On the Structure of Oriented Exchange Graphs

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Dedication

To my parents, Bonnie Garver and Christian Garver.

Abstract

The exchange graph of a quiver is the graph of mutation-equivalent quivers whose edges correspond to mutations. The exchange graph admits a natural acyclic orientation called the oriented exchange graph. Oriented exchange graphs arise in many areas of mathematics including representation theory, algebraic combinatorics, and noncommutative algebraic geometry. In representation theory, an oriented exchange graph is isomorphic to a poset of certain torsion classes of a finite dimensional algebra.

Of particular interest to mathematicians and string theorists are the finite length maximal directed paths in oriented exchange graphs, which are known as maximal green sequences. Maximal green sequences were introduced to obtain quantum dilogarithm identities and combinatorial formulas for refined Donaldson-Thomas invariants. They were also used in supersymmetric gauge theory to compute the complete spectrum of BPS states. For quivers mutation-equivalent to an orientation of a type A_n Dynkin diagram, we show that the oriented exchange graphs can realized as quotients of other posets of representation theoretic objects. For the same class of quivers we also show how to explicitly construct some of their maximal green sequences.

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Chapter 1

Introduction

Cluster algebras were discovered by Fomin and Zelevinsky [FZ02] in 2002 in order to develop a combinatorial framework with which to study total positivity and canonical bases of algebraic groups. Since their introduction, cluster algebras have been the object of intense mathematical research. Their study has brought together mathematicians and physicists from across world. In mathematics, cluster algebras appear in areas such as the representation theory of quivers [CCS06b, BMR$^+$06a], noncommutative algebraic geometry [Bri07, Nag13], Poisson geometry [GSV03], and Teichmüller theory [FG06, FT12]. In physics, cluster algebras appear in areas such as discrete integrable systems [DFK11] and string theory [ACC$^+$13].

Cluster algebras are a class of combinatorially defined commutative rings. A cluster algebra comes equipped with a family a overlapping subsets of its generators called **clusters** and a family of rules for moving between these clusters called **mutations**. Any cluster algebra $\mathcal{A}$ defines a graph called the **exchange graph** of $\mathcal{A}$ whose vertices are the clusters of $\mathcal{A}$ and whose edges indicate which two clusters are reachable from each other by a single mutation. The exchange graph thus describes the combinatorics of mutation in cluster algebras.

Given a cluster algebra $\mathcal{A}$ and a choice of **initial cluster** $\mathbf{x}$, the exchange graph of $\mathcal{A}$ admits an acyclic orientation with $\mathbf{x}$ as its unique source. This acyclic orientation is known as the **oriented exchange graph** of $\mathcal{A}$ [BDP14] with respect to $\mathbf{x}$, and is the focus of this thesis.

By viewing cluster algebras through the lens of representation theory of finite dimensional algebras, these oriented exchange graphs are very natural objects. The oriented exchange graph turns out to be the Hasse diagram of many important posets in representation theory [BY13] related to **Jacobian algebras** Λ [DWZ08] (an algebra naturally associated with the initial cluster $\mathbf{x}$). These include posets of **support τ-tilting modules** [AIR14, AS81] and **torsion pairs** [HRS96] in the module category of Λ and posets of **bounded t-structures** [BBD83, HRS96] in the bounded derived category of Λ. Examples of oriented exchange graphs also include flip graphs of triangulations of Riemann surfaces with oriented edges [FT12].

An especially important part of the data of an oriented exchange graph is its set of maximal directed paths. These maximal directed paths are known as **maximal green sequences**, which were introduced by Keller [Kel12] to study the **refined Donaldson-Thomas invariants** of Kontsevich and Soibelman [KS08]. Maximal green sequences were also discovered independently in the context supersymmetric gauge theory where they were used to compute the complete spectrum of BPS states [ACC$^+$13].

In this thesis, we focus on understanding both the global and local structure of oriented exchange graphs. To study these objects globally, we use techniques from representation theory of finite dimensional algebras and combinatorics. More specifically, we regard oriented exchange graphs as posets of certain torsion classes in the module category of Λ. When Λ has finitely many indecomposable modules, the oriented exchange graph is a finite lattice [IRTT13]. In this situation, we make use of lattice theory to understand the global lattice theoretic structure of oriented exchange graphs (see Chapter 3).

We also investigate the local structure of oriented exchange graphs. More specifically, in Chapter 4 we show how one can construct maximal green sequences for particular families of quivers. In particular, we show how given any initial seed in a type $\mathbb{A}$ cluster algebra, one can explicitly construct some of the maximal green sequences in the corresponding oriented exchange graph.

All of our work in this thesis, begins with the input of an initial seed $\mathbf{x}$ of a cluster algebra. For the family of cluster algebras that we consider here, this initial data is equivalent to a choice of **quiver** (i.e. directed graph). The quivers that we work with in Chapters 3 and 4 are **type $\mathbb{A}$ quivers** (i.e. those obtained by a finite sequence of mutations from a quiver whose underlying graph is a type $\mathbb{A}$ Dynkin diagram).

In Chapter 5 we restrict our study to only the quivers whose underlying graph is a type $\mathbb{A}$ Dynkin diagram, also known as type $\mathbb{A}$ Dynkin quivers. By restricting to such quivers, we are able to study the connection between cluster algebras and **complete exceptional sequences** of quiver representations, which are only defined for acyclic quivers. Exceptional sequences were originally used by algebraic geometers [GR87, BK89, Rud90] to study exceptional vector bundles on projective spaces. When one is given the initial data of an acyclic quiver there is a bijection between vertices of the corresponding oriented exchange graph (i.e. seeds) and **complete exceptional sequences** of quiver representations as shown by Speyer and Thomas in [ST13]. In Chapter 5, we interpret this correspondence combinatorially. Along the way, we obtain a diagrammatic classification of all exceptional sequences associated with Dynkin quivers of type $\mathbb{A}$.

1.1 Summary of Results

1.1.1 Results in Chapter 3

The results presented in this chapter are joint work with T. McConville. Our first main results describe the lattice structure of all oriented exchange graphs with finitely many vertices. The class of such oriented exchange graphs are exactly those defined by Dynkin quivers of type $\mathbb{A}$, $\mathbb{D}$, or $\mathbb{E}$.

- We establish that lattices of torsion classes of representation finite $\Bbbk$-algebras are **semidistributive lattices** (Theorem 3.4.5 and Lemma 3.4.10), which relies on a description of the join of two torsion classes (Lemma 3.4.9).
- By regarding oriented exchange graphs defined by type $\mathbb{A}$ quivers as posets of torsion classes, we obtain that these have the structure of a semidistributive lattice (Corollary 3.4.7).

In order to further describe the lattice structure of oriented exchange graphs, we define an auxiliary poset of **biclosed sets** of paths in a graph, denoted Bic(AP), from which we will obtain the oriented exchange graph by a **lattice quotient map**. Identifying the oriented exchange graph as a lattice quotient implies that it will inherit nice lattice properties from Bic(AP).

- The poset Bic(AP) has the structure of a semidistributive, congruence-uniform, and polygonal lattice (Theorem 3.5.4).
- The lattice Bic(AP) has the property that every interval containing exactly two maximal chains is either a square or a hexagon, as in Figure 3.5 (Corollary 3.5.5).
- When considering a quiver Q that is mutation-equivalent to a type $\mathbb{A}$ Dynkin quiver or one that is an oriented cycle, the corresponding lattice $\text{Bic}(Q) :=$ Bic(AP) is canonically isomorphic to a lattice of **biclosed subcategories** of the module category of Λ, denoted $\mathcal{BIC}(Q)$ (Proposition 3.6.5).
- There is a lattice quotient map $\pi_\downarrow : \mathcal{BIC}(Q) \to \text{tors}(\Lambda)$ where the target of this map is the lattice of torsion classes of Λ, and this map identifies the oriented exchange graph of Q as a lattice quotient of $\mathcal{BIC}(Q)$ (Theorem 3.6.9). Using this map, the oriented exchange graph of Q inherits the semidistributive, congruence-uniform, and polygonal properties from $\mathcal{BIC}(Q)$.
- Using this lattice quotient map, we obtain that every interval of the oriented exchange graph of Q containing exactly two maximal chains is either a square or a pentagon. That the oriented exchange graph of Q is polygonal implies that its maximal chains (i.e. the maximal green sequences of Q) are connected by a finite sequence local moves using only squares and pentagons (Corollary 3.6.10).
- The oriented exchange graph of Q has the property that if Q has a maximal green sequence of length i and one of length j, then for any $k \in \mathbb{N}$ where $i \leqslant k \leqslant j$ there exists a maximal green sequence of Q of length k (Corollary 3.6.11).

Lastly, we obtain some additional lattice theoretic information about torsion classes corresponding to vertices of oriented exchange graphs defined by a quiver Q that is mutation-equivalent to a type $\mathbb{A}$ Dynkin quiver or one that is an oriented cycle.

- We describe the canonical join and canonical meet representations of torsion classes (Theorem 3.8.3 and Corollary 3.8.4).

1.1.2 Result in Chapter 4

In this chapter, we show how one can actually construct some of the maximal green sequences of certain types of quivers. The results presented here are joint work with G. Musiker.

- Suppose that a quiver Q can be obtained from two quivers Q_1 and Q_2 by adding finitely many arrows starting at vertices of Q_1 and ending at vertices of Q_2 in such a way that for any vertex a in Q_1 and any vertex b in Q_2, at most one arrow starts at a and ends at b in Q. If $\mathbf{i}_1$ is a maximal green sequence of Q_1 and $\mathbf{i}_2$ is a maximal green sequence of Q_2, then $\mathbf{i}_2 \circ \mathbf{i}_1$ is a maximal green sequence of Q (Theorem 4.3.12).

- If Q is a quiver that is mutation-equivalent to a type $\mathbb{A}$ Dynkin quiver, we show how to construct at least one of its maximal green sequences (Definition 4.6.3 and Theorem 4.6.5).

1.1.3 Results in Chapter 5

We define a family of objects called **strand diagrams**, which are collections of non-crossing curves in the plane. We use these strand diagrams in a combinatorial model for exceptional sequences of representations of type $\mathbb{A}$ Dynkin quivers. The results presented here are joint work with J. P. Matherne, K. Igusa, and J. Ostroff.

- We show that strand diagrams are in bijection with **exceptional collections**, which are sets of exceptional representations that can be linearly ordered so that they define an exceptional sequence (Theorem 5.3.6).

- We show that strand diagrams whose curves have a **good labeling** are in bijection with exceptional sequences (Theorem 5.3.9).

- We show that by allowing certain orientations of the curves in strand diagrams, the resulting class of **oriented strand diagrams** are in bijection with **mixed cobinary trees**, a family of combinatorial objects appearing in the theory of semi-invariants of quiver representations (Theorem 5.4.2).

- Using the latter correspondence, we obtain a combinatorial classification of the vertices of the oriented exchange graph in terms of oriented strand diagrams (Theorem 5.3.15).

- As an application of Theorem 5.3.15, we provide a combinatorial proof that oriented exchange graphs defined by type $\mathbb{A}$ Dynkin quivers have a unique source and a unique sink, previous proofs of which all involve algebraic and geometric methods (Theorem 5.6.3).

Chapter 2

Preliminaries

2.1 Quiver mutation

A **quiver** Q is a directed graph without loops or 2-cycles. In other words, Q is a 4-tuple (Q_0, Q_1, s, t), where $Q_0 = [m] := \{1, 2, \ldots, m\}$ is a set of **vertices**, Q_1 is a set of **arrows**, and two functions $s, t : Q_1 \to Q_0$ defined so that for every $\alpha \in Q_1$, we have $s(\alpha) \xrightarrow{\alpha} t(\alpha)$. An **ice quiver** is a pair (Q, F) with Q a quiver and $F \subset Q_0$ a set of **frozen vertices** with the additional restriction that any $i, j \in F$ have no arrows of Q connecting them. We refer to the elements of $Q_0 \backslash F$ as **mutable vertices**. By convention, we assume $Q_0 \backslash F = [n]$ and $F = [n+1, m] := \{n+1, n+2, \ldots, m\}$. Any quiver Q can be regarded as an ice quiver by setting $Q = (Q, \varnothing)$.

The **mutation** of an ice quiver (Q, F) at mutable vertex k, denoted μ_k, produces a new ice quiver $(\mu_k Q, F)$ by the three step process:

(1) For every 2-path $i \to k \to j$ in Q, adjoin a new arrow $i \to j$.

(2) Reverse the direction of all arrows incident to k in Q.

(3) Delete any 2-cycles created during the first step.

We show an example of mutation below depicting the mutable (resp. frozen) vertices in black (resp. blue).

$$\begin{tikzcd}
 & & & 2 \arrow[dr] & & & & 2 \arrow[dl, shift left] \arrow[dl, shift right] & & & \\
(Q,F) & = & 1 \arrow[ur, shift left] \arrow[ur, shift right] & & 3 & \overset{\mu_2}{\longmapsto} & 1 \arrow[rr, shift left] \arrow[rr, shift right] & & 3 \arrow[ul] & = & (\mu_2 Q, F) \\
 & & & {\color{blue}4} \arrow[ur] & & & & {\color{blue}4} \arrow[ur] & & &
\end{tikzcd}$$

The information of an ice quiver can be equivalently described by its skew-symmetric **exchange matrix**. Given (Q, F), we define $B = B_{(Q,F)} = (b_{ij}) \in \mathbb{Z}^{n\times m} := \{n \times m$ integer matrices$\}$ by $b_{ij} := \#\{i \xrightarrow{\alpha} j \in Q_1\} - \#\{j \xrightarrow{\alpha} i \in Q_1\}$. Furthermore, ice quiver mutation can equivalently be defined as **matrix mutation** of the corresponding exchange matrix. Given an exchange matrix $B \in \mathbb{Z}^{n\times m}$, the **mutation** of B at $k \in [n]$, also denoted μ_k, produces a new exchange matrix $\mu_k(B) = (b'_{ij})$ with entries

$$b'_{ij} := \begin{cases} -b_{ij} & : \text{ if } i = k \text{ or } j = k \\ b_{ij} + \operatorname{sgn}(b_{ik})[b_{ik}b_{kj}]_+ & : \text{ otherwise} \end{cases}$$

where $[x]_+ = \max(x, 0)$. For example, the mutation of the ice quiver above (here $m = 4$ and $n = 3$) translates into the following matrix mutation. Note that mutation of matrices (or of ice quivers) is an involution (i.e. $\mu_k\mu_k(B) = B$). Let $\mathrm{Mut}((Q, F))$ denote the collection of ice quivers obtainable from (Q, F) by finitely many mutations.

$$B_{(Q,F)} \;=\; \left[\begin{array}{ccc|c} 0 & 2 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \end{array}\right] \;\xmapsto{\mu_2}\; \left[\begin{array}{ccc|c} 0 & -2 & 2 & 0 \\ 2 & 0 & -1 & 0 \\ -2 & 1 & 0 & -1 \end{array}\right] \;=\; B_{(\mu_2 Q,F)}.$$

Given a quiver Q, we define its **framed** (resp. **coframed**) quiver to be the ice quiver $\widehat{Q}$ (resp. $\check{Q}$) where $\widehat{Q}_0$ $(= \check{Q}_0) := Q_0 \sqcup [n+1, 2n]$, $F = [n+1, 2n]$, and $\widehat{Q}_1 := Q_1 \sqcup \{i \to n+i : i \in [n]\}$ (resp. $\check{Q}_1 := Q_1 \sqcup \{n+i \to i : i \in [n]\}$). Now given $\widehat{Q}$ we define the **exchange tree** of $\widehat{Q}$, denoted $ET(\widehat{Q})$, to be the (a priori infinite) graph whose vertex set is $\mathrm{Mut}(\widehat{Q})$ and with an edge between two vertices if and only if the quivers corresponding to those vertices are obtained from each other by a single mutation. Similarly, define the **exchange graph** of $\widehat{Q}$, denoted $EG(\widehat{Q})$, to be the quotient of $ET(\widehat{Q})$ where two vertices are identified if and only if there is a **frozen isomorphism** of the corresponding quivers (i.e. an isomorphism of quivers that fixes the frozen vertices). Such an isomorphism is equivalent to a simultaneous permutation of the rows and first n columns of the corresponding exchange matrices.

In this paper, we focus our attention on **type $\mathbb{A}$ quivers** (i.e. quivers $R \in \mathrm{Mut}(1 \leftarrow 2 \leftarrow \cdots \leftarrow n)$ for some positive integer n.) We will use the following classification due to Buan and Vatne in our study of type $\mathbb{A}$ quivers.

Lemma 2.1.1. *[BV08, Prop. 2.4] A quiver Q is of type $\mathbb{A}$ if and only if Q satisfies the following:*

i) All non-trivial cycles in the underlying graph of Q are oriented and of length 3.

ii) Any vertex has at most four neighbors.

iii) If a vertex has four neighbors, then two of its adjacent arrows belong to one 3-cycle, and the other two belong to another 3-cycle.

iv) If a vertex has exactly three neighbors, then two of its adjacent arrows belong to a 3-cycle and the third arrow does not belong to any 3-cycle.

2.2 Oriented exchange graphs

In this brief section, we recall the definitions of **c**-vectors and their sign-coherence property. We use these notions to explain how to orient the edges of $EG(\widehat{Q})$ for a given quiver Q to obtain the **oriented exchange graph** of Q, denoted $\overrightarrow{EG}(\widehat{Q})$. Oriented exchange graphs were introduced in [BDP14] and were shown to be isomorphic to many important partially-ordered sets in representation theory in [BY13].

Given $\widehat{Q}$, we define the **c-matrix** $C(n) = C_R(n)$ (resp. $C = C_R$) of $R \in ET(\widehat{Q})$ (resp. $R \in EG(\widehat{Q})$) to be the submatrix of B_R where $C(n) := (b_{ij})_{i\in[n], j\in[n+1,2n]}$ (resp. $C := (b_{ij})_{i\in[n], j\in[n+1,2n]}$). We let $\mathbf{c}\text{-mat}(Q) := \{C_R : R \in EG(\widehat{Q})\}$. By definition, B_R (resp. C) is only defined up to simultaneous permutations of its rows and its first n columns (resp. up to permutations of its rows) for any $R \in EG(\widehat{Q})$.

A row vector of a **c**-matrix, $\mathbf{c}_i$, is known as a **c-vector**. We will denote the set of **c**-vectors of Q by $\mathbf{c}\text{-vec}(Q)$. The celebrated theorem of Derksen, Weyman, and Zelevinsky [DWZ10, Theorem 1.7], known as the sign-coherence of **c**-vectors, states that for any $R \in ET(\widehat{Q})$ and $i \in [n]$ the **c**-vector $\mathbf{c}_i$ is a nonzero element of $\mathbb{Z}^n_{\geqslant 0}$ or $\mathbb{Z}^n_{\leqslant 0}$. Thus we say a **c**-vector is either **positive** or **negative**. A mutable vertex i of an ice quiver $(R, F) \in \text{Mut}(\widehat{Q})$ is said to be **green** (resp. **red**) if all arrows of (R, F) connecting an element of F and i point away from (resp. towards) i. Note that all vertices of $\widehat{Q}$ are green and all vertices of $\check{Q}$ are red. We use the notion of green and red vertices to orient the edges of $EG(\widehat{Q})$ to obtain $\overrightarrow{EG}(\widehat{Q})$.

Definition 2.2.1. *[BDP14] Let Q be a quiver. The **oriented exchange graph** of Q is the directed graph whose underlying unoriented graph is $EG(\widehat{Q})$ with its edges oriented* *as follows. If (R^1, F) and (R^2, F) are connected by an edge in $EG(\widehat{Q})$, then there is a directed edge $(R^1, F) \longrightarrow (R^2, F)$ if $R^2 = \mu_k R^1$ where $k \in R^1_0$ is green, otherwise there is a directed edge $(R^1, F) \longleftarrow (R^2, F)$.*

*We define a **maximal green sequence** of Q, denoted $\boldsymbol{i} = (i_1, \ldots, i_k)$, to be a sequence of mutable vertices of $\widehat{Q}$ where*

a) i_j is green in $\mu_{i_{j-1}} \circ \cdots \circ \mu_{i_1}(\widehat{Q})$ for each $j \in [k]$,

b) $\mu_{i_k} \circ \cdots \circ \mu_{i_1}(\widehat{Q})$ has only red vertices.

*Let green(Q) denote the set of maximal green sequences of Q. By [BDP14, Proposition 2.13], maximal green sequences of Q are in bijection with the maximal chains in the poset $\overrightarrow{EG}(\widehat{Q})$. Furthermore, if we define $\ell en(\boldsymbol{i}) := k$ to be the **length** of $\boldsymbol{i}$, then the maximal chain $C_{\boldsymbol{i}}$ of $\overrightarrow{EG}(\widehat{Q})$ corresponding to $\boldsymbol{i}$ also has length k (here the length of a chain $C = c_1 < \cdots < c_d$ in a poset is defined to be $d - 1$).*

Example 2.2.2. *Let $Q = 1 \longrightarrow 2$. Below we show $\overrightarrow{EG}(\widehat{Q})$ and we also show all of the $\boldsymbol{c}$-matrices in $\boldsymbol{c}$-mat(Q). Additionally, we note that $\boldsymbol{i}_1 = (1, 2)$ and $\boldsymbol{i}_2 = (2, 1, 2)$ are the two maximal green sequences of Q.*

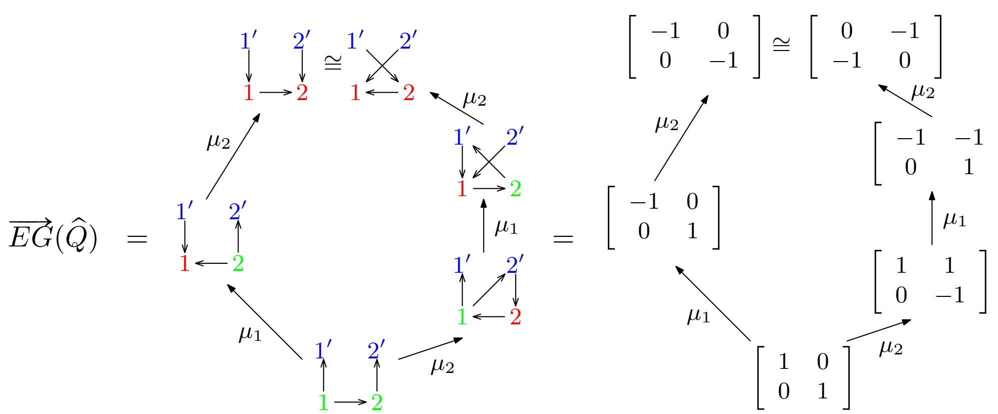


Figure 2.1: The oriented exchange graph of $Q = 1 \to 2$.

2.3 Path algebras and quiver representations

Following [ASS06], let Q be a given quiver. We define a **path** of **length** $\ell \geqslant 1$ to be an expression $\alpha_1\alpha_2 \cdots \alpha_\ell$ where $\alpha_i \in Q_1$ for all $i \in [\ell]$ and $s(\alpha_i) = t(\alpha_{i+1})$ for all $i \in [\ell - 1]$.

We may visualize such a path in the following way

$$\cdot \xleftarrow{\alpha_1} \cdot \xleftarrow{\alpha_2} \cdot \longleftarrow \cdot \quad \cdots \quad \cdot \longleftarrow \cdot \xleftarrow{\alpha_\ell} \cdot .$$

Furthermore, the **source** (resp. **target**) of the path $\alpha_1\alpha_2\cdots\alpha_\ell$ is $s(\alpha_\ell)$ (resp. $t(\alpha_1)$). Let Q_ℓ denote the set of all paths in Q of length ℓ. We also associate to each vertex $i \in Q_0$ a path of length $\ell = 0$, denoted ε_i, that we will refer to as the **lazy path** at i.

Definition 2.3.1. *Let Q be a quiver. The **path algebra** of Q is the $\Bbbk$-algebra generated by all paths of length $\ell \geqslant 0$. Throughout this paper, we assume that $\Bbbk$ is algebraically closed. The multiplication of two paths $\alpha_1\cdots\alpha_\ell \in Q_\ell$ and $\beta_1\cdots\beta_k \in Q_k$ is given by the following rule*

$$\alpha_1\cdots\alpha_\ell \cdot \beta_1\cdots\beta_k = \begin{cases} \alpha_1\cdots\alpha_\ell\beta_1\cdots\beta_k \in Q_{\ell+k} & : \quad s(\alpha_\ell) = t(\beta_1) \\ 0 & : \quad s(\alpha_\ell) \neq t(\beta_1). \end{cases}$$

We will denote the path algebra of Q by $\Bbbk Q$. Note also that as $\Bbbk$-vector spaces we have

$$\Bbbk Q = \bigoplus_{\ell=0}^{\infty} \Bbbk Q_\ell$$

where $\Bbbk Q_\ell$ is the $\Bbbk$-vector space of all paths of length ℓ.

In this paper, we study certain quivers Q which have **oriented cycles**. We say a path of length $\ell \geqslant 0$ $\alpha_1\cdots\alpha_\ell \in Q_\ell$ is an **oriented cycle** if $t(\alpha_1) = s(\alpha_\ell)$. We denote by $\Bbbk Q_{\ell,\mathrm{cyc}} \subset \Bbbk Q_\ell$ the subspace of all oriented cycles of length $\ell \geqslant 0$. If a quiver Q possesses any oriented cycles of length $\ell \geqslant 1$, we see that $\Bbbk Q$ is infinite dimensional. If Q has no oriented cycles, we say that Q is **acyclic**.

In order to avoid studying infinite dimensional algebras, we will add relations to path algebras whose quivers contain oriented cycles in such a way that we obtain finite dimensional quotients of path algebras. The relations we add are those coming from an **admissible** ideal I of $\Bbbk Q$ meaning that

$$I \subset \bigoplus_{\ell=2}^{\infty} \Bbbk Q.$$

If I is an admissible ideal of $\Bbbk Q$, we say that (Q, I) is a **bound quiver** and that $\Bbbk Q/I$ is a **bound quiver algebra**.

In this paper, we study modules over a bound quiver algebra $\Bbbk Q/I$ by studying certain quiver representations of Q that are "compatible" with the relations coming from I. A **representation** $V = ((V_i)_{i\in Q_0}, (\varphi_\alpha)_{\alpha\in Q_1})$ of a quiver Q is an assignment of a $\Bbbk$-vector space V_i to each vertex i and a $\Bbbk$-linear map $\varphi_\alpha : V_{s(\alpha)} \to V_{t(\alpha)}$ to each arrow $\alpha \in Q_1$. If $\rho \in \Bbbk Q$, it can be expressed as

$$\rho = \sum_{i=1}^{m} c_i \alpha_{i_1}^{(i)} \cdots \alpha_{i_k}^{(i)}$$

where $c_i \in \Bbbk$ and $\alpha_{i_1}^{(i)} \cdots \alpha_{i_k}^{(i)} \in Q_{i_k}$ so when considering a representation V of Q, we define

$$\varphi_\rho := \sum_{i=1}^{m} c_i \varphi_{\alpha_{i_1}^{(i)}} \cdots \varphi_{\alpha_{i_k}^{(i)}}.$$

If we have a bound quiver (Q, I), we define a representation of Q **bound by** I to be a representation of Q where $\varphi_\rho = 0$ if $\rho \in I$. We say a representation of Q bound by I is **finite dimensional** if $\dim_\Bbbk V_i < \infty$ for all $i \in Q_0$.

Let $V = ((V_i)_{i\in Q_0}, (\varphi_a)_{a\in Q_1})$ and $W = ((W_i)_{i\in Q_0}, (\varrho_a)_{a\in Q_1})$ be two representations of a quiver Q bound by I. A **morphism** $\theta : V \to W$ consists of a collection of linear maps $\theta_i : V_i \to W_i$ that are compatible with each of the linear maps in V and W. That is, for each arrow $a \in Q_1$, we have $\theta_{t(a)} \circ \varphi_a = \varrho_a \circ \theta_{s(a)}$. An **isomorphism** of quiver representations is a morphism $\theta : V \to W$ where θ_i is a $\Bbbk$-vector space isomorphism for all $i \in Q_0$. We define $V \oplus W := ((V_i \oplus W_i)_{i\in Q_0}, (\varphi_a \oplus \varrho_a)_{a\in Q_1})$ to be the **direct sum** of V and W. We say that a nonzero representation V is **indecomposable** if it is not isomorphic to a direct sum of two nonzero representations. Note that the collection of finite dimensional representations of a quiver Q bound by I and the morphisms between them form an abelian category denoted $\mathrm{rep}_\Bbbk(Q, I)$, with the indecomposable representations forming a full subcategory called $\mathrm{ind}(\mathrm{rep}_\Bbbk(Q, I))$. We define $\mathrm{rep}_\Bbbk(Q) := \mathrm{rep}_\Bbbk(Q, I)$ when $I = 0$.

It turns out that one has a $\Bbbk$-linear equivalence of categories

$$\Bbbk Q/I\text{-mod} \xrightarrow{\simeq} \mathrm{rep}_\Bbbk(Q, I).$$

In the sequel, we use this fact without mentioning it further. Additionally, the **dimension vector** of $V \in \Bbbk Q/I$-mod is the vector $\underline{\dim}(V) := (\dim_\Bbbk V_i)_{i\in Q_0}$ and the **dimension** of V is defined as $\dim_\Bbbk(V) = \sum_{i\in Q_0} \dim_\Bbbk V_i$. The **support** of $V \in \Bbbk Q/I$-mod is the set $\mathrm{supp}(V) := \{i \in Q_0 : V_i \neq 0\}$.

Chapter 3

Lattice Properties of Oriented Exchange Graphs and Torsion Classes

3.1 Introduction

The exchange graph defined by a quiver Q admits a natural acyclic orientation called the oriented exchange graph defined by Q (see [BY13]). If Q is an orientation of a Dynkin diagram of type $\mathbb{A}$, $\mathbb{D}$, or $\mathbb{E}$, then its oriented exchange graph is a Cambrian lattice of the same type (see [Rea06]). In the Dynkin case, we may extract combinatorial information about oriented exchange graphs from the Cambrian lattice structure. The purpose of this chapter is to uncover similar information about oriented exchange graphs associated to some non-Dynkin quivers of finite type. We summarize our approach below and distinguish it from other approaches. Most of the definitions will be given in later sections.

The orientation of an exchange graph associated to a quiver is known to be acyclic. However, this fact is far from obvious (see [Nag13]). One approach to proving this fact invokes a surprising connection to representation theory. Given a quiver Q, there is an associative algebra $\Lambda = \Bbbk Q/I$ whose functorially finite torsion classes are in natural bijection with vertices of the oriented exchange graph of Q. Ordering by inclusion, the covering relations among functorially finite torsion classes correspond to edges of the exchange graph in such a way that the orientation is preserved. As an inclusion order on any family of sets is acyclic, it follows that the oriented exchange graph is acyclic.

The set of torsion classes of Λ is known to form a lattice, of which the functorially finite torsion classes form a sublattice when Λ has certain algebraic properties [IRTT13] (we will focus on algebras Λ having finite lattices of torsion classes so we do not need to distinguish between lattices and complete lattices). In the situation where the functorially finite torsion classes of Λ form a lattice, we use a formula for the join of two torsion classes found by Hugh Thomas [Tho] to prove that these functorially finite torsion classes form a semidistributive lattice (see Theorem 3.4.5). We believe that the lattices of functorially finite torsion classes of Λ are congruence-uniform, though we do not have a proof in general.

A Cambrian lattice can be constructed either as a special lattice quotient of the weak order of a finite Coxeter group or as an oriented exchange graph of a Dynkin quiver. We construct a similar lattice quotient description when Q is an oriented cycle or mutation equivalent to a path quiver. First, we define a closure operator on its set of positive $\mathbf{c}$-vectors. We then show that biclosed sets of $\mathbf{c}$-vectors can be interpreted algebraically as what we call biclosed subcategories of the module category of Λ. After that, we construct a map $\pi_{\downarrow}$ from biclosed subcategories of Λ-mod to functorially finite torsion classes of Λ. We prove that the set of biclosed sets ordered by inclusion forms a congruence-uniform lattice, and the above map has the structure of a lattice quotient map. Moreover, the Hasse diagram of the lattice structure on functorially finite torsion classes of Λ induced by this map is the oriented exchange graph. Thus the oriented exchange graph of Q inherits the congruence-uniform property via the quotient map $\pi_{\downarrow}$.

The chapter is organized as follows. In Section 3.2, we introduce the class of path algebras with relations that we consider in this chapter. They are the cluster-tilted algebras of type $\mathbb{A}$ (i.e. these algebras are defined by quivers that are mutation-equivalent to a path quiver) and the cluster-tilted algebras defined by a quiver that is an oriented cycle. We carefully describe these algebras and their properties in Sections 3.2.1 and 3.2.2. Throughout this chapter, when we speak about one of these algebras, we denote it by Λ.

In Section 3.3, we review basic notions related to lattice theory that will be useful to us. Of particular importance to us will be the notions of semidistributive, congruence-uniform, and polygonal lattices.

In Section 3.4, we review the concepts of torsion classes and torsion-free classes. Using the fact that the lattice of functorially finite torsion classes of Λ is isomorphic to the oriented exchange graph of Q, we prove that when Q is mutation-equivalent to a Dynkin quiver its oriented exchange graph is a semidistributive lattice.

In Section 3.5, we develop the theory of biclosed sets. We introduce the notion of biclosed sets of acyclic paths in a graph, denoted Bic(AP). We prove that Bic(AP) is a semidistributive, congruence-uniform, and polygonal lattice (see Theorem 3.5.4). When Q is mutation-equivalent to a path quiver or is an oriented cycle, we can identify the **c**-vectors of Q with acyclic paths in Q. In this way, we can consider the lattice of biclosed sets of **c**-vectors of Q, denoted $\mathrm{Bic}(Q)$, and conclude that this lattice is semidistributive, congruence-uniform, and polygonal.

In Section 3.6, we show that $\mathrm{Bic}(Q)$ is isomorphic to what we call the lattice of biclosed subcategories of Λ-mod, denoted $\mathcal{BIC}(Q)$. Using this categorification, we define maps $\pi_\downarrow$ and $\pi^\uparrow$ on $\mathcal{BIC}(Q)$. Our main Theorem is that $\pi_\downarrow : \mathcal{BIC}(Q) \to \mathrm{tors}(\Lambda)$ is a lattice quotient map (see Theorem 3.6.9). We remark that it is not clear a priori that the image of $\pi_\downarrow$ is contained in $\mathrm{tors}(\Lambda)$. We conclude this section by giving an affirmative answer to a conjecture of Brüstle, Dupont, and Pérotin (see [BDP14, Conjecture 2.22]) when Q is mutation-equivalent to a path quiver or is an oriented cycle (see Corollary 3.6.11).

In Section 3.7, we prove several important properties of $\pi_\downarrow$ and $\pi^\uparrow$ that are needed for the proof of Theorem 3.6.9. Much of this section is dedicated to proving that the image of $\pi_\downarrow$ is contained in $\mathrm{tors}(\Lambda)$. A crucial step in this argument is the use of a basis for the equivalence classes of extensions of one indecomposable Λ-module by another given in [CS14].

In Section 3.8, we apply our results about biclosed subcategories to classify canonical join and canonical meet representations of torsion classes.

In Section 3.9, we record a few necessary results whose statements and proofs do not fit with the exposition in other sections.

In this chapter, we only present a lattice quotient description of oriented exchange graphs defined by quivers that are mutation-equivalent to a path quiver or are an oriented cycle. We believe that one needs a more refined notion of biclosed subcategories in order to produce a lattice quotient description of oriented exchange graphs defined by any finite type quiver.

3.2 Preliminaries

3.2.1 Cluster-tilted algebras and c-vectors

In this section, we review the definition of cluster-tilted algebras [BMR07] and their connections with **c**-vectors [Cha13]. As we will focus on cluster-tilted algebras of type $\mathbb{A}$, we recall a useful description of these algebras as bound quiver algebras and we recall a useful classification of the indecomposable modules over these algebras.

To define cluster-tilted algebras, we need to recall the definition of the cluster category of an acyclic quiver Q, which was introduced in [BMR$^+$06a]. Let Q be an acyclic quiver. Let $\mathcal{D} := \mathcal{D}^b(\Bbbk Q\text{-mod})$ denote the bounded derived category of $\Bbbk Q$-mod. Let $\tau : \mathcal{D} \to \mathcal{D}$ denote the **Auslander-Reiten translation** and let $[1] : \mathcal{D} \to \mathcal{D}$ denote the shift functor. We define the **cluster category** of Q, denoted $\mathcal{C}_Q$, to be the orbit category $\mathcal{D}/\tau^{-1}[1]$. The objects of $\mathcal{C}_Q$ are $\tau^{-1}[1]$-orbits of modules, denoted $\overline{M} := ((\tau^{-1}[1])^i M)_{i\in\mathbb{Z}}$, where $M \in \Bbbk Q$-mod. The morphisms between $\overline{M}, \overline{N} \in \mathcal{C}_Q$ are given by

$$\mathrm{Hom}_{\mathcal{C}_Q}\left(\overline{M}, \overline{N}\right) := \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{D}}(M, (\tau^{-1}[1])^i N).$$

Cluster categories were invented to provide an additive categorification of cluster algebras. We will not discuss cluster algebras in this chapter, but we remark that **cluster-tilting objects** in $\mathcal{C}_Q$, which we will define shortly, are in bijection with clusters of the cluster algebra $\mathcal{A}_Q$ associated to Q.

Definition 3.2.1. *([BMR$^+$06a]) We say $T \in \mathcal{C}_Q$ is a **cluster-tilting object** if*
(1) $\mathrm{Ext}^1_{\mathcal{C}_Q}(T, T) = 0$ and
(2) $T = \bigoplus_{i=1}^n T_i$ where $\{T_i\}_{i=1}^n$ is a maximal collection of pairwise non-isomorphic indecomposable objects in $\mathcal{C}_Q$.

Now we define a **cluster-tilted algebra** to be the endomorphism algebra $\Lambda := \mathrm{End}_{\mathcal{C}_Q}(T)^{\mathrm{op}}$ where $T = \bigoplus_{i=1}^n T_i$ is a cluster-tilting object in $\mathcal{C}_Q$. If the Q is a **Dynkin**

quiver, (i.e. the underlying graph of Q is the Dynkin graph $\Delta \in \{\mathbb{A}_n, \mathbb{D}_m, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8\}$ with $n \geqslant 1$ and $m \geqslant 4$) we say that Λ is of **type** Δ or of **Dynkin type**. It follows from [Buan-Marsh-Reiten, Cor. 2.4] that Λ is representation finite if and only if $\Bbbk Q$ is representation finite. Thus a cluster-tilted algebra Λ is representation finite if and only if Q is of Dynkin type.

Cluster-tilted algebras of Dynkin type can be described explicitly as bound quiver algebras (see [BMR06b]). In the sequel, we use the following description of cluster-tilted algebras of type $\mathbb{A}$ as bound quiver algebras. The following result appeared in [CCS06b] and was generalized in [CCS06a] and [BMR06b].

Lemma 3.2.2. *A cluster-tilted algebra Λ is of type $\mathbb{A}$ if and only if $\Lambda \cong \Bbbk Q/I$ where Q is a type $\mathbb{A}$ quiver and I is generated by all 2-paths $\alpha\beta \in Q_2$ where α and β are two of the arrows of a 3-cycle of Q.*

Using Lemma 3.2.2 and the language of string modules, we can explicitly parameterize the indecomposable modules of a type $\mathbb{A}$ cluster-tilted algebra. A **string algebra** $\Lambda = \Bbbk Q/I$ is a bound quiver algebra where:

i) for each vertex i of Q at most two arrows of Q start at i and at most two arrows of Q end at i,

ii) for each arrow $\beta \in Q_1$ there is at most one arrow $\alpha \in Q_1$ and at most one arrow $\gamma \in Q_1$ such that $\alpha\beta \notin I$ and $\beta\gamma \notin I$.

A **string** in Λ is a sequence

$$w = x_1 \xleftrightarrow{\alpha_1} x_2 \xleftrightarrow{\alpha_2} \cdots \xleftrightarrow{\alpha_m} x_{m+1}$$

where each $x_i \in Q_0$ and each $\alpha_i \in Q_1$ or $\alpha_i \in Q_1^{-1} := \{\text{formal inverses of arrows of Q}\}$. We require that each α_i **connects** x_i and x_{i+1} (i.e. either $s(\alpha_i) = x_i$ and $t(\alpha_i) = x_{i+1}$ or $s(\alpha_i) = x_{i+1}$ and $t(\alpha_i) = x_i$ where if $\alpha_i \in Q_1^{-1}$ we define $s(\alpha_i) := t(\alpha_i^{-1})$ and $t(\alpha_i) := s(\alpha_i^{-1})$) and that w contains no **substrings** of w of the following forms:

i) $x \xrightarrow{\beta} y \xleftarrow{\beta^{-1}} x$ or $x \xleftarrow{\beta} y \xrightarrow{\beta^{-1}} x$,

ii) $x_{i_1} \xrightarrow{\beta_1} x_{i_2} \cdots x_{i_s} \xrightarrow{\beta_s} x_{i_{s+1}}$ or $x_{i_1} \xleftarrow{\gamma_1} x_{i_2} \cdots x_{i_s} \xleftarrow{\gamma_s} x_{i_{s+1}}$ where $\beta_s \cdots \beta_1 \in I$ or $\gamma_1 \cdots \gamma_s \in I$.

In other words, w is an irredundant walk in Q that avoids the relations imposed by I. Additionally, as in [CS14], we consider strings up to inverses.

Let w be a string in Λ. The **string module** defined by w is the bound quiver representation $M(w) := ((V_i)_{i\in Q_0}, (\varphi_\alpha)_{\alpha\in Q_1})$ where

$$V_i \quad := \quad \begin{cases} \Bbbk^{s_j} & : \quad i = x_j \text{ for some } j \in [m+1] \\ 0 & : \quad \text{otherwise} \end{cases}$$

with $s_j := \#\{k \in [m+1] : \; x_k = x_j\}$ and the action of φ_α is induced by the relevant identity morphisms if α lies on w and is zero otherwise.

If $\Bbbk Q/I$ is a representation finite string algebra, it follows from [BR87], that

$$\operatorname{ind}(\Bbbk Q/I\text{-mod}) := \{\text{indecomposable } \Bbbk Q/I\text{-modules}\}$$

consists of exactly the string modules over $\Bbbk Q/I$ so

$$\operatorname{ind}(\Bbbk Q/I\text{-mod}) = \{M(w) : w \text{ is a string in } \Bbbk Q/I \text{ where } w \sim w^{-1}\}.$$

Furthermore, if $M(w) = ((V_i)_{i\in Q_0}, (\varphi_\alpha)_{\alpha\in Q_1})$ is a string module over $\Bbbk Q/I$ and Q is a type $\mathbb{A}$ quiver, then the relations in $\Bbbk Q/I$ require that $\dim_{\Bbbk} V_i \leqslant 1$ for all $i \in Q_0$.

Example 3.2.3. *Let Q denote the type $\mathbb{A}$ quiver shown below. Then $\Bbbk Q/I$ is a string algebra where $I = \langle \beta\alpha, \gamma\beta, \alpha\gamma\rangle$.*

$$Q \quad = \quad \begin{tikzcd} & 2 \arrow[dr, "\beta"] & \\ 1 \arrow[ur, "\alpha"] & & 3 \arrow[ll, "\gamma"] \end{tikzcd}$$

The algebra $\Bbbk Q/I$ has the following (indecomposable) string modules.

$$M(1) \quad = \quad \begin{tikzcd} & 0 \arrow[dr, "0"] & \\ \Bbbk \arrow[ur, "0"] & & 0 \arrow[ll, "0"] \end{tikzcd} \qquad M(1 \xrightarrow{\alpha} 2) \quad = \quad \begin{tikzcd} & \Bbbk \arrow[dr, "0"] & \\ \Bbbk \arrow[ur, "1"] & & 0 \arrow[ll, "0"] \end{tikzcd}$$

$$M(2) \quad = \quad \begin{tikzcd} & \Bbbk \arrow[dr, "0"] & \\ 0 \arrow[ur, "0"] & & 0 \arrow[ll, "0"] \end{tikzcd} \qquad M(2 \xrightarrow{\beta} 3) \quad = \quad \begin{tikzcd} & \Bbbk \arrow[dr, "1"] & \\ 0 \arrow[ur, "0"] & & \Bbbk \arrow[ll, "0"] \end{tikzcd}$$

$$M(3) \quad = \quad \begin{tikzcd} & 0 \arrow[dr, "0"] & \\ 0 \arrow[ur, "0"] & & \Bbbk \arrow[ll, "0"] \end{tikzcd} \qquad M(3 \xrightarrow{\gamma} 1) \quad = \quad \begin{tikzcd} & 0 \arrow[dr, "0"] & \\ \Bbbk \arrow[ur, "0"] & & \Bbbk \arrow[ll, "1"] \end{tikzcd}$$

The final result that we present in this section allows us to connect the representation theory of cluster-tilted algebras $\Lambda = \Bbbk Q/I$ of finite representation type with the combinatorics of the $\mathbf{c}$-vectors of Q. The following result appears in [Cha13].

Proposition 3.2.4. *[Cha13, Thm. 6] Let Q be a quiver that is mutation-equivalent to a Dynkin quiver and let $\Lambda \cong \Bbbk Q/I$ denote the cluster-tilted algebra associated to Q. Then we have a bijection*

$$\begin{array}{rcl} ind(\Bbbk Q/I\text{-}mod) & \longrightarrow & \{\boldsymbol{c} \in \boldsymbol{c}\text{-}vec(Q): \ \boldsymbol{c} \text{ is positive}\} \\ V & \longmapsto & \underline{\dim}(V). \end{array}$$

Thus $\boldsymbol{c}$-vec$(Q) = \{\underline{\dim}(V) : V \in ind(\Bbbk Q/I\text{-}mod)\} \bigsqcup \{-\underline{\dim}(V) : V \in ind(\Bbbk Q/I\text{-}mod)\}$.

Example 3.2.5. *Let Q be the quiver appearing in Example 3.2.3. By Proposition 3.2.4, we have that*

$$\boldsymbol{c}\text{-}vec(Q) = \{\pm(1,0,0), \pm(0,1,0), \pm(0,0,1), \pm(1,1,0), \pm(0,1,1), \pm(1,0,1)\}.$$

3.2.2 Cyclic quivers

In this section, we describe the second family of bound quiver algebras that we will study. To begin, let $Q(n)$ denote the quiver with $Q(n)_0 := [n]$ and $Q(n)_1 := \{i \to i+1 : i \in [n-1]\} \sqcup \{n \to 1\}$. For example, when $n = 4$ we have

$$\text{Q}(4) \quad = \quad \begin{tikzcd} 1 \arrow[r] & 2 \arrow[d] \\ 4 \arrow[u] & 3. \arrow[l] \end{tikzcd}$$

As discussed in [BMR06b, Prop. 2.6, Prop. 2.7], the algebra

$$\Lambda = \Bbbk Q(n)/\langle \alpha_1 \cdots \alpha_{n-1} : \ \alpha_i \in Q(n)_1 \rangle$$

is cluster-tilted of type $\mathbb{D}_n$. As such, Λ is representation finite. Furthermore, one observes that Λ is a string algebra and thus the indecomposables Λ-modules are string modules. One can verify the following lemma.

Lemma 3.2.6. *Let $w_2 = x_1^{(2)} \leftarrow \cdots \leftarrow x_{k_2}^{(2)}$ and $w_1 = x_1^{(1)} \leftarrow \cdots \leftarrow x_{k_1}^{(1)}$ be strings in Λ. Then $\text{Hom}(M(w_2), M(w_1)) \neq 0$ if and only if $x_{k_2}^{(2)} = x_i^{(1)}$ for some $i \in [k_1]$ and*

$x_1^{(2)} \notin \{x_2^{(1)}, \ldots, x_{i-1}^{(1)}\}$. *Furthermore, if* $\mathrm{Hom}(M(w_2), M(w_1)) \neq 0$*, then* $\{\theta\}$ *is a* $\Bbbk$*-basis for* $\mathrm{Hom}_\Lambda(M(w_2), M(w_1))$ *where*

$$\theta_{x_j^{(1)}} \quad = \quad \begin{cases} 1 & : \quad \textit{if } j \in [i] \\ 0 & : \quad \textit{otherwise.} \end{cases}$$

It will be convenient to introduce an alternative notation for the indecomposable Λ-modules. Let $X(i,j)$ where $i \in [n]$ and $j \in [n-1]$ denote the unique indecomposable Λ-module containing $M(i)$ and whose length is j. For example, if $\Lambda = \Bbbk Q(n)/\langle \alpha_1 \cdots \alpha_{n-1} : \alpha_i \in Q(n)_1 \rangle$, then $X(n,i) = M(n \leftarrow \cdots \leftarrow n-i+1)$.

Example 3.2.7. *Let* $\Lambda = \Bbbk Q(4)/\langle \alpha_1\alpha_2\alpha_3 : \ \alpha_i \in Q(4)_1 \rangle$*. Then the* ***Auslander-Reiten quiver*** *of* Λ *is*

$$\Gamma(\Lambda\text{-}mod) \quad = \quad \begin{tikzcd}[column sep=small]
X(1,3) \arrow[dr, two heads] & & X(4,3) \arrow[dr, two heads] & & X(3,3) \arrow[dr, two heads] & & X(2,3) \arrow[dr, two heads] & & X(1,3)\,. \\
& X(4,2) \arrow[ur, hook] \arrow[dr, two heads] & & X(3,2) \arrow[ll, "\tau"] \arrow[ur, hook] \arrow[dr, two heads] & & X(2,2) \arrow[ll, "\tau"] \arrow[ur, hook] \arrow[dr, two heads] & & X(1,2) \arrow[ll, "\tau"] \arrow[ur, hook] \arrow[dr, two heads] & \\
X(4,1) \arrow[ur, hook] & & X(3,1) \arrow[ll, "\tau"] \arrow[ur, hook] & & X(2,1) \arrow[ll, "\tau"] \arrow[ur, hook] & & X(1,1) \arrow[ll, "\tau"] \arrow[ur, hook] & & X(4,1) \arrow[ll, "\tau"]
\end{tikzcd}$$

Remark 3.2.8. *For any quiver* $Q(n)$*, the Auslander-Reiten quiver of the corresponding cluster-tilted algebra may be embedded on a cylinder. In general, the irreducible morphisms between indecomposable* Λ*-modules are exactly those of the form*

$$X(i,j) \hookrightarrow X(i, j+1) \qquad \textit{and} \qquad X(i,j) \twoheadrightarrow X(i-1, j-1).$$

Also, if $X(i,j) \in ind(\Lambda\text{-}mod)$ *and* $j \in [n-2]$*, then* $\tau^{\pm 1}X(i,j) = X(i \pm 1, j)$ *where we agree that* $\tau X(n,j) = X(1,j)$ *and* $\tau^{-1}X(1,j) = X(n,j)$*. Roughly speaking,* τ *acts on non-projective modules by rotation of dimension vectors. The modules* $\{X(i, n-1)\}_{i \in [n]}$ *are both the indecomposable projective and indecomposable injective modules since* Λ *is self-injective. Thus* $\tau^{\pm 1}X(i, n-1) = 0$ *for any* $i \in [n]$*.*

We conclude this section by classifying extensions of indecomposable modules $M(w_1)$, $M(w_2) \in \mathrm{ind}(\Lambda\text{-mod})$ where $\Lambda = \Bbbk Q(n)/\langle \alpha_1 \cdots \alpha_{n-1} : \ \alpha_i \in Q(n)_1 \rangle$ (i.e. extensions of the form $0 \to M(w_2) \to Z \to M(w_1) \to 0$ where $Z \in \mathrm{ind}(\Lambda\text{-mod})$.) This classification will be an important tool in the proofs of our main results. The first Lemma we present can be easily verified by considering the structure of the Auslander-Reiten quiver of Λ. Recall that

$$\underline{\mathrm{Hom}}_\Lambda(M,N) := \{f \in \mathrm{Hom}_\Lambda(M,N) : \ f \text{ factors through a projective } \Lambda\text{-module}\}.$$

Lemma 3.2.9. *Let $i \in \mathbb{Z}/n\mathbb{Z}$ and $1 \leqslant j < n-1$.*

a) The set of indecomposable Λ-modules X satisfying $\mathrm{Hom}_\Lambda(X(i,j),X) \neq 0$ has the form

$$\left\{ X(s,t) \in \mathit{ind}(\Lambda\text{-}\mathit{mod}) : \begin{array}{c} s \in [i-j+1, i]_n, \\ j - d(i,s) \leqslant t \leqslant n-1 \end{array} \right\}.$$

b) The set of indecomposable Λ-modules X satisfying $\underline{\mathrm{Hom}}_\Lambda(X(i,j),X) \neq 0$ has the form

$$\left\{ X(s,t) \in \mathit{ind}(\Lambda\text{-}\mathit{mod}) : \begin{array}{c} s \in [i-j+1, i]_n, \\ j - d(i,s) \leqslant t \leqslant n-2-d(i,s) \end{array} \right\}.$$

*Here we define $d(a,b) := \#\{\text{arrows in the string } w = a \leftarrow \cdots \leftarrow b\}$ and $[i-j+1,i]_n$ to be the **cyclic interval** in $[n]$ (i.e. there is a string $i \leftarrow (i-1) \leftarrow \cdots \leftarrow (i-j+2) \leftarrow (i-j+1)$ and the arithmetic is carried out mod n).*

We now use Lemma 3.2.9 to classify extensions. By Lemma 3.9.4, the dimension of $\mathrm{Ext}^1_\Lambda(X(k,\ell),X(i,j))$ is at most 1 for any indecomposables $X(k,\ell)$ and $X(i,j)$. Thus there is at most one nonsplit extension of the form $0 \to X(i,j) \to Z \to X(k,\ell) \to 0$ up to equivalence of extensions.

Proposition 3.2.10. *Let $i,k \in \mathbb{Z}/n\mathbb{Z}$ and $1 \leqslant j,\ell < n-1$. If $\mathrm{Ext}^1_\Lambda(X(k,\ell),X(i,j)) \neq 0$, then*

i) if $\mathrm{supp}(X(i,j) \cap X(k,\ell)) = \varnothing$, then the unique nonsplit extension is of the form $0 \to X(i,j) \to X(i,j+\ell) \to X(k,\ell) \to 0$,

ii) if $\mathrm{supp}(X(i,j) \cap X(k,\ell)) \neq \varnothing$, then the unique nonsplit extension is of the form $0 \to X(i,j) \to X(i,d(i,k)+\ell) \oplus X(k,j-d(i,k)) \to X(k,\ell) \to 0$.

Proof. By the Auslander-Reiten formula,

$$\begin{aligned} \dim \mathrm{Ext}^1_\Lambda(X(k,\ell),X(i,j)) &= \dim \underline{\mathrm{Hom}}_\Lambda(\tau^{-1}X(i,j),X(k,\ell)) \\ &= \dim \underline{\mathrm{Hom}}_\Lambda(X(i-1,j),X(k,\ell)). \end{aligned}$$

Hence, if $\mathrm{Ext}^1_\Lambda(X(k,\ell), X(i,j)) \neq 0$ then $\mathrm{supp}(X(i-1,j)) \cap \mathrm{supp}(X(k,\ell)) \neq \varnothing$.

i) If $\mathrm{supp}(X(i,j)) \cap \mathrm{supp}(X(k,\ell)) = \varnothing$, then $\mathrm{supp}(X(i-1,j)) \cap \mathrm{supp}(X(k,\ell)) = \{i-j\}$ since $\mathrm{supp}(X(i-1,j)) \backslash \mathrm{supp}(X(i,j)) = \{i-j\}$. Since there is a nonzero morphism $X(i-1,j) \to X(k,\ell)$, we must have $k = i-j$ by the description of morphisms in Lemma 3.2.9 a). Then $X(k,\ell) = X(i-j,\ell)$ and the extension must be of the form

$$0 \to X(i,j) \to X(i,j+\ell) \to X(k,\ell) \to 0.$$

ii) Assume $\mathrm{supp}(X(i,j)) \cap \mathrm{supp}(X(k,\ell)) \neq \varnothing$. Here it is enough to show that there are inclusions (resp. surjections) $X(i,j) \hookrightarrow X(i,d(i,k)+\ell)$ and $X(k,j-d(i,k)) \hookrightarrow X(k,\ell)$ (resp. $X(i,j) \twoheadrightarrow X(k,j-d(i,k))$ and $X(i,d(i,k)+\ell) \twoheadrightarrow X(k,\ell)$).

To do so, we first show that $d(i,k)+\ell \leqslant n-1$. Observe that $\ell \leqslant n-2-d(i-1,k)$ by Lemma 3.2.9 b) since $\underline{\mathrm{Hom}}_\Lambda(X(i-1,j), X(k,\ell)) \neq 0$. Since $d(i,k) - d(i-1,k) = 1$, we have that $d(i,k)+\ell \leqslant n-1$. Thus $X(i,d(i,k)+\ell) \in \mathrm{ind}(\Lambda\text{-mod})$ and therefore there is an inclusion $X(i,j) \hookrightarrow X(i,d(i,k)+\ell)$.

Next, we show that and $j - d(i,k) \geqslant 1$. By Lemma 3.2.9 a), we deduce that $k \in [i-j+1, i]_n$ since $k \neq i-j$. Therefore, we conclude that $d(i,k) \leqslant j-1$ so $j-d(i,k) \geqslant 1$. Thus $X(k,j-d(i,k)) \in \mathrm{ind}(\Lambda\text{-mod})$ so there is an inclusion $X(k,j-d(i,k)) \hookrightarrow X(k,\ell)$.

Lastly, we show that the desired surjections exist. Observe that since $d(i,k) = \#\{\text{arrows in the string } i \leftarrow (i-1) \leftarrow \cdots \leftarrow (k+1) \leftarrow k\}$ we have that $i - d(i,k) = k$ where this equation holds mod n. Thus, by composing surjective irreducible morphisms, we obtain the desired surjections

$$X(i,j) \twoheadrightarrow X(i-1,j-1) \twoheadrightarrow \cdots \twoheadrightarrow X(k+1,j+1-d(i,k)) \twoheadrightarrow X(k,j-d(i,k))$$

and

$$X(i,d(i,k)+\ell) \twoheadrightarrow X(i-1,d(i,k)+\ell-1) \twoheadrightarrow \cdots \twoheadrightarrow X(k+1,\ell+1) \twoheadrightarrow X(k,\ell).$$

Hence, the unique extension is of the form

$$0 \to X(i,j) \to X(i,d(i,k)+\ell) \oplus X(k,j-d(i,k)) \to X(k,\ell) \to 0.$$

□

When we use this classification of extensions to prove our main results, we will want to use only the notation for string modules. Thus we give the following translation of Proposition 3.2.10 using the notation for string modules.

Lemma 3.2.11. *Let $Q = Q(n)$ for some $n \geqslant 3$ and let Λ denote the corresponding cluster-tilted algebra. Let $M(w_2), M(w_1) \in ind(\Lambda\text{-}mod)$ where $\operatorname{Ext}^1_\Lambda(M(w_1), M(w_2)) \neq 0$. Let*

$$\xi = 0 \to M(w_2) \to Z \to M(w_1) \to 0$$

denote the unique nonsplit extension up to equivalence of extensions with $Z \in \Lambda$-mod. Then either

i) $supp(M(w_2)) \cap supp(M(w_1)) = \varnothing$ and $Z = M(w_2 \longleftarrow w_1)$ or

ii) $w_2 = u \longleftarrow w$ with $w_1 = w \longleftarrow v$ for some strings u, v, and w where $w = x_1 \xleftarrow{\alpha_1} x_2 \cdots x_{k-1} \xleftarrow{\alpha_{k-1}} x_k$ satisfies $supp(M(w_2)) \cap supp(M(w_1)) = \{x_i\}_{i \in [k]}$, $(Q(n))_0 \backslash (supp(M(w_2)) \cap supp(M(w_1))) \neq \varnothing$, and $Z = M(u \leftarrow w \leftarrow v) \oplus M(w)$.

3.3 Lattice properties

In this section, we give some background on lattices. After establishing notation in Section 3.3.1, we discuss semidistributive, congruence-uniform, and polygonal lattices in the remaining sections.

3.3.1 Basic notions

A **lattice** L is a poset for which every pair of elements $x, y \in L$ has a least upper bound $x \vee y$ and greatest lower bound $x \wedge y$, called the **join** and **meet**, respectively. A lattice is **complete** if meets and joins exist for arbitrary subsets of L. As we will mainly deal with finite lattices, these two conditions coincide. Any complete lattice has a **top element** $\bigvee L$ and **bottom element** $\bigwedge L$, which we denote $\hat{1}$ and $\hat{0}$, respectively.

Many properties of posets come in dual pairs. Given a poset $(P, \leqslant)$, its **dual poset** $(P^{\text{op}}, \leqslant^{\text{op}})$ has the same underlying set and $x \leqslant^{\text{op}} y$ if and only if $y \leqslant x$. If P is a lattice, then P^{op} has the same lattice structure with $\wedge$ and $\vee$ swapped.

A **lattice congruence** Θ on a lattice L is an equivalence relation that respects the lattice operations; i.e. for $x, y, z \in L$, $x \equiv y \mod \Theta$ implies $(x \vee z) \equiv (y \vee z) \mod \Theta$ and $(x \wedge z) \equiv (y \wedge z) \mod \Theta$. The lattice operations on L induce a lattice structure on the set of equivalence classes of Θ, which we denote L/Θ. The natural map $L \to L/\Theta$ is a **lattice quotient map**.

Figure 3.1 contains two examples of lattice quotient maps. The BLUE arrows in each of the upper lattices are contracted to form the lower lattices.

To prove that a given equivalence relation is a lattice congruence, we will make use of the following well-known result.

Lemma 3.3.1. *Let L be a finite lattice with idempotent, order-preserving maps $\pi_\downarrow, \pi^\uparrow : L \to L$. Let Θ be the equivalence relation $x \equiv y \mod \Theta$ if $\pi_\downarrow(x) = \pi_\downarrow(y)$. If $\pi_\downarrow \circ \pi^\uparrow = \pi_\downarrow$ and $\pi^\uparrow \circ \pi_\downarrow = \pi^\uparrow$, then Θ is a lattice congruence of L.*

The maps in part (2) of the above lemma are typically called $\pi_\downarrow$ and $\pi^\uparrow$. Both of these maps are idempotent endomorphisms on L. However, we may identify $\pi_\downarrow$ with the natural lattice quotient map $L \to L/\Theta$ when convenient.

An element j of a lattice L is **join-irreducible** (dually, **meet-irreducible**) if $j \neq \hat{0}$ and for $x, y \in L$, $j = x \vee y$ implies $j = x$ or $j = y$. For finite lattices, an element j is join-irreducible exactly when it covers a unique element, denoted j_*. Dually, a meet-irreducible element m is covered by a unique element m^*. We let $J(L)$ (resp. $M(L)$) be the set of join-irreducible (resp. meet-irreducible) elements of L.

3.3.2 Semidistributive lattices

A lattice L is **meet-semidistributive** if for $x, y, z \in L$,

$$(x \vee y) \wedge z = x \wedge z \qquad \text{holds whenever} \qquad x \wedge z = y \wedge z.$$

A lattice is **join-semidistributive** if its dual is meet-semidistributive. It is **semidistributive** if it is both join-semidistributive and meet-semidistributive. Clearly, every distributive lattice is semidistributive. On the other hand, the five-element lattice of Figure 3.1 is semidistributive but not distributive. As semidistributivity is defined by equations in the lattice operations, it is preserved under lattice quotients.

For a finite lattice L, any element x admits a representation of the form $x = \bigvee A$ where A is a subset of $J(L)$. The representation is **irredundant** if $x > \bigvee A'$ for any proper subset A' of A. Given antichains $A, B \subseteq J(L)$, we say $A \leqslant B$ if every element of A is less than some element of B. A join-representation $x = \bigvee A$ for $x \in L$, $A \subseteq J(L)$ is called a **canonical join-representation** if it is irredundant and $A \leqslant B$ whenever $B \subseteq J(L)$ with $x = \bigvee B$. **Canonical meet-representations** are defined dually. The following Lemma gives an explicit definition of canonical-meet representations.

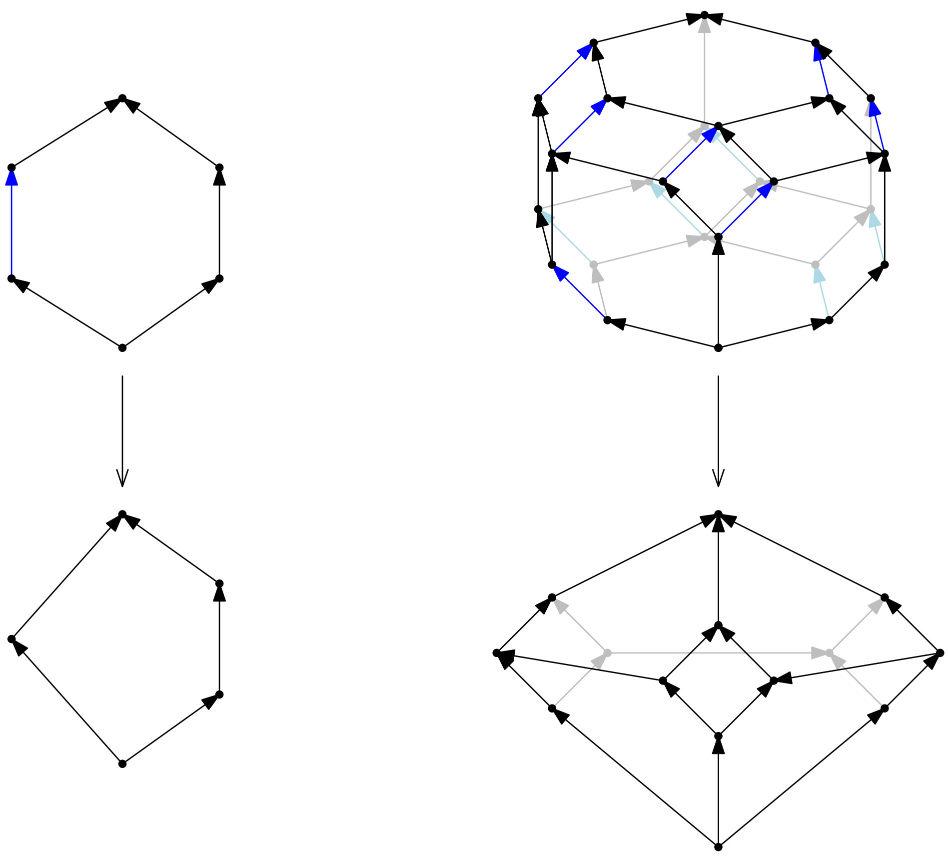


Figure 3.1: Two examples of lattice quotient maps.

Lemma 3.3.2. *Given an element x of a lattice L, the expression $x = \bigvee_{i=1}^{l} j_i$ is a canonical join-representation of x in L if and only if $x^{\mathrm{op}} = \bigwedge_{i=1}^{l} j_i^{\mathrm{op}}$ is a canonical meet-representation in the dual lattice L^{op}.*

A finite semidistributive lattice admits canonical join-representations for all of its elements [FJN95, Theorem 2.24]. These canonical join-representations often take a very nice form. One of our main applications of semidistributivity is a description of canonical join-representations and canonical meet-representations of torsion classes (see Theorem 3.8.3 and Corollary 3.8.4).

3.3.3 Congruence-uniform lattices

Given a closed interval $I = [x, y]$ of a poset P, the **doubling** $P[I]$ is the induced subposet of $P \times \{0, 1\}$ with elements

$$P[I] = (P_{\leqslant y} \times \{0\}) \sqcup [(P - P_{\leqslant y}) \cup I] \times \{1\},$$

where $P_{\leqslant y} = \{z \in P : \ z \leqslant y\}$. If P is a lattice, then $P[I]$ is a lattice. A finite lattice L is **congruence-uniform** (or **bounded**) if there exists a sequence of lattices $L_1, \ldots, L_t$ such that L_1 is the one-element lattice, $L_t = L$, and for all i, there exists a closed interval I_i of L_i such that $L_{i+1} \cong L_i[I_i]$.

As interval doublings preserve semidistributivity, finite congruence-uniform lattices are always semidistributive. Congruence-uniform lattices admit other characterizations in terms of lattice congruences [Day94], edge-labelings [Rea03], or as "bounded" quotients of free lattices.

3.3.4 Polygonal lattices

A finite lattice is a **polygon** if contains exactly two maximal chains and those chains only agree at the bottom and top elements. A finite lattice L is **polygonal** if for all $x \in L$:

- if y and z are distinct elements covering x, then $[x, y \vee z]$ is a polygon, and
- if y and z are distinct elements covered by x, then $[y \wedge z, x]$ is a polygon.

Given two maximal chains C, C' in a lattice L, we say C and C' differ by a **polygonal flip** if there is a polygon $[x, y]$ such that $C \cap [\hat{0}, x] = C' \cap [\hat{0}, x]$, $C \cap [y, \hat{1}] = C' \cap [y, \hat{1}]$ and $C \cap [x, y]$ and $C' \cap [x, y]$ are distinct maximal chains of $[x, y]$.

Our main use of polygonal lattices is the following connectivity result.

Lemma 3.3.3 (Lemma 1-6.3 [Reang])**.** *Let L be a polygonal lattice. If C and C' are maximal chains of L, then there is a sequence of maximal chains $C = C_0, C_1, \ldots, C_N = C'$ such that C_i and C_{i+1} differ by a polygonal flip for all i.*

3.4 Semidistributivity of oriented exchange graphs

In this section, we prove that if Q is mutation-equivalent to a Dynkin quiver, then $\overrightarrow{EG}(\widehat{Q})$ is a semidistributive lattice. To do so, we begin by identifying $\overrightarrow{EG}(\widehat{Q})$ with the lattice of functorially finite torsion classes of the cluster-tilted algebra $\Lambda = \Bbbk Q/I$ (see [BY13]). After that, we prove that the lattice of torsion classes of any finite dimensional algebra of finite representation type is semidistributive. Since Λ is representation finite, all torsion classes are functorially finite and thus we conclude that $\overrightarrow{EG}(\widehat{Q})$ is semidistributive.

3.4.1 Torsion classes and oriented exchange graphs

In this brief section, we recall the definition of torsion classes and the connection between torsion classes and oriented exchange graphs.

Let Λ be a finite dimensional $\Bbbk$-algebra. A full, additive subcategory $\mathcal{C} \subset \Lambda$-mod is **extension closed** if for any objects $X, Y \in \mathcal{C}$ satisfying $0 \to X \to Z \to Y \to 0$ one has $Z \in \mathcal{C}$. We say $\mathcal{C}$ is **quotient closed** (resp. **submodule closed**) if for any $X \in \mathcal{C}$ satisfying $X \xrightarrow{\alpha} Z$ where α is a surjection (resp. $Z \xrightarrow{\beta} X$ where β is an injection), then $Z \in \mathcal{C}$. A full, additive subcategory $\mathcal{T} \subset \Lambda$-mod is called a **torsion class** if $\mathcal{T}$ is quotient closed and extension closed. Dually, a full, additive subcategory $\mathcal{F} \subset \Lambda$-mod is called a **torsion-free class** if $\mathcal{F}$ is extension closed and submodule closed.

Let tors(Λ) (resp. torsf(Λ)) denote the lattice of torsion classes (resp. of torsion-free classes) of Λ ordered by inclusion. We have the following Proposition, which shows that a torsion class of Λ uniquely determines a torsion-free class of Λ and vice versa.

Proposition 3.4.1. *[IRTT13, Prop. 1.1 a)] The maps*

$$\begin{array}{rcl} tors(\Lambda) & \xrightarrow{(-)^{\perp}} & torsf(\Lambda) \\ \mathcal{T} & \longmapsto & \mathcal{T}^{\perp} := \{X \in \Lambda\text{-}mod : \ \mathrm{Hom}_{\Lambda}(\mathcal{T}, X) = 0\} \end{array}$$

and

$$\begin{array}{rcl} torsf(\Lambda) & \xrightarrow{{}^{\perp}(-)} & tors(\Lambda) \\ \mathcal{F} & \longmapsto & {}^{\perp}\mathcal{F} := \{X \in \Lambda\text{-}mod : \ \mathrm{Hom}_{\Lambda}(X, \mathcal{F}) = 0\} \end{array}$$

are inverse bijections.

The lattices tors(Λ) and torsf(Λ) have the following description of the meet and join operations. In Lemma 3.4.9, we present an alternative description of the join operation.

Proposition 3.4.2. *[IRTT13, Prop. 1.3] Let Λ be a finite dimensional algebra. Then tors(Λ) and torsf(Λ) are complete lattices. The join and meet operations are described as follows*

a) Let $\{\mathcal{T}_i\}_{i \in I} \subset tors(\Lambda)$ be a collection of torsion classes. Then we have $\bigwedge_{i\in I} \mathcal{T}_i = \bigcap_{i \in I} \mathcal{T}_i$ and $\bigvee_{i \in I} \mathcal{T}_i = {}^{\perp}\left(\bigcap_{i\in I} \mathcal{T}_i^{\perp}\right)$.

b) Let $\{\mathcal{F}_i\}_{i\in I} \subset torsf(\Lambda)$ be a collection of torsion-free classes. Then we have $\bigwedge_{i\in I} \mathcal{F}_i = \bigcap_{i\in I} \mathcal{F}_i$ and $\bigvee_{i\in I} \mathcal{F}_i = \left(\bigcap_{i\in I} {}^{\perp}\mathcal{F}_i\right)^{\perp}$.

An important subset of tors(Λ) is the set of functorially finite torsion classes, denoted f-tors(Λ). By definition, $\mathcal{T} \in$ tors(Λ) is a **functorially finite** torsion class if there exists $X \in \Lambda$-mod such that

$$\mathcal{T} = \mathrm{Fac}(X) := \{Y \in \Lambda\text{-mod} : \exists X^m \twoheadrightarrow Y \text{ for some } m \in \mathbb{N}\}.$$

Dually, a torsion-free class $\mathcal{F} \in$ tors(Λ) is **functorially finite** if there exists $X \in \Lambda$-mod such that

$$\mathcal{F} = \mathrm{Sub}(X) := \{Y \in \Lambda\text{-mod} : \exists Y \hookrightarrow X^m \text{ for some } m \in \mathbb{N}\}.$$

We let f-torsf(Λ) denote the set functorially finite torsion-free classes of Λ. The sets f-tors(Λ) and f-torsf(Λ) are clearly partially-ordered by inclusion. The bijection given in Proposition 3.4.1 restricts to a bijection f-tors(Λ) $\to$ f-torsf(Λ). We also have the following useful Lemma.

Lemma 3.4.3. *[IRTT13, Prop. 1.4 a), c)] The maps*

$$\begin{array}{ccc} tors(\Lambda) & \xrightarrow{D(-)} & torsf(\Lambda^{op}) \\ \mathcal{T} & \longmapsto & D\mathcal{T} \end{array}$$

and

$$\begin{array}{ccc} torsf(\Lambda) & \xrightarrow{D(-)} & tors(\Lambda^{op}) \\ \mathcal{F} & \longmapsto & D\mathcal{F} \end{array}$$

are isomorphisms of lattices where $D(-) := \mathrm{Hom}_\Lambda(-, \mathbb{k})$ *is the* ***standard duality****. Furthermore, the functor* $D((-)^\perp) : tors(\Lambda) \to tors(\Lambda^{\mathrm{op}}) \cong tors(\Lambda)^{\mathrm{op}}$ *is an anti-isomorphism of posets.*

Let $\Lambda = \mathbb{k}Q/I$ be a cluster-tilted algebra. The next result, which appears in [BY13] in much greater generality, shows that oriented exchange graphs can be studied using functorially finite torsion classes of Λ.

Proposition 3.4.4. *Let Q be a quiver that is mutation equivalent to a Dynkin quiver and let $\Lambda = \mathbb{k}Q/I$ denote the associated cluster-tilted algebra. Then $\overrightarrow{EG}(\widehat{Q}) \cong$ f-tors(Λ) as posets.*

Theorem 3.4.5. *Let Λ be a finite dimensional $\mathbb{k}$-algebra where tors(Λ) = f-tors(Λ). Then f-tors(Λ) is a semisdistributive lattice. In particular, if Λ is a finite dimensional $\mathbb{k}$-algebra of finite representation type, then f-tors(Λ) is a semidistributive lattice.*

Proof. It is enough to show f-tors(Λ) = tors(Λ) is **meet-semidistributive** (i.e. for any $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \in$ tors(Λ) satisfying $\mathcal{T}_1 \wedge \mathcal{T}_2 = \mathcal{T}_2 \wedge \mathcal{T}_3$ we have that $(\mathcal{T}_1 \vee \mathcal{T}_2) \wedge \mathcal{T}_3 = \mathcal{T}_1 \wedge \mathcal{T}_3$) since tors$(\Lambda)$ is join-semidistributive if and only if tors(Λ^{op}) is meet-semidistributive. This is proved in the next section (see Lemma 3.4.10).

It is well-known that f-tors(Λ) = tors(Λ) holds when Λ is a finite dimensional, representation finite $\mathbb{k}$-algebra. Thus the second assertion holds. □

Remark 3.4.6. *In [DIJ15, Thm. 1.2], finite dimension algebras Λ satisfying tors(Λ) = f-tors(Λ) are shown to be exactly those algebras that are* τ-***rigid finite*** *algebras (i.e. Λ has only finitely many indecomposable modules M satisfying* $\mathrm{Hom}_\Lambda(M, \tau M) = 0$.*) Additionally, in [IRTT13, Thm. 0.2] it is shown that f-tors(Λ) is a complete lattice if and only if Λ is a τ-rigid finite algebra.*

Corollary 3.4.7. *Let Q be a quiver that is mutation equivalent to a Dynkin quiver. Then $\overrightarrow{EG}(\widehat{Q})$ is a semidistributive lattice.*

Proof. Since Q defines a representation finite cluster-tilted algebra Λ, we know that f-tors(Λ) = tors(Λ). By Proposition 3.4.4, we have that $\overrightarrow{EG}(\widehat{Q}) \cong \text{tors}(\Lambda)$. By Theorem 3.4.5, we now have that $\overrightarrow{EG}(\widehat{Q})$ is semidistributive. □

Example 3.4.8. *Let Q be the quiver appearing in Example 3.2.3. Note that $Q = Q(3)$. We show the Auslander-Reiten quiver of the cluster-tilted algebra $\Lambda = \Bbbk Q(3)/\langle \alpha_1\alpha_2 : \alpha_i \in Q(3)_1\rangle$ below (see Figure 3.3) and use it to describe the oriented exchange graph of Q as the lattice of torsion classes and torsion-free classes of Λ. Any $\mathcal{T} \in tors(\Lambda)$ or $\mathcal{F} \in torsf(\Lambda)$ is **additively generated** (i.e. a full, additive subcategory $\mathcal{A}$ of Λ-mod is **additively generated** if $\mathcal{A} = add(\oplus_{i\in[k]}M(w_i))$ for some finite subset $\{M(w_i)\}_{i\in[k]} \subset ind(\Lambda\text{-}mod)$ where $add(\oplus_{i\in[k]}M(w_i))$ is the smallest, full additive subcategory of Λ-mod that contains $\{M(w_i)\}_{i\in[k]}$) so $\mathcal{T}$ and $\mathcal{F}$ are completely determined by the set of indecomposable modules they contain. Using this fact, we show the torsion classes of Λ (resp. torsion-free classes of Λ) below in blue (resp. red) in Figure 3.4. For example, $\mathcal{T} = add(X(3,2) \oplus X(2,1))$ and its corresponding torsion-free class $\mathcal{F} = add(X(1,1) \oplus X(1,2) \oplus X(3,1))$ are depicted in Figure 3.2. Recall that $\overrightarrow{EG}(\widehat{Q(3)})$ is oriented by inclusion of torsion classes.*

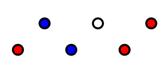


Figure 3.2: $\mathcal{T} = \text{add}(X(3,2) \oplus X(2,1))$ and $\mathcal{F} = \text{add}(X(1,1) \oplus X(1,2) \oplus X(3,1))$

$$\Gamma(\Lambda\text{-mod}) \quad = \quad \begin{tikzcd} & X(3,2) \arrow[dr] & & X(2,2) \arrow[dr] & & X(1,2) \arrow[dr] & \\ X(3,1) \arrow[ur, hook] & & X(2,1) \arrow[ll, "\tau"'] \arrow[ur, hook] & & X(1,1) \arrow[ll, "\tau"'] \arrow[ur, hook] & & X(3,1) \arrow[ll, "\tau"'] \end{tikzcd}$$

Figure 3.3: The Auslander-Reiten quiver of Λ.

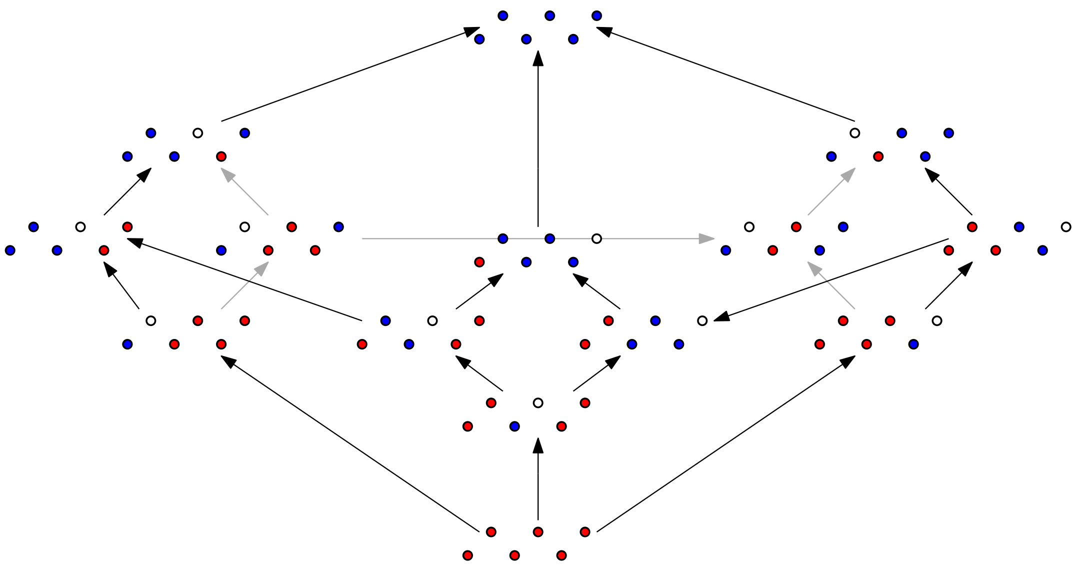


Figure 3.4: The oriented exchange graph of $Q(3)$ modeled using tors(Λ) and torsf(Λ).

3.4.2 Meet semidistributivity of tors(Λ)

In this section, we prove that the lattice of torsion classes of a finite dimensional $\Bbbk$-algebra Λ is meet semidistributive. As a prelminary step, we give an explicit description of the join of two torsion classes (see Lemma 3.4.9). We thank Hugh Thomas for mentioning this description of the join to us [Tho].

Lemma 3.4.9. *If $\mathcal{T},\mathcal{U} \in tors(\Lambda)$. Then*

$$\mathcal{T} \vee \mathcal{U} = \mathcal{F}ilt(\mathcal{T},\mathcal{U})$$

where $\mathcal{F}ilt(\mathcal{T},\mathcal{U})$ is defined to be the subcategory of all Λ-modules X with a filtration $0 = X_0 \subset X_1 \subset \cdots \subset X_n = X$ with the property that X_j/X_{j-1} belongs to $\mathcal{T}$ or $\mathcal{U}$ for any $j \in [n]$.

Proof. We can express $\mathcal{T} \vee \mathcal{U}$ as

$$\mathcal{T} \vee \mathcal{U} = \bigwedge_{\substack{\mathcal{T},\mathcal{U} \subset \mathcal{T}_\alpha \\ \mathcal{T}_\alpha \in \text{tors}(\Lambda)}} \mathcal{T}_\alpha$$

and the expression on the right hand side makes sense since $\mathrm{tors}(\Lambda)$ is a complete lattice by Proposition 3.4.2. If $X \in \mathcal{T}$ or $X \in \mathcal{U}$, then the filtration $0 \subset X$ of X shows that $X \in \mathcal{F}ilt(\mathcal{T},\mathcal{U})$. If $X \in \mathcal{F}ilt(\mathcal{T},\mathcal{U})$, and $0 = X_0 \subset X_1 \subset \cdots \subset X_n = X$ is a filtration that witnesses this, one obtains an extension $0 \to X_1 \to X \to X/X_{n-1} \to 0$ where X_1 and $X/X_{n-1} = X_n/X_{n-1}$ each belong to one of $\mathcal{T}$ or $\mathcal{U}$. Thus X must belong to any torsion class $\mathcal{T}_\alpha$ containing both $\mathcal{T}$ and $\mathcal{U}$ as torsion classes are extension closed. This implies that $\mathcal{F}ilt(\mathcal{T},\mathcal{U}) \subset \mathcal{T}_\alpha$. Thus it is enough to show that $\mathcal{F}ilt(\mathcal{T},\mathcal{U})$ is a torsion class.

We first show that $\mathcal{F}ilt(\mathcal{T},\mathcal{U})$ is additive. Assume $X, Y \in \mathcal{F}ilt(\mathcal{T},\mathcal{U})$. Let $0 = X_0 \subset X_1 \subset \cdots \subset X_n = X$ and $0 = Y_0 \subset Y_1 \subset \cdots \subset Y_m = Y$ be filtrations where each of the quotients X_i/X_{i-1} and Y_i/Y_{i-1} belong $\mathcal{T}$ or $\mathcal{U}$. Consider the filtration

$$0 = X_0 \subset X_1 \subset \cdots \subset X_n \subset X_n \oplus Y_1 \subset \cdots \subset X_n \oplus Y_m = X \oplus Y.$$

We have that $(X_n \oplus Y_1)/X_n \cong Y_1$ belongs to $\mathcal{T}$ or $\mathcal{U}$. We also have that for each $i \in [m]$ the quotient $(X_n \oplus Y_i)/(X_n \oplus Y_{i-1}) \cong Y_i/Y_{i-1}$ belongs to $\mathcal{T}$ or $\mathcal{U}$. Thus $\mathcal{F}ilt(\mathcal{T},\mathcal{U})$ is additive.

Next, suppose $X \to Y$ is a surjection and suppose that X has filtration $0 = X_0 \subset X_1 \subset \cdots \subset X_n = X$ where for each $i \in [n]$ the quotient X_i/X_{i-1} belongs to $\mathcal{T}$ or $\mathcal{U}$. Then we have a short exact sequence

$$0 \to K \to X \to Y \to 0.$$

We see that $(X_n + K)/K \cong Y$ because $X_n + K = X_n$ as $K \subset X_n$. We claim that

$$0 \subset (X_1 + K)/K \subset \cdots \subset (X_n + K)/K \cong Y$$

is a filtration of Y where for each $i \in [n]$ we have

$$((X_i + K)/K) \big/ ((X_{i-1} + K)/K) \cong (X_i + K)/(X_{i-1} + K) \in \mathcal{T} \text{ or } \mathcal{U}$$

and where the isomorphism is obtained from the Third Isomorphism Theorem. As $X_i \subset X_i + K$ for any $i \in [n]$, the map $X_i/X_{i-1} \to (X_i + K)/(X_{i-1} + K)$ defined by $a + X_{i-1} \mapsto a + X_{i-1} + K$ is well-defined and surjective. Since $X_i/X_{i-1} \in \mathcal{T}$ or $\mathcal{U}$, we have that $(X_i + K)/(X_{i-1} + K) \in \mathcal{T}$ or $\mathcal{U}$ so $\mathcal{F}ilt(\mathcal{T},\mathcal{U})$ is quotient closed.

Lastly, we show that $\mathcal{F}ilt(\mathcal{T},\mathcal{U})$ is extension closed. Suppose that $Z \in \Lambda$-mod is of minimal length (we denote the length of a module M by $\ell(M)$) with the property that there is an extension $0 \to X \to Z \to Y \to 0$ where $X, Y \in \mathcal{F}ilt(\mathcal{T},\mathcal{U})$, but $Z \notin \mathcal{F}ilt(\mathcal{T},\mathcal{U})$. Let $0 = X_0 \subset X_1 \subset \cdots \subset X_n = X$ be a filtration witnessing that $X \in \mathcal{F}ilt(\mathcal{T},\mathcal{U})$. The assumption that $Z \notin \mathcal{F}ilt(\mathcal{T},\mathcal{U})$ implies that the filtration $0 = X_0 \subset X_1 \subset \cdots \subset X_n = X \subset Z$ has the property that $Z/X \cong Y$ does not belong to $\mathcal{T}$ and does not belong to $\mathcal{U}$.

Now observe that there exists $Z^{(1)} \in \Lambda$-mod such that $X \subsetneq Z^{(1)} \subsetneq Z$, otherwise $Z/X \cong Y$ is simple and thus $Y \in \mathcal{T}$ or $Y \in \mathcal{U}$. Since $\ell(Z) > \ell(Z^{(1)})$, we have that $Z^{(1)} \in \mathcal{F}ilt(\mathcal{T},\mathcal{U})$. Let $0 = Z_0^{(1)} \subset Z_1^{(1)} \subset \cdots \subset Z_{n_1}^{(1)} = Z^{(1)}$ be a filtration that witnesses this. Since $Z \notin \mathcal{F}ilt(\mathcal{T},\mathcal{U})$, we have that the filtration $0 = Z_0^{(1)} \subset Z_1^{(1)} \subset \cdots \subset Z_{n_1}^{(1)} = Z^{(1)} \subset Z^{(1)}$ satisfies $Z/Z^{(1)} \notin \mathcal{T}$ and $Z/Z^{(1)} \notin \mathcal{U}$. However, since $\mathcal{F}ilt(\mathcal{T},\mathcal{U})$ is quotient closed and since we have a surjection $Y \cong Z/X \twoheadrightarrow Z/Z^{(1)}$, we know that $Z/Z^{(1)} \in \mathcal{F}ilt(\mathcal{T},\mathcal{U})$. This implies that there exists $Z^{(2)} \in \Lambda$-mod such that $Z^{(1)} \subsetneq Z^{(2)} \subsetneq Z$, otherwise $Z/Z^{(1)}$ is simple and therefore must belong to $\mathcal{T}$ or $\mathcal{U}$. Since $\ell(Z) > \ell(Z^{(2)})$, we have that $Z^{(2)} \in \mathcal{F}ilt(\mathcal{T},\mathcal{U})$. If $0 = Z_0^{(2)} \subset Z_1^{(2)} \subset \cdots \subset Z_{n_2}^{(2)} = Z^{(2)}$ is a filtration witnessing that $Z^{(2)} \in \mathcal{F}ilt(\mathcal{T},\mathcal{U})$, then the filtration $0 = Z_0^{(2)} \subset Z_1^{(2)} \subset \cdots \subset Z_{n_2}^{(2)} = Z^{(2)} \subset Z$ has the property that $Z/Z^{(2)} \notin \mathcal{T}$ and $Z/Z^{(2)} \notin \mathcal{U}$. However, $Z/Z^{(2)} \in \mathcal{F}ilt(\mathcal{T},\mathcal{U})$ since we have a surjection $Y \cong Z/X \twoheadrightarrow Z/Z^{(2)}$.

Inductively, one obtains a Λ-module $Z^{(k)} \in \mathcal{F}ilt(\mathcal{T},\mathcal{U})$ with $1 \leqslant k \leqslant \ell(Z/X)$ and where $Z/Z^{(k)} \in \mathcal{F}ilt(\mathcal{T},\mathcal{U})$ is simple. Since $Z/Z^{(k)}$ is simple, it belongs to either $\mathcal{T}$ or $\mathcal{U}$. If $0 = Z_0^{(k)} \subset Z_1^{(k)} \subset \cdots \subset Z_{n_k}^{(k)} = Z^{(k)}$ witnessing that $Z^{(k)} \in \mathcal{F}ilt(\mathcal{T},\mathcal{U})$, then the filtration $0 = Z_0^{(k)} \subset Z_1^{(k)} \subset \cdots \subset Z_{n_k}^{(k)} = Z^{(k)} \subset Z$ shows that $Z \in \mathcal{F}ilt(\mathcal{T},\mathcal{U})$. This contradicts our assumption that $Z \notin \mathcal{F}ilt(\mathcal{T},\mathcal{U})$. □

Next, we complete the proof that tors(Λ) is meet semidistributive using Lemma 3.4.9.

Lemma 3.4.10. *The lattice tors(Λ) is meet semidistributive.*

Proof. Let $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \in$ tors(Λ). We show that if $\mathcal{T}_1 \wedge \mathcal{T}_3 = \mathcal{T}_2 \wedge \mathcal{T}_3$, then $(\mathcal{T}_1 \vee \mathcal{T}_2) \wedge \mathcal{T}_3 = \mathcal{T}_1 \wedge \mathcal{T}_3$. It is clear that $(\mathcal{T}_1 \vee \mathcal{T}_2) \wedge \mathcal{T}_3 \supset \mathcal{T}_1 \wedge \mathcal{T}_3$ so it is enough to show $(\mathcal{T}_1 \vee \mathcal{T}_2) \wedge \mathcal{T}_3 \subset \mathcal{T}_1 \wedge \mathcal{T}_3$.

Let $X \in (\mathcal{T}_1 \vee \mathcal{T}_2) \wedge \mathcal{T}_3$. By Lemma 3.4.9, we see that X has a filtration $0 = X_0 \subset X_1 \subset \cdots \subset X_n = X$ where for each $i \in [n]$ the quotient X_i/X_{i-1} belongs $\mathcal{T}_1$ or $\mathcal{T}_2$.

To complete the proof, it is enough to show that the module $X/X_{n-j} \in \mathcal{T}_1 \wedge \mathcal{T}_2 \wedge \mathcal{T}_3$ for any $j \in [n]$. Since $\mathcal{T}_3$ is a torsion class and since there is a surjection $X \to X/X_{n-j}$ for any $j \in [n]$, it is clear that $X/X_{n-j} \in \mathcal{T}_3$ for any $j \in [n]$. Thus we need to show that $X/X_{n-j} \in \mathcal{T}_1 \wedge \mathcal{T}_2$ for any $j \in [n]$.

To show that $X/X_{n-j} \in \mathcal{T}_1 \wedge \mathcal{T}_2$ for any $j \in [n]$, we induct on j. Let $j = 1$. We observe $X/X_{n-1} \in \mathcal{T}_1$ or $\mathcal{T}_2$ by the properties of the filtration of X. Since $X/X_{n-1} \in \mathcal{T}_3$ and since we assume $\mathcal{T}_1 \wedge \mathcal{T}_3 = \mathcal{T}_2 \wedge \mathcal{T}_3$, we obtain that $X/X_{n-1} \in \mathcal{T}_1 \wedge \mathcal{T}_2$.

Now suppose $X/X_{n-j} \in \mathcal{T}_1 \wedge \mathcal{T}_2$ and consider X/X_{n-j-1}. We have a short exact sequence

$$0 \to X_{n-j}/X_{n-j-1} \to X/X_{n-j-1} \to X/X_{n-j} \to 0$$

where X_{n-j}/X_{n-j-1} belongs to $\mathcal{T}_1$ or $\mathcal{T}_2$ by the properties of the filtration of X and $X/X_{n-j} \in \mathcal{T}_1 \wedge \mathcal{T}_2$ by induction. Thus $X/X_{n-j-1} \in \mathcal{T}_1$ or $\mathcal{T}_2$ since torsion classes are extension closed. By assumption, $\mathcal{T}_1 \wedge \mathcal{T}_3 = \mathcal{T}_2 \wedge \mathcal{T}_3$ so $X/X_{n-j-1} \in \mathcal{T}_1 \wedge \mathcal{T}_2$. We conclude that $X/X_{n-j} \in \mathcal{T}_1 \wedge \mathcal{T}_2$ for any $j \in [n]$. □

3.5 Biclosed sets

A **closure operator** on a set C is an operator $X \mapsto \overline{X}$ on subsets of C such that for $X, Y \subseteq S$:

- $X \subseteq \overline{X}$,
- $\overline{\overline{X}} = \overline{X}$, and
- if $X \subseteq Y$, then $\overline{X} \subseteq \overline{Y}$.

In addition, we assume that $\overline{\varnothing} = \varnothing$. A subset X of C is **closed** if $X = \overline{X}$. It is **co-closed** (or **open**) if $C - X$ is closed. We say X is **biclosed** (or **clopen**) if it is both closed and co-closed. We let $\mathrm{Bic}(C)$ denote the poset of biclosed subsets of C, ordered by inclusion.

For many closure operators, the poset of biclosed sets is not a lattice. However, in some special cases, $\mathrm{Bic}(C)$ is a lattice with a semidistributive or congruence-uniform structure. For example, if C is the set of positive roots of a finite root system endowed with the convex closure, then $\mathrm{Bic}(C)$ is a congruence-uniform lattice [Rea03]. In this setting, biclosed sets of positive roots are inversion sets of elements of the associated Coxeter group, so $\mathrm{Bic}(C)$ may be identified with the weak order.

Some sufficient criteria for semidistributivity, congruence-uniformity, and polygonality was given in [McC15]. However, these criteria are not necessary. We say a collection $\mathcal{B}$ of subsets of C is **ordered by single-step inclusion** if whenever $X, Y \in \mathcal{B}$ with $X \subsetneq Y$, there exists $c \in Y - X$ such that $X \cup \{c\} \in \mathcal{B}$.

Theorem 3.5.1 ([McC15] Theorem 5.5)**.** *Let C be a set with a closure operator. Assume that*

1. $\mathrm{Bic}(C)$ *is ordered by single-step inclusion, and*

2. $W \cup \overline{(X \cup Y) - W}$ *is biclosed for* $W, X, Y \in \mathrm{Bic}(C)$ *with* $W \subseteq X \cap Y$.

   Then $\mathrm{Bic}(C)$ *is a semidistributive lattice.*

   $\mathrm{Bic}(C)$ *is congruence-uniform if it satisfies (1), (2), and it has a poset structure* $(\mathrm{Bic}(C), \prec)$ *such that*

3. *if* $x, y, z \in C$ *with* $z \in \overline{\{x, y\}} - \{x, y\}$ *then* $x \prec z$ *and* $y \prec z$.

   $\mathrm{Bic}(C)$ *is polygonal if it satisfies (1), (2), and*

4. *for distinct* $x, y \in C$, $\mathrm{Bic}(\overline{\{x, y\}})$ *is a polygon.*

Example 3.5.2. *For $X \subseteq \binom{[n]}{2}$, say X is closed if $\{i, k\} \in X$ holds whenever $\{i, j\} \in X$ and $\{j, k\} \in X$ for $1 \leqslant i < j < k \leqslant n$. It is easy to check that biclosed subsets of $\binom{[n]}{2}$ are inversion sets of permutations. Moreover, ordering $\{j, k\} \preceq \{i, l\}$ if $i \leqslant j < k \leqslant l$, this closure space satisfies the hypotheses of Theorem 3.5.1. Hence, one may deduce that the weak order is a congruence-uniform and polygonal lattice. We refer to [McC15] for more examples.*

3.5.1 Biclosed sets of paths

A **path** in a graph $G = (V, E)$ is a finite sequence of vertices $(v_0, \ldots, v_t)$ such that v_i is adjacent to v_{i+1} for $i < t$. If $p = (v_0, \ldots, v_t)$ is a path, its **reverse** $p_{\mathrm{rev}} = (v_t, \ldots, v_0)$ is also a path. We say a path $(v_0, \ldots, v_t)$ is **acyclic** if v_i and v_j are not adjacent whenever

$|i-j| \geqslant 2$. Given paths $p = (v_0, \ldots, v_t)$ and $p' = (v'_0, \ldots, v'_s)$, if v_t is adjacent to v'_0 then we say p and p' are **composable**, and their concatenation is the sequence

$$p \circ p' = (v_0, \ldots, v_t, v'_0, \ldots, v'_s).$$

Observe that an acyclic path is determined up to reversal by its set of vertices. For our purposes, we will not distinguish between an acyclic path and its reversal.

Let AP be the collection of acyclic paths of G. For $X \subseteq \mathrm{AP}$, we say X is **closed** if for $p, p' \in X$, if $p \circ p' \in \mathrm{AP}$ then $p \circ p' \in X$. As before, we say X is biclosed if both X and $\mathrm{AP} - X$ are closed.

The closure of any subset of acyclic paths may be computed by successively concatenating paths. We record this useful fact in the following lemma.

Lemma 3.5.3. *Let $X \subseteq \mathrm{AP}$. If $p \in \overline{X}$, then there exist paths $q_1, \ldots, q_t \in X$ such that $p = q_1 \circ \cdots \circ q_t$.*

Theorem 3.5.4. $\mathrm{Bic(AP)}$ *is a semidistributive, congruence-uniform, and polygonal lattice.*

Proof. To prove this result, we verify properties (1)-(4) of Theorem 3.5.1.

Let $X, Y \in \mathrm{Bic(AP)}$ such that $X \subsetneq Y$. If p, q, q' are paths such that $p \in Y$ and $q \circ q' = p$, then either q or q' is in Y. If $p \in Y - X$ is chosen of minimum length, then either q or q' must be in X.

Among the elements p of $Y - X$ such that if $p = q \circ q'$ then either $q \in X$ or $q' \in X$, choose p_0 to be of maximum length. We prove that $X \cup \{p_0\}$ is biclosed. By the choice of p_0, it is immediate that $X \cup \{p_0\}$ is co-closed.

Assume that $X \cup \{p_0\}$ is not closed. Then there exists $p \in X$ such that $p \circ p_0$ is an acyclic path but is not in X. Among such paths, we assume p is of minimum length. Since Y is closed, $p \circ p_0$ is in Y. By the maximality of p_0, there exist acyclic paths q, q' both not in X such that $p \circ p_0 = q \circ q'$. Let $p \circ p_0 = (v_0, \ldots, v_t)$. Up to path reversal, we may assume $p = (v_0, \ldots, v_{i-1}),\ p_0 = (v_i, \ldots, v_t),\ q = (v_0, \ldots, v_{j-1}),\ q' = (v_j, \ldots, v_t)$ for some distinct indices i, j.

If $i < j$, then $p \in X,\ q \notin X$ implies $(v_i, \ldots, v_{j-1}) \notin X$ since X is closed. But, $(v_i, \ldots, v_{j-1}) \circ q' = p_0$, contradicting the choice of p_0.

If $j < i$, then $p \in X$, $q \notin X$ implies $(v_j, \ldots, v_{i-1}) \in X$ since X is co-closed. But $(v_j, \ldots, v_{i-1}) \circ p_0 = q'$, contradicting the minimality of p.

We conclude that $X \cup \{p_0\}$ is biclosed. Hence, Bic(AP) is ordered by single-step inclusion. This completes the proof of (1).

Now let $W, X, Y \in \text{Bic(AP)}$ such that $W \subseteq X \cap Y$.

Assume $W \cup \overline{(X \cup Y) - W}$ is not closed. Choose $p, q \in W \cup \overline{(X \cup Y) - W}$ such that $p \circ q$ is of minimum length with $p \circ q \notin W \cup \overline{(X \cup Y) - W}$. As W and $\overline{(X \cup Y) - W}$ are both closed, we may assume $p \in W$ and $q \in \overline{(X \cup Y) - W}$. If $q \in (X \cup Y) - W$, then $p \circ q \in X \cup Y$ as X and Y are both closed. Otherwise, $q = q' \circ q_1$ where $q' \in \overline{(X \cup Y) - W}$, $q_1 \in (X \cup Y) - W$. By the minimality hypothesis, $p \circ q' \in W \cup \overline{(X \cup Y) - W}$. If $p \circ q' \in W$, then $p \circ q = p \circ q' \circ q_1 \in X \cup Y$ as X and Y are both closed. If $p \circ q' \in \overline{(X \cup Y) - W}$, then so is $p \circ q' \circ q_1$. Either case contradicts the assumption that $p \circ q \notin W \cup \overline{(X \cup Y) - W}$. Hence, this set is closed. Using properties of closure operators, we deduce

$$W \cup \overline{(X \cup Y) - W} = \overline{W \cup \overline{(X \cup Y) - W}} = \overline{W \cup ((X \cup Y) - W)} = \overline{X \cup Y}.$$

Now assume $\overline{X \cup Y}$ is not co-closed. Choose $p \in \overline{X \cup Y}$ of minimum length such that $p = q \circ q'$ for some paths q, q' not in $\overline{X \cup Y}$. If $p \in X \cup Y$, then either $q \in X \cup Y$ or $q' \in X \cup Y$ since both X and Y are co-closed. Otherwise, there exist $p_1 \in X \cup Y$, $p' \in \overline{X \cup Y}$ such that $p = p_1 \circ p'$.

Suppose p_1 is a subpath of q and let $r \in \text{AP}$ such that $p_1 \circ r = q$. Then $r \circ q' = p'$, so either $r \in \overline{X \cup Y}$ or $q' \in \overline{X \cup Y}$ by the minimality of p. But if r is in $\overline{X \cup Y}$ then so is $q = p_1 \circ r$. This contradicts the hypothesis on q.

Suppose q is a subpath of p_1 and let $r \in \text{AP}$ such that $q \circ r = p_1$. This implies $r \circ p' = q'$. Since X and Y are co-closed, either $r \in X \cup Y$ or $q \in X \cup Y$. But if $r \in X \cup Y$, then $q' = r \circ p' \in X \cup Y$ holds. This contradicts the hypothesis on q'.

Hence, $\overline{X \cup Y}$ is co-closed. Putting this together, we deduce that $W \cup \overline{(X \cup Y) - W}$ is biclosed, establishing (2).

Partially order the set of acyclic paths by inclusion; that is, for $p, q \in \text{AP}$ set $p \preceq q$ if p is a subpath of q.

For $p, q \in \text{AP}$, the set $\overline{\{p, q\}}$ is $\{p, q\}$ if they are not composable and is $\{p, q, p \circ q\}$ otherwise. In either case, it is easy to verify both (3) and (4). $\square$

Using the properties of Theorem 3.5.1, we may determine the structure of all polygons in Bic(AP).

Corollary 3.5.5. *Every polygon of* Bic(AP) *is either a square or hexagon as in Figure 3.5.*

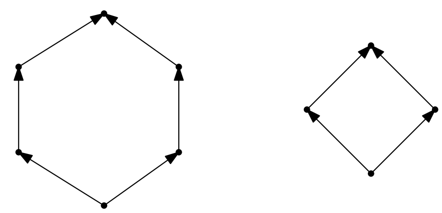


Figure 3.5: The polygons of Bic(AP).

Proof. A polygon of Bic(AP) is an interval of the form $[W, X \vee Y]$ where $W, X, Y \in$ Bic(AP) such that $W \lessdot X$ and $W \lessdot Y$. By (1), there exist unique paths $p \in X - W$ and $q \in Y - W$. By (2), $X \vee Y = W \cup \overline{\{p, q\}}$. If p and q are not composable, then $\overline{\{p, q\}} = \{p, q\}$, which implies that the interval $[W, X \vee Y]$ is a square. Otherwise, $\overline{\{p, q\}} = \{p, q, p \circ q\}$, and the interval $[W, X \vee Y]$ is a hexagon as in Figure 3.5. $\square$

Let $\widehat{Q}$ be the framed quiver of Q with positive **c**-vectors **c**-vec$^+(Q)$. We say that a subset X of **c**-vec$^+(Q)$ is **closed** if $x + y \in X$ whenever $x, y \in X$ and $x + y \in$ **c**-vec$^+(Q)$.

The relation to the previous closure operator is that if Q is of type $\mathbb{A}$ or is an oriented cycle, then the positive **c**-vectors of Q are in natural bijection with acyclic paths on Q. Moreover, the closure operators are identified via this bijection. Thus we define Bic(Q) to be the lattice of **biclosed sets of c-vectors** of Q.

3.6 Biclosed subcategories

Throughout this section, we assume that $\Lambda = \Bbbk Q/I$ is the cluster-tilted algebra define by a quiver Q, which is either a cyclic quiver or of type $\mathbb{A}$. In this section, we show how to translate the information of Bic(Q) into a lattice of **biclosed** subcategories of Λ-mod that we will denote by $\mathcal{BIC}(Q)$. More specifically, each biclosed set $B \in$

$\mathrm{Bic}(Q)$ will determine a unique subcategory $\mathcal{B}$ of Λ-mod and an inclusion of biclosed sets $B_1 \subset B_2$ will translate into an inclusion of biclosed subcategories $\mathcal{B}_1 \subset \mathcal{B}_2$. Using the additional algebraic data that accompanies these subcategories, we prove that the oriented exchange graph defined by Q is a lattice quotient of $\mathcal{BIC}(Q)$.

Definition 3.6.1. *Let $\mathcal{C}$ be a subcategory of Λ-mod. We say that $\mathcal{C}$ is* ***biclosed*** *if*

i) $\mathcal{C} = add\big(\oplus_{i\in[k]}M(w_i)\big)$ for some set of Λ-modules $\{M(w_i)\}_{i\in[k]}$ ($add(\oplus_{i\in[k]}M(w_i))$ denotes the smallest full, additive subcategory of Λ-mod containing each $M(w_i)$).

ii) $\mathcal{C}$ is ***weakly extension closed*** *(i.e. if $0 \to M(w_1) \to M(w_3) \to M(w_2) \to 0$ is an exact sequence with $M(w_1), M(w_2) \in \mathcal{C}$, then $M(w_3) \in \mathcal{C}$).*

iii) $\mathcal{C}$ is ***weakly extension coclosed*** *(i.e. if $0 \to M(w_1) \to M(w_3) \to M(w_2) \to 0$ is an exact sequence with $M(w_1), M(w_2) \notin \mathcal{C}$, then $M(w_3) \notin \mathcal{C}$).*

Let $\mathcal{BIC}(Q)$ denote the collection of biclosed subcategories of Λ-mod ordered by inclusion.

Part *i)* in Definition 3.6.1 says that the elements of $\mathcal{BIC}(Q)$ are additively generated subcategories $\mathcal{C}$ of Λ-mod. Since we are restricting our attention to representation finite algebras and thus to module categories with finitely many indecomposable objects, there is an obvious self-duality defined on the collection of additively generated subcategories of Λ-mod. Let $\mathcal{ADD}(Q)$ denote the collection of additively generated subcategories of Λ-mod. Let $A := \{M(w_i)\}_{i\in[k]}$ be a set of indecomposable Λ-modules. We define the **complementation** of an additively generated subcategory by

$$\begin{array}{rcl} \mathcal{ADD}(Q) & \xrightarrow{(-)^c} & \mathcal{ADD}(Q) \\ \mathcal{A} := \mathrm{add}(\oplus M(w_i) : M(w_i) \in A) & \longmapsto & \mathcal{A}^c := \mathrm{add}(\oplus M(w_i) : M(w_i) \notin A). \end{array}$$

Clearly, $(\mathcal{A}^c)^c = \mathcal{A}$. It is also clear from the definition of biclosed subcategories of Λ-mod that complementation restricts to a duality $(-)^c : \mathcal{BIC}(Q) \to \mathcal{BIC}(Q)$.

Additionally, we remark that the standard duality (i.e. $D(-) := \mathrm{Hom}_\Lambda(-, \Bbbk)$) gives us the following bijection

$$\begin{array}{rcl} \mathcal{ADD}(Q) & \xrightarrow{D(-)} & \mathcal{ADD}(Q^{\mathrm{op}}) \\ \mathcal{A} := \mathrm{add}(\oplus M(w_i) : M(w_i) \in A) & \longmapsto & D\mathcal{A} := \mathrm{add}(\oplus DM(w_i) : M(w_i) \in A). \end{array}$$

As with the complementation functor, one has $D(D\mathcal{A}) = \mathcal{A}$. The following obvious lemma shows that the standard duality and the complementation functor interact nicely.

Lemma 3.6.2. *For any $\mathcal{A} \in \mathcal{ADD}(Q)$, we have that $(D\mathcal{A})^c = D(\mathcal{A}^c)$.*

Lemma 3.6.3. *Let $B \in Bic(Q)$. Then $\mathcal{B} = add(\oplus_{\boldsymbol{c} \in B} M(w(\boldsymbol{c}))) \in \mathcal{BIC}(Q)$ where $M(w(\boldsymbol{c}))$ denotes the string module satisfying $\underline{dim}(M(w(\boldsymbol{c}))) = \boldsymbol{c}$.*

Proof. Given $B \in \mathrm{Bic}(Q)$ it is clear that $\mathcal{B}$ is additive. Suppose that $0 \to Y \to X \to Z \to 0$ is an exact sequence where $X, Z \in \mathrm{ind}(\mathcal{B})$ and $Y \in \mathrm{ind}(\Lambda\text{-mod})$. Thus $Y = M(w(\mathbf{c}_1))$, $Z = M(w(\mathbf{c}_2))$, and $X = M(w(\mathbf{c}_3))$ for some $\mathbf{c}_1, \mathbf{c}_2 \in B$ and some $\mathbf{c}_3 \in \mathbf{c}\text{-vec}(Q)$. Then, by exactness, we have that $\underline{\mathrm{dim}}(M(w(\mathbf{c}_3))) = \underline{\mathrm{dim}}(M(w(\mathbf{c}_1))) + \underline{\mathrm{dim}}(M(w(\mathbf{c}_2)))$. This implies that $\mathbf{c}_3 = \mathbf{c}_1 + \mathbf{c}_2$. Thus $\mathbf{c}_3 \in B$ so $X = M(w(\mathbf{c}_3)) \in \mathcal{B}$. We conclude that $\mathcal{B}$ is weakly extension closed. An analogous argument shows that $\mathcal{B}$ is weakly extension coclosed. Thus we have that $\mathcal{B} \in \mathcal{BIC}(Q)$. □

Lemma 3.6.4. *Let $\mathcal{B} \in \mathcal{BIC}(Q)$. Then $B := \{\underline{\mathrm{dim}}(M(w)) \in \mathbb{Z}^n : \ M(w) \in ind(\mathcal{B})\} \in Bic(Q)$ (here $ind(\mathcal{B})$ denotes the indecomposable Λ-modules that belong to $\mathcal{B}$).*

Proof. Assume that $\mathbf{c}_1, \mathbf{c}_2 \in B$ and that $\mathbf{c}_3 = \mathbf{c}_1 + \mathbf{c}_2 \in \mathbf{c}\text{-vec}(Q)$. We have that $M(w(\mathbf{c}_1)), M(w(\mathbf{c}_2)) \in \mathrm{ind}(\mathcal{B})$ and $M(w(\mathbf{c}_3)) \in \mathrm{ind}(\Lambda\text{-mod})$. Without loss of generality, we have that $0 \to M(w(\mathbf{c}_1)) \to M(w(\mathbf{c}_3)) \to M(w(\mathbf{c}_2)) \to 0$ is exact. Since $\mathcal{B} \in \mathcal{BIC}(Q)$, we have that $M(w(\mathbf{c}_3)) \in \mathrm{ind}(\mathcal{B})$. Thus $\mathbf{c}_3 \in B$ so B is closed. An analogous argument shows that B is coclosed. □

Proposition 3.6.5. *We have the following isomorphism of posets*

$$\begin{array}{rcl} Bic(Q) & \xrightarrow{\sim} & \mathcal{BIC}(Q) \\ B & \longmapsto & \mathcal{B} := add(\oplus_{\boldsymbol{c} \in B} M(w(\boldsymbol{c}))) \\ B := \{\underline{\mathrm{dim}}(M(w)) \in \mathbb{Z}^n : \ M(w) \in ind(\mathcal{B})\} & \longleftarrow & \mathcal{B}. \end{array}$$

In particular, $\mathcal{BIC}(Q)$ is a lattice.

Proof. By Lemmas 3.6.3 and 3.6.4, we have that the maps in the statement of the proposition map biclosed sets to biclosed subcategories and vice versa. These maps are clearly order-preserving bijections. That $\mathcal{BIC}(Q)$ is a lattice now follows immediately from the fact that $\mathrm{Bic}(Q)$ is a lattice. □

Let $\mathcal{A}_1 := \text{add}(\oplus_{i\in[n]}M(w_i)), \mathcal{A}_2 := \text{add}(\oplus_{j\in[m]}M(v_j)) \in \mathcal{ADD}(Q)$. We define

$$\mathcal{A}_1 \cup \mathcal{A}_2 := \text{add}\left(\bigoplus_{i\in[n]} M(w_i) \oplus \bigoplus_{j\in[m]} M(v_j)\right) \in \mathcal{ADD}(Q)$$

and

$$\mathcal{A}_1 \backslash \mathcal{A}_2 := \text{add}\left(\bigoplus_{i\in[n]} M(w_i) : \ M(w_i) \notin \{M(v_j)\}_{j\in[m]}\right) \in \mathcal{ADD}(Q).$$

If M is a module in $\mathcal{A} = \text{add}(\oplus_{i\in[n]}M(w_i)) \in \mathcal{ADD}(Q)$, we define $\mathcal{A}\backslash M$ to be the largest additively generated subcategory of $\mathcal{A}$ not containing any summands of M. Additionally, we define $\overline{\mathcal{A}} \in \mathcal{ADD}(Q)$ to be the smallest additively generated subcategory of Λ-mod containing $\mathcal{A}$ that is weakly extension closed.

We can now translate the formula for the join of two biclosed sets of **c**-vectors into a formula for the join of two biclosed subcategories.

Corollary 3.6.6. *If $\mathcal{B}_1, \mathcal{B}_2 \in \mathcal{BIC}(Q)$, then $\mathcal{B}_1 \vee \mathcal{B}_2 = \overline{\mathcal{B}_1 \cup \mathcal{B}_2}$.*

Proof. We prove this identity by showing that $\text{Bic}(Q)$ and $\mathcal{BIC}(Q)$ are isomorphic as closure spaces. A subcategory $\mathcal{A}$ of $\mathcal{BIC}(Q)$ is weakly extension closed exactly when $M(w) \in \mathcal{A}$ whenever there exists a short exact sequence $0 \to M(u) \to M(w) \to M(v) \to 0$ for some $M(u), M(v) \in \mathcal{A}$. But this short exact sequence exists exactly when $\underline{\dim}(M(w)) = \underline{\dim}(M(u)) + \underline{\dim}(M(v))$, which is the condition for the corresponding subset of $\text{Bic}(Q)$ to be closed. $\square$

Lemma 3.6.7. *There is an inclusion of posets tors$(\Lambda) \hookrightarrow \mathcal{BIC}(Q)$.*

Proof. Let $\mathcal{T} \in \text{tors}(\Lambda)$. Since Λ is representation finite and since $\mathcal{T}$ is a torsion class, $\mathcal{T} = \text{add}(\oplus_{i\in[k]}M(w_i))$ for some collection of indecomposables $\{M(w_i)\}_{i\in[k]}$. Since $\mathcal{T}$ is extension closed, it is weakly extension closed.

Assume $0 \to X \to Z \to Y \to 0$ is an exact sequence with $X, Y \notin \mathcal{T}$. Suppose $Z \in \mathcal{T}$. Then since $\mathcal{T}$ is quotient closed, $Y \in \mathcal{T}$, a contradiction. Thus $\mathcal{T}$ is weakly extension coclosed so $\mathcal{T} \in \mathcal{BIC}(Q)$. $\square$

Let $\mathcal{B} \in \mathcal{BIC}(Q)$. Define $\mathbb{X}(\mathcal{B})$ to be the set of objects X of $\mathcal{B}$ up to isomorphism with the property that if one has a surjection $X \twoheadrightarrow Y$ then $Y \in \mathcal{B}$. Observe that $\mathcal{B}$ is quotient closed if and only if $\mathbb{X}(\mathcal{B}) = \mathcal{B}$. Also, define $\mathbb{Y}(\mathcal{B})$ to be the set of objects X

of $\mathcal{B}$ up to isomorphism with the property that there exists an object Y of $\mathcal{B}$ such that $Y \hookrightarrow X$. Now define maps $\pi_\downarrow, \pi^\uparrow : \mathcal{BIC}(Q) \to \mathcal{ADD}(Q)$ by

$$\pi_\downarrow(\mathcal{B}) := \operatorname{add}\left(\bigoplus M : \ M \in \operatorname{ind}(\mathbb{X}(\mathcal{B}))\right)$$

and

$$\pi^\uparrow(\mathcal{B}) := \operatorname{add}\left(\bigoplus M : \ M \in \operatorname{ind}(\mathbb{Y}(\mathcal{B}))\right).$$

Clearly, $\pi_\downarrow(\mathcal{B}) \subset \mathcal{B} \subset \pi^\uparrow(\mathcal{B})$, $\pi_\downarrow \circ \pi_\downarrow = \pi_\downarrow$, and $\pi^\uparrow \circ \pi^\uparrow = \pi^\uparrow$.

Proposition 3.6.8. *If $\mathcal{B} \in \mathcal{BIC}(Q)$, then $\pi_\downarrow(\mathcal{B}) \in tors(\Lambda)$. Furthermore, $\pi_\downarrow(\mathcal{BIC}(Q)) = tors(\Lambda)$.*

Proof. Given $\mathcal{B} \in \mathcal{BIC}(Q)$, Lemma 3.7.1 shows that $\pi_\downarrow(\mathcal{B})$ is a full, additive, quotient closed subcategory of Λ-mod. That $\pi_\downarrow(\mathcal{B})$ is extension closed follows from Lemma 3.7.3. Thus $\pi_\downarrow(\mathcal{B}) \in \operatorname{tors}(\Lambda)$. The second assertion now follows from Lemma 3.6.7. □

Theorem 3.6.9. *Let Θ be the equivalence relation on $\mathcal{BIC}(Q)$ where $\mathcal{B}_1 \equiv \mathcal{B}_2$ mod Θ if and only if $\pi_\downarrow(\mathcal{B}_1) = \pi_\downarrow(\mathcal{B}_2)$. Then $\pi_\downarrow : \mathcal{BIC}(Q) \to tors(\Lambda)$ is a lattice quotient map. In particular, $\overrightarrow{EG}(\widehat{Q}) \cong \mathcal{BIC}(Q)/\Theta$.*

Proof. We prove this Theorem by appealing to Lemma 3.3.1. By definition, $\pi_\downarrow$ and $\pi^\uparrow$ are idempotent. By Proposition 3.6.8 and Lemma 3.6.7, we know that $\pi_\downarrow(\mathcal{B}) \in \mathcal{BIC}(Q)$ for any $\mathcal{B} \in \mathcal{BIC}(Q)$. By Lemma 3.7.6 *a*), we have that $\pi^\uparrow(\mathcal{B}) \in \mathcal{BIC}(Q)$ for any $\mathcal{B} \in \mathcal{BIC}(Q)$. By Lemma 3.7.4 and Lemma 3.7.6 *b*), we know that both $\pi_\downarrow$ and $\pi^\uparrow$ are order-preserving. Lastly, by Lemma 3.7.7 *a*) and *b*), we know that $\pi_\downarrow \circ \pi^\uparrow = \pi_\downarrow$ and $\pi^\uparrow \circ \pi_\downarrow = \pi^\uparrow$. By Lemma 3.3.1, we obtain that $\pi_\downarrow$ is a lattice quotient map. The last assertion immediately follows from the fact that $\operatorname{tors}(\Lambda) \cong \overrightarrow{EG}(\widehat{Q})$. □

Corollary 3.6.10. *Let Q be either a type $\mathbb{A}$ quiver or a cyclic quiver. Then any two maximal green sequences of Q are connected by a sequence of polygonal flips. Moreover, every polygon in $\overrightarrow{EG}(\widehat{Q})$ is either a square or pentagon.*

Proof. Since $\mathcal{BIC}(Q)$ is polygonal by Theorem 3.5.4 and polygonality is preserved by lattice quotients, Theorem 3.6.9 implies that $\overrightarrow{EG}(\widehat{Q})$ is a polygonal lattice. Since maximal green sequences correspond to maximal chains in $\overrightarrow{EG}(\widehat{Q})$, Lemma 3.3.3 implies that any two maximal green sequences are connected by polygonal flips.

Let $\overline{Q}, \overline{Q}_1, \overline{Q}_2 \in \overrightarrow{EG}(\widehat{Q})$ such that $\overline{Q}_1$ and $\overline{Q}_2$ are distinct ice quivers covering $\overline{Q}$. Let $\mathcal{B} \in \mathcal{BIC}(Q)$ such that $\pi_\downarrow(\mathcal{B})$ is the torsion class corresponding to $\overline{Q}$ and $\pi^\uparrow(\mathcal{B}) = \mathcal{B}$. Then there exist $\mathcal{B}_1, \mathcal{B}_2 \in \mathcal{BIC}(Q)$ both covering $\mathcal{B}$ such that $\pi_\downarrow(\mathcal{B}_i)$ is the torsion class corresponding to $\overline{Q}_i$ for $i = 1, 2$. Then $[\mathcal{B}, \mathcal{B}_1 \vee \mathcal{B}_2]$ is a polygon of $\mathcal{BIC}(Q)$, so it is either a square or hexagon. Restricting Θ to the interval $[\mathcal{B}, \mathcal{B}_1 \vee \mathcal{B}_2]$, the polygon $[\overline{Q}, \overline{Q}_1 \vee \overline{Q}_2]$ is a lattice quotient of a square or hexagon as in Figure 3.5. Hence, this interval is either a square, pentagon, or hexagon.

Suppose $[\mathcal{B}, \mathcal{B}_1 \vee \mathcal{B}_2]$ is a hexagon, and let $M(u_1)$ and $M(u_2)$ be the unique indecomposables in $\mathcal{B}\backslash\mathcal{B}_1$ and $\mathcal{B}\backslash\mathcal{B}_2$, respectively. By the description of polygons in the proof of Corollary 3.5.5, there exists an extension of the form $0 \to M(u_1) \to M(w) \to M(u_2) \to 0$. Then the covering relation $\mathcal{B}_1 \lessdot \mathcal{B}_1 \cup \mathrm{add}(M(w))$ is contracted by Θ. Hence, $[\overline{Q}, \overline{Q}_1 \vee \overline{Q}_2]$ cannot be a hexagon. □

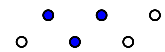
We now address a conjecture on the lengths of maximal green sequences (see [BDP14, Conjecture 2.22]) and give an affirmative answer when Q is a type $\mathbb{A}$ quiver or a cyclic quiver. Let $\mathrm{green}_\ell(Q) := \{\mathbf{i} \in \mathrm{green}(Q) : \ \ell en(\mathbf{i}) = \ell\}$ be the set of maximal green sequences of length ℓ.

Corollary 3.6.11. *Let Q be either a type $\mathbb{A}$ quiver or a cyclic quiver. Then the set $\{\ell \in \mathbb{N} : green_\ell(Q) \neq \varnothing\}$ is an interval in $\mathbb{N}$.*

Proof. Since $\overrightarrow{EG}(\widehat{Q})$ is a finite lattice, it has at least one maximal chain. Moreover, it has only finitely many maximal chains. Thus Q has only finitely many maximal green sequences. Let $\mathbf{i}_{\min}$ (resp. $\mathbf{i}_{\max}$) be a maximal green sequence of Q of smallest (resp. largest) length. Let $\ell_{\min} := \ell en(\mathbf{i}_{\min})$ and $\ell_{\max} := \ell en(\mathbf{i}_{\max})$. By Lemma 3.3.3 and by regarding maximal green sequences as maximal chains in $\overrightarrow{EG}(\widehat{Q})$, there exists maximal green sequences $\mathbf{i}_{\min} = \mathbf{i}_0, \mathbf{i}_1, \ldots, \mathbf{i}_k = \mathbf{i}_{\max}$ where $\mathbf{i}_j \in \mathrm{green}(Q)$ and where $\mathbf{i}_j$ and $\mathbf{i}_{j+1}$ differ by a polygonal flip for all j. By Corollary 3.6.10, $|\ell en(\mathbf{i}_j) - \ell en(\mathbf{i}_{j-1})| \leqslant 1$ for each $j \in [k]$. Thus for each $\ell \in [\ell_{\min}, \ell_{\max}]$ there exists $\mathbf{i} \in \mathrm{green}(Q)$ such that $\ell en(\mathbf{i}) = \ell$. □

Remark 3.6.12. *In [Kas15], it is shown that if Q is a path quiver, then $\{\ell \in \mathbb{N} : green_\ell(Q) \neq \varnothing\} = \left[n, \frac{n(n+1)}{2}\right]$. If Q is mutation-equivalent to a path quiver, we only know that $\{\ell \in \mathbb{N} : green_\ell(Q) \neq \varnothing\}$ is an interval in $\mathbb{N}$ that is contained in $\left[n, \frac{n(n+1)}{2}\right]$.*

For example, if Q is the cyclic quiver appearing in Example 3.2.3, then its maximal green sequences are of length 4 or 5.

Example 3.6.13. *Let $Q = Q(3)$ and let $\Lambda = \Bbbk Q(3)/\langle \alpha_1\alpha_2 : \alpha_i \in Q(3)_1\rangle$. In Figure 3.6, we show how $\pi_\downarrow$ maps elements of $\mathcal{BIC}(Q(3))$ to elements of tors(Λ) using the notation in Example 3.4.8 for additively generated subcatgories of Λ-mod. For instance, add$(X(3,2) \oplus X(2,1) \oplus X(2,2)) \in \mathcal{BIC}(Q(3))$ is represented as*  *. As in Figure 3.1, blue edges of $\mathcal{BIC}(Q(3))$ indicate edges that will be contracted to form tors(Λ).*

3.7 Properties of $\pi_\downarrow$ and $\pi^\uparrow$

In this section, we prove several Lemmas that establish important properties satisfied by $\pi_\downarrow$ and $\pi^\uparrow$. Throughout this section, we assume that $\Lambda = \Bbbk Q/I$ is the cluster-tilted algebra defined by a quiver Q, which is either a cyclic quiver or of type $\mathbb{A}$. Before presenting these Lemmas and their proofs, we introduce some additional notation for string modules. Let $M(w) \in \text{ind}(\Lambda\text{-mod})$ be a string module with

$$w = x_1 \xleftrightarrow{\alpha_1} x_2 \cdots x_i \xleftrightarrow{\alpha_i} x_{i+1} \cdots x_m \xleftrightarrow{\alpha_m} x_{m+1}.$$

Define $\text{Pred}(\alpha_i) := x_1 \xleftrightarrow{\alpha_1} x_2 \cdots x_{i-1} \xleftrightarrow{\alpha_{i-1}} x_i$ and $\text{Succ}(\alpha_i) := x_{i+1} \xleftrightarrow{\alpha_{i+1}} x_{i+2} \cdots x_m \xleftrightarrow{\alpha_m} x_{m+1}$.

Lemma 3.7.1. *If $\mathcal{B} \in \mathcal{BIC}(Q)$, then $\pi_\downarrow(\mathcal{B})$ is a full, additive, quotient closed subcategory of Λ-mod.*

Proof. By the definition of $\pi_\downarrow(\mathcal{B})$, it is clear that $\pi_\downarrow(\mathcal{B})$ is a full, additive subcategory of Λ-mod.

Next, we show that $\pi_\downarrow(\mathcal{B})$ is quotient closed. Since

$$\text{Hom}_\Lambda(\oplus_{i=1}^{m_1} X_i, \oplus_{j=1}^{m_2} Y_j) \cong \bigoplus_{i=1}^{m_1}\bigoplus_{j=1}^{m_2} \text{Hom}_\Lambda(X_i, Y_j),$$

it is enough to show that if $M(w_1) \twoheadrightarrow M(w_2)$ and $M(w_1) \in \pi_\downarrow(\mathcal{B})$, then $M(w_2) \in \pi_\downarrow(\mathcal{B})$. The latter statement is clear, by the definition of $\pi_\downarrow(\mathcal{B})$. Thus $\pi_\downarrow(\mathcal{B})$ is quotient closed. $\square$

Lemma 3.7.2. *If $\mathcal{B} \in \mathcal{BIC}(Q)$, then $\pi_{\downarrow}(\mathcal{B})$ is weakly extension closed.*

Proof. Let $0 \to M(w_2) \to Z \to M(w_1) \to 0$ be an extension where $M(w_2), M(w_1) \in \pi_{\downarrow}(\mathcal{B})$ and where $Z \in \operatorname{ind}(\mathcal{B})$ since $\mathcal{B}$ is weakly extension closed. Thus it is easy to see that $Z = M(w_2 \xleftarrow{\alpha} w_1)$. Let $w_2 = x_1^{(2)} \xleftrightarrow{\alpha_1^{(2)}} x_2^{(2)} \cdots x_{n_2-1}^{(2)} \xleftrightarrow{\alpha_{n_2-1}^{(2)}} x_{n_2}^{(2)}$ and let $w_1 = x_1^{(1)} \xleftrightarrow{\alpha_1^{(1)}} x_2^{(1)} \cdots x_{n_1-1}^{(1)} \xleftrightarrow{\alpha_{n_1-1}^{(1)}} x_{n_1}^{(1)}$.

To show that $\pi_{\downarrow}(\mathcal{B})$ is weakly extension closed we must show that for any surjection $p : M(w_2 \xleftarrow{\alpha} w_1) \twoheadrightarrow M(u)$ one has $M(u) \in \mathcal{B}$. Suppose we have such a surjection and suppose that u is substring of w_1. Then we have

$$u = x_i^{(1)} \xleftrightarrow{\alpha_i^{(1)}} x_{i+1}^{(1)} \cdots x_{j-1}^{(1)} \xleftrightarrow{\alpha_{j-1}^{(1)}} x_j^{(1)}$$

where $i, j \in [n_1]$ and $i \leqslant j$. Let $\beta \in Q_1$ be an arrow that appears in $w_2 \xleftarrow{\alpha} w_1$ with exactly one of its vertices belonging to u. Such an arrow β belongs to the set $\{\alpha, \alpha_1^{(1)}, \ldots, \alpha_{n_1-1}^{(1)}\}$. Since p is a surjection, the unique vertex of β that belongs to u is the source of β. Thus we have that $M(w_1) \twoheadrightarrow M(u)$. Therefore, $M(u) \in \pi_{\downarrow}(\mathcal{B})$ by Lemma 3.7.1 so $M(u) \in \mathcal{B}$. An analogous proof shows that $M(u) \in \mathcal{B}$ if u is a substring of $M(w_2)$.

To complete the proof, we need to show that if $p : M(w_2 \xleftarrow{\alpha} w_1) \twoheadrightarrow M(u)$ for some string $u = x_i^{(2)} \longleftrightarrow \cdots \longleftrightarrow x_j^{(1)}$ with $i \in [n_2]$ and $j \in [n_1]$, then $M(u) \in \mathcal{B}$. Suppose to the contrary that $M(w_2 \xleftarrow{\alpha} w_1)$ is of minimal dimension with the property that $p : M(w_2 \xleftarrow{\alpha} w_1) \twoheadrightarrow M(u)$ is a surjection where u is a string of the above form, but $M(u) \notin \mathcal{B}$. Since $M(w_2 \xleftarrow{\alpha} w_1) \in \mathcal{B}$, we can assume that $\dim_{\Bbbk}(M(u)) < \dim_{\Bbbk}(M(w_2 \xleftarrow{\alpha} w_1))$. Thus $i \neq 1$ or $j \neq n_1$. We will assume that $i \neq 1$ and $j \neq n_1$ and the proof in the case where exactly one of these conditions is satisfied is analogous.

Now, since p is a surjection, we know that $i \neq 1$ implies that $s(\alpha_{i-1}^{(2)}) = x_i^{(2)}$ and $t(\alpha_{i-1}^{(2)}) = x_{i-1}^{(2)}$. Similarly, $j \neq n_1$ implies that $s(\alpha_j^{(1)}) = x_j^{(1)}$ and $t(\alpha_j^{(1)}) = x_{j+1}^{(1)}$. Observe that we have the exact sequence

$$0 \to M(x_1^{(2)} \leftrightarrow \cdots \leftrightarrow x_{i-1}^{(2)}) \oplus M(x_{j+1}^{(1)} \leftrightarrow \cdots \leftrightarrow x_{n_1}^{(1)}) \to M(w_2 \xleftarrow{\alpha} w_1) \to M(u) \to 0.$$

From these facts, we deduce that we have the following two exact sequences

$$0 \to M(x_1^{(2)} \leftrightarrow \cdots \leftrightarrow x_{i-1}^{(2)}) \to M(w_2) \to M(x_i^{(2)} \leftrightarrow \cdots \leftrightarrow x_{n_2}^{(2)}) \to 0$$

$$0 \to M(x_{j+1}^{(1)} \leftrightarrow \cdots \leftrightarrow x_{n_1}^{(1)}) \to M(w_1) \to M(x_1^{(1)} \leftrightarrow \cdots \leftrightarrow x_j^{(1)}) \to 0.$$

By Lemma 3.7.1, we have that $M(x_i^{(2)} \leftrightarrow \cdots \leftrightarrow x_{n_2}^{(2)}), M(x_1^{(1)} \leftrightarrow \cdots \leftrightarrow x_j^{(1)}) \in \pi_\downarrow(\mathcal{B})$. Now notice that we have the exact sequence

$$0 \to M(x_i^{(2)} \leftrightarrow \cdots \leftrightarrow x_{n_2}^{(2)}) \to M(u) \to M(x_1^{(1)} \leftrightarrow \cdots \leftrightarrow x_j^{(1)}) \to 0.$$

Since $\dim_{\Bbbk}(M(u)) < \dim_{\Bbbk}(M(w_2 \xleftarrow{\alpha} w_1))$ and since $M(w_2 \xleftarrow{\alpha} w_1)$ was a counterexample of minimal dimension, we have that $M(u) \in \mathcal{B}$, a contradiction.

We conclude that $M(w_2 \xleftarrow{\alpha} w_1) \in \pi_\downarrow(\mathcal{B})$. Thus $\pi_\downarrow(\mathcal{B})$ is weakly extension closed. □

Lemma 3.7.3. *If $\mathcal{B} \in \mathcal{BIC}(Q)$, then $\pi_\downarrow(\mathcal{B})$ is extension closed.*

Proof. Since

$$\mathrm{Ext}^1_\Lambda(\oplus_{i=1}^{m_1} X_i, \oplus_{j=1}^{m_2} Y_j) \cong \bigoplus_{i=1}^{m_1} \bigoplus_{j=1}^{m_2} \mathrm{Ext}^1_\Lambda(X_i, Y_j)$$

and since elements of $\mathrm{Ext}^1_\Lambda(X, Y)$ are in natural bijection with extensions of the form $0 \to Y \to Z \to X \to 0$ for some $Z \in \Lambda$-mod (up to equivalence of extensions), it is enough to prove that $\pi_\downarrow(\mathcal{B})$ is closed under extensions of the form $0 \to M(w_2) \to Z \to M(w_1) \to 0$.

Suppose $\xi = 0 \to M(w_2) \to Z \to M(w_1) \to 0$ is an extension and $M(w_1), M(w_2) \in \pi_\downarrow(\mathcal{B})$. We know that $\pi_\downarrow(\mathcal{B})$ is additive so we can assume that ξ is a nonsplit extension. By Lemma 3.9.4, we know that $\dim_{\Bbbk} \mathrm{Ext}^1_\Lambda(M(w_1), M(w_2)) = 1$ so up to equivalence of extensions ξ is the unique nonsplit extension of $M(w_1)$ by $M(w_2)$. We can assume $Z \notin \mathrm{ind}(\Lambda\text{-mod})$ since $\mathcal{B}$ is weakly extension closed by Lemma 3.7.2.

If Q is a type $\mathbb{A}$ quiver, then by [CS14, Thm 3.5, Thm 4.2] ξ has the following form

$$\xi = 0 \to M(w_2) \to M(w_3) \oplus M(w_4) \to M(w_1) \to 0.$$

Here $M(w_2) = \mathrm{Pred}(\gamma) \xleftarrow{\gamma} w \xrightarrow{\delta} \mathrm{Succ}(\delta)$, $M(w_1) = \mathrm{Pred}(\alpha) \xrightarrow{\alpha} w \xleftarrow{\beta} \mathrm{Succ}(\beta)$ where $w = x_1 \xleftrightarrow{\alpha_1} x_2 \xleftrightarrow{\alpha_2} \cdots \xleftrightarrow{\alpha_m} x_{m+1}$ such that $\mathrm{supp}(M(w_1)) \cap \mathrm{supp}(M(w_2)) = \{x_i\}_{i \in [m+1]}$. Furthermore,

$$M(w_3) = \mathrm{Pred}(\alpha) \xrightarrow{\alpha} w \xrightarrow{\delta} \mathrm{Succ}(\delta)$$

and

$$M(w_4) = \mathrm{Pred}(\gamma) \xleftarrow{\gamma} w \xleftarrow{\beta} \mathrm{Succ}(\beta).$$

Thus it is enough to show that $M(w_3), M(w_4) \in \pi_\downarrow(\mathcal{B})$.

First, we show that $M(w_3) \in \pi_\downarrow(\mathcal{B})$. Observe that $M(w_2) \twoheadrightarrow M(w \xrightarrow{\delta} \mathrm{Succ}(\delta))$ so $M(w \xrightarrow{\delta} \mathrm{Succ}(\delta)) \in \pi_\downarrow(\mathcal{B})$ by Lemma 3.7.1. Similarly, $M(w_1) \twoheadrightarrow M(\mathrm{Pred}(\alpha))$ so $M(\mathrm{Pred}(\alpha)) \in \pi_\downarrow(\mathcal{B})$. Now notice that we have the extension

$$0 \to M(w \xrightarrow{\delta} \mathrm{Succ}(\delta)) \to M(w_3) \to M(\mathrm{Pred}(\alpha)) \to 0$$

where $M(w \xrightarrow{\delta} \mathrm{Succ}(\delta)), M(\mathrm{Pred}(\alpha)) \in \pi_\downarrow(\mathcal{B})$ so $M(w_3) \in \pi_\downarrow(\mathcal{B})$ by Lemma 3.7.2.

Next, we show that $M(w_4) \in \pi_\downarrow(\mathcal{B})$. Observe that $M(w_2) \twoheadrightarrow M(\mathrm{Pred}(\gamma) \xleftarrow{\gamma} w)$ so $M(\mathrm{Pred}(\gamma) \xleftarrow{\gamma} w) \in \pi_\downarrow(\mathcal{B})$ by Lemma 3.7.1. Similarly, $M(w_1) \twoheadrightarrow M(\mathrm{Succ}(\beta))$ so $M(\mathrm{Succ}(\beta)) \in \pi_\downarrow(\mathcal{B})$. Now notice that we have the extension

$$0 \to M(\mathrm{Pred}(\alpha) \xleftarrow{\gamma} w) \to M(w_4) \to M(\mathrm{Succ}(\beta)) \to 0$$

where $M(\mathrm{Pred}(\gamma) \xleftarrow{\gamma} w), M(\mathrm{Succ}(\beta)) \in \pi_\downarrow(\mathcal{B})$ so $M(w_4) \in \pi_\downarrow(\mathcal{B})$ by Lemma 3.7.2. Thus we conclude that if Q is a type $\mathbb{A}$ quiver, $\pi_\downarrow(\mathcal{B})$ is extension closed.

Now suppose that $Q = Q(n)$ for some positive integer $n \geqslant 3$. Since we can assume that $Z \notin \mathrm{ind}(\Lambda\text{-mod})$ and that ξ is nonsplit, we know that $\mathrm{supp}(M(w_2)) \cap \mathrm{supp}(M(w_1)) \neq \varnothing$. By Lemma 3.2.11, we have that the nonsplit extension ξ is of the following form

$$\xi = 0 \to M(w_2) \to M(u \leftarrow w \leftarrow v) \oplus M(w) \to M(w_1) \to 0$$

where $w_2 = u \longleftarrow w$ with $w_1 = w \longleftarrow v$ for some strings $u, v,$ and w where $w = x_1 \xleftarrow{\alpha_1} x_2 \cdots x_{k-1} \xleftarrow{\alpha_{k-1}} x_k$ satisfies $\mathrm{supp}(M(w_2)) \cap \mathrm{supp}(M(w_1)) = \{x_i\}_{i \in [k]}$.

Here we show that $M(u \leftarrow w \leftarrow v), M(w) \in \pi_\downarrow(\mathcal{B})$. Since $M(w_1) \twoheadrightarrow M(v)$ and $M(w_2) \twoheadrightarrow M(w)$, we have that $M(v), M(w) \in \pi_\downarrow(\mathcal{B})$ by Lemma 3.7.1. Thus we have the extension

$$0 \to M(w_2) \to M(u \leftarrow w \leftarrow v) \to M(v) \to 0$$

and so Lemma 3.7.2 implies that $M(u \leftarrow w \leftarrow v) \in \pi_\downarrow(\mathcal{B})$. Thus we conclude that if $Q = Q(n)$ for some $n \geqslant 3$, then $\pi_\downarrow(\mathcal{B})$ is extension closed. $\square$

Lemma 3.7.4. *The map* $\pi_\downarrow : \mathcal{BIC}(Q) \to \mathit{tors}(\Lambda)$ *is order-preserving.*

Proof. Let $\mathcal{B}, \mathcal{B}' \in \mathcal{BIC}(Q)$ where $\mathcal{B} \subset \mathcal{B}'$. Let $X \in \mathrm{ind}(\mathbb{X}(\mathcal{B}))$ and let $X \twoheadrightarrow Y$ be a surjection. Then $Y \in \mathcal{B} \subset \mathcal{B}'$ so $X \in \mathrm{ind}(\mathbb{X}(\mathcal{B}'))$. Thus $\pi_\downarrow(\mathcal{B}) \subset \pi_\downarrow(\mathcal{B}')$ so $\pi_\downarrow : \mathcal{BIC}(Q) \to \mathrm{tors}(\Lambda)$ is order-preserving. $\square$

Lemma 3.7.5. *The maps $\pi_\downarrow$ and $\pi^\uparrow$ satisfy $D\pi^\uparrow(\mathcal{B})^c = \pi_\downarrow(D\mathcal{B}^c)$ for any $\mathcal{B} \in \mathcal{BIC}(Q)$.*

Proof. We have that

$$\begin{aligned} D\pi^\uparrow(\mathcal{B})^c &= D\left(\{X \in \mathcal{B}^c \in \mathcal{BIC}(Q) : \ Y \hookrightarrow X \implies Y \in \mathcal{B}^c\}\right) \\ &= \{DX \in D\mathcal{B}^c \in \mathcal{BIC}(Q^{\mathrm{op}}) : \ DX \twoheadrightarrow DY \implies DY \in D\mathcal{B}^c\} \\ &= \pi_\downarrow(D\mathcal{B}^c). \end{aligned}$$

□

Lemma 3.7.6. *The map $\pi_\downarrow : \mathcal{BIC}(Q) \to \mathcal{ADD}(Q)$ satisfies the following*

a) $\pi^\uparrow(\mathcal{B}) \in \mathcal{BIC}(Q)$,

b) $\pi^\uparrow$ *is order-preserving.*

Proof. To prove both *a*) and *b*), we use that $\pi^\uparrow(\mathcal{B}) = D(\pi_\downarrow(D\mathcal{B}^c))^c$ for any $\mathcal{B} \in \mathcal{BIC}(Q)$, which follows from Lemma 3.7.5. Let $\mathcal{B} \in \mathcal{BIC}(Q)$, then we have that $D\mathcal{B}^c \in \mathcal{BIC}(Q^{\mathrm{op}})$. By Proposition 3.6.8 and Lemma 3.6.7, we have that $\pi_\downarrow(D\mathcal{B}^c) \in \mathcal{BIC}(Q^{\mathrm{op}})$. Now it follows that $\pi^\uparrow(\mathcal{B}) = D(\pi_\downarrow(D\mathcal{B}^c))^c \in \mathcal{BIC}(Q)$.

To prove *b*), let $\mathcal{B}_1, \mathcal{B}_2 \in \mathcal{BIC}(Q)$. Then we have

$$\begin{aligned} \mathcal{B}_1 \subset \mathcal{B}_2 &\implies \mathcal{B}_1^c \supset \mathcal{B}_2^c \\ &\implies D\mathcal{B}_1^c \subset D\mathcal{B}_2^c \\ &\implies \pi_\downarrow(D\mathcal{B}_1^c) \subset \pi_\downarrow(D\mathcal{B}_2^c) \qquad \text{(by Lemma 3.7.4)} \\ &\implies (\pi_\downarrow(D\mathcal{B}_1^c))^c \supset (\pi_\downarrow(D\mathcal{B}_2^c))^c \\ &\implies D(\pi_\downarrow(D\mathcal{B}_1^c))^c \subset D(\pi_\downarrow(D\mathcal{B}_2^c))^c \\ &\implies \pi^\uparrow(\mathcal{B}_1) \subset \pi^\uparrow(\mathcal{B}_2). \end{aligned}$$

Thus $\pi^\uparrow$ is order-preserving. □

Lemma 3.7.7. *The maps $\pi_\downarrow$ and $\pi^\uparrow$ satisfy the following*

a) $\pi_\downarrow(\mathcal{B}) = (\pi_\downarrow \circ \pi^\uparrow)(\mathcal{B})$ for any $\mathcal{B} \in \mathcal{BIC}(Q)$,

b) $\pi_\downarrow(\mathcal{B}) = (\pi_\downarrow \circ \pi^\uparrow)(\mathcal{B})$ for any $\mathcal{B} \in \mathcal{BIC}(Q)$ if and only if $\pi^\uparrow(\mathcal{B}) = (\pi^\uparrow \circ \pi_\downarrow)(\mathcal{B})$ for any $\mathcal{B} \in \mathcal{BIC}(Q)$.

Proof. We first prove *a*). Since $\mathcal{B} \subset \pi^\uparrow(\mathcal{B})$, by Lemma 3.7.4 we know that $\pi_\downarrow(\mathcal{B}) \subset \pi_\downarrow(\pi^\uparrow(\mathcal{B}))$. Thus we need to show that $\pi_\downarrow(\pi^\uparrow(\mathcal{B})) \subset \pi_\downarrow(\mathcal{B})$. To do so, let $M(u) \in$

$\mathrm{ind}(\mathbb{X}(\pi^{\uparrow}(\mathcal{B})))$ and suppose that $M(u) \twoheadrightarrow M(w)$ is a surjection where $M(w) \notin \mathcal{B}$ such that any other such indecomposable $M(w')$ with $\dim(M(w')) < \dim(M(w))$ belongs to $\mathcal{B}$.

Since $M(w) \in \pi^{\uparrow}(\mathcal{B})$, there exists $M(w_1) \in \mathcal{B}$ and an inclusion $M(w_1) \hookrightarrow M(w)$. This inclusion gives rise to an exact sequence

$$0 \to M(w_1) \to M(w) \to M(w)/M(w_1) \to 0.$$

Note that $\dim(M(w)/M(w_1)) < \dim(M(w))$ and we have a surjection $M(u) \twoheadrightarrow M(w) \twoheadrightarrow M(w)/M(w_1)$ so by assumption $M(w)/M(w_1) \in \mathcal{B}$. If $M(w)/M(w_1)$ is indecomposable, then by the fact that $\mathcal{B}$ is biclosed, $M(w) \in \mathcal{B}$, a contradiction. Thus we can assume $M(w)/M(w_1)$ is not indecomposable.

Observe that since $M(w_1)$ is indecomposable and since w_1 is a substring of w, we have that $M(w)/M(w_1) = M(w_2) \oplus M(w_3)$ for some substrings of w, denoted w_2 and w_3. Now observe that we obtain an exact sequence

$$0 \to M(w_1) \to M(w_1 \leftarrow w_2) \to M(w_2) \to 0.$$

Since $M(w)/M(w_1) \in \mathcal{B}$, we know that $M(w_2), M(w_3) \in \mathrm{ind}(\mathcal{B})$. By the fact that $\mathcal{B}$ is biclosed, we have that $M(w_1 \leftarrow w_2) \in \mathrm{ind}(\mathcal{B})$. We now notice that $w = w_3 \to w_1 \leftarrow w_2$ so $M(w_1 \leftarrow w_2) \hookrightarrow M(w)$ and thus we have the exact sequence

$$0 \to M(w_1 \leftarrow w_2) \to M(w) \to M(w_3) \to 0.$$

Now by the fact that $\mathcal{B}$ is biclosed, we obtain that $M(w) \in \mathcal{B}$, a contradiction. Thus $M(u) \in \mathrm{ind}(\mathbb{X}(\mathcal{B}))$ and so $\pi_{\downarrow}(\pi^{\uparrow}(\mathcal{B})) \subset \pi_{\downarrow}(\mathcal{B})$.

To prove b), assume $\pi_{\downarrow}(\mathcal{B}) = \pi_{\downarrow}(\pi^{\uparrow}(\mathcal{B}))$ for any $\mathcal{B} \in \mathcal{BIC}(Q)$. Then we have that

$$\begin{aligned} D(\pi^{\uparrow}(\mathcal{B}))^c &= D(\pi_{\downarrow}(D\mathcal{B}^c)) && \text{(by Lemma 3.7.5)} \\ &= \pi_{\downarrow}(\pi^{\uparrow}(D\mathcal{B}^c)) && \text{(by assumption)} \\ &= \pi_{\downarrow}\left(D(\pi_{\downarrow}(\mathcal{B}))^c\right) && \text{(by Lemma 3.7.5)} \\ &= D(\pi^{\uparrow}(\pi_{\downarrow}(\mathcal{B})))^c && \text{(by Lemma 3.7.5).} \end{aligned}$$

Thus we have that $\pi^{\uparrow}(\mathcal{B}) = (\pi^{\uparrow} \circ \pi_{\downarrow})(\mathcal{B})$. The converse is proved analogously. □

3.8 Canonical Join Representations

In this section, we use our previous results to classify canonical join and meet representations of torsion classes $\mathcal{T} \in \mathrm{tors}(\Lambda)$. Throughout this section, we assume that $\Lambda = \Bbbk Q/I$ is the cluster-tilted algebra defined by a quiver Q, which is either a cyclic quiver or of type $\mathbb{A}$.

Lemma 3.8.1. *Let $M(w) \in ind(\Lambda\text{-}mod)$. Then*

a) there are no extensions of the form $0 \to M(w_1) \to M(v) \to M(w_2) \to 0$ where $M(w_i) \in Fac(M(w))$ for $i = 1, 2$,

b) $Fac(M(w)) \in tors(\Lambda)$.

Proof. a) Suppose we have an extension $0 \to M(w_1) \to M(v) \to M(w_2) \to 0$ where $M(w_i) \in \mathrm{Fac}(M(w))$ for $i = 1, 2$. Then by exactness one has that $v = w_1 \xleftarrow{\alpha} w_2$. Since $M(w_i) \in \mathrm{Fac}(M(w))$ and since $M(w_i)$ is indecomposable, $M(w) \twoheadrightarrow M(w_i)$. Thus $w_2 \xleftarrow{\alpha} w_1$ is a substring of w. However, the orientation of α contradicts that $\mathrm{Hom}_\Lambda(M(w), M(w_1)) \neq 0$.

b) We observe that since $\mathrm{Fac}(M(w))$ is quotient closed, one has $\pi_\downarrow(\mathrm{Fac}(M(w))) = \mathrm{Fac}(M(w))$. Also,

$$\mathrm{Fac}(M(w)) = \mathrm{add}(\bigoplus M(v_i) : \exists M(w) \twoheadrightarrow M(v_i))$$

so $\mathrm{Fac}(M(w))$ is additively generated. Thus, by Lemma 3.6.7, it remains to show that $\mathrm{Fac}(M(w)) \in \mathcal{BIC}(Q)$. By part a), $\mathrm{Fac}(M(w))$ vacuously is weakly extension closed. Since $\mathrm{Fac}(M(w))$ is quotient closed, any extension $0 \to M(w_1) \to M(v) \to M(w_2) \to 0$ with $M(v) \in \mathrm{Fac}(M(w))$ has $M(w_2) \in \mathrm{Fac}(M(w))$. This means there are no extensions of the form $0 \to M(w_1) \to M(w) \to M(w_2) \to 0$ with $M(w_i) \notin \mathrm{Fac}(M(w))$ for $i = 1, 2$. Thus $\mathrm{Fac}(M(w))$ is weakly extension coclosed. We conclude that $\mathrm{Fac}(M(w)) \in \mathcal{BIC}(Q)$. $\square$

Lemma 3.8.2. *A torsion class $\mathcal{T} \in tors(\Lambda)$ is join-irreducible if and only if $\mathcal{T} = Fac(M(w))$ for some $M(w) \in ind(\Lambda\text{-}mod)$.*

Proof. Suppose $\mathcal{T} = \mathrm{Fac}(M(w))$ for some $M(w) \in \mathrm{ind}(\Lambda\text{-mod})$. Let $\mathcal{T}_1, \mathcal{T}_2 \subsetneq \mathrm{Fac}(M(w))$. This means that $M(w) \notin \mathcal{T}_1$ and $M(w) \notin \mathcal{T}_2$. Recall that by regarding $\mathcal{T}_1$ and $\mathcal{T}_2$ as elements of $\mathcal{BIC}(Q)$, we have that $\mathcal{T}_1 \vee \mathcal{T}_2 = \overline{\mathcal{T}_1 \cup \mathcal{T}_2}$ by Corollary 3.6.6. By part Lemma 3.8.1 a), every additively generated subcategory $\mathcal{A} \subset \mathrm{Fac}(M(w))$ is weakly extension closed. Now since $\mathcal{T}_1 \vee \mathcal{T}_2 \subset \mathrm{Fac}(M(w))$, we have that $\mathcal{T}_1 \vee \mathcal{T}_2 = \mathcal{T}_1 \cup \mathcal{T}_2$. Thus $M(w) \notin \mathcal{T}_1 \vee \mathcal{T}_2$ so $\mathcal{T}_1 \vee \mathcal{T}_2 \subsetneq \mathrm{Fac}(M(w))$.

Conversely, suppose that $\mathcal{T} \in \mathrm{tors}(\Lambda)$ is join-irreducible. Since $\mathrm{tors}(\Lambda) = \text{f-tors}(\Lambda)$, we have that $\mathcal{T} = \mathrm{Fac}(X)$ for some $X \in \Lambda$-mod. Let $X = \oplus_{i=1}^{\ell} M(w_i)^{a_i}$ for some positive integers $a_i \in \mathbb{N}$.

We claim that $\mathrm{Fac}(X) = \bigvee_{i=1}^{\ell} \mathrm{Fac}(M(w_i))$. Observe that for any $j \in [\ell]$, we have $M(w_j) \in \bigvee_{i=1}^{\ell} \mathrm{Fac}(M(w_i))$. Since $\bigvee_{i=1}^{\ell} \mathrm{Fac}(M(w_i)) \in \mathrm{tors}(\Lambda)$, it is additive and thus $X \in \bigvee_{i=1}^{\ell} \mathrm{Fac}(M(w_i))$. We conclude that $\mathrm{Fac}(X) \subset \bigvee_{i=1}^{\ell} \mathrm{Fac}(M(w_i))$. On the other hand, $M(w_j) \in \mathrm{Fac}(X)$ for any $j \in [\ell]$ since $\mathrm{Fac}(X)$ is quotient closed so we have that $\mathrm{Fac}(M(w_j)) \subset \mathrm{Fac}(X)$ for any $j \in [\ell]$. Since $\mathrm{Fac}(X) \in \mathrm{tors}(\Lambda)$, we have that $\bigvee_{i=1}^{\ell} \mathrm{Fac}(M(w_i)) \subset \mathrm{Fac}(X)$.

Since $\mathcal{T} = \bigvee_{i=1}^{\ell} \mathrm{Fac}(M(w_i))$ and since $\mathcal{T}$ is join-irreducible, we know that $\mathcal{T} = \mathrm{Fac}(M(w_j))$ for some $j \in [\ell]$. □

Theorem 3.8.3. *Let $\mathcal{T} \in$ tors(Λ). Let $M(w_1), \ldots, M(w_\ell)$ be a complete list of non-isomorphic indecomposables such that for all $i \in [\ell]$,*

1. *$M(w_i)$ is in $\mathcal{T}$ and no proper submodule of $M(w_i)$ is in $\mathcal{T}$, and*
2. *if $M(w_i) \in$ Fac$(M(w))$ and $M(w) \in \mathcal{T}$ then $M(w)$ has a proper submodule $M(u)$ in $\mathcal{T}$.*

Then $\mathcal{T} = \bigvee_{i=1}^{\ell}$ Fac$(M(w_i))$ is a canonical join representation of $\mathcal{T}$.

Proof. We first prove that the equality $\mathcal{T} = \bigvee_{i=1}^{\ell} \mathrm{Fac}(M(w_i))$ holds. Since $\mathrm{Fac}(M(w_i))$ is contained in $\mathcal{T}$ for all i, it is clear that $\mathcal{T}$ contains $\bigvee_{i=1}^{\ell} \mathrm{Fac}(M(w_i))$. Suppose this containment is proper, and let $M(w) \in \mathcal{T}$ be an indecomposable of minimum dimension such that $M(w) \notin \bigvee_{i=1}^{\ell} \mathrm{Fac}(M(w_i))$. Suppose first that $M(w)$ contains no proper submodule in $\mathcal{T}$. Then there must exist some $M(w') \in \mathcal{T}$ such that $M(w) \in \mathrm{Fac}(M(w'))$ but $M(w')$ has no proper submodule in $\mathcal{T}$. Choosing such an $M(w')$ of maximal dimension, we have $w' = w_i$ for some i and $M(w) \in \mathrm{Fac}(M(w_i))$, contrary to the assumption that $M(w) \notin \bigvee_{i=1}^{\ell} \mathrm{Fac}(M(w_i))$. Hence, $M(w)$ contains a submodule M such that $M \in \mathcal{T}$. Since $\mathcal{T}$ is quotient-closed, $M(w)/M \in \mathcal{T}$. But $M(w)/M$ decomposes into a direct sum of indecomposables, each of smaller dimension than $M(w)$. By the minimality hypothesis, $M(w)/M \in \bigvee_{i=1}^{\ell} \mathrm{Fac}(M(w_i))$. As $\bigvee_{i=1}^{\ell} \mathrm{Fac}(M(w_i))$ is extension-closed, it must contain $M(w)$, contrary to our assumption.

Next, we show that $\mathcal{T}\backslash M(w_i)$ is in $\mathcal{BIC}(Q)$. It is clear by (1) that $\mathcal{T}\backslash M(w_i)$ is weakly extension closed. Assume that it is not co-closed. Let $M(w)$ be an indecomposable in $\mathcal{T}\backslash M(w_i)$ of minimum dimension such that there exists an extension $0 \to M(u) \to M(w) \to M(w') \to 0$ for which $M(u)$ and $M(w')$ are not in $\mathcal{T}\backslash M(w_i)$. Since $\mathcal{T}$ is quotient-closed, we deduce $M(w') = M(w_i)$. By (2), there exists some $M(u') \in \mathcal{T}$ such that $M(u')$ is a submodule of $M(w)$. By (1), the composition $M(u') \to M(w) \to M(w_i)$ must be 0. Hence, there is an inclusion $M(u') \to M(u)$, which gives an exact sequence of the form

$$0 \to M(u)/M(u') \to M(w)/M(u') \to M(w_i) \to 0.$$

Since $\mathcal{T}$ is extension closed, $M(u)/M(u')$ is not in $\mathcal{T}$. But, since $\mathcal{T}$ is quotient closed, $M(w)/M(u')$ is in $\mathcal{T}$.

If $M(w)/M(u')$ is an indecomposable, then so is $M(u)/M(u')$, and we obtain a contradiction to the minimality of $M(w)$. Otherwise, $M(w)/M(u')$ is a direct sum of two string modules $M(v) \oplus M(v')$. In this case, w_i must be a substring of one of these strings, so we may assume $M(v) \to M(w_i)$ is a quotient map. Since there is an extension of the form $0 \to M(u) \to M(w) \to M(w_i) \to 0$, the string v must decompose into two strings w_i and u'' where $0 \to M(u'') \to M(v) \to M(w_i) \to 0$ is exact. But this implies $M(u)/M(u') \cong M(u'') \oplus M(v')$, so $M(u'') \notin \mathcal{T}$ while $M(v) \in \mathcal{T}$. Again, this contradicts the minimality of $M(w)$. Hence, we conclude that $\mathcal{T}\backslash M(w_i)$ is in $\mathcal{BIC}(Q)$.

Now suppose $\mathcal{T} = \bigvee_{j=1}^{m} \mathrm{Fac}(M(w'_j))$ is some other join representation of $\mathcal{T}$. For a given $i \in [\ell]$, if none of the factors $\mathrm{Fac}(M(w'_j))$ contains $M(w_i)$, then $\bigvee_{j=1}^{m} \mathrm{Fac}(M(w'_j)) \subseteq \mathcal{T}\backslash M(w_i)$, in contradiction with our assumption. Hence, for all $i \in [\ell]$, there exists $j \in [m]$ such that $\mathrm{Fac}(M(w_i)) \subseteq \mathrm{Fac}(M(w'_j))$. This means that our join representation $\mathcal{T} = \bigvee_{i=1}^{\ell} \mathrm{Fac}(M(w_i))$ is canonical. □

Dually, every torsion class has a canonical meet-representation.

Corollary 3.8.4. *Let $\mathcal{T} \in$ tors(Λ). Let $M(w_1), \ldots, M(w_\ell)$ be Λ-modules such that $D(\mathcal{T}^{\perp}) = \bigvee_{j=1}^{\ell} Fac(DM(w_i))$ is a canonical join representation of $D(\mathcal{T}^{\perp})$. Then $\mathcal{T} = \bigwedge_{i=1}^{\ell} {}^{\perp}Sub(M(w_i))$ is a canonical meet representation of $\mathcal{T}$.*

Proof. We first show that $\mathcal{T} = \bigwedge_{i=1}^{\ell} {}^{\perp}\mathrm{Sub}(M(w_i))$. Observe that

$$\begin{aligned}
\mathcal{T} &= {}^{\perp}(D(D(\mathcal{T}^{\perp}))) \\
&= {}^{\perp}\left(D\left(\bigvee_{j=1}^{\ell} \mathrm{Fac}(DM(w_i))\right)\right) \\
&= {}^{\perp}\left(D\left(\bigvee_{j=1}^{\ell} D\mathrm{Sub}(M(w_i))\right)\right) \\
&= {}^{\perp}\left(\left(\bigvee_{j=1}^{\ell} DD\mathrm{Sub}(M(w_i))\right)\right) \\
&= {}^{\perp}\left(\left(\bigcap_{j=1}^{\ell} {}^{\perp}\mathrm{Sub}(M(w_i))\right)^{\perp}\right) \qquad \text{(by Propostion 3.4.2 } b) \text{)} \\
&= \bigwedge_{j=1}^{\ell} {}^{\perp}\mathrm{Sub}(M(w_i)) \qquad \text{(by Propositions 3.4.1 and 3.4.2 } a) \text{).}
\end{aligned}$$

Since the functor $D((-)^{\perp}) : \mathrm{tors}(\Lambda) \to \mathrm{tors}(\Lambda^{\mathrm{op}}) \cong \mathrm{tors}(\Lambda)^{\mathrm{op}}$ is an anti-isomorphism by Lemma 3.4.3 and since $\bigvee_{j=1}^{\ell} \mathrm{Fac}(DM(w_i))$ is a canonical join representation of $D(\mathcal{T}^{\perp})$, we have by Lemma 3.3.2 that $\bigwedge_{i=1}^{\ell} {}^{\perp}\mathrm{Sub}(M(w_i))$ is a canonical meet representation of $\mathcal{T}$. □

3.9 Some Additional Lemmas

In this section, unless otherwise stated, we let Q be a type $\mathbb{A}$ quiver and let $\Lambda = \Bbbk Q/I$ denote the cluster-tilted algebra corresonding to Q.

Lemma 3.9.1. *Let $M(u), M(v) \in$ ind$(\Lambda$-mod$)$ with supp$(M(u)) \cap$ supp$(M(v)) \neq \varnothing$. Then there is a unique string $w = x_1 \leftrightarrow x_2 \cdots x_{k-1} \leftrightarrow x_k$ in Λ such that supp$(M(u)) \cap$ supp$(M(v)) = \{x_i\}_{i\in[k]}$.*

Proof. By the classification of type $\mathbb{A}$ quivers in Lemma 2.1.1 and the relations in Λ, any string in Λ includes at most two vertices from any 3-cycle in Q. Thus a string $u = y_1 \leftrightarrow y_2 \cdots y_{s-1} \leftrightarrow y_s$ is the shortest path connecting y_1 and y_s in the underlying graph of Q. This implies that for any y_i and y_j appearing in u, the string $y_i \leftrightarrow y_{i+1} \cdots y_{j-1} \leftrightarrow y_j$ is the shortest path connecting y_i and y_j in the underlying graph of Q. Therefore if $\mathrm{supp}(M(u)) \cap \mathrm{supp}(M(v)) \neq \varnothing$, there is a unique string $w = x_1 \leftrightarrow x_2 \cdots x_{k-1} \leftrightarrow x_k$ in Λ such that $\mathrm{supp}(M(u)) \cap \mathrm{supp}(M(v)) = \{x_i\}_{i\in[k]}$. □

Lemma 3.9.2. *Let $M(u), M(v) \in$ ind$(\Lambda$-mod$)$. If $M(u) \hookrightarrow M(v)$ or $M(u) \twoheadrightarrow M(v)$, then*

$$\dim_{\Bbbk} \mathrm{Hom}_{\Lambda}(M(u), M(v)) = 1.$$

Proof. Let $M(u) = ((U_i)_{i\in Q_0}, (\varphi_\alpha)_{\alpha\in Q_1})$ and $M(v) = ((V_i)_{i\in Q_0}, (\varrho_\alpha)_{\alpha\in Q_1})$. We prove the result in the case that $M(u) \twoheadrightarrow M(v)$. The case where $M(u) \hookrightarrow M(v)$ is similar.

Since $M(u) \twoheadrightarrow M(v)$, v is a substring of u. Let $v = x_1 \xleftrightarrow{\alpha_1} x_2 \xleftrightarrow{\alpha_2} \cdots \xleftrightarrow{\alpha_m} x_{m+1}$. Now let $\theta : M(u) \to M(v)$ be a surjection. Clearly, $\theta_s = 0$ if $s \notin \{x_i : i \in [m+1]\}$. If $\theta_s \neq 0$ for some $s \in \{x_i : i \in [m+1]\}$, then $\theta_s = \lambda$ for some $\lambda \in \mathbb{k}^*$ (i.e. θ_s is a nonzero scalar transformation). As θ is a morphism of representations, it must satisfy that for any $\alpha \in (Q)_1$ the equality $\theta_{t(\alpha)}\varphi_\alpha = \varrho_\alpha\theta_{s(\alpha)}$ holds. Thus for any $\alpha \in \{\alpha_i : i \in [m]\}$, we have $\theta_{t(\alpha)} = \theta_{s(\alpha)}$. We conclude that $\{\theta\}$ is a $\mathbb{k}$-basis for $\mathrm{Hom}_\Lambda(M(u), M(v))$. □

Lemma 3.9.3. *Let* $M(u), M(v) \in$ *ind*$(\Lambda$*-mod*$)$. *Then* $\dim_{\mathbb{k}} \mathrm{Hom}_\Lambda(M(u), M(v)) \leqslant 1$.

Proof. We can assume that $\mathrm{Hom}_\Lambda(M(u), M(v)) \neq 0$. Thus, by Lemma 3.9.1 there exists a unique string

$$w = x_1 \xleftrightarrow{\alpha_1} x_2 \xleftrightarrow{\alpha_2} \cdots \xleftrightarrow{\alpha_m} x_{m+1}$$

that is a substring of both u and v such that $\pi : M(u) \twoheadrightarrow M(w)$ and $\iota : M(w) \hookrightarrow M(v)$. It is easy to see that any map $\theta : M(u) \to M(v)$ factors as $\theta = c\iota\pi$ where $c \in \mathbb{k}$. Combining this with Lemma 3.9.2, we have that $\dim_{\mathbb{k}} \mathrm{Hom}_\Lambda(M(u), M(v)) = 1$. □

Lemma 3.9.4. *Assume Q is of type $\mathbb{A}$ or of the form $Q = Q(n)$ and let $\Lambda = \mathbb{k}Q/I$ denote the corresponding cluster-tilted algebra. Let* $M(u), M(v) \in$ *ind*$(\Lambda$*-mod*$)$. *Then* $\dim_{\mathbb{k}} \mathrm{Ext}^1_\Lambda(M(u), M(v)) \leqslant 1$.

Proof. By the Auslander-Reiten Formula (see [ASS06]), we have that

$$\begin{aligned}
\dim_{\mathbb{k}} \mathrm{Ext}^1_\Lambda(M(u), M(v)) &= \dim_{\mathbb{k}} \underline{\mathrm{Hom}}_\Lambda(\tau^{-1}M(v), M(u)) \\
&\leqslant \dim_{\mathbb{k}} \mathrm{Hom}_\Lambda(\tau^{-1}M(v), M(u)) \\
&\leqslant 1
\end{aligned}$$

where the last inequality follows from Lemma 3.9.3 if Q is of type $\mathbb{A}$ and Lemma 3.2.6 if $Q = Q(n)$ and the fact that $\tau^{-1}M(v)$ is either zero or indecomposable. □

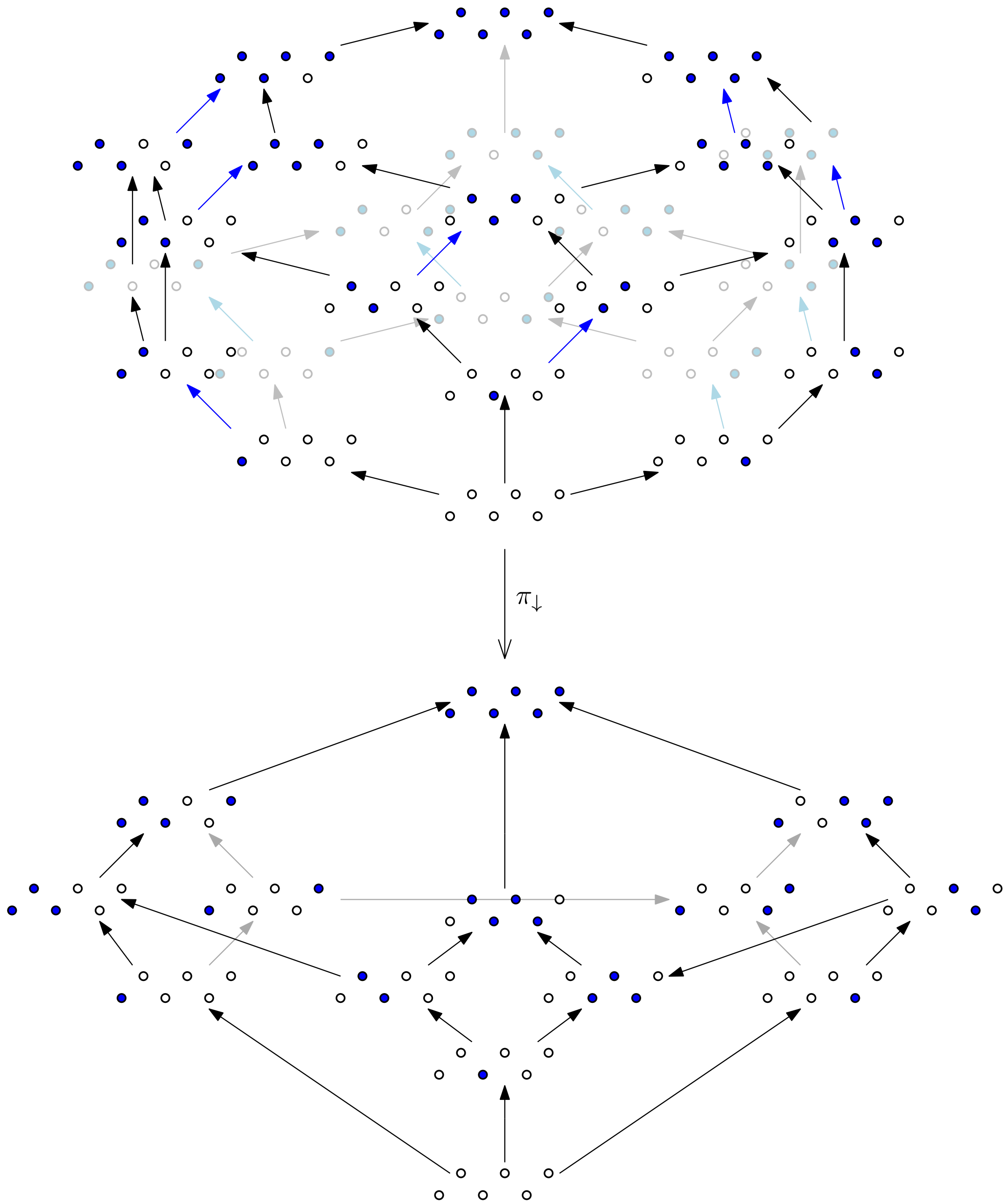


Figure 3.6: The map $\pi_\downarrow : \mathcal{BIC}(Q(3)) \to \mathrm{tors}(\Lambda)$.

Chapter 4

On Maximal Green Sequences for Type $\mathbb{A}$ Quivers

4.1 Introduction

A very important problem in cluster algebra theory, with connections to polyhedral combinatorics and the enumeration of BPS states in string theory, is to determine when a given quiver has a maximal green sequence. In particular, it is open to decide which quivers arising from triangulations of surfaces admit a maximal green sequence, although progress for surfaces has been made in [Lad13], [Buc14], [BM15] and in the physics literature in [ACC$^+$13]. In [ACC$^+$13], they give heuristics for exhibiting maximal green sequences for quivers arising from triangulations of surfaces with boundary and present examples of this for spheres with at least 4 punctures and tori with at least 2 punctures. They write down a particular triangulation of such a surface and show that the quiver defined by this triangulation has a maximal green sequence. In [Buc14, BM15], this same approach is used on surfaces of any genus with at least 2 punctures. In [Lad13], it is shown that there do not exist maximal green sequences for a quiver arising from any triangulation of a closed once-punctured genus g surface. It is still unknown the exact set of surfaces with the property that each of its triangulations defines a quiver admitting a maximal green sequence.

Outside the class of quivers defined by triangulated surfaces there has also been progress in proving that certain quivers do not have maximal green sequences. In

[BDP14], it is shown that if a quiver has non-degenerate Jacobi-infinite potential, then the quiver has no maximal green sequences. This is used in [BDP14] to show that a certain McKay quiver has no maximal green sequences, and in [Sev14] it is shown that the $\mathbb{X}_7$ quiver has no maximal green sequences. Other work [Mul15] illustrates that it is possible to have two mutation-infinite quivers that are mutation equivalent to one another where only one of the two admits a maximal green sequence.

Even for cases where the existence of maximal green sequences is known (e.g. for *quivers of type* $\mathbb{A}$), the problem of exhibiting, classifying or counting maximal green sequences has been challenging and serves as our motivation. By a quiver of type $\mathbb{A}$, we mean any quiver that is *mutation-equivalent* to an orientation of a type $\mathbb{A}$ Dynkin diagram. In the case where Q is acyclic, one can find a maximal green sequence whose length is the number of vertices of Q, by mutating at sources and iterating until all vertices have been mutated exactly once. In general, maximal green sequences must have length at least the number of vertices of Q. However, even for the smallest non-acyclic quiver, i.e. the oriented 3-cycle (of type $\mathbb{A}_3$), a shortest maximal green sequence is of length 4. (While we were in the process of revising this chapter, it was shown in [CDR$^+$15] that the shortest possible length of a maximal green sequence for a quiver Q of type $\mathbb{A}_n$ is $n+t$ where $t = \#\{\text{3-cycles of } Q\}$. See Remark 4.6.7 and Sections 4.8.1 for more details.) With a goal of gaining a better understanding of such sequences, in this chapter we *explicitly* construct a maximal green sequence for every quiver of type $\mathbb{A}$. As any triangulation of the disk with $n+3$ marked points on the boundary defines a quiver of type $\mathbb{A}_n$, our construction shows that the disk belongs to the set of surfaces each of whose triangulations define a quiver admitting a maximal green sequence. We remark that the latter result has also been proved in [CDR$^+$15] by constructing maximal green sequences of type $\mathbb{A}_n$ quivers of shortest possible length. Additionally, the maximal green sequences constructed in [CDR$^+$15] are almost never the same as the maximal green sequences constructed in this chapter.

In Section 4.2, we present preliminary notions on mutation sequences and maximal green sequences. Section 4.3 describes how to decompose quivers into *direct sums* of strongly connected components, which we call irreducible quivers. We remark that this definition of direct sum of quivers, which is based on a quiver glueing rule from [ACC$^+$13], coincides with the definition of a triangular extension of quivers appearing in [Ami09]. As shown in Theorem 4.3.12, to construct a maximal green sequence of a quiver, it suffices to construct maximal green sequences for each of its irreducible components. We refer to the class of such quivers for which Theorem 4.3.12 holds as *t-colored direct sums* of quivers (see Definition 4.3.1). In Section 4.4, we show that almost all quivers arising from triangulated surfaces (with 1 connected component) which are a direct sum of at least 2 irreducible components are in fact a t-colored directed sum.

For type $\mathbb{A}$ quivers, irreducible quivers have an especially nice form as trees of 3-cycles, as described by Corollary 4.5.1. This allows us to restrict our attention to *signed irreducible* quivers of *type* $\mathbb{A}$, which are constructed in detail in Section 4.5. We then construct a special maximal green sequence for every signed irreducible quiver of type $\mathbb{A}$ in Section 4.6, which we call an *associated mutation sequence*. This brings us to the main theorem of the chapter, Theorem 4.6.5, which states that this associated mutation sequence is a maximal green sequence. Section 4.6 also highlights how the results of Section 4.3 can be combined with Theorem 4.6.5 to get maximal green sequences for *any* quiver of type $\mathbb{A}$ (see Corollary 4.6.8).

The proof of Theorem 4.6.5 is somewhat involved. The proof of Theorem 4.6.5 essentially follows from two important lemmas (see Lemma 4.7.2 and Lemma 4.7.3). Our proof begins by attaching *frozen vertices* to a signed irreducible type $\mathbb{A}$ quiver Q to get a *framed quiver* $\widehat{Q}$ (see Section 4.2 for more details). We then apply the associated mutation sequence $\underline{\mu}$ alluded to above, which is constructed in Section 4.6, but decompose it into certain subsequences as $\underline{\mu} = \underline{\mu}_n \circ \cdots \circ \underline{\mu}_1 \circ \underline{\mu}_0$ and apply each mutation subsequence $\underline{\mu}_k$ one after the other. In Lemma 4.7.2 we explicitly describe, for the resulting intermediate quivers, the full subquiver that will be affected by the next iteration of mutations $\underline{\mu}_k$. We will refer to this full subquiver of $\underline{\mu}_{k-1} \circ \cdots \circ \underline{\mu}_1 \circ \underline{\mu}_0(\widehat{Q})$ affected by $\underline{\mu}_k$ as $\overline{R}_k$. Lemma 4.7.3 then explicitly describes how each of these full subquivers, $\overline{R}_k$, is affected by the mutation sequence $\underline{\mu}_k$. Together these lemmas lead us to conclude that the associated mutation sequence $\underline{\mu} = \underline{\mu}_n \circ \cdots \circ \underline{\mu}_1 \circ \underline{\mu}_0$ is a maximal green sequence.

Furthermore, these two lemmas imply that the final quiver $\underline{\mu}_n \circ \cdots \circ \underline{\mu}_1 \circ \underline{\mu}_0(\widehat{Q})$ is isomorphic (as a directed graph) to $\check{Q}$, the *co-framed* quiver where the directions of arrows between vertices of Q and frozen vertices have all been reversed. In particular, such an isomorphism is known as a *frozen isomorphism* since it permutes the vertices of Q while leaving the frozen vertices fixed. We refer to this permutation, of vertices of Q, as the *permutation induced* by a maximal green sequence (we refer the reader to Section 4.2 for precise definitions of these notions). One of the benefits of proving Theorem 4.6.5 using the two lemmas mentioned in the previous paragraph is that we exactly describe the permutation that is induced by an associated mutation sequence of a signed irreducible quiver of type $\mathbb{A}$. (See the last paragraph in Section 4.2 and Definition 4.7.1.) This is a result that may be of independent interest.

Finally, Section 4.8 ends with further remarks and ideas for future directions, including extensions to quivers arising from triangulations of surfaces besides the disk with marked points on the boundary.

4.2 Preliminaries and Notation

In this chapter, we focus on successively applying mutations to a fixed ice quiver. As such, if (Q, F) is a given ice quiver we define an **admissible sequence** of (Q, F), denoted $\mathbf{i} = (i_1, \ldots, i_d)$, to be a sequence of mutable vertices of (Q, F) such that $i_j \neq i_{j+1}$ for all $j \in [d-1]$. An admissible sequence $\mathbf{i} = (i_1, \ldots, i_d)$ also gives rise to a **mutation sequence**, which we consider to be an expression $\underline{\mu} = \mu_{i_d} \circ \cdots \circ \mu_{i_1}$ with $i_j \neq i_{j+1}$ for all $j \in [d-1]$ that maps an ice quiver (Q, F) to a mutation-equivalent one. Let $\mathrm{Mut}((Q, F))$ denote the collection of ice quivers obtainable from (Q, F) by a mutation sequence of **finite length** where the **length** of a mutation sequence is defined to be d, the number of vertices appearing in the associated admissible sequence $\mathbf{i} = (i_1, \ldots, i_d)$. Given a mutation sequence $\underline{\nu}$ of $\overline{Q} \in \mathrm{Mut}(\widehat{Q})$ we define the **support** of $\underline{\nu}$, denoted $\mathrm{supp}(\underline{\nu})$, to be the set of mutable vertices of $\overline{Q}$ appearing in the admissible sequence which gives rise to $\underline{\nu}$.

Let $\underline{\mu} = \mu_{i_d} \circ \cdots \circ \mu_{i_1}$ be a mutation sequence of $\widehat{Q}$. Define $\{\overline{Q}(k)\}_{0 \leqslant k \leqslant d}$ to be the sequence of ice quivers where $\overline{Q}(0) := \widehat{Q}$ and $\overline{Q}(j) := \left(\mu_{i_j} \circ \cdots \circ \mu_{i_1}\right)\left(\widehat{Q}\right)$. (In particular, throughout this chapter, we apply a sequence of mutations in order from right-to-left.) A **green sequence** of Q is an admissible sequence $\mathbf{i} = (i_1, i_2, \ldots i_d)$ of $\widehat{Q}$ such that i_j is a green vertex of $\overline{Q}(j-1)$ for each $1 \leqslant j \leqslant d$. The admissible sequence $\mathbf{i}$ is a **maximal green sequence** of Q if it is a green sequence of Q such that in the final quiver $\overline{Q}(d)$, the vertices $1, 2, \ldots, N$ are all red. Note that maximal green sequences also induce maximal green mutation sequences, as they are referred to in [Qiu15]. In other words, $\overline{Q}(d)$ contains no green vertices.

Proposition 2.10 of [BDP14] shows that given any maximal green sequence $\underline{\mu}$ of Q, one has a frozen isomorphism $\overline{Q}(d) \cong \check{Q}$. Such an isomorphism amounts to a permutation of the mutable vertices of $\check{Q}$, (i.e. $\overline{Q}(d) = \widecheck{Q\sigma}$ for some permutation $\sigma \in \mathfrak{S}_N$ where $\widecheck{Q\sigma}$ is defined by the exchange matrix $B\sigma = B_{\check{Q}}\sigma$ that has entries $(B\sigma)_{i,j} = B_{i\cdot\sigma, j\cdot\sigma}$). We call this the **permutation induced** by $\underline{\mu}$. Note that we can regard σ as an element $\mathfrak{S}_{2N}$ where $i \cdot \sigma = i$ for any $i \in [N+1, 2N]$.

4.3 Direct Sums of Quivers

In this section, we define a direct sum of quivers based on notation appearing in [ACC$^+$13, Section 4.2]. We also show that, under certain restrictions, if a quiver Q can be written as a direct sum of quivers where each summand has a maximal green sequence, then the maximal green sequences of the summands can be concatenated in some way to give a maximal green sequence for Q. Throughout this section, we let (Q_1, F_1) and (Q_2, F_2) be finite ice quivers with N_1 and N_2 vertices, respectively. Furthermore, we assume $(Q_1)_0 \backslash F_1 = [N_1]$ and $(Q_2)_0 \backslash F_2 = [N_1 + 1, N_1 + N_2]$.

Definition 4.3.1. *Let $(a_1, \ldots, a_k)$ denote a k-tuple of elements from $(Q_1)_0 \backslash F_1$ and $(b_1, \ldots, b_k)$ a k-tuple of elements from $(Q_2)_0 \backslash F_2$. (By convention, we assume that the k-tuple $(a_1, \ldots, a_k)$ is ordered so that $a_i \leqslant a_j$ if $i < j$ unless stated otherwise.) Additionally, let $(R_1, F_1) \in Mut((Q_1, F_1))$ and $(R_2, F_2) \in Mut((Q_2, F_2))$. We define the **direct sum** of (R_1, F_1) and (R_2, F_2), denoted $(R_1, F_1) \oplus_{(a_1,\ldots,a_k)}^{(b_1,\ldots,b_k)} (R_2, F_2)$, to be the ice quiver with vertices*

$$\left((R_1, F_1) \oplus_{(a_1,\ldots,a_k)}^{(b_1,\ldots,b_k)} (R_2, F_2)\right)_0 := (R_1)_0 \sqcup (R_2)_0 = (Q_1)_0 \sqcup (Q_2)_0 = [N_1 + N_2] \sqcup F_1 \sqcup F_2$$

and arrows

$$\left((R_1, F_1) \oplus_{(a_1,\ldots,a_k)}^{(b_1,\ldots,b_k)} (R_2, F_2)\right)_1 := (R_1, F_1)_1 \sqcup (R_2, F_2)_1 \sqcup \left\{a_i \xrightarrow{\alpha_i} b_i : i \in [k]\right\}.$$

Observe that we have the identification of ice quivers

$$\widehat{Q_1 \oplus_{(a_1,\ldots,a_k)}^{(b_1,\ldots,b_k)} Q_2} \cong \widehat{Q_1} \oplus_{(a_1,\ldots,a_k)}^{(b_1,\ldots,b_k)} \widehat{Q_2},$$

where the total number of vertices is $M = 2(N_1 + N_2)$ in both cases.

*We say that $(R_1, F_1) \oplus_{(a_1,\ldots,a_k)}^{(b_1,\ldots,b_k)} (R_2, F_2)$ is a **t-colored direct sum** if*

$$t = \#\{\textit{distinct elements of } \{a_1 \ldots, a_k\}\}$$

and there does not exist i and j such that

$$\#\{a_i \xrightarrow{\alpha} b_j\} \geqslant 2.$$

Remark 4.3.2. *Our definition of the direct sum of two quivers coincides with the definition of a **triangular extension** of two quivers introduced by C. Amiot in [Ami09], except that we consider quivers as opposed to quivers with potential. We thank S. Ladkani for bringing this to our attention. He uses this terminology to study the representation theory of a related class of quivers with potential, called class $\mathcal{P}$ by M. Kontsevich and Y. Soibelman [KS08, Section 8.4].*

Remark 4.3.3. *The direct sum of two ice quivers is a non-associative operation as is shown in Example 4.3.5.*

Definition 4.3.4. *We say that a quiver Q is **irreducible** if*

$$Q = Q_1 \oplus_{(a_1,\ldots,a_k)}^{(b_1,\ldots,b_k)} Q_2$$

for some k-tuple $(a_1 \ldots, a_k)$ on $(Q_1)_0$ and some k-tuple $(b_1, \ldots, b_k)$ on $(Q_2)_0$ implies that Q_1 or Q_2 is the empty quiver. Note that we define irreducibility only for quivers rather than for ice quivers because we later only study reducibility when $F = \varnothing$.

Example 4.3.5. *Let Q denote the quiver shown in Figure 4.1. Define Q_1 to be the full subquiver of Q on the vertices $1, \ldots, 4$, Q_2 to be the full subquiver of Q on the vertices $6, \ldots, 11$, and Q_3 to be the full subquiver of Q on the vertex 5. Note that Q_1, Q_2, and Q_3 are each irreducible. Then*

$$Q = Q_1 \oplus_{(1,1,1,3,4,4)}^{(5,8,11,8,9,11)} Q_{23}$$

where $Q_{23} = Q_2 \oplus_{(6)}^{(5)} Q_3$ so Q is a 3-colored direct sum. On the other hand, we could write

$$Q = Q_{12} \oplus_{(1,6)}^{(5,5)} Q_3$$

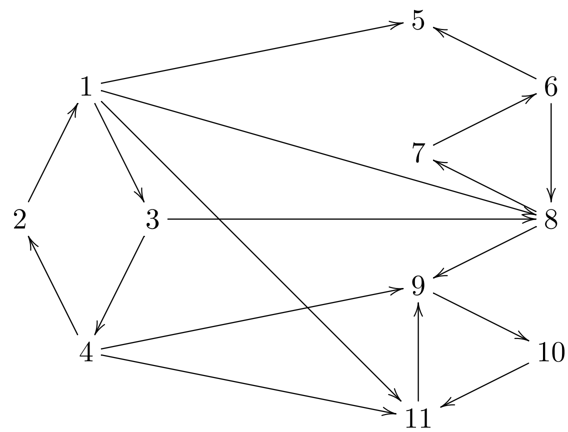


Figure 4.1: The quiver Q used in Example 4.3.5.

where $Q_{12} = Q_1 \oplus_{(1,1,3,4,4)}^{(8,11,8,9,11)} Q_2$ *so* Q *is a 2-colored direct sum. Additionally, note that*

$$Q_1 \oplus_{(1,1,1,3,4,4)}^{(5,8,11,8,9,11)} Q_{23} = Q_1 \oplus_{(1,1,1,3,4,4)}^{(5,8,11,8,9,11)} \left(Q_2 \oplus_{(6)}^{(5)} Q_3\right) \neq \left(Q_1 \oplus_{(1,1,1,3,4,4)}^{(5,8,11,8,9,11)} Q_2\right) \oplus_{(6)}^{(5)} Q_3$$

where the last equality does not hold because $Q_1 \oplus_{(1,1,1,3,4,4)}^{(5,8,11,8,9,11)} Q_2$ *is not defined as* 5 *is not a vertex of* Q_2*. This shows that the direct sum of two quivers, in the sense of this chapter, is not associative.*

Our next goal is to prove that Q has a maximal green sequence if Q is a t-colored direct sum and each of its summands has a maximal green sequence (see Proposition 4.3.12). Before proving this, we introduce a standard form of t-colored direct sums of ice quivers from which we will work:

$$(R, F) \quad = \quad \widehat{Q_1} \oplus_{(a_1,\ldots,a_1,\ldots,a_t,\ldots,a_t)}^{(b_1^{(1)},\ldots,b_{r_1}^{(1)},\ldots,b_1^{(t)},\ldots,b_{r_t}^{(t)})} \overline{Q}_2 \tag{4.1}$$

where $\overline{Q}_2 \in \mathrm{Mut}(\widehat{Q_2})$, $a_1, \ldots, a_t \in (Q_1)_0 \backslash [N_1]'$, $b_1^{(j)}, \ldots, b_{r_j}^{(j)} \in (Q_2)_0 \backslash [N_2]'$, and $\underline{\mu}$ is a fixed mutation sequence $\mu_{i_d} \circ \cdots \circ \mu_{i_1}$ where $\mathrm{supp}(\underline{\mu}) \subset (Q_1)_0$.

We consider the sequence of mutated quivers $(R^{(k)}, F)$, for each $k \in [0, d]$, where $R^{(k)} := \mu_{i_k} \circ \cdots \circ \mu_{i_1} R$. By convention, $k = 0$ implies that the empty mutation sequence has been applied to (R, F) so $R^{(0)} = R$. For every $k \in [0, d]$, we define $\overline{Q}_1(k) :=$

$(\mu_{i_k} \circ \cdots \circ \mu_{i_1})(\widehat{Q}_1)$ and the following set of arrows

$$A(k) := \left\{ \alpha \in (R^{(k)}, F)_1 : \begin{array}{c} s(\alpha) \text{ or } t(\alpha) \in (\overline{Q_1}(k))_0 \backslash [N_1]' \text{ and the other end of} \\ \alpha \text{ is in } \{a_1', \ldots, a_t'\} \cup \{b_1^{(1)}, \ldots, b_{r_1}^{(1)}, \ldots, b_1^{(t)}, \ldots, b_{r_t}^{(t)}\} \end{array} \right\}.$$

Observe that the sets $A(k)$ only contain arrows in the partially mutated quivers which have exactly one of their two ends incident to a vertex in $(Q_1)_0$. The next lemma illustrates how the set of arrows $A(k-1)$ transforms into the set $A(k)$.

Lemma 4.3.6. *If $(i \xrightarrow{\alpha} j) \in A(k)$, but $\alpha, \alpha^{op} \notin A(k-1)$, then there is a 2-path $i \xrightarrow{\alpha_1} i_k \xrightarrow{\alpha_2} j$ in $(R^{(k-1)}, F)$ and exactly one of the arrows $\alpha_1, \alpha_2 \in (R^{(k-1)}, F)_1$ belongs to $A(k-1)$.*

Proof. By the definition of quiver mutation, the arrow $(i \xrightarrow{\alpha} j) \in A(k) \subset (R^{(k)}, F)_1 = (\mu_{i_k} R^{(k-1)}, F)_1$ was originally in $A(k-1)$, was the reversal of an arrow originally in $A(k-1)$, or resulted from a 2-path.

By hypothesis, we must be in the last case. By the definition of $A(k)$, either the source or target of α is in $(\overline{Q_1}(k))_0 \backslash [N_1']$ but not both. Hence the 2-path $i \xrightarrow{\alpha_1} i_k \xrightarrow{\alpha_2} j$ must contain one arrow from $(\overline{Q_1}(k))_0 \backslash [N_1']$ to itself and one arrow in $A(k-1)$. $\square$

In the context of this lemma, we refer to this unique arrow in $A(k-1)$ as $\overline{\alpha}$. We use Lemma 4.3.6 to define a **coloring function** to stratify the set of arrows $A(k)$. This will allow us to keep track of their orientations as will be needed to prove a crucial lemma (see Lemma 4.3.10).

Definition 4.3.7. *Let (R, F) be a t-colored direct sum with a direct sum decomposition of the form shown in (4.1) and let $\underline{\mu}$ be a mutation sequence where $supp(\underline{\mu}) \subset (Q_1)_0$. Define a **coloring function** with respect to Q_1 by*

$$\begin{array}{rcl} f^0 : A(0) & \longrightarrow & \{a_1, \ldots, a_t\} \\ \alpha & \longmapsto & s(\alpha). \end{array}$$

*We say that $\alpha \in A(0)$ has **color** $f^0(\alpha)$ in (R, F). Now, inductively we define a coloring function on each ice quiver $(R^{(k)}, F)$ where $k \in [0, d]$. Define $f^k : A(k) \to \{a_1, \ldots, a_t\}$ by*

$$f^k(\alpha) \;=\; \left\{ \begin{array}{rcl} f^{k-1}(\overline{\alpha}) & : & \text{if } \alpha, \alpha^{op} \notin A(k-1) \\ f^{k-1}(\alpha^{op}) & : & \text{if } \alpha \notin A(k-1),\ \alpha^{op} \in A(k-1), \\ f^{k-1}(\alpha) & : & \text{if } \alpha \in A(k-1). \end{array} \right.$$

We say that $\alpha \in A(k)$ has ***color*** *$f^k(\alpha)$ in $(R^{(k)}, F)$.*

Example 4.3.8. *Using the notation from Example 4.3.5 and writing*

$$\widehat{Q} = \widehat{Q}_1 \oplus_{(1,1,1,3,4,4)}^{(5,8,11,8,9,11)} \left(\widehat{Q}_2 \oplus_{(6)}^{(5)} \widehat{Q}_3\right),$$

we have $a_1 = 1$ and $b_1^{(1)} = 5, b_2^{(1)} = 8, b_3^{(1)} = 11$, $a_2 = 3$ and $b_1^{(2)} = 8$, and $a_3 = 4$ and $b_1^{(3)} = 9, b_2^{(3)} = 11$. In Figure 4.2, we show $\widehat{Q}$ and $\mu_3\widehat{Q}$. The label written on an arrow α of $\widehat{Q}$ or $\mu_3\widehat{Q}$ indicates its color with respect to Q_1.

Our first result shows how the coloring functions $\{f^k\}_{0 \leqslant k \leqslant d}$ defined by an ice quiver (R, F) of the form in (4.1) and a mutation sequence $\underline{\mu} = \mu_{i_d} \circ \cdots \circ \mu_{i_1}$ partition the arrows connecting a mutable vertex $x \in (Q_1)_0$ and a vertex in $\{b_i^{(j)} : i \in [r_j]\} \sqcup \{a'_j\}$.

Lemma 4.3.9. *Let (R, F) be a t-colored direct sum with a direct sum decomposition of the form shown in (4.1) and let $\underline{\mu}$ be a mutation sequence where supp($\underline{\mu}$) $\subset (Q_1)_0$. For any $k \in [0, d]$, we have that the coloring function f^k is defined on each $\alpha \in A(k)$.*

Proof. We proceed by induction on k. If $k = 0$, no mutations have been applied so the desired result holds. Suppose the result holds for $(R^{(k-1)}, F)$ and we will show that the result also holds for $(R^{(k)}, F)$. We can write $(R^{(k)}, F) = (\mu_y R^{(k-1)}, F)$ for some $y \in (Q_1)_0$. Let $\alpha \in A(k)$ such that $s(\alpha) = x \in (Q_1)_0$ and $t(\alpha) = z \in \{b_i^{(j)} : i \in [r_j]\} \sqcup \{a'_j\}$ or vice-versa. There are three cases to consider:

a) $x = y$,

b) x is connected to y and there is a 2-path $x \to y \to z$ or $x \leftarrow y \leftarrow z$ in $(R^{(k-1)}, F)$,

c) x does not satisfy $a)$ or $b)$.

In *Case a)*, we have that all arrows $\alpha \in A(k-1)$ connecting x and z are replaced by $\alpha^{\text{op}} \in A(k)$. By the definition of the coloring functions, these reversed arrows obtain color $f^k(\alpha) = f^{k-1}(\alpha^{\text{op}})$.

In *Case b)*, it follows by Lemma 4.3.6 that an arrow $\alpha \in A(k)$ resulting from mutation of the middle of a 2-path has a well-defined color given by $f^{k-1}(\overline{\alpha})$. Further, mutation at y would reverse both arrows of such a 2-path hence vertex y is in the middle of a 2-path in $A(k)$ if and only if it is in the middle of a 2-path in $A(k-1)$.

Finally, in *Case c)*, the mutation at y does not affect the arrows α connecting x and z and therefore the colors of such an arrow is inherited from its color as an arrow in $A(k-1)$. Note that an arrow between x and y would connect vertices of $(Q_1)_0$ and thus has no color. $\square$

For the proofs in the remainder of this section, we denote the exchange matrix of $(R^{(k)}, F)$, as $B_{(R^{(k)},F)} = (b^k(x,y))_{x\in[N_1+N_2],y\in[2(N_1+N_2)]}$. Here $b^k(x,y) := \#\{(x \xrightarrow{\alpha} y) \in (R^{(k)}, F)_1\} - \#\{(y \xrightarrow{\alpha} x) \in (R^{(k)}, F)_1\}$. (This differs from the notation of Section 4.2 to differentiate it from our notation for the set of vertices $\{b_i^{(j)} : i \in [r_j]\}$.) Furthermore, we refine this enumeration according to color using the following terminology.

$$\begin{aligned} b^k(x,y,\ell) \;\; := \;\; & \#\{(x \xrightarrow{\alpha} y) \in (R^{(k)}, F)_1 : \ \alpha \text{ has color } \ell\} \\ & -\#\{(y \xrightarrow{\alpha} x) \in (R^{(k)}, F)_1 : \ \alpha \text{ has color } \ell\}. \end{aligned}$$

We proceed with the following two technical lemmas.

Lemma 4.3.10. *Let (R,F) be a t-colored direct sum with a direct sum decomposition of the form shown in (4.1) and let $\underline{\mu}$ be a mutation sequence of (R,F) where supp$(\underline{\mu}) \subset (Q_1)_0$. For any $k \in [0,d]$, $\ell \in [t]$, and $x \in (Q_1)_0$, all of the arrows of $A(k)$ with color a_ℓ and incident to vertex x either all point towards vertex x or all point away from vertex x. Moreover they do so with the same multiplicity.*

Proof. We need to show that for any $x \in (Q_1)_0$, $k \in [0,d]$, $j \in [t]$, and $\ell \in \{a_1, \dots, a_t\}$ we have that $b^k(x, b_i^{(j)}, \ell) = b^k(x, a_j', \ell)$ for all $i \in [r_j]$. We proceed by induction on k. If $k = 0$, no mutations have been applied so the desired results holds. Suppose the result holds for $(R^{(k-1)}, F)$ and we will show that the result also holds for $(R^{(k)}, F)$. We can write $(R^{(k)}, F) = (\mu_y R^{(k-1)}, F)$ for some $y \in (Q_1)_0$. Let $x \in (Q_1)_0$ and $z \in \{b_i^{(j)} : i \in [r_j]\} \sqcup \{a_j'\}$ be given. There are three cases to consider:

a) $x = y$,

b) x is connected to y and $\mathrm{sgn}(b^{k-1}(x,y)) = \mathrm{sgn}(b^{k-1}(y,z)) \neq 0$,

c) x does not satisfy *a)* or *b)*.

By Lemma 4.3.9, we know that

$$b^k(x,z) = \sum_{\ell\in\{a_i : i\in[t]\}} b^k(x,z,\ell)$$

and

$$b^{k-1}(y,z) = \sum_{\ell \in \{a_i : i \in [t]\}} b^{k-1}(y,z,\ell).$$

Thus, from the definition of μ_y and the proof of Lemma 4.3.9, we have that

$$b^k(x,z,\ell) \quad = \quad \begin{cases} -b^{k-1}(x,z,\ell) & : \ \textit{Case a)} \\ b^{k-1}(x,y)b^{k-1}(y,z,\ell) + b^{k-1}(x,z,\ell) & : \ \textit{Case b)} \\ b^{k-1}(x,z,\ell) & : \ \textit{Case c)}. \end{cases}$$

By induction, each expression on the right hand side of the equality is independent of the choice of $z \in \{b_i^{(j)} : i \in [r_j]\} \sqcup \{a'_j\}$. Thus $b^k(x,z,\ell)$ is independent of of the choice of $z \in \{b_i^{(j)} : i \in [r_j]\} \sqcup \{a'_j\}$. □

Lemma 4.3.11. *Let (R,F) be a t-colored direct sum with a direct sum decomposition of the form shown in (4.1), let $\underline{\mu}$ be a mutation sequence of (R,F) where supp($\underline{\mu}$) $\subset (Q_1)_0$, and let $k \in [0,d]$. In any $(R^{(k)},F)$, the arrows incident to the frozen vertex a'_i (for all $i \in [t]$) have color a_i.*

Proof. Let $a'_i \in \{a'_1,\ldots,a'_t\}$ be given. We proceed by induction on k. If $k = 0$, no mutations have been applied so the desired result holds. Suppose the result holds $(R^{(k-1)},F)$ and we will show that the result holds for $(R^{(k)},F)$. We can write $(R^{(k)},F) = (\mu_y R^{(k-1)},F)$ for some $y \in (Q_1)_0$. As $y \neq a'_i$, there are only two cases to consider:

b) a'_i is connected to y and there is a 2-path $a'_i \to y \to z$ or $a'_i \leftarrow y \leftarrow z$ in $(R^{(k-1)},F)$,

c) a'_i does not satisfy $b)$.

First, in *Case b)*, if there is a 2-path $a'_i \to y \to z$ in $(R^{(k-1)},F)$ (resp. $a'_i \leftarrow y \leftarrow z$ in $(R^{(k-1)},F)$), then by induction the arrow $(a'_i \to y) \in (R^{(k-1)},F)_1$ (resp. $(a'_i \leftarrow y) \in (R^{(k-1)},F)_1$) has color a_i. Thus if there is a 2-path $a'_i \to y \to z$ in $(R^{(k-1)},F)$ (resp. $a'_i \leftarrow y \leftarrow z$ in $(R^{(k-1)},F)$), then there is an arrow $a'_i \to z \in (R^{(k)},F)_1$ (resp. $a'_i \leftarrow z \in (R^{(k)},F)_1$) of color a_i.

In *Case c)*, the mutation at y does not affect the arrows α connecting a'_i and any vertex $z \in (R^{(k)},F)_0$. Therefore the color of such an arrow is inherited from its color as an arrow in $A(k-1)$. By induction, such arrows have color a_i. □

We now arrive at the main result of this section. It shows that if $\widehat{Q}$ is a t-colored direct sum each of whose summands has a maximal green sequence, then one can build a maximal green sequence for Q using the maximal green sequences for each of its summands.

Theorem 4.3.12. *If* $\underline{\mu}_1 \in \text{green}(Q_1)$ *and* $\underline{\mu}_2 \in \text{green}(Q_2)$*, then* $\underline{\mu}_2 \circ \underline{\mu}_1 \in \text{green}(Q)$ *where*

$$Q = Q_1 \oplus_{(a_1,\ldots,a_1,\ldots,a_t,\ldots,a_t)}^{(b_1^{(1)},\ldots,b_{r_1}^{(1)},\ldots,b_1^{(t)},\ldots,b_{r_t}^{(t)})} Q_2.$$

Proof of Theorem 4.3.12. Let σ_i denote the permutation of the vertices of Q_i induced by $\underline{\mu}_i$. Observe that under the identification in Definition 4.3.1, we let

$$\widehat{Q} \quad = \quad \widehat{Q_1} \oplus_{(a_1,\ldots,a_1,\ldots,a_t,\ldots,a_t)}^{(b_1^{(1)},\ldots,b_{r_1}^{(1)},\ldots,b_1^{(t)},\ldots,b_{r_t}^{(t)})} \widehat{Q_2}.$$

We also have that $\widehat{Q_1} \oplus_{(a_1,\ldots,a_1,\ldots,a_t,\ldots,a_t)}^{(b_1^{(1)},\ldots,b_{r_1}^{(1)},\ldots,b_1^{(t)},\ldots,b_{r_t}^{(t)})} \widehat{Q_2}$ is a t-colored direct sum of the form shown in (4.1).

We first show that $\underline{\mu}_1\widehat{Q}$ is a s-colored direct sum (for some s). Let $\underline{\mu}_1 = \mu_{i_d} \circ \cdots \circ \mu_{i_1}$. Since $\underline{\mu}_1 \in \text{green}(Q_1)$, we have that $\underline{\mu}_1\widehat{Q_1} = \widetilde{Q_1\sigma_1}$ and so for each frozen vertex a_j' with $j \in [t]$, we obtain that $x_j := a_j \cdot \sigma_1 \in (Q_1)_0$ is the unique mutable vertex of $\widehat{Q}$ that is connected to a_j' by an arrow. Furthemore, $(x_j \overset{\alpha}{\leftarrow} a_j') \in (\underline{\mu}_1\widehat{Q})_1$ is the unique arrow of $\underline{\mu}_1\widehat{Q}$ connecting these two vertices.

By Lemma 4.3.9, for any a_j' we have that $b^d(x_j, a_j') = \sum_{\ell \in \{a_i:\ i \in [t]\}} b^d(x_j, a_j', \ell)$. Since $\widetilde{Q_1\sigma_1}$ has no 2-cycles, $\text{sgn}(b^d(x_j, a_j', \ell)) \leqslant 0$ for any $\ell \in \{a_i :\ i \in [t]\}$. By Lemma 4.3.11, α_j has color a_j so $b^d(x_j, a_j') = b^d(x_j, a_j', a_j)$. By Lemma 4.3.10, given any $x_j := a_j \cdot \sigma_1 \in (Q_1)_0$ we have that $b^d(x_j, z) = b^d(x_j, z, a_j) = -1$ for any $z \in \{b_i^{(j)} :\ i \in [r_j]\} \sqcup \{a_j'\}$. Thus we have that $\underline{\mu}_1\widehat{Q} = \widehat{Q_2} \oplus_{(b_1^{(1)},\ldots,b_{r_1}^{(1)},\ldots,b_1^{(t)},\ldots,b_{r_t}^{(t)})}^{(x_1,\ldots,x_1,\ldots,x_t,\ldots,x_t)} \widetilde{Q_1\sigma_1}$ is a s-colored direct sum where $\{b_1^{(1)}, \ldots, b_{r_1}^{(1)}, \ldots, b_1^{(t)}, \ldots, b_{r_t}^{(t)}\}$ is a multiset on $(Q_2)_0 \backslash F_2$ (with s distinct elements) and $\{x_1, \ldots, x_1, \ldots, x_t, \ldots, x_t\}$ is a multiset on $(Q_1)_0 \backslash F_1$. Note that in this s-colored direct sum, the $b_i^{(j)}$'s are not necessarily given in increasing order.

Next, we show that $\underline{\mu}_2(\underline{\mu}_1(\widehat{Q}))$ is a t-colored direct sum. Since $\underline{\mu}_1\widehat{Q}$ is a s-colored direct sum and $\underline{\mu}_2 = \mu_{j_{d'}} \circ \cdots \circ \mu_{j_1}$ is a mutation sequence with $\text{supp}(\underline{\mu}_2) \subset (Q_2)_0$, one defines coloring functions $\{g^k\}_{0 \leqslant k \leqslant d'}$ on $\underline{\mu}_1\widehat{Q}$ with respect to Q_2 in the sense of Definition 4.3.7. Now an analogous argument to that of the previous two paragraphs shows that

$$\underline{\mu}_2(\underline{\mu}_1(\widehat{Q})) = \widetilde{Q_1\sigma_1} \oplus_{(x_1,\ldots,x_1,\ldots,x_t,\ldots,x_t)}^{(y_1^{(1)},\ldots,y_{r_1}^{(1)},\ldots,y_1^{(t)},\ldots,y_{r_t}^{(t)})} \widetilde{Q_2\sigma_2}$$

where $y_j^{(i)} := b_j^{(i)} \cdot \sigma_2$ with $i \in [t]$, $j \in [r_i]$. One now observes that

$$(\underline{\mu}_2 \circ \underline{\mu}_1)(\widehat{Q}) = \widetilde{Q_1\sigma_1} \oplus_{(x_1,\ldots,x_1,\ldots,x_t,\ldots,x_t)}^{(y_1^{(1)},\ldots,y_{r_1}^{(1)},\ldots,y_1^{(t)},\ldots,y_{r_t}^{(t)})} \widetilde{Q_2\sigma_2} \cong \check{Q}$$

and thus all mutable vertices of $(\underline{\mu}_2 \circ \underline{\mu}_1)(\widehat{Q})$ are red.

Finally, since $\underline{\mu}_i \in \text{green}(Q_i)$ for $i = 1, 2$, each mutation of $\widehat{Q}$ along $\underline{\mu}_2 \circ \underline{\mu}_1$ takes place at a green vertex. Thus $\underline{\mu}_2 \circ \underline{\mu}_1 \in \text{green}(Q)$. $\square$

Remark 4.3.13. *We believe that Theorem 4.3.12 holds for any quiver that can be realized as the direct sum of two non-empty quivers, but we do not have a proof.*

4.4 Quivers Arising from Triangulated Surfaces

In this section, we show that Theorem 4.3.12 can be applied to quivers that arise from triangulated surfaces. Our main result of this section is that quivers Q arising from triangulated surfaces can be realized as t-colored direct sums (see Corollary 4.4.5). Before presenting this result and its proof, we recall for the reader how a triangulated surface defines a quiver. For more details on this construction, we refer the reader to [FST08].

Let $\mathbf{S}$ denote an oriented Riemann surface that may or may not have a boundary and let $\mathbf{M} \subset \mathbf{S}$ be a finite subset of $\mathbf{S}$ where we require that for each component $\mathbf{B}$ of $\partial\mathbf{S}$ we have $\mathbf{B} \cap \mathbf{M} \neq \varnothing$. We call the elements of $\mathbf{M}$ **marked points**, we call the elements of $\mathbf{M}\backslash(\mathbf{M} \cap \partial\mathbf{S})$ **punctures**, and we call the pair $(\mathbf{S}, \mathbf{M})$ a **marked surface**. We require that $(\mathbf{S}, \mathbf{M})$ is not one of the following **degenerate marked surfaces**: a sphere with one, two, or three punctures; a disc with one, two, or three marked points on the boundary; or a punctured disc with one marked point on the boundary.

Given a marked surface $(\mathbf{S}, \mathbf{M})$, we consider curves on $\mathbf{S}$ up to isotopy. We define an **arc** on $\mathbf{S}$ to be a simple curve γ in $\mathbf{S}$ whose endpoints are marked points and which is not isotopic to a boundary component of $\mathbf{S}$. We say two arcs γ_1 and γ_2 on $\mathbf{S}$ are **compatible** if they are isotopic relative to their endpoints to curves that are nonintersecting except possibly at their endpoints. A **triangulation** of $\mathbf{S}$ is defined to be a maximal collection of pairwise compatible arcs, denoted $\mathbf{T}$. Each triangulation $\mathbf{T}$ of $\mathbf{S}$ defines a quiver $Q_{\mathbf{T}}$ by associating vertices to arcs and arrows based on oriented adjacencies (see Figure 4.3).

One can also move between different triangulations of a given marked surface $(\mathbf{S}, \mathbf{M})$. Define the **flip** of an arc $\gamma \in \mathbf{T}$ to be the unique arc $\gamma' \neq \gamma$ that produces a triangulation of $(\mathbf{S}, \mathbf{M})$ given by $\mathbf{T}' = (\mathbf{T} \backslash \{\gamma\}) \sqcup \{\gamma'\}$ (see Figure 4.4). If $(\mathbf{S}, \mathbf{M})$ is a marked surface where $\mathbf{M}$ contains punctures, there will be triangulations of $\mathbf{S}$ that contain **self-folded triangles** (the region of $\mathbf{S}$ bounded by γ_3 and γ_4 in Figure 4.5 is an example of a self-folded triangle). We refer to the arc γ_3 (resp. γ_4) shown in the triangulation in Figure 4.5 as a **loop** (a **radius**). As the flip of a radius of a self-folded triangle is not defined, Fomin, Shapiro, and Thurston introduced **tagged arcs**, a generalization of arcs, in order to develop such a notion.

We will not review the details of tagged arcs in this chapter, but we remark that any triangulation can be regarded as a **tagged triangulation** of $(\mathbf{S}, \mathbf{M})$ (i.e. a maximal collection of pairwise compatible tagged arcs). In Figure 4.5, we show how one regards a triangulation of $(\mathbf{S}, \mathbf{M})$ as a tagged triangulation of $(\mathbf{S}, \mathbf{M})$. We also note that any tagged triangulation $\mathbf{T}$ of $(\mathbf{S}, \mathbf{M})$ gives rise to a quiver $Q_{\mathbf{T}}$ (see Example 4.8.1 for a quiver defined by a tagged triangulation or see [FST08] for more examples and details).

We now review the notion of blocks, which was introduced in [FST08] and used to classify quivers defined by a triangulation of some surface.

Definition 4.4.1. *[FST08, Def. 13.1] A **block** is a directed graph isomorphic to one of the graphs shown in Figure 4.6. Depending on which graph it is, we call it a block of type I, II, III, IV, or V. The vertices marked by unfilled circles in Figure 4.6 are called outlets. A directed graph Γ is called **block-decomposable** if it can be obtained from a collection of disjoint blocks by the following procedure. Take a partial matching of the combined set of outlets; matching an outlet to itself or to another outlet from the same block is not allowed. Identify (or "glue") the vertices within each pair of the matching. We require that the resulting graph Γ' be connected. If Γ' contains a pair of edges connecting the same pair of vertices but going in opposite directions, then remove each such a pair of edges. The result is a block-decomposable graph Γ.*

As quivers are examples of directed graphs, one can ask if there is a description of the class of block-decomposable quivers. The following theorem answers this question completely.

Theorem 4.4.2. *[FST08, Thm. 13.3] Block-decomposable quivers are exactly those quivers defined by a triangulation of some surface.*

Remark 4.4.3. *Let $Q_{\boldsymbol{T}}$ be a quiver defined by a triangulated surface with no frozen vertices. In other words, we are assuming that every $v \in (Q_{\boldsymbol{T}})_0$ is a mutable vertex. Then*

$$\begin{aligned} \#\{\alpha \in (Q_{\boldsymbol{T}})_1 : \ x \xrightarrow{\alpha} y \text{ for some } y \in (Q_{\boldsymbol{T}})_0\} &\leqslant 2 \\ \#\{\alpha \in (Q_{\boldsymbol{T}})_1 : \ y \xrightarrow{\alpha} x \text{ for some } y \in (Q_{\boldsymbol{T}})_0\} &\leqslant 2. \end{aligned}$$

We now consider the quivers that are defined by triangulations, but are not irreducible. We show that any such quiver is a t-colored direct sum. The following lemma is a crucial step in showing that a quiver defined by a triangulation that is not irreducible will not have a double arrow connecting two summands of Q.

Lemma 4.4.4. *Assume that Q is defined by a triangulated surface (with 1 connected component) and that* $a \overset{\alpha_1}{\underset{\alpha_2}{\rightrightarrows}} b$ *is a proper subquiver of Q. Then there exists a path of length 2 from b to a.*

Proof. Since Q is defined by a triangulated surface, there exists a block decomposition $\{R_j\}_{j\in[m]}$ of Q by Theorem 4.4.2. By definition of the blocks, α_1 and α_2 come from distinct blocks. Without loss of generality, α_1 is an arrow of R_1 and α_2 is an arrow of R_2. Furthermore, in R_i with $i = 1, 2$ we must have that $s(\alpha_i)$ and $t(\alpha_i)$ are outlets. Thus R_i with $i = 1, 2$ is of type I, II, or IV, but by assumption R_1 and R_2 are not both of type I. When we glue the R_1 to R_2 to using the identifications associated with Q, a case by case analysis shows that there exists a path of length 2 from b to a. Furthermore, the vertices corresponding to a and b are no longer outlets. Thus attaching the remaining R_j's will not delete any arrows from this path. $\square$

Corollary 4.4.5. *Let Q be a quiver defined by a triangulated surface (with 1 connected component) that is not irreducible. If $Q \neq$* $a \overset{\alpha_1}{\underset{\alpha_2}{\rightrightarrows}} b$ *, then Q is a t-colored direct sum for some $t \in \mathbb{N}$.*

Proof. Since we are assuming that Q is not irreducible, there exists subquivers Q_1 and Q_2 of Q such that we can write $Q = Q_1 \oplus^{(b_1,\ldots,b_k)}_{(a_1,\ldots,a_k)} Q_2$ where $\{a_1, \ldots, a_k\}$ is a multiset on $(Q_1)_0$ and $\{b_1, \ldots, b_k\}$ is a multiset on $(Q_2)_0$. Let $a_i \in \{a_1, \ldots, a_k\}$ and $b_j \in \{b_1, \ldots, b_k\}$

be given. We claim that $\#\{\alpha \in (Q)_1 : \ a_i \xrightarrow{\alpha} b_j\} \leqslant 1$. Suppose this were not the case, then Q would have a proper subquiver of the form $\begin{tikzcd} a_i \arrow[r, shift left, "\alpha_1"] \arrow[r, shift right, "\alpha_2"'] & b_j \end{tikzcd}$. By Lemma 4.4.4, there must be a path of length 2 from b_j to a_i. This contradicts the fact that all arrows between $\{a_1, \ldots, a_k\}$ and $\{b_1, \ldots, b_k\}$ point towards the latter. Hence, Q is not only a direct sum but is a t-colored direct sum. $\square$

4.5 Signed Irreducible Type $\mathbb{A}$ Quivers

In this section, we focus our attention on **type $\mathbb{A}_n$ quivers**, which are defined to be quivers $R \in \mathrm{Mut}(1 \leftarrow 2 \leftarrow \cdots \leftarrow n)$ where $n \geqslant 1$ is a positive integer. We begin by classifying irreducible type $\mathbb{A}_n$ quivers. After that, we explain how almost any irreducible type $\mathbb{A}_n$ quiver carries the structure of a binary tree of 3-cycles. In section 4.6, we will show how regarding irreducible type $\mathbb{A}_n$ quivers as trees of 3-cycles allows us to construct maximal green sequences for such quivers. The next result follows from Lemma 2.1.1.

Corollary 4.5.1. *Besides the quiver of type $\mathbb{A}_1$, the irreducible quivers of type $\mathbb{A}$ are exactly those quivers Q obtained by gluing together a finite number of Type II blocks $\{S_\alpha\}_{\alpha\in[n]}$ in such a way that the cycles in the underlying graph of Q are in bijection with the elements of $\{S_\alpha\}_{\alpha\in[n]}$. Additionally, each S_α shares a vertex with at most three other S_β's. (We say that S_α is **connected** to S_β in such a situation.)*

Proof. Assume that Q is a quiver obtained by gluing together a finite number of Type II blocks $\{S_\alpha\}_{\alpha\in[n]}$ in such a way that the cycles in the underlying graph of Q are in bijection with the elements of $\{S_\alpha\}_{\alpha\in[n]}$. Then Q satisfies *i)* in Lemma 2.1.1. By the rules for gluing blocks together, each vertex $i \in (Q)_0$ has either two or four neighbors so *ii)* and *iv)* in Lemma 2.1.1 hold. It also follows from the gluing rules that if i has four neighbors, then two of its adjacent arrows belong to one 3-cycle and the other two belong to another 3-cycle so *iii)* in Lemma 2.1.1 holds. Additionally, since each arrow of Q is contained in an oriented 3-cycle, there is no way to partition the vertices into two components so that the arrows connecting them coherently point from one to the other. Thus the quiver Q is irreducible.

Conversely, let Q be an irreducible type $\mathbb{A}$ quiver that is not the quiver of type

$\mathbb{A}_1$. We first show that any arrow of Q belongs to a (necessarily) oriented 3-cycle of Q. Suppose $(i \xrightarrow{\alpha} j) \in (Q)_1$ does not belong to an oriented 3-cycle of Q. Then there exist nonempty full subquivers Q_1 and Q_2 of Q such that $Q = Q_1 \oplus_{(i)}^{(j)} Q_2$. (By property *i)*, there cannot be an (undirected) cycle of length larger than 3.) This contradicts the fact that Q is irreducible.

Not only is it true that every arrow of Q belongs to an oriented 3-cycle of Q, property *i)* also ensures that Q is obtained by identifying certain vertices of Type II blocks in a finite set of Type II blocks $\{S_\alpha\}_{\alpha\in[n]}$. Furthermore, property *ii)* in Lemma 2.1.1 implies these identifications are such that all vertices have two or four neighbors. By properties *i)* and *iii)*, these identifications do not create any new cycles in the underlying graph of Q. Thus Q is obtained by gluing together a finite number of Type II blocks $\{S_\alpha\}_{\alpha\in[n]}$ in such a way that the cycles in the underlying graph of Q are in bijection with the elements of $\{S_\alpha\}_{\alpha\in[n]}$. □

Definition 4.5.2. *Let Q be an irreducible type $\mathbb{A}$ quiver with at least one 3-cycle. Define a **leaf** 3-cycle S_α in Q to be a 3-cycle in Q that is connected to at most one other 3-cycle in Q. We define a **root** 3-cycle to be a chosen leaf 3-cycle.*

Lemma 4.5.3. *Suppose Q is an irreducible type $\mathbb{A}$ quiver with at least one 3-cycle. Then Q has a leaf 3-cycle.*

Proof. If Q has exactly one 3-cycle R, then $Q = R$ is a leaf 3-cycle. If Q is obtained from the Type II blocks $\{S_i\}_{i\in[n]}$, consider the block S_{i_1}. If S_{i_1} is connected to only one other 3-cycle, then S_{i_1} is a leaf 3-cycle. If S_{i_1} is connected to more than one 3-cycle, let S_{i_2} denote one of the 3-cycles to which S_{i_1} is connected. If S_{i_2} is only connected to S_{i_1}, then S_{i_2} is a leaf 3-cycle. Otherwise, there exists a 3-cycle $S_{i_3} \neq S_{i_1}$ connected to S_{i_2}. By Lemma 2.1.1 there are no non-trivial cycles in the underlying graph of Q besides those determined by the blocks $\{S_i\}_{i\in[n]}$ so this process will end. Thus Q has a leaf 3-cycle. □

Consider a pair, (Q, S) where Q is an irreducible type $\mathbb{A}$ quiver Q with at least one 3-cycle, and S denotes a root 3-cycle in Q. We now define a labeling of the arrows of Q, an ordering of the 3-cycles, and a sign function on the set of 3-cycles of Q. Adding this additional data to (Q, S) yields a binary tree structure on the set of 3-cycles $\{S_\alpha\}_{\alpha\in[n]}$.

We begin by letting $S_1 := S$ denote the chosen root 3-cycle, S_2 denote the unique 3-cycle connected to S_1, and z_1 denote the vertex shared by S_1 and S_2. (In the event that Q is a single 3-cycle, we choose z_1 to be a vertex of S_1 arbitrarily.) Next, we let α_1, β_1 and γ_1 denote the three arrows of S_1 in cyclic order such that $s(\gamma_1) = z_1 = t(\beta_1)$, $s(\beta_1) = t(\alpha_1)$, and $s(\alpha_1) = t(\gamma_1)$. We next label the arrows of S_2 such that $s(\alpha_2) = z_1 = t(\gamma_2)$, $t(\alpha_2) = s(\beta_2)$, and $t(\beta_2) = s(\gamma_2)$. See Figure 4.7 for examples of this labeling.

For $i \geqslant 2$, we order the remaining 3-cycles by a depth-first ordering where we

(1) inductively define S_{i+1} to be the 3-cycle attached to the vertex $t(\alpha_i)$,

(2) define α_{i+1} such that $s(\alpha_{i+1}) = t(\alpha_i)$ and then β_{i+1}, γ_{i+1} follow α_{i+1} in cyclic order,

(3) if no 3-cycle is attached to $t(\alpha_i)$, define S_{i+1} to be the 3-cycle attached to $t(\beta_i)$ and $s(\alpha_{i+1}) = t(\beta_i)$ instead, and finally

(4) minimally backtrack and continue the depth-first ordering until all arrows and 3-cycles have been labeled.

Given a 3-cycle S_i in the block decomposition of Q, define $x_i := s(\alpha_i)$, $y_i := s(\beta_i)$, and $z_i = s(\gamma_i)$. The vertex z_1 of S_1 was already defined in the previous paragraph and that definition of z_1 clearly agrees with this one. We say that a 3-cycle S_i is **positive** (resp. **negative**) if $s(\alpha_i) = t(\alpha_j)$ (resp. $s(\alpha_i) = t(\beta_j)$) for some $j < i$. We define $\mathrm{sgn}(S_i) := +$ (resp. $-$) if S_i is positive (resp. negative). We define $T_i := (S_i, \mathrm{sgn}(S_i))$ to be a 3-cycle in the block decomposition of Q and its sign. We will refer to T_i where $i \in [n]$ as a **signed 3-cycle** of Q. For graphical convenience, we will consistently draw 3-cycles as shown in Figure 4.8 with the convention that $\mathrm{sgn}(S_i) = +$ (resp. $-$) in the former figure (resp. latter figure). We refer to the data $\mathcal{Q} := (Q, S, \{T_i\}_{i\in[n]})$ as a **signed** irreducible type $\mathbb{A}$ quiver.

Remark 4.5.4. *If Q is an irreducible type $\mathbb{A}$ quiver with more than one 3-cycle, then the choice of a root 3-cycle completely determines the sign of each 3-cycle of Q. Thus $\mathcal{Q} = (Q, S, \{T_i\}_{i\in[n]})$ depends only on (Q, S) and thus it makes sense to refer to the signed irreducible type $\mathbb{A}$ quiver defined by (Q, S).*

The next lemma follows immediately from Corollary 4.5.1 and from our definition of the sign of a 3-cycle S_i in Q.

Lemma 4.5.5. *If Q is an irreducible type $\mathbb{A}$ quiver with at least one 3-cycle, S is a root 3-cycle of Q and $\mathcal{Q} = (Q, S, \{T_i\}_{i\in[n]})$ is a signed irreducible type $\mathbb{A}$ quiver defined by (Q, S), then $\mathcal{Q}$ is equivalent to a labeled binary tree with vertex set $\{S_i\}_{i\in[n]}$ where S_i is connected to S_j by an edge if and only if S_i is connected to S_j (i.e. S_i and S_j share a vertex). Furthermore, a 3-cycle $S_j \in \{S_i\}_{i\in[n]}$ has a right child (resp. left child) if and only if S_j shares the vertex y_j (resp. z_j) with another 3-cycle, .*

For the remainder of this section, we assume that Q is a given irreducible type $\mathbb{A}$ quiver and S a root 3-cycle of Q. We also assume $\mathcal{Q}$ is a signed irreducible type $\mathbb{A}$ defined by the data (Q, S). For convenience, we will abuse notation and refer to the **vertices**, **arrows**, **3-cycles**, etc. of $\mathcal{Q}$ with the understanding that we are referring to the vertices, arrows, 3-cycles, etc. of Q, respectively. Since we will often work with $\widehat{Q}$, the framed quiver of Q, it will also be useful to define $\widehat{\mathcal{Q}}$ to be framed quiver of Q with the additional data of S, the root 3-cycle of Q, and the data of a sign associated with each 3-cycle of Q. Now for convenience, we will abuse notation and refer to the **mutable vertices**, **frozen vertices**, **arrows**, and **3-cycles** of $\widehat{\mathcal{Q}}$ with the understanding that we are referring to the mutable vertices, frozen vertices, arrows, and 3-cycles of $\widehat{Q}$, respectively. We will refer to $\widehat{\mathcal{Q}}$ as a **signed irreducible type $\mathbb{A}$ framed quiver**. Additionally, we define a **full subquiver** $\mathcal{R}$ of $\mathcal{Q}$ or $\widehat{\mathcal{Q}}$ to be a full subquiver of Q or $\widehat{Q}$, respectively, with the property that the sign of any 3-cycle C of $\mathcal{R}$ is the same as the sign of C when regarded as a 3-cycle of $\mathcal{Q}$ or $\widehat{\mathcal{Q}}$.

Example 4.5.6. *In Figure 4.9, we show an example of a signed irreducible type $\mathbb{A}_{23}$ quiver, which we denote by $\mathcal{Q}$. The positive 3-cycles of $\mathcal{Q}$ are T_1, T_3, T_4, T_5, T_7. For clarity, we have labeled the arrows of $\mathcal{Q}$ in Figure 4.9, but we will often suppress these labels in later examples. We also note that many of the vertices, e.g. $z_1, y_2, y_3, \dots, z_3$ could also be labeled as $x_2, x_3, x_4, \dots, x_{11}$, but we suppress the vertex labels x_i except for x_1.*

It will be helpful to define an ordering on the vertices of $\widehat{\mathcal{Q}}$. We label the mutable vertices of $\widehat{\mathcal{Q}}$ according to the linear order

$$1 = s(\alpha_1) \quad < t(\alpha_1) < t(\beta_1) \quad < t(\alpha_2) < t(\beta_2) \quad < \dots \quad < t(\alpha_n) < t(\beta_n) = N$$

and the frozen vertices of $\widehat{\mathcal{Q}}$ according to the linear order

$$N+1 = s(\alpha_1)' \quad < t(\alpha_1)' < t(\beta_1)' \quad < t(\alpha_2)' < t(\beta_2)' \quad < \dots \quad < t(\alpha_n)' < t(\beta_n)' = 2N.$$

We call this the **standard ordering** of the vertices of $\widehat{\mathcal{Q}}$.

Example 4.5.7. *Let $\widehat{\mathcal{Q}}$ denote the signed irreducible type $\mathbb{A}_{23}$ framed quiver shown in Figure 4.10. We have labeled the vertices of $\widehat{\mathcal{Q}}$ in Figure 4.10 according to the standard ordering. Note that we have suppressed the arrow labels in Figure 4.10.*

4.6 Associated Mutation Sequences

Throughout this section we work with a given signed irreducible type $\mathbb{A}$ quiver $\mathcal{Q}$ with respect to a fixed root 3-cycle S. Based on the data defining the signed irreducible type $\mathbb{A}$ quiver $\mathcal{Q}$, we construct a mutation sequence of Q that we will call the **associated mutation sequence** of $\mathcal{Q}$. After that we state our main theorem which says that the associated mutation sequence of $\mathcal{Q}$ is a maximal green sequence (see Theorem 4.6.5). We then apply our main theorem to construct a maximal green sequence for any type $\mathbb{A}$ quiver Q (see Corollary 4.6.8).

4.6.1 Definition of Associated Mutation Sequences

Before defining the associated mutation sequence of $\mathcal{Q}$, we need to develop some terminology.

Definition 4.6.1. *Let T_k be a signed 3-cycle of $\mathcal{Q}$. Define the sequence of vertices $(x(0,k), x(1,k), \dots, x(d,k))$ of $\mathcal{Q}$*

$$x(j,k) \quad := \quad \begin{cases} z_k & : \ \text{if } j = 0, \\ t(\gamma_{m_j}) & : \ \gamma_{m_j} \in (Q)_1 \text{ satisfies } s(\gamma_{m_j}) = x(j-1,k). \end{cases}$$

Note that such a sequence is necessarily finite, and we choose d to be maximal, or equivalently so that $sgn(S_{m_d}) = +$. When k is clear from context, we abbreviate $x(s,k)$ as $x(s)$. It follows from the definition of $x(j)$ that $x(j) = x_{m_j}$ for any $j \in [d]$, and that $x(d) = x_1$ or y_{m_d-1}. However, $x(0)$ can be expressed as x_s for some $s \in [n]$ only if $\deg(x(0)) = \deg(z_k) = 4$. *See Figure 4.11. Note that if $sgn(S_k) = +$, then this sequence of vertices is simply $(x(0), x(1))$.*

Definition 4.6.2. *For any vertex v of Q which can be expressed as $v = y_k$, i.e. as a point of some signed 3-cycle T_k of $\mathcal{Q}$, we define the* **transport** *of y_k by the following procedure. We will denote the image of the transport as $tr(v)$. Consider the full subquiver of $\mathcal{Q}$ on the vertices of the signed 3-cycles T_1,T_2,...,T_k, which we denote by $\mathcal{Q}_k$. Inside this subquiver,*

i) move from y_k along β_k to $t(\beta_k)$,

ii) move from $t(\beta_k)$ along the sequence of arrows $\gamma_{m_1}, \gamma_{m_2}, \ldots, \gamma_{m_d}$ of maximal length to $t(\gamma_{m_d})$ where the integers $\{m_i\}_{i\in[d]}$ are those defined by the signed 3-cycle T_k (see Definition 4.6.1),

iii) if possible, move from $t(\gamma_{m_d})$ to $tr(y_k) := t(\beta_{k_s})$ along the sequence of arrows of the form shown in (4.2) each of which belongs to a signed 3-cycle T_i for some $i < k$, under the assumption that the the subsequences A_1 and A_2 are of maximal length, and A_2 must be nonempty. If no such sequence exists of this form, we instead define $tr(y_k) := t(\gamma_{m_d})$.

$$\underbrace{\alpha_{k_1}, \beta_{k_1}, \alpha_{k_2}, \beta_{k_2}, \ldots, \alpha_{k_{\ell-1}}, \beta_{k_{\ell-1}}}_{A_1}, \alpha_{k_\ell}, \underbrace{\alpha_{k_{\ell+1}}, \beta_{k_{\ell+1}}, \alpha_{k_{\ell+2}}, \beta_{k_{\ell+2}}, \ldots, \alpha_{k_s}, \beta_{k_s}}_{A_2} \tag{4.2}$$

See Figures 4.11, 4.12, and 4.13.

We now use the above notation to define the **associated mutation sequence** of $\mathcal{Q}$.

Definition 4.6.3. *Let $\mathcal{Q} = (Q, S, \{T_i\}_{i\in[n]})$ be a signed irreducible type $\mathbb{A}$ quiver. Define $\underline{\mu}_0 := \mu_{x_1}$. For each $k \in [n]$ we define a sequence of mutations, denoted $\underline{\mu}_k$, as follows. Note that when we write $\varnothing$ below we mean the empty mutation sequence. We define*

$$\underline{\mu}_k := \underline{\mu}_A \circ \underline{\mu}_B \circ \underline{\mu}_C \circ \underline{\mu}_D$$

where $\underline{\mu}_A, \underline{\mu}_B, \underline{\mu}_C$, *and* $\underline{\mu}_D$ *are mutation sequences defined in the following way*

$$\begin{aligned} \underline{\mu}_D &:= \mu_{y_k} \\ \underline{\mu}_C &:= \mu_{x(d-1)} \circ \cdots \circ \mu_{x(1)} \circ \mu_{x(0)} \\ \underline{\mu}_B &:= \begin{cases} \mu_{tr(x(d))} & : \text{ if } x(d) \neq x_1 \\ \varnothing & : \text{ if } x(d) = x_1 \end{cases} \\ \underline{\mu}_A &:= \mu_{tr(y_k)}. \end{aligned}$$

Note that $x(d) = x_1$ *or* $y_{m_d - 1}$ *so the transport* $tr(x(d))$ *in* $\underline{\mu}_B$ *is well-defined. Now define the* ***associated mutation sequence*** *of* $\mathcal{Q}$ *to be* $\underline{\mu} := \underline{\mu}_n \circ \cdots \circ \underline{\mu}_1 \circ \underline{\mu}_0$. *We will denote the associated mutation sequence of* $\mathcal{Q}$ *by* $\underline{\mu}$ *or by* $\underline{\mu}^{\mathcal{Q}}$ *if it is not clear from context which signed irreducible type* $\mathbb{A}$ *quiver defines* $\underline{\mu}$. *At times it will be useful to write* $\underline{\mu}_k = \underline{\mu}_{A(k)} \circ \underline{\mu}_{B(k)} \circ \underline{\mu}_{C(k)} \circ \underline{\mu}_{D(k)}$.

Example 4.6.4. *Let* $\mathcal{Q}$ *denote the signed irreducible type* $\mathbb{A}_{31}$ *quiver appearing in Figure 4.14. In the table in Figure 4.15, we describe* $\underline{\mu}_i$ *for each* $0 \leqslant i \leqslant 15$. *Thus, the associated mutation sequence defined by* $\mathcal{Q}$ *is* $\underline{\mu}_{15} \circ \underline{\mu}_{14} \circ \cdots \circ \underline{\mu}_1 \circ \underline{\mu}_0$.

We now arrive at the main result of this chapter.

Theorem 4.6.5. *If* $\mathcal{Q} = (Q, S, \{T_i\}_{i \in [n]})$ *is a signed irreducible type* $\mathbb{A}$ *quiver with associated mutation sequence* $\underline{\mu}$, *then we have* $\underline{\mu} \in \mathit{green}(Q)$.

We present the proof Theorem 4.6.5 in the next section, as the argument requires some additional tools.

Remark 4.6.6. *For a given irreducible type* $\mathbb{A}$ *quiver with at least one 3-cycle, the length of* $\underline{\mu}$ *can vary depending on the choice of leaf 3-cycle. Let* Q *denote the irreducible type* $\mathbb{A}_7$ *quiver shown in Figure 4.16. By choosing the 3-cycle 1,2,3 (resp. 5,6,7) to be the root 3-cycle, one obtains the signed irreducible type* $\mathbb{A}$ *quiver* $\mathcal{Q}_1$ *(resp.* $\mathcal{Q}_2$*) shown in Figure 4.17. Then the associated mutations of* $\mathcal{Q}_1$ *and* $\mathcal{Q}_2$ *are*

$$\begin{aligned} \underline{\mu}^{\mathcal{Q}_1} &= \mu_1 \circ \mu_3 \circ \mu_5 \circ \mu_7 \circ \mu_6 \circ \mu_1 \circ \mu_3 \circ \mu_5 \circ \mu_4 \circ \mu_1 \circ \mu_3 \circ \mu_2 \circ \mu_1 \\ \underline{\mu}^{\mathcal{Q}_2} &= \mu_3 \circ \mu_6 \circ \mu_2 \circ \mu_1 \circ \mu_6 \circ \mu_5 \circ \mu_4 \circ \mu_3 \circ \mu_6 \circ \mu_5 \circ \mu_7 \circ \mu_6. \end{aligned}$$

Furthermore, the maximal green sequence produced by Theorem 4.6.5, i.e. the associated mutation sequence of a each signed irreducible type $\mathbb{A}$ *quiver associated to* Q, *is not* *necessarily a minimal length maximal green sequence. For example, it is easy to check that* $\underline{\nu} = \mu_3 \circ \mu_1 \circ \mu_4 \circ \mu_3 \circ \mu_7 \circ \mu_6 \circ \mu_2 \circ \mu_5 \circ \mu_1 \circ \mu_4 \circ \mu_7$ *is a maximal green sequence of* Q*, which is of length less than that of* $\underline{\mu}^{\mathcal{Q}_1}$ *or* $\underline{\mu}^{\mathcal{Q}_2}$*.*

Remark 4.6.7. *Note that while we were revising this chapter, Cormier, Dillery, Resh, Serhiyenko, and Whelan [CDR+15] found a construction of minimal length maximal green sequences for type* $\mathbb{A}$ *quivers. Therein, they construct a maximal green sequence for any irreducible type* $\mathbb{A}$ *quiver* Q *with at least one 3-cycle by mutating first at all leaf 3-cycles of* Q*, then mutating at the 3-cycles connected to the leaf 3-cycles of* Q*, continuing this process, and then mutating a subsequence of the vertices in reverse. This contrasts with the maximal green sequences we construct in this paper, which involve some extraneous steps but whose process can be defined locally and inductively, akin to writing down the reduced word for a permutation using bubble sort.*

We conclude this section by using Theorem 4.6.5 to show that any type $\mathbb{A}$ quiver has at least one maximal green sequence.

Corollary 4.6.8. *Let* $Q \in Mut(1 \to 2 \to \cdots \to n)$*. Then* Q *has a maximal green sequence.*

Proof. By Corollary 4.4.5, Q can be expressed as a direct sum of irreducible type $\mathbb{A}$ quivers $\{Q_1, Q_2, \ldots, Q_k\}$. In other words, $Q = Q_1 \oplus^{(b_{(1,1)}, b_{(2,1)}, \ldots, b_{(d_1,1)})}_{(a_{(1,1)}, a_{(2,1)}, \ldots, a_{(d_1,1)})} Q'_2$ where $Q'_j = Q_j \oplus^{(b_{(1,j)}, b_{(2,j)}, \ldots, b_{(d_j,j)})}_{(a_{(1,j)}, a_{(2,j)}, \ldots, a_{(d_j,j)})} Q'_{j+1}$ for $2 \leqslant j \leqslant k-1$, and $Q'_k = Q_k$.

If Q_i is of type $\mathbb{A}_1$ and a_i denotes the unique vertex of Q_i, then $\underline{\mu}^{(i)} := \mu_{a_i}$ is a maximal green sequence of Q_i. If Q_i is not of type $\mathbb{A}_1$, then we form a signed irreducible type $\mathbb{A}$ quiver, $\mathcal{Q}^{(i)}$, associated to Q_i by picking a leaf 3-cycle. Now by Theorem 4.6.5, the associated mutation sequence of $\mathcal{Q}^{(i)}$, denoted $\underline{\mu}^{(i)}$, is a maximal green sequence of Q_i. By applying Proposition 4.3.12 iteratively, we obtain $\underline{\mu} = \underline{\mu}^{(k)} \circ \cdots \circ \underline{\mu}^{(2)} \circ \underline{\mu}^{(1)}$ is a maximal green sequence of $\widehat{Q}$. □

4.7 Proof of Theorem 4.6.5

In this section, we work with a fixed signed irreducible type $\mathbb{A}$ quiver $\mathcal{Q} = (Q, S, \{T_i\}_{i \in [n]})$ with N vertices. We write $\underline{\mu} = \underline{\mu}_n \circ \cdots \circ \underline{\mu}_1 \circ \underline{\mu}_0$ for the associated mutation sequence of $\mathcal{Q}$.

Definition 4.7.1. *For each $\underline{\mu}_i$ appearing in $\underline{\mu}$ we define a permutation $\tau_i \in \mathfrak{S}_{(\mathcal{Q})_0} \cong \mathfrak{S}_N$ where $\mathfrak{S}_{(\mathcal{Q})_0}$ denotes the symmetric group on the vertices of $\mathcal{Q}$. In the special case where $i = 0$, we define τ_0 to be the identity permutation. Then for $i \in [n]$ where $\underline{\mu}_i = \mu_{i_d} \circ \cdots \circ \mu_{i_1}$ we define $\tau_i := (i_2, \ldots, i_d)$ in cycle notation (i.e. $i_j \cdot \tau_i = i_{j+1}$ for $j \in [d-1]$ and $i_d \cdot \tau_i = i_2$). Note that $i_1 = y_i$. We also define*

$$\begin{aligned} \sigma_i &:= \tau_i \cdots \tau_1 \tau_0 \\ &= \tau_i \cdots \tau_1 \end{aligned}$$

*where the last equality holds since τ_0 is the identity permutation. We say that σ_n is the **associated permutation** corresponding to $\mathcal{Q}$.*

Theorem 4.6.5 will imply that σ_n is exactly the permutation induced by $\underline{\mu}$ (see the last paragraph of Section 4.2).

Let T_k and T_t where $k \leqslant t$ be signed 3-cycles of $\mathcal{Q}$. Let $\mathcal{Q}_{k,t}$ denote the full subquiver of $\mathcal{Q}$ on the vertices of $T_1, \ldots, T_k$ and the vertices of $T_{m_1}, \ldots, T_{m_d}$ where the integers $m_1, \ldots, m_d \in [n]$ are those defined by T_t as in Definition 4.6.1. For example, $\mathcal{Q}_{k,k}$ is the full subquiver of $\mathcal{Q}$ on the vertices of the signed 3-cycles $T_1, \ldots, T_k$. By convention, we also define $\mathcal{Q}_{0,0}$ to be the full subquiver of $\mathcal{Q}$ consisting of only the vertex x_1. Now define $\mathrm{tr}|_{k,t}$ to be the restriction of the transport to $\mathcal{Q}_{k,t}$.

Lemma 4.7.2. *For each $k \in [n]$ there is an ice quiver $\overline{R}_k$ that is a full subquiver of $\underline{\mu}_{k-1} \circ \cdots \circ \underline{\mu}_1 \circ \underline{\mu}_0(\widehat{Q})$ of the form shown in Figure 4.18 (resp. Figure 4.19) where the vertices $z_k = x(0), x(1), \ldots, x(d-1), tr(x(d))$, and $tr(y_k)$ (resp. $z_k = x(0), x(1), \ldots, x(d-1)$, and $tr(y_k)$) are those appearing in the mutation sequence $\underline{\mu}_{A(k)} \circ \underline{\mu}_{B(k)} \circ \underline{\mu}_{C(k)}$ and the integers $m_1, m_2, \ldots, m_d$ are those defined by T_k in Definition 4.6.1. Note that we only mutate at $tr(x(d))$ if $x(d) \neq x_1$. Furthermore, the ice quiver $\overline{R}_k$ has the following properties:*

- *$\overline{R}_k$ includes every frozen vertex that is connected to a mutable vertex appearing in Figure 4.18 (resp. Figure 4.19) by at least one arrow in $\underline{\mu}_{k-1} \circ \cdots \circ \underline{\mu}_1 \circ \underline{\mu}_0(\widehat{Q})$ where $\widetilde{x}(1) := z'_{m_d-1}$, $\widetilde{tr}(x(d)) := x'_{m_2}, \widetilde{tr}(y_k) := x'_{m_1}$ and $\widetilde{x}(s) := x'_{m_{d-s+2}}$ for[1] $s \in [2, d-1]$ (resp. $\widetilde{tr}(y_k) := x'_{m_1}$ and $\widetilde{x}(s) := x'_{m_{d-s+1}}$ for[2] $s \in [1, d-1]$),*
- *vertices y_m, y'_m, z_m, and z'_m appear in $\overline{R}_k$ if and only if $\deg(y_k) = 4$ in Q,*

[1] *If $d = 1$, then $\widetilde{tr}(x(d)) = x'_{m_1} = x_k$.*
[2] *If $d = 1$, then $\widetilde{tr}(x(d)) = x'_{m_1} = x_k$. Furthermore, $d = 1$, in this case, if and only if $k = 1$.*

- *vertices* y_ℓ, y'_ℓ, z_ℓ, *and* z'_ℓ *appear in* $\overline{R}_k$ *if and only if* $\deg(z_k) = 4$ *in* Q,
- *vertices* y_t *and* y'_t *appear in* $\overline{R}_k$ *if and only if there exists a signed 3-cycle* T_t *in* $\mathcal{Q}$ *with* $k < t$ *such that* $tr|_{k,t}(y_t) = z_k$ *and such that in* $\underline{\mu}_{k-1} \circ \cdots \circ \underline{\mu}_1 \circ \underline{\mu}_0(\widehat{Q})$ *the vertex* x_t *has been mutated exactly once*[3] *, and*
- *in Figure 4.18 (resp. Figure 4.19)* $C_{1,k} := x_{m_d-1} \cdot \sigma_{k-1}^{-1}$, $\widetilde{C_{1,k}} := x'_{m_d-1}$, $C_{s,k} := y_{m_j} \cdot \sigma_{k-1}^{-1}$, *and* $\widetilde{C_{s,k}} := y'_{m_j}$ *for* $s \in [2, d]$ *and* $j = d - s + 2$ *(resp.* $C_{s,k} := y_{m_j} \cdot \sigma_{k-1}^{-1}$ *and* $\widetilde{C_{s,k}} := y'_{m_j}$ *for* $s \in [1, d-1]$ *and* $j = d - s + 2$*).*

Additionally, for each $k \in [n]$ *we have* $\underline{\mu}_k \circ \cdots \circ \underline{\mu}_1 \circ \underline{\mu}_0(\widehat{\mathcal{Q}_{k,k}}) = \widetilde{\mathcal{Q}_{k,k}} \cdot \sigma_k$.

We will prove Lemma 4.7.2 in the case where the vertex $\mathrm{tr}(x(d))$ appears in the mutation sequence $\underline{\mu}_k$ (i.e. when $x(d) \neq x_1$). Under this assumption, the following lemma will allow us to prove Lemma 4.7.2 inductively. The proof of Lemma 4.7.2 when $\mathrm{tr}(x(d))$ does not appear in $\underline{\mu}_k$ is very similar so we omit it.

Lemma 4.7.3. *Let* $k \in [n]$ *be given and let* $\overline{R}_k$ *be the ice quiver described in Lemma 4.7.2. (See Figure 4.18.) Then*

- $\underline{\mu}_k(\overline{R}_k)$ *has the form shown in Figure 4.21 (here, the vertices* y_m, y'_m, z_m, z'_m, y_ℓ, y'_ℓ, z_ℓ, z'_ℓ, y_t, *and* y'_t *appear in* $\underline{\mu}_k(\overline{R}_k)$ *if and only if they appear in* $\overline{R}_k$*),*
- $\underline{\mu}_k(\overline{R}_k)$ *is a full subquiver of* $\underline{\mu}_k \circ \cdots \circ \underline{\mu}_1 \circ \underline{\mu}_0(\widehat{Q})$,
- *as one mutates* $\overline{R}_k$ *along* $\underline{\mu}_k$, *one does so only at green vertices,*
- $\underline{\mu}_k(\overline{R}_k)$ *includes every frozen vertex that is connected to a mutable vertex appearing in Figure 4.21 by at least one arrow in* $\underline{\mu}_k \circ \cdots \circ \underline{\mu}_1 \circ \underline{\mu}_0(\widehat{Q})$,
- *the full subquiver of* $\underline{\mu}_{k-1} \circ \cdots \circ \underline{\mu}_1 \circ \underline{\mu}_0(\widehat{Q})$ *on the vertices* $(\widehat{Q})_0 \backslash (\overline{R}_k)_0$ *is unchanged by the mutation sequence* $\underline{\mu}_k$.
- *the vertices* z_ℓ *and* z_m *(rather than* z_k*) are the only mutable vertices in* $\underline{\mu}_k(\overline{R}_k)$ *that are incident to multiple frozen vertices.*

[3] *Note that this can only happen if there exists* $j < k$ *such that* $z_j = x_t$ *and* $x(d,k) = y_j$ *as in Definition 4.6.1.*

Additionally, for each $k \in [n]$, the full subquiver of $\underline{\mu}_k \circ \cdots \circ \underline{\mu}_1 \circ \underline{\mu}_0(\widehat{Q})$ restricted to the green mutable vertices outside of $(\overline{R}_k)_0$, as well as the incident frozen vertices, equals the original framed quiver $\widehat{Q}$ restricted to those vertices.

Proof of Theorem 4.6.5. By the third assertion in Lemma 4.7.3, the associated mutation sequence $\underline{\mu} = \underline{\mu}_n \circ \cdots \underline{\mu}_1 \circ \underline{\mu}_0$ of $\mathcal{Q}$ is a green mutation sequence of Q. By Lemma 4.7.2,

$$\underline{\mu}_n \circ \cdots \circ \underline{\mu}_1 \circ \underline{\mu}_0(\widehat{Q}) = \underline{\mu}_n \circ \cdots \circ \underline{\mu}_1 \circ \underline{\mu}_0(\widehat{\mathcal{Q}}_{n,n}) = \check{\mathcal{Q}}_{n,n} \cdot \sigma_n = \check{Q} \cdot \sigma_n$$

and so every mutable vertex of $\underline{\mu}_n \circ \cdots \circ \underline{\mu}_1 \circ \underline{\mu}_0(\widehat{Q})$ is red. Thus $\underline{\mu} = \underline{\mu}_n \circ \cdots \underline{\mu}_1 \circ \underline{\mu}_0 \in \text{green}(Q)$. □

Remark 4.7.4. *It follows from Lemma 2.1.1 that as one mutates $\overline{R}_k$ along $\underline{\mu}_k = \mu_{i_r} \circ \cdots \circ \mu_{i_1}$, we have that i_j in $\mu_{i_{j-1}} \circ \cdots \circ \mu_{i_1}(\overline{R}_k)$ is incident to at most four other mutable vertices.*

Proof of Lemma 4.7.3. The first assertion follows inductively by mutating the vertices of $\overline{R}_k$ in the specified order $\underline{\mu}_k = \mu_{\text{tr}(y_k)} \circ \mu_{\text{tr}(x(d))} \circ \mu_{x(d-1)} \circ \mu_{x(d-2)} \circ \cdots \mu_{x(1)} \circ \mu_{x(0)} \circ \mu_{y_k}$, reading right-to-left. In particular, as this mutation sequence is applied to $\overline{R}_k$, Remark 4.7.4 shows that the mutable vertices incident to y_ℓ are located further and further to the right in Figure 19 until we see that they are $\text{tr}(y_k)$ and z_m at the end of the sequence. In fact, we observe after mutating $\overline{R}_k$ at y_k that z_k is the unique green vertex of $\overline{R}_k$ (with the exception of the vertices y_ℓ, z_ℓ, y_t, y_m and z_m, if they appear in $\overline{R}_k$). As we continue to mutate $\mu_{y_k}(\overline{R}_k)$ along the remaining mutations in $\underline{\mu}_k$, the unique green vertex is $x(s)$ for some $s \in [0, d-1]$ or as $\text{tr}(x(d))$ or $\text{tr}(y_k)$ (as before, with the exception of the vertices y_ℓ, z_ℓ, y_t, y_m and z_m). Iteratively mutating at this unique green vertex exactly corresponds to performing the mutation sequence $\underline{\mu}_{A(k)} \circ \underline{\mu}_{B(k)} \circ \underline{\mu}_{C(k)}$ on $\underline{\mu}_{D(k)}(\overline{R}_k)$.

The second assertion, $\underline{\mu}_k(\overline{R}_k)$ is a full subquiver of $\underline{\mu}_k \circ \cdots \circ \underline{\mu}_1 \circ \underline{\mu}_0(\widehat{Q})$, follows since the vertices of Q at which one mutates when applying $\underline{\mu}_k$ are all vertices of $\overline{R}_k$. One can see that the third assertion follows from the above observation that a unique vertex becomes green as we iteratively mutate. The fourth assertion holds for $\underline{\mu}_k(\overline{R}_k)$ since it holds for $\overline{R}_k$. The fifth assertion follows since the vertices in the support of $\underline{\mu}_k$ are all disconnected from the vertices in $(\underline{\mu}_{k-1} \circ \cdots \circ \underline{\mu}_1 \circ \underline{\mu}_0(\widehat{Q}))_0 \backslash (\overline{R}_k)_0$. Further, the sixth assertion is demonstrated inductively as we mutate $x(s)$ for $s \in [0, d-1]$. Lastly, by restricting to the green mutable vertices outside of $(\overline{R}_k)_0$ and the incident frozen vertices, it is clear that the mutation sequence $\underline{\mu}_k$ leaves this full subquiver unaffected. $\square$

Proof of Lemma 4.7.2. We prove the lemma by induction. For $k = 1$, observe that $\underline{\mu}_0(\widehat{Q})$ has the full subquiver $\overline{R}$ shown in Figure 4.22 where we assume that $n > 1$. We show that $\overline{R}$ has all of the properties that $\overline{R}_1$ must satisfy. Note that $\mathrm{tr}(y_1) = x_1$ and for $k = 1$ one has that $x(1) = x_{m_1} = x_1$. Since $\deg(y_1) = 2$, no vertices y_m, y'_m, z_m, and z'_m appear in $\overline{R}$, as desired. Since only vertex x_1 has been mutated to obtain $\underline{\mu}_0(\widehat{Q})$, no arrows between vertices of a signed 3-cycle T_t with $1 < t$ and vertices of signed 3-cycle T_i with $i \leqslant 1$ have been created. Furthermore, there is no signed 3-cycle T_t of $\mathcal{Q}$ with $1 < t$ where $\mathrm{tr}|_{1,t}(y_t) = z_1$. Note that in this degenerate case, $\mathrm{tr}(y_1) = x(1)$ and so no $C_{i,1}$'s or $\widetilde{C_{i,1}}$'s appear in $\overline{R}_1$, and $\widetilde{x}(1) = \widetilde{\mathrm{tr}}(y_1) = x'_1$. Thus the quiver $\overline{R}$ satisfies all of the properties that $\overline{R}_1$ must satisfy. Further, in this special case $\widehat{\mathcal{Q}_{0,0}}$ contains only the vertex x_1 and x'_1 and σ_0 is the identity permutation. Thus $\underline{\mu}_0\widehat{Q_{0,0}}$ indeed equals $\widetilde{Q_{0,0}} \cdot \sigma_0$.

Now assume that $k > 1$ and that $\underline{\mu}_{k-1} \circ \cdots \circ \underline{\mu}_1 \circ \underline{\mu}_0(\widehat{Q})$ has a full subquiver $\overline{R}_k$ with the properties in the statement of the lemma. To show that $\underline{\mu}_k \circ \cdots \circ \underline{\mu}_1 \circ \underline{\mu}_0(\widehat{Q})$ has the desired full subquiver $\overline{R}_{k+1}$, we consider four cases:

i) $\deg(y_k) = 2$ and $\deg(z_k) = 4$,

ii) $\deg(y_k) = 4$ and $\deg(z_k) = 2$,

iii) $\deg(y_k) = 4$ and $\deg(z_k) = 4$, and

iv) $\deg(y_k) = 2$ and $\deg(z_k) = 2$.

Suppose that we are in Case i). By the properties of the ice quiver $\overline{R}_k$, this means that vertices y_m, y'_m, z_m, and z'_m do not appear in $\overline{R}_k$. This also implies that $\ell = k + 1$. Now Lemma 4.7.3 implies that $\underline{\mu}_k(\overline{R}_k)$ has the form shown in Figure 4.23 where the vertices y_t and y'_t appear $\underline{\mu}_k(\overline{R}_k)$ if and only if they appear in $\overline{R}_k$. Note that the quiver in Figure 4.23 is the same as the quiver in Figure 4.21 with the notation updated accordingly. In particular, the integers $m_1^{(k+1)}, m_2^{(k+1)}, \ldots, m_{d+1}^{(k+1)} \in [n]$ and the vertices $x(1, k+1), \ldots, x(d+1, k+1) \in (Q)_0$ are those defined by T_{k+1} following Definition 4.6.1.

Since the signed 3-cycles T_k and T_{k+1} share the vertex z_k (i.e. $z_k = x_{k+1}$), we have that

$$m_1^{(k+1)} = k+1,\ m_2^{(k+1)} = m_1,\ \ldots,\ m_j^{(k+1)} = m_{j-1},\ \ldots,\ m_{d+1}^{(k+1)} = m_d \tag{4.3}$$

and

$$x(1,k+1) = z_k,\ x(2,k+1) = x(1),\ \ldots,\ x(s,k+1) = x(s-1),\ \ldots, x(d+1,k+1) = x(d).$$

This implies that $\mathrm{tr}(x(d+1,k+1)) = \mathrm{tr}(x(d,k))$ and $\mathrm{tr}(y_{k+1}) = \mathrm{tr}(y_k)$. Now we also obtain that

$$\widetilde{x}(1,k+1) = z'_{m_{d+1}^{(k+1)}-1} = z'_{m_d-1} = \widetilde{x}(1,k)$$

and

$$\widetilde{x}(s,k+1) = x'_{m_{d+1-s+2}^{(k+1)}} = x'_{m_j^{(k+1)}} = x'_{m_{j-1}} = \widetilde{x}(s,k)$$

for $s \in [2,d]$ where $j = (d+1) - s + 2$ and that

$$\widetilde{\mathrm{tr}}(x(d+1,k+1)) = x'_{m_2^{(k+1)}} = x'_{m_1} = \widetilde{\mathrm{tr}}(y_k)$$

and

$$\widetilde{\mathrm{tr}}(y_{k+1}) = x'_{m_1^{(k+1)}} = x'_{k+1} = z'_k$$

where the last equality follows from the fact that T_k and T_{k+1} share the vertex z_k. Thus we have labeled the vertices of $\underline{\mu}_k(\overline{R}_k)$ accordingly in Figure 4.23. Furthermore, that the signed 3-cycles T_k and T_{k+1} share the vertex z_k implies that $z_{k+1} = \mathrm{tr}|_{k+1,t}(y_t)$ if and only if $z_k = \mathrm{tr}|_{k,t}(y_t)$.

Next, observe that for any $s \in [d]$ we have $C_{s,k} \cdot \tau_k^{-1} = C_{s,k}$ since we do not mutate $C_{s,k}$ when applying $\underline{\mu}_k$. Additionally $x_{m_{d+1}^{(k+1)}-1} = x_{m_d-1}$ and $y_{m_{j+1}^{(k+1)}} = y_{m_j}$ (for $j = (d+1)-s+2$ where $s \in [2,d]$) follows from (4.3). Comparing with the fifth bullet point of Lemma 4.7.2, we obtain $C_{s,k} = C_{s,k} \cdot \tau_k^{-1} = C_{s,k+1}$ and $\widetilde{C_{s,k}} = \widetilde{C_{s,k}} \cdot \tau_k^{-1} = \widetilde{C_{s,k+1}}$ for any $s \in [d]$.

Now let $C_{d+1,k+1} = y_k$ and $\widetilde{C_{d+1,k+1}} = y'_k$. Note that $y_k \cdot \sigma_{k-1}^{-1} = y_k$ since y_k has not been mutated in $\underline{\mu}_{k-1} \circ \cdots \circ \underline{\mu}_0(\widehat{Q})$. Furthermore, $y_k \cdot \sigma_{k-1}^{-1}\tau_k^{-1} = y_k \cdot \tau_k^{-1} = y_k$ by the definition of τ_k so $C_{d+1,k+1} = y_k \cdot \sigma_k^{-1}$, as desired.

We now construct an ice quiver $\overline{R}$ that is the full subquiver of $\underline{\mu}_k \circ \cdots \circ \underline{\mu}_1 \circ \underline{\mu}_0(\widehat{Q})$ on the vertices of $\underline{\mu}_k(\overline{R}_k)$, as well as the vertices x_r, y_r, z_r and corresponding frozen vertices

x'_r, y'_r, z'_r of any signed 3-cycles T_r of $\mathcal{Q}$ where $k < r$ and z_{k+1} or y_{k+1} is a vertex if T_r. Comparing this construction of $\overline{R}$ to the quiver $\overline{R}_{k+1}$ appearing in Figure 4.18, we verify that $\overline{R}$ indeed equals $\overline{R}_{k+1}$ and satisfies the five properties listed as bullet points in Lemma 4.7.2.

Next, suppose that we are in Case ii). In this situation, we have that $m = k+1$ and the vertices y_ℓ, y'_ℓ, z_ℓ, and z'_ℓ do not belong to $\overline{R}_k$. Now Lemma 4.7.3 implies that $\underline{\mu}_k(\overline{R}_k)$ has the form shown in Figure 4.24. We let T_p (resp. T_q) be the signed 3-cycle not equal to T_{k+1} that contains z_{k+1} (resp. y_{k+1}), if they exist. Define $\overline{R}$ to be the ice quiver that is a full subquiver of $\underline{\mu}_k \circ \cdots \circ \underline{\mu}_1 \circ \underline{\mu}_0(\widehat{Q})$ on the vertices

$$y_{k+1}, y'_{k+1}, z_{k+1}, z'_{k+1}, \mathrm{tr}(y_k), z'_k, y_k, y'_k, \mathrm{tr}(x(d)), x'_k, y_p, y'_p, z_p, z'_p, y_q, y'_q, z_q, z'_q$$

where we include y_p and y'_p (resp. z_p, z'_p, y_q, y'_q, z_q, and z'_q) in $\overline{R}$ if and only if y_p and y'_p (resp. z_q, z'_q, y_q, y'_q, z_q, and z'_q) appear in $\underline{\mu}_k(\overline{R}_k)$, i.e. depending on if $\deg(y_{k+1}) = 4$ and if $\deg(z_{k+1}) = 4$. See Figure 4.25.

Just as above, we claim that the ice quiver $\overline{R}$ equals $\overline{R}_{k+1}$ and satifies the five bullet points in the statement of Lemma 4.7.2. It is easy to see that $\overline{R}$ is a full subquiver of $\underline{\mu}_k \circ \cdots \circ \underline{\mu}_1 \circ \underline{\mu}_0(\widehat{Q})$ that includes every frozen vertex that is connected to a mutable vertex appearing in Figure 4.25 by at least one arrow in $\underline{\mu}_k \circ \cdots \circ \underline{\mu}_1 \circ \underline{\mu}_0(\widehat{Q})$. In particular, $\underline{\mu}_k(\overline{R}_k)$ has this property and no vertices of T_p or T_q and neither y_{k+1} nor z_{k+1} have been mutated in $\underline{\mu}_k \circ \cdots \circ \underline{\mu}_1 \circ \underline{\mu}_0(\widehat{Q})$. Furthermore, defining $m_1^{(k+1)} \in [n]$, $x(0, k+1)$, and $x(1, k+1) \in (\widehat{Q})_0$ just as we did in Case i), following Definition 4.6.1, and using the fact that $\mathrm{sgn}(T_{k+1}) = +$, we have $m_1^{(k+1)} = k+1$, $x(0, k+1) = z_{k+1}$, and $x(1, k+1) = x_{k+1}$. Hence we obtain that

$$\mathrm{tr}(x(1, k+1)) = \mathrm{tr}(x_{k+1}) = \mathrm{tr}(y_k) \text{ and } z'_{m_1^{(k+1)}-1} = z'_k,$$

as desired. Additionally, the fact that $\mathrm{sgn}(T_{k+1}) = +$ also implies that

$$\mathrm{tr}(y_{k+1}) = x_{k+1} = y_k \text{ and } x'_{m_1^{(k+1)}} = x'_k = y'_k,$$

as desired. These calculations are reflected in the quiver $\overline{R}$ shown in Figure 4.25, thus verifying the first three bullet points of Lemma 4.7.2.

Furthermore, since $\deg(z_k) = 2$, there is no signed 3-cycle T_t with $k+1 < t$ such that $\mathrm{tr}|_{k+1,t}(y_t) = z_{k+1}$ in $\underline{\mu}_k \circ \cdots \circ \underline{\mu}_1 \circ \underline{\mu}_0(\widehat{Q})$ vertex x_t has been mutated exactly once.

The fourth bullet point follows. Now observe that $\widetilde{\mathrm{tr}}(y_k) = x'_{m_1} = x'_k$. Since we have applied a maximal green sequence to $\mathcal{Q}_k$ and since $\mathrm{tr}(x(d,k))$ is only connected to the frozen vertex x'_k, Proposition 2.10 of [BDP14] implies that $\mathrm{tr}(x(d,k)) = x_k \cdot \sigma_k^{-1}$. We thus have the fifth bullet point.

Case iii) is similar to Case ii), but with some key differences. In this situation, we again have that $m = k+1$ but this time both y_ℓ and z_ℓ are relevant. Now Lemma 4.7.3 implies that $\underline{\mu}_k(\overline{R}_k)$ has the form shown in Figure 4.26. We let T_p (resp. T_q) be the signed 3-cycles incident to z_{k+1} (resp. y_{k+1}) if they exist. Define $\overline{R}$ to be the ice quiver that is a full subquiver of $\underline{\mu}_k \circ \cdots \circ \underline{\mu}_1 \circ \underline{\mu}_0(\widehat{Q})$ on the vertices

$$y_{k+1}, y'_{k+1}, z_{k+1}, z'_{k+1}, \mathrm{tr}(y_k), z'_k, y_k, y'_k, \mathrm{tr}(x(d)), x'_k, y_\ell, y'_\ell, y_p, y'_p, z_p, z'_p, y_q, y'_q, z_q, z'_q$$

where we include y_p and y'_p (resp. z_p, z'_p, y_q, y'_q, z_q, and z'_q) in $\overline{R}$ if and only if y_p and y'_p (resp. z_q, z'_q, y_q, y'_q, z_q, and z'_q) appear in $\underline{\mu}_k(\overline{R}_k)$, i.e. depending on if $\deg(y_{k+1}) = 4$ and if $\deg(z_{k+1}) = 4$. See Figure 4.27.

We claim that the ice quiver $\overline{R}$ has the properties in the statement of Lemma 4.7.2. It is easy to see that $\overline{R}$ is a full subquiver of $\underline{\mu}_k \circ \cdots \circ \underline{\mu}_1 \circ \underline{\mu}_0(\widehat{Q})$. That $\overline{R}$ includes every frozen vertex that is connected to a mutable vertex appearing in Figure 4.27 by at least one arrow in $\underline{\mu}_k \circ \cdots \circ \underline{\mu}_1 \circ \underline{\mu}_0(\widehat{Q})$ follows from the fact that $\underline{\mu}_k(\overline{R}_k)$ has this property and from the fact that no vertices of T_p or T_q and neither y_{k+1} nor z_{k+1} have been mutated in $\underline{\mu}_k \circ \cdots \circ \underline{\mu}_1 \circ \underline{\mu}_0(\widehat{Q})$. Now observe that $\widetilde{\mathrm{tr}}(y_k) = x'_{m_1} = x'_k$. As in Case ii), Proposition 2.10 of [BDP14] implies that $\mathrm{tr}(x(d,k)) = x_k \cdot \sigma_k^{-1}$.

Let $m_1^{(k+1)} \in [n]$ be the integer from the definition of $\underline{\mu}_{k+1}$, and let $x(0,k+1), x(1,k+1) \in (Q)_0$ be the vertices from the definition of $\underline{\mu}_{k+1}$. As in Case ii), we are using the fact that $\mathrm{sgn}(T_{k+1}) = +$. Now notice that $m_1^{(k+1)} = m_d^{(k+1)} = k+1$, $x(0,k+1) = z_{k+1}$, and $x(1,k+1) = x_{k+1}$. We now obtain that

$$\mathrm{tr}(x(d,k+1)) = \mathrm{tr}(x_{k+1}) = \mathrm{tr}(y_k),$$

as desired. The fact that $\mathrm{sgn}(T_{k+1}) = +$ implies that

$$\mathrm{tr}(y_{k+1}) = x_{k+1} = y_k.$$

Since $\deg(z_{k+1}) = 4$, the vertices y_ℓ and y'_ℓ both appear in $\overline{R}$. Now it is clear that $\deg(z_{k+1}) = 4$ if and only if $\mathrm{tr}|_{k+1,\ell}(y_\ell) = z_{k+1}$ and the signed 3-cycle T_ℓ has the property that vertex x_ℓ has been mutated exactly once in $\underline{\mu}_k \circ \cdots \circ \underline{\mu}_1 \circ \underline{\mu}_0(\widehat{Q})$. Hence we see that the vertex y_ℓ is positioned in $\overline{R}$ exactly where y_t is positioned in $\overline{R}_{k+1}$, see Figure 4.18. These calculations are reflected in the quiver $\overline{R}$ shown in Figure 4.27, and we see that this quiver has the properties that the desired quiver $\overline{R}_{k+1}$ should have. The proof of the five bullet points of Lemma 4.7.2 in Case iii) concludes in the same way as the proof for Case ii).

In addition, we illustrate how in Case iii), for each $c \in [n]$ satisfying $k < c \leqslant \ell$ there is an ice quiver $\overline{R}_{c,\ell}$ that is isomorphic to $\overline{R}_\ell$ and that appears as a full subquiver of $\underline{\mu}_{c-1} \circ \cdots \circ \underline{\mu}_1 \circ \underline{\mu}_0(\widehat{Q})$. Furthermore, we show that $\overline{R}_{\ell,\ell} = \overline{R}_\ell$. This analysis will be used in the argument for Case iv), which is given below.

As we are in Case iii), we know that both vertices y_k and z_k are of degree 4 and the signed 3-cycles T_k, T_{k+1}, and T_ℓ appear in a full subquiver of $\mathcal{Q}$ of the form shown in Figure 4.28 or 4.29. It follows that y_ℓ and z_ℓ, which are incident to z_k in $\underline{\mu}_{k-1} \circ \cdots \circ \underline{\mu}_1 \circ \underline{\mu}_0(\widehat{Q})$, will not be mutated until after applying the mutation sequences $\underline{\mu}_k, \underline{\mu}_{k+1}, \ldots, \underline{\mu}_r$ where $k \leqslant r < \ell$ (see Figure 4.30). To be precise, the quiver in Figure 4.30 is a full subquiver of $\underline{\mu}_{k-1} \circ \cdots \circ \underline{\mu}_1 \circ \underline{\mu}_0(\widehat{Q})$, which we define as follows. Letting $k < r < \ell$ be the integer such that $z_r = \mathrm{tr}(y_\ell)$ and e such that $x(e,r) = x_{k+1} = y_k$, this full subquiver includes the vertices of $\overline{R}_{k-1}$ as well as the mutable vertices of the signed 3-cycles $T_{m_e^{(r)}} = T_{k+1}, T_{m_{e-1}^{(r)}}, \ldots, T_{m_2^{(r)}}, T_{m_1^{(r)}} = T_r$, as in Definition 4.6.1, and their corresponding frozen vertices.

We now mutate the quiver shown in Figure 4.30 along $\underline{\mu}_k$. By Lemma 4.7.3, this does not affect the full subquiver of $\underline{\mu}_{k-1} \circ \cdots \circ \underline{\mu}_1 \circ \underline{\mu}_0(\widehat{Q})$ on the vertices $(\widehat{Q})_0 \backslash (\overline{R}_k)_0$. Thus we conclude that $\underline{\mu}_k \circ \cdots \circ \underline{\mu}_1 \circ \underline{\mu}_0(\widehat{Q})$ has the quiver shown in Figure 4.31 as a full subquiver. We observe that the permutation σ_{k-1}^{-1} has the vertices y_k and z_k as fixed points. However, τ_k^{-1} maps $z_k \mapsto \mathrm{tr}(y_k)$ and fixes y_k. These equalities are illustrated in Figure 4.31.

Next, we relabel the vertices of the quiver in Figure 4.31 to obtain the quiver shown in Figure 4.32. In particular, since $\mathrm{sgn}(T_\ell) = -$ with $x_\ell = z_k$, note that $z_k = x(1,\ell)$, $x(s,k) = x(s+1,\ell)$, and $\mathrm{tr}(y_k) = \mathrm{tr}|_{k,\ell}(y_\ell)$. Define $\overline{R}_{k+1,\ell}$ to be the full subquiver of $\underline{\mu}_k \circ \cdots \circ \underline{\mu}_1 \circ \underline{\mu}_0(\widehat{Q})$ on the red vertices appearing in Figure 4.32, the neighbors of y_ℓ and z_ℓ, as well as the frozen vertices to which these all are connected. One observes that $\overline{R}_{k+1,\ell}$ and $\overline{R}_\ell$ are isomorphic as ice quivers. Furthermore, we will see that $\overline{R}_{k+1,\ell}$ has the same vertices as $\overline{R}_\ell$ with the exceptions of y_k and $\mathrm{tr}|_{k,\ell}(y_\ell)$.

For $k < c \leqslant \ell$, we define $\overline{R}_{c,\ell}$ analogously as the full subquiver of $\underline{\mu}_{c-1} \circ \cdots \circ \underline{\mu}_1 \circ \underline{\mu}_0(\widehat{Q})$ on the set of vertices $(\overline{R}_{k+1,\ell})_0 \cdot \tau_{k+1}^{-1}\tau_{k+2}^{-1} \cdots \tau_{c-1}^{-1}$. With this definition, we observe that $\overline{R}_{c+1,\ell}$ is identical to $\overline{R}_{c,\ell}$ except possibly at two vertices. In particular, for $k < c \leqslant \ell$, if T_c does not appear in Figure 4.28 (resp. Figure 4.29), then the mutation sequence $\underline{\mu}_c$ does not involve any vertices that appear in $\overline{R}_{c,\ell}$. Consequently, after mutation by $\underline{\mu}_c$, we obtain $\overline{R}_{c+1,\ell} = \overline{R}_{c,\ell}$

On the other hand, when T_c, for $k < c \leqslant \ell$, i.e. $c = m_s^{(r)}$ for some s, does appear in Figure 4.28 (resp. Figure 4.29), then the mutation sequence $\underline{\mu}_c$, as indicated by bold arrows in Figures 4.32 and 4.33, involves vertices $y_k \cdot \sigma_{c-1}^{-1}$ and $z_k \cdot \sigma_{c-1}^{-1}$. In this case, $\overline{R}_{c+1,\ell} \cong \overline{R}_{c,\ell}$ with the relabeling $y_k \cdot \sigma_{c-1}^{-1} \mapsto y_k \cdot \sigma_c^{-1}$ and $z_k \cdot \sigma_{c-1}^{-1} \mapsto z_k \cdot \sigma_c^{-1}$ since each application of $\underline{\mu}_c$ permutes these two vertices by τ_c^{-1}. This isomorphism of full subquivers follows from Lemma 4.7.3.

We obtain the identity $z_k \cdot \sigma_c^{-1} = \mathrm{tr}|_{m_s^{(r)},\ell}(y_\ell)$ for $m_s^{(r)} \leqslant c < m_{s-1}^{(r)}$ when $s \in [2, e]$ or $r = m_1^{(r)} \leqslant c < \ell$ when $s = 1$, which is implicit in Figure 4.33, by Lemma 4.7.5. We leave this argument until after completing the proof of Lemma 4.7.2 (see below). We also observe, by the specialization $c = \ell - 1$, that $y_k \cdot \sigma_{\ell-1}^{-1} = C_{d+1,\ell}$ and $z_k \cdot \sigma_{\ell-1}^{-1} = \mathrm{tr}(y_\ell)$. Consequently, we eventually arrive at the configuration in Figure 4.34 with configurations of the form as in Figure 4.33 as intermediate steps. In summary, we conclude that $\overline{R}_{\ell,\ell} = \overline{R}_\ell$ as desired.

Next, suppose we are in Case iv). Since $\deg(z_k) = 2$, this case is similar to Case i). However, here we have $\deg(y_k) = 2$ as well, and so the quiver $\underline{\mu}_k(\overline{R}_k)$ looks like Figure 4.21, but without y_ℓ, y_ℓ', z_ℓ, z_ℓ', y_m, y_m', z_m, nor z_m'. The green vertex y_t and y_t' may or may not appear in the quiver $\underline{\mu}_k(\overline{R}_k)$. In the latter case, $k = n$ and we have applied the entire mutation sequence $\underline{\mu}$ to $\widehat{Q}$. In the former case, we see that $t = k+1$, and T_t can be realized as a signed 3-cycle T_ℓ appearing in one of Figures 4.28 or 4.29. Now by the argument at the end of Case iii), $\overline{R}_{t,t} = \overline{R}_t$ is indeed a full subquiver of Figure 4.34 with the desired properties. The five bullet points of Lemma 4.7.2 follow immediately.

Lastly, for all four cases, we wish to describe the quiver obtained by $\underline{\mu}_k \circ \cdots \circ \underline{\mu}_1 \circ \underline{\mu}_0(\widehat{\mathcal{Q}_{k,k}})$. To this end, we decompose the vertices of $\widehat{\mathcal{Q}_{k,k}}$ into two sets: (1) $(\widehat{\mathcal{Q}_{k,k}})_0 \backslash (\overline{R}_k)_0$

and (2) $(\overline{R}_k)_0 \cap (\widehat{\mathcal{Q}_{k,k}})_0$. By induction, we have

$$\underline{\mu}_{k-1} \circ \cdots \circ \underline{\mu}_1 \circ \underline{\mu}_0(\widehat{\mathcal{Q}_{k-1,k-1}}) = \widetilde{\mathcal{Q}_{k-1,k-1}} \cdot \sigma_{k-1}$$

and we observe that $(\widehat{\mathcal{Q}_{k,k}})_0 \backslash (\overline{R}_k)_0 \subset (\widehat{\mathcal{Q}_{k-1,k-1}})_0$. The fifth bullet point of Lemma 4.7.3 implies that the quiver $\underline{\mu}_{k-1} \circ \cdots \circ \underline{\mu}_1 \circ \underline{\mu}_0 \left(\widehat{\mathcal{Q}_{k,k}}|_{(\widehat{\mathcal{Q}_{k,k}})_0 \backslash (\overline{R}_k)_0}\right)$[4] is unchanged by the mutation sequence $\underline{\mu}_k$, and the permutation τ_k fixes all vertices in $(\widehat{\mathcal{Q}_{k,k}})_0 \backslash (\overline{R}_k)_0$. It follows that

$$\underline{\mu}_k \circ \cdots \circ \underline{\mu}_1 \circ \underline{\mu}_0 \left(\widehat{\mathcal{Q}_{k,k}}|_{(\widehat{\mathcal{Q}_{k,k}})_0 \backslash (\overline{R}_k)_0}\right) = \left(\widetilde{\mathcal{Q}_{k,k}}|_{(\widehat{\mathcal{Q}_{k,k}})_0 \backslash (\overline{R}_k)_0}\right) \cdot \sigma_k.$$

Additionally, the first bullet point of Lemma 4.7.3 indicates how the vertices of the second set, i.e. $(\overline{R}_k)_0 \cap (\widehat{\mathcal{Q}_{k,k}})_0$, is affected by $\underline{\mu}_k$. Comparing Figures 4.18 and 4.20, we see that the vertices of $\overline{R}_k$ have been permuted cyclically exactly as described by τ_k. We conclude that

$$\underline{\mu}_k \circ \cdots \circ \underline{\mu}_1 \circ \underline{\mu}_0(\widehat{\mathcal{Q}_{k,k}}) = \widetilde{\mathcal{Q}_{k,k}} \cdot \tau_k \sigma_{k-1} = \widetilde{\mathcal{Q}_{k,k}} \cdot \sigma_k$$

which completes the proof of Lemma 4.7.2.

□

Lemma 4.7.5. *Using the notation from the proof of Lemma 4.7.2, for any $s \in [2, e]$ and any $c \in [n]$ satisfying $m_s^{(r)} \leqslant c < m_{s-1}^{(r)}$, one has $z_k \cdot \sigma_c^{-1} = tr|_{m_s^{(r)},\ell}(y_\ell)$ (see Figure 4.33). Additionally, for any $c \in [n]$ satisfying $r = m_1^{(r)} \leqslant c < \ell$ we have $z_k \cdot \sigma_c^{-1} = tr(y_\ell)$.*

Proof. For $c = k + 1$, we have

$$\begin{aligned}
z_k \cdot \sigma_{k+1}^{-1} &= z_k \cdot \sigma_k^{-1}\tau_{k+1}^{-1} \\
&= \mathrm{tr}(y_k) \cdot \tau_{k+1}^{-1} && \text{(see Figure 4.31)} \\
&= \mathrm{tr}(x(1, k+1)) \cdot \tau_{k+1}^{-1} && \text{(using that } \mathrm{sgn}(T_{k+1}) = +) \\
&= x(0, k+1) && \text{(by the definition of } \tau_{k+1}) \\
&= z_k && \text{(by Definition 4.6.1)} \\
&= \mathrm{tr}|_{k+1,\ell}(y_\ell), && \text{(by Definition 4.6.2)}
\end{aligned}$$

[4] We define $\widehat{\mathcal{Q}_{k,k}}|_{(\widehat{\mathcal{Q}_{k,k}})_0 \backslash (\overline{R}_k)_0}$ (resp. $\widetilde{\mathcal{Q}_{k,k}}|_{(\widehat{\mathcal{Q}_{k,k}})_0 \backslash (\overline{R}_k)_0}$) to be the ice quiver that is a full subquiver of $\widehat{\mathcal{Q}_{k,k}}$ (resp. $\widetilde{\mathcal{Q}_{k,k}}$) on the vertices of $(\widehat{\mathcal{Q}_{k,k}})_0 \backslash (\overline{R}_k)_0$.

as desired. Now suppose that $z_k \cdot \sigma_c^{-1} = \mathrm{tr}|_{m_s^{(r)},\ell}(y_\ell)$ where $s \in [e]$ and $m_s^{(r)} \leqslant c < m_{s-1}^{(r)}$. Then for $c \in [n]$ satisfying $m_{s-1}^{(r)} \leqslant c < m_{s-2}^{(r)}$ we have

$$\begin{aligned}
z_k \cdot \sigma_c^{-1} &= z_k \cdot \sigma^{-1}_{m_{s-1}^{(r)}-1} \tau^{-1}_{m_{s-1}^{(r)}} \cdots \tau_c^{-1} \\
&= \mathrm{tr}|_{m_s^{(r)},\ell}(y_\ell) \cdot \tau^{-1}_{m_{s-1}^{(r)}} \cdots \tau_c^{-1} && \text{(by induction)} \\
&= x(1, m_{s-1}^{(r)}) \cdot \tau^{-1}_{m_{s-1}^{(r)}} \tau^{-1}_{m_{s-1}^{(r)}+1} \cdots \tau_c^{-1} && \text{(note that } x(1, m_{s-1}^{(r)}) = x(s-1, r)\text{)} \\
&= x(0, m_{s-1}^{(r)}) \cdot \tau^{-1}_{m_{s-1}^{(r)}+1} \cdots \tau_c^{-1} && \text{(note that } x(0, m_{s-1}^{(r)}) = x(s-2, r)\text{)} \\
&= \mathrm{tr}|_{m_{s-1}^{(r)},\ell}(y_\ell) \cdot \tau^{-1}_{m_{s-1}^{(r)}+1} \cdots \tau_c^{-1} && \text{(note that } x(0, m_{s-1}^{(r)}) = \mathrm{tr}|_{m_{s-1}^{(r)},\ell}(y_\ell)\text{)} \\
&= \mathrm{tr}|_{m_{s-1}^{(r)},\ell}(y_\ell),
\end{aligned}$$

as desired. We remark that the last equality in the previous computation follows from observing that $\mathrm{tr}|_{m_{s-1}^{(r)},\ell}(y_\ell)$ is not mutated in any of the mutation sequences $\underline{\mu}_{m_{s-1}^{(r)}+1}, \ldots, \underline{\mu}_c$, and thus it is unaffected by any of the permutations $\tau^{-1}_{m_{s-1}^{(r)}+1}, \ldots, \tau_c^{-1}$. By induction, this completes the proof. □

4.8 Additional Questions and Remarks

In this section, we give an example to show how our results provide explicit maximal green sequences for quivers that are not of type $\mathbb{A}$. We also discuss ideas we have for further research.

4.8.1 Maximal Green Sequences for Quivers Arising from Surface Triangulations

The following example shows how our formulas for maximal green sequences for type $\mathbb{A}$ quivers can be used to give explicit formulas for maximal green sequences for quivers arising from other types of triangulated surfaces.

Example 4.8.1. *Consider the marked surface $(\boldsymbol{S}, \boldsymbol{M})$ with the triangulation $\boldsymbol{T}$ shown in Figure 4.35 on the left. The surface $\boldsymbol{S}$ is a once-punctured pair of pants with triangulation*

$$\boldsymbol{T} = \boldsymbol{T}_1 \sqcup \boldsymbol{T}_2 \sqcup \{\eta, \epsilon, \zeta\}$$

where $\alpha_1, \alpha_2, \alpha_3 \in \boldsymbol{T}_1$ *and* $\beta_1, \beta_2, \beta_3, \nu \in \boldsymbol{T}_2$. *We assume that the boundary arcs* b_i *with* $i \in [5]$ *contain no marked points except for those shown in Figure 4.35. The other boundary arcs may contain any number of marked points. As in Section 4.4, let* $Q_{\boldsymbol{T}}$ *be the quiver determined by* $\boldsymbol{T}$ *and let* $v_\delta \in (Q)_0$ *denote the vertex corresponding to arc* $\delta \in \boldsymbol{T}$.

We can think of the marked surface $(\boldsymbol{S}_1, \boldsymbol{M}_1)$ *determined by* $c_1, \beta_1, b_1, c_2, \beta_2, b_2, \beta_3, b_3$ *as an* m_1*-gon where* $m_1 = \#\boldsymbol{M}_1$ *and we can think of* $\boldsymbol{T}_1$ *as a triangulation of* $\boldsymbol{S}_1$. *Similarly, we can think of the marked surface* $(\boldsymbol{S}_2, \boldsymbol{M}_2)$ *determined by* $\alpha_1, c_3, \eta, b_5, c_6, \alpha_3, c_5, b_2, \alpha_2, c_4, b_1$ *as an* m_2*-gon where* $m_2 = \#\boldsymbol{M}_2$ *and we can think of* $\boldsymbol{T}_2$ *as a triangulation of* $\boldsymbol{S}_2$. *Thus quiver* $Q_{\boldsymbol{T}_i}$, *determined by* $\boldsymbol{T}_i$, *is a type* $\mathbb{A}$ *quiver for* $i = 1, 2$. *Furthermore, we have*

$$Q_{\boldsymbol{T}} = Q_{\boldsymbol{T}_1} \oplus_{(v_{\alpha_1}, v_{\alpha_2}, v_{\alpha_3})}^{(v_{\beta_1}, v_{\beta_2}, v_{\beta_3})} Q_{\boldsymbol{T}_2} \oplus_{(v_\nu)}^{(v_\eta)} R$$

where

$$R \quad = \quad \begin{tikzcd} & v_\zeta \\ v_\eta \arrow[ur] \arrow[dr] & \\ & v_\epsilon. \end{tikzcd}$$

By Corollary 4.6.8, $Q_{\boldsymbol{T}_1}$ *and* $Q_{\boldsymbol{T}_2}$ *each have a maximal green sequence* $\underline{\mu}^{Q_{\boldsymbol{T}_i}}$ *for* $i = 1, 2$. *Since* R *is acyclic, we can define* $\underline{\mu}^R$ *to be any mutation sequence of* $\widehat{R}$ *where each mutation occurs at a source (for instance, put* $\underline{\mu}^R = \mu_{v_\epsilon} \circ \mu_{v_\zeta} \circ \mu_{v_\eta}$*). Then* $\underline{\mu}^R$ *is clearly a maximal green sequence of* R. *Now Theorem 4.3.12 implies that* $\underline{\mu}^R \circ \underline{\mu}^{Q_{\boldsymbol{T}_2}} \circ \underline{\mu}^{Q_{\boldsymbol{T}_1}}$ *is a maximal green sequence of* $Q_{\boldsymbol{T}}$.

Suppose that $\boldsymbol{T}_1$ *and* $\boldsymbol{T}_2$ *are given by the triangulations shown in Figure 4.35 on the right. Then we have that* $Q_{\boldsymbol{T}_1}$ *and* $Q_{\boldsymbol{T}_2}$ *are the quivers shown in Figure 4.36 where we think of the irreducible parts of* $Q_{\boldsymbol{T}_1}$ *and* $Q_{\boldsymbol{T}_2}$ *as signed irreducible type* $\mathbb{A}$ *quivers with respect to the root 3-cycles* $S_1^{(1)}$ *and* $S_1^{(2)}$, *respectively. In this situation,* $Q_{\boldsymbol{T}_1}$ *and* $Q_{\boldsymbol{T}_2}$ *have the maximal green sequences*

$$\begin{aligned} \underline{\mu}^{Q_{\boldsymbol{T}_1}} &= \mu_{w_1} \circ \mu_{w_2} \circ \mu_{v_{\alpha_1}} \circ \underline{\mu}_3^{(1)} \circ \underline{\mu}_2^{(1)} \circ \underline{\mu}_1^{(1)} \circ \underline{\mu}_0^{(1)} \\ \underline{\mu}^{Q_{\boldsymbol{T}_2}} &= \mu_{w_4} \circ \mu_{w_3} \circ \underline{\mu}_5^{(2)} \circ \underline{\mu}_4^{(2)} \circ \underline{\mu}_3^{(2)} \circ \underline{\mu}_2^{(2)} \circ \underline{\mu}_1^{(2)} \circ \underline{\mu}_0^{(2)} \end{aligned}$$

respectively where

$$\begin{aligned}
\underline{\mu}_0^{(1)} &= \mu_{x_1^{(1)}} \\
\underline{\mu}_1^{(1)} &= \mu_{x_1^{(1)}} \circ \mu_{z_1^{(1)}} \circ \mu_{y_1^{(1)}} \\
\underline{\mu}_2^{(1)} &= \mu_{x_1^{(1)}} \circ \mu_{z_1^{(1)}} \circ \mu_{z_2^{(1)}} \circ \mu_{y_2^{(1)}} \\
\underline{\mu}_3^{(1)} &= \mu_{y_2^{(1)}} \circ \mu_{x_1^{(1)}} \circ \mu_{z_3^{(1)}} \circ \mu_{y_3^{(1)}}
\end{aligned}
\qquad
\begin{aligned}
\underline{\mu}_0^{(2)} &= \mu_{x_1^{(2)}} \\
\underline{\mu}_1^{(2)} &= \mu_{x_1^{(2)}} \circ \mu_{z_1^{(2)}} \circ \mu_{y_1^{(2)}} \\
\underline{\mu}_2^{(2)} &= \mu_{x_1^{(2)}} \circ \mu_{z_1^{(2)}} \circ \mu_{z_2^{(2)}} \circ \mu_{y_2^{(2)}} \\
\underline{\mu}_3^{(2)} &= \mu_{y_2^{(2)}} \circ \mu_{x_1^{(2)}} \circ \mu_{z_3^{(2)}} \circ \mu_{y_3^{(2)}} \\
\underline{\mu}_4^{(2)} &= \mu_{y_3^{(2)}} \circ \mu_{y_2^{(2)}} \circ \mu_{z_4^{(2)}} \circ \mu_{y_4^{(2)}} \\
\underline{\mu}_5^{(2)} &= \mu_{z_4^{(2)}} \circ \mu_{x_1^{(2)}} \circ \mu_{z_3^{(2)}} \circ \mu_{z_5^{(2)}} \circ \mu_{y_5^{(2)}}.
\end{aligned}$$

and $\underline{\mu}^R \circ \underline{\mu}^{Q_{T_2}} \circ \underline{\mu}^{Q_{T_1}}$ *is a maximal green sequence of* $Q_{\boldsymbol{T}}$*. In general, if we have a quiver* $Q_{\boldsymbol{T}}$ *that can be realized as a direct sum of type* $\mathbb{A}$ *quivers and acyclic quivers, we can write an explicit formula for a maximal green sequence of* $Q_{\boldsymbol{T}}$*.*

Problem 4.8.2. *Find explicit formulas for maximal green sequences for quivers arising from triangulations of surfaces.*

Using Corollary 4.4.5, we can reduce Problem 4.8.2 to the problem of finding explicit formulas for maximal green sequences of irreducible quivers that arise from a triangulated surface. In [ACC$^+$13], the authors sketch an argument showing the existence of maximal green sequences for quivers arising from triangulated surfaces. However, we would like to prove the existence of maximal green sequences by giving explicit formulas for maximal green sequences of such quivers.

Some progress has already been made in answering Problem 4.8.2. In [Lad13], Ladkani shows that quivers arising from triangulations of once-punctured closed surfaces of genus $g \geqslant 1$ have no maximal green sequences. In [Buc14, BM15], explicit formulas for maximal green sequences are given for specific triangulations of closed genus $g \geqslant 1$ surfaces. In [CDR$^+$15], a formula is given for the minimal length maximal green sequences of quivers defined by polygon triangulations. It would be interesting to understand, in general, what are the possible lengths that can be achieved by maximal green sequences of a given quiver.

4.8.2 Trees of Cycles

Our study of signed irreducible type $\mathbb{A}$ quivers was made possible by the fact that such quivers are equivalent to labeled binary trees of 3-cycles (see Lemma 4.5.5). It is therefore reasonable to ask if one can find explicit formulas for maximal green sequences of quivers that are trees of cycles where each cycle has length at least $k \geqslant 3$. In our construction, we define a total ordering and a sign function on the set of 3-cycles of an irreducible type $\mathbb{A}$ quiver (with at least one 3-cycle), and this data was important in discovering and describing the associated mutation sequence. One could use a similar technique to construct an analogue of the associated mutation sequence for quivers that are trees of oriented cycles.

Problem 4.8.3. *Find a construction of maximal green sequences for quivers that are trees of oriented cycles.*

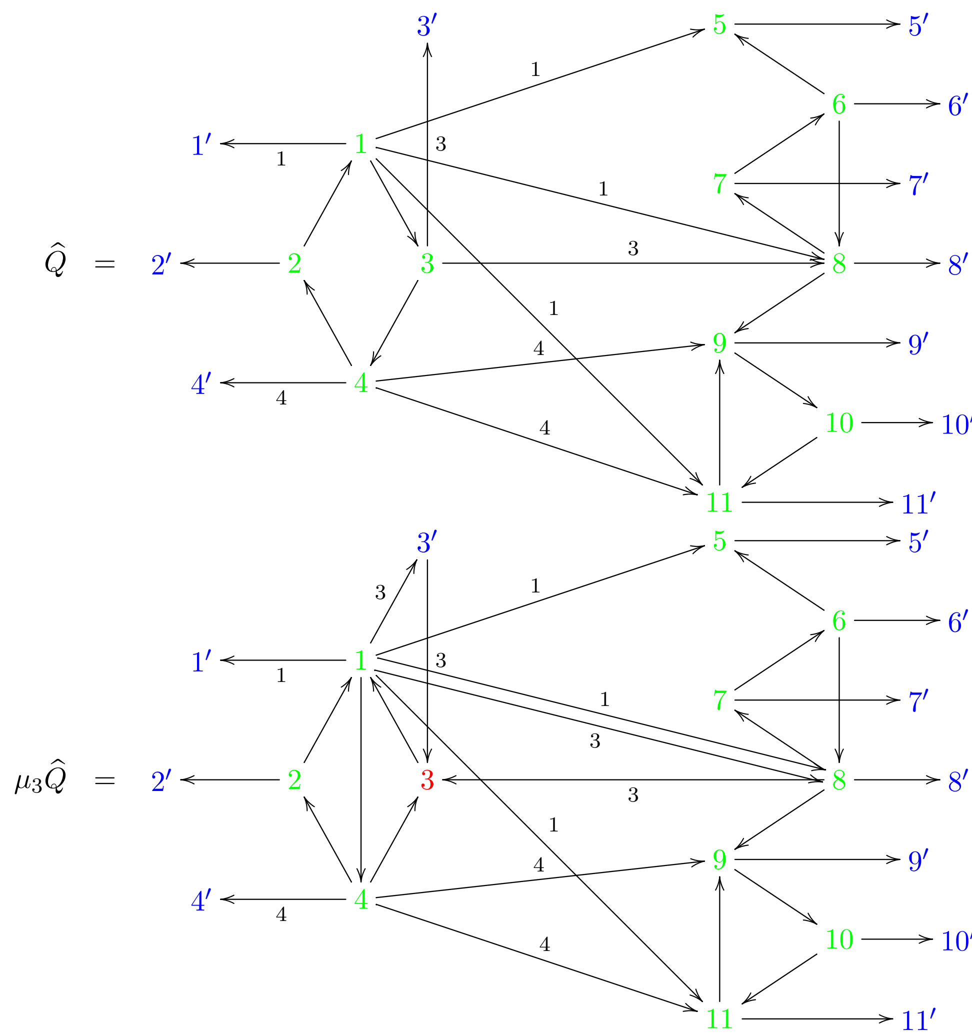


Figure 4.2: The quivers $\widehat{Q}$ and $\mu_3\widehat{Q}$ with the coloring functions f^1 and f^2, respectively.

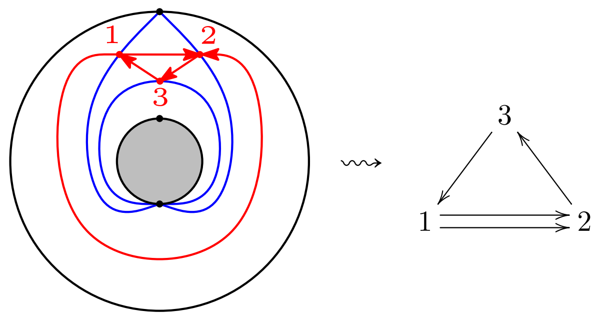


Figure 4.3: The quiver $Q_{\mathbf{T}}$ defined by a triangulation $\mathbf{T}$.

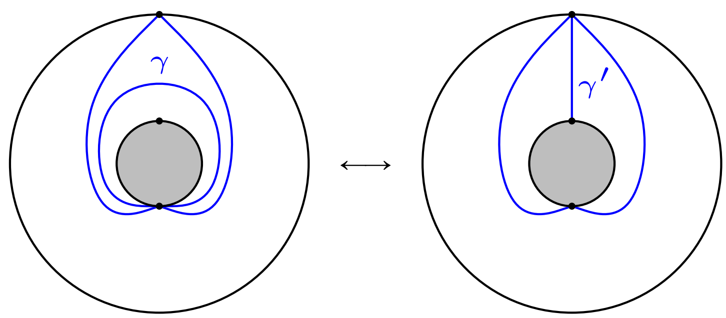


Figure 4.4: A flip connecting two triangulations of an annulus.

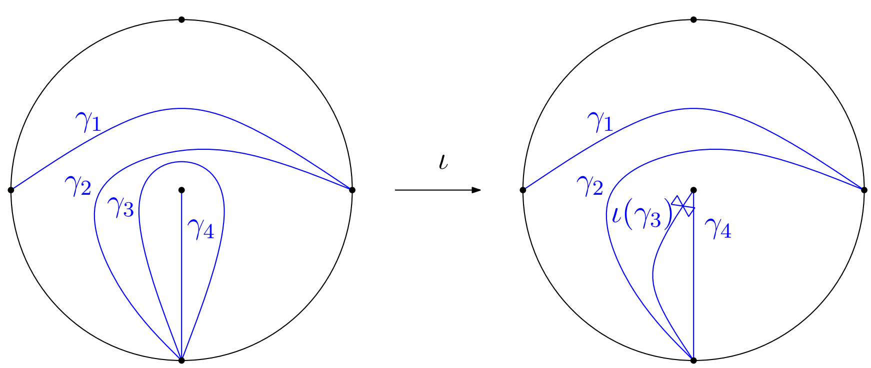


Figure 4.5: The map identifying a triangulation of a punctured disk as a tagged triangulation of a punctured disk.

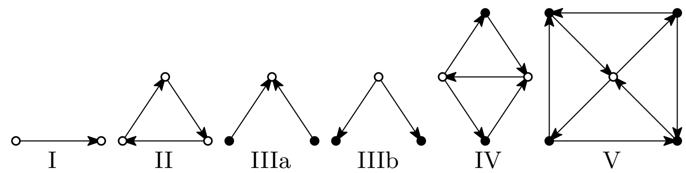


Figure 4.6: The Fomin-Shapiro-Thurston blocks.

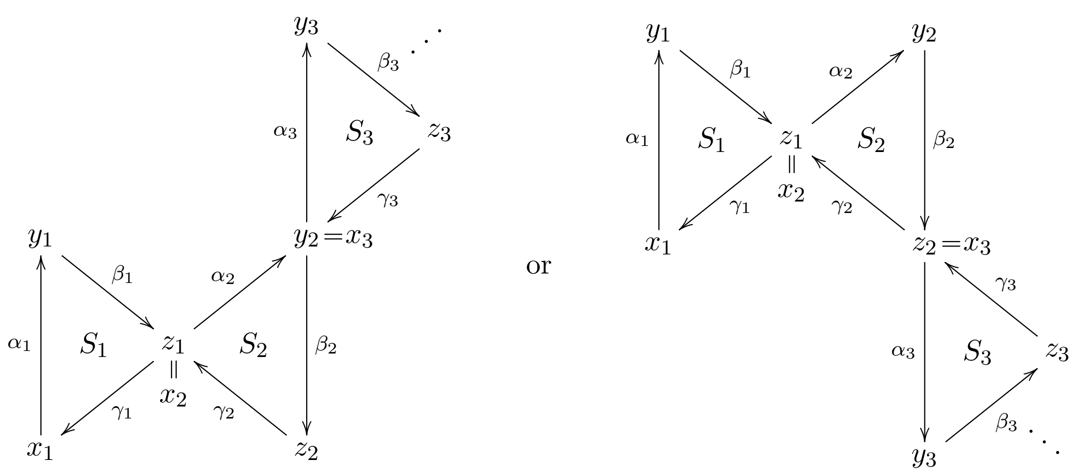


Figure 4.7: Labeling arrows of an irreducible quiver of type $\mathbb{A}$.

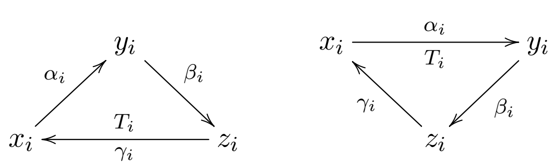


Figure 4.8: A positive (resp. negative) 3-cycle is shown on the left (resp. right).

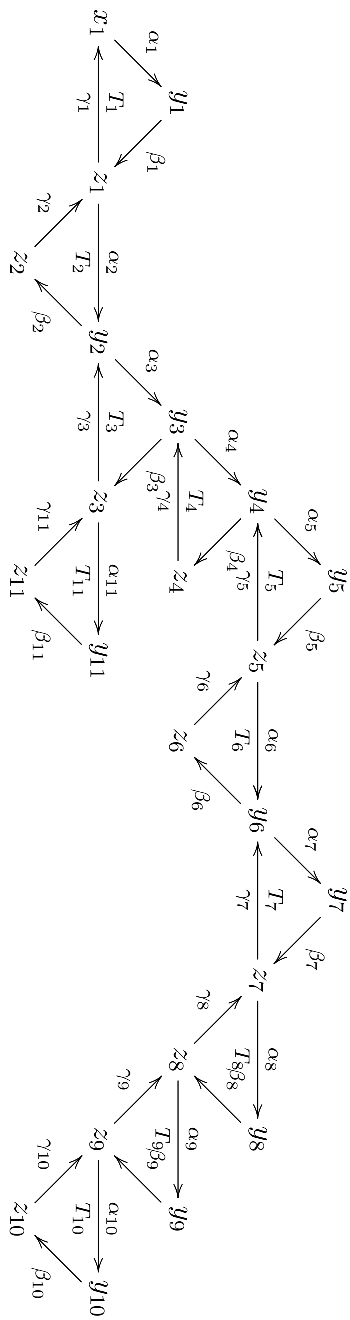


Figure 4.9: A signed irreducible type $\mathbb{A}_{23}$ quiver.

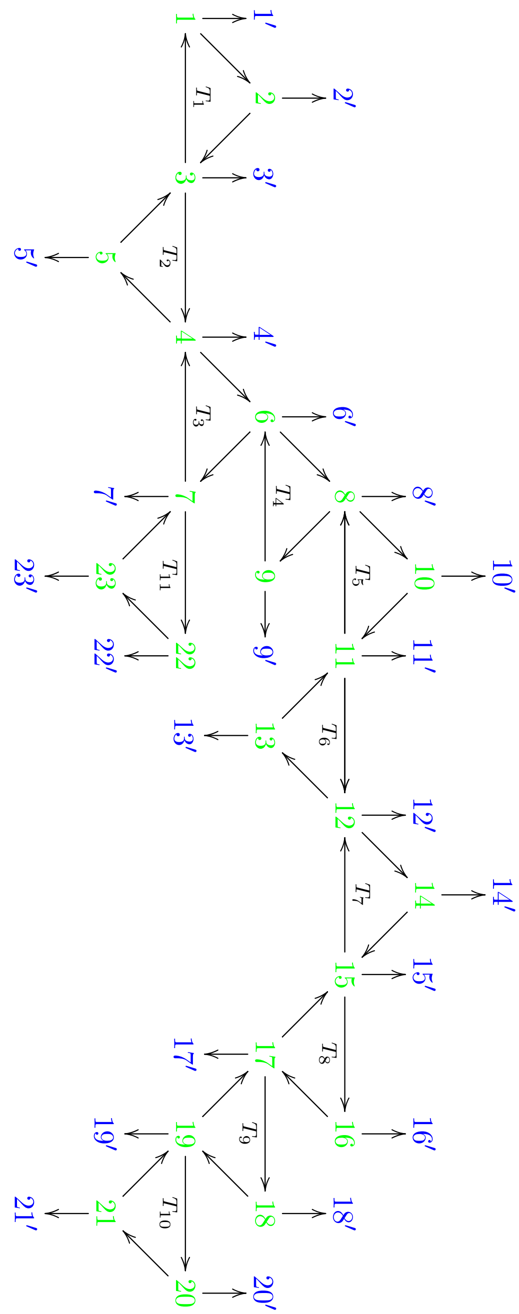


Figure 4.10: The framed quiver of a signed irreducible type $\mathbb{A}_{23}$ quiver with vertices labeled using the standard ordering.

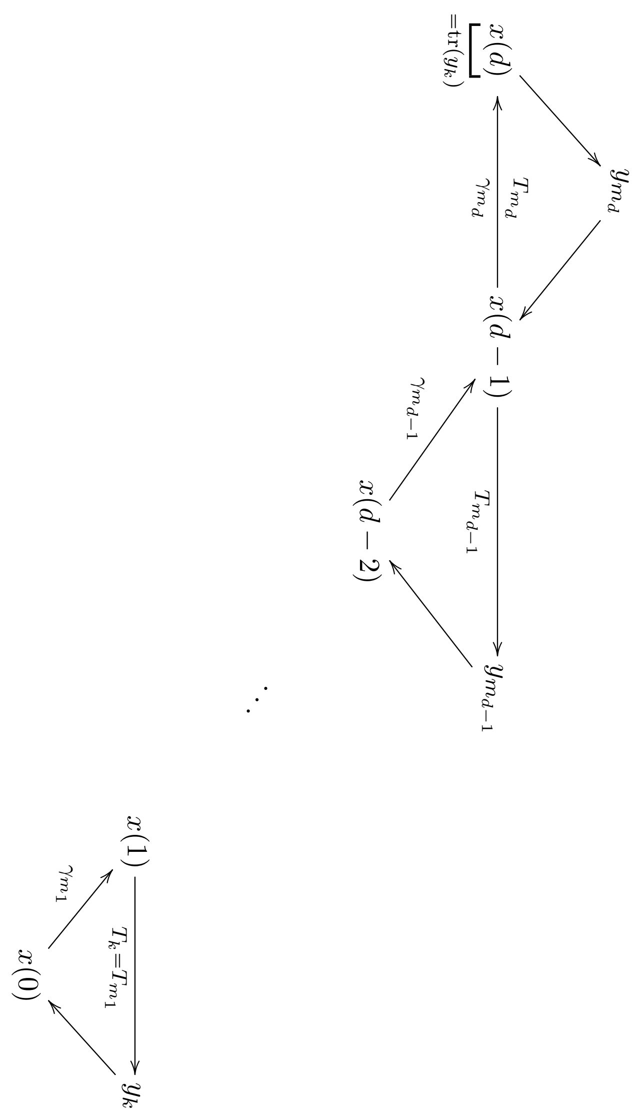


Figure 4.11: The sequence $(x(0), x(1), \ldots, x(d))$ defined by T_k where $\mathrm{sgn}(S_k) = -$. The transport of y_k is also illustrated for quivers where there is no sequence of the form in (4.2) of Definition 4.6.2.

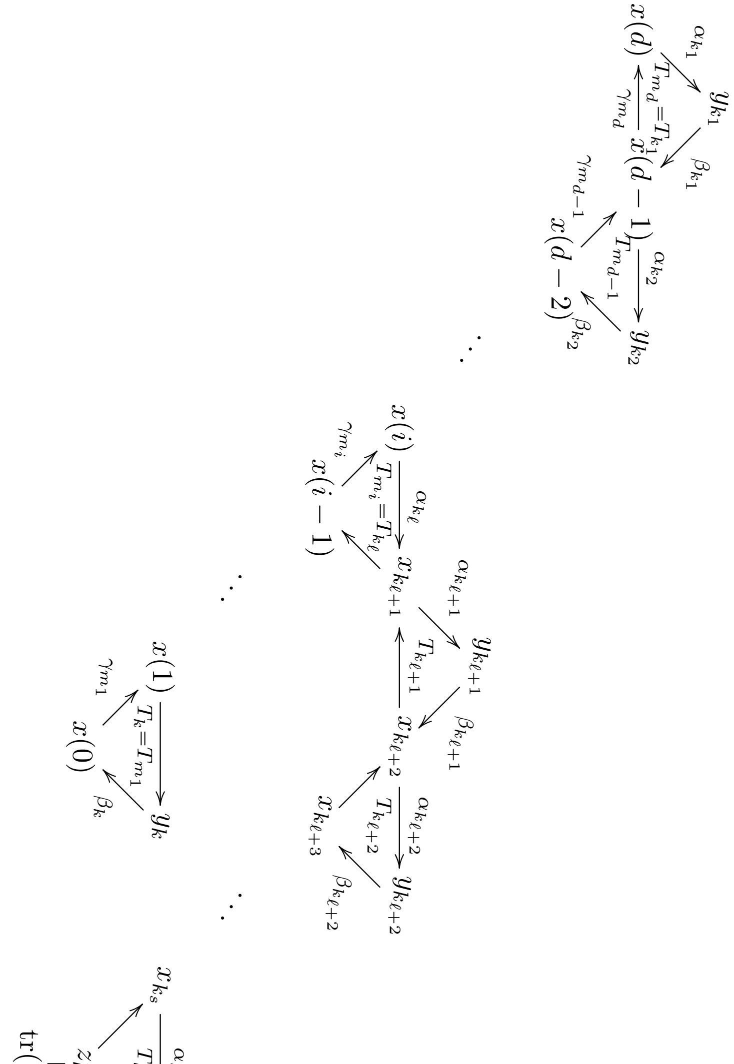


Figure 4.12: The sequence of arrows one follows to compute the transport of y_k. Note that in this case, the sequence A_1 is non-empty.

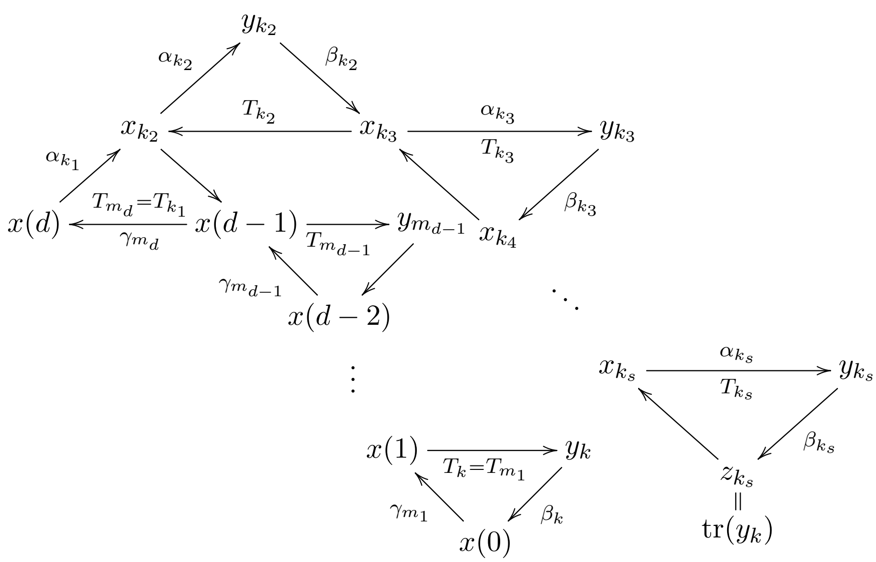


Figure 4.13: The sequence of arrows one follows to compute the transport of y_k. Note that in this case, the sequence A_1 is empty.

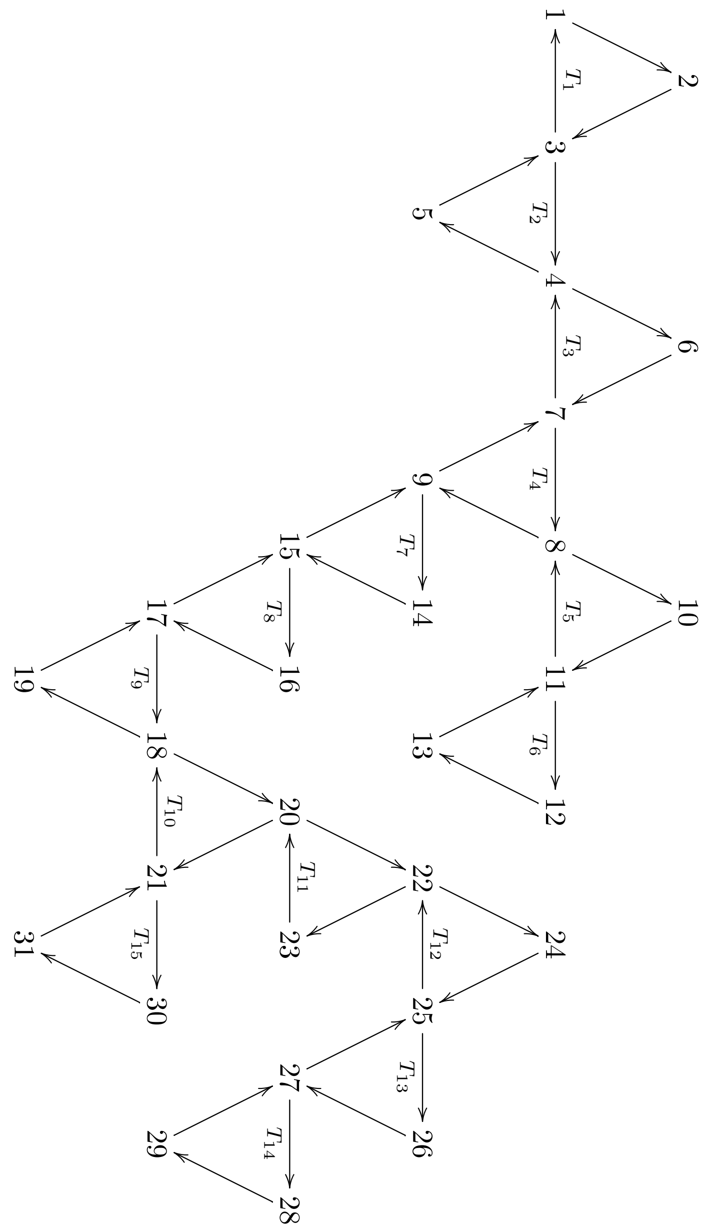


Figure 4.14: The signed irreducible type $\mathbb{A}_{31}$ quiver described in Example 4.6.4.

i	$\underline{\mu}_i$	i	$\underline{\mu}_i$
0	μ_1	8	$\mu_{13} \circ \mu_1 \circ \mu_7 \circ \mu_9 \circ \mu_{15} \circ \mu_{17} \circ \mu_{16}$
1	$\mu_1 \circ \mu_3 \circ \mu_2$	9	$\mu_{13} \circ \mu_1 \circ \mu_7 \circ \mu_9 \circ \mu_{15} \circ \mu_{17} \circ \mu_{19} \circ \mu_{18}$
2	$\mu_1 \circ \mu_3 \circ \mu_5 \circ \mu_4$	10	$\mu_{18} \circ \mu_{13} \circ \mu_{21} \circ \mu_{20}$
3	$\mu_4 \circ \mu_1 \circ \mu_7 \circ \mu_6$	11	$\mu_{20} \circ \mu_{18} \circ \mu_{23} \circ \mu_{22}$
4	$\mu_4 \circ \mu_1 \circ \mu_7 \circ \mu_9 \circ \mu_8$	12	$\mu_{22} \circ \mu_{20} \circ \mu_{25} \circ \mu_{24}$
5	$\mu_8 \circ \mu_4 \circ \mu_{11} \circ \mu_{10}$	13	$\mu_{22} \circ \mu_{20} \circ \mu_{25} \circ \mu_{27} \circ \mu_{26}$
6	$\mu_8 \circ \mu_4 \circ \mu_{11} \circ \mu_{13} \circ \mu_{12}$	14	$\mu_{22} \circ \mu_{20} \circ \mu_{25} \circ \mu_{27} \circ \mu_{29} \circ \mu_{28}$
7	$\mu_{13} \circ \mu_1 \circ \mu_7 \circ \mu_9 \circ \mu_{15} \circ \mu_{14}$	15	$\mu_{23} \circ \mu_{13} \circ \mu_{21} \circ \mu_{31} \circ \mu_{30}$

Figure 4.15: The associated mutation of the signed irreducible type $\mathbb{A}_{31}$ quiver in Figure 4.14.

$$Q \quad = \quad \begin{tikzcd} & 2 \arrow[dr] & & 4 \arrow[dr] & & 6 \arrow[dr] & \\ 1 \arrow[ur] & & 3 \arrow[ll] \arrow[ur] & & 5 \arrow[ll] \arrow[ur] & & 7. \arrow[ll] \end{tikzcd}$$

Figure 4.16: The type $\mathbb{A}_7$ quiver from Remark 4.6.6.

$$\mathcal{Q}_1 \quad = \quad \begin{tikzcd} & 2 \arrow[dr] & & & & \\ 1 \arrow[ur] & & 3 \arrow[ll] \arrow[rr] & & 4 \arrow[dl] & \\ & & & 5 \arrow[ul] \arrow[rr] & & 6 \arrow[dl] \\ & & & & 7 \arrow[ul] & \end{tikzcd} \qquad \begin{tikzcd} & 7 \arrow[dr] & & & & 1 \arrow[dr] & \\ 6 \arrow[ur] & & 5 \arrow[ll] \arrow[rr] & & 3 \arrow[dl] \arrow[ur] & & 2 \arrow[ll] \\ & & & 4 \arrow[ul] & & & \end{tikzcd} \quad = \quad \mathcal{Q}_2$$

Figure 4.17: The two signed irreducible type $\mathbb{A}$ quivers that can be obtained from Q.

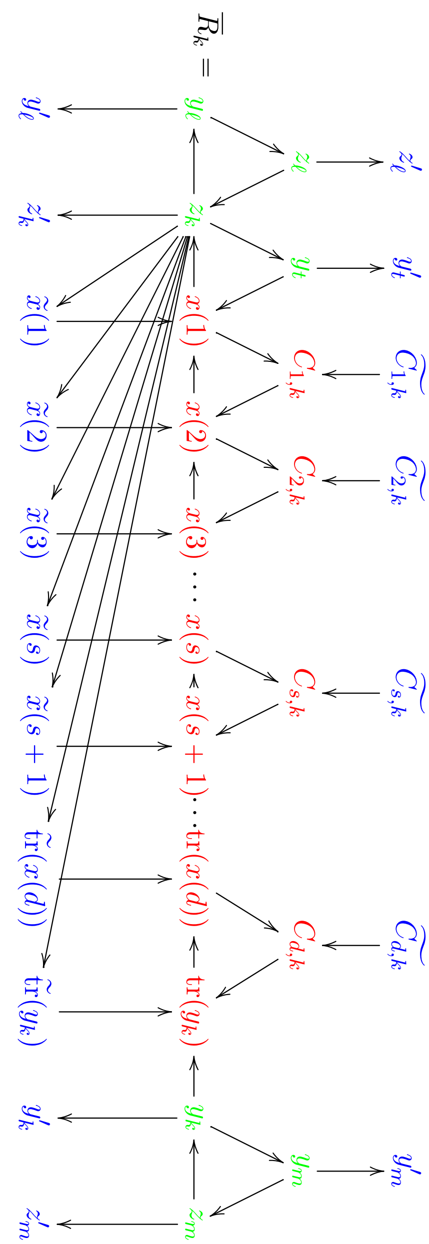


Figure 4.18: The local configuration around y_k and z_k just before $\underline{\mu}_k$ is applied.

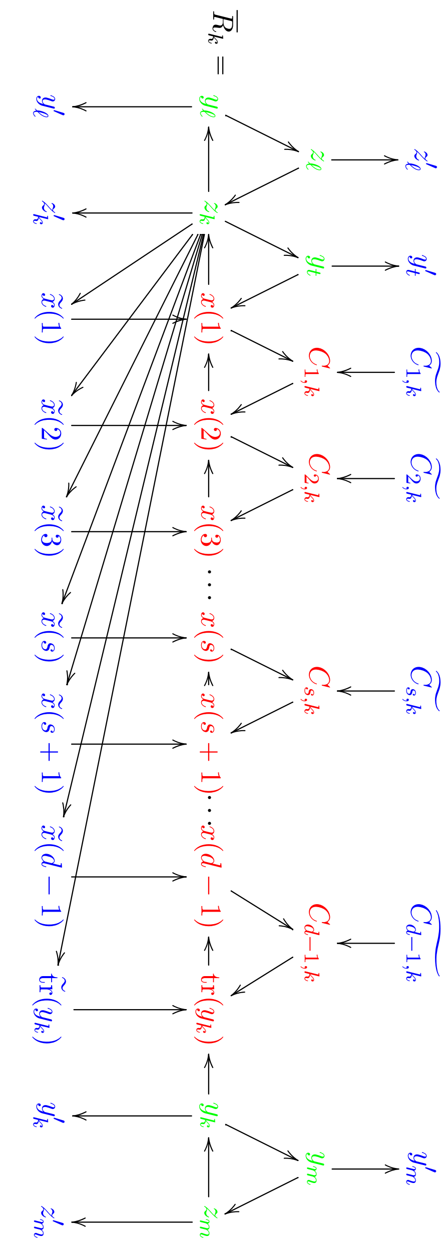


Figure 4.19: The local configuration in the special case when $x(d) = x_1$ since $\mathrm{tr}(x_1)$ is not defined.

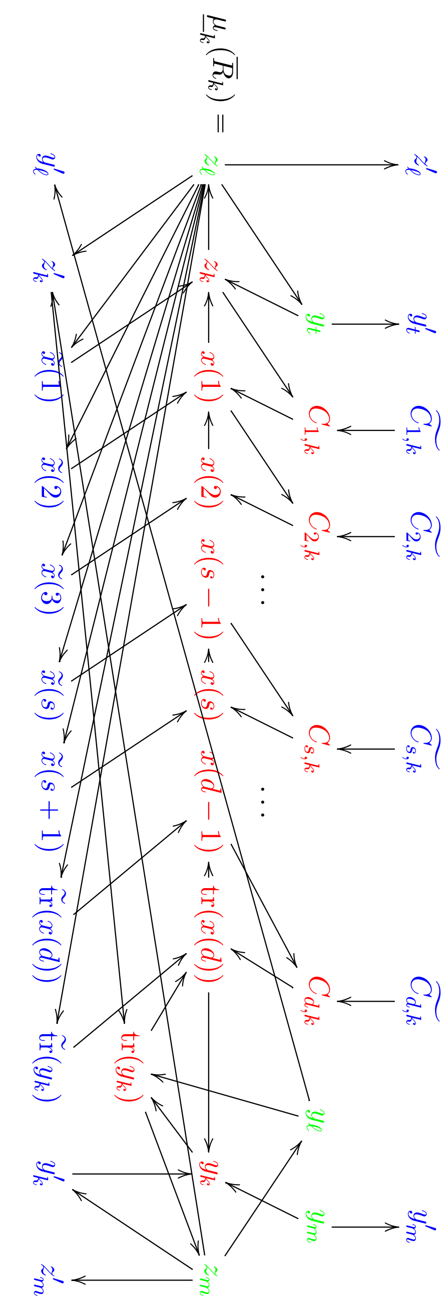


Figure 4.20: The quiver $\underline{\mu}_k(\overline{R}_k)$ before rearrangement.

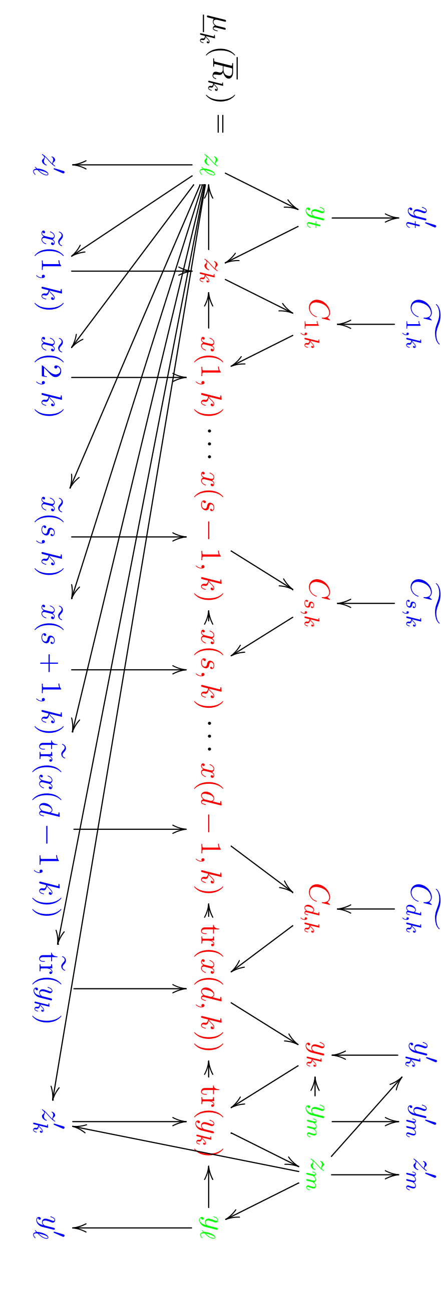


Figure 4.21: The quiver $\underline{\mu}_k(\overline{R}_k)$ rearranged to look more like $\overline{R}_{k+1}$.

$$
\begin{tikzcd}
 & & z_2' & & & & & \\
 & & z_2 \arrow[u] \arrow[dr] & & & & & \\
R = & y_2 \arrow[ur] \arrow[d] & & z_1 \arrow[ll] \arrow[d] \arrow[drr] & & x_1 \arrow[ll] & & y_1 \arrow[ll] \arrow[d] \\
 & y_2' & & z_1' & & x_1' \arrow[u] & & y_1'
\end{tikzcd}
$$

Figure 4.22: The subquiver $R = \overline{R}_1$ of $\underline{\mu}_0(\widehat{Q})$.

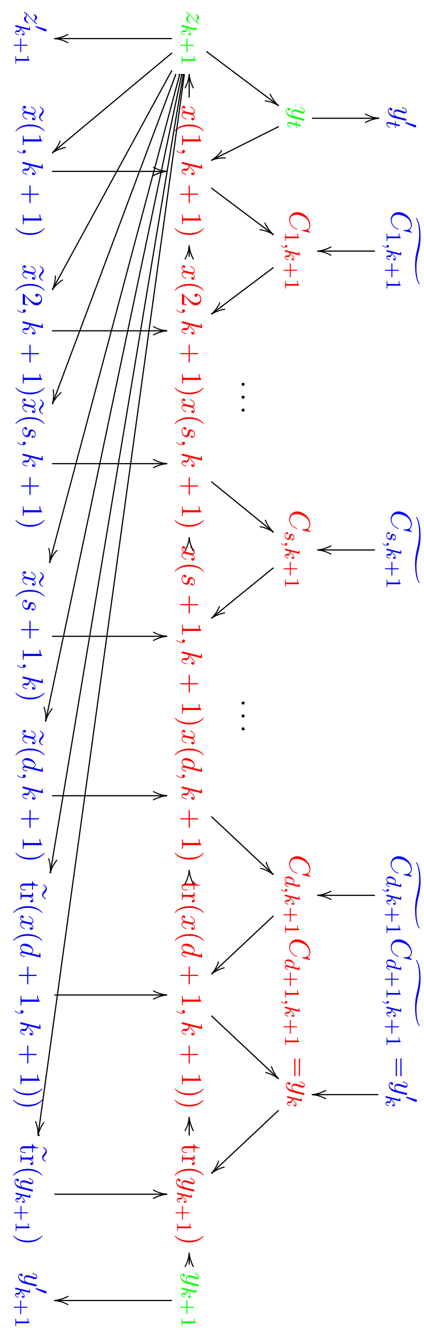


Figure 4.23: The quiver $\underline{\mu}_k(\overline{R}_k)$ obtained by mutating $\overline{R}_k$ in Case i).

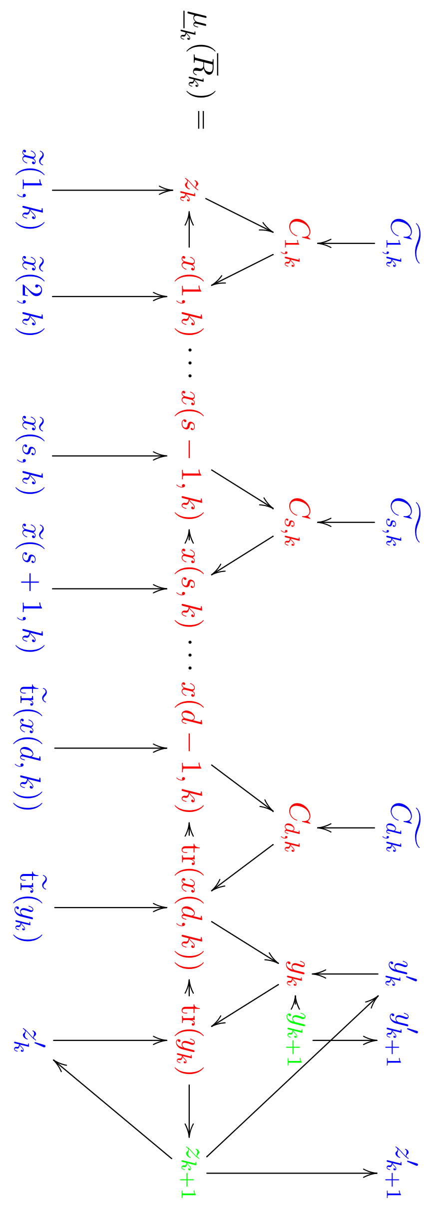


Figure 4.24: The quiver obtained by mutating $\overline{R}_k$ in Case ii).

$$
\begin{tikzcd}
 & z'_p & & & \widetilde{C_{1,k+1}} = x'_k \arrow[d] & & & y'_q & \\
 & z_p \arrow[u] \arrow[dr] & & & C_{1,k+1} = x_k \cdot \sigma_k^{-1} \arrow[dr] & & & y_q \arrow[u] \arrow[dr] & \\
R' = y_p \arrow[ur] \arrow[d] & & z_{k+1} \arrow[ll] \arrow[d] \arrow[dr] \arrow[drrr] & \mathrm{tr}(x(1,k+1)) \arrow[l] \arrow[ur] & & \mathrm{tr}(y_{k+1}) \arrow[ll] & y_{k+1} \arrow[l] \arrow[ur] \arrow[d] & & z_q \arrow[ll] \arrow[d] \\
y'_p & & z'_{k+1} & z'_k \arrow[u] & & y'_k \arrow[u] & y'_{k+1} & & z'_q
\end{tikzcd}
$$

Figure 4.25: The quiver $R' = \overline{R}_{k+1}$ that we obtain in Case ii).

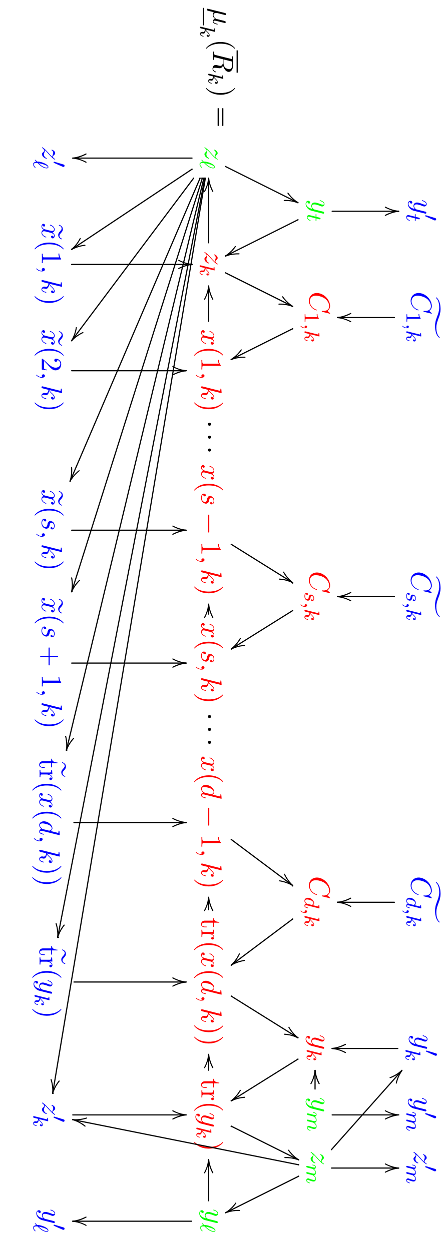


Figure 4.26: The quiver obtained by mutating $\overline{R}_k$ in Case iii).

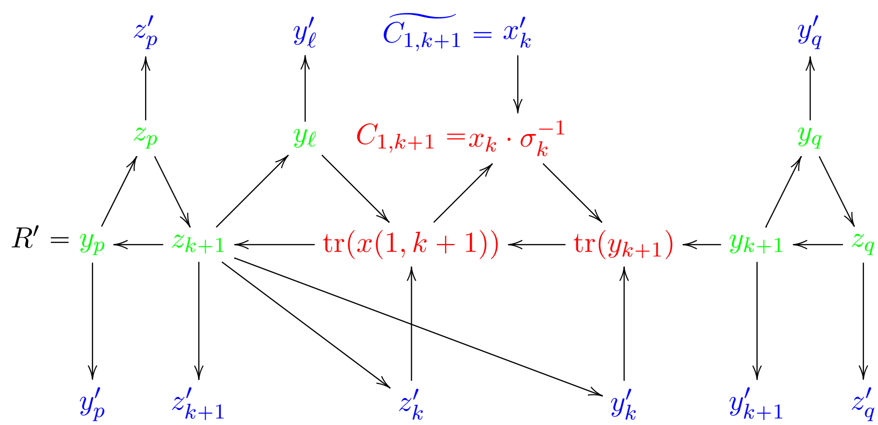


Figure 4.27: The quiver $\overline{R} = \overline{R}_{k+1}$ that we obtain in Case iii).

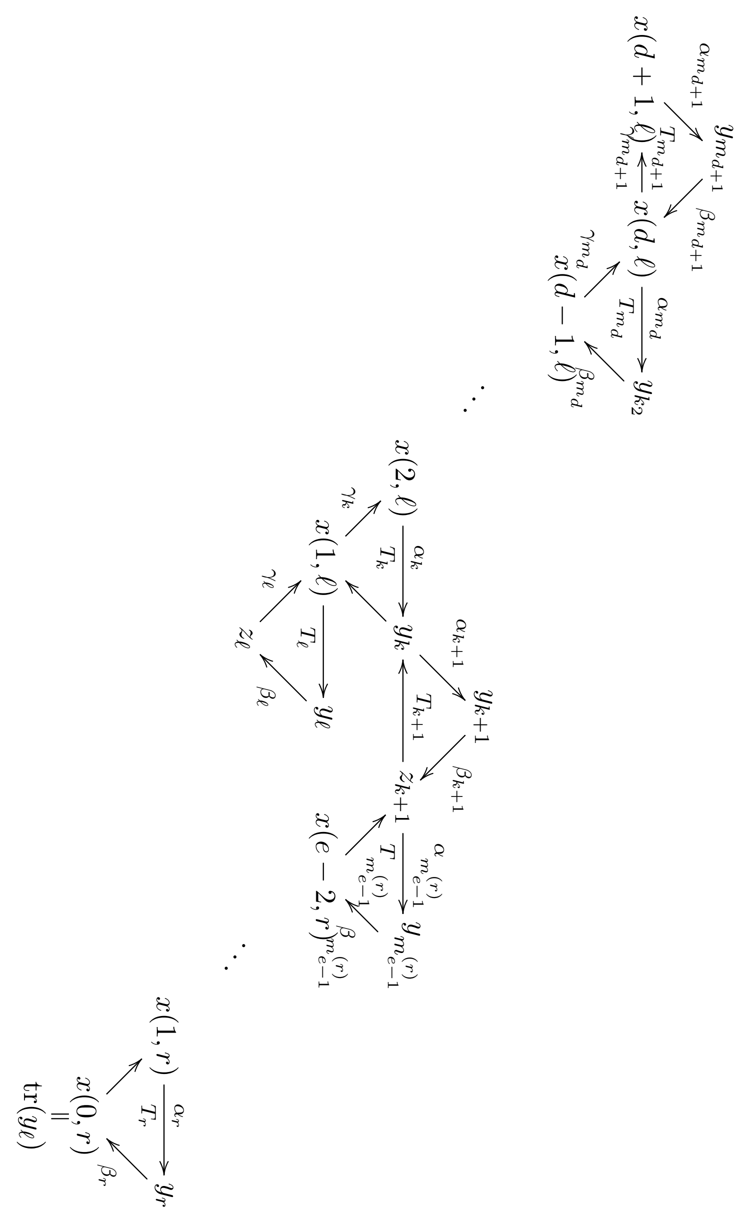


Figure 4.28: A full subquiver of $\mathcal{Q}$ showing one possible configuration of the signed 3-cycles T_k, T_{k+1}, and T_ℓ, as described in the proof of Lemma 4.7.2 at the end of Case iii).

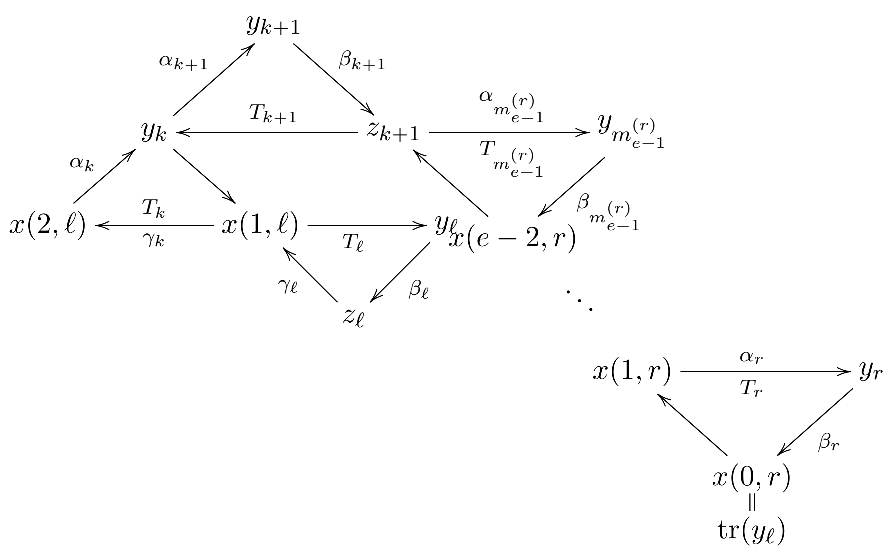


Figure 4.29: A full subquiver of $\mathcal{Q}$ showing the other possible configuration of the signed 3-cycles T_k, T_{k+1}, and T_ℓ, as described in the proof of Lemma 4.7.2 at the end of Case iii).

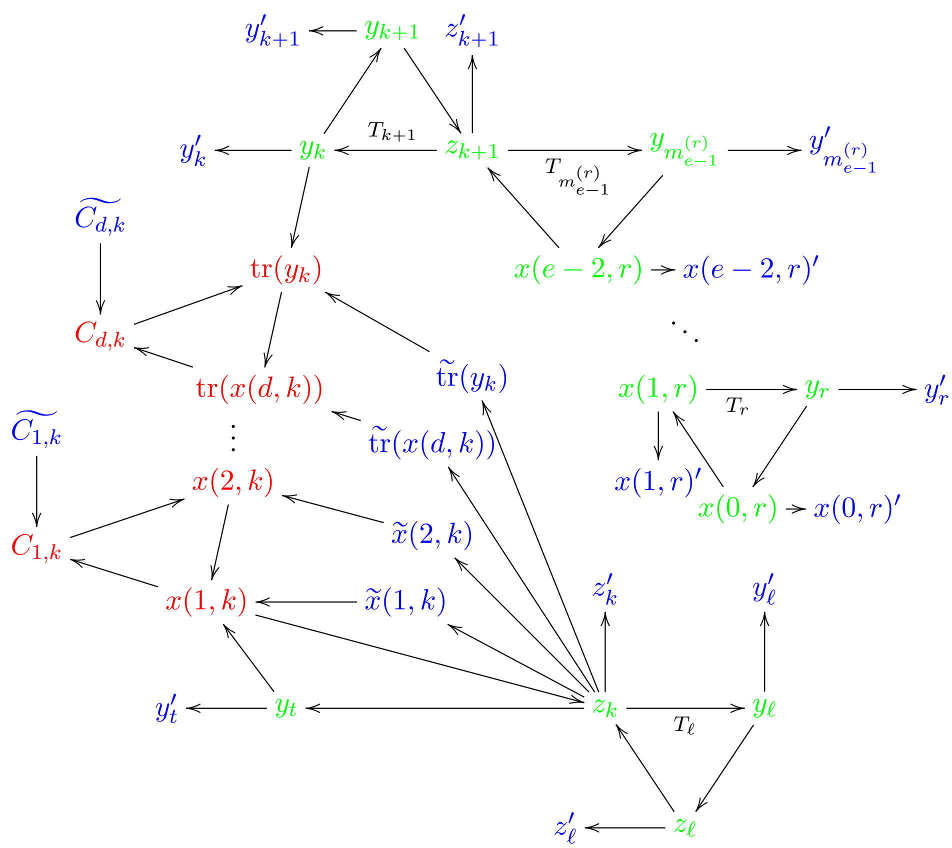


Figure 4.30: The full subquiver of $\underline{\mu}_{k-1} \circ \cdots \circ \underline{\mu}_1 \circ \underline{\mu}_0(\widehat{Q})$ on the vertices and frozen vertices shown here.

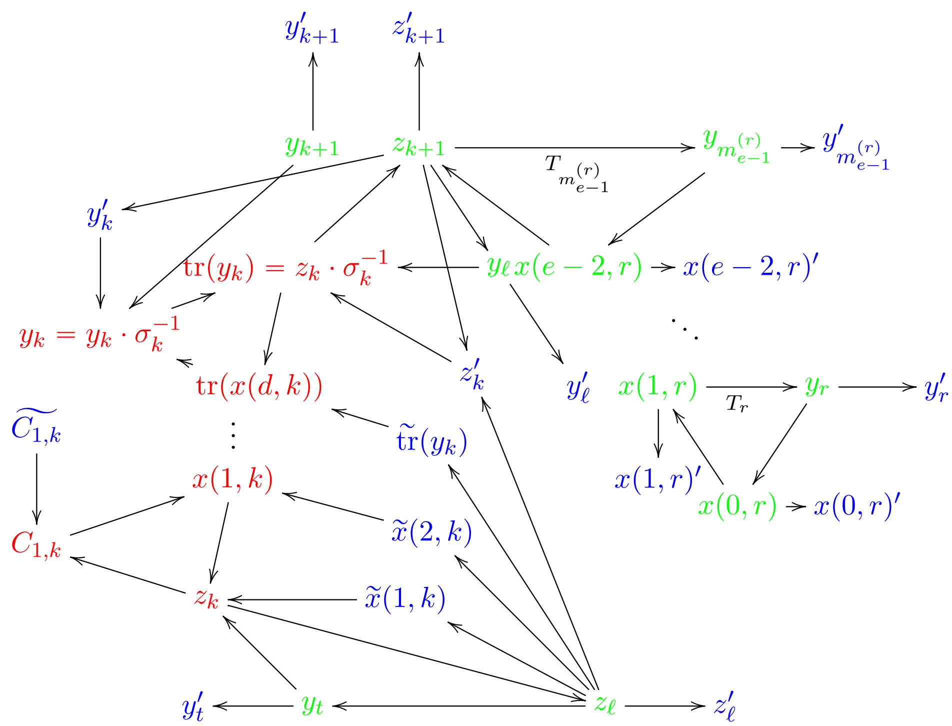


Figure 4.31: The full subquiver of $\underline{\mu}_k \circ \cdots \circ \underline{\mu}_1 \circ \underline{\mu}_0(\widehat{Q})$ on the vertices and frozen vertices shown here.

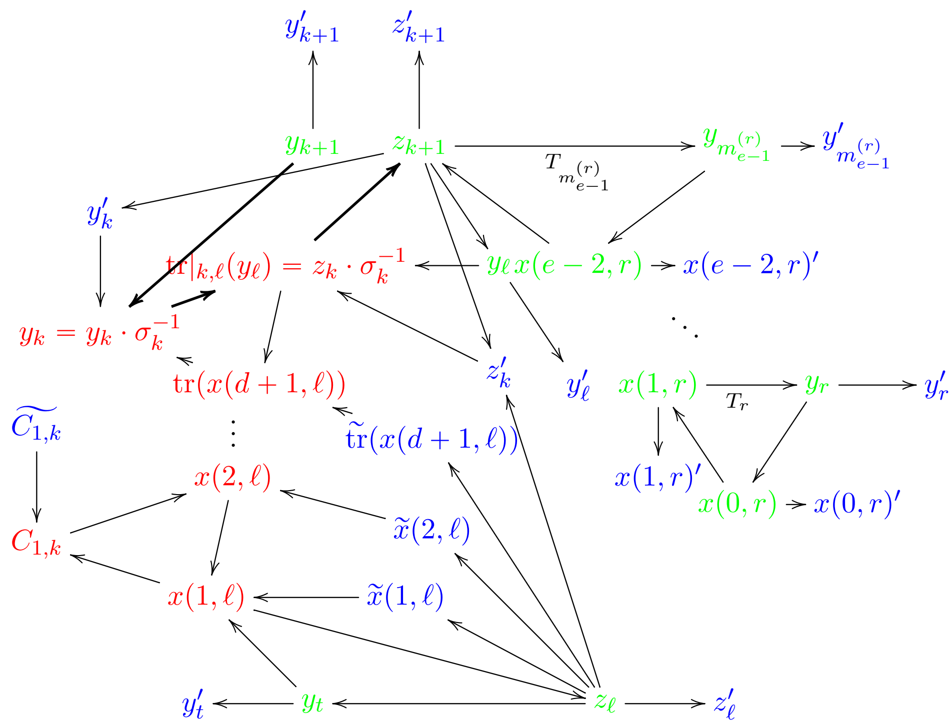


Figure 4.32: The quiver that appears in Figure 4.31 with its vertex labels updated so that the part of $\overline{R}_{k+1,\ell}$ that appears here looks like the corresponding part of the quiver $\overline{R}_\ell$.

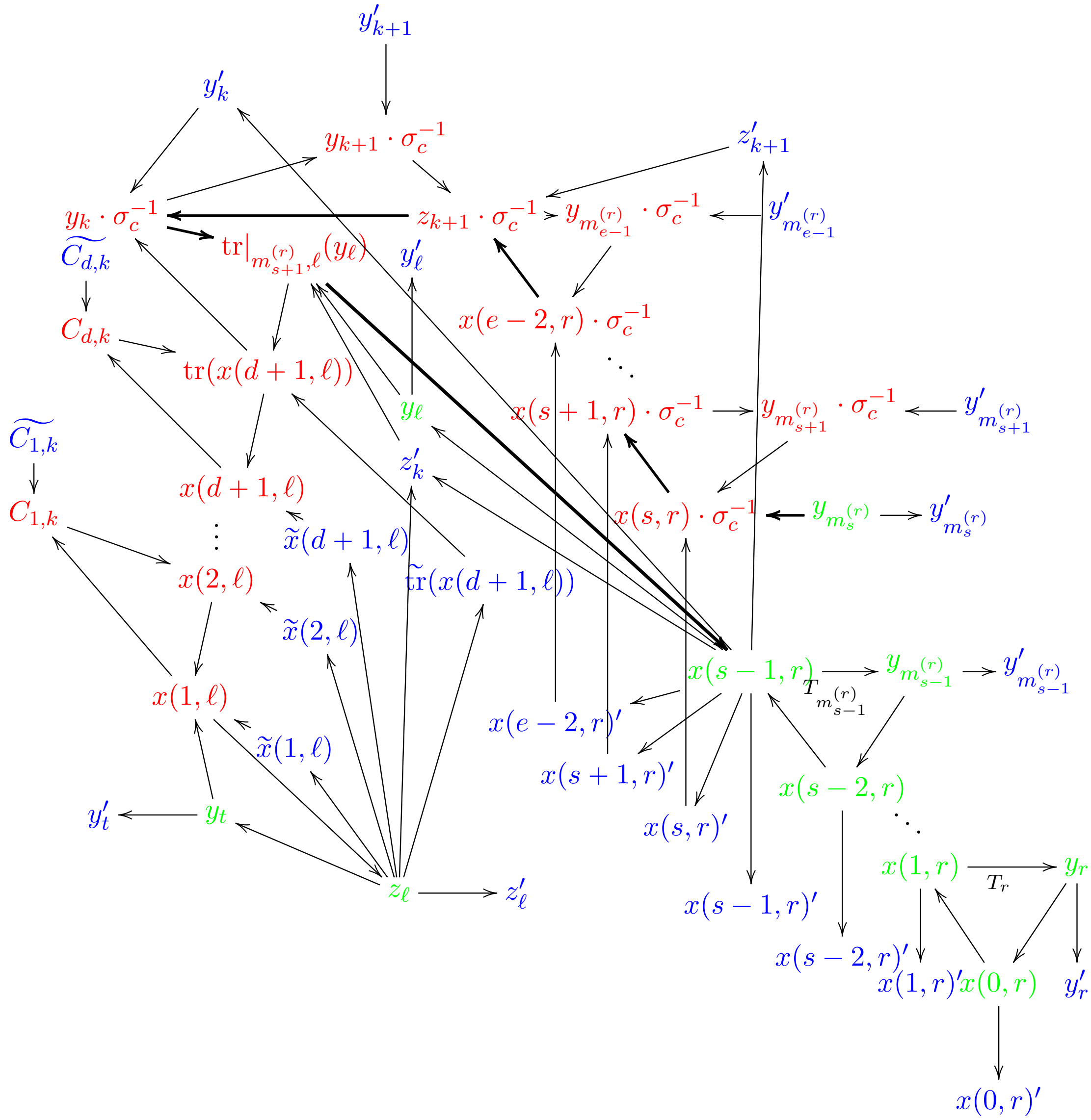


Figure 4.33: The effect of applying $\underline{\mu}_c \circ \cdots \circ \underline{\mu}_{k+1}$ to $\underline{\mu}_k \circ \cdots \circ \underline{\mu}_1 \circ \underline{\mu}_0(\widehat{Q})$ where $m_{s+1}^{(r)} \leqslant c < m_s^{(r)}$. If $c = m_s^{(r)} - 1$, the mutation sequence $\underline{\mu}_{m_s^{(r)}}$ is indicated by the bold arrows. Note that, as in the statement of Lemma 4.7.5, we have that $\mathrm{tr}|_{m_{s+1}^{(r)},\ell}(y_\ell) = z_k \cdot \sigma_c^{-1}$.

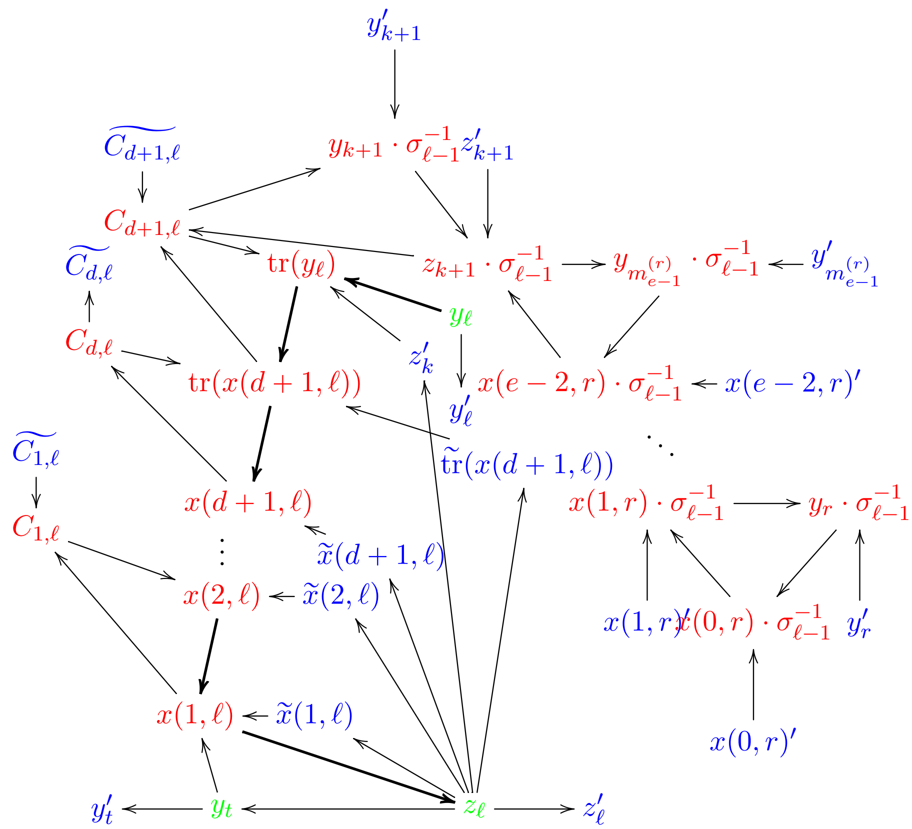


Figure 4.34: The effect of applying $\underline{\mu}_{\ell-1} \circ \cdots \circ \underline{\mu}_{k+1}$ to $\underline{\mu}_k \circ \cdots \circ \underline{\mu}_1 \circ \underline{\mu}_0(\widehat{Q})$ where $C_{d+1,\ell} = y_k \cdot \sigma_{\ell-1}^{-1}$ and $\widetilde{C_{d+1,\ell}} = y'_k$, as desired. The mutation sequence $\underline{\mu}_\ell$ is indicated by the bold arrows.

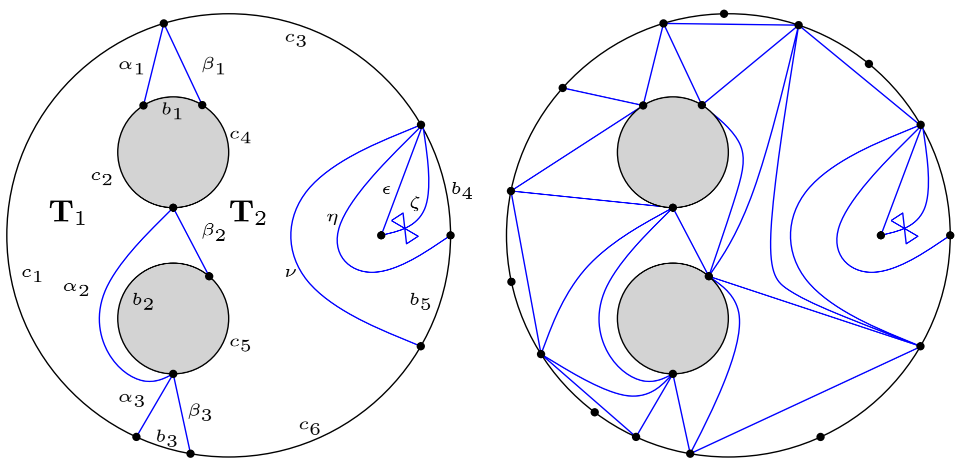


Figure 4.35:

$$Q_{\mathbf{T}_1} \quad = \quad \begin{tikzcd}
 & y_1^{(1)} \arrow[dr] & & & & y_3^{(1)} \arrow[dr] & & & v_{\alpha_1} \arrow[d] \\
x_1^{(1)} \arrow[ur] & & z_1^{(1)} \arrow[ll, "T_1^{(1)}"] \arrow[rr, "T_2^{(1)}"'] & & y_2^{(1)} \arrow[ur] \arrow[dl] & & z_3^{(1)} \arrow[ll, "T_3^{(1)}"] \arrow[r] & w_1 & w_2 \arrow[l] \\
 & & & z_2^{(1)} \arrow[ul] & & & & &
\end{tikzcd}$$

$$Q_{\mathbf{T}_2} \quad = \quad \begin{tikzcd}
 & & & & & & y_4^{(2)} \arrow[dr] & & & \\
 & y_1^{(2)} \arrow[dr] & & & & y_3^{(2)} \arrow[ur] \arrow[dr] & & z_4^{(2)} \arrow[ll, "T_4^{(2)}"] & & \\
x_1^{(2)} \arrow[ur] & & z_1^{(2)} \arrow[ll, "T_1^{(2)}"] \arrow[rr, "T_2^{(2)}"'] & & y_2^{(2)} \arrow[ur] \arrow[dl] & & z_3^{(2)} \arrow[ll, "T_3^{(2)}"] \arrow[rr, "T_5^{(2)}"'] & & y_5^{(2)} \arrow[dl] & \\
 & & & z_2^{(2)} \arrow[ul] \arrow[rr] & & w_3 & & z_5^{(2)} \arrow[ul] \arrow[rr] & & w_4
\end{tikzcd}$$

Figure 4.36:

Chapter 5

Combinatorics of Exceptional Sequences

5.1 Introduction

We now shift our focus to the connection between vertices of the oriented exchange graphs defined by type $\mathbb{A}$ Dynkin quivers and exceptional sequences of quiver representations. Exceptional sequences are certain sequences of quiver representations with strong homological properties. They were introduced in [GR87] to study exceptional vector bundles on $\mathbb{P}^2$, and more recently, Crawley-Boevey showed that the braid group acts transitively on the set of complete exceptional sequences (exceptional sequences of maximal length) [CB93]. This result was generalized to hereditary Artin algebras by Ringel [Rin94]. Since that time, Meltzer has also studied exceptional sequences for weighted projective lines [Mel04], and Araya for Cohen-Macaulay modules over one dimensional graded Gorenstein rings with a simple singularity [Ara99]. Exceptional sequences have been shown to be related to many other areas of mathematics since their invention:

- chains in the lattice of noncrossing partitions [Bes03, HK13, IT09],
- factorizations of Coxeter elements [IS10], and
- t-structures and derived categories [Bez03, BK89, Rud90].

Despite their ubiquity, very little work has been done to concretely describe exceptional sequences, even for path algebras of Dynkin quivers [Ara13, GM15]. In this paper, we give a concrete classification of exceptional sequences of representations of quivers whose underlying graph is a type $\mathbb{A}_n$ Dynkin diagram. We will refer to such quivers as type $\mathbb{A}_n$ Dynkin quivers. This work extends a classification of exceptional sequences for the linearly-ordered quiver obtained in [GM15] by the first and third authors.

Exceptional sequences are composed of indecomposable representations which have a particularly nice description. For a Dynkin quiver Q_ϵ of type $\mathbb{A}_n$, where ϵ is a vector that encodes the orientation of the quiver, the indecomposable representations are completely determined by their dimension vectors, which are of the form $(0, ..., 0, 1, ..., 1, 0, ..., 0)$. Let us denote such a representation by $X^\epsilon_{i,j}$, where $i+1$ and j are the positions where the string of 1's begins and ends, respectively. The path algebra $\Bbbk Q_\epsilon$ is an example of a string algebra and so $X^\epsilon_{i,j}$ is a string module. However, it is simpler in this setting to use the notation $X^\epsilon_{i,j}$ rather than expressing this module as $M(w)$ for some string w.

This simple description allows us to view exceptional sequences as combinatorial objects. We define a map Φ_ϵ which associates to each indecomposable representation $X^\epsilon_{i,j}$ an isotopy class of simple curves in the plane whose endpoints we think of as i and j and whose path between these points is dictated by ϵ. We refer to such curves as **strands**.

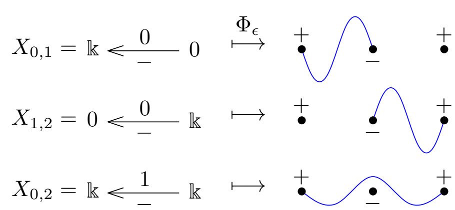


Figure 5.1: The indecomposable representations of $Q = 1 \leftarrow 2$ and their representations as strands

As exceptional sequences are collections of representations, the map Φ_ϵ allows one to regard them as collections of strands. The following lemma is the foundation for all of our results in this paper (it characterizes the homological data encoded by a pair of strands and thus by a pair of representations). Since exceptional sequences are sequences of representations, each pair of which satisfy certain homological properties, Lemma 5.3.5 allows us to completely classify exceptional sequences using strand diagrams.

Lemma 5.1.1. *Let Q_ϵ be a Dynkin quiver of type $\mathbb{A}_n$ and let U and V be two distinct indecomposable representations of Q_ϵ.*

a) The strands $\Phi_\epsilon(U)$ and $\Phi_\epsilon(V)$ intersect nontrivially if and only if neither (U, V) nor (V, U) are exceptional pairs.

b) The strand $\Phi_\epsilon(U)$ is clockwise from $\Phi_\epsilon(V)$ if and only if (U, V) is an exceptional pair and (V, U) is not an exceptional pair.

c) The strands $\Phi_\epsilon(U)$ and $\Phi_\epsilon(V)$ do not intersect at any of their endpoints and they do not intersect nontrivially if and only if (U, V) and (V, U) are both exceptional pairs.

The chapter is organized in the following way. In Section 5.2, we give the preliminaries on exceptional sequences of quiver representations which are needed for the rest of the paper.

In Section 5.3.1, we introduce strand diagrams, which we will use to model collections of indecomposable representations. We will decorate our strand diagrams with strand-labelings and oriented edges so that they can keep track of both the ordering of the representations in a complete exceptional sequence as well as the signs of the rows in the **c**-matrix it came from. While unlabeled diagrams classify complete exceptional collections (Theorem 5.3.6), we show that the new decorated diagrams classify more complicated objects called exceptional sequences (Theorem 5.3.9). Although Lemma 5.3.5 is the main tool that allows us to obtain these results, we delay its proof to Section 5.3.2.

The work of Speyer and Thomas (see [ST13]) allows complete exceptional sequences to be obtained from **c**-matrices. In [ONA$^+$13], the number of complete exceptional sequences in type $\mathbb{A}_n$ is given, and there are more of these than there are **c**-matrices. Thus, it is natural to ask exactly which **c**-matrices appear as strand diagrams. By establishing a bijection between the mixed cobinary trees of Igusa and Ostroff [IO13] and a certain subcollection of strand diagrams, we give an answer to this question in Section 5.4.

In Section 5.5, we ask how many complete exceptional sequences can be formed using the representations in a complete exceptional collection. We interpret this number as the number of linear extensions of the poset determined by the chord diagram of the complete exceptional collection. This also gives an interpretation of complete exceptional sequences as linear extensions.

In Section 5.6, we give several applications of the theory in type $\mathbb{A}$, including combinatorial proofs that two reddening sequences produce isomorphic ice quivers (see [Kel12] for a general proof in all types using deep category-theoretic techniques) and that there is a bijection between exceptional sequences and certain chains in the lattice of noncrossing partitions.

5.2 Preliminaries

In this section, we recall some basic definitions. We will be interested in the connection between exceptional sequences and the **c**-matrices of an acyclic quiver Q so we begin by defining exceptional sequences. which serve as the starting point in our study of exceptional sequences. We then explain the notation we will use to discuss exceptional representations of quivers that are orientations of a type $\mathbb{A}_n$ Dynkin diagram.

5.2.1 Exceptional sequences of representations

An **exceptional sequence** $\xi = (V_1, \ldots, V_k)$ $(k \leqslant n := \#Q_0)$ is an ordered *list* of **exceptional representations** V_j of Q (i.e. V_j is indecomposable and $\mathrm{Ext}^s_{\Bbbk Q}(V_j, V_j) = 0$ for all $s \geqslant 1$) satisfying $\mathrm{Hom}_{\Bbbk Q}(V_j, V_i) = 0$ and $\mathrm{Ext}^s_{\Bbbk Q}(V_j, V_i) = 0$ if $i < j$ for all $s \geqslant 1$. We define an **exceptional collection** $\overline{\xi} = \{V_1, \ldots, V_k\}$ to be a *set* of exceptional representations V_j of Q that can be ordered in such a way that they define an exceptional sequence. When $k = n$, we say ξ (resp. $\overline{\xi}$) is a **complete exceptional sequence** (CES) (resp. **complete exceptional collection** (CEC)). For Dynkin quivers, a representation is exceptional if and only if it is indecomposable.

The following result of Speyer and Thomas gives a beautiful connection between **c**-matrices of an acyclic quiver Q and CESs. It serves as motivation for our work. Before stating it we remark that for any $R \in ET(\widehat{Q})$ and any $i \in [n]$ where Q is an acyclic quiver, the **c**-vector $\overrightarrow{c_i} = \overrightarrow{c_i}(R) = \pm\underline{\dim}(V_i)$ for some exceptional representation of Q (see [Cha12]). In general, not all indecomposable representations are exceptional. The **c**-vectors are exactly the dimension vectors of the exceptional modules and their negatives.

Notation 5.2.1. *Let $\overrightarrow{c}$ be a **c**-vector of an acyclic quiver Q. Define*

$$|\overrightarrow{c}| \quad := \quad \begin{cases} \overrightarrow{c} & : \text{ if } \overrightarrow{c} \text{ is positive} \\ -\overrightarrow{c} & : \text{ if } \overrightarrow{c} \text{ is negative.} \end{cases}$$

Theorem 5.2.2 ([ST13])**.** *Let $C \in$ **c**-mat(Q), let $\{\overrightarrow{c_i}\}_{i\in[n]}$ denote the **c**-vectors of C, and let $|\overrightarrow{c_i}| = \underline{\dim}(V_i)$ for some indecomposable representation of Q. There exists a permutation $\sigma \in \mathfrak{S}_n$ such that $(V_{\sigma(1)}, ..., V_{\sigma(n)})$ is a CES with the property that if there exist positive **c**-vectors in C, then there exists $k \in [n]$ such that $\overrightarrow{c_{\sigma(i)}}$ is positive if and only if $i \in [k, n]$, and $\mathrm{Hom}_{\Bbbk Q}(V_i, V_j) = 0$ if $\overrightarrow{c_i}, \overrightarrow{c_j}$ have the same sign. Conversely, any set of n vectors having these properties defines a **c**-matrix whose rows are $\{\overrightarrow{c_i}\}_{i\in[n]}$.*

5.2.2 Quivers of Dynkin type $\mathbb{A}_n$

For the purposes of this paper, we will only be concerned with quivers of Dynkin type $\mathbb{A}_n$. We say a quiver Q is of **Dynkin type** $\mathbb{A}_n$ if the underlying graph of Q is a Dynkin diagram of type $\mathbb{A}_n$. By convention, two vertices i and j with $i < j$ in a type $\mathbb{A}_n$ Dynkin quiver Q are connected by an arrow if and only if $j = i + 1$ and $i \in [n-1]$.

It will be convenient to denote a given type $\mathbb{A}_n$ Dynkin quiver Q using the notation Q_ϵ, which we now define. Let $\epsilon = (\epsilon_0, \epsilon_1, \ldots, \epsilon_n) \in \{+,-\}^{n+1}$ and for $i \in [n-1]$ define $a_i^{\epsilon_i} \in Q_1$ by

$$a_i^{\epsilon_i} \quad := \quad \begin{cases} i \leftarrow i+1 & : \quad \epsilon_i = - \\ i \rightarrow i+1 & : \quad \epsilon_i = +. \end{cases}$$

Then $Q_\epsilon := ((Q_\epsilon)_0 := [n], (Q_\epsilon)_1 := \{a_i^{\epsilon_i}\}_{i\in[n-1]}) = Q$. One observes that the values of ϵ_0 and ϵ_n do not affect Q_ϵ.

Example 5.2.3. *Let $n = 5$ and $\epsilon = (-,+,-,+,-,+)$ so that $Q_\epsilon = 1 \xrightarrow{a_1^+} 2 \xleftarrow{a_2^-} 3 \xrightarrow{a_3^+} 4 \xleftarrow{a_4^-} 5$. Below we show its framed quiver $\widehat{Q}_\epsilon$.*

Let Q_ϵ be the quiver where $\epsilon = (\epsilon_0, \epsilon_1, \ldots, \epsilon_n) \in \{+,-\}^{n+1}$. Let $i, j \in [0, n] := \{0, 1, \ldots, n\}$ where $i < j$ and let $X_{i,j}^\epsilon = ((V_\ell)_{\ell\in(Q_\epsilon)_0}, (\varphi_a^{i,j})_{a\in(Q_\epsilon)_1}) \in \mathrm{rep}_\Bbbk(Q_\epsilon)$ be the indecomposable representation defined by

$$\widehat{Q}_\epsilon = \begin{tikzcd} 6 & 7 & 8 & 9 & 10 \\ 1 \arrow[u] \arrow[r] & 2 \arrow[u] & 3 \arrow[u] \arrow[l] \arrow[r] & 4 \arrow[u] & 5 \arrow[u] \arrow[l] \end{tikzcd}$$

Figure 5.2: An example of a framed quiver

$$V_\ell := \begin{cases} \mathbb{k} & : \quad i+1 \leqslant \ell \leqslant j \\ 0 & : \quad \text{otherwise} \end{cases}$$
$$\varphi_a^{i,j} := \begin{cases} 1 & : \quad a = a_k^{\epsilon_k} \text{ where } i+1 \leqslant k \leqslant j-1 \\ 0 & : \quad \text{otherwise.} \end{cases}$$

The objects of $\text{ind}(\text{rep}_{\mathbb{k}}(Q_\epsilon))$ are those of the form $X_{i,j}^\epsilon$ where $0 \leqslant i < j \leqslant n$, up to isomorphism.

5.3 Strand diagrams

In this section, we define three different types of combinatorial objects: strand diagrams, labeled strand diagrams, and oriented strand diagrams. We will use these objects to classify exceptional collections, exceptional sequences, and **c**-matrices of Dynkin type $\mathbb{A}_n$ quivers. Throughout this section, we work with a given Dynkin type $\mathbb{A}_n$ quiver Q_ϵ.

5.3.1 Exceptional sequences and strand diagrams

Let $\mathcal{S}_{n,\epsilon} \subset \mathbb{R}^2$ be a collection of $n+1$ points arranged in a horizontal line. We identify these points with $\epsilon_0, \epsilon_1, \ldots, \epsilon_n$ where ϵ_j appears to the right of ϵ_i for any $i, j \in [0, n] := \{0, 1, 2, \ldots, n\}$ where $i < j$. Using this identification, we can write $\epsilon_i = (x_i, y_i) \in \mathbb{R}^2$.

Definition 5.3.1. *Let $i, j \in [0, n]$ where $i \neq j$. A **strand** $c(i, j)$ on $\mathcal{S}_{n,\epsilon}$ is an isotopy class of simple curves in $\mathbb{R}^2$ where any $\gamma \in c(i, j)$ satisfies:*

a) the endpoints of γ are ϵ_i and ϵ_j,

b) as a subset of $\mathbb{R}^2$, $\gamma \subset \{(x, y) \in \mathbb{R}^2 : x_i \leqslant x \leqslant x_j\} \backslash \{\epsilon_{i+1}, \epsilon_{i+2}, \ldots, \epsilon_{j-1}\}$,

c) if $k \in [0, n]$ satisfies $i \leqslant k \leqslant j$ and $\epsilon_k = +$ (resp. $\epsilon_k = -$), then γ is locally below (resp. above) ϵ_k.

There is a natural map Φ_ϵ from $\text{ind}(\text{rep}_{\mathbb{k}}(Q_\epsilon))$ to the set of strands on $\mathcal{S}_{n,\epsilon}$ given by $\Phi_\epsilon(X_{i,j}^\epsilon) := c(i, j)$.

Remark 5.3.2. *It is clear that any strand can be represented by a monotone curve* $\gamma \in c(i,j)$ *(i.e. if* $t, s \in [0,1]$ *and* $t < s$*, then* $\gamma^{(1)}(t) < \gamma^{(1)}(s)$ *where* $\gamma^{(1)}$ *denotes the* ***x-coordinate function*** *of* γ*.).*

We say that two strands $c(i_1, j_1)$ and $c(i_2, j_2)$ **intersect nontrivially** if any two curves $\gamma_\ell \in c(i_\ell, j_\ell)$ with $\ell \in \{1, 2\}$ have at least one transversal crossing. Otherwise, we say that $c(i_1, j_1)$ and $c(i_2, j_2)$ **do not intersect nontrivially**. For example, $c(1,3), c(2,4)$ intersect nontrivially if and only if $\epsilon_2 = \epsilon_3$. Additionally, we say that $c(i_2, j_2)$ is **clockwise** from $c(i_1, j_1)$ (or, equivalently, $c(i_1, j_1)$ is **counterclockwise** from $c(i_2, j_2)$) if and only if any $\gamma_1 \in c(i_1, j_1)$ and $\gamma_2 \in c(i_2, j_2)$ share an endpoint ϵ_k and locally appear in one of the following six configurations up to isotopy.

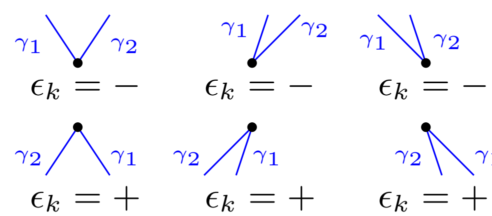


Figure 5.3: Configurations for the strand $c(i_2, j_2)$ to be clockwise from $c(i_1, j_1)$

Definition 5.3.3. *A* ***strand diagram*** $d = \{c(i_\ell, j_\ell)\}_{\ell \in [k]}$ *(*$k \leqslant n$*) on* $\mathcal{S}_{n,\epsilon}$ *is a collection of strands on* $\mathcal{S}_{n,\epsilon}$ *that satisfies the following conditions:*

a) distinct strands do not intersect nontrivially, and

b) the graph determined by d *is a* ***forest*** *(i.e. a disjoint union of trees)*

Let $\mathcal{D}_{k,\epsilon}$ *denote the set of strand diagrams on* $\mathcal{S}_{n,\epsilon}$ *with* k *strands and let* $\mathcal{D}_\epsilon$ *denote the set of strand diagrams with any positive number of strands. Then*

$$\mathcal{D}_\epsilon = \bigsqcup_{k \in [n]} \mathcal{D}_{k,\epsilon}.$$

Example 5.3.4. *Let* $n = 4$ *and* $\epsilon = (-, +, -, +, +)$ *so that* $Q_\epsilon = 1 \xrightarrow{a_1^+} 2 \xleftarrow{a_2^-} 3 \xrightarrow{a_3^+} 4$*. Then we have that* $d_1 = \{c(0,1), c(0,2), c(2,3), c(2,4)\} \in \mathcal{D}_{4,\epsilon}$ *and* $d_2 = \{c(0,4), c(1,3), c(2,4)\} \in \mathcal{D}_{3,\epsilon}$*. We draw these strand diagrams below.*

The following technical lemma classifies when two distinct indecomposable representations of Q_ϵ define 0, 1, or 2 exceptional pairs. Its proof appears in Section 5.3.2.

Figure 5.4: Two examples of strand diagrams

Lemma 5.3.5. *Let Q_ϵ be given. Fix two distinct indecomposable representations $U, V \in ind(rep_{\Bbbk}(Q_\epsilon))$.*

a) The strands $\Phi_\epsilon(U)$ and $\Phi_\epsilon(V)$ intersect nontrivially if and only if neither (U, V) nor (V, U) are exceptional pairs.

b) The strand $\Phi_\epsilon(U)$ is clockwise from $\Phi_\epsilon(V)$ if and only if (U, V) is an exceptional pair and (V, U) is not an exceptional pair.

c) The strands $\Phi_\epsilon(U)$ and $\Phi_\epsilon(V)$ do not intersect at any of their endpoints and they do not intersect nontrivially if and only if (U, V) and (V, U) are both exceptional pairs.

Using Lemma 5.3.5 we obtain our first main result. The following theorem says that the data of an exceptional collection is completely encoded in the strand diagram it defines.

Theorem 5.3.6. *Let $\overline{\mathcal{E}}_\epsilon :=$ {exceptional collections of Q_ϵ}. There is a bijection $\overline{\mathcal{E}}_\epsilon \to \mathcal{D}_\epsilon$ defined by*

$$\overline{\xi}_\epsilon = \{X^\epsilon_{i_\ell, j_\ell}\}_{\ell \in [k]} \mapsto \{\Phi_\epsilon(X^\epsilon_{i_\ell, j_\ell})\}_{\ell \in [k]}.$$

Proof. Let $\overline{\xi}_\epsilon = \{X^\epsilon_{i_\ell, j_\ell}\}_{\ell \in [k]}$ be an exceptional collection of Q_ϵ. Let ξ_ϵ be an exceptional sequence gotten from $\overline{\xi}_\epsilon$ by reordering its representations. Without loss of generality, assume $\xi_\epsilon = (X^\epsilon_{i_\ell, j_\ell})_{\ell \in [k]}$ is an exceptional sequence. Thus, $(X^\epsilon_{i_\ell, j_\ell}, X^\epsilon_{i_p, j_p})$ is an exceptional pair for all $\ell < p$. Lemma 5.3.5 a) implies that distinct strands of $\{\Phi_\epsilon(X^\epsilon_{i_\ell, j_\ell})\}_{\ell \in [k]}$ do not intersect nontrivially.

Now we will show that $\{\Phi_\epsilon(X^\epsilon_{i_\ell, j_\ell})\}_{\ell \in [k]}$ has no cycles. Suppose that $\Phi_\epsilon(X^\epsilon_{i_{\ell_1}, j_{\ell_1}}), ..., \Phi_\epsilon(X^\epsilon_{i_{\ell_p}, j_{\ell_p}})$ is a cycle of length $p \leqslant k$ in $\Phi_\epsilon(\xi_\epsilon)$. Then, there exist $\ell_a, \ell_b \in [k]$ with $\ell_b > \ell_a$ such that $\Phi_\epsilon(X^\epsilon_{i_{\ell_b}, j_{\ell_b}})$ is clockwise from $\Phi_\epsilon(X^\epsilon_{i_{\ell_a}, j_{\ell_a}})$. Thus, by Lemma 5.3.5 b), $(X^\epsilon_{i_{\ell_a}, j_{\ell_a}}, X^\epsilon_{i_{\ell_b}, j_{\ell_b}})$ is not an exceptional pair. This contradicts the fact that $(X^{\epsilon}_{i_{\ell_1},j_{\ell_1}},...,X^{\epsilon}_{i_{\ell_p},j_{\ell_p}})$ is an exceptional sequence. Hence, the graph determined by $\{\Phi_\epsilon(X^{\epsilon}_{i_\ell,j_\ell})\}_{\ell\in[k]}$ is a tree. We have shown that $\Phi_\epsilon(\overline{\xi}_\epsilon) \in \mathcal{D}_{k,\epsilon}$.

Now let $d = \{c(i_\ell, j_\ell)\}_{\ell\in[k]} \in \mathcal{D}_{k,\epsilon}$. Since $c(i_\ell, j_\ell)$ and $c(i_m, j_m)$ do not intersect nontrivially, it follows that for every $\ell \neq m$, either

$$(\Phi_\epsilon^{-1}(c(i_\ell, j_\ell)), \Phi_\epsilon^{-1}(c(i_m, j_m)))$$

or

$$(\Phi_\epsilon^{-1}(c(i_m, j_m)), \Phi_\epsilon^{-1}(c(i_\ell, j_\ell)))$$

is an exceptional pair. Notice that there exists $c(i_{\ell_1}, j_{\ell_1}) \in d$ such that

$$(\Phi_\epsilon^{-1}(c(i_{\ell_1}, j_{\ell_1})), \Phi_\epsilon^{-1}(c(i_\ell, j_\ell)))$$

is an exceptional pair for every $c(i_\ell, j_\ell) \in d\backslash\{c(i_{\ell_1}, j_{\ell_1})\}$. This is true because if such $c(i_{\ell_1}, j_{\ell_1})$ did not exist, then d must have a cycle. Set $E_1 = \Phi_\epsilon^{-1}(c(i_{\ell_1}, j_{\ell_1}))$. Now, choose $c(i_{\ell_p}, j_{\ell_p})$ such that $(\Phi_\epsilon^{-1}(c(i_{\ell_p}, j_{\ell_p})), \Phi_\epsilon^{-1}(c(i_\ell, j_\ell)))$ is an exceptional pair for every $c(i_\ell, j_\ell) \in d\backslash\{c(i_{\ell_1}, j_{\ell_1}), ..., c(i_{\ell_p}, j_{\ell_p})\}$ inductively and put $E_p = \Phi_\epsilon^{-1}(c(i_{\ell_p}, j_{\ell_p}))$. By construction, $(E_1, ..., E_k)$ is a complete exceptional sequence, as desired. □

Our next step is to add distinct integer labels to each strand in a given strand diagram d. When these labels have what we call a **good** labeling, these labels will describe exactly the order in which to put the representations corresponding to strands of d so that the resulting sequence of representations is an exceptional sequence.

Definition 5.3.7. *A* ***labeled diagram*** $d(k) = \{(c(i_\ell, j_\ell), s_\ell)\}_{\ell\in[k]}$ *on* $\mathcal{S}_{n,\epsilon}$ *is a strand diagram on* $\mathcal{S}_{n,\epsilon}$ *whose strands are labeled by integers* $s_\ell \in [k]$ *bijectively.*

Let $\epsilon_i \in \mathcal{S}_{n,\epsilon}$ *and let* $((c(i, j_1), s_1), \ldots, (c(i, j_r), s_r))$ *be the complete list of labeled strands of* $d(k)$ *that involve* ϵ_i *and ordered so that strand* $c(i, j_k)$ *is clockwise from* $c(i, j_{k'})$ *if* $k' < k$. *We say the strand labeling of* $d(k)$ *is* ***good*** *if for each point* $\epsilon_i \in \mathcal{S}_{n,\epsilon}$ *one has* $s_1 < \cdots < s_r$. *Let* $\mathcal{D}_{k,\epsilon}(k)$ *denote the set of labeled strand diagrams on* $\mathcal{S}_{n,\epsilon}$ *with* k *strands and with good strand labelings.*

Example 5.3.8. *Let* $n = 4$ *and* $\epsilon = (-, +, -, +, +)$ *so that* $Q_\epsilon = 1 \xrightarrow{a_1^+} 2 \xleftarrow{a_2^-} 3 \xrightarrow{a_3^+} 4$. *Below we show the labeled diagrams*
$d_1(4) = \{(c(0, 1), 1), (c(0, 2), 2), (c(2, 3), 3), (c(2, 4), 4)\}$ *and*
$d_2(3) = \{(c(0, 4), 1), (c(2, 4), 2), (c(1, 3), 3)\}$.

Figure 5.5: Two examples of labeled strand diagrams

We have that $d_1(4) \in \mathcal{D}_{4,\epsilon}(4)$*, but* $d_2(3) \notin \mathcal{D}_{3,\epsilon}(3)$.

Theorem 5.3.9. *Let* $k \in [n]$ *and let* $\mathcal{E}_\epsilon(k)$ *denote the set of exceptional sequences of* Q_ϵ *of length* k*. There is a bijection* $\widetilde{\Phi}_\epsilon : \mathcal{E}_\epsilon(k) \to \mathcal{D}_{k,\epsilon}(k)$ *defined by*

$$\xi_\epsilon = (X^\epsilon_{i_\ell,j_\ell})_{\ell\in[k]} \longmapsto \{(c(i_\ell, j_\ell), k+1-\ell)\}_{\ell\in[k]}.$$

Proof. Let $\xi_\epsilon := (V_1, ..., V_k) \in \mathcal{E}_\epsilon(k)$. By Lemma 5.3.5 a), $\widetilde{\Phi}_\epsilon(\xi_\epsilon)$ has no strands that intersect nontrivially. Let (V_1, V_2) be an exceptional pair appearing in ξ_ϵ with V_i corresponding to strand c_i in $\widetilde{\Phi}_\epsilon(\xi_\epsilon)$ for $i = 1, 2$, where c_1 and c_2 intersect only at one of their endpoints. Note that by the definition of $\widetilde{\Phi}_\epsilon$, the strand label of c_1 is larger than that of c_2. From Lemma 5.3.5 b), strand c_1 is clockwise from c_2 in $\widetilde{\Phi}_\epsilon(\xi_\epsilon)$. Thus the strand-labeling of $\widetilde{\Phi}_\epsilon(\xi_\epsilon)$ is good, so $\widetilde{\Phi}_\epsilon(\xi_\epsilon) \in \mathcal{D}_{k,\epsilon}(k)$ for any $\xi_\epsilon \in \mathcal{E}_\epsilon(k)$.

Let $\widetilde{\Psi}_\epsilon : \mathcal{D}_{k,\epsilon}(k) \to \mathcal{E}_\epsilon(k)$ be defined by the formula

$$\{(c(i_\ell, j_\ell), \ell)\}_{\ell\in[k]} \mapsto (X^\epsilon_{i_k,j_k}, X^\epsilon_{i_{k-1},j_{k-1}}, ..., X^\epsilon_{i_1,j_1}).$$

We will show that $\widetilde{\Psi}_\epsilon(d(k)) \in \mathcal{E}_\epsilon(k)$ for any $d(k) \in \mathcal{D}_{k,\epsilon}(k)$ and that $\widetilde{\Psi}_\epsilon = \widetilde{\Phi}_\epsilon^{-1}$. Let $\widetilde{\Psi}_\epsilon(\{(c(i_\ell, j_\ell), \ell)\}_{\ell\in[k]}) = (X^\epsilon_{i_k,j_k}, X^\epsilon_{i_{k-1},j_{k-1}}, \ldots, X^\epsilon_{i_1,j_1})$. Consider the pair $(X^\epsilon_{i_s,j_s}, X^\epsilon_{i_{s'},j_{s'}})$ with $s > s'$. We will prove that the pair $(X^\epsilon_{i_s,j_s}, X^\epsilon_{i_{s'},j_{s'}})$ is an exceptional pair and conclude that $\widetilde{\Psi}_\epsilon(\{(c(i_\ell, j_\ell), \ell)\}_{\ell\in[k]}) \in \mathcal{E}_\epsilon(k)$ for any $d(k) \in \mathcal{D}_{k,\epsilon}(k)$. Clearly, $c(i_s, j_s)$ and $c(i_{s'}, j_{s'})$ do not intersect nontrivially. If $c(i_s, j_s)$ and $c(i_{s'}, j_{s'})$ do not intersect at one of their endpoints, then by Lemma 5.3.5 c) $(X^\epsilon_{i_s,j_s}, X^\epsilon_{i_{s'},j_{s'}})$ is exceptional. Now suppose $c(i_s, j_s)$ and $c(i_{s'}, j_{s'})$ intersect at one of their endpoints. Because the strand-labeling of $\{(c(i_\ell, j_\ell), \ell)\}_{\ell\in[k]}$ is good, $c(i_s, j_s)$ is clockwise from $c(i_{s'}, j_{s'})$. By Lemma 5.3.5 b), we have that $(X^\epsilon_{i_s,j_s}, X^\epsilon_{i_{s'},j_{s'}})$ is exceptional.

To see that $\widetilde{\Psi}_\epsilon = \widetilde{\Phi}_\epsilon^{-1}$, observe that

$$\begin{aligned}
\widetilde{\Phi}_\epsilon\left(\widetilde{\Psi}_\epsilon(\{(c(i_\ell, j_\ell), \ell)\}_{\ell\in[k]})\right) &= \widetilde{\Phi}_\epsilon\left((X^\epsilon_{i_k,j_k}, X^\epsilon_{i_{k-1},j_{k-1}}, \ldots, X^\epsilon_{i_1,j_1})\right) \\
&= \{(c(i_\ell, j_\ell), k+1-(k+1-\ell))\}_{\ell\in[k]} \\
&= \{(c(i_\ell, j_\ell), \ell)\}_{\ell\in[k]}.
\end{aligned}$$

Thus $\widetilde{\Phi}_\epsilon \circ \widetilde{\Psi}_\epsilon = 1_{\mathcal{D}_{n,\epsilon}(k)}$. Similarly, one shows that $\widetilde{\Psi}_\epsilon \circ \widetilde{\Phi}_\epsilon = 1_{\mathcal{E}_\epsilon(k)}$. Thus $\widetilde{\Phi}_\epsilon$ is a bijection. $\square$

The last combinatorial objects we discuss in this section are called **oriented diagrams**. These are strand diagrams whose strands have a direction. We will use these to classify **c**-matrices of Dynkin type $\mathbb{A}_n$ quivers Q_ϵ.

Definition 5.3.10. *An* ***oriented diagram*** $\overrightarrow{d} = \{\overrightarrow{c}(i_\ell, j_\ell)\}_{\ell\in[k]}$ *on* $\mathcal{S}_{n,\epsilon}$ *is a strand diagram on* $\mathcal{S}_{n,\epsilon}$ *whose strands* $\overrightarrow{c}(i_\ell, j_\ell)$ *are oriented from* ϵ_{i_ℓ} *to* ϵ_{j_ℓ}.

Remark 5.3.11. *When it is clear from the context what the values of* n *and* ϵ *are, we will often refer to a strand diagram on* $\mathcal{S}_{n,\epsilon}$ *simply as a* ***diagram****. Similarly, we will often refer to labeled diagrams (resp. oriented diagrams) on* $\mathcal{S}_{n,\epsilon}$ *as* ***labeled diagrams*** *(resp.* ***oriented diagrams****).*

We now define a special subset of the oriented diagrams on $\mathcal{S}_{n,\epsilon}$. As we will see, each element in this subset of oriented diagrams, denoted $\overrightarrow{\mathcal{D}}_{n,\epsilon}$, will correspond to a unique **c**-matrix $C \in$ **c**-mat(Q_ϵ) and vice versa. Thus we obtain a diagrammatic classification of **c**-matrices (see Theorem 5.3.15).

Definition 5.3.12. *Let* $\overrightarrow{\mathcal{D}}_{n,\epsilon}$ *be the set of oriented diagrams* $\overrightarrow{d} = \{\overrightarrow{c}(i_\ell, j_\ell)\}_{\ell\in[n]}$ *on* $\mathcal{S}_{n,\epsilon}$ *with the property that any oriented subdiagram* $\overrightarrow{d}_1$ *of* $\overrightarrow{d}$ *consisting only of oriented strands connected to* ϵ_k *in* $\mathcal{S}_{n,\epsilon}$ *for some* $k \in [0, n]$ *is a subdiagram of one of the following:*

i) $\{\overrightarrow{c}(k, i_1), \overrightarrow{c}(k, i_2), \overrightarrow{c}(j, k)\}$ *where* $i_1 < k < i_2$ *and* $\epsilon_k = +$ *(shown in Figure 5.6 (left)),*

ii) $\{\overrightarrow{c}(i_1, k), \overrightarrow{c}(i_2, k), \overrightarrow{c}(k, j)\}$ *where* $i_1 < k < i_2$ *and* $\epsilon_k = -$ *(shown in Figure 5.6 (right)).*

Figure 5.6: Allowable subdiagrams in $\overrightarrow{\mathcal{D}}_{n,\epsilon}$

Lemma 5.3.13. *Let* $\{\overrightarrow{c_i}\}_{i\in[k]}$ *be a collection of* k **c***-vectors of* Q_ϵ *where* $k \leqslant n$*. Let* $\overrightarrow{c_i} = \pm\underline{\dim}(X^\epsilon_{i_1,i_2})$ *where the sign is determined by* $\overrightarrow{c_i}$*. If* $\{\overrightarrow{c_i}\}_{i\in[k]}$ *is a* ***noncrossing***

collection *of* **c***-vectors (i.e.* $\Phi_\epsilon(X^\epsilon_{i_1,i_2})$ *and* $\Phi_\epsilon(X^\epsilon_{i'_1,i'_2})$ *do not intersect nontrivially for any* $i, i' \in [k]$*), there is an injective map*

$$\left\{ \begin{array}{c} \textit{noncrossing collections} \\ \overrightarrow{c_i}_{\,i\in[k]} \textit{ of } Q_\epsilon \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \textit{oriented diagrams} \\ \overrightarrow{d} = \{\overrightarrow{c}(i_\ell, j_\ell)\}_{\ell\in[k]} \end{array} \right\}$$

defined by

$$\overrightarrow{c_i} \longmapsto \begin{cases} \overrightarrow{c}(i_1, i_2) & : \overrightarrow{c_i} \textit{ is positive} \\ \overrightarrow{c}(i_2, i_1) & : \overrightarrow{c_i} \textit{ is negative.} \end{cases}$$

In particular, each **c***-matrix* $C_\epsilon \in$ **c***-mat*(Q_ϵ) *determines a unique oriented diagram denoted* $\overrightarrow{d}_{C_\epsilon}$ *with* n *oriented strands.*

Example 5.3.14. *Let* $n = 4$ *and* $\epsilon = (+, +, -, +, -)$ *so that* $Q_\epsilon = 1 \xrightarrow{a_1^+} 2 \xleftarrow{a_2^-} 3 \xrightarrow{a_3^+} 4$*. After performing the mutation sequence* $\mu_3 \circ \mu_2$ *to the corresponding framed quiver, we have the* **c***-matrix with its oriented diagram.*

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

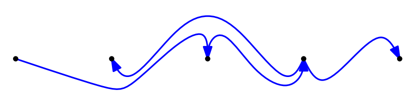


Figure 5.7: A **c**-matrix and its oriented diagram

The following theorem shows oriented diagrams belonging to $\overrightarrow{\mathcal{D}}_{n,\epsilon}$ are in bijection with **c**-matrices of Q_ϵ. We delay its proof until Section 5.4 because it makes heavy use of the concept of a mixed cobinary tree.

Theorem 5.3.15. *The map* **c***-mat*$(Q_\epsilon) \to \overrightarrow{\mathcal{D}}_{n,\epsilon}$ *induced by the map defined in Lemma 5.3.13 is a bijection.*

5.3.2 Proof of Lemma 5.3.5

The proof of Lemma 5.3.5 requires some notions from representation theory of finite dimensional algebras, which we now briefly review. For a more comprehensive treatment of the following notions, we refer the reader to [ASS06].

Definition 5.3.16. *Given a quiver Q with $\#Q_0 = n$, the* ***Euler characteristic*** *(of Q) is the $\mathbb{Z}$-bilinear (nonsymmetric) form $\mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ defined by*

$$\langle \underline{\dim}(V), \underline{\dim}(W) \rangle = \sum_{i \geqslant 0} (-1)^i \dim \operatorname{Ext}^i_{\Bbbk Q}(V, W)$$

for every $V, W \in \operatorname{rep}_{\Bbbk}(Q)$.

For hereditary algebras A (e.g. path algebras of acyclic quivers), it is known that $\operatorname{Ext}^i_A(V, W) = 0$ for $i \geqslant 2$ and the formula reduces to

$$\langle \underline{\dim}(V), \underline{\dim}(W) \rangle = \dim \operatorname{Hom}_{\Bbbk Q}(V, W) - \dim \operatorname{Ext}^1_{\Bbbk Q}(V, W)$$

The following result gives a simple formula for the Euler characteristic. We note that this formula is independent of the orientation of the arrows of Q.

Lemma 5.3.17 ([ASS06, Lemma VII.4.1])**.** *Given an acyclic quiver Q with $\#Q_0 = n$ and integral vectors $x = (x_1, x_2, ..., x_n), y = (y_1, y_2, ..., y_n) \in \mathbb{Z}^n$, the Euler characteristic of Q has the form*

$$\langle x, y \rangle = \sum_{i \in Q_0} x_i y_i - \sum_{\alpha \in Q_1} x_{s(\alpha)} y_{t(\alpha)}$$

Next, we give a slight simplification of the previous formula. Recall that the support of $V \in \operatorname{rep}_{\Bbbk}(Q)$ is the set $\operatorname{supp}(V) := \{i \in Q_0 : V_i \neq 0\}$. Thus for quivers of the form Q_ϵ, any representation $X^\epsilon_{i,j} \in \operatorname{ind}(\operatorname{rep}_{\Bbbk}(Q_\epsilon))$ has $\operatorname{supp}(X^\epsilon_{i,j}) = [i+1, j]$.

Lemma 5.3.18. *Let $X^\epsilon_{k,\ell}, X^\epsilon_{i,j} \in ind(rep_{\Bbbk}(Q_\epsilon))$ and $A := \{a \in (Q_\epsilon)_1 : s(a), t(a) \in supp(X^\epsilon_{k,\ell}) \cap supp(X^\epsilon_{i,j})\}$. Then $\langle \underline{\dim}(X^\epsilon_{k,\ell}), \underline{\dim}(X^\epsilon_{i,j}) \rangle$ is given by the formula*

$$\chi_{supp(X^\epsilon_{k,\ell}) \cap supp(X^\epsilon_{i,j})} - \#\left(\left\{ a \in (Q_\epsilon)_1 : \begin{array}{l} s(a) \in supp(X^\epsilon_{k,\ell}), \\ t(a) \in supp(X^\epsilon_{i,j}) \end{array} \right\} \setminus A \right)$$

where $\chi_{supp(X^\epsilon_{k,\ell}) \cap supp(X^\epsilon_{i,j})} = 1$ if $supp(X^\epsilon_{k,\ell}) \cap supp(X^\epsilon_{i,j}) \neq \varnothing$ and 0 otherwise.

Proof. We have that $\langle \underline{\dim}(X^{\epsilon}_{k,\ell}), \underline{\dim}(X^{\epsilon}_{i,j})\rangle$ is equal to

$$\begin{aligned}
&\sum_{m\in(Q_\epsilon)_0} \underline{\dim}(X^{\epsilon}_{k,\ell})_m \underline{\dim}(X^{\epsilon}_{i,j})_m - \sum_{a\in(Q_\epsilon)_1} \underline{\dim}(X^{\epsilon}_{k,\ell})_{s(a)} \underline{\dim}(X^{\epsilon}_{i,j})_{t(a)} \\
&= \#\left(\mathrm{supp}(X^{\epsilon}_{k,\ell}) \cap \mathrm{supp}(X^{\epsilon}_{i,j})\right) - \#\left\{\alpha \in (Q_\epsilon)_1 : \begin{array}{l} s(a) \in \mathrm{supp}(X^{\epsilon}_{k,\ell}), \\ t(a) \in \mathrm{supp}(X^{\epsilon}_{i,j}) \end{array}\right\} \\
&= \#\left(\mathrm{supp}(X^{\epsilon}_{k,\ell}) \cap \mathrm{supp}(X^{\epsilon}_{i,j})\right) - \#A \\
&\quad - \#\left(\left\{a \in (Q_\epsilon)_1 : \begin{array}{l} s(a) \in \mathrm{supp}(X^{\epsilon}_{k,\ell}), \\ t(a) \in \mathrm{supp}(X^{\epsilon}_{i,j}) \end{array}\right\} \backslash A\right).
\end{aligned}$$

Observe that if $\mathrm{supp}(X^{\epsilon}_{k,\ell}) \cap \mathrm{supp}(X^{\epsilon}_{i,j}) \neq \varnothing$, then $\#A = \#(\mathrm{supp}(X^{\epsilon}_{k,\ell}) \cap \mathrm{supp}(X^{\epsilon}_{i,j})) - 1$. Otherwise $\#A = 0$. Thus $\langle \underline{\dim}(X^{\epsilon}_{k,\ell}), \underline{\dim}(X^{\epsilon}_{i,j})\rangle$ is given by

$$\chi_{\mathrm{supp}(X^{\epsilon}_{k,\ell}) \cap \mathrm{supp}(X^{\epsilon}_{i,j})} - \#\left(\left\{a \in (Q_\epsilon)_1 : \begin{array}{l} s(a) \in \mathrm{supp}(X^{\epsilon}_{k,\ell}), \\ t(a) \in \mathrm{supp}(X^{\epsilon}_{i,j}) \end{array}\right\} \backslash A\right)$$

as desired. □

In the sequel, we will use this formula for the Euler characteristic without further comment. We now present several lemmas that will be useful in the proof of Lemma 5.3.5. The proofs of the next four lemmas use very similar techniques so we only prove Lemma 5.3.19. The following four lemmas characterize when $\mathrm{Hom}_{\Bbbk Q_\epsilon}(-,-)$ and $\mathrm{Ext}^1_{\Bbbk Q_\epsilon}(-,-)$ vanish for a given Dynkin type $\mathbb{A}_n$ quiver Q_ϵ. The conditions describing when $\mathrm{Hom}_{\Bbbk Q_\epsilon}(-,-)$ and $\mathrm{Ext}^1_{\Bbbk Q_\epsilon}(-,-)$ vanish are given in terms of inequalities satisfied by the indices that describe a pair of indecomposable representations of Q_ϵ and the entries of ϵ.

Lemma 5.3.19. *Let $X^{\epsilon}_{k,\ell}, X^{\epsilon}_{i,j} \in ind(rep_{\Bbbk}(Q_\epsilon))$. Assume $0 \leqslant i < k < j < \ell \leqslant n$.*

i) $\mathrm{Hom}_{\Bbbk Q_\epsilon}(X^{\epsilon}_{i,j}, X^{\epsilon}_{k,\ell}) \neq 0$ *if and only if* $\epsilon_k = -$ *and* $\epsilon_j = -$.

ii) $\mathrm{Hom}_{\Bbbk Q_\epsilon}(X^{\epsilon}_{k,\ell}, X^{\epsilon}_{i,j}) \neq 0$ *if and only if* $\epsilon_k = +$ *and* $\epsilon_j = +$.

iii) $\mathrm{Ext}^1_{\Bbbk Q_\epsilon}(X^{\epsilon}_{i,j}, X^{\epsilon}_{k,\ell}) \neq 0$ *if and only if* $\epsilon_k = +$ *and* $\epsilon_j = +$.

iv) $\mathrm{Ext}^1_{\Bbbk Q_\epsilon}(X^{\epsilon}_{k,\ell}, X^{\epsilon}_{i,j}) \neq 0$ *if and only if* $\epsilon_k = -$ *and* $\epsilon_j = -$.

Proof. We only prove *i)* and *iv)* as the proofs of *ii)* is very similar to that of *i)* the proof of *iii)* is very similar to that of *iv)*. To prove *i)*, first assume there is a nonzero morphism $\theta : X^{\epsilon}_{i,j} \to X^{\epsilon}_{k,\ell}$. Clearly, $\theta_s = 0$ if $s \notin [k+1, j]$. If $\theta_s \neq 0$ for some $s \in [n]$, then

$\theta_s = \lambda$ for some $\lambda \in \Bbbk^*$ (i.e. θ_s is a nonzero scalar transformation). As θ is a morphism of representations, it must satisfy that for any $a \in (Q_\epsilon)_1$ the equality $\theta_{t(a)}\varphi_a^{i,j} = \varphi_a^{k,\ell}\theta_{s(a)}$ holds. Thus for any $a \in \{a_{k+1}^{\epsilon_{k+1}}, \ldots, a_{j-1}^{\epsilon_{j-1}}\}$, we have $\theta_{t(a)} = \theta_{s(a)}$. As θ is nonzero, this implies that $\theta_s = \lambda$ for any $s \in [k+1, j]$. If $a = a_k^{\epsilon_k}$, then we have

$$\begin{aligned} \theta_{t(a)}\varphi_a^{i,j} &= \varphi_a^{k,\ell}\theta_{s(a)} \\ \theta_{t(a)} &= 0. \end{aligned}$$

Thus $\epsilon_k = -$. Similarly, $\epsilon_j = -$.

Conversely, it is easy to see that if $\epsilon_k = \epsilon_j = -$, then $\theta : X_{i,j}^\epsilon \to X_{k,\ell}^\epsilon$ defined by $\theta_s = 0$ if $s \notin [k+1, j]$ and $\theta_s = 1$ otherwise is a nonzero morphism.

Next, we prove *iv*). Observe that by Lemma 5.3.18 we have

$$\begin{aligned} \dim \operatorname{Ext}^1_{\Bbbk Q_\epsilon}(X_{k,\ell}^\epsilon, X_{i,j}^\epsilon) &= \dim \operatorname{Hom}_{\Bbbk Q_\epsilon}(X_{k,\ell}^\epsilon, X_{i,j}^\epsilon) - \langle \underline{\dim}(X_{k,\ell}^\epsilon), \underline{\dim}(X_{i,j}^\epsilon)\rangle \\ &= \dim \operatorname{Hom}_{\Bbbk Q_\epsilon}(X_{k,\ell}^\epsilon, X_{i,j}^\epsilon) - 1 \\ &\quad + \#\left(\left\{ b \in (Q_\epsilon)_1 : \begin{array}{l} s(b) \in \operatorname{supp}(X_{k,\ell}^\epsilon), \\ t(b) \in \operatorname{supp}(X_{i,j}^\epsilon) \end{array}\right\} \backslash A\right). \end{aligned}$$

Note that $\#\Big(\{b \in (Q_\epsilon)_1 : s(b) \in \operatorname{supp}(X_{k,\ell}^\epsilon),\ t(b) \in \operatorname{supp}(X_{i,j}^\epsilon)\}\backslash A\Big) \leqslant 2$. In addition, the argument in the first paragraph of the proof shows that $\dim \operatorname{Hom}_{\Bbbk Q_\epsilon}(X_{k,\ell}^\epsilon, X_{i,j}^\epsilon) \leqslant 1$. By *ii*), we conclude that $\operatorname{Ext}^1_{\Bbbk Q_\epsilon}(X_{k,\ell}^\epsilon, X_{i,j}^\epsilon) \neq 0$ if and only if $\epsilon_k = \epsilon_j = -$. □

Lemma 5.3.20. *Let* $X_{k,\ell}^\epsilon, X_{i,j}^\epsilon \in ind(rep_\Bbbk(Q_\epsilon))$. *Assume* $0 \leqslant i < k < \ell < j \leqslant n$.

i) $\operatorname{Hom}_{\Bbbk Q_\epsilon}(X_{i,j}^\epsilon, X_{k,\ell}^\epsilon) \neq 0$ *if and only if* $\epsilon_k = -$ *and* $\epsilon_\ell = +$.

ii) $\operatorname{Hom}_{\Bbbk Q_\epsilon}(X_{k,\ell}^\epsilon, X_{i,j}^\epsilon) \neq 0$ *if and only if* $\epsilon_k = +$ *and* $\epsilon_\ell = -$.

iii) $\operatorname{Ext}^1_{\Bbbk Q_\epsilon}(X_{i,j}^\epsilon, X_{k,\ell}^\epsilon) \neq 0$ *if and only if* $\epsilon_k = +$ *and* $\epsilon_\ell = -$.

iv) $\operatorname{Ext}^1_{\Bbbk Q_\epsilon}(X_{k,\ell}^\epsilon, X_{i,j}^\epsilon) \neq 0$ *if and only if* $\epsilon_k = -$ *and* $\epsilon_\ell = +$.

Lemma 5.3.21. *Assume* $0 \leqslant i < k < j \leqslant n$. *Then* *i)* $\mathrm{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{i,k}, X^\epsilon_{k,j}) = 0$ *and* $\mathrm{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{k,j}, X^\epsilon_{i,k}) = 0$.

ii) $\mathrm{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{i,k}, X^\epsilon_{k,j}) \neq 0$ *if and only if* $\epsilon_k = +$.

iii) $\mathrm{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{k,j}, X^\epsilon_{i,k}) \neq 0$ *if and only if* $\epsilon_k = -$.

iv) $\mathrm{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{i,k}, X^\epsilon_{i,j}) \neq 0$ *if and only if* $\epsilon_k = -$.

v) $\mathrm{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{i,j}, X^\epsilon_{i,k}) \neq 0$ *if and only if* $\epsilon_k = +$.

vi) $\mathrm{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{i,k}, X^\epsilon_{i,j}) = 0$ *and* $\mathrm{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{i,j}, X^\epsilon_{i,k}) = 0$.

vii) $\mathrm{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{k,j}, X^\epsilon_{i,j}) \neq 0$ *if and only if* $\epsilon_k = +$.

viii) $\mathrm{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{i,j}, X^\epsilon_{k,j}) \neq 0$ *if and only if* $\epsilon_k = -$.

ix) $\mathrm{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{k,j}, X^\epsilon_{i,j}) = 0$ *and* $\mathrm{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{i,j}, X^\epsilon_{k,j}) = 0$.

Lemma 5.3.22. *Let* $X^\epsilon_{k,\ell}, X^\epsilon_{i,j} \in ind(rep_\Bbbk(Q_\epsilon))$. *Assume* $0 \leqslant i < j < k < \ell \leqslant n$. *Then*

i) $\mathrm{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{i,j}, X^\epsilon_{k,\ell}) = 0$, $\mathrm{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{k,\ell}, X^\epsilon_{i,j}) = 0$,

ii) $\mathrm{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{i,j}, X^\epsilon_{k,\ell}) = 0$, $\mathrm{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{k,\ell}, X^\epsilon_{i,j}) = 0$.

Next, we present three geometric facts about pairs of distinct strands. These geometric facts will be crucial in our proof of Lemma 5.3.5.

Lemma 5.3.23. *If two distinct strands* $c(i_1, j_1)$ *and* $c(i_2, j_2)$ *on* $\mathcal{S}_{n,\epsilon}$ *intersect nontrivially, then* $c(i_1, j_1)$ *and* $c(i_2, j_2)$ *can be represented by a pair of monotone curves that have a unique transversal crossing.*

Proof. Suppose $c(i_1, j_1)$ and $c(i_2, j_2)$ intersect nontrivially. Without loss of generality, we assume $i_1 \leqslant i_2$. Let $\gamma_k \in c(i_k, j_k)$ with $k \in [2]$ be monotone curves. There are two cases:

a) $i_1 \leqslant i_2 < j_1 \leqslant j_2$

b) $i_1 \leqslant i_2 < j_2 \leqslant j_1$.

Suppose that case *a)* holds. Let $(x', y') \in \{(x, y) \in \mathbb{R}^2 : x_{i_2} \leqslant x \leqslant x_{j_1}\}$ denote a point where γ_1 crosses γ_2 transversally. If $\epsilon_{i_2} = -$ (resp. $\epsilon_{i_2} = +$), isotope γ_1 relative to ϵ_{i_1} and (x', y') in such a way that the monotonocity of γ_1 is preserved and so that γ_1 lies strictly above (resp. strictly below) γ_2 on $\{(x, y) \in \mathbb{R}^2 : x_{i_2} \leqslant x < x'\}$.

Next, if $\epsilon_{j_1} = -$ (resp. $\epsilon_{j_1} = +$), isotope γ_2 relative to (x', y') and ϵ_{j_2} in such a way that the monotonicity of γ_2 is preserved and so that γ_2 lies strictly above (resp. strictly below) γ_1 on $\{(x, y) \in \mathbb{R}^2 : x' < x \leqslant x_{j_1}\}$. This process produces two monotone curves $\gamma_1 \in c(i_1, j_1)$ and $\gamma_2 \in c(i_2, j_2)$ that have a unique transversal crossing. The proof in case *b)* is very similar. $\square$

Lemma 5.3.24. *Let $c(i_1, j_1)$ and $c(i_2, j_2)$ be distinct strands on $\mathcal{S}_{n,\epsilon}$ that intersect nontrivially. Then $c(i_1, j_1)$ and $c(i_2, j_2)$ do not share an endpoint.*

Proof. Suppose $c(i_1, j_1)$ and $c(i_2, j_2)$ share an endpoint. Since $c(i_1, j_1)$ and $c(i_2, j_2)$ intersect nontrivially, then there exist curves $\gamma_k \in c(i_k, j_k)$ with $k \in \{1, 2\}$ that have a unique transversal crossing. However, since $c(i_1, j_1)$ and $c(i_2, j_2)$ share an endpoint, γ_1 and γ_2 are isotopic relative to their endpoints to curves with no transversal crossing. This contradicts that $c(i_1, j_1)$ and $c(i_2, j_2)$ share an endpoint. □

Remark 5.3.25. *If $c(i_1, j_1)$ and $c(i_2, j_2)$ are two distinct strands on $\mathcal{S}_{n,\epsilon}$ that do not intersect nontrivially, then $c(i_1, j_1)$ and $c(i_2, j_2)$ can be represented by a pair of monotone curves $\gamma_\ell \in c(i_\ell, j_\ell)$ where $\ell \in [2]$ that are nonintersecting, except possibly at their endpoints.*

We now arrive at the proof of Lemma 5.3.5. The proof is a case by case analysis where the cases are given in terms of inequalities satisfied by the indices that describe a pair of indecomposable representations of Q_ϵ and the entries of ϵ.

Proof of Lemma 5.3.5 a). Let $X^\epsilon_{i,j} := U$ and $X^\epsilon_{k,\ell} := V$. Assume that the strands $\Phi_\epsilon(X^\epsilon_{i,j})$ and $\Phi_\epsilon(X^\epsilon_{k,\ell})$ intersect nontrivially. By Lemma 5.3.24, we can assume without loss of generality that either $0 \leqslant i < k < j < \ell \leqslant n$ or $0 \leqslant i < k < \ell < j \leqslant n$. By Lemma 5.3.23, we can represent $\Phi_\epsilon(X^\epsilon_{i,j})$ and $\Phi_\epsilon(X^\epsilon_{k,\ell})$ by monotone curves $\gamma_{i,j}$ and $\gamma_{k,\ell}$ that have a unique transversal crossing. Furthermore, we can assume that this unique crossing occurs between ϵ_k and ϵ_{k+1}. There are four possible cases:

i) $\epsilon_k = \epsilon_{k+1} = -$,

ii) $\epsilon_k = -$ and $\epsilon_{k+1} = +$,

iii) $\epsilon_k = \epsilon_{k+1} = +$,

iv) $\epsilon_k = +$ and $\epsilon_{k+1} = -$.

We illustrate these cases up to isotopy in Figure 5.8. We see that in cases *i)* and *ii)* (resp. *iii)* and *iv)*) $\gamma_{k,\ell}$ lies
above (resp. below) $\gamma_{i,j}$ inside of $\{(x, y) \in \mathbb{R}^2 : x_{k+1} \leqslant x \leqslant x_{\min\{\ell,j\}}\}$.

Suppose $\gamma_{k,\ell}$ lies above $\gamma_{i,j}$ inside $\{(x, y) \in \mathbb{R}^2 : x_{k+1} \leqslant x \leqslant x_{\min\{\ell,j\}}\}$. Then

$$\epsilon_{\min\{\ell,j\}} = \begin{cases} + & : \min\{\ell, j\} = \ell \\ - & : \min\{\ell, j\} = j \end{cases}$$

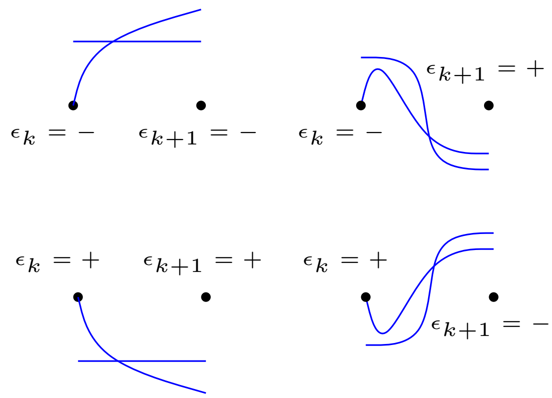


Figure 5.8: The four types of crossings

otherwise $\gamma_{k,\ell}$ and $\gamma_{i,j}$ would have a nonunique transversal crossing. If $\min\{\ell, j\} = \ell$, then $0 \leqslant i < k < \ell < j \leqslant n$, $\epsilon_k = -$, and $\epsilon_\ell = +$. Now by Lemma 5.3.20, we have that $\mathrm{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{i,j}, X^\epsilon_{k,\ell}) \neq 0$ and $\mathrm{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{k,\ell}, X^\epsilon_{i,j}) \neq 0$. If $\min\{\ell, j\} = j$, then $0 \leqslant i < k < j < \ell \leqslant n$, $\epsilon_k = -$, and $\epsilon_j = -$. Thus, by Lemma 5.3.19, we have that $\mathrm{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{i,j}, X^\epsilon_{k,\ell}) \neq 0$ and $\mathrm{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{k,\ell}, X^\epsilon_{i,j}) \neq 0$.

Similarly, if $\gamma_{i,j}$ lies above $\gamma_{k,\ell}$ inside $\{(x, y) \in \mathbb{R}^2 : x_{k+1} \leqslant x \leqslant x_{\min\{\ell,j\}}\}$, it follows that

$$\epsilon_{\min\{\ell,j\}} = \begin{cases} - & : \quad \min\{\ell, j\} = \ell \\ + & : \quad \min\{\ell, j\} = j. \end{cases}$$

If $\min\{\ell, j\} = \ell$, then Lemma 5.3.20 implies $\mathrm{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{k,\ell}, X^\epsilon_{i,j})$ and $\mathrm{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{i,j}, X^\epsilon_{k,\ell})$ are nonzero. If $\min\{\ell, j\} = j$, then Lemma 5.3.19 implies that $\mathrm{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{k,\ell}, X^\epsilon_{i,j})$ and $\mathrm{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{i,j}, X^\epsilon_{k,\ell})$ are nonzero. Thus we conclude that neither $(X^\epsilon_{i,j}, X^\epsilon_{k,\ell})$ nor $(X^\epsilon_{k,\ell}, X^\epsilon_{i,j})$ are exceptional pairs.

Conversely, assume that neither (U, V) nor (V, U) are exceptional pairs where $X^\epsilon_{i,j} := U$ and $X^\epsilon_{k,\ell} := V$. Then at least one of the following is true:

a) $\mathrm{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{i,j}, X^\epsilon_{k,\ell}) \neq 0$ and $\mathrm{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{k,\ell}, X^\epsilon_{i,j}) \neq 0$,

b) $\mathrm{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{i,j}, X^\epsilon_{k,\ell}) \neq 0$ and $\mathrm{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{k,\ell}, X^\epsilon_{i,j}) \neq 0$,

c) $\mathrm{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{i,j}, X^\epsilon_{k,\ell}) \neq 0$ and $\mathrm{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{k,\ell}, X^\epsilon_{i,j}) \neq 0$,

d) $\mathrm{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{i,j}, X^\epsilon_{k,\ell}) \neq 0$ and $\mathrm{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{k,\ell}, X^\epsilon_{i,j}) \neq 0$.

As $X^\epsilon_{i,j}$ and $X^\epsilon_{k,\ell}$ are indecomposable and distinct, we know $\mathrm{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{i,j}, X^\epsilon_{k,\ell})$ or

$\mathrm{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{k,\ell}, X^\epsilon_{i,j})$ is zero. Without loss of generality, assume $\mathrm{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{k,\ell}, X^\epsilon_{i,j}) = 0$. Thus *b*) or *d*) hold, so we have $\mathrm{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{k,\ell}, X^\epsilon_{i,j}) \neq 0$. Then Lemma 5.3.21 and Lemma 5.3.22 imply that $0 \leqslant i < k < j < \ell \leqslant n$ or $0 \leqslant i < k < \ell < j \leqslant n$.

If $0 \leqslant i < k < j < \ell < n$, then $\epsilon_k = \epsilon_j = -$ by Lemma 5.3.19 as $\mathrm{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{i,j}, X^\epsilon_{k,\ell})$ and $\mathrm{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{k,\ell}, X^\epsilon_{i,j})$ are nonzero. Let $\gamma_{i,j} \in \Phi_\epsilon(X^\epsilon_{i,j})$ and $\gamma_{k,\ell} \in \Phi_\epsilon(X^\epsilon_{k,\ell})$. We can assume that there exists $\delta(k) > 0$ such that $\gamma_{i,j}$ and $\gamma_{k,\ell}$ have no transversal crossing inside $\{(x,y) \in \mathbb{R}^2 : x_k \leqslant x \leqslant x_k + \delta(k)\}$. This implies that $\gamma_{i,j}$ lies above $\gamma_{k,\ell}$ inside $\{(x,y) \in \mathbb{R}^2 : x_k \leqslant x \leqslant x_k + \delta(k)\}$. Similarly, we can assume there exists $\delta(j) > 0$ such that $\gamma_{i,j}$ and $\gamma_{k,\ell}$ have no transversal crossing inside $\{(x,y) \in \mathbb{R}^2 : x_j - \delta(j) \leqslant x \leqslant x_j\}$. Thus $\gamma_{i,j}$ lies below $\gamma_{k,\ell}$ inside $\{(x,y) \in \mathbb{R}^2 : x_j - \delta(j) \leqslant x \leqslant x_j\}$. This means $\gamma_{i,j}$ and $\gamma_{k,\ell}$ must have at least one transversal crossing. Thus $\Phi_\epsilon(X^\epsilon_{i,j})$ and $\Phi_\epsilon(X^\epsilon_{k,\ell})$ intersect nontrivially. An analogous argument shows that if $0 \leqslant i < k < \ell < j \leqslant n$, then $\Phi_\epsilon(X^\epsilon_{i,j})$ and $\Phi_\epsilon(X^\epsilon_{k,\ell})$ intersect nontrivially. □

Proof of Lemma 5.3.5 b). Assume that $\Phi_\epsilon(U)$ is clockwise from $\Phi_\epsilon(V)$. Then we have that one of the following holds:

a) $X^\epsilon_{k,j} = U$ and $X^\epsilon_{i,k} = V$ for some $0 \leqslant i < k < j \leqslant n$,
b) $X^\epsilon_{i,k} = U$ and $X^\epsilon_{k,j} = V$ for some $0 \leqslant i < k < j \leqslant n$,
c) $X^\epsilon_{i,j} = U$ and $X^\epsilon_{i,k} = V$ for some $0 \leqslant i < j \leqslant n$ and $0 \leqslant i < k \leqslant n$,
d) $X^\epsilon_{i,j} = U$ and $X^\epsilon_{k,j} = V$ for some $0 \leqslant i < j \leqslant n$ and $0 \leqslant k < j \leqslant n$.

In Case *a*), we have that $\epsilon_k = -$ since $\Phi_\epsilon(X^\epsilon_{k,j})$ is clockwise from $\Phi_\epsilon(X^\epsilon_{i,k})$. By Lemma 5.3.21 *i*) and *ii*), we have that $\mathrm{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{i,k}, X^\epsilon_{k,j})$ and $\mathrm{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{i,k}, X^\epsilon_{k,j})$ are zero. Thus $(X^\epsilon_{k,j}, X^\epsilon_{i,k})$ is an exceptional pair. By Lemma 5.3.21 *iii*), we have that $\mathrm{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{k,j}, X^\epsilon_{i,k}) \neq 0$. Thus $(X^\epsilon_{i,k}, X^\epsilon_{k,j})$ is not an exceptional pair.

In Case *b*), we have that $\epsilon_k = +$ since $\Phi_\epsilon(X^\epsilon_{i,k})$ is clockwise from $\Phi_\epsilon(X^\epsilon_{k,j})$. By Lemma 5.3.21 *i*) and *iii*), we have that $\mathrm{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{k,j}, X^\epsilon_{i,k})$ and $\mathrm{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{k,j}, X^\epsilon_{i,k})$ are zero. Thus $(X^\epsilon_{i,k}, X^\epsilon_{k,j})$ is an exceptional pair. By Lemma 5.3.21 *ii*), we have that $\mathrm{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{i,k}, X^\epsilon_{k,j}) \neq 0$. Thus $(X^\epsilon_{k,j}, X^\epsilon_{i,k})$ is not an exceptional pair.

In Case *c*), if $j < k$, it follows that $\epsilon_j = -$. Indeed, $\Phi_\epsilon(X^\epsilon_{i,j})$ is clockwise from $\Phi_\epsilon(X^\epsilon_{i,k})$ and so by Lemma 5.3.24 the two do not intersect nontrivially. Now by Remark 5.3.25, we can choose monotone curves $\gamma_{i,k} \in \Phi_\epsilon(X^\epsilon_{i,k})$ and $\gamma_{i,j} \in \Phi_\epsilon(X^\epsilon_{i,j})$ such that $\gamma_{i,k}$ lies strictly above $\gamma_{i,j}$ on $\{(x,y) \in \mathbb{R}^2 : x_i < x \leqslant x_j\}$. Thus $\epsilon_j = -$. By Lemma 5.3.21 *v*) and *vi*), we have that $\mathrm{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{i,k}, X^\epsilon_{i,j}) = 0$ and $\mathrm{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{i,k}, X^\epsilon_{i,j}) =$

0 so that $(X^\epsilon_{i,j}, X^\epsilon_{i,k})$ is an exceptional pair. By Lemma 5.3.21 *iv*), we have that $\mathrm{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{i,j}, X^\epsilon_{i,k}) \neq 0$. Thus $(X^\epsilon_{i,k}, X^\epsilon_{i,j})$ is not an exceptional pair.

Similarly, one shows that if $k < j$, then $\epsilon_k = +$. By Lemma 5.3.21 *iv*) and *vi*), we have that $\mathrm{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{i,k}, X^\epsilon_{i,j}) = 0$ and $\mathrm{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{i,k}, X^\epsilon_{i,j}) = 0$. It follows that $(X^\epsilon_{i,j}, X^\epsilon_{i,k})$ is an exceptional pair. By Lemma 5.3.21 *v*), we know $\mathrm{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{i,j}, X^\epsilon_{i,k}) \neq 0$. Thus $(X^\epsilon_{i,k}, X^\epsilon_{i,j})$ is not an exceptional pair. The proof in Case *d*) is completely analogous to the proof in Case *c*) so we omit it.

Conversely, let $U = X^\epsilon_{i,j}$ and $V = X^\epsilon_{k,\ell}$ and assume that $(X^\epsilon_{i,j}, X^\epsilon_{k,\ell})$ is an exceptional pair and $(X^\epsilon_{k,\ell}, X^\epsilon_{i,j})$ is not an exceptional pair. This implies that at least one of the following holds:

1) $\mathrm{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{k,\ell}, X^\epsilon_{i,j}) = 0, \mathrm{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{k,\ell}, X^\epsilon_{i,j}) = 0,$ and $\mathrm{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{i,j}, X^\epsilon_{k,\ell}) \neq 0,$
2) $\mathrm{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{k,\ell}, X^\epsilon_{i,j}) = 0, \mathrm{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{k,\ell}, X^\epsilon_{i,j}) = 0,$ and $\mathrm{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{i,j}, X^\epsilon_{k,\ell}) \neq 0.$

By Lemma 5.3.22, we know that $[i, j] \cap [k, \ell] \neq \varnothing$. This implies that either

i) $\Phi_\epsilon(X^\epsilon_{i,j})$ and $\Phi_\epsilon(X^\epsilon_{k,\ell})$ share an endpoint,
ii) $0 \leqslant i < k < j < \ell \leqslant n$,
iii) $0 \leqslant i < k < \ell < j \leqslant n$,
iv) $0 \leqslant k < i < \ell < j \leqslant n$,
v) $0 \leqslant k < i < j < \ell \leqslant n$.

We will show that $\Phi_\epsilon(X^\epsilon_{i,j})$ and $\Phi_\epsilon(X^\epsilon_{k,\ell})$ share an endpoint.

Suppose $0 \leqslant i < k < j < \ell \leqslant n$. Since $\mathrm{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{k,\ell}, X^\epsilon_{i,j}) = 0, \mathrm{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{k,\ell}, X^\epsilon_{i,j}) = 0$, we have by Lemma 5.3.19 *ii*) and *iv*) that either $\epsilon_k = -$ and $\epsilon_j = +$ or $\epsilon_k = +$ and $\epsilon_j = -$. However, as $\mathrm{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{i,j}, X^\epsilon_{k,\ell}) \neq 0$ or $\mathrm{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{i,j}, X^\epsilon_{k,\ell}) \neq 0$, Lemma 5.3.19 *i*) and *iii*) we have that $\epsilon_k = \epsilon_j = -$ or $\epsilon_k = \epsilon_j = +$. This contradicts that $0 \leqslant i < k < j < \ell \leqslant n$. An analogous argument shows that i, j, k, ℓ do not satisfy $0 \leqslant k < i < \ell < j \leqslant n$.

Suppose $0 \leqslant i < k < \ell < j \leqslant n$. Since $\mathrm{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{k,\ell}, X^\epsilon_{i,j}) = 0, \mathrm{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{k,\ell}, X^\epsilon_{i,j}) = 0$, we have by Lemma 5.3.20 *ii*) and *iv*) that either $\epsilon_k = \epsilon_\ell = +$ or $\epsilon_k = \epsilon_\ell = -$. However, as $\mathrm{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{i,j}, X^\epsilon_{k,\ell}) \neq 0$ or $\mathrm{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{i,j}, X^\epsilon_{k,\ell}) \neq 0$, Lemma 5.3.20 *i*) and *iii*) we have that $\epsilon_k = -$ and $\epsilon_\ell = +$ or $\epsilon_k = +$ and $\epsilon_\ell = -$. This contradicts that $0 \leqslant i < k < \ell < j \leqslant n$. An analogous argmuent shows that i, j, k, ℓ do not satisfy $0 \leqslant k < i < j < \ell \leqslant n$.

We conclude that $\Phi_\epsilon(U)$ and $\Phi_\epsilon(V)$ share an endpoint. Thus we have that one of the following holds where we forget the previous roles played by i, j, k:

a) $X^\epsilon_{k,j} = U$ and $X^\epsilon_{i,k} = V$ for some $0 \leqslant i < k < j \leqslant n$,

b) $X^\epsilon_{i,k} = U$ and $X^\epsilon_{k,j} = V$ for some $0 \leqslant i < k < j \leqslant n$,

c) $X^\epsilon_{i,j} = U$ and $X^\epsilon_{i,k} = V$ for some $0 \leqslant i < j \leqslant n$ and $0 \leqslant i < k \leqslant n$,

d) $X^\epsilon_{i,j} = U$ and $X^\epsilon_{k,j} = V$ for some $0 \leqslant i < j \leqslant n$ and $0 \leqslant k < j \leqslant n$.

Suppose Case *a*) holds. Then since (U, V) is an exceptional pair, $\mathrm{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{i,k}, X^\epsilon_{k,j}) = 0$. By Lemma 5.3.21 *ii*), we have that $\epsilon_k = -$. Thus $\Phi_\epsilon(U)$ is clockwise from $\Phi_\epsilon(V)$.

Suppose Case *b*) holds. Then since (U, V) is an exceptional pair, $\mathrm{Ext}^1_{\Bbbk Q_\epsilon}(X^\epsilon_{k,j}, X^\epsilon_{i,k}) = 0$. By Lemma 5.3.21 *iii*), we have that $\epsilon_k = +$. Thus $\Phi_\epsilon(U)$ is clockwise from $\Phi_\epsilon(V)$.

Suppose Case *c*) holds. Assume $k < j$. Then Lemma 5.3.21 *iv*) and the fact that $\mathrm{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{i,k}, X^\epsilon_{i,j}) = 0$ imply that $\epsilon_k = +$. Thus we have that $\Phi_\epsilon(U) = \Phi_\epsilon(X^\epsilon_{i,j})$ is clockwise from $\Phi_\epsilon(V) = \Phi_\epsilon(X^\epsilon_{i,k})$. Now suppose $j < k$. Then Lemma 5.3.21 *v*) and $\mathrm{Hom}_{\Bbbk Q_\epsilon}(X^\epsilon_{i,k}, X^\epsilon_{i,j}) = 0$ imply that $\epsilon_j = -$. Thus we have that $\Phi_\epsilon(U) = \Phi_\epsilon(X^\epsilon_{i,j})$ is clockwise from $\Phi_\epsilon(V) = \Phi_\epsilon(X^\epsilon_{i,k})$. The proof in Case *d*) is very similar so we omit it. □

Proof of Lemma 5.3.5 c). Observe that two strands $c(i_1, j_1)$ and $c(i_2, j_2)$ share and endpoint if and only if one of the two strands is clockwise from the other. Thus Lemma 5.3.5 *a*) and *b*) implies that $\Phi_\epsilon(U)$ and $\Phi_\epsilon(V)$ do not intersect at any of their endpoints and they do not intersect nontrivially if and only if both (U, V) and (V, U) are exceptional pairs. □

5.4 Mixed cobinary trees

We recall the definition of an ϵ-mixed cobinary tree and construct a bijection between the set of (isomorphism classes of) such trees and the set of maximal oriented strand diagrams on $\mathcal{S}_{n,\epsilon}$.

Definition 5.4.1 ([IO13]). *Given a sign function $\epsilon : [0, n] \to \{+, -\}$, an **$\epsilon$-mixed cobinary tree** (MCT) is a tree T embedded in $\mathbb{R}^2$ with vertex set $\{(i, y_i) | i \in [0, n]\}$ and edges straight line segments and satisfying the following conditions.*

a) *None of the edges is horizontal.*

b) *If $\epsilon_i = +$ then $y_i \geqslant z$ for any $(i,z) \in T$. So, the tree goes under (i,y_i).*

c) *If $\epsilon_i = -$ then $y_i \leqslant z$ for any $(i,z) \in T$. So, the tree goes over (i,y_i).*

d) *If $\epsilon_i = +$ then there is at most one edge descending from (i,y_i) and at most two edges ascending from (i,y_i) and not on the same side.*

e) *If $\epsilon_i = -$ then there is at most one edge ascending from (i,y_i) and at most two edges descending from (i,y_i) and not on the same side.*

Two MCT's T,T' are ***isomorphic*** *as MCT's if there is a graph isomorphism $T \cong T'$ which sends (i,y_i) to (i,y_i') and so that corresponding edges have the same sign of their slopes.*

Given a MCT T, there is a partial ordering on $[0,n]$ given by $i <_T j$ if the unique path from (i,y_i) to (j,y_j) in T is monotonically increasing. Isomorphic MCTs give the same partial ordering by definition. Conversely, the partial ordering $<_T$ determines T uniquely up to isomorphism since T is the Hasse diagram of the partial ordering $<_T$. We sometimes decorate MCTs with **leaves** at vertices so that the result is **trivalent**, i.e., with three edges incident to each vertex. See, e.g., Figure 5.10. The ends of these leaves are not considered to be vertices. In that case, each vertex with $\epsilon = +$ forms a "Y" and this pattern is vertically inverted for $\epsilon = -$. The position of the leaves is uniquely determined.

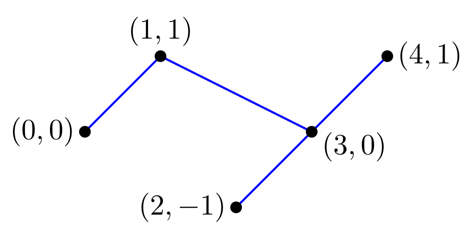


Figure 5.9: A MCT with $\epsilon_1 = \epsilon_2 = -$, $\epsilon_3 = +$ and any value for ϵ_0, ϵ_4

In Figure 5.10, the four vertices have coordinates $(0,y_0),(1,y_1),(2,y_2),(3,y_3)$ where y_i can be any real numbers so that $y_0 < y_1 < y_2 < y_3$. This inequality defines an open subset of $\mathbb{R}^4$ which is called the region of this tree T. More generally, for any MCT T, the **region** of T, denoted $\mathcal{R}_\epsilon(T)$, is the set of all points $y \in \mathbb{R}^{n+1}$ with the property that

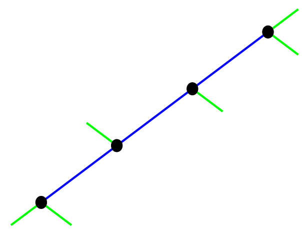


Figure 5.10: This MCT (in blue) has added green leaves showing that $\epsilon = (-,+,-,-)$

there exists a mixed cobinary tree T' which is isomorphic to T so that the vertex set of T' is $\{(i, y_i) \mid i \in [n]\}$.

Theorem 5.4.2 ([IO13])**.** *Let n and $\epsilon : [n] \to \{+,-\}$ be fixed. Then, for every MCT T, the region $\mathcal{R}_\epsilon(T)$ is convex and nonempty. Furthermore, every point $y = (y_0, \cdots, y_n)$ in $\mathbb{R}^{n+1}$ with distinct coordinates lies in $\mathcal{R}_\epsilon(T)$ for a unique T (up to isomorphism). In particular these regions are disjoint and their union is dense in $\mathbb{R}^{n+1}$.*

For a fixed n and $\epsilon : [n] \to \{+,-\}$ we will construct a bijection between the set $\mathcal{T}_\epsilon$ of isomorphism classes of mixed cobinary trees with sign function ϵ and the set $\overrightarrow{\mathcal{D}}_{n,\epsilon}$ defined in Definition 5.3.12.

Lemma 5.4.3. *Let $\overrightarrow{d} = \{\overrightarrow{c}(i_\ell, j_\ell)\}_{\ell \in [n]} \in \overrightarrow{\mathcal{D}}_{n,\epsilon}$. Let p, q be two points on this graph so that q lies directly above p. Then each edge of $\overrightarrow{d}$ in the unique path γ from p to q is oriented in the same direction as γ.*

Proof. The proof will be by induction on the number m of internal vertices of the path γ. If $m = 1$ with internal vertex ϵ_i then the path γ has only two edges of $\overrightarrow{d}$: one going from p to ϵ_i, say to the left, and the other going from ϵ_i back to q. Since $\overrightarrow{d} \in \overrightarrow{\mathcal{D}}_{n,\epsilon}$, the edge coming into ϵ_i from its right is below the edge going out from ϵ_i to q. Therefore the orientation of these two edges in $\overrightarrow{d}$ matches that of γ.

Now suppose that $m \geqslant 2$ and the lemma holds for smaller m. There are two cases. Case 1: The path γ lies entirely on one side of p and q (as in the case $m = 1$). Case 2: γ has internal vertices on both sides of p, q.

<u>Case 1</u>: Suppose by symmetry that γ lies entirely on the left side of p and q. Let j be maximal so that ϵ_j is an internal vertex of γ. Then γ contains an edge connecting ϵ_j

to either p or q, say p. And the edge of γ ending in q contains a unique point r which lies above ϵ_j. This forces the sign to be $\epsilon_j = -$. By induction on m, the rest of the path γ, which goes from ϵ_j to r has orientation compatible with that of $\overrightarrow{d}$. So, it must be oriented outward from ϵ_j. Any other edge at ϵ is oriented inward. So, the edge from p to ϵ_j is oriented from p to ϵ_j as required. The edge coming into r from the left is oriented to the right (by induction). So, this same edge continues to be oriented to the right as it goes from r to q. The other subcases (when ϵ_j is connected to q instead of p and when γ lies to the right of p and q) are analogous.

<u>Case 2</u>: Suppose that γ on both sides of p and q. Then γ passes through a third point, say r, on the same vertical line containing p and q. Let γ_0 and γ_1 denote the parts of γ going from p to r and from r to q respectively. Then γ_0, γ_1 each have at least one internal vertex. So, the lemma holds for each of them separately. There are three subcases: either (a) r lies below p, (b) r lies above q or (c) r lies between p and q. In subcase (a), we have, by induction on m, that γ_0, γ_1 are both oriented away from r. So, $r = \epsilon_k = +$ which contradicts the assumption that q lies above r. Similarly, subcase (c) is not possible. In subcase (b), we have by induction on m that the orientations of the edges of $\overrightarrow{d}$ are compatible with the orientations of γ_0 and γ_1. So, the lemma holds in subcase (b), which is the only subcase of Case 2 which is possible. Therefore, the lemma holds in all cases. □

Theorem 5.4.4. *For each $\overrightarrow{d} = \{\overrightarrow{c}(i_\ell, j_\ell)\}_{\ell \in [n]} \in \overrightarrow{\mathcal{D}}_{n,\epsilon}$, let $\mathcal{R}(\overrightarrow{d})$ denote the set of all $y \in \mathbb{R}^{n+1}$ so that $y_i < y_j$ for any $\overrightarrow{c}(i,j)$ in $\overrightarrow{d}$. Then $\mathcal{R}(\overrightarrow{d}) = \mathcal{R}_\epsilon(T)$ for a uniquely determined mixed cobinary tree T. Furthermore, this gives a bijection*

$$\overrightarrow{\mathcal{D}}_{n,\epsilon} \cong \mathcal{T}_\epsilon.$$

Proof. We first verify the existence of a mixed cobinary tree T for every choice of $y \in \mathcal{R}(\overrightarrow{d})$. Since the strand diagram is a tree, the vector y is uniquely determined by $y_0 \in \mathbb{R}$ and $y_{j_\ell} - y_{i_\ell} > 0$, $\ell \in [n]$, which are arbitrary. Given such a y, we need to verify that the n line segments L_ℓ in $\mathbb{R}^2$ connecting the pairs of points $(i_\ell, y_{i_\ell}), (j_\ell, y_{j_\ell})$ meet only on their endpoints. This follows from the lemma above. If two of these line segments, say L_k, L_ℓ, meet then they come from two distinct points $p \in \overrightarrow{c}(i_k, j_k)$ and $q = \overrightarrow{c}(i_\ell, j_\ell)$ in the strand diagram which lie one the same vertical line. If q lies above p in the strand diagram then, by Lemma 5.4.3, the unique path γ from p to q is oriented positively. This implies that the y coordinate of the point in L_k corresponding to p is less that the y coordinate of the point in L_ℓ corresponding to q. Thus, this intersection is not possible. So, T is a linearly embedded tree. The lemma also implies that the tree T lies above all negative vertices and below all positive vertices. The other parts of Definition 5.4.1 follow from the definition of an oriented strand diagram. Therefore $T \in \mathcal{T}_\epsilon$. Since this argument works for every $y \in \mathcal{R}(\overrightarrow{d})$, we see that $\mathcal{R}(\overrightarrow{d}) = \mathcal{R}_\epsilon(T)$ as claimed.

A description of the inverse mapping $\mathcal{T}_\epsilon \to \overrightarrow{\mathcal{D}}_{n,\epsilon}$ is given as follows. Take any MCT T and deform the tree by moving all vertices vertically to the subset $[n] \times 0$ on the x-axis and deforming the edges in such a way that they are always embedded in the plane with no vertical tangents and so that their interiors do not meet. The result is an oriented strand diagram $\overrightarrow{d}$ with $\mathcal{R}(\overrightarrow{d}) = \mathcal{R}_\epsilon(T)$.

It is clear that these are inverse mappings giving the desired bijection $\overrightarrow{\mathcal{D}}_{n,\epsilon} \cong \mathcal{T}_\epsilon$. $\square$

Example 5.4.5. *The MCTs in Figures 5.9 and 5.10 above give the oriented strand diagrams:*

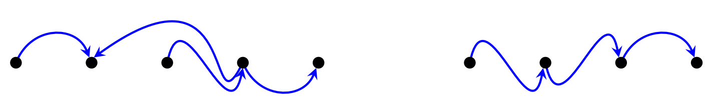


Figure 5.11: Oriented strand diagrams gotten from the MCTs in Figures 5.9 and 5.10

and the oriented strand diagram in Example 5.3.14 gives the MCT:

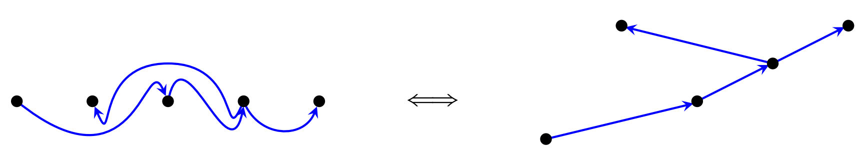


Figure 5.12: An example of the bijection given in Theorem 5.4.4

We now arrive at the proof of Theorem 5.3.15. This theorem follows from the fact that oriented diagrams belonging to $\overrightarrow{\mathcal{D}}_{n,\epsilon}$ can be regarded as mixed cobinary trees by Theorem 5.4.4.

Proof of Theorem 5.3.15. Let f be the map $\mathbf{c}\text{-mat}(Q_\epsilon) \to \overrightarrow{\mathcal{D}}_{n,\epsilon}$ induced by the map defined in Lemma 5.3.13, and let g be the bijective map $\mathcal{T}_\epsilon \to \overrightarrow{\mathcal{D}}_{n,\epsilon}$ defined in Theorem 5.4.4. We will assert the existence of a map $h : \mathbf{c}\text{-mat}(Q_\epsilon) \to \mathcal{T}_\epsilon$ which fits into the diagram

$$\begin{tikzcd} \mathbf{c}\text{-mat}(Q_\epsilon) \arrow[rr, "h"] \arrow[dr, "f"'] & & \mathcal{T}_\epsilon \arrow[dl, "\sim"', "g"] \\ & \overrightarrow{\mathcal{D}}_{n,\epsilon} & \end{tikzcd}$$

The theorem will follow after verifying that h is a bijection and that $f = g \circ h$.

We will define two new notions of $\mathbf{c}$-matrix, one for MCTs and one for oriented strand diagrams. Let $T \in \mathcal{T}_\epsilon$ with internal edges ℓ_i having endpoints (i_1, y_{i_1}) and (i_2, y_{i_2}). For each ℓ_i, define the '$\mathbf{c}$-vector' of ℓ_i to be $c_i(T) := \sum_{i_1 < j \leqslant i_2} \text{sgn}(\ell_i) e_j$, where $\text{sgn}(\ell_i)$ is the sign of the slope of ℓ_i. Define $c(T)$ to be the '$\mathbf{c}$-matrix' of T whose rows are the $\mathbf{c}$-vectors $c_i(T)$. Now, let $\overrightarrow{d} = \{\overrightarrow{c}(i_\ell, j_\ell)\}_{\ell \in [n]} \in \overrightarrow{\mathcal{D}}_{n,\epsilon}$. For each oriented strand $\overrightarrow{c}(i_\ell, j_\ell)$, define the '$\mathbf{c}$-vector' of $\overrightarrow{c}(i_\ell, j_\ell)$ to be

$$c_\ell(\overrightarrow{d}) \quad := \quad \begin{cases} \sum_{i_\ell < k \leqslant j_\ell} \text{sgn}(\overrightarrow{c}(i_\ell, j_\ell)) e_k & : \quad i_\ell < j_\ell \\ \sum_{j_\ell < k \leqslant i_\ell} \text{sgn}(\overrightarrow{c}(i_\ell, j_\ell)) e_k & : \quad i_\ell > j_\ell \end{cases}$$

where $\text{sgn}(\overrightarrow{c}(i_\ell, j_\ell))$ is positive if $i_\ell < j_\ell$ and negative if $i_\ell > j_\ell$. Define $c(\overrightarrow{d})$ to be the '$\mathbf{c}$-matrix' of $\overrightarrow{d}$ whose rows are the $\mathbf{c}$-vectors $c_\ell(\overrightarrow{d})$.

It is known that the notion of $\mathbf{c}$-matrix for MCT's coincides with the original notion of $\mathbf{c}$-matrix defined in Section 2.1, and that there is a bijection between $\mathbf{c}\text{-mat}(Q_\epsilon)$ and $\mathcal{T}_\epsilon$ which preserves $\mathbf{c}$-matrices (see [IO13, Remarks 2 and 4] for details). Thus, we have a bijective map $h : \mathbf{c}\text{-mat}(Q_\epsilon) \to \mathcal{T}_\epsilon$. On the other hand, the bijection $g : \mathcal{T}_\epsilon \to \overrightarrow{\mathcal{D}}_{n,\epsilon}$ defined in Theorem 5.4.4 also preserves $\mathbf{c}$-matrices. The map $f : \mathbf{c}\text{-mat}(Q_\epsilon) \to \mathcal{T}_\epsilon$ preserves $\mathbf{c}$-matrices by definition. Hence, we have $f = g \circ h$ and f is a bijection, as desired. $\square$

Remark 5.4.6. *For linearly-ordered quivers (those with either $\epsilon = (+, ..., +)$ or $\epsilon = (-, ..., -)$), this bijection was established by the first and third authors in [GM15] using a different approach. The bijection was given by hand without going through MCTs. This was more tedious, and the authors feel that some aspects (such as mutation) are better phrased in terms of MCTs.*

5.5 Exceptional sequences and linear extensions

In this section, we consider the problem of counting the number of CESs arising from a given CEC. We show that this problem can be restated as the problem of counting the number of linear extensions of certain posets. Throughout this section we fix a strand diagram $d = \{c(i_\ell, j_\ell)\}_{\ell \in [n]}$ on $\mathcal{S}_{n,\epsilon}$.

Definition 5.5.1. *We define the* ***poset*** $\mathcal{P}_d = (\{c(i_\ell, j_\ell)\}_{\ell\in[n]}, \leqslant)$ ***associated to*** d *as the partially ordered set whose elements are the strands of d with covering relations given by $c(i,j) \lessdot c(k,\ell)$ if and only if the strand $c(k,\ell)$ is clockwise from $c(i,j)$ and there does not exist another strand $c(i',j')$ distinct from $c(i,j)$ and $c(k,\ell)$ such that $c(i',j')$ is clockwise from $c(i,j)$ and counterclockwise from $c(k,\ell)$.*

The construction defines a poset because any oriented cycle in the Hasse diagram of $\mathcal{P}_d$ arises from a cycle in the underlying graph of d. Since the underlying graph of d is a tree, the diagram d has no cycles. In Figure 5.13, we show a diagram $d \in \mathcal{D}_{4,\epsilon}$ where $\epsilon := (-, +, -, +, +)$ and its poset $\mathcal{P}_d$.

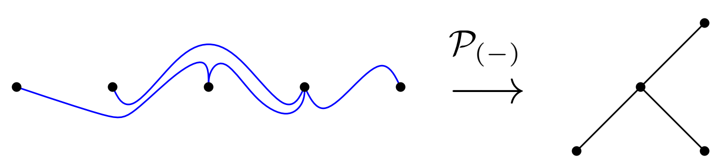


Figure 5.13: A diagram and its poset

Let $\mathcal{P}$ be a finite poset with $m = \#\mathcal{P}$. Let $f : \mathcal{P} \to \mathbf{m}$ be an injective, order-preserving map (i.e. $x \leqslant y$ implies $f(x) \leqslant f(y)$ for all $x, y \in \mathcal{P}$) where $\mathbf{m}$ is the linearly-ordered poset with m elements. We call f a **linear extension** of $\mathcal{P}$. We

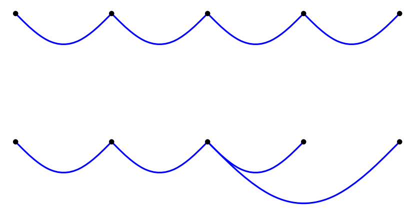


Figure 5.14: Two diagrams with the same poset

denote the set of linear extensions of $\mathcal{P}$ by $\mathscr{L}(\mathcal{P})$. Note that since f is an injective map between sets of the same cardinality, f is a bijective map between those sets.

In general, the map $\mathcal{D}_{n,\epsilon} \to \mathscr{P}(\mathcal{D}_{n,\epsilon}) := \{\mathcal{P}_d : d \in \mathcal{D}_{n,\epsilon}\}$ is not injective. For instance, each of the two diagrams in Figure 5.14 have $\mathcal{P}_d = \mathbf{4}$ where $\mathbf{4}$ denotes the linearly-ordered poset with 4 elements. It is thus natural to ask which posets are obtained from strand diagrams.

Our next result describes the posets arising from diagrams in $\mathcal{D}_{n,\epsilon}$ where $\epsilon = (-,\ldots,-)$ or $\epsilon = (+,\ldots,+)$. Before we state it, we remark that diagrams in $\mathcal{D}_{n,\epsilon}$ where $\epsilon = (-,\ldots,-)$ or $\epsilon = (+,\ldots,+)$ can be regarded as **chord diagrams**.[1] The following example shows the simple bijection.

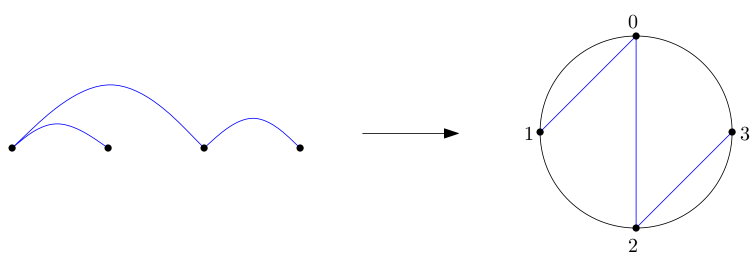


Figure 5.15: An example of the bijection between strand diagrams with $\epsilon = (+,\ldots,+)$ or $\epsilon = (-,\ldots,-)$ and chord diagrams

Let $d \in \mathcal{D}_{n,\epsilon}$ where $\epsilon = (-,\ldots,-)$ or $\epsilon = (+,\ldots,+)$. Let $c(i,j)$ be a strand of d. There is an obvious action of $\mathbb{Z}/(n+1)\mathbb{Z}$ on chord diagrams. Let $\tau \in \mathbb{Z}/(n+1)\mathbb{Z}$ denote a generator and define $\tau c(i,j) := c(i-1,j-1)$ and $\tau^{-1}c(i,j) := c(i+1,j+1)$

[1] These noncrossing trees embedded in a disk with vertices lying on the boundary have been studied by Araya in [Ara13], Goulden and Yong in [GY02], and the first and third authors in [GM15].

where we consider $i \pm 1$ and $j \pm 1$ mod $n+1$. We also define $\tau d := \{\tau c(i_\ell, j_\ell)\}_{\ell \in [n]}$ and $\tau^{-1} d := \{\tau^{-1} c(i_\ell, j_\ell)\}_{\ell \in [n]}$. The next lemma, which is easily verified, shows that the order-theoretic properties of CECs are invariant under the action of $\tau^{\pm 1}$.

Lemma 5.5.2. *Let $d \in \mathcal{D}_{n,\epsilon}$ where $\epsilon = (-, \ldots, -)$ or $\epsilon = (+, \ldots, +)$. Then we have the following isomorphisms of posets $\mathcal{P}_d \cong \mathcal{P}_{\tau d}$ and $\mathcal{P}_d \cong \mathcal{P}_{\tau^{-1} d}$.*

Theorem 5.5.3. *Let $\epsilon = (-, \ldots, -)$ or let $\epsilon = (+, \ldots, +)$. Then a poset $\mathcal{P} \in \mathscr{P}(\mathcal{D}_{n,\epsilon})$ if and only if*

i) each $x \in \mathcal{P}$ has at most two covers and covers at most two elements,

ii) the underlying graph of the Hasse diagram of $\mathcal{P}$ has no cycles,

iii) the Hasse diagram of $\mathcal{P}$ is connected.

Proof. Let $\mathcal{P}_d \in \mathscr{P}(\mathcal{D}_{n,\epsilon})$. By definition, $\mathcal{P}_d$ satisfies $i)$. It is also clear that the Hasse diagram of $\mathcal{P}_d$ is connected since d is a connected graph. To see that $\mathcal{P}_d$ satisfies $ii)$, suppose that C is a full subposet of $\mathcal{P}_d$ whose Hasse diagram is a **minimal cycle** (i.e. the underlying graph of C is a cycle, but does not contain a proper subgraph that is a cycle). Thus there exists $x_C \in \mathcal{P}_d$ such that $x_C \in C$ is covered by two distinct elements $y, z \in C$. Observe that C can be regarded as a sequence of chords $\{c_i\}_{i=0}^{\ell}$ of d in which y and z appear exactly once and where for all $i \in [0, \ell]$ c_i and c_{i+1} (we consider the indices modulo $\ell + 1$) share a marked point j and no chord adjacent to j appears between c_i and c_{i+1}. Since the chords of d are noncrossing, such a sequence cannot exist. Thus the Hasse diagram of $\mathcal{P}_d$ has no cycles.

To prove the converse, we proceed by induction on the number of elements of $\mathcal{P}$ where $\mathcal{P}$ is a poset satisfying conditions $i), ii), iii)$. If $\#\mathcal{P} = 1$, then $\mathcal{P}$ is the unique poset with one element and $\mathcal{P} = \mathcal{P}_d$ where d is the unique chord diagram associated to the disk with two marked points that is a spanning tree. Assume that for any poset $\mathcal{P}$ satisfying conditions $i), ii), iii)$ with $\#\mathcal{P} = r$ for any positive integer $r < n+1$ there exists a chord diagram d such that $\mathcal{P} = \mathcal{P}_d$. Let $\mathcal{Q}$ be a poset satisfying the above conditions and assume $\#\mathcal{Q} = n+1$. Let $x \in \mathcal{Q}$ be a maximal element.

Assume x covers two elements $y, z \in \mathcal{Q}$. Then the poset $\mathcal{Q} - \{x\} = \mathcal{Q}_1 + \mathcal{Q}_2$ where $y \in \mathcal{Q}_1$, $z \in \mathcal{Q}_2$, and $\mathcal{Q}_i$ satisfies $i), ii), iii)$ for $i \in [2]$. By induction, there exists positive integers k_1, k_2 satisfying $k_1 + k_2 = n$ and diagrams

$$d_i \in \mathcal{D}_{k_i, \epsilon^{(i)}} := \{\text{chord diagrams } \{c_i(i_\ell, j_\ell)\}_{\ell \in [k_i]} \text{ with } k_i + 1 \text{ marked points}\}$$

where $\mathcal{Q}_i = \mathcal{P}_{d_i}$ for $i \in [2]$ and where $\epsilon^{(i)} \in \{+,-\}^{k_i+1}$ has all of its entries equal to the entries of ϵ. By Lemma 5.5.2, we can further assume that the chord corresponding to $y \in \mathcal{Q}_1$ (resp. $z \in \mathcal{Q}_2$) is $c_1(i(y), k_1) \in d_1$ for some $i(y) \in [0, k_1-1]$ (resp. $c_2(j(z), k_2) \in d_2$ for some $j(z) \in [1, k_2]$). Define $d_1 \sqcup d_2 := \{c'(i'_\ell, j'_\ell)\}_{\ell\in[n]}$ to be the diagram in the disk with $n+2$ marked points as follows:

$$c'(i'_\ell, j'_\ell) \quad := \quad \begin{cases} c_1(i_\ell, j_\ell) & : \text{ if } \ell \in [k_1] \\ \tau^{-(k_1+1)} c_2(i_{\ell-k_1}, j_{\ell-k_1}) & : \text{ if } \ell \in [k_1+1, n]. \end{cases}$$

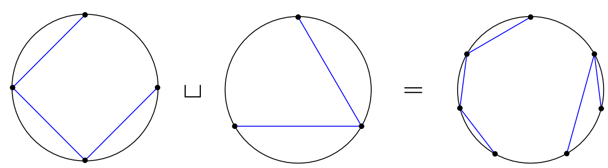


Figure 5.16: An example with $k_1 = 3$ and $k_2 = 2$ so that $n = k_1 + k_2 = 5$

Define $c'(i'_{n+1}, j'_{n+1}) := c(k_1, n+1)$ and then $d := \{c'(i'_\ell, j'_\ell)\}_{\ell\in[n+1]}$ satisfies $i), ii), iii)$, and $\mathcal{Q} = \mathcal{P}_d$.

If the Hasse diagram of $\mathcal{Q} - \{x\}$ is connected, then by induction the poset $\mathcal{Q} - \{x\} = \mathcal{P}_d$ for some diagram $d = \{c(i_\ell, j_\ell)\}_{\ell\in[n]} \in \mathcal{D}_{n,\epsilon}$ where we assume $i_\ell < j_\ell$. Since the Hasse diagram of $\mathcal{Q} - \{x\}$ is connected, it follows that x covers a unique element in $\mathcal{Q}$. Let $y = c(i(y), j(y)) \in \mathcal{Q} - \{x\}$ $(i(y) < j(y))$ denote the unique element that is covered by x in $\mathcal{Q}$. This means that there are no chords in d obtained by a clockwise rotation of $c(i(y), j(y))$ about $i(y)$ or there are no chords in d obtained by a clockwise rotation of $c(i(y), j(y))$ about $j(y)$. Without loss of generality, we assume that there are no chords in d obtained by a clockwise rotation of $c(i(y), j(y))$ about $i(y)$.

Regard d as an element of $\mathcal{D}_{n+1,\epsilon'}$ where $\epsilon' \in \{+,-\}^{n+2}$ has all of its entries equal to the entries of ϵ as follows. Replace it with $\widetilde{d} := \{c'(i'_\ell, j'_\ell)\}_{\ell\in[n]} \in \mathcal{D}_{n+1,\epsilon'}$ defined by (we give an example of this operation below with $n = 6$)

$$c'(i'_\ell, j'_\ell) \quad := \quad \begin{cases} \rho^{-1} c(i_\ell, j_\ell) & : \text{ if } i_\ell \leqslant i(y) \text{ and } j(y) \leqslant j_\ell, \\ \tau^{-1} c(i_\ell, j_\ell) & : \text{ if } j(y) \leqslant i_\ell, \\ c(i_\ell, j_\ell) & : \text{ otherwise.} \end{cases}$$

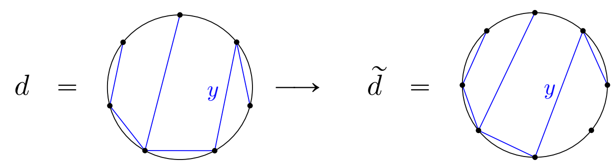


Figure 5.17: An example with $n = 6$

Define $c'(i'_{n+1}, j'_{n+1}) := c(i(y), i(y)+1)$ and put $d' := \{c'(i'_\ell, j'_\ell)\}_{\ell \in [n+1]}$. As $\mathcal{Q} - \{x\}$ satisfies $i), ii)$, and $iii)$, it is clear that the resulting chord diagram d' satisfies $\mathcal{P} = \mathcal{P}_{d'}$. $\square$

Theorem 5.5.4. *Let $d = \{c(i_\ell, j_\ell)\}_{\ell \in [n]} \in \mathcal{D}_{n,\epsilon}$ and let $\overline{\xi}_\epsilon$ denote the corresponding complete exceptional collection. Let $CES(\overline{\xi}_\epsilon)$ denote the set of CESs that can be formed using only the representations appearing in $\overline{\xi}_\epsilon$. Then the map $\chi : CES(\overline{\xi}_\epsilon) \to \mathscr{L}(\mathcal{P}_d)$ defined by $(X^\epsilon_{i_1,j_1}, \ldots, X^\epsilon_{i_n,j_n}) \xmapsto{\chi_2} \{(c(i_\ell, j_\ell), n+1-\ell)\}_{\ell \in [n]} \xmapsto{\chi_1} (f(c(i_\ell, j_\ell)) := n+1-\ell)$ is a bijection.*

Proof. The map $\chi_2 = \Phi : \mathrm{CES}(\overline{\xi}_\epsilon) \to \mathcal{D}_{n,\epsilon}(n)$ is a bijection by Theorem 5.3.9. Thus it is enough to prove that $\chi_1 : \mathcal{D}_{n,\epsilon}(n) \to \mathscr{L}(\mathcal{P}_d)$ is a bijection.

First, we show that $\chi_1(d(n)) \in \mathscr{L}(\mathcal{P}_d)$ for any $d(n) \in \mathcal{D}_{n,\epsilon}(n)$. Let $d(n) \in \mathcal{D}_{n,\epsilon}(n)$ and let $f := \chi_1(d(n))$. Since the strand-labeling of $d(n)$ is good, if (c_1, ℓ_1) and (c_2, ℓ_2) are two labeled strands of $d(n)$ satisfying $c_1 \leqslant c_2$, then $f(c_1) = \ell_1 \leqslant \ell_2 = f(c_2)$. Thus f is order-preserving. As the strands of $d(n)$ are bijectively labeled by $[n]$, we have that f is bijective so $f \in \mathscr{L}(\mathcal{P}_d)$.

Next, define a map

$$\begin{array}{rcl} \mathscr{L}(\mathcal{P}_d) & \xrightarrow{\varphi} & \mathcal{D}_{n,\epsilon}(n) \\ f & \longmapsto & \{(c(i_\ell, j_\ell), f(c(i_\ell, j_\ell)))\}_{\ell \in [n]}. \end{array}$$

To see that $\varphi(f) \in \mathcal{D}_{n,\epsilon}(n)$ for any $f \in \mathscr{L}(\mathcal{P}_d)$, consider two labeled strands $(c_1, f(c_1))$ and $(c_2, f(c_2))$ belonging to $\varphi(f)$ where $c_1 \leqslant c_2$. Since f is order-preserving, $f(c_1) \leqslant f(c_2)$. Thus the strand-labeling of $\varphi(f)$ is good so $\varphi(f) \in \mathscr{L}(\mathcal{P}_d)$.

Lastly, we have that

$$\chi_1(\varphi(f)) = \chi_1(\{(c(i_\ell, j_\ell), f(c(i_\ell, j_\ell)))\}_{\ell \in [n]}) = f$$

and

$$\varphi(\chi_1(\{(c(i_\ell, j_\ell), \ell)\}_{\ell\in[n]})) = \varphi(f(c(i_\ell, j_\ell)) := \ell) = \{(c(i_\ell, j_\ell), \ell)\}_{\ell\in[n]}$$

so $\varphi = \chi_1^{-1}$. Thus χ_1 is a bijection. □

5.6 Applications

Here we showcase some interesting results that follow easily from our main theorems.

5.6.1 Labeled trees

In [SW86, p. 67], Stanton and White gave a nonpositive formula for the number of vertex-labeled trees with a fixed number of leaves. By connecting our work with that of Goulden and Yong [GY02], we obtain a positive expression for this number. Here we consider diagrams in $\mathcal{D}_{n,\epsilon}$ where $\epsilon = (-, \ldots, -)$ or $\epsilon = (+, \ldots, +)$. We regard these as chord diagrams to make clear the connection between our work and that of [GY02].

Theorem 5.6.1. *Let $T_{n+1}(r) := \{$trees on $[n+1]$ with r leaves$\}$. Then*

$$\#T_{n+1}(r) \quad = \sum_{d \in \mathcal{D}_{n,\epsilon}\,:\ d \text{ has } r \text{ chords } c(i_j, i_j+1)} \#\mathscr{L}(\mathcal{P}_d).$$

Proof. Observe that

$$\begin{aligned}
\sum_{\substack{d \in \mathcal{D}_{n,\epsilon}\,:\, d \text{ has } r\\ \text{chords } c(i_j, i_j+1)}} \#\mathscr{L}(\mathcal{P}_d) &= \sum_{\substack{d \in \mathcal{D}_{n,\epsilon}\,:\, d \text{ has } r\\ \text{chords } c(i_j, i_j+1)}} \#\{\text{good labelings of } d\} \\
&= \#\left\{ d(n) \in \mathcal{D}_{n,\epsilon}(n) : \begin{array}{l} d(n) \text{ has } r \text{ chords} \\ c(i_j, i_j+1) \text{ for some} \\ i_1, \ldots, i_r \in [0, n] \end{array} \right\}
\end{aligned}$$

where we consider $i_j+1 \bmod n+1$. By [GY02, Theorem 1.1], we have a bijection between diagrams $d \in \mathcal{D}_{n,\epsilon}$ with r chords of the form $c(i_j, i_j+1)$ for some $i_1, \ldots, i_r \in [0, n]$ with good labelings and elements of $T_{n+1}(r)$. □

Corollary 5.6.2. *We have $(n+1)^{n-1} = \sum_{d\in\mathcal{D}_{n,\epsilon}} \#\mathscr{L}(\mathcal{P}_d)$.*

Proof. Let $T_{n+1} := \{\text{trees on [n+1]}\}$. One has that

$$\begin{aligned}
(n+1)^{n-1} &= \#T_{n+1} \\
&= \sum_{r \geqslant 0} \#T_{n+1}(r) \\
&= \sum_{r \geqslant 0} \sum_{\substack{d \in \mathcal{D}_{n,\epsilon} \,:\, d \text{ has } r \\ \text{chords } c(i_j, i_j+1)}} \#\mathscr{L}(\mathcal{P}_d) \quad \text{(by Theorem 5.6.1)} \\
&= \sum_{d \in \mathcal{D}_{n,\epsilon}} \#\mathscr{L}(\mathcal{P}_d).
\end{aligned}$$

□

5.6.2 Reddening sequences

In [Kel12], Keller proves that for any quiver Q, any two reddening mutation sequences applied to $\widehat{Q}$ produce isomorphic ice quivers. As mentioned in [Kel13], his proof is highly dependent on representation theory and geometry, but the statement is purely combinatorial–we give a combinatorial proof of this result for the linearly-ordered quiver Q.

Let $R \in EG(\widehat{Q})$. A mutable vertex $i \in R_0$ is called **green** if there are no arrows $j \to i$ in R with $j \in [n+1, m]$. Otherwise, i is called **red**. A sequence of mutations $\mu_{i_r} \circ \cdots \circ \mu_{i_1}$ is **reddening** if all mutable vertices of the quiver $\mu_{i_r} \circ \cdots \circ \mu_{i_1}(\widehat{Q})$ are red. Recall that an isomorphism of quivers that fixes the frozen vertices is called a **frozen isomorphism**. We now state the theorem.

Theorem 5.6.3. *If $\mu_{i_r} \circ \cdots \circ \mu_{i_1}$ and $\mu_{j_s} \circ \cdots \circ \mu_{j_1}$ are two reddening sequences of $\widehat{Q}_\epsilon$ for some $\epsilon \in \{+,-\}^{n+1}$, then there is a frozen isomorphism $\mu_{i_r} \circ \cdots \circ \mu_{i_1}(\widehat{Q}_\epsilon) \cong \mu_{j_s} \circ \cdots \circ \mu_{j_1}(\widehat{Q}_\epsilon)$.*

Proof. Let $\mu_{i_r} \circ \cdots \circ \mu_{i_1}$ be any reddening sequence. Denote by C the **c**-matrix of $\mu_{i_r} \circ \cdots \circ \mu_{i_1}(\widehat{Q}_\epsilon)$. By Corollary 5.3.15, C corresponds to an oriented strand diagram $\overrightarrow{d}_C \in \overrightarrow{\mathcal{D}}_{n,\epsilon}$ with all chords of the form $\overrightarrow{c}(j,i)$ for some i and j satisfying $i < j$. As $\overrightarrow{d}_C$ avoids the configurations described in Defintion 5.3.13, we conclude that $\overrightarrow{d}_C = \{\overrightarrow{c}(i,i-1)\}_{i \in [n]}$ and $C = -I_n$. Since **c**-matrices are in bijection with ice quivers in $EG(\widehat{Q}_\epsilon)$ (see [NZ12, Thm 1.2]) and since $\check{Q}_\epsilon$ is an ice quiver in $EG(\widehat{Q}_\epsilon)$ whose **c**-matrix is $-I_n$, we obtain the desired result. □

5.6.3 Noncrossing partitions and exceptional sequences

In this section, we give a combinatorial proof of Ingalls' and Thomas' result that complete exceptional sequences are in bijection with maximal chains in the lattice of noncrossing partitions [IT09]. We remark that their result is more general than that which we present here. Throughout this section, we assume that Q_ϵ has $\epsilon = (-, \ldots, -)$ and we regard the strand diagrams of Q_ϵ as chord diagrams.

A **partition** of $[n]$ is a collection $\pi = \{B_\alpha\}_{\alpha \in I} \in 2^{[n]}$ of subsets of $[n]$ called **blocks** that are nonempty, pairwise disjoint, and whose union is $[n]$. We denote the lattice of set partitions of $[n]$ by Π_n. A set partition $\pi = \{B_\alpha\}_{\alpha \in I} \in \Pi_n$ is called **noncrossing** if for any $i < j < k < \ell$ where $i, k \in B_{\alpha_1}$ and $j, \ell \in B_{\alpha_2}$, one has $B_{\alpha_1} = B_{\alpha_2}$. We denote the lattice of noncrossing partitions of $[n]$ by $NC^{\mathbb{A}}(n)$.

Label the vertices of a convex n-gon $\mathcal{S}$ with elements of $[n]$ so that reading the vertices of $\mathcal{S}$ counterclockwise determines an increasing sequence mod n. We can thus regard $\pi = \{B_\alpha\}_{\alpha \in I} \in NC^{\mathbb{A}}(n)$ as a collection of convex hulls B_α of vertices of $\mathcal{S}$ where B_α has empty intersection with any other block $B_{\alpha'}$.

Let $n = 5$. The following partitions all belong to Π_5, but only $\pi_1, \pi_2, \pi_3 \in NC^{\mathbb{A}}(5)$.

$$\pi_1 = \{\{1\}, \{2, 4, 5\}, \{3\}\},\ \pi_2 = \{\{1, 4\}, \{2, 3\}, \{5\}\},$$

$$\pi_3 = \{\{1, 2, 3\}, \{4, 5\}\},\ \pi_4 = \{\{1, 3, 4\}, \{2, 5\}\}$$

Below we represent the partitions $\pi_1, \ldots, \pi_4$ as convex hulls of sets of vertices of a convex pentagon. We see from this representation that $\pi_4 \notin NC^{\mathbb{A}}(5)$.

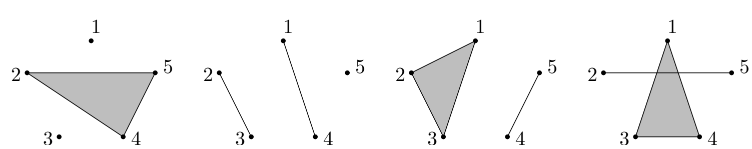


Figure 5.18: Some partitions of [5] represented geometrically as convex hulls to illustrate the 'noncrossing' condition

Theorem 5.6.4. *Let $k \in [n]$. Also, let $\mathcal{S}(k+1)$ denote the set of chains*

$$\{\{i\}\}_{i \in [n+1]} < \pi_1 < \cdots < \pi_k \in (NC^{\mathbb{A}}(n+1))^{k+1}$$

such that $\pi_j = (\pi_{j-1} \backslash \{B_\alpha, B_\beta\}) \sqcup \{B_\alpha \sqcup B_\beta\}$ *for some* $B_\alpha \neq B_\beta$ *in* π_{j-1}. *There is a natural bijection between* $\mathcal{D}_{k,\epsilon}(k)$ *and* $\mathcal{S}(k+1)$.

In particular, when $k = n$, *there is a bijection between* $\mathcal{D}_{n,\epsilon}(n)$ *and maximal chains in* $NC^{\mathbb{A}}(n+1)$. *We remark that each chain in* $\mathcal{S}(k+1)$ *is* ***saturated*** *(i.e. each inequality appearing in* $\{\{i\}\}_{i\in[n+1]} < \pi_1 < \cdots < \pi_k$ *is a covering relation).*

Proof. Let $d(k) = \{(c(i_\ell, j_\ell), \ell)\}_{\ell\in[k]} \in \mathcal{D}_{k,\epsilon}(k)$. Define $\pi_{d(k),1} := \{\{i\}\}_{i\in[n+1]} \in \Pi_{n+1}$. Next, define $\pi_{d(k),2} := \left(\pi_{d(k),1} \backslash \{\{i_1+1\}, \{j_1+1\}\}\right) \sqcup \{i_1+1, j_1+1\}$. Now assume that $\pi_{d(k),s}$ has been defined for some $s \in [k]$. Define $\pi_{d(k),s+1}$ to be the partition obtained by merging the blocks of $\pi_{d(k),s}$ containing i_s+1 and j_s+1. Now define $f(d(k)) := \{\pi_{d(k),s} : s \in [k+1]\}$.

It is clear that $f(d(k))$ is a chain in Π_{n+1} with the desired property as $\pi_1 \lessdot \pi_2$ in Π_{n+1} if and only if π_2 is obtained from π_1 by merging exactly two distinct blocks of π_1. To see that each $\pi_{d(k),s} \in NC^{\mathbb{A}}(n+1)$ for all $s \in [k+1]$, suppose a crossing of two blocks occurs in a partition appearing in $f(d(k))$. Let $\pi_{d(k),s}$ be the smallest partition of $f(d(k))$ (with respect to the partial order on set partitions) with two blocks crossing blocks B_1 and B_2. Without loss of generality, we assume that $B_2 \in \pi_{d(k),s}$ is obtained by merging the blocks $B_{\alpha_1}, B_{\alpha_2} \in \pi_{d(k),s-1}$ containing $i_{s-1}+1$ and $j_{s-1}+1$, respectively. This means that $d(k)$ has a chord $c(i_{s-1}, j_{s-1})$ that crosses at least one other chord of $d(k)$. This contradicts that $d(k) \in \mathcal{D}_{k,\epsilon}(k)$. Thus $f(d(k))$ is a chain in $NC^{\mathbb{A}}(n+1)$ with the desired property.

Next, we define a map g that is the inverse of f. To this end, let $C = (\pi_1 = \{\{i\}\}_{i\in[n+1]} < \pi_2 < \cdots < \pi_{k+1}) \in (NC^{\mathbb{A}}(n+1))^{k+1}$ be a chain where each partition in C satisfies $\pi_j = (\pi_{j-1} \backslash \{B_\alpha, B_\beta\}) \sqcup \{B_\alpha \sqcup B_\beta\}$ for some $B_\alpha \neq B_\beta$ in π_{j-1}. As $\pi_2 = (\pi_1 \backslash \{\{s_1\}, \{t_1\}\}) \sqcup \{s_1, t_1\}$, define $c(i_1, j_1) := c(s_1 - 1, t_1 - 1)$ where we consider $s_1 - 1$ and $t_1 - 1$ mod $n+1$. Assume $s_1 < t_1$. If t_1 is in a block of size 3 in π_3, let t denote the element of this block where $t \neq s_1, t_1$. If t satisfies $s_1 < t < t_1$, define $c(i_2, j_2) := c(s_1 - 1, t - 1)$. Otherwise, define $c(i_2, j_2) := c(t_1 - 1, t - 1)$. If there is no block of size 3 in π_3, define $c(i_2, j_2) := c(s_2 - 1, t_2 - 1)$ where $\{s_2\}$ and $\{t_2\}$ were singleton blocks in π_2 and $\{s_2, t_2\}$ is a block in π_3.

Now suppose we have defined $c(i_r, j_r)$. Let B denote the block of π_{r+2} obtained by merging two blocks of π_{r+1}. If B is obtained by merging two singleton blocks $\{s_{r+1}\}, \{t_{r+1}\} \in \pi_{r+1}$, define $c(i_{r+1}, j_{r+1}) := c(s_{r+1}-1, t_{r+1}-1)$. Otherwise, $B = B_1 \sqcup B_2$

where $B_1, B_2 \in \pi_{r+1}$. Now note that, up to rotation and up to adding or deleting elements of $[n+1]$ for B_1 and B_2, $B_1 \sqcup B_2$ appears in π_{r+2} as follows.

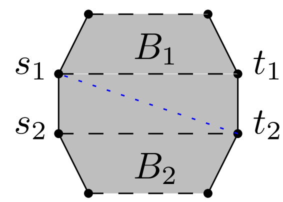


Figure 5.19: An illustration of how the union of two blocks appears in a partition

Thus define $c(i_{r+1}, j_{r+1}) := c(s_1 - 1, t_2 - 1)$. Finally, put $g(C) := \{(c(i_\ell, j_\ell), \ell) : \ell \in [k]\}$.

We claim that $g(C)$ has no crossing chords. Suppose that $(c(s_i, t_i), i)$ and $(c(s_j, t_j), j)$ are crossing chords in $g(C)$ with $i < j$ and $i, j \in [k]$. We further assume that

$$j = \min\{j' \in [i+1, k] : (c(s_{j'}, t_{j'}), j') \text{ crosses } (c(s_i, t_i), i) \text{ in } g(C)\}.$$

We observe that $s_i + 1, t_i + 1 \in B_1$ for some block $B_1 \in \pi_j$ and that $s_j + 1, t_j + 1 \in B_2$ for some block $B_2 \in \pi_{j+1}$. We further observe that $s_j + 1, t_j + 1 \notin B_1$ otherwise, by the definition of the map g, the chords $(c(s_i, t_i), i)$ and $(c(s_j, t_j), j)$ would be noncrossing. Thus $B_1, B_2 \in \pi_{j+1}$ are distinct blocks that cross so $\pi_{j+1} \notin NC^{\mathbb{A}}(n+1)$. We conclude that $g(C)$ has no crossing chords so $g(C) \in \mathcal{D}_{k,\epsilon}(k)$.

To complete the proof, we show that $g \circ f = 1_{\mathcal{D}_{k,\epsilon}(k)}$. The proof that $f \circ g$ is the identity map is similar. Let $d(k) \in \mathcal{D}_{k,\epsilon}(k)$. Then $f(d(k)) = \{\{i\}\}_{i \in [n+1]} < \pi_1 < \cdots < \pi_k$ where for any $s \in [k]$ we have

$$\pi_s = (\pi_{s-1} \backslash \{B_\alpha, B_\beta\}) \sqcup \{B_\alpha, B_\beta\}$$

where $i_{s-1} + 1 \in B_\alpha$ and $j_{s-1} + 1 \in B_\beta$. Then we have

$$g(f(d(k))) = \{c((i_\ell + 1) - 1, (j_\ell + 1) - 1), \ell)\}_{\ell \in [k]} = \{(c(i_\ell, j_\ell), \ell)\}_{\ell \in [k]}.$$

□

Corollary 5.6.5. *If $\epsilon = (-, \ldots, -)$, then the exceptional sequences of Q_ϵ of size k are in bijection with the elements of $\mathcal{S}(k+1)$.*

Example 5.6.6. *Here we give examples of the bijection from the previous theorem with $k = 4$.*

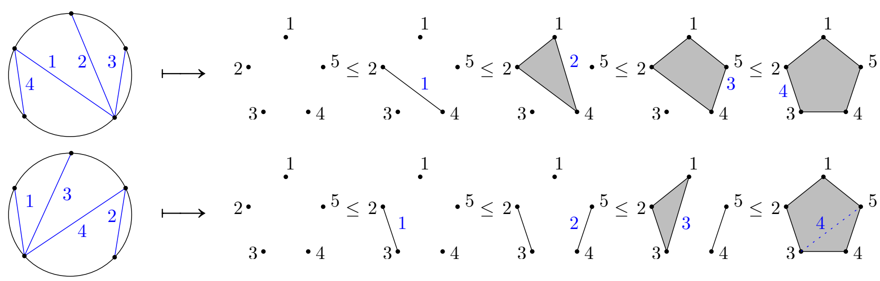


Figure 5.20: Two examples of the bijection between $\mathcal{D}_{4,\epsilon}(4)$ and $\mathcal{S}(5)$

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