are categories. An *adjunction of two variables* from $\mathcal{C} \times \mathcal{D}$ to $\mathcal{E}$ is a quintuple $(\otimes, \mathbf{hom}_{\mathcal{C}}, \mathcal{M}ap_{\mathcal{C}}, \phi, \psi)$, where

$$\otimes : \mathcal{C} \times \mathcal{D} \to \mathcal{E}, \quad \mathbf{hom}_{\mathcal{C}} : \mathcal{D}^{op} \times \mathcal{E} \to \mathcal{C}, \text{ and } \quad \mathcal{M}ap_{\mathcal{C}} : \mathcal{C}^{op} \times \mathcal{E} \to \mathcal{D}$$

are functors and ϕ, ψ are the following natural transformations

$$\mathcal{C}(C, \mathbf{hom}_{\mathcal{C}}(D, E)) \underset{\cong}{\overset{\phi^{-1}}{\to}} \mathcal{E}(\mathcal{C} \otimes D, E) \underset{\cong}{\overset{\psi}{\to}} \mathcal{D}(D, \mathcal{M}ap_{\mathcal{C}}(C, E)).$$

The following definition is based on Quillen's $SM7$ axiom, see [Qui67], and is also found in [**?**].

Definition E.2. Given model categories $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$, an adjunction of two variables, $(\otimes, \mathbf{hom}_{\mathcal{C}}, \mathcal{M}ap_{\mathcal{C}}, \phi, \psi) : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$, is called a *Quillen adjunction of two variables*, if, given a cofibration $f : U \to V$ in $\mathcal{C}$ and a cofibration $g : W \to X$ in $\mathcal{D}$, the induced map

$$f \square g : (V \otimes W) \coprod_{U \otimes W} (U \otimes X) \to V \otimes X$$

is a cofibration in $\mathcal{E}$ that is trivial if either f or g is. We will refer to the left adjoint of a Quillen adjunction of two variables as a *Quillen bifunctor.*

The following lemma provides three equivalent characterizations of the notion of a Quillen bifunctor. These will be useful in this paper in establishing enriched model category structures.

Lemma E.3. *[Hov99, Lemma 4.2.2] Given model categories $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$, an adjunction of two variables, $(\otimes, \mathbf{hom}_{\mathcal{C}}, \mathcal{M}ap_{\mathcal{C}}, \phi, \psi) : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$. Then the following conditions are equivalent:*

(1) $\otimes : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ is a Quillen bifunctor.

(2) Given a cofibration $g : W \to X$ in $\mathcal{D}$ and a fibration $p : Y \to Z$ in $\mathcal{E}$, the induced map

$$\mathbf{hom}^{\square}_{\mathcal{C}}(g,p) : \mathbf{hom}_{\mathcal{C}}(X,Y) \to \mathbf{hom}_{\mathcal{C}}(X,Z) \underset{\mathbf{hom}_{\mathcal{C}}(W,Z)}{\times} \mathbf{hom}_{\mathcal{C}}(W,Y)$$

is a fibration in $\mathcal{C}$ that is trivial if either g or p is a weak equivalence in their respective model categories.

(3) Given a cofibration $f : U \to V$ in $\mathcal{C}$ and a fibration $p : Y \to Z$ in $\mathcal{E}$, the induced map

$$\mathcal{M}ap^{\square}_{\mathcal{C}}(f,p) : \mathcal{M}ap_{\mathcal{C}}(V,Y) \to \mathcal{M}ap_{\mathcal{C}}(V,Z) \underset{\mathcal{M}ap_{\mathcal{C}}(W,Z)}{\times} \mathcal{M}ap_{\mathcal{C}}(W,Y)$$

is a fibration in $\mathcal{C}$ that is trivial if either f or p is a weak equivalence in their respective model categories.

We will give a proof of the last isomorphism and leave the proof of the other two isomorphisms as an excercise for the interested reader.

Proposition E.4. *For any pointed simplicial set $K \in \mathbf{sSets}_\bullet$ and any pair of Γ-spaces $(F,G) \in \Gamma\mathcal{S} \times \Gamma\mathcal{S}$, there is an isomorphism of Γ-spaces*

$$\mathbf{hom}_{\Gamma\mathcal{S}}(K, Hom(F,G)) \xrightarrow{\cong} Hom(F, \mathbf{hom}_{\Gamma\mathcal{S}}(K,G))$$

Proof. We begin the proof by observing that, for any Γ-space G, the pair of functors $\mathbf{hom}(-,G) : \mathbf{sSets}_\bullet^{op} \to \Gamma\mathcal{S}$ and $hom(-,B) : \Gamma\mathcal{S} \to \mathbf{sSets}_\bullet^{op}$ are adjoints of an adjunction $< \mathbf{hom}(-,G), hom(-,G), \eta^{1^+}, \epsilon^{1^+} >: \Gamma\mathcal{S} \to \mathbf{sSets}_\bullet^{op}$. The *counit* of this adjunction, which is a natural transformation of simplicial categories, assigns to each Γ-space F, a morphism of Γ-spaces

$$\epsilon_F^{1^+} : F \to \mathbf{hom}(hom(F,G),G).$$

The categories $\mathbf{sSets}_\bullet$ and $\Gamma\mathcal{S}$ can be enriched over the category of pointed simplicial sets $\mathbf{sSets}_\bullet$. The functor $\mathbf{hom}(-,G)$ extends to a functor of categories enriched over pointed simplicial sets. We will refer to such functors as *pointed simplicial functors*. This implies that for every pair, (K,L), of pointed simplicial sets, the (extended) functor $\mathbf{hom}(-,G)$ provides a morphism of pointed simplicial sets

$$\mathbf{hom}(-,G)_{K,L} : (K)_\bullet^L \to Hom(\mathbf{hom}(L,G), \mathbf{hom}(K,G))(1^+)$$

Further, we observe that the composite map

$$(\epsilon_F^{1^+})^* \circ \mathbf{hom}(-,G)_{K,Hom(F,G)(1^+)} : \mathbf{hom}(K, Hom(F,G))(1^+) \to Hom(F, \mathbf{hom}(K,G))(1^+)$$

is an isomorphim of pointed simplicial sets.

We claim that the isomorphism of pointed simplicial sets $(\epsilon_F^{1^+})^* \circ \mathbf{hom}(-,G)_{K,Hom(F,G)(1^+)}$ extends to an isomorphism of Γ-spaces

$$(\epsilon_F^\bullet)^* \circ \mathbf{hom}(-,G)_{K,Hom(F,G)(\bullet)} : \mathbf{hom}(K, Hom(F,G)) \to Hom(F, \mathbf{hom}(K,G)).$$

In order to prove this claim we observe that for each $k^+ \in \Gamma^{op}$, we have an adjunction $< \mathbf{hom}(-, G(k^+ \wedge -)), hom(-, G(k^+ \wedge -)), \eta^{k^+}, \epsilon^{k^+} >: \Gamma\mathcal{S} \to \mathbf{sSets}_\bullet^{op}$. The counit of each of these adjunctions gives, for each $F \in \Gamma\mathcal{S}$, a morphism of Γ-spaces

$$\epsilon_F^{k^+} : F \to \mathbf{hom}(hom(F, G(k^+ \wedge -)), G(k^+ \wedge -)).$$

As above, each of these adjunctions determine an isomorphism of pointed simplicial sets

$$(\epsilon_F^{k^+})^* \circ \mathbf{hom}(-, G)_{K, Hom(F,G)(k^+)} : \mathbf{hom}(K, Hom(F, G))(k^+) \to Hom(F, \mathbf{hom}(K, G))(k^+)$$

The following collection of morphisms of pointed simplicial sets,

$$\{(\epsilon_F^{k^+})^* \circ \mathbf{hom}(-, G)_{K, Hom(F,G)(k^+)}; k^+ \in \Gamma^{op}\},$$

determines a morphism of Γ-spaces. This follows from the isomorphism $Hom(F, G(k^+ \wedge -)) \cong Hom(F, G)(k^+ \wedge -)$ and the fact that any morphism $k^+ \to l^+$ in Γ^{op} determines a natural transformation of (pointed) simplicial functors $\mathbf{hom}(-, G(k^+ \wedge -)) \dot{\to} \mathbf{hom}(-, G(l^+ \wedge -))$. In other words, we define the morphism $(\epsilon_F^{\bullet})^* \circ \mathbf{hom}(-, G)_{K, Hom(F,G)(\bullet)}$ as follows

$$(\epsilon_F^{\bullet})^* \circ \mathbf{hom}(-, G)_{K, Hom(F,G)(\bullet)}(k^+) := \epsilon_F^{k^+} \circ \mathbf{hom}(-, G)_{K, Hom(F,G)(k^+)}.$$

This morphism of Γ-spaces is an isomorphism of pointed simplicial sets in each degree, therefore $(\epsilon_F^{\bullet})^* \circ \mathbf{hom}(-, G)_{K, Hom(F,G)(\bullet)}$ is an isomorphism of Γ-spaces.

□

We will need the following lemma in order to prove E.7. This lemma indicated the idea that a map of Γ-spaces, $f : F \to G$, extends to a morphism of Γ-spaces from the n-fold (homotopy) product of F to the n-fold (homotopy) product of G.

Lemma E.5. *Each $n^+ \in \Gamma^{op}$ determines a functor of simplicial categories*

$$\prod_{,}^{n,1} : \Gamma\mathcal{S} \to \Gamma\mathcal{S},$$

whose object function is defined by the assignment $F \mapsto F(n^+ \wedge -)$.

Proof. For every pair, F', F, of Γ-spaces, we have to define a simplicial function

$$\prod_{F',F}^{n,1} : hom(F', F) \to hom(F'(n^+ \wedge -), (F'(n^+ \wedge -)).$$

For any $n^+ \in \Gamma^{op}$, we get a functor

$$n^+ \wedge - : \Gamma^{op} \to \Gamma^{op},$$

where $- \wedge - : \Gamma^{op} \times \Gamma^{op} \to \Gamma^{op}$ is the bifunctor defined in appendix **??**. Recall that any morphism of Γ-spaces is a natural transformation. In degree k, we define this function as follows:

$$\left(\prod_{F',F}^{n,1}(k)\right)(f) := f \circ id_{n^+ \wedge -}.$$

The right side of the above definition is the horizontal composition of the two natural transformations $id_{n^+ \wedge -}$ and $f \in hom(F', F)_k$. Clearly, this is a simplicial map. When F' is the same as F, it is easy to see that

$$\left(\prod_{F',F}^{n,1}(0)\right)(id_F) = id_{F(n^+ \wedge -)}.$$

□

Corollary E.6. *For any pair of* Γ*-spaces* QF', QF *and all* $n^+, m^+ \in Ob(\Gamma^{op})$*, we have a morphism of simplicial sets*

$$\prod_{F',F}^{n,m} : hom(QF', QF(m^+ \wedge -) \to hom(QF'(n^+ \wedge -), QF((m^+ \wedge n^+) \wedge -)).$$

Further, this simplicial morphism is natural in both m^+ *and* n^+*.*

Proof. The existence follows from the lemma above. □

Theorem E.7. *For a given triple* F', F *and* G *of* Γ*-spaces, the simplicial composition morphism*

$$- \circ - : Hom_{\mathbf{D}(\Gamma\mathcal{S}^f)}(F', F) \times Hom_{\mathbf{D}(\Gamma\mathcal{S}^f)}(F, G) \to Hom_{\mathbf{D}(\Gamma\mathcal{S}^f)}(F', G)$$

extends to a morphism of Γ*-spaces*

$$- \underset{\mathbf{D}(\Gamma\mathcal{S}^f)}{\circ} - : \underline{Hom}_{\mathbf{D}(\Gamma\mathcal{S}^f)}(F', F) \wedge \underline{Hom}_{\mathbf{D}(\Gamma\mathcal{S}^f)}(F, G) \to \underline{Hom}_{\mathbf{D}(\Gamma\mathcal{S}^f)}(F', G)$$

Proof. The mapping space $Hom_{\mathbf{D}(\Gamma\mathcal{S}^f)}(F', F)$ is a pointed simplicial set for every pair $F', F \in Ob(\Gamma\mathcal{S}) \times Ob(\Gamma\mathcal{S})$. It is easy to see that the simplicial composition map, $- \circ -$:, is a morphism of pointed simplicial sets and induces a morphism on the tensor product of domain spaces. In other words the composition map induces the following morphism of pointed simplicial sets, which we also denote by $- \circ -$,

$$- \circ - : Hom_{\mathbf{D}(\Gamma\mathcal{S}^f)}(F', F) \wedge Hom_{\mathbf{D}(\Gamma\mathcal{S}^f)}(F, G) \to Hom_{\mathbf{D}(\Gamma\mathcal{S}^f)}(F', G).$$

Recall that a smash product of two Γ-spaces F' and F is defined as follows:

$$F' \wedge F := L(F' \widetilde{\wedge} F),$$

where L is the left adjoint to the smash product inclusion functor, see 2.7. This implies that we can prove this theorem by exibiting a morphism of $\Gamma \times \Gamma$-spaces

$$- \underset{\mathbf{D}(\Gamma\mathcal{S}^f)}{\widetilde{\circ}} - : \underline{Hom}_{\mathbf{D}(\Gamma\mathcal{S}^f)}(F', F) \widetilde{\wedge} \underline{Hom}_{\mathbf{D}(\Gamma\mathcal{S}^f)}(F, G) \to RHom_{\mathbf{D}(\Gamma\mathcal{S}^f)}(F', G),$$

such that this morphism restricts, in degree $(1^+, 1^+)$, to the (simplicial) composition map in $\mathbf{D}(\Gamma\mathcal{S}^f)$. Now we construct this adjoint morphism of $\Gamma \times \Gamma$-spaces.

For any object $(m^+, n^+) \in Ob(\Gamma^{op} \times \Gamma^{op})$, we define the morphism of pointed simplicial sets

$$(- \underset{\mathbf{D}(\Gamma\mathcal{S}^f)}{\widetilde{\circ}} -)(m^+, n^+) : \underline{Hom}_{\mathbf{D}(\Gamma\mathcal{S}^f)}(F', F)(m^+) \wedge \underline{Hom}_{\mathbf{D}(\Gamma\mathcal{S}^f)}(F, G)(n^+)$$
$$\longrightarrow \underline{Hom}_{\mathbf{D}(\Gamma\mathcal{S}^f)}(F', F)(m^+ \wedge n^+)$$

as the following composite morphism:

$$(- \underset{\mathbf{D}(\Gamma\mathcal{S}^f)}{\widetilde{\circ}} -)(m^+, n^+) := (- \circ -) \circ (id \wedge \prod_{F,G}^{m,n})$$

□

Appendix F

$\mathbf{2}\mathcal{P}\boldsymbol{ic}$ Categories

In this appendix we give the definition of a mathematical structure which is built on a bicategory but whose mapping categories have the structure of Picard groupoids.

Definition F.1. A $\mathbf{2}\mathcal{P}ic$-category $\mathcal{C}$ consists of the following data

1. A small set, $Ob(\mathcal{C})$, whose elements will be called the *objects* of $\mathcal{C}$.

2. A function $\mathcal{C}(-,-) : Ob(\mathcal{C}) \times Ob(\mathcal{C}) \to Ob(\mathbf{2}\mathcal{P}ic)$, where $Ob(\mathbf{2}\mathcal{P}ic)$ is the set of all Picard groupoids.

3. For each object $s \in Ob(\mathcal{C})$, a homomorphism $id_s : * \to \mathcal{C}(s,s)$, where $*$ is the terminal Picard groupoid.

4. For each triple of objects $s, t, u \in Ob(\mathcal{C})$, a *composition bifunctor* $- \circ - : \mathcal{C}(t,u) \times \mathcal{C}(s,t) \to \mathcal{C}(s,u)$ which is subject to the following conditions

   (a) For each $h \in Ob(\mathcal{C}(t,u))$, the functor

   $$h \circ - : \mathcal{C}(s,t) \to \mathcal{C}(s,u).$$

   is a homomorphism

(b) For each $g \in Ob(\mathcal{C}(s,t))$, the functor

$$- \circ g : \mathcal{C}(t,u) \to \mathcal{C}(s,u).$$

is a homomorphism.

5. For each triple of objects $s, t, u \in Ob(\mathcal{C})$ and each pair of morphisms $g_1, g_2 \in Ob(\mathcal{C}(s,t))$, a monoidal natural transformation $\phi^{-}_{g_1,g_2} : - \circ g_1 + - \circ g_2 \Rightarrow - \circ g_1 + g_2$, where the homomorphism $- \circ g_1 + - \circ g_2 : \mathcal{C}(t,u) \to \mathcal{C}(s,u)$ is defined pointwise.

6. For each triple of objects $s, t, u \in Ob(\mathcal{C})$ and each pair of morphisms $h_1, h_2 \in Ob(\mathcal{C}(t,u))$, a monoidal natural transformation $\psi^{h_1,h_2}_{-} : h_1 \circ - + h_2 \circ - \Rightarrow h_1 + h_2 \circ -$, where the homomorphism $h_1 \circ - + h_2 \circ - : \mathcal{C}(s,t) \to \mathcal{C}(s,u)$ is defined pointwise.

7. For each quadruple of objects $s, t, u, v \in Ob(\mathcal{C})$, a natural transformation, α called the *associator*, between functors defined in the following diagram

$$\begin{tikzcd}
\mathcal{C}(u,v) \times \mathcal{C}(t,u) \times \mathcal{C}(s,t) \arrow[r, "id \times - \circ -"] \arrow[d, "- \circ - \times id"'] \arrow[dr, phantom, "\overset{\Leftarrow}{\alpha}"] & \mathcal{C}(u,v) \times \mathcal{C}(s,u) , \arrow[d, "- \circ -"] \\
\mathcal{C}(t,v) \times \mathcal{C}(s,t) \arrow[r] & \mathcal{C}(s,v).
\end{tikzcd}$$

and which is subject to the following conditions:

(a) For each pair $(g,h) \in Ob(\mathcal{C}(t,u)) \times Ob(\mathcal{C}(u,v))$, the natural transformation $\alpha_{(h,g,-)}$ as in the following diagram

$$\begin{tikzcd}
\mathcal{C}(s,t) \arrow[r, "g \circ -"'] \arrow[rr, bend left, "(h \circ g) \circ -"] & \mathcal{C}(s,u) \arrow[r, "h \circ -"'] \arrow[u, phantom, "\alpha_{(h,g,-)} \Uparrow"] & \mathcal{C}(s,v)
\end{tikzcd}$$

is a monoidal natural transformation.

(b) For each pair $(f, h) \in Ob(\mathcal{C}(s,t)) \times Ob(\mathcal{C}(u,v))$, the natural transformation $\alpha_{(h,-,f)}$ as in the following diagram

$$\begin{tikzcd}
\mathcal{C}(t,u) \arrow[r, "h\circ -"] \arrow[d, "-\circ f"'] & \mathcal{C}(t,v) \arrow[d, "-\circ f"] \arrow[dl, phantom, "\overset{\Leftarrow}{\alpha_{(f,-,h)}}"] \\
\mathcal{C}(s,u) \arrow[r, "h\circ -"'] & \mathcal{C}(s,v)
\end{tikzcd}$$

is a monoidal natural transformation.

(c) For each pair $(f, g) \in Ob(\mathcal{C}(s,t)) \times Ob(\mathcal{C}(t,u))$, the natural transformation $\alpha_{(-,g,f)}$ as in the following diagram

$$\begin{tikzcd}
\mathcal{C}(u,v) \arrow[r, "-\circ g"] \arrow[rr, bend right, "-\circ(g\circ f)"'] \arrow[rr, phantom, bend right=20, "\alpha_{(-,g,f)}\Uparrow"] & \mathcal{C}(t,v) \arrow[r, "-\circ f"] & \mathcal{C}(s,v)
\end{tikzcd}$$

is a monoidal natural transformation.

8. For each pair of objects $s, t \in Ob(\mathcal{C})$, two monoidal natural transformations

$$\begin{tikzcd}
\mathcal{C}(s,t) \arrow[r, equal] \arrow[r, bend right=60, "id_t\circ -"'] \arrow[r, phantom, bend right=30, "\lambda\Uparrow"] & \mathcal{C}(s,t)
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathcal{C}(s,t) \arrow[r, equal] \arrow[r, bend right=60, "-\circ id_s"'] \arrow[r, phantom, bend right=30, "\rho\Uparrow"] & \mathcal{C}(s,t)
\end{tikzcd}$$

Let $\mathcal{C}$ and $\mathcal{D}$ be two $\mathbf{2}\mathcal{P}ic$-categories, A functor of $\mathbf{2}\mathcal{P}ic$-categories $F : \mathcal{C} \to \mathcal{D}$ is a functor of bicategories which respects the additional structure on the morphism categories of $\mathcal{C}$ and $\mathcal{D}$. We will skip a precise definition of a functor of $\mathbf{2}\mathcal{P}ic$-categories but an interested reader can define these functors rigorously using our definition of $\mathbf{2}\mathcal{P}ic$-categories.

Appendix G

Hom 2-Chain complex

In this section we define the Hom 2 - chain complex and a tensor product in $\mathbf{2Ch}(SCG)$. We recall that given any two Picard groupoids $\mathcal{A}, \mathcal{B} \in Ob(SCG)$, $Hom_{SCG}(\mathcal{A}, \mathcal{B})$ inherits a Picard groupoid structure, i.e. the category *(SCG)* is enriched over itself .Let $A_\bullet, \mathcal{B}_\bullet \in Ob(\mathbf{2Ch}(SCG))$. Then, $(Hom_{\mathbf{2Ch}(SCG)}(\mathcal{A}_\bullet, \mathcal{B}_\bullet), d, \phi)$ is a chain complex whose $nth.$ degree is defined as follows:
$Hom_{\mathbf{2Ch}(SCG)}(\mathcal{A}_\bullet, \mathcal{B}_\bullet)_n = \prod\limits_p Hom_{SCG}(\mathcal{A}_p, \mathcal{B}_{p+n})$.
The differential $d : Hom_{\mathbf{2Ch}(SCG)}(\mathcal{A}_\bullet, \mathcal{B}_\bullet)_n \to Hom_{\mathbf{2Ch}(SCG)}(\mathcal{A}_\bullet, \mathcal{B}_\bullet)_{n-1}$ is given by $(df)_p = df_p + (-1)^{p+1} f_{p-1} d$ and a composition of 2 - morphisms $dd(f) \Rightarrow d^2 f + f d^2 \Rightarrow 0$, where the first 2 - morphism comes from the distributivity law on each degree of the Hom complex which is the consequence of the enrichment of SCG over itself and the second 2 - morphism in the composition is obvious. Similarly, we may define the Hom 2 - chain complex of chain maps between a 2-cochain complex and a 2-chain complex. Let $\mathcal{C}^\bullet$ be a 2-cochain complex of Picard groupoids, let $\mathcal{B}_\bullet \in Ob(\mathbf{2Ch}(SCG))$, then the 2-chain complex $([\mathcal{C}^\bullet, \mathcal{B}_\bullet], d, \phi)$ is defined as the 2 - chain complex $(Hom_{\mathbf{2Ch}(SCG)}(\mathcal{C}^{-\bullet}, \mathcal{B}_\bullet), d, \phi)$, where $\mathcal{C}^{-\bullet} \in Ob(\mathbf{2Ch}(SCG))$ is the 2-chain complex obtained by negatively regrading $\mathcal{C}^\bullet$, its degree n is $[\mathcal{C}^\bullet, \mathcal{B}_\bullet]_n = \prod\limits_p Hom_{SCG}(\mathcal{C}^{-p}, \mathcal{B}_{p+n})$. The tensor product of two chain

complexes could be defined similarly.

Appendix H

Index of Notations

Table H.1: Index of Notations

Notation	Meaning
$\mathcal{K}_\bullet$	The quasicategory of pointed Kan complexes.
$\mathbf{Kan}_\bullet$	The simplicial category of Kan complexes.
$\mathbf{sSets}$	The category of simplicial sets.
$\mathbf{sSets}_\bullet$	The category of pointed simplicial sets.
$\mathbf{D}(\Gamma\mathcal{S}^f)$	The (simplicial) derived category of Picard groupoids.
$\Gamma\mathcal{S}$	The category of Γ-spaces.
$\mathcal{P}ic$	The ∞-category of higher Picard groupoids.
$(\mathbf{sSets}, \mathbf{Q})$	The model category of simplicial sets whose fibrant objects are quasicategories.
$(\mathbf{sSets}, \mathbf{Kan})$	The model category of simplicial sets whose fibrant objects are Kan complexes.

Continued on next page

Table H.1 – continued from previous page

Notation	Meaning
$(\mathbf{sSets}_\bullet, \mathbf{Q})$	The model category of (pointed) simplicial sets whose fibrant objects are quasicategories.
$(\mathbf{sSets}_\bullet, \mathbf{Kan})$	The model category of (pointed) simplicial sets whose fibrant objects are Kan complexes.
$\mathcal{M}^{cf}$	The full (simplicial) subcategory of a simplicial model category $\mathcal{M}$ whose objects are both cofibrant and fibrant in $\mathcal{M}$.
$\mathbf{F}\mathfrak{C}(N(\Gamma^{op}))$	A cofibrant replacement of Γ^{op} in the model category $\mathbf{Cat}_{(\mathbf{sSets},\mathbf{Q})}$.
$\underline{\mathcal{P}ic}^{\oplus}$	A JQ-fibrant Γ-space modelling $\mathcal{P}ic$.
$\underline{\mathcal{P}ic}^{\oplus}_w$	A simplicial diagram encoding a coherently commutative monoidal structure on $\mathcal{P}ic$.
$\mathcal{L}_\infty$	The quasi-category of weak Picard groupoids.
Vect	The category of (complex) vector spaces.
$\mathbf{Cob}(n+1)$	The category of n+1 - cobordisms.
$\mathcal{L}$	The Picard groupoid of hermetian lines and linear isometries between them.
$\mathcal{L}_Y$	local system on the space $Map(Y, BG)$ with values in the Picard groupoid $\mathcal{L}$.
$Sing_\bullet X$	The singular simplicial set of the topological space X.
$H^n(X_\bullet; \mathcal{A})$	The nth cohomology Picard groupoid of the simplicial set $X_\bullet$ with coefficients in $\mathcal{A}$.
$H^n(X; \mathcal{L})$	The nth cohomology Picard groupoid of the simplicial set $Sing_\bullet X$ with coefficients in $\mathcal{L}$.

Continued on next page

Table H.1 – continued from previous page

Notation	Meaning
$H_n(X;\mathcal{L})$	The nth homology Picard groupoid of the simplicial set $Sing_\bullet X$ with coefficients in $\mathcal{L}$.
$\mathbb{Z}[0]$	The discrete Picard groupoid of Integers.
$C^\bullet(X_\bullet;\mathcal{A})$	The 2-cochain complex of the simplicial set $X_\bullet$ with coefficients in a Picard groupoid $\mathcal{A}$.
$C_\bullet(X_\bullet;\mathcal{A})$	The 2-chain complex of the simplicial set $X_\bullet$ with coefficients in a Picard groupoid $\mathcal{A}$.
$Ho(\mathcal{M})$	The homotopy category of a model category $\mathcal{M}$.