Modeling and control of collective dynamics: From Schrödinger bridges to Optimal Mass Transport

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Dedication

To my parents

Abstract

We study modeling and control of collective dynamics. More specifically, we consider the problem of steering a particle system from an initial distribution to a final one with minimum energy control during some finite time window. It turns out that this problem is closely related to Optimal Mass Transport (OMT) and the Schrödinger bridge problem (SBP). OMT is concerned with reallocating mass from a specified starting distribution to a final one while incurring minimum cost. The SBP, on the other hand, seeks a most likely density flow to reconcile two marginal distributions with a prior probabilistic model for the flow. Both of these problems can be reformulated as those of controlling a density flow that may represent either a model for the distribution of a collection of dynamical systems or, a model for the uncertainty of the state of single dynamical system. This thesis is concerned with extensions of and point of contact between these two subjects, OMT and SBP. The aim of the work is to provide theory and tools for modeling and control of collections of dynamical systems. The SBP can be seen as a stochastic counterpart of OMT and, as a result, OMT can be recovered as the limit of the SBP as the stochastic excitation vanishes. The link between these two problems gives rise to a novel and fast algorithm to compute solutions of OMT as a suitable limit of SBP. For the special case where the marginal distributions are Gaussian and the underlying dynamics linear, the solution to either problem can be expressed as linear state feedback and computed explicitly in closed form.

A natural extension of the work in the thesis concerns OMT and the SBP on discrete spaces and graphs in particular. Along this line we develop a framework to schedule transportation of mass over networks. Control in this context amounts to selecting a transition mechanism that is consistent with initial and final marginal distributions. The SBP on graphs on the other hand can be viewed as an atypical stochastic control problem where, once again, the control consists in suitably modifying the prior transition mechanism. By taking the Ruelle-Bowen random walk as a prior, we obtain scheduling that tends to utilize all paths as uniformly as the topology allows. Effectively, a consequence of such a choice is reduced congestion and increased robustness. The paradigm of Schrödinger bridges as a mechanism for scheduling transport on networks can be adapted to weighted graphs. Thus, our approach may be used to design transportation plans that represent a suitable compromise between robustness and cost of transport.

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Nomenclature

Abbreviations and Acronyms

a.s.	almost surely
ARE	Algebraic Riccati Equation
DP	Dynamic Programming
LHS	Left-Hand Side
OMT	Optimal Mass Transport
OMT-wpd	Optimal Mass Transport with prior dynamics
RB	Ruelle-Bowen
RHS	Right-Hand Side
SBP	Schrödinger Bridge Problem
SDE	Stochastic Differential Equation
SDP	Semidefinite Programming

List of Symbols

A	Dynamic in state space model
B	Control input channel in state space model
B_1	Noise channel in state space model
$\mathbb{D}(\mu_0, \mu_1)$	Set of measures on $C([0,1], \mathbb{R}^n)$ with marginals μ_0 and μ_1
$H(\cdot, \cdot)$	Relative entropy
$L(t, x, u)$	Lagrangian
μ_0	Initial distribution
μ_1	Terminal distribution
φ, $\hat{\varphi}$	Factors of Schrödinger bridges
Ω	Path space $C([0,1], \mathbb{R}^n)$
$P_2(\mathbb{R}^n)$	Space of all distributions on $\mathbb{R}^n$ with finite second order moment
$\Pi(\mu_0, \mu_1)$	Set of joint distributions on $\mathbb{R}^n \times \mathbb{R}^n$ with marginals μ_0 and μ_1
$\Psi(x)$	Terminal cost function
$\mathbb{R}^n$	Euclidean space of dimension n
ρ_0, ρ_1	Initial and terminal densities
$V(t, x)$	Value function
$\\| \cdot \\|$	Euclidean distance

Chapter 1

Introduction

Collective dynamics refers to the group behavior of collections of systems. In the present work, we are interested in the modeling and control of such collections. The main object of interest is the (probability) distribution of the states of the underlying dynamical systems. Imagine a cloud of particles (smoke, dust, pollutants, etc.) that diffuse in the air, or the ground, through porous media. Evidently, it is of great practical interest to be able to interpolate flow and density measurements and to control the spread of such distributions. Controlling the distribution of particles is easier to envisage in the context of focusing particle beams or directing swarms of robots, a topic with a rapidly growing literature. Other paradigms include the control of thermodynamic systems as well as quality control of industrial and manufacturing processes. The former relates to suppressing thermodynamic stochastic excitation, effecting "active cooling" for the purpose of e.g., improving performance in certain high-resolution instruments. In quality control, it is the state-uncertainty of a stochastic dynamical process that is to be modeled and regulated.

A distribution may equally well represent the state of a collection of dynamical systems or the uncertainty in the state of a single dynamical process. Either way, in broad terms, our aim is (i) to model the corresponding density-flow and reconcile priors with measurements, and (ii) to shape and regulate the flow by a suitable control action. These seemingly different questions, the first pertaining to estimation and the second to control, turn out to be closely related. Indeed, the problem of shaping the distribution of a collection of particles with minimum energy control is equivalent to the problem of constructing a model that reconciles observed experimental marginal distributions and is closest to a given prior in the sense of a large-deviation principle. More specifically, the experimentally observed marginals may represent a rare event for the given prior probability law, and modeling the flow amounts to determining the most likely trajectories of the particles. These two problems exemplify the relation between Optimal Mass Transport (OMT) and Schrödinger bridges, and fit within the broader framework of Stochastic Optimal control.

The OMT problem goes back to 1781, when Gaspard Monge asked for the most efficient way to transport mass between two locations. An account of the historical context and extensive literature can be found in [1]. The problem formulation calls for an efficient transportation plan to move mass between two given end-point marginal distributions. A satisfactory approach to solving such problems withstood the efforts of many brilliant mathematical minds, until 1942, when Leonid Kantorovich introduced duality and linear programming along with a suitable "relaxation" of Monge's problem to optimize resource allocation. This area is currently undergoing a new, rapidly developing phase, starting from mid 1990's when contributions by Brenier, McCann, Gangbo, Rachev, Cullen, Mather and others, led to new insights and techniques for a range of problems in physics (fluids, porous media, galaxy dynamics, etc.), probability, financial mathematics, economics, and operations research (see the two recent monographs [1, 2] by Villani).

The Schrödinger bridge problem was first posed in 1931/32 by Erwin Schrödinger [3, 4], one of the founders of Quantum Mechanics. The question was to identify the most likely paths that diffusive particles may have taken, given experimentally observed marginal distributions at the beginning and end-point of a time interval. In asking such a question, Schrödinger, wanted to gain insights for a new interpretation of Quantum Mechanics and thereby explain the famous equation that bears his name by purely classical means. As a side note, Stochastic Mechanics, born out of this line of research, has to some degree fulfilled Schrödinger's original aim [5, 6]. Thus, the probability structure of diffusions that interpolate given marginals at two points in time are known as Schrödinger bridges; the mathematical problem consists of specifying a suitable probability law on the space of all possible paths that is consistent with the marginals and is closest in the relative entropy sense to the given prior. In Schrödinger's work the prior was simply the law of Brownian motion.

As we will see, both of the above topics amount to problems in stochastic optimal control where the goal is to steer density functions. Indeed, they can be reformulated as the optimal control problems to minimize the cost of reallocating distributions with and without stochastic excitation, respectively. More specifically, for the Schrödinger problem, particles are diffusive whereas for the OMT problem, they obey deterministic "inertia-less" dynamics; most of the technical issues relate to the nature of the cost and the geometry of the underlying space. The initial focus on "non-degenerate" diffusions and on "inertia-less" particles in the original formulation of the Schrödinger and OMT problems, respectively, is evidently limiting. Herein, we consider particles with inertia and, possibly, "degenerate" diffusions, and we address some of the same issues at the interface between the two topics. As it turned out, this has been a fruitful endeavor with many potential applications. By studying OMT and SBP from a controls perspective, we are able to address a range of problems that pertain to collective dynamics. SBP can be seen as a stochastic version of the OMT problem – the former converges to the latter as the stochastic excitation vanishes. As a byproduct, a fast algorithm that we have developed for the Schrödinger problem can be applied to obtain approximate solutions to OMT. This should prove advantageous in high dimensional spaces where computations are challenging, and compare favorably to state of the art techniques [7–9].

A special case of density steering problems of particular interest is when the underlying dynamics are linear, the cost functional quadratic, and the goal is to drive from one Gaussian distribution to another with minimum cost. In this case we are able to obtain optimal control laws in closed form, as state feedback, very much like in standard linear quadratic optimal control. The flow of densities remains within the family of Gaussian distributions. Thus, this linear-quadratic version of the theory (OMT and SBP) can be seen as a covariance control problem, that is, the problem to control the covariance of the Gaussian distribution of the state vector (e.g., see [10] for a stationary version of such a problem). In this direction, we also consider cases where noise and control channels differ as well as where the cost functional has a quadratic penalty on the state vector.

The last part of the thesis is concerned with OMT and SBP on discrete spaces. Motivation is provided by problems of mass transport and of resource allocation over networks. The goal is to schedule a robust transport plan that carries mass (resources, goods, etc.) from a starting distribution to a final one (target distribution), and it does so by utilizing all alternative paths in an efficient manner. More specifically, our aim is to devise transport scheduling that is robust and economical in the sense of reducing fragility (sensitivity to failures of links), and at the same time, reducing wear that may be caused by excessive usage of a few key links. The approach we take is based on SBP on graphs. The solution to such a problem provides a transport plan that effects the desired mass transport and it is the closest to the prior transport mechanism in the sense of relative entropy. Therefore, the solution may inherit important properties from the prior mechanism. The prior is now seen as a design parameter. By selecting as prior the Ruelle-Bowen (RB) random walk [11,12], which has the striking property of giving equal probability to all paths of equal length (a type of uniform measure on paths), we can achieve transport plans that utilize all alternatives as much as the topology allows. The choice of this particular prior leads to low probability of conflict and congestion, and ensures a certain degree of inherent robustness of the resulting transport plan. Our approach to consider SBP on graphs with respect to the RB random walk can be generalized to weighted graphs. In fact, the choice of a prior distribution may be used to ensure that the resulting transportation achieves a compromise between robustness and cost. It appears attractive with respect to robustness when compared to OMT strategies that optimize cost by solving a linear program. Thus, the approach to scheduling transport based on Schrödinger bridges affords great flexibility and potential practical advantages.

Overview of the contents We first give a brief introduction to the three main areas, optimal control, optimal mass transport, and Schrödinger bridges in Chapters 2, 3 and 4, respectively. In Chapter 4, we also provide a proof of existence and uniqueness of solutions to SBP from which we derive a fast computational algorithm. In Chapter 5, we formulate the problems to model and control collective dynamics. In Chapter 6, we explore the case of linear dynamics and Gaussian marginals. Explicit solutions are presented and several extensions are studied. In Chapter 7 we consider similar issues on graphs and networks. In Chapter 8 we provide some concluding thoughts and discuss future directions.

Chapter 2

Optimal control

Optimal control theory concerns the problem of finding an optimal control strategy for driving a dynamical system while minimizing a given cost function. Depending on the underlying dynamics, the problem can be deterministic or stochastic. In this chapter, we review basic results in optimal control theory that are relevant to our work.

2.1 Deterministic optimal control problem

Consider a dynamical system

$$\dot{x}(t) = f(t, x(t), u(t)), \quad x(0) = x_0,$$

where $x \in \mathbb{R}^n$ denotes the state vector and $u \in \mathbb{R}^m$ denotes the control input. The finite horizon optimal control problem is to minimize the cost function [1]

$$J(u) = \int_0^1 L(t, x(t), u(t))dt + \Psi(x(1)).$$

We refer to L as the *running cost*, or the *Lagrangian*, and Ψ as the *terminal cost*. We require the optimization variable $u(\cdot)$ to be piecewise continuous [13]. The control inputs are often restricted to a subset of $\mathbb{R}^m$. The terminal state $x(1)$ may have to satisfy additional constraints. In this chapter, we consider only the simple case

[1] Here, without loss of generality, we choose the time interval to be [0, 1] because general time interval $[t_0,\, t_1]$ can be converted into this case by scaling.

without all these complexities. We refer the interested reader to [13–16].

Two main methods to solve optimal control problems are *Pontryagin's principle* [13] due to Lev Pontryagin and his students, and *dynamic programming* (DP) [13] due to Richard Bellman. We discuss only the latter as it will be needed in this work.

2.1.1 Dynamic programming

Compared to Pontryagin's principle, which gives a necessary condition for optimality, dynamic programming provides a sufficient and necessary condition. A key ingredient in dynamic programming is the *value function*, or cost-to-go function, defined by

$$V(t,x) = \inf_u \left\{ \int_t^1 L(t,x(t),u(t))dt + \Psi(x(1)) \mid x(t) = x \right\}$$

for $t \in [0,1]$ and $x \in \mathbb{R}^n$. Here the minimization is taken over all possible control inputs from t to 1 and the state satisfies $x(t) = x$. Clearly, V satisfies the boundary condition

$$V(1,x) = \Psi(x), \quad \forall x \in \mathbb{R}^n.$$

Due to the additivity of the cost function, one can show that for any control input $u(\cdot)$ the inequality

$$V(t,x) \le \int_t^r L(s,x(s),u(s))ds + V(r,x(r)),$$

holds for all $0 \le t \le r \le 1$. In fact, the minimum value of the RHS is equal to the LHS; this is the famous *dynamic programming principle* [13].

Lemma 1 *For any* $0 \le t \le r \le 1$

$$V(t,x) = \inf_u \left\{ \int_t^r L(s,x(s),u(s))ds + V(r,x(r)) \mid x(t) = x \right\}.$$

The DP principle provides a criterion to verify if a given control policy is optimal or not. In the above, taking $r = t+h$ yields

$$\inf_u \left\{ \frac{1}{h} \int_t^{t+h} L(s,x(s),u(s))ds + \frac{1}{h}[V(t+h,x(t+h)) - V(t,x)] \mid x(t) = x \right\} = 0.$$

Under some technical assumptions, we can let $h \to 0$ and take the limit, which leads to the *dynamic programming equation*

$$\frac{\partial}{\partial t}V(t,x) + \inf_{u\in\mathbb{R}^m}\left\{L(t,x,u) + \frac{\partial}{\partial x}V(t,x)\cdot f(t,x,u)\right\} = 0.$$

The biggest difference between the DP equation and the DP principle is that the infimum is taken over $u \in \mathbb{R}^m$ in DP equation while it is taken over all the control law $u(\cdot)$ in DP principle. Therefore, the dynamic programming equation is easier to check. One, however, has to note that much stronger assumptions are needed for the dynamic programming equation to hold. Define the Hamiltonian

$$H(t,x,p) = \inf_{u\in\mathbb{R}^m}\left\{L(t,x,u) + p\cdot f(t,x,u)\right\}.$$

Then the DP equation becomes

$$\frac{\partial}{\partial t}V(t,x) + H(t,x,\frac{\partial}{\partial x}V(t,x)) = 0.$$

This is the celebrated *Hamilton-Jacobi-Bellman (HJB)* equation. In principle, under some technical assumptions, solving the HJB equation with boundary condition $V(1,x) = \Psi(x)$ yields an optimal control policy, which in fact is given in the feedback form

$$u(t,x) = \mathrm{argmin}_{u\in\mathbb{R}^m}\left\{L(t,x,u) + \frac{\partial}{\partial x}V(t,x)\cdot f(t,x,u)\right\}.$$

However, in general, the HJB equation rarely has a solution in the classical sense. One has to resort to weak solutions. The highly fine-tuned theory of *viscosity solutions* [13, 17] is designed especially for this purpose [18].

2.2 Stochastic optimal control

We next move to optimal control problems with stochastic disturbances, i.e., stochastic optimal control. This is a highly technical subject. Before going into the problem formulation, we briefly recall the basis of stochastic calculus (Itô calculus). For more details, see [13, 18, 19].

2.2.1 Probability theory, Stochastic processes and the Wiener process

Probability theory is a branch of mathematics concerned with the analysis of random phenomena. Modern theory of probability is built on the concept of probability space [18, 19], which is a triple $(\Omega, \mathbb{F}, \mathbb{P})$. Here Ω is the sample space, $\mathbb{F}$ is a σ-algebra on Ω and $\mathbb{P}$ denotes the probability measure, which is a map from $\mathbb{F}$ to $[0,\ 1]$. An $\mathbb{R}^n$-valued *random vector* is a map x from Ω to $\mathbb{R}^n$ satisfying some measurability requirements. This map induces a measure on $\mathbb{R}^n$. An $\mathbb{R}^n$-valued *stochastic process* [18, 19] is a family of maps $x(t)$ from Ω to $\mathbb{R}^n$ satisfying some measurability requirements. In our work, we are interested in diffusion processes, that is, $\mathbb{R}^n$-valued stochastic processes with continuous sample paths over the time interval $[0,\ T_f]$. In this case, the sample space can be identified with the space of continuous, n-dimensional sample paths, namely, $\Omega = C([0, T_f], \mathbb{R}^n)$.

An extremely important concept in the study of diffusion processes is Brownian motion, or equivalently, the Wiener process [18, 19]. This provides the theoretical foundation on how molecular dynamics affect large scale properties of ensembles, which was layed down more than a hundred years ago. Brownian motion is named after the Scottish botanist Robert Brown, who first observed this phenomenon in 1827. Many decades later, Albert Einstein published a paper [20] in 1905 that explained Brownian motion in detail. Brownian motion is captured by the mathematical model of the Wiener process, denoted by $w(t)$ in this thesis. The probability measure induced by the Wiener process is referred to as the Wiener measure.

2.2.2 Stochastic differential equations

Although the Wiener process itself is a good model for molecular dynamics, its significance rests on its role as a building block for more complex models, more specifically, as a driving force for the differential equations corresponding to the underlying dynamics. This class of differential equations, which we refer to as stochastic differential equations (SDEs) [19], is perhaps the most important and widely used form of stochastic modeling in continuous time setting.

A commonly used SDE in control is

$$dx(t) = f(t, x(t))dt + \sigma(t, x(t))dw(t),\ \ x(0) = x_0,\ \text{ a.s.} \tag{2.1}$$

where $dw(t)$, known as white noise, represents the driving force induced by Brownian motion and a.s. means almost surely. Under some proper assumptions on f, σ and x_0 [19], equation (2.1) has a solution. There are two types of solutions, strong and weak [19]. Strong solutions have the property that $x(\cdot)$ causally depends on the past, while requiring very strong assumptions on f, σ. The assumptions for weak solutions to exist are much weaker. Even though weak solutions do not possess the causal dependence property of strong solutions, it is a powerful tool widely used in stochastic control theory [13, 18].

For linear dynamics, the associated SDE

$$dx(t) = A(t)x(t)dt + B(t)dw(t), \; x(0) = x_0. \tag{2.2}$$

always has a unique solution, given explicitly by

$$x(t) = \Phi(t, 0)x_0 + \int_0^t \Phi(t, \tau)B(\tau)dw(\tau).$$

Here $\Phi(t, s)$ is the state-transition matrix of (2.2) determined by

$$\frac{\partial}{\partial t}\Phi(t, s) = A(t)\Phi(t, s) \;\text{ and }\; \Phi(t, t) = I,$$

with I denoting the identity matrix.

2.2.3 Stochastic optimal control

In stochastic control theory the basic object of interest is a SDE with a control input, namely

$$dx(t) = f(t, x(t), u(t))dt + \sigma(t, x(t), u(t))dw(t), \; x(0) = x_0.$$

A control policy is said to be *admissible* if the resulting controlled process admits a (weak) solution. The control input $u(t)$ is assumed to be adapted[2] to the process $x(t)$. Here we consider the Markov control strategy $u = \alpha(t, x(t))$ only. For the problems we are interested in, the optimal controls are indeed of the form $u = \alpha(t, x(t))$.

[2] One process is adapted to another process if the current value of the former depends only on the past values of the latter.

The goal of stochastic optimal control is to obtain an admissible control strategy u to achieve a particular purpose by minimizing a suitable cost functional that is attached to each admissible control strategy u. Similar to the deterministic case, we consider the cost function

$$J(u) = \mathbb{E}\left\{\int_0^1 L(s, x(s), u(s))ds + \Psi(x(1))\right\}, \tag{2.3}$$

where we again refer to L as the running cost and Ψ as the terminal cost.

The two main approaches to solve stochastic optimal control problems are the stochastic counterparts of Pontryagin's principle and DP. The stochastic Pontryagin's principle involves the so-called backward stochastic differential equation [21], which is a very delicate object to handle. We shall only discuss the stochastic version of DP, which shares the same name.

2.2.4 Dynamic programming

The DP approach for stochastic control resembles that for deterministic optimal control. Define the value function

$$V(t, x) = \inf_u \mathbb{E}\left\{\int_t^1 L(t, x(t), u(t))dt + \Psi(x(1)) \mid x(t) = x\right\},$$

where the minimum on the RHS is taken over all the admissible Markov control strategy. Then, we have the DP principle

$$V(t, x) = \inf_u \mathbb{E}\left\{\int_t^r L(s, x(s), u(s))ds + V(r, x(r)) \mid x(t) = x\right\}$$

for $0 \leq t \leq r \leq 1$. In other words, the process

$$\int_0^t L(s, x(t), u(t))ds + V(t, x(t))$$

is a martingale [18] when u is the optimal control policy. Taking $r = t+h$, multiplying both sides by $1/h$ and letting $h \to 0$ we obtain, at least formally, the DP equation

$$\frac{\partial}{\partial t}V(t, x) + \inf_{u \in \mathbb{R}^m}\left\{L(t, x, u) + f \cdot \frac{\partial}{\partial x}V(t, x) + \frac{1}{2}\,\mathrm{trace}[\sigma\sigma' \frac{\partial^2}{\partial x^2}V(t, x)]\right\} = 0.$$

Now define

$$H(t,x,p,G) = \inf_{u\in\mathbb{R}^m} \left\{ L(t,x,u) + f\cdot p + \frac{1}{2}\,\mathrm{trace}[\sigma\sigma' G] \right\}.$$

Then the dynamic programming equation becomes

$$\frac{\partial}{\partial t}V(t,x) + H(t,x,\frac{\partial}{\partial x}V(t,x),\frac{\partial^2}{\partial x^2}V(t,x)) = 0,$$

which is a second-order HJB equation. Recall that the DP equation is a first-order HJB equation in the deterministic setting. In general, the second-order HJB equation is more likely to have a classical solution because of the second-order Laplacian-like term. The optimal control strategy is

$$u(t,x) = \mathrm{argmin}_{u\in\mathbb{R}^m} \left\{ L(t,x,u) + \frac{\partial}{\partial x}V(t,x)\cdot f(t,x,u) + \frac{1}{2}\,\mathrm{trace}[\sigma\sigma'\frac{\partial^2}{\partial x^2}V(t,x)] \right\},$$

where $V(\cdot,\cdot)$ is the solution of the HJB equation with the boundary condition $V(1,x) = \Psi(x)$.

2.3 Linear Quadratic case

The Linear Quadratic Gaussian (LQG) [22] problems are a special class of optimal control that can be solved in closed form. In these problems, the underlying dynamics are linear and the cost function is quadratic. In deterministic setting, we consider the dynamics

$$\dot{x}(t) = A(t)x(t) + B(t)u(t),$$

and the cost function

$$J(u) = \int_0^1 [x(t)'Q(t)x(t) + u(t)'R(t)u(t)]dt + x(1)'Fx(1),$$

where $R(\cdot)$ is positive definite and $Q(\cdot)$, F are positive semi-definite. Similarly, for the stochastic case, the problem has the dynamics

$$dx(t) = A(t)x(t)dt + B(t)u(t)dt + B_1(t)dw(t)$$

and the cost function

$$J(u) = \mathbb{E}\left\{\int_0^1 [x(t)'Q(t)x(t) + u(t)'R(t)u(t)]dt + x(1)'Fx(1)\right\},$$

with $R(\cdot), Q(\cdot)$ and F satisfying the same assumptions as in deterministic setting. In both cases, the systems are assumed to be controllable in the sense that the reachability Gramian

$$M(t_1, t_0) := \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)B(\tau)'\Phi(t_1, \tau)'d\tau,$$

is nonsingular for all $0 \leq t_0 < t_1 \leq 1$. Here, as usual, $\Phi(t, s)$ denotes the state-transition matrix of the dynamics $A(\cdot)$.

The optimal control for both the deterministic and the stochastic problems can be established by solving the corresponding Riccati equations. In fact, they share the same optimal control strategy

$$u(t) = -R(t)^{-1}B(t)'P(t)x(t),$$

where $P(\cdot)$ satisfies the Riccati equation

$$-\dot{P}(t) = A(t)'P(t) + P(t)A(t) - P(t)B(t)R(t)^{-1}B(t)'P(t) + Q(t)$$

with the boundary condition

$$P(1) = F.$$

To derive this, one can use either Pontryagin's principle or DP. An easier approach is completion of square [23, 24] (see Chapter 6).

2.3.1 Infinite horizon optimal control

For the linear quadratic problems, we will also need the results for the infinite horizon optimal control, where the cost function becomes

$$J(u) = \int_0^\infty [x(t)'Qx(t) + u(t)'Ru(t)]dt$$

in deterministic setting, and

$$J(u) = \lim_{T_f \to \infty} \mathbb{E}\left\{ \frac{1}{T_f} \int_0^{T_f} [x(t)'Qx(t) + u(t)'Ru(t)]dt \right\}$$

in the stochastic setting. Here R is positive definite and Q is positive semi-definite. Furthermore, the underlying dynamics are assumed to be time-invariant, namely, $A(\cdot), B(\cdot), B_1(\cdot)$ are constant. These two problems (deterministic and stochastic) share the same optimal control policy

$$u(t) = -R^{-1}B'Px(t),$$

where P is the unique positive definite solution to the Algebraic Riccati Equation (ARE)

$$A'P + PA - PBR^{-1}B'P + Q = 0.$$

The infinite horizon optimal control problem can also be studied for general dynamics, but is not pursued in this thesis. Interested reader is referred to [13, 18].

Chapter 3

Optimal Mass Transport

The Optimal Mass Transport (OMT) theory is concerned with moving mass (e.g., soil, goods) from an initial distribution to a final one with the least amount of cost. The problem was first proposed by the French mathematician Gaspard Monge [25] in 1781. Later on, Leonid Kantorovich studied the problem and applied it to resource allocation [26]. The OMT theory was further developed by many great mathematicians, e.g., Brenier, Gangbo, McCann, Otto, Rachev, Villani, and it has a range of applications in physics, mathematics and economics. In this chapter, we introduce some basic results of OMT that are relevant to our work. Interested reader who wishes to learn more about OMT is referred to [1, 27].

3.1 Monge's optimal mass transport problem

Consider two nonnegative measures μ_0, μ_1 on $\mathbb{R}^n$ with equal total mass. These may represent probability distributions, distribution of resources, etc. Without loss of generality, we take μ_0 and μ_1 to be probability distributions, i.e.,

$$\mu_0(\mathbb{R}^n) = \mu_1(\mathbb{R}^n) = 1.$$

We seek a transport (measurable) map

$$T \,:\, \mathbb{R}^n \to \mathbb{R}^n \,:\, x \mapsto T(x)$$

that specifies where mass μ_0 at x must be transported so as to match the final distribution in the sense that $T_\sharp\mu_0 = \mu_1$ [28]. That is, μ_1 is the push-forward of μ_0 under T, meaning

$$\mu_1(B) = \mu_0(T^{-1}(B)) \tag{3.1}$$

for every Borel set [29] B in $\mathbb{R}^n$. In other words, we require

$$\int_{\mathbb{R}^n} h(T(x))\mu_0(dx) = \int_{\mathbb{R}^n} h(y)\mu_1(dy)$$

for every continuous function h.

A transport map T is called optimal if it achieves the minimum cost of transportation

$$\int_{\mathbb{R}^n} c(x, T(x))\mu_0(dx).$$

Here, $c(x, y)$ represents the transportation cost per unit mass from point x to point y. Let $\mathcal{T}(\mu_0, \mu_1)$ denote the set of all the feasible transport maps satisfying (3.1), then Monge's OMT problem is described as follows.

Problem 2

$$\inf_{T\in\mathcal{T}(\mu_0,\mu_1)} \quad \int_{\mathbb{R}^n} c(x, T(x))\mu_0(dx).$$

In the original formulation of OMT, due to Gaspard Monge, the cost $c(x, y)$ was chosen to be the distance $\|x - y\|$. Later on, the problem was generalized to cover more general costs like $\|x - y\|^p$ for $1 \leq p < \infty$. The special case where $p = 2$ possesses many nice properties and has been extensively studied [2].

It is a difficult task to solve Problem 2 directly due to the highly nonlinear dependence of the transportation cost on T. This fact complicated early analyses of the problem [1]. To see the difficulty, assume μ_0 and μ_1 have smooth densities ρ_0 and ρ_1, respectively. Then condition (3.1) reduces to

$$\rho_0(x) = \rho_1(T(x)) \det(DT(x)), \quad x \in \mathbb{R}^n, \tag{3.2}$$

where DT is the Jacobian matrix of the map T. If we write $T = (T^1, \ldots, T^n)'$, then

$$DT(x) = \begin{bmatrix} \frac{\partial T^1}{\partial x_1} & \cdots & \frac{\partial T^1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial T^n}{\partial x_1} & \cdots & \frac{\partial T^n}{\partial x_n} \end{bmatrix}.$$

It is not even straightforward to see that there exists a map satisfying the constraint (3.2), let alone the existence of the minimizer of Problem 2.

3.2 Kantorovich relaxation and solutions

A new chapter of OMT was opened in 1942 when Leonid Kantorovich presented a relaxed formulation [26]. Therein, instead of a transport map, we seek a joint distribution π on the product space $\mathbb{R}^n \times \mathbb{R}^n$ so that the marginal distributions coincide with μ_0 and μ_1 respectively. Namely, π satisfies

$$\int_{B\times\mathbb{R}^n} \pi(dxdy) = \int_B \mu_0(dx) \quad \text{and} \quad \int_{\mathbb{R}^n\times B} \pi(dxdy) = \int_B \mu_1(dy)$$

for any measurable $B \subset \mathbb{R}^n$. The joint distribution π is referred to as the "coupling" of μ_0 and μ_1. Let $\Pi(\mu_0, \mu_1)$ denote the set of all couplings of μ_0 and μ_1, then we arrive at the Kantorovich formulation of OMT:

Problem 3

$$\inf_{\pi\in\Pi(\mu_0,\mu_1)} \int_{\mathbb{R}^n\times\mathbb{R}^n} c(x, y)\ \pi(dxdy). \tag{3.3}$$

Note that Problem 3 is a linear program. Therefore, standard duality theory can be applied to solve it. This duality result is referred to as the *Kantorovich duality theorem* [1, 28], which plays an important role in OMT.

3.2.1 Solutions

We focus on the special case where

$$c(x, y) = \|x - y\|^2. \tag{3.4}$$

We assume that μ_0 and μ_1 are absolutely continuous [29] with densities ρ_0 and ρ_1 respectively with respect to the Lebesgue measure. Moreover, we assume μ_0 and μ_1

possess finite second moment[1], namely, $\mu_0, \mu_1 \in P_2(\mathbb{R}^n)$, the space of all probability measures with finite second order moment. Interested reader is referred to [1] for more details.

Under the above assumptions, one can show that Problem 3 has a unique solution [1,2,30] using the Kantorovich duality theorem. The optimal coupling π concentrates on the graph of the optimal mass transport map T, namely,

$$\pi = (\mathrm{Id} \times T)_\sharp \mu_0.$$

Here Id stands for the identity map. The unique optimal transport map T is the solution of Problem 2. In addition, it is the gradient of a convex function ϕ, i.e.,

$$y = T(x) = \nabla\phi(x). \tag{3.5}$$

By virtue of the fact that the push-forward of μ_0 under $T = \nabla\phi$ is μ_1 (see (3.2)), the function ϕ satisfies a particular case of the Monge-Ampère equation [1, p.126], [7, p.377]

$$\det(H\phi(x))\rho_1(\nabla\phi(x)) = \rho_0(x),$$

where $H\phi$ is the Hessian matrix of ϕ. This is a fully nonlinear second-order elliptic equation. Numerical schemes for computing ϕ have been recently developed [7], [8], [9]. However, computation of the optimal transport plan T in high dimensional spaces is still a challenge.

3.2.2 The Wasserstein distance

The OMT problem with quadratic cost (3.4) induces an important metric [1] on the space $P_2(\mathbb{R}^n)$ via

$$W_2(\mu_0, \mu_1) := \left\{ \inf_{\pi \in \Pi(\mu_0,\mu_1)} \int_{\mathbb{R}^n \times \mathbb{R}^n} \|x - y\|^2 \, \pi(dxdy) \right\}^{1/2}.$$

This metric is the so-called Wasserstein metric (distance). It induces the weak topology [31] on $P_2(\mathbb{R}^n)$. Another nice property of W_2 is that the gradient flow on $P_2(\mathbb{R}^n)$ with respect to this metric is a useful tool to study many partial differential equations. For instance, the gradient flow of the entropy functional $\int \rho \log \rho$ is the heat equation

[1] Otherwise the minimum of the transport cost could be infinity.

$\partial\rho/\partial t = \Delta\rho$. See [1, 2, 28, 32] for details.

3.2.3 Displacement interpolation

After determining the amount of mass to be moved from where to where via the optimal transport map T, we have many options on how to move mass from x to $T(x)$. For instance, one can choose to move the mass with constant speed. If so, the displacement of the mass along the path from $t = 0$ to $t = 1$ is

$$\mu_t = (T_t)_\sharp \mu_0, \quad T_t(x) = (1-t)x + tT(x). \tag{3.6a}$$

Moreover, one can show that μ_t is absolutely continuous with density

$$\rho(t, x) = \frac{d\mu_t}{dx}(x). \tag{3.6b}$$

The distribution flow μ_t, $0 \le t \le 1$ is referred to as the *displacement interpolation* [33] between μ_0 and μ_1.

The displacement interpolation is in fact the geodesic between μ_0 and μ_1 with respect to the Wassertein metric W_2; it is the unique path between μ_0 and μ_1 with minimal length and

$$W_2(\mu_t, \mu_s) = (s-t)W_2(\mu_0, \mu_1)$$

for any $0 \le t \le s \le 1$.

Another remarkable property of the displacement interpolation is that the entropy functional $\int \rho \log \rho$ is convex along the path [33]. This, in fact, opened the door to study the connection between OMT and Riemannian geometry, and led to the discover of the celebrated Lott-Sturm-Villani theory [2, 34–36].

3.3 Fluid dynamics formulation of OMT

In the previous section on displacement interpolation we have already seen the usefulness of taking the time dimension into account. Indeed, in [7], Benamou and Brenier noticed that by considering the problem during a time interval, the OMT problem can be fitted into the framework of continuum mechanics. More specifically, the OMT problem can be recast as a "fluid-dynamics" problem [7]:

$$\inf_{(\rho,v)} \int_{\mathbb{R}^n}\int_0^1 \|v(t,x)\|^2\rho(t,x)dtdx, \tag{3.7a}$$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(v\rho) = 0, \tag{3.7b}$$

$$\rho(0,x) = \rho_0(x), \quad \rho(1,y) = \rho_1(y). \tag{3.7c}$$

It turns out that the displacement interpolation $\rho(t,\cdot)$ between ρ_0 and ρ_1 is the optimal density flow for the above optimization problem. The basic idea behind this new formulation is quite intuitive. The space-time function $v(t,x)$ can be viewed as a time-varying velocity field that drives the fluid from ρ_0 to ρ_1. In the Lagrangian coordinates [37], we can define the pathline as

$$X(0,x) = x, \quad \frac{\partial}{\partial t}X(t,x) = v(t, X(t,x)).$$

This induces a transport plan $x \mapsto X(1,x)$ from ρ_0 to ρ_1. Assume ρ and v are sufficiently smooth, then

$$\begin{aligned}
\int_{\mathbb{R}^n}\int_0^1 \|v(t,x)\|^2\rho(t,x)dtdx &= \int_{\mathbb{R}^n}\int_0^1 \|v(t,X(t,x))\|^2\rho_0(x)dtdx \\
&= \int_{\mathbb{R}^n}\int_0^1 \|\frac{\partial}{\partial t}X(t,x)\|^2\rho_0(x)dtdx \\
&\geq \int_{\mathbb{R}^n} \|X(1,x)-x\|^2\rho_0(x)dx \\
&\geq \int_{\mathbb{R}^n} \|\nabla\phi(x)-x\|^2\rho_0(x)dx,
\end{aligned}$$

where $\nabla\phi(x)$ is the optimal transport map as in (3.5). On the other hand, when

$$v(t,x) = \nabla\phi(x) - x,$$

equalities are achieved in the above since the corresponding

$$X(t,x) = x + t(\nabla\phi(x) - x)$$

is a constant speed path for all x.

Note that, in the above, we made very strong assumptions on the smoothness of

ρ and v. These assumptions are not necessary. In general, the minimum in (3.7) is taken over all the pairs ρ, v satisfying (3.7) and other weaker assumptions. It is not an easy task to rigorously show the equivalence between (3.7) and the OMT problem (see [1, Theorem 8.1], [27, Chapter 8]).

Chapter 4

Schrödinger bridge problem

In this chapter, we briefly review some background of the Schrödinger bridge problem (SBP). In 1931/32, Schrödinger [3, 4] considered the following problem. A large number N of independent Brownian particles in $\mathbb{R}^n$ are observed to have an empirical distribution approximately equal to $\mu_0(dx) = \rho_0(x)dx$ at time $t = 0$, and an empirical distribution approximately equal to $\mu_1(dy) = \rho_1(y)dy$ at some later time $t = 1$. Suppose that $\rho_1(y)$ is considerably different from what it should be according to probability theory, namely

$$\int_{\mathbb{R}^n} q^B(0, x, 1, y)\rho_0(x)dx,$$

where

$$q^B(s, x, t, y) = (2\pi)^{-n/2}(t-s)^{-n/2}\exp\left(-\frac{\|x-y\|^2}{2(t-s)}\right)$$

denotes the Brownian (Gaussian) transition probability kernel. Apparently, the particles have been transported in an unlikely way. Of all the many unlikely ways in which this could have happened, which one is the most likely?

By discretization and passage to the limit, Schrödinger [3, 4] computed the most likely probability law that is consistent with the empirical marginal distributions as $N \to \infty$. He derived that the corresponding density flow $\rho(t, \cdot)$ from ρ_0 to ρ_1 is of the form

$$\rho(t, x) = \varphi(t, x)\hat{\varphi}(t, x),$$

where the factors φ and $\hat{\varphi}$ satisfy

$$\varphi(t,x) = \int_{\mathbb{R}^n} q^B(t,x,1,y)\varphi(1,y)dy, \tag{4.1a}$$

$$\hat{\varphi}(t,x) = \int_{\mathbb{R}^n} q^B(0,y,t,x)\hat{\varphi}(0,y)dy, \tag{4.1b}$$

respectively. To match the wanted marginals ρ_0 and ρ_1, the two factors also have to fulfill the boundary conditions

$$\rho_0(x) = \varphi(0,x)\hat{\varphi}(0,x), \tag{4.1c}$$

$$\rho_1(x) = \varphi(1,x)\hat{\varphi}(1,x). \tag{4.1d}$$

The coupled system (4.1) is referred to as the *Schrödinger system*. It is not obvious that this nonlinearly coupled system admits a solution, not to mention the uniqueness. In fact, it is a highly nontrivial problem. Even though Schrödinger obtained the solution to his problem and he strongly believed its validity based on his intuition, he could not mathematically prove its existence. It was almost 10 years later when Fortet [38] firstly proved the existence of the solution to the Schrödinger system for one dimensional case, i.e., $n = 1$, under some strong assumptions on the marginal distributions ρ_0 and ρ_1. Later on, Beurling [39], Jamison [40], and Föllmer [41] generalized the results to various degrees.

4.1 Problem formulation and solutions

In [41], Föllmer showed that, in the language of large deviations, Schrödinger's problem amounts to seeking a probability law on the path space that agrees with the observed empirical marginals and is the closest to the prior law of the Brownian diffusion in the sense of relative entropy [41, 42]. That is, determining a probability measure $\mathcal{P}$ on the space of continuous functions on $[0,1]$, denoted by $\Omega = C([0,1],\mathbb{R}^n)$, which minimizes the relative entropy[1]

$$H(\mathcal{P},\mathcal{W}) := \int_\Omega \log\left(\frac{d\mathcal{P}}{d\mathcal{W}}\right) d\mathcal{P}$$

over all the probability measures that are absolutely continuous [29] with respect to the stationary Wiener measure $\mathcal{W}$ [41, 42], and have prescribed marginals ρ_0 and ρ_1.

[1] $\frac{d\mathcal{P}}{d\mathcal{W}}$ denotes the Radon-Nikodym derivative.

The stationary Wiener measure $\mathcal{W}$ is the unbounded measure is induced by a Wiener process (Brownian motion) with Lebesgue measure as initial measure, namely,

$$\mathcal{W} := \int_{\mathbb{R}^n} \mathcal{W}^x \, dx$$

where $\mathcal{W}^x$ is the measure induced by the Brownian motion starting from x at $t = 0$. The prior measure can also be replaced by $\int_{\mathbb{R}^n} \mathcal{W}^x \, \eta(dx)$ for some measure η with respect to which μ_0 is absolutely continuous, i.e., $\mu_0 << \eta$. While the relative entropy between two probability measures is nonnegative [43], this is not the case for general measures. See [42, Appendix A] for more precise definition of relative entropy with respect to an unbounded measure.

In fact, the Schrödinger bridge problem solved in Beurling [39], Jamison [40], and Föllmer [41] involved general reference measures other then the Wiener measure. Let $\mathbb{D}(\mu_0, \mu_1)$ be the set of probability measures on the path space Ω with marginal distributions μ_0 and μ_1, and $\mathcal{Q}$ the prior reference measure, then the general SBP can be formulated as follows.

Problem 4 (Schrödinger bridge problem)

$$\text{minimize} \quad H(\mathcal{P}, \mathcal{Q}) \quad \text{s.t.} \quad \mathcal{P} \in \mathbb{D}(\mu_0, \mu_1).$$

Although this is an abstract problem on an infinite-dimensional space, it is a convex optimization problem since $H(\cdot, \mathcal{Q})$ is a strictly convex function. We refer to the solution of Problem 4 as the *Schrödinger bridge*. As mentioned earlier, the existence of the minimizer has been proven in various degrees of generality by Fortet [38], Beurling [39], Jamison [40], Föllmer [41]. In particular, the proof in [41] is based on the convexity of the SBP together with some compactness arguments (see also [44]). The uniqueness of the solution follows from the strict convexity of $H(\cdot, \mathcal{Q})$.

In the present work, we focus on the case where the prior reference measure $\mathcal{Q}$ is induced by certain Markovian evolution with kernel q. The result is summarized below in Theorem 5. See [42, 45, and the references therein] for results with more general reference measures other than Markovian evolution, and more general underlying spaces other than the Euclidean space $\mathbb{R}^n$.

Theorem 5 *Consider two probability measures $\mu_0(dx) = \rho_0(x)dx$ and $\mu_1(dy) = \rho_1(y)dy$ on $\mathbb{R}^n$ and a continuous, everywhere positive Markov kernel $q(s,x,t,y)$. Then there exists a unique pair of nonnegative functions $(\hat\varphi_0, \varphi_1)$ on $\mathbb{R}^n$ such that the measure $\mathcal{P}_{01}$ on $\mathbb{R}^n \times \mathbb{R}^n$ defined by*

$$\mathcal{P}_{01}(E) = \int_E q(0,x,1,y)\hat\varphi_0(x)\varphi_1(y)dxdy, \quad \forall E \in \mathbb{R}^n \times \mathbb{R}^n \tag{4.2}$$

has marginals μ_0 and μ_1. Furthermore, the Schrödinger bridge from μ_0 to μ_1 has the one-time marginal distribution flow

$$\mathcal{P}_t(dx) = \varphi(t,x)\hat\varphi(t,x)dx, \tag{4.3a}$$

where

$$\begin{aligned} \varphi(t,x) &= \int_{\mathbb{R}^n} q(t,x,1,y)\varphi_1(y)dy, && (4.3b)\\ \hat\varphi(t,x) &= \int_{\mathbb{R}^n} q(0,y,t,x)\hat\varphi_0(y)dy. && (4.3c) \end{aligned}$$

The distribution flow (4.3) is referred to as the *entropic interpolation with prior q* between μ_0 and μ_1, or simply entropic interpolation, when it is clear from the context what the Markov kernel q is.

4.2 Static Schrödinger bridge problem

Any measure on the path space $\Omega = C([0,1], \mathbb{R}^n)$ can be disintegrated as [27,42]

$$\mathcal{P}(\cdot) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \mathcal{P}^{xy}(\cdot)\, \mathcal{P}_{01}(dxdy),$$

where $\mathcal{P}_{01} \in \Pi(\mu_0, \mu_1)$ is the projection of $\mathcal{P}$ to the boundaries of the path $x(\cdot) \in \Omega$ at $t = 0$ and $t = 1$, and $\mathcal{P}^{xy} \in \mathbb{D}(\delta_x, \delta_y)$, which we refer to as the *pinned bridge* [46], is the probability measure on Ω conditioned on $x(0) = x$, $x(1) = y$, defined as

$$\mathcal{P}^{xy} = \mathcal{P}(\cdot \mid x(0) = x,\ x(1) = y).$$

Hence, the relative entropy $H(\mathcal{P}, \mathcal{Q})$ of $\mathcal{P}$ with respect to $\mathcal{Q}$ has the following form

$$H(\mathcal{P}, \mathcal{Q}) = \int_{\mathbb{R}^n \times \mathbb{R}^n} H(\mathcal{P}^{xy}, \mathcal{Q}^{xy}) \mathcal{P}_{01}(dxdy) + H(\mathcal{P}_{01}, \mathcal{Q}_{01}).$$

The first term of the RHS is nonnegative since both $\mathcal{P}^{xy}$ and $\mathcal{Q}^{xy}$ are probability measures and the relative entropy between two probability measures is nonnegative. Moreover, the two terms on the RHS are decoupled, therefore, these can be minimized independently. Intuitively, the minimum of the first term is zero by taking $\mathcal{P}^{xy}$ to be $\mathcal{Q}^{xy}$ for all x, y. That is, the Schrödinger bridge $\mathcal{P}$ shares the same pinned bridges with the prior measure $\mathcal{Q}$. The second term on the RHS is minimized over the set $\Pi(\mu_0, \mu_1)$ of all couplings between the marginal distributions μ_0 and μ_1. Namely, the optimal coupling $\mathcal{P}_{01}$ in (4.2) solves the *static Schrödinger bridge problem* as follows [42].

Problem 6 (static Schrödinger bridge problem)

$$\text{minimize} \quad H(\mathcal{P}_{01}, \mathcal{Q}_{01}) \quad \text{s.t.} \quad \mathcal{P}_{01} \in \Pi(\mu_0, \mu_1).$$

It turns out that Problem 6 and Problem 4 are indeed equivalent, as stated next [41, 42].

Proposition 1 *Problem 4 and Problem 6 admit respectively at most one solution $\hat{\mathcal{P}}$ and $\hat{\pi}$. If Problem 4 admits the solution $\hat{\mathcal{P}}$, then $\hat{\pi} = \hat{\mathcal{P}}_{01}$ solves Problem 6. Conversely, if $\hat{\pi}$ solves Problem 6, then Problem 4 admits the solution*

$$\hat{\mathcal{P}}(\cdot) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \mathcal{Q}^{xy}(\cdot)\, \hat{\pi}(dxdy), \tag{4.4}$$

which means

$$\hat{\mathcal{P}}_{01} = \hat{\pi},$$

and $\hat{\mathcal{P}}$ shares its pinned bridge with $\mathcal{Q}$, namely,

$$\hat{\mathcal{P}}^{xy} = \mathcal{Q}^{xy}, \quad \forall (x, y) \quad \hat{\pi} \text{ a.s..}$$

Problem 6 is simpler than Problem 4 since the optimization variable is a distribution on the finite dimensional space $\mathbb{R}^n \times \mathbb{R}^n$ rather than the infinite dimensional space Ω. The equivalence between these two problems implies that one needs only to worry about the two marginals when solving a SBP. It also provides the intuition for the fact that the Schrödinger bridge is in the same reciprocal class [40] as the prior process.

4.3 Fluid dynamics formulation

In Problem 6, when the prior is Markovian, the solution is also a Markov process. In particular, if the prior corresponds to a Wiener process, the solution is a Markov diffusion process with the generator

$$\nabla \log \varphi \cdot \nabla + \Delta/2, \tag{4.5}$$

where φ is one of the factor of the Schrödinger bridge as in (4.3). In other words, the Schrödinger bridge is a diffusion process with an extra drift term $\nabla \log \varphi$ compared to the Wiener process. By Girsanov's theorem, the relative entropy between the Schröndinger bridge $\mathcal{P}$ and the prior measure $\mathcal{W}$ is equal to [42, 47]

$$\int_{\mathbb{R}^n} \int_0^1 \frac{1}{2} \|\nabla \log \varphi(t,x)\|^2 \rho(t,x) dt dx$$

plus some constant. On the other hand, since the Schrödinger bridge has generator (4.5), its density flow satisfies the Fokker-Planck equation [48]

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \nabla \log \varphi) - \frac{1}{2} \Delta \rho = 0 \tag{4.6}$$

This leads to the fluid dynamic formulation of the SBP [42, 47] with prior $\mathcal{W}$

$$\inf_{(\rho,v)} \int_{\mathbb{R}^n} \int_0^1 \|v(t,x)\|^2 \rho(t,x) dt dx, \tag{4.7a}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (v\rho) - \frac{1}{2} \Delta \rho = 0, \tag{4.7b}$$

$$\rho(0,x) = \rho_0(x), \quad \rho(1,y) = \rho_1(y). \tag{4.7c}$$

The minimizer is given by

$$v(t,x) = \nabla \log \varphi(t,x).$$

An alternative equivalent reformulation of (4.7) is [49]

$$\inf_{(\rho,v)} \int_{\mathbb{R}^n} \int_0^1 \left[\|v(t,x)\|^2 + \frac{1}{4}\|\nabla \log \rho(t,x)\|^2 \right] \rho(t,x) dt dx, \tag{4.8a}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (v\rho) = 0, \tag{4.8b}$$

$$\rho(0,x) = \rho_0(x), \quad \rho(1,y) = \rho_1(y). \tag{4.8c}$$

In the above the Laplacian in the dynamical constraint is traded for a "Fisher information" regularization term in the cost functional. Note that now the constraint (4.8b) is a continuity equation. This formulation answers at once a question posed by E. Carlen in 2006 investigating the connections between optimal transport and Nelson's stochastic mechanics [50].

4.4 Alternative proof and algorithm based on Hilbert metric

As we already mentioned, the SBP (Problem 6) is a convex optimization problem, thus it can be solved using standard optimization tools after discretization. However, the complexity increases very fast as we increase the resolution of discretization. Herein, we provide an alternative proof [51] of the existence and uniqueness of Schrödinger bridge based on the Hilbert metric. Naturally, this proof leads to a simple and efficient algorithm to obtain the pair $(\hat{\varphi}_0, \varphi_1)$ in Theorem 5. This algorithm turns out to be a continuous counterpart of Sinkhorn iteration [52–54] in discrete setting, which is widely used in statistics to study contingency tables.

4.4.1 Hilbert metric

The Hilbert metric was firstly introduced by David Hilbert in 1895 [55]. The form that the metric takes to quantify distance between rays in positive cones, as used herein, is due to Garrett Birkhoff [56]. The importance of the metric and subsequent developments has been discussed in [57]. See also a recent survey on its applications in analysis [58]. The Hilbert metric and certain key facts are presented next.

Definition 7 *Let $\mathcal{S}$ be a real Banach space and let $\mathcal{K}$ be a closed solid cone in $\mathcal{S}$, i.e., $\mathcal{K}$ is a closed subset of $\mathcal{S}$ with nonempty interior, and has the properties (i) $\mathcal{K}+\mathcal{K} \subset \mathcal{K}$, (ii) $\alpha\mathcal{K} \subset \mathcal{K}$ for all real $\alpha \geq 0$ and (iii) $\mathcal{K} \cap -\mathcal{K} = \{0\}$. The cone $\mathcal{K}$*

induces a partial order relation $\preceq$ *in* $\mathcal{S}$

$$\mathbf{x} \preceq \mathbf{y} \Leftrightarrow \mathbf{y} - \mathbf{x} \in \mathcal{K}.$$

For any $\mathbf{x}, \mathbf{y} \in \mathcal{K}^+ := \mathcal{K} \backslash \{0\}$, *define*

$$M(\mathbf{x}, \mathbf{y}) \;\; := \;\; \inf\{\lambda \mid \mathbf{x} \preceq \lambda \mathbf{y}\}, \tag{4.9a}$$

$$m(\mathbf{x}, \mathbf{y}) \;\; := \;\; \sup\{\lambda \mid \lambda \mathbf{y} \preceq \mathbf{x}\} \tag{4.9b}$$

with the convention $\inf \emptyset = \infty$. *Then the Hilbert metric is defined on* $\mathcal{K}^+$ *by*

$$d_H(\mathbf{x}, \mathbf{y}) := \log \left(\frac{M(\mathbf{x}, \mathbf{y})}{m(\mathbf{x}, \mathbf{y})} \right). \tag{4.10}$$

Strictly speaking, the Hilbert metric is a projective metric since it remains invariant under scaling by positive constants, i.e., $d_H(\mathbf{x}, \mathbf{y}) = d_H(\lambda \mathbf{x}, \mathbf{y}) = d_H(\mathbf{x}, \lambda \mathbf{y})$ for any $\lambda > 0$. Thus, it actually measures the distance between rays while not elements.

A map $\mathcal{E} : \mathcal{K} \to \mathcal{K}$ is said to be *positive* if $\mathcal{E}$ maps $\mathcal{K}^+$ to $\mathcal{K}^+$. For such a map define its projective diameter

$$\Delta(\mathcal{E}) := \sup\{d_H(\mathcal{E}(\mathbf{x}), \mathcal{E}(\mathbf{y})) \mid \mathbf{x}, \mathbf{y} \in \mathcal{K}^+\}$$

and contraction ratio

$$\kappa(\mathcal{E}) := \inf\{\lambda \mid d_H(\mathcal{E}(\mathbf{x}), \mathcal{E}(\mathbf{y})) \le \lambda d_H(\mathbf{x}, \mathbf{y}), \quad \forall \mathbf{x}, \mathbf{y} \in \mathcal{K}^+\}.$$

For a positive map $\mathcal{E}$ which is also linear, we have $\kappa(\mathcal{E}) \le 1$. In fact, the Birkhoff-Bushell theorem [56–59] gives the relation between $\Delta(\mathcal{E})$ and $\kappa(\mathcal{E})$ as

$$\kappa(\mathcal{E}) = \tanh(\frac{1}{4}\Delta(\mathcal{E})). \tag{4.11}$$

Another important relation is

$$\Delta(\mathcal{E}) \le 2 \sup\{d_H(\mathcal{E}(\mathbf{x}), \mathbf{x}_0) \mid \mathbf{x} \in \mathcal{K}^+\}, \tag{4.12}$$

where $\mathbf{x}_0$ denotes an arbitrary element in the interior of $\mathcal{K}$. This follows directly from the definition of $\Delta(\mathcal{E})$ and the triangular inequality

$$d_H(\mathcal{E}(\mathbf{x}), \mathcal{E}(\mathbf{y})) \leq d_H(\mathcal{E}(\mathbf{x}), \mathbf{x}_0) + d_H(\mathcal{E}(\mathbf{y}), \mathbf{x}_0), \quad \mathbf{x}, \mathbf{y} \in \mathcal{K}^+.$$

The main objects of SBP are probability distributions. These are nonnegative, by definition, and thereby belong to a convex set (simplex) or a cone (if we dispense of the normalization). According to the Birkhoff-Bushell theorem (4.11), as we noted, linear endomorphisms of a positive cone are contractive; a fact which is often the key in obtaining solutions of the corresponding equations. Thus, the geometry underlying the SBP is expected to be impacted by endowing distributions with a suitable version of Hilbert metric.

4.4.2 Alternative proof

Herein, we summarized an alternative proof presented in [51] for the existence and uniqueness of the solution of Problem 6 based on the Hilbert metric. We focus on the case where ρ_0 and ρ_1 have compact supports. See [51] for the general case without this assumption.

Proposition 2 *Suppose that, for $i \in \{0,1\}$, $S_i \subset \mathbb{R}^n$ is a compact set, $\rho_i(x_i)dx_i$ is absolutely continuous probability measure (with respect to the Lebesgue measure) on the σ-field Σ_i of Borel sets of S_i, and that $q(0,\cdot,1,\cdot)$ is a continuous, everywhere positive function on $S_0 \times S_1$. Then, there exist nonnegative functions $\varphi(0,\cdot), \hat{\varphi}(0,\cdot)$ defined on S_0 and $\varphi(1,\cdot), \hat{\varphi}(1,\cdot)$ defined on S_1 satisfying the following* Schrödinger system *of equations:*

$$\begin{aligned}
\varphi(0,x_0) &= \int_{S_1} q(0,x_0,1,x_1)\varphi(1,x_1)dx_1, && \text{(4.13a)}\\
\hat{\varphi}(1,x_1) &= \int_{S_0} q(0,x_0,1,x_1)\hat{\varphi}(0,x_0)dx_0, && \text{(4.13b)}\\
\rho_0(x_0) &= \varphi(0,x_0)\hat{\varphi}(0,x_0), && \text{(4.13c)}\\
\rho_1(x_1) &= \varphi(1,x_1)\hat{\varphi}(1,x_1). && \text{(4.13d)}
\end{aligned}$$

Moreover, this solution is unique up to multiplication of $\varphi(0,\cdot)$ and $\varphi(1,\cdot)$ and division of $\hat{\varphi}(0,\cdot)$ and $\hat{\varphi}(1,\cdot)$ by the same positive constant.

In order to study the Schrödinger system (4.13) we consider

$$\mathcal{E}: \varphi(1,x_1) \mapsto \varphi(0,x_0) = \int_{S_1} q(0,x_0,1,x_1)\varphi(1,x_1)dx_1 \tag{4.14a}$$

$$\mathcal{E}^\dagger: \hat\varphi(0,x_0) \mapsto \hat\varphi(1,x_1) = \int_{S_0} q(0,x_0,1,x_1)\hat\varphi(0,x_0)dx_0 \tag{4.14b}$$

$$\hat{\mathcal{D}}_{\rho_0}: \varphi(0,x_0) \mapsto \hat\varphi(0,x_0) = \rho_0(x_0)/\varphi(0,x_0) \tag{4.14c}$$

$$\mathcal{D}_{\rho_1}: \hat\varphi(1,x_1) \mapsto \varphi(1,x_1) = \rho_1(x_1)/\hat\varphi(1,x_1), \tag{4.14d}$$

on appropriate domains we describe next. Define

$$\mathcal{L}^p_\epsilon(S) := \{f \in \mathcal{L}^p(S) \mid f(x) \geq \epsilon, \forall x \in S\},$$

for $\epsilon \geq 0$, and

$$\mathcal{L}^p_+(S) := \bigcup_{\epsilon>0} \mathcal{L}^p_\epsilon(S),$$

for $p \in \{1,2,\infty\}$ and $S \in \{S_0, S_1\}$, and endow $\mathcal{L}^\infty_+(S)$ with the Hilbert metric with respect to the natural partial order of inequality between elements (almost everywhere). It is noted that its closure $\mathcal{L}^\infty_0(S)$ is a closed convex cone satisfying the condition (non-empty interior) in Definition 7 of the Hilbert metric.

Since q is positive and continuous on the compact set $S_0 \times S_1$, it must be bounded from below and above, i.e., there exist $0 < \alpha \leq \beta < \infty$ such that

$$\alpha \leq q(0,x,1,y) \leq \beta, \ \forall (x,y) \in S_0 \times S_1. \tag{4.15}$$

It follows that $\mathcal{E}, \mathcal{E}^\dagger$ map nonnegative integrable functions ($\mathcal{L}^1_0$), except the zero function, to functions that are bounded below by a positive constant ($\mathcal{L}^\infty_+$). Conversely, since ρ_0 and ρ_1 are nonnegative and integrate to 1 (though, possibly unbounded), $\hat{\mathcal{D}}_{\rho_0}, \mathcal{D}_{\rho_1}$ map $\mathcal{L}^\infty_+$ to $\mathcal{L}^1_0$. Thus, the Schrödinger system relates to the following circular diagram

$$\begin{array}{ccc}
\hat\varphi(0,x_0) & \xrightarrow{\mathcal{E}^\dagger} & \hat\varphi(1,x_1) \\
\hat{\mathcal{D}}_{\rho_0} \Big\uparrow & & \Big\downarrow \mathcal{D}_{\rho_1} \\
\varphi(0,x_0) & \xleftarrow{\mathcal{E}} & \varphi(1,x_1)
\end{array}$$

where $\varphi(0,x_0), \hat\varphi(1,x_1) \in \mathcal{L}^\infty_+$, while $\varphi(1,x_1), \hat\varphi(0,x_0) \in \mathcal{L}^1_0$ on the corresponding domains S_0, S_1, i.e., the circular diagram provides a correspondence between spaces as follows,

$$\begin{array}{ccc}
\mathcal{L}_0^1(S_0) & \xrightarrow{\mathcal{E}^\dagger} & \mathcal{L}_+^\infty(S_1) \\
\hat{\mathcal{D}}_{\rho_0} \Big\uparrow & & \Big\downarrow \mathcal{D}_{\rho_1} \\
\mathcal{L}_+^\infty(S_0) & \xleftarrow{\mathcal{E}} & \mathcal{L}_0^1(S_1).
\end{array}$$

We will focus on the composition $\mathcal{C} := \mathcal{E}^\dagger \circ \hat{\mathcal{D}}_{\rho_0} \circ \mathcal{E} \circ \mathcal{D}_{\rho_1}$, that is,

$$\begin{aligned}
\mathcal{C} : \;& \mathcal{L}_+^\infty(S_1) \to \mathcal{L}_+^\infty(S_1) \\
: \;& \hat{\varphi}(1, x_1) \xrightarrow{\mathcal{E}^\dagger \circ \hat{\mathcal{D}}_{\rho_0} \circ \mathcal{E} \circ \mathcal{D}_{\rho_1}} (\hat{\varphi}(1, x_1))_{\mathrm{next}}
\end{aligned}$$

and establish the following key lemma.

Lemma 8 *There exists a positive constant $\gamma < 1$ such that*

$$d_H(\mathcal{C}(f_1), \mathcal{C}(f_2)) \leq \gamma d_H(f_1, f_2)$$

for any $f_1, f_2 \in \mathcal{L}_+^\infty$.

The rest of the proof is omitted. It lies on the strict contractiveness of $\mathcal{C}$. See [51] for more details.

4.4.3 Computational algorithm

Given marginal probability measures $\mu_0(dx) = \rho_0(x)dx$ and $\mu_1(dx) = \rho_1(x)dx$ on $\mathbb{R}^n$, we begin by specifying a compact subset $S \subset \mathbb{R}^n$ that supports most of the two densities, i.e., such that $\mu_0(S)$ and $\mu_1(S)$ are both $\geq 1 - \delta$, for sufficiently small value $\delta > 0$. We treat the restriction to S for both, after normalization so that they integrate to 1, as the end-point marginals for which we wish to construct the corresponding entropic interpolation. Thus, for the purposes of this section and subsequent examples/applications both ρ_0 and ρ_1 are supported on a compact subset $S \in \mathbb{R}^n$.

It is important to consider the potential spread of the mass along the entropic interpolation and the need for S to support the flow without "excessive" constraints at the boundary. Thus, a slightly larger compact set S, beyond what is suggested in the previous paragraph, might be necessary in some applications.

Next, we discretize in space and represent functions $\varphi(i,x)$, $\hat{\varphi}(i,x)$ ($i \in \{0,1\}$) using (column) vectors ϕ_i, $\hat{\phi}_i$, e.g., $\phi_i(k) = \varphi(i,x_k)$ for a choice of sample points $x_k \in S$, $k = 1,\dots,N$ and, likewise, ρ_0, ρ_1 (column) vectors representing the sampled values of the two densities. Then, we recast (4.14) as operations on these vectors. Accordingly,

$$\begin{aligned}
\mathcal{E}:\ & \phi_1 \mapsto \phi_0 = Q\phi_1 && \text{(4.16a)}\\
\mathcal{E}^\dagger:\ & \hat{\phi}_0 \mapsto \hat{\phi}_1 = Q^\dagger\hat{\phi}_0 && \text{(4.16b)}\\
\hat{\mathcal{D}}_{\rho_0}:\ & \phi_0 \mapsto \hat{\phi}_0 = \rho_0 \oslash \phi_0 && \text{(4.16c)}\\
\mathcal{D}_{\rho_1}:\ & \hat{\phi}_1 \mapsto \phi_1 = \rho_1 \oslash \hat{\phi}_1, && \text{(4.16d)}
\end{aligned}$$

using the same symbols for the corresponding operators, and using $\oslash$ to denote entry-wise division between two vectors, i.e., $a \oslash b := [a_i/b_i]$. Here, Q represents a matrix. The values of its entries depend on the prior kernel q. By iterating the discretized action of $\mathcal{C}$, we obtain a fixed-point pair of vectors $(\phi_i, \hat{\phi}_i)$. From this we can readily construct the entropic interpolation between the marginals by discretizing for intermediate points in time. By Lemma 8, the speed of convergence is linear with rate of convergence γ. The rate also depends on the compact sets we select for ρ_0 and ρ_1, see (4.15).

We note that, when the noise intensity is too small, numerical issues may arise due to limited machine precision. One way to alleviate this effect, especially regarding (4.16c-4.16d), is to store and operate with the logarithms of elements in $Q, \rho_i, \phi_i, \hat{\phi}_i$, denoted by $lQ, l\rho_i, l\phi_i, l\hat{\phi}_i$ ($i \in \{0,1\}$). More specifically, let $(lQ)_{jk} = \log Q_{jk}$ and set

$$(l\rho_i)_j = \begin{cases} \log(\rho_i)_j & \text{if } \ (\rho_i)_j > 0, \\ -10000 & \text{otherwise,} \end{cases}$$

(since, e.g., in double precision floating point numbers $< 10^{-308}$ are taken to be zero). Carrying out operations in (4.16) in logarithmic coordinates, $\hat{\mathcal{D}}_{\rho_0}$ becomes

$$(l\hat{\phi}_0)_j = (l\rho_0)_j - (l\phi_0)_j,$$

and similarly for $\mathcal{D}_{\rho_1}$. The map $\mathcal{E}^\dagger$ becomes

$$
\begin{aligned}
(l\hat{\phi}_1)_k &= \log \sum_j \exp(lQ_{jk} + (l\hat{\phi}_0)_j) \\
&= M_k + \log \sum_j \exp(lQ_{jk} + (l\hat{\phi}_0)_j - M_k),
\end{aligned}
$$

(and similarly for $\mathcal{E}$), where $M_k = \max_j\{lQ_{jk} + (l\hat{\phi}_0)_j\}$. In this way the dominant part of the power index, which is M_k, is respected.

Chapter 5

Steering of densities

In this chapter, we study the general modeling and control problems of collective dynamics. More specifically, we are interested in stochastic control problems of steering the probability density of the state vector of a linear system between an initial and a final distribution for two cases, i) with and ii) without stochastic disturbance. That is, we consider the linear dynamics

$$dx(t) = A(t)x(t)dt + B(t)u(t)dt + \sqrt{\epsilon}B(t)dw(t) \tag{5.1}$$

where w is a standard Wiener process, $u \in \mathbb{R}^m$ is a control input, $x \in \mathbb{R}^n$ is the state process, and A, B is a controllable pair of continuous matrices, for the two cases where i) $\epsilon > 0$ and ii) $\epsilon = 0$. In either case, the state is a random vector with an initial distribution μ_0. Our goal is to determine a minimum energy control that drives the system to a final state distribution μ_1 over the time interval $[0,\ 1]$[1], that is, the minimum of

$$\mathbb{E}\{\int_0^1 \|u(t)\|^2 dt\} \tag{5.2}$$

subject to μ_1 being the probability distribution of the state vector at the terminal time $t = 1$. The distribution of the state vector may represent either the probability distribution of the state of a single dynamical system where the goal is to reduce the uncertainty of the state, or the distribution of the states of a collection of identical systems where the goal is to control the collective dynamics. Since the main object

[1] Without loss of generality, we consider the problem during the time window $[0,\ 1]$ because any finite time window can be reduced to this case by rescalling.

is density, we also refer to the problem as the density steering problem. Several generalizations of this problem will also be briefly mentioned at the end of this chapter.

When the state distribution represents the density of particles whose position obeys $\dot{x}(t) = u(t)$ (i.e., $A(t) \equiv 0$, $B(t) \equiv I$, and $\epsilon = 0$) the problem reduces to the classical Optimal Mass Transport (OMT) (see Chapter 3) with a quadratic cost [1,7]. Thus, the above problem, for $\epsilon = 0$, represents a generalization of OMT to deal with particles obeying known "prior" *non-trivial* dynamics while being steered between two end-point distributions – we refer to this as the problem of *OMT with prior dynamics* (OMT-wpd). Applications are envisioned in the steering of particle beams through time-varying potential, the steering of swarms (UAV's, large collection of microsatelites, etc.), as well as in the modeling of the flow and collective motion of particles, clouds, platoons, flocking of insects, birds, fish, etc. between end-point distributions [60] and the interpolation/morphing of distributions [8]. From a controls perspective, "important limitations standing in the way of the wider use of optimal control can be circumvented by explicitly acknowledging that in most situations the apparatus implementing the control policy will judged on its ability to cope with a distribution of initial states, rather than a single state" as pointed out by R. Brockett in [61, page 23]. A similar problem has been studied in [62] from another angle.

For the case where $\epsilon > 0$, the flow of "particles" is dictated by dynamics as well as by Brownian diffusion. The corresponding stochastic control problem to steer the state density function between the marginal distributions has been shown to be equivalent to the Schrödinger bridge problem (SBP) [63]. In its original formulation [3, 4, 64], SBP seeks a probability law on path space with given two end-point marginals which is close to a Markovian *prior* in the sense of relative entropy. Important contributions were due to Fortet, Beurling, Jamison and Föllmer [38–41]. Another closely related area of research has been that of *reciprocal processes,* with important engineering applications in, e.g., image processing and other fields [40,65–70].

Renewed interest in SBP was sparked after a close relationship to stochastic control was recognized [63, 71, 72]. The Schrödinger bridge problem can be seen as a stochastic version of OMT due to the presence of the diffusive term in the dynamics ($\varepsilon > 0$). As a result, its solution is more well behaved due to the smoothing property of the Laplacian. On the other hand, it follows from [42, 73–75] that for the special case $A(t) \equiv 0$ and $B(t) \equiv I$, the solution of the SBP converges to that of the OMT when "slowing down" the diffusion by taking $\epsilon \to 0$. Recalling the algorithm in Section 4.4.3, these two facts suggest the SBP as means to construct approximate solutions for both the standard OMT and the problem of OMT-wpd.

5.1 Optimal control formulation of OMT

Recall the OMT problem (see Chapter 3)

$$\inf_{T\in\mathcal{T}(\mu_0,\mu_1)} \int_{\mathbb{R}^n} c(x,T(x))\mu_0(dx), \tag{5.3}$$

and the relaxed version, namely, the Kantorovich problem

$$\inf_{\pi\in\Pi(\mu_0,\mu_1)} \int_{\mathbb{R}^n\times\mathbb{R}^n} c(x,y)\pi(dxdy). \tag{5.4}$$

The above formulations represent a "static" end-point formulation, i.e., focusing on "what goes where". In contrast, Benamou and Brenier [7] provides a fluid dynamic formulation of OMT (see Section 3.3) which captures the moving trajectories of the mass. We next present another closely related formulation [49], where the OMT problem is cast as a stochastic control problem with atypical boundary constraints:

$$\inf_{v\in\mathcal{V}} \mathbb{E}\left\{\int_0^1 \|v(t,x(t))\|^2 dt\right\}, \tag{5.5a}$$

$$\dot{x}(t) = v(t,x(t)), \tag{5.5b}$$

$$x(0)\sim\mu_0, \quad x(1)\sim\mu_1. \tag{5.5c}$$

Here $\mathcal{V}$ represents the family of admissible Markov feedback control laws. We call a control law $v(t,x)$ admissible if the corresponding controlled system (5.5b) has a unique solution for almost every deterministic initial condition at $t=0$. Note that requiring $v(t,\cdot)$ to be uniformly Lipschitz continuous on $[0,1]$ is a sufficient but not necessary condition for v to be admissible.

This optimal control formulation of OMT (5.5) leads to an intuitive derivation of the fluid dynamic formulation (3.7) of OMT as follows. Assume μ_0 and μ_1 are absolutely continuous with density functions ρ_0, ρ_1 and $x(t)$ has an absolutely continuous distribution, namely, $x(t)\sim\rho(t,x)dx$. Then ρ satisfies weakly[2] the continuity

[2] In the sense $\int_{[0,1]\times\mathbb{R}^n}[(\partial f/\partial t + v\cdot\nabla f)\rho]dtdx = 0$ for smooth functions f with compact support.

equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (v\rho) = 0 \tag{5.6}$$

expressing the conservation of probability mass, where $\nabla\cdot$ denotes the divergence of a vector field, and

$$\mathbb{E}\left\{\int_0^1 \|v(t,x(t))\|^2 dt\right\} = \int_{\mathbb{R}^n}\int_0^1 \|v(t,x)\|^2 \rho(t,x) dt dx.$$

As a result, (5.5) is recast as:

$$\inf_{(\rho,v)} \int_{\mathbb{R}^n}\int_0^1 \|v(t,x)\|^2 \rho(t,x) dt dx, \tag{5.7a}$$
$$\frac{\partial \rho}{\partial t} + \nabla \cdot (v\rho) = 0, \tag{5.7b}$$
$$\rho(0,x) = \rho_0(x), \quad \rho(1,y) = \rho_1(y), \tag{5.7c}$$

which is the same as the fluid dynamic formulation (3.7) in Chapter 3.

Recall the optimal transport plan, namely, the minimizer of (5.3) is

$$T(x) = \nabla \phi(x) \tag{5.8}$$

the gradient of convex function ϕ (see (3.5)). Consequently, the optimal control strategy of (5.5) is given by

$$v(t,x) = T \circ T_t^{-1}(x) - T_t^{-1}(x), \tag{5.9}$$

where $\circ$ denotes the composition of maps and

$$T_t(x) = (1-t)x + tT(x).$$

Apparently, T_t is the gradient of a uniformly convex function for $0 \le t < 0$, so T_t is injective and therefore (5.9) is well-defined on $T_t(\mathbb{R}^n)$, the codomain of T_t. The values $v(t,x)$ outside $T_t(\mathbb{R}^n)$ do not play any role.

An alternative expression for the optimal control (5.9) can be established using standard optimal control theory, and this is summarized in the following proposition [1, Theorem 5.51].

Proposition 3 *Given marginal distributions* $\mu_0(dx) = \rho_0(x)dx, \mu_1(dx) = \rho_1(x)dx$, *let* $\psi(t,x)$ *be defined by the Hopf-Lax representation*

$$\psi(t,x) = \inf_y \left\{ \psi(0,y) + \frac{\|x-y\|^2}{2t} \right\}, \quad t \in (0,1]$$

with

$$\psi(0,x) = \phi(x) - \frac{1}{2}\|x\|^2$$

and ϕ *as in* (5.8). *Then* $v(t,x) := \nabla\psi(t,x)$ *exists almost everywhere and it solves* (5.5).

5.2 Optimal mass transport with prior dynamics

The OMT problems ((5.3) or (5.4)) have been studied for general cost $c(x,y)$ that derives from an action functional

$$c(x,y) = \inf_{x(\cdot)\in\mathcal{X}_{xy}} \int_0^1 L(t,x(t),\dot{x}(t))dt, \tag{5.10}$$

where the Lagrangian $L(t,x,p)$ is strictly convex and superlinear in the velocity variable p, see [2, Chapter 7], [76, Chapter 1], [77] and $\mathcal{X}_{xy}$ is the family of absolutely continuous paths with $x(0)=x$ and $x(1)=y$. Existence and uniqueness of an optimal transport map T has been established for general cost functionals as in (5.10). It is easy to see that the choice $c(x,y) = \|x-y\|^2$ corresponds to the special case where

$$L(t,x,p) = \|p\|^2.$$

Another interesting special case is when

$$L(t,x,p) = \|p - v(t,x)\|^2. \tag{5.11}$$

This has been motivated by a transport problem "with prior" associated to the velocity field $v(t,x)$ [49, Section VII], where the prior was thought to reflect a solution to a "nearby" problem that needs to be adjusted so as to be consistent with updated estimates for marginals.

An alternative motivation for (5.11) is to address transport in an ambient flow field $v(t,x)$. In this case, assuming the control has the ability to steer particles in all directions, transport will be effected according to dynamics

$$\dot{x}(t) = v(t, x) + u(t)$$

where $u(t)$ represents control effort and

$$\int_0^1 \|u(t)\|^2 dt = \int_0^1 \|\dot{x}(t) - v(t, x)\|^2 dt$$

represents corresponding quadratic cost (energy). Thus, it is of interest to consider more general dynamics where the control does not affect directly all state directions. One such example is the problem to steer inertial particles in phase space through force input.

Therefore, herein, we consider a natural generalization of OMT where the transport paths are required to satisfy dynamical constraints. We focus our attention on linear dynamics and, consequently, cost of the form

$$\begin{aligned} c(x, y) \quad = \quad & \inf_{u \in \mathcal{U}} \int_0^1 \tilde{L}(t, x(t), u(t)) dt, \text{ where} && (5.12a)\\ & \dot{x}(t) = A(t)x(t) + B(t)u(t), && (5.12b)\\ & x(0) = x, \quad x(1) = y, && (5.12c) \end{aligned}$$

and $\mathcal{U}$ is a suitable class of controls. Apparently, (5.11) corresponds to $A(t) \equiv 0$ and $B(t) \equiv I$ in (5.12). When $B(t)$ is invertible, (5.12) reduces to (5.10) by a change of variables, taking

$$L(t, x, p) = \tilde{L}(t, x, B(t)^{-1}(p - A(t)x)).$$

However, when $B(t)$ is not invertible, an analogous change of variables leads to a Lagrangian $L(t, x, p)$ that fails to satisfy the classical conditions (strict convexity and superlinearity in p). Therefore, in this case, the existence and uniqueness of an optimal transport map T has to be established independently. We do this for the case where $\tilde{L}(t, x, u) = \|u\|^2$ corresponding to control energy.

We now formulate the corresponding stochastic control problem. The system dynamics

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \tag{5.13}$$

are assumed to be controllable and the initial state $x(0)$ is a random vector with probability density ρ_0. Here, A and B are continuous maps taking values in $\mathbb{R}^{n\times n}$ and $\mathbb{R}^{n\times m}$, respectively. We seek a minimum energy feedback control law $u(t,x)$ that steers the system to a final state $x(1)$ having distribution $\rho_1(x)dx$. That is, we address the following:

$$\inf_{u\in\mathcal{U}} \mathbb{E}\left\{\int_0^1 \|u(t,x^u)\|^2 dt\right\}, \tag{5.14a}$$

$$\dot{x}^u(t) = A(t)x^u(t) + B(t)u(t,x^u(t)), \tag{5.14b}$$

$$x^u(0) \sim \rho_0(x)dx, \quad x^u(1) \sim \rho_1(y)dy, \tag{5.14c}$$

where $\mathcal{U}$ is the family of admissible Markov feedback control laws. We call a control law $u(t,x)$ admissible if the corresponding controlled system (5.14b) has a unique solution for almost every deterministic initial condition at $t=0$.

We next show that (5.14) is indeed a reformulation of (5.4) with generalized cost (5.12) when $\tilde{L}(t,x,u) = \|u\|^2$. First we note the cost is equal to

$$c(x,y) = \min_{x(\cdot)\in\mathcal{X}_{xy}} \int_0^1 \hat{L}(t,x(t),\dot{x}(t))dt, \tag{5.15}$$

where

$$\hat{L}(t,x,v) = \begin{cases} (v-A(t)x)'(B(t)B(t)')^{\dagger}(v-A(t)x), & \text{if } v-A(t)x \in \mathcal{R}(B(t)), \\ \infty, & \text{otherwise} \end{cases}$$

with $\dagger$ denoting pseudo-inverse and $\mathcal{R}(\cdot)$ "the range of". If the minimizer of (5.15) exists, which will be denoted as $x^*(\cdot)$, then any probabilistic average of the action relative to absolutely continuous trajectories starting at x at time 0 and ending in y at time 1 cannot give a lower value. Thus, the probability measure on $\mathcal{X}_{xy}$ concentrated on the path $x^*(\cdot)$ solves the following problem

$$\inf_{\mathcal{P}^{xy}\in\mathbb{D}(\delta_x,\delta_y)} \mathbb{E}_{\mathcal{P}^{xy}}\left\{\int_0^1 \hat{L}(t,x(t),\dot{x}(t))dt\right\}, \tag{5.16}$$

where $\mathbb{D}(\delta_x,\delta_y)$ are the probability measures on the space $\Omega = C([0,1],\mathbb{R}^n)$ of continuous paths whose initial and final one-time marginals are Dirac's deltas concentrated at x and y, respectively.

Let u be a feasible control strategy in (5.14), and $x^u(\cdot)$ be the corresponding controlled process. This process induces a probability measure $\mathcal{P}$ in $\mathbb{D}(\mu_0, \mu_1)$, namely a measure on the path space Ω whose one-time marginals at 0 and 1 are μ_0 and μ_1, respectively. The measure $\mathcal{P}$ can be disintgrated as [27, 42]

$$\mathcal{P} = \int_{\mathbb{R}^n \times \mathbb{R}^n} \mathcal{P}^{xy} \, \pi(dxdy), \tag{5.17}$$

where $\mathcal{P}^{xy} \in \mathbb{D}(\delta_x, \delta_y)$ and $\pi \in \Pi(\mu_0, \mu_1)$. Then the control energy in (5.14) is greater than or equal to

$$\begin{aligned} & \mathbb{E}_{\mathcal{P}} \left\{ \int_0^1 \hat{L}(t, x(t), \dot{x}(t)) dt \right\} \\ = & \int_{\mathbb{R}^n \times \mathbb{R}^n} \mathbb{E}_{\mathcal{P}^{xy}} \left\{ \int_0^1 \hat{L}(t, x(t), \dot{x}(t)) dt \right\} \pi(dxdy) \\ \geq & \int_{\mathbb{R}^n \times \mathbb{R}^n} c(x, y) \pi(dxdy), \end{aligned} \tag{5.18}$$

which shows that the minimum of (5.14) is bounded below by the minimum of (5.4) with cost in (5.12) or equivalently (5.15). In Section 5.2.1 we will construct a control strategy such that the joint measure π in (5.17) solves (5.4) and $\mathcal{P}^{xy}$ is concentrated on the path $x^*(\cdot)$ for π-almost every pair of intial position x and terminal position y. Therefore, the stochastic optimal control problem (5.14) is indeed a reformulation of the OMT (5.4) with the general cost in (5.12), and we refer to both of them as OMT-wpd.

Formally, the stochastic control formulation (5.14) suggests the "fluid-dynamics" version:

$$\inf_{(\rho, u)} \int_{\mathbb{R}^n} \int_0^1 \|u(t, x)\|^2 \rho(t, x) dt dx, \tag{5.19a}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot ((A(t)x + B(t)u)\rho) = 0, \tag{5.19b}$$

$$\rho(0, x) = \rho_0(x), \quad \rho(1, y) = \rho_1(y). \tag{5.19c}$$

Establishing rigorously the equivalence between (5.19) and OMT-wpd (5.14) is a delicate issue. We expect the equivalence can be shown along the lines of [1, Theorem 8.1], [27, Chapter 8].

Naturally, for the trivial prior dynamics $A(t) \equiv 0$ and $B(t) \equiv I$, the OMT-wpd reduces to the classical OMT and the solution $\{\rho(t, \cdot) \mid 0 \leq t \leq 1\}$ is the *displacement interpolation* of the two marginals [33]. In Section 5.2.1, we show directly that Problem (5.14) has a unique solution.

5.2.1 Solutions to OMT-wpd

As usual, let $\Phi(t, s)$ be the state transition matrix of (5.13) from s to t, and

$$M(t, s) = \int_s^t \Phi(t, \tau)B(\tau)B(\tau)'\Phi(t, \tau)'d\tau$$

be the reachability Gramian of the system which, by the controllability assumption, is positive definite for all $0 \leq s < t \leq 1$; we denote $\Phi_{10} := \Phi(1, 0)$ and $M_{10} := M(1, 0)$. Recall [78,79] that for linear dynamics (5.13) and given boundary conditions $x(0) = x$, $x(1) = y$, the least energy $c(x, y)$ and the corresponding optimal control input can be given in closed-form, namely

$$c(x, y) = \int_0^1 \|u^*(t)\|^2 dt = (y - \Phi_{10}x)'M_{10}^{-1}(y - \Phi_{10}x) \tag{5.20}$$

where

$$u^*(t) = B(t)'\Phi(1, t)'M_{10}^{-1}(y - \Phi_{10}x). \tag{5.21}$$

The corresponding optimal trajectory is

$$x^*(t) = \Phi(t, 1)M(1, t)M_{10}^{-1}\Phi_{10}x + M(t, 0)\Phi(1, t)'M_{10}^{-1}y. \tag{5.22}$$

The OMT-wpd problem with this cost is

$$\inf_\pi \int_{\mathbb{R}^n \times \mathbb{R}^n} (y - \Phi_{10}x)'M_{10}^{-1}(y - \Phi_{10}x)\pi(dxdy), \tag{5.23a}$$

$$\pi(dx \times \mathbb{R}^n) = \rho_0(x)dx, \; \pi(\mathbb{R}^n \times dy) = \rho_1(y)dy, \tag{5.23b}$$

where π is a measure on $\mathbb{R}^n \times \mathbb{R}^n$.

Problem (5.23) can be converted to the standard Kantorovich formulation (5.4) of the OMT by a transformation of coordinates. Indeed, consider the linear map

$$C : \begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} = \begin{bmatrix} M_{10}^{-1/2}\Phi_{10}x \\ M_{10}^{-1/2}y \end{bmatrix} \tag{5.24}$$

and set

$$\hat{\pi} = C_\sharp \pi.$$

Clearly, (5.23a-5.23b) become

$$\inf_{\hat{\pi}} \int_{\mathbb{R}^n \times \mathbb{R}^n} \|\hat{y} - \hat{x}\|^2 \hat{\pi}(d\hat{x} d\hat{y}), \tag{5.25a}$$

$$\hat{\pi}(d\hat{x} \times \mathbb{R}^n) = \hat{\rho}_0(\hat{x}) d\hat{x}, \; \hat{\pi}(\mathbb{R}^n \times d\hat{y}) = \hat{\rho}_1(\hat{y}) d\hat{y}, \tag{5.25b}$$

where

$$\begin{aligned} \hat{\rho}_0(\hat{x}) &= |M_{10}|^{1/2} |\Phi_{10}|^{-1} \rho_0(\Phi_{10}^{-1} M_{10}^{1/2} \hat{x}), \\ \hat{\rho}_1(\hat{y}) &= |M_{10}|^{1/2} \rho_1(M_{10}^{1/2} \hat{y}). \end{aligned}$$

Problem (5.25) is now a standard OMT with quadratic cost function and we know that the optimal transport map $\hat{T}$ for this problem exists [1]. It is the gradient of a convex function ϕ, i.e.,

$$\hat{T} = \nabla \phi, \tag{5.26}$$

and the optimal $\hat{\pi}$ is concentrated on the graph of $\hat{T}$ [30]. The solution to the original problem (5.23) can now be determined using $\hat{T}$, and it is

$$\pi = (\mathrm{Id} \times T)_\sharp \mu_0$$

with

$$y = T(x) = M_{10}^{1/2} \hat{T}(M_{10}^{-1/2} \Phi_{10} x). \tag{5.27}$$

From the above argument we can see that, with cost function (5.12) and $\tilde{L}(t, x, u) = \|u\|^2$, the OMT problem (5.3) and its relaxed version (5.4) are equivalent.

Having the optimal map T, the one-time marginals can be readily computed as the push-forward

$$\mu_t = (T_t)_\sharp \mu_0, \tag{5.28a}$$

where

$$T_t(x) = \Phi(t,1) M(1,t) M_{10}^{-1} \Phi_{10} x + M(t,0) \Phi(1,t)' M_{10}^{-1} T(x), \tag{5.28b}$$

and

$$\rho(t,x) = \frac{d\mu_t}{dx}(x). \tag{5.28c}$$

In this case, we refer to the parametric family of one-time marginals as *displacement interpolation with prior dynamics.* Combining the optimal map T with (5.21), we obtain the optimal control strategy

$$u(t,x) = B(t)'\Phi(1,t)'M_{10}^{-1}[T\circ T_t^{-1}(x) - T_t^{-1}(x)]. \tag{5.29}$$

Again T_t is injective for $0 \leq t < 1$, so the above control strategy is well-defined on $T_t(\mathbb{R}^n)$. $T_0 = \mathrm{Id}$ is of course injective. To see T_t is indeed an injection for $0 < t < 1$, assume that there are two different points $x \neq y$ such that $T_t(x) = T_t(y)$. Then

$$\begin{aligned} 0 &= (x-y)'\Phi(t,0)'M(t,0)^{-1}(T_t(x) - T_t(y))' \\ &= (x-y)'\Phi(t,0)'M(t,0)^{-1}\Phi(t,1)M(1,t)M_{10}^{-1}\Phi_{10}(x-y) + \\ &\quad (x-y)'\Phi_{10}'M_{10}^{-1/2}(\nabla\phi(M_{10}^{-1/2}\Phi_{10}x) - \nabla\phi(M_{10}^{-1/2}\Phi_{10}y)). \end{aligned}$$

The second term is nonnegative due to the convexity of ϕ. The first term is equal to

$$(x-y)'\left(\Phi(t,0)'M(t,0)^{-1}\Phi(t,0) - \Phi_{10}'M_{10}^{-1}\Phi_{10}\right)(x-y),$$

which is positive since

$$\begin{aligned} \Phi(t,0)'M(t,0)^{-1}\Phi(t,0) - \Phi_{10}'M_{10}^{-1}\Phi_{10} &= \left(\int_0^t \Phi(0,\tau)B(\tau)B(\tau)'\Phi(0,\tau)'d\tau\right)^{-1} \\ &\quad - \left(\int_0^1 \Phi(0,\tau)B(\tau)B(\tau)'\Phi(0,\tau)'d\tau\right)^{-1} \end{aligned}$$

is positive definite for all $0 < t < 1$. Since the control (5.29) is consistent with both the optimal coupling π and the optimal trajectories (5.22), it achieves the minimum of (5.4), which is of course is the minimum of (5.14) based on (5.18).

An alternative expression for the optimal control (5.29) can be derived as follows using standard optimal control theory (see Chapter 2). Consider the following deterministic optimal control problem

$$\inf_{u\in\mathcal{U}} \int_0^1 \frac{1}{2}\|u(t,x^u)\|^2 dt - \psi_1(x^u(1)), \tag{5.30a}$$
$$\dot{x}^u(t) = A(t)x^u(t) + B(t)u(t) \tag{5.30b}$$

for some terminal cost $-\psi_1$. The dynamic programming principle [13] gives the value function (cost-to-go function) $-\psi(t,x)$ as

$$-\psi(t,x) = \inf_{u:x(t)=x} \int_t^1 \frac{1}{2}\|u(t,x^u)\|^2 dt - \psi_1(x^u(1)). \tag{5.31}$$

The associated dynamic programming equation is

$$\inf_{u\in\mathbb{R}^m} \left[\frac{1}{2}\|u\|^2 - \frac{\partial\psi}{\partial t} - \nabla\psi\cdot(A(t)x + B(t)u)\right] = 0. \tag{5.32}$$

Point-wise minimization yields the Hamilton-Jacobi-Bellman equation

$$\frac{\partial\psi}{\partial t} + x'A(t)'\nabla\psi + \frac{1}{2}\nabla\psi' B(t)B(t)'\nabla\psi = 0 \tag{5.33a}$$

with boundary condition

$$\psi(1,y) = \psi_1(y), \tag{5.33b}$$

and the corresponding optimal control is

$$u(t,x) = B(t)'\nabla\psi(t,x). \tag{5.34}$$

When the value function $-\psi(t,x)$ is smooth, it solves the Hamilton-Jacobi-Bellman equation (5.33). In this case, if the optimal control (5.34) drives the controlled process from initial distribution μ_0 to terminal distribution μ_1, then this u in fact solves the OMT-wpd (5.14). In general, one cannot expect (5.33) to have a classical solution and has to be content with viscosity solutions [13,17]. Here, however, it is possible to give an explicit expression for the value function based only on the dynamic programming principle (5.31). This is summarized in the following proposition. The proof is given in Appendix A.1.

Proposition 4 *Given marginal distributions* $\mu_0(dx) = \rho_0(x)dx, \mu_1(dx) = \rho_1(x)dx,$

let $\psi(t,x)$ be defined by the formula

$$\psi(t,x) = \inf_y \left\{ \psi(0,y) + \frac{1}{2}(x - \Phi(t,0)y)' M(t,0)^{-1}(x - \Phi(t,0)y) \right\} \tag{5.35}$$

with

$$\psi(0,x) = \phi(M_{10}^{-1/2}\Phi_{10}x) - \frac{1}{2}x'\Phi_{10}' M_{10}^{-1}\Phi_{10}x$$

and ϕ as in (5.26). *Then $u(t,x) := B(t)'\nabla\psi(t,x)$ exists almost everywhere and it solves* (5.14).

5.3 Optimal control formulation of Schrödinger bridges

For the case of nondegenerate Markov processes, a connection between the SBP and stochastic control has been drawn in [63], see also [71] and [80]. In particular, for the case of a Gaussian (Brownian) kernel, it was shown there that the one-time marginals $\rho(t,x)$ for SBP are the densities of the optimal state vector in the stochastic control problem

$$\inf_{v\in\mathcal{V}} \mathbb{E}\left\{ \int_0^1 \|v(t,x^v)\|^2 dt \right\}, \tag{5.36a}$$

$$dx^v(t) = v(t,x^v(t))dt + dw(t), \tag{5.36b}$$

$$x^v(0) \sim \rho_0, \quad x^v(1) \sim \rho_1. \tag{5.36c}$$

Here $\mathcal{V}$ is the class of admissible Markov controls with finite energy. In particular, it implies the controlled process has a weak solution [19, p. 129] in $[0,1]$. This reformulation reckons on the fact that the relative entropy between x^v and x^0 (zero control) in (5.36b) is bounded above by the control energy, namely,

$$H(\mathcal{P}_{x^v}, \mathcal{P}_{x^0}) \le \frac{1}{2}\mathbb{E}\left\{ \int_0^1 \|v(t,x^v(t))\|^2 dt \right\},$$

where $\mathcal{P}_{x^v}, \mathcal{P}_{x^0}$ denote the measures induced by x^v and x^0, respectively. The proof is based on Girsanov theorem (see [19,63]). The optimal control to (5.36) is then given by

$$v(t,x) = \nabla \log \varphi(t,x)$$

with φ is one of the factor of the Schrödinger bridge as in (4.3b), see [63]. The stochastic control problem (5.36) has the following equivalent formulation [81,82]:

$$\inf_{(\rho,v)} \int_{\mathbb{R}^n} \int_0^1 \|v(t,x)\|^2 \rho(t,x) dt dx, \tag{5.37a}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (v\rho) - \frac{1}{2}\Delta\rho = 0, \tag{5.37b}$$

$$\rho(0,x) = \rho_0(x), \quad \rho(1,y) = \rho_1(y). \tag{5.37c}$$

Here, the infimum is over smooth fields v and ρ solves weakly of the corresponding Fokker-Planck equation (5.37b). The entropic interpolation is $\mathcal{P}_t(dx) = \rho(t,x)dx$.

We next present a more general result on the connection between SBP and optimal control. We consider the Markov kernel

$$q^\epsilon(s,x,t,y) = (2\pi\epsilon)^{-n/2} |M(t,s)|^{-1/2} \exp\left(-\frac{1}{2\epsilon} (y - \Phi(t,s)x)' M(t,s)^{-1} (y - \Phi(t,s)x) \right) \tag{5.38}$$

corresponding to the process

$$dx(t) = A(t)x(t)dt + \sqrt{\epsilon} B(t) dw(t), \tag{5.39}$$

and provide a stochastic control formulation of the Schrödinger bridge problem with this kernel. Note that the kernel q in (5.38) is everywhere positive because of the controllability assumption. As a consequence, $\varphi(t,x)$ for $0 \le t < 1$ satisfying (4.3b) is also everywhere positive and smooth.

Motivated by the nondegenerate case in (5.36), we consider the following stochastic control problem,

$$\inf_{u \in \mathcal{U}} \mathbb{E} \left\{ \int_0^1 \|u(t,x^u)\|^2 dt \right\}, \tag{5.40a}$$

$$dx^u(t) = A(t)x^u(t)dt + B(t)u(t,x^u)dt + \sqrt{\epsilon} B(t) dw(t), \tag{5.40b}$$

$$x^u(0) \sim \rho_0, \quad x^u(1) \sim \rho_1. \tag{5.40c}$$

Here, $\mathcal{U}$ is the set of admissible Markov controls such that for each $u \in \mathcal{U}$ the controlled process admits a weak solution in [0, 1] and the control has finite energy. By a general version of Girsanov theorem [48, Chapter IV.4] and the contraction property of relative entropy [43], we have

$$H(\mathcal{P}_{x^u}, \mathcal{P}_{x^0}) \leq \mathbb{E}\left\{\int_0^1 \frac{1}{2\epsilon}\|u(t, x^u(t))\|^2 dt\right\},$$

where $\mathcal{P}_{x^u}, \mathcal{P}_{x^0}$ denote the measures induced by x^u and x^0 (zero control) on Ω, respectively.

Let $\varphi, \hat{\varphi}$ be as in (4.3b) with the Markov kernel corresponding to (5.39). We claim that, under the technical assumptions that

i) $\int_{\mathbb{R}^n} \varphi(0, x)\mu_0(dx) < \infty$

ii) $H(\mu_1, \hat{\varphi}(1, \cdot)) < \infty$,

the optimal solution to (5.40) is

$$u(t, x) = \epsilon B(t)' \nabla \log \varphi(t, x). \tag{5.41}$$

The assumption i) guarantees that the local martingale $\varphi(t, x(t))$, where x is the uncontrolled evolution (5.39), is actually a martingale. The assumption ii) implies that the control (5.41) has finite energy. For both statements see [63, Theorem 3.2], whose proof carries verbatim. While these conditions i) and ii) are difficult to verify in general, they are satisfied when both μ_0 and μ_1 have compact support (c.f. [63, Proposition 3.1]).

Then, by the argument in [63, Theorem 2.1] and in view of the equivalence between existence of weak solutions to stochastic differential equations (SDEs) and solutions to the martingale problem (see [83, Theorem 4.2], [19, p. 314]), it follows that with $u(t, x)$ as in (5.41) the SDE (5.40b) has a weak solution. By substituting (5.41) in the Fokker-Planck equation it can be seen that the corresponding controlled process satisfies the marginals (5.40c). In fact, the density flow ρ coincides with the one-time marginals of the Schrödinger bridge (4.3a).

Finally, to see that (5.41) is optimal, we use a completion of squares argument. To this end, consider the equivalent problem of minimizing the cost functional

$$J(u) = \mathbb{E}\left\{\int_0^1 \frac{1}{2\epsilon}\|u(t)\|^2 dt - \log \varphi(1, x(1)) + \log(\varphi(0, x(0))\right\}$$

in (5.40a) (the boundary terms are constant over the admissible path-space probability distributions, cf. (5.40c)). Using Itô's rule, and the fact that u in (5.41) has finite energy, a standard calculation [84] shows

$$\begin{aligned} J(u) &= \mathbb{E}\left\{\int_0^1 \frac{1}{2\epsilon}\|u(t)\|^2 dt - d\log\varphi(t, x(t))\right\} \\ &= \mathbb{E}\left\{\int_0^1 \frac{1}{2\epsilon}\|u(t) - \epsilon B(t)'\nabla\log\varphi(t, x(t))\|^2 dt\right\}, \end{aligned}$$

from which we readily conclude that (5.41) is the optimal control law. On the other hand, the Schrödinger bridge has the generator

$$x'A'\nabla(\cdot) + \epsilon\nabla\log\varphi(t, x)\cdot BB'\nabla(\cdot) + \frac{\epsilon}{2}\operatorname{trace} BB'H(\cdot),$$

which corresponds to exactly the controlled process (5.40b) with control in (5.41). Therefore, we conclude that the SBP with prior Markov kernel (5.39) is equivalent to the optimal control problem (5.40).

As a consequence, the entropic interpolation $\mathcal{P}_t(dx) = \rho(t, x)dx$ can now be obtained by solving

$$\inf_{(\rho,u)} \int_{\mathbb{R}^n}\int_0^1 \frac{1}{2}\|u(t, x)\|^2\rho(t, x)dtdx, \tag{5.42a}$$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot((A(t)x + B(t)u)\rho) - \frac{\epsilon}{2}\sum_{i,j=1}^n \frac{\partial^2(a(t)_{ij}\rho)}{\partial x_i\partial x_j} = 0, \tag{5.42b}$$

$$\rho(0, x) = \rho_0(x), \quad \rho(1, y) = \rho_1(y), \tag{5.42c}$$

where $a(t) = B(t)B(t)'$, see [47, 81]. Comparing (5.42) with (5.19) we see that the only difference is the extra term

$$\frac{\epsilon}{2}\sum_{i,j=1}^n \frac{\partial^2(a(t)_{ij}\rho)}{\partial x_i\partial x_j}$$

in (5.42b) as compared to (5.19b).

5.4 Zero-noise limit

In this section we study the relation between OMT (OMT-wpd) and SBP. In particular, we show that the OMT problem is the limit of the SBP as the noise intensity goes to zero. We start with the case when the prior dynamics is Brownian motion. Formulation (5.37) is quite similar to OMT (5.7) except for the presence of the Laplacian in (5.37b). It has been shown [42, 73–75] that the OMT problem is, in a suitable sense, indeed the limit of the Schrödinger problem when the diffusion coefficient of the reference Brownian motion goes to zero. In particular, the minimizers of the SBP converge to the unique solution of OMT as below.

Theorem 9 *Given two probability measures $\mu_0(dx) = \rho_0(x)dx, \mu_1(dy) = \rho_1(y)dy$ on $\mathbb{R}^n$ with finite second moment, let $\mathcal{P}_{01}^{B,\epsilon}$ be the solution of the Schrödinger problem with Markov kernel*

$$q^{B,\epsilon}(s,x,t,y) = (2\pi)^{-n/2}((t-s)\epsilon)^{-n/2}\exp\left(-\frac{\|x-y\|^2}{2(t-s)\epsilon}\right) \tag{5.43}$$

and marginals μ_0, μ_1, and let $\mathcal{P}_t^{B,\epsilon}$ be the corresponding entropic interpolation. Similarly, let π be the solution to the OMT problem (3.3) *with the same marginal distributions, and μ_t the corresponding displacement interpolation. Then, $\mathcal{P}_{01}^{B,\epsilon}$ converges weakly*[3] *to π and $\mathcal{P}_t^{B,\epsilon}$ converges weakly to μ_t, as ϵ goes to* 0.

To see the intuition, consider

$$dx(t) = \sqrt{\epsilon}dw(t)$$

whose Markov kernel is $q^{B,\epsilon}$ in (5.43). Here $w(t)$ is the standard Wiener process. The SBP with the law of $x(t)$ as prior, is equivalent to

$$\inf_{(\rho,v)} \int_{\mathbb{R}^n}\int_0^1 \frac{1}{2\epsilon}\|v(t,x)\|^2\rho(t,x)dtdx, \tag{5.44a}$$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(v\rho) - \frac{\epsilon}{2}\Delta\rho = 0, \tag{5.44b}$$

$$\rho(0,x) = \rho_0(x), \quad \rho(1,y) = \rho_1(y). \tag{5.44c}$$

[3] A sequence $\{P_n\}$ of probability measures on a metric space $\mathcal{S}$ converges weakly to a measure P if $\int_{\mathcal{S}} fdP_n \to \int_{\mathcal{S}} fdP$ for every bounded, continuous function f on the space.

Note that the solution exists for all $\epsilon > 0$ and coincides with the solution of the problem to minimize the cost functional

$$\int_{\mathbb{R}^n} \int_0^1 \|v(t,x)\|^2 \rho(t,x) dt dx$$

instead, i.e., "rescaling" (5.44a) by removing the factor $1/2\epsilon$. Now observe that the only difference between (5.44) after removing the scaling $1/\epsilon$ in the cost functional and the OMT formulation (5.7) is the regularization term $\frac{\epsilon}{2}\Delta\rho$ in (5.44b). Thus, formally, the constraint (5.44b) becomes (5.7b) as ϵ goes to 0.

Next we present a general result that includes the case when the zero-noise limit of SBP corresponds to OMT-wpd. This problem has been studied in [73] in a more abstract setting based on Large Deviation Theory [85]. Here we consider the special case that is connected to our OMT-wpd formulation.

Formally, (5.42b) converges to (5.19b) as ϵ goes to 0. This suggests that the minimizer of the OMT-wpd might be obtained as the limit of the joint initial-final time distribution of solutions to the Schrödinger bridge problems as the disturbance vanishes. This result is stated next and can be proved based on the result in [73] together with the Freidlin-Wentzell Theory [85, Section 5.6] (a large deviation principle on sample path space). In Appendix A.2, we provide an alternative proof which doesn't require a large deviation principle directly.

Theorem 10 *Given two probability measures $\mu_0(dx) = \rho_0(x)dx, \mu_1(dy) = \rho_1(y)dy$ on $\mathbb{R}^n$ with finite second moment, let $\mathcal{P}_{01}^\epsilon$ be the solution of the Schrödinger problem with reference Markov evolution* (5.39) *and marginals μ_0, μ_1, and let $\mathcal{P}_t^\epsilon$ be the corresponding entropic interpolation. Similarly, let π be the solution to* (5.23) *with the same marginal distributions, and μ_t the corresponding displacement interpolation. Then, $\mathcal{P}_{01}^\epsilon$ converges weakly to π and $\mathcal{P}_t^\epsilon$ converges weakly to μ_t as ϵ goes to* 0.

An important consequence of this theorem is that one can now develop numerical algorithms for the general problem of OMT with prior dynamics, and in particular for the standard OMT, by solving the Schrödinger problem for a vanishing ϵ. This approach appears particular promising in view of recent work [86] that provides an effective computational scheme to solve the Schrödinger problem by computing the pair $(\hat{\varphi}_0, \varphi_1)$ in Theorem 5 as the fixed point of an iteration. This is now being developed for diffusion processes in [51] (see Chapter 4). See also [87,88] for similar works in discrete space setting, which has a wide range of applications. This approach to obtain approximate solutions to general OMT problems, via solutions to Schrödinger problems with vanishing noise, is illustrated in the examples of Section 5.6. It should also be noted that OMT problems are known to be computationally challenging in high dimensions, and specialized algorithms have been developed [7], [8]. The present approach suggests a totally new computational scheme.

5.5 Generalizations

The density steering problems can be generalized along several different directions. We next briefly mention the cases where i) the cost function contains a state penalty term, and ii) the prior dynamics is nonlinear. Interested reader is referred to [81] for i) and [47] for ii).

5.5.1 State penalty

Consider the optimal steering problem (5.40) with however, the cost function

$$\mathbb{E}\left\{\int_0^1 [\|u(t)\|^2 + U(t, x(t))]dt\right\}. \tag{5.45}$$

The term $U(\cdot, \cdot)$ represents the penalty on the state vector. The case of particular interest to us is when

$$U(t, x) = x' S(t) x$$

for some positive semi-definite matrix $S(\cdot)$. This special case will be discussed in further details in Chapter 6.

When $\epsilon = 0$, the problem resembles the OMT-wpd and the solution can be obtained through coordinate transformation as in (5.24), see also [62]. When $\epsilon > 0$, the problem is shown to be equivalent to a SBP with killing, that is, the particles have a possibility to disappear. Therefore, the optimal control strategy can be established using the theory of Schrödinger bridges. See [64, 81] for more details.

5.5.2 Nonlinear dynamics

When the particles are governed by the general nonlinear dynamics

$$dx(t) = f(t, x, u)dt + \sigma(t, x, u)dw(t),$$

how do we optimally steer the particles between two marginal distributions with minimum control energy (5.2) or more general cost (5.45)? Apparently this problem is more difficult than the case of linear dynamics. One cannot expect a solution to always exist. In fact, it is not even clear if it is possible to steer a system from one distribution ρ_0 to another distribution ρ_1.

In [47], we studied a special case with application to the cooling of oscillators. The oscillators are governed by the generalized Ornstein-Uhlenbeck model of physical Brownian motion [5] with external control u

$$\begin{aligned} dx(t) &= v(t)\,dt, \quad x(t_0) = x_0 \\ M dv(t) &= -Dv(t)\,dt + u(t)dt - \nabla_x V(x(t))dt + \sigma dw(t), \quad v(t_0) = v_0 \end{aligned}$$

where $x(t), v(t)$ represent the position and velocity of the oscillators respectively, and V is a potential function that plays an important role in this coupled system of oscillators. The positive definite matrix M represents the mass and D is the damping. Again, as before, $w(t)$ is the white noise that models the thermal disturbance.

When the control input u is zero, the distribution of the state (x, v) satisfies the Boltzmann distribution [89]

$$\rho_B(x, v) = Z_B^{-1} \exp\left[-\frac{2V(x) + v'Mv}{2kT_a}\right],$$

where T_a represents the temperature and Z_B is a normalization factor. The goal of cooling is to reduce the temperature T_a of the oscillators system by adding proper feedback control u. This is achieved by controlling the distribution of the state (x, v). See [47] for details.

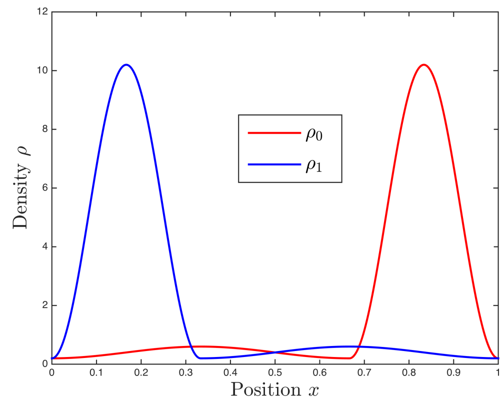


Figure 5.1: Marginal distributions

5.6 Examples

Several examples are provided here to highlight the theoretical framework. The first one is on density interpolation on one dimensional space and the second one is on imagine processing.

5.6.1 Densities interpolation

Consider now a large collection of particles obeying

$$dx(t) = -2x(t)dt + u(t)dt$$

in 1-dimensional state space with marginal distributions

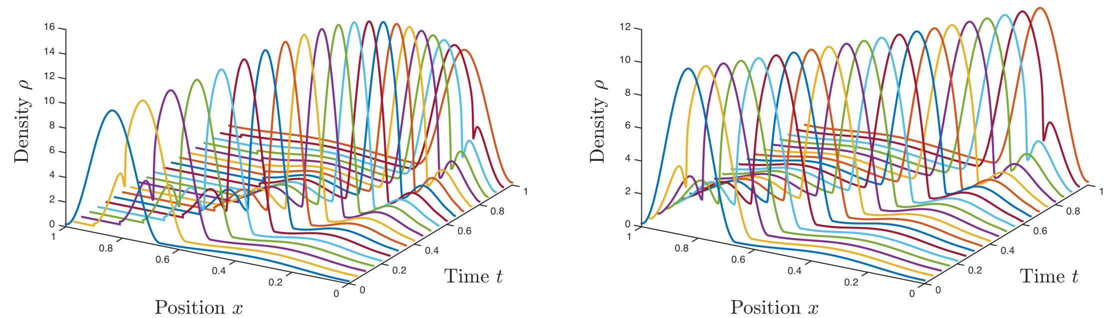
$$\rho_0(x) = \begin{cases} 0.2 - 0.2\cos(3\pi x) + 0.2 & \text{if } 0 \le x < 2/3 \\ 5 - 5\cos(6\pi x - 4\pi) + 0.2 & \text{if } 2/3 \le x \le 1, \end{cases}$$

and

$$\rho_1(x) = \rho_0(1 - x).$$

These are shown in Figure 5.1. Our goal is to steer the state of the system (equivalently, the particles) from the initial distribution ρ_0 to the final ρ_1 using minimum energy control. That is, we need to solve the problem of OMT-wpd. In this 1-dimensional case, just like in the classical OMT problem, the optimal transport map

[h]



(a) OMT-wpd

(b) OMT

Figure 5.2: Interpolations based on OMT-wpd and OMT

$y = T(x)$ between the two end-points can be determined from[4]

$$\int_{-\infty}^{x} \rho_0(y)dy = \int_{-\infty}^{T(x)} \rho_1(y)dy.$$

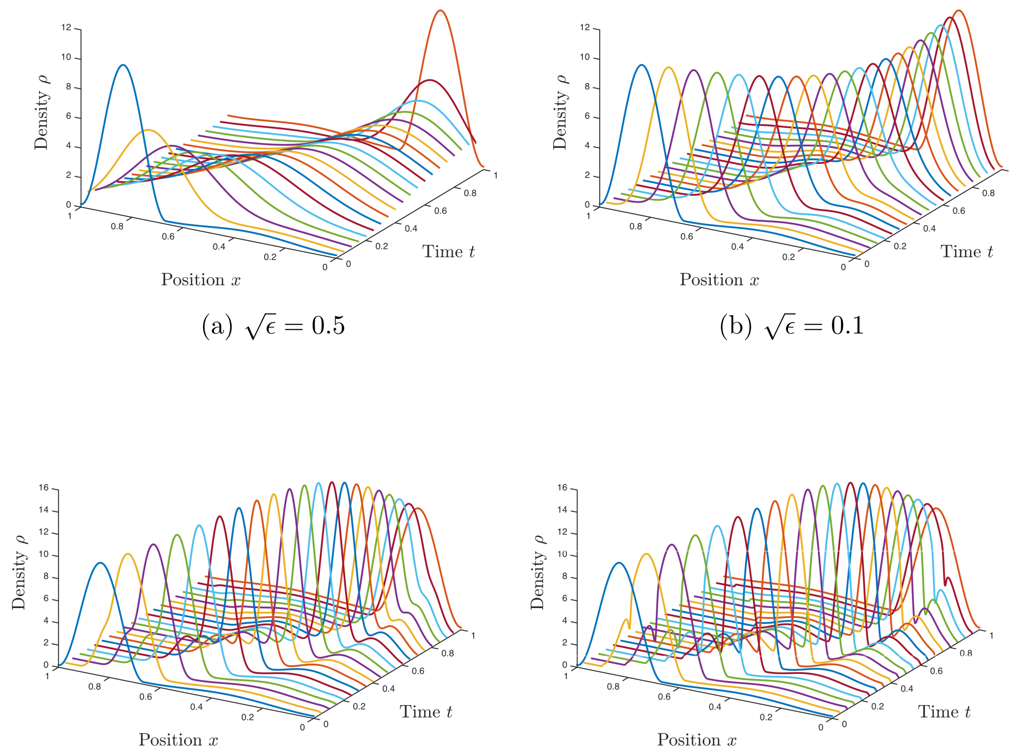
The interpolation flow $\rho_t,\ 0 \le t \le 1$ can then be obtained using (5.28). Figure 5.2a depicts the solution of OMT-wpd. For comparison, we also show the solution of the classical OMT in figure 5.2b where the particles move on straight lines.

Finally, we assume a stochastic disturbance,

$$dx(t) = -2x(t)dt + u(t)dt + \sqrt{\epsilon}dw(t),$$

with $\epsilon > 0$. Figure 5.3 depicts minimum energy flows for diffusion coefficients $\sqrt{\epsilon} = 0.5,\ 0.15,\ 0.05,\ 0.01$, respectively. As $\epsilon \to 0$, it is seen that the solution to the Schrödinger problem converges to the solution of the problem of OMT-wpd as expected.

[4] In this 1-dimensional case, (5.27) is a simple rescaling and, therefore, $T(\cdot)$ inherits the monotonicity of $\hat{T}(\cdot)$.



[h]

(a) $\sqrt{\epsilon} = 0.5$ (b) $\sqrt{\epsilon} = 0.1$

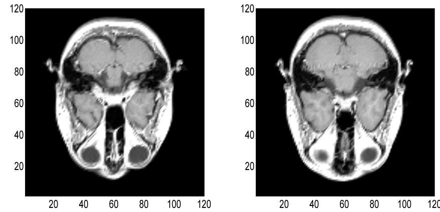
(c) $\sqrt{\epsilon} = 0.05$ (d) $\sqrt{\epsilon} = 0.01$

Figure 5.3: Interpolation with different ϵ

5.6.2 Images interpolation

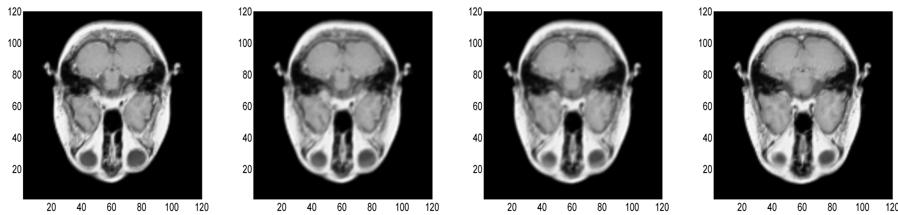
We consider interpolation/morphing of 2D images. When suitably normalized, these can be viewed as probability densities on $\mathbb{R}^2$. Interpolation is important in many applications. One such application is Magnetic Resonance Imaging (MRI) where due to cost and time limitations, a limited number of slices are scanned. Suitable interpolation between the 2D-slices may yield a better 3D reconstruction.

Figure 5.4 shows the two brain images that we seek to interpolate. The data is available as `mri.dat` in Matlab®. Figure 5.5 compares displacement interpolants with trivial prior dynamics at $t = 0.2,\ 0.4,\ 0.6,\ 0.8$, respectively, based on solving a Schrödinger bridge problem with Brownian prior with diffusivity $\epsilon = 0.01$ using our numerical algorithm. For comparison, we display in Figure 5.6 another set of interpolants corresponding to larger diffusivity, namely, $\epsilon = 0.04$. As expected, we



(a) $t = 0$ (b) $t = 1$

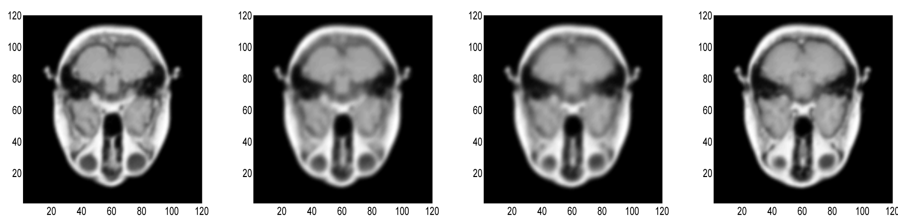
Figure 5.4: MRI slices at two different points



(a) $t = 0.2$ (b) $t = 0.4$ (c) $t = 0.6$ (d) $t = 0.8$

Figure 5.5: Interpolation with $\epsilon = 0.01$

observe a more blurry set of interpolants due to the larger diffusivity.



(a) $t = 0.2$ (b) $t = 0.4$ (c) $t = 0.6$ (d) $t = 0.8$

Figure 5.6: Interpolation with $\epsilon = 0.04$

Chapter 6

Linear Quadratic case: covariance control

In this chapter we specialize the density steering problems studied in Chapter 5 to the case where both the initial distribution ρ_0 and the terminal distribution ρ_1 are Gaussian. Since in this setting we focus on the covariance of the distributions, we refer to it as the covariance control problem. We provide explicit expressions for the solutions and examine their structure.

The density steering problems we are interested in have the dynamics

$$dx(t) = A(t)x(t)dt + B(t)u(t)dt + \sqrt{\epsilon}B(t)dw(t), \tag{6.1}$$

where the pair $(A(\cdot),\, B(\cdot))$ is assumed to be controllable. The goal is to find a control strategy with minimum energy

$$\mathbb{E}\{\int_0^1 \|u(t)\|^2 dt\} \tag{6.2}$$

driving the particles from initial distribution ρ_0 at $t = 0$ to terminal distribution ρ_1 at $t = 1$. For the special case when both of the marginal distributions are Gaussian, the existence and uniqueness of the solution follows directly from the results in Chapter 5. Here, however, we developed direct approaches to study this special case without appealing to the material in Chapter 5. These new approaches also lead to explicit expressions for the solutions.

In Section 6.1 we study the case where $\epsilon \neq 0$. This corresponds to the Schrödinger bridge problem (SBP). The case of $\epsilon = 0$, which can be viewed as a special example of the optimal mass transport with prior dynamics (OMT-wpd), is discussed in Section 6.2. We also consider several extensions in the rest of this chapter.

6.1 Schrödinger bridge problem: $\epsilon \neq 0$

Without loss of generality, we discuss the case $\epsilon = 1$, that is, the controlled evolution

$$dx^u(t) = A(t)x^u(t)dt + B(t)u(t)dt + B(t)dw(t). \tag{6.3}$$

The same method carries through for any $\epsilon \neq 0$. For simplicity, the marginal distributions are assumed to have zero mean, namely,

$$\rho_0(x) = (2\pi)^{-n/2}\det(\Sigma_0)^{-1/2}\exp\left(-\frac{1}{2}x'\Sigma_0^{-1}x\right), \tag{6.4}$$

and

$$\rho_1(x) = (2\pi)^{-n/2}\det(\Sigma_1)^{-1/2}\exp\left(-\frac{1}{2}x'\Sigma_1^{-1}x\right). \tag{6.5}$$

The result for nonzero mean case is provided in Remark 17. For the ease of reference, we provide the stochastic differential equation (SDE) formula for the prior dynamics

$$dx(t) = A(t)x(t)dt + B(t)dw(t) \quad \text{with } x(0) = x_0 \tag{6.6}$$

where x_0 is a random vector with distribution ρ_0.

Let $\mathcal{U}$ be the family of *adapted*, *finite-energy* control functions such that (6.3) has a strong solution and $x^u(1)$ is distributed according to (6.5). More precisely, $u \in \mathcal{U}$ is such that $u(t)$ only depends on t and $\{x^u(s); 0 \leq s \leq t\}$ for each $t \in [0,1]$, satisfies

$$\mathbb{E}\left\{\int_0^1 \|u(t)\|^2\, dt\right\} < \infty,$$

and affects $x^u(1)$ to be distributed according to (6.5). The family $\mathcal{U}$ represents *admissible* control inputs which achieve the desired probability density transfer from ρ_0 to ρ_1. Thence, we formulate the following optimal steering problem:

Problem 11 *Determine whether* $\mathcal{U}$ *is non-empty and if so, determine*

$$u^* := \operatorname{argmin}_{u\in\mathcal{U}} J(u)$$

where

$$J(u) := \mathbb{E}\left\{\int_0^1 \|u(t)\|^2\, dt\right\}.$$

From the result in Chapter 5 we know u^* always exists. Here we give an alternative proof by taking advantage of the Gaussian structure.

6.1.1 Optimality conditions

First we identify a candidate structure for the optimal controls, which reduces the problem to an algebraic condition involving two differential Lyapunov equations that are nonlinearly coupled through split boundary conditions.

Let us start by observing that this problem resembles a standard stochastic linear quadratic regulator problem except for the boundary conditions. The usual variational analysis can in fact be carried out, up to a point, namely the expression for the optimal control, in a similar fashion. Of the several ways in which the form of the optimal control can be obtained, we choose a most familiar one, namely the so-called [24] "completion of squares"[1]. Let $\{\Pi(t) \mid 0 \le t \le 1\}$ be a differentiable function taking values in the set of symmetric, $n \times n$ matrices satisfying the matrix Riccati equation

$$\dot{\Pi}(t) = -A(t)'\Pi(t) - \Pi(t)A(t) + \Pi(t)B(t)B(t)'\Pi(t). \tag{6.7}$$

Observe that Problem 11 is equivalent to minimizing over $\mathcal{U}$ the modified cost functional

$$\tilde{J}(u) = \mathbb{E}\left\{\int_0^1 u(t)'u(t)\, dt + x(1)'\Pi(1)x(1) - x(0)'\Pi(0)x(0)\right\}. \tag{6.8}$$

Indeed, as the two end-point marginal densities ρ_0 and ρ_1 are fixed when u varies in

[1]Although it might be the most familiar to control engineers, the completion of the square argument for stochastic linear quadratic regulator control is not the most elementary. A derivation which does not employ Itô's rule was presented in [90].

$\mathcal{U}$, the two boundary terms are constant over $\mathcal{U}$. We can now rewrite $\tilde{J}(u)$ as

$$\tilde{J}(u) = \mathbb{E}\left\{ \int_0^1 u(t)'u(t)\,dt + \int_0^1 d\left(x(t)'\Pi(t)x(t)\right) \right\}.$$

Applying Itô's rule (e.g., see [15]) we obtain

$$\tilde{J}(u) = \mathbb{E}\left\{ \int_0^1 \|u(t) + B(t)'\Pi(t)x(t)\|^2\,dt + \int_0^1 \operatorname{trace}\left(\Pi(t)B(t)B(t)'\right)dt \right\}. \quad (6.9)$$

Observe that the second integral is finite and invariant over $\mathcal{U}$. Hence, a candidate for the optimal control is

$$u^*(t) = -B(t)'\Pi(t)x(t). \quad (6.10)$$

Such a choice of control will be possible provided we can find a solution $\Pi(t)$ of (6.7) such that the process

$$dx^*(t) = \left(A(t) - B(t)B(t)'\Pi(t)\right)x^*(t)dt + B(t)dw(t), \quad (6.11)$$
$$\text{with } x^*(0) = x_0 \text{ a.s.}$$

leads to $x^*(1)$ with density ρ_1. If this is indeed possible, then we have solved Problem 11. It is important to observe that the optimal control, if it exists, is in a *state feedback* form. Consequently, the new optimal evolution is a *Gauss-Markov* process just as the prior evolution (6.6).

Finding the solution of the Riccati equation (6.7) which achieves the density transfer is nontrivial. In the classical linear quadratic regulator theory, the terminal cost of the index would provide the boundary value $\Pi(1)$ for (6.7). However, here there is no boundary value and the two analyses sharply bifurcate. Therefore, we need to resort to something quite different as we have information concerning both initial and final densities, namely Σ_0 and Σ_1.

Let $\Sigma(t) := \mathbb{E}\left\{x^*(t)x^*(t)'\right\}$ be the state covariance of the sought optimal evolution. From (6.11) we have that $\Sigma(t)$ satisfies

$$\dot{\Sigma}(t) = \left(A(t) - B(t)B(t)'\Pi(t)\right)\Sigma(t) + \Sigma(t)\left(A(t) - B(t)B(t)'\Pi(t)\right)' + B(t)B(t)'. \quad (6.12)$$

It must also satisfy the two boundary conditions

$$\Sigma(0) = \Sigma_0, \quad \Sigma(1) = \Sigma_1 \tag{6.13}$$

and that $\Sigma(t)$ is positive definite on $[0, 1]$. Thus, we seek a solution pair $(\Pi(t), \Sigma(t))$ of the *coupled* system of equations (6.7) and (6.12) with split boundary conditions (6.13).

Interestingly, if we define the new matrix-valued function

$$\mathrm{H}(t) := \Sigma(t)^{-1} - \Pi(t),$$

then a direct calculation using (6.12) and (6.7) shows that $\mathrm{H}(t)$ satisfies the Riccati equation

$$\dot{\mathrm{H}}(t) = -A(t)'\mathrm{H}(t) - \mathrm{H}(t)A(t) - \mathrm{H}(t)B(t)B(t)'\mathrm{H}(t). \tag{6.14}$$

This equation is dual to (6.7) and the system of coupled matrix equations (6.7) and (6.12) can be replaced by (6.7) and (6.14). The new system is *decoupled*, except for the coupling through their boundary conditions

$$\Sigma_0^{-1} = \Pi(0) + \mathrm{H}(0) \tag{6.15a}$$

$$\Sigma_1^{-1} = \Pi(1) + \mathrm{H}(1). \tag{6.15b}$$

We refer to these coupled equations as the *Schrödinger system*. Boundary conditions (6.15) are sufficient for meeting the two end-point marginals ρ_0 and ρ_1 provided of course that $\Pi(t)$ remains finite. We have therefore established the following result.

Proposition 5 *Suppose* $\Pi(t)$ *and* $\mathrm{H}(t)$ *satisfy equations* (6.7)-(6.14) *on* $[0, 1]$ *with boundary conditions* (6.15). *Then the feedback control* u^* *given in* (6.10) *is the solution to Problem 11.*

Since (6.7) and (6.14) are homogeneous, they always admit the zero solution. The case $\Pi(t) \equiv 0$ corresponds to the situation where the prior evolution satisfies the boundary marginal conditions and, in that case, $\mathrm{H}(t)^{-1}$ is simply the prior state covariance. In addition, it is also possible that $\Pi(t)$ vanishes in certain directions. Clearly, such directions remain invariant in that, if $\Pi(t)v = 0$ for a value of $t \in [0,\ 1]$, then $\Pi(t)v = 0$ for all $t \in [0,\ 1]$ as well. In such cases, it suffices to consider (6.7) and (6.14) in the orthogonal complement of null directions.

Thus, in general, Problem 11 reduces to the atypical situation of *two* Riccati equations (6.7) and (6.14) coupled through their boundary values. This might still, at first glance, appear to be a formidable problem. However, (6.7)-(6.14) are homogeneous and, as far as their non singular solutions, they reduce to *linear* differential Lyapunov equations. The latter, however, are still coupled through their boundary values in a *nonlinear* way. Indeed, suppose $\Pi(t)$ exists on the time interval $[0, 1]$ and is invertible. Then $Q(t) = \Pi(t)^{-1}$ satisfies the linear equation

$$\dot{Q}(t) = A(t)Q(t) + Q(t)A(t)' - B(t)B(t)'. \tag{6.16a}$$

Likewise, if $\mathrm{H}(t)$ exists on the time interval $[0, 1]$ and is invertible, $P(t) = \mathrm{H}(t)^{-1}$ satisfies the linear equation

$$\dot{P}(t) = A(t)P(t) + P(t)A(t)' + B(t)B(t)'. \tag{6.16b}$$

The boundary conditions (6.15) for this new pair $(P(t), Q(t))$ now read

$$\Sigma_0^{-1} = P(0)^{-1} + Q(0)^{-1} \tag{6.17a}$$

$$\Sigma_1^{-1} = P(1)^{-1} + Q(1)^{-1}. \tag{6.17b}$$

Conversely, if $Q(t)$ solves (6.16a) and is nonsingular on $[0, 1]$, then $Q(t)^{-1}$ is a solution of (6.7), and similarly for $P(t)$. We record the following immediate consequence of Proposition 5.

Corollary 12 *Suppose $P(t)$ and $Q(t)$ are nonsingular on $[0, 1]$ and satisfy the equations* (6.16) *with boundary conditions* (6.17)*. Then the feedback control*

$$u^*(t) = -B(t)'Q(t)^{-1}x(t) \tag{6.18}$$

is optimal for Problem 11.

Thus, the system (6.16)-(6.17) or, equivalently, the system (6.7), (6.14), and (6.15), appears as the *bottleneck* of the SBP. In Section 6.1.2, we prove that this Schrödinger system always has a solution $(\Pi(t), H(t))$, with both $\Pi(t)$ and $H(t)$ bounded on $[0, 1]$, that satisfies (6.7), (6.14), and (6.15) and that this solution is unique.

6.1.2 Existence and uniqueness of optimal control

Since $(A(\cdot), B(\cdot))$ is controllable, the reachability Gramian

$$M(s,t) := \int_t^s \Phi(s,\tau)B(\tau)B(\tau)'\Phi(s,\tau)'d\tau,$$

is nonsingular for all $t < s$ (with $t, s \in [0,1]$). As usual, $\Phi(t,s)$ denotes the state-transition matrix of the dynamics $A(\cdot)$ determined via

$$\frac{\partial}{\partial t}\Phi(t,s) = A(t)\Phi(t,s) \quad \text{and} \quad \Phi(t,t) = I,$$

and this is nonsingular for all $t, s \in [0,1]$. It is worth noting that for $s > 0$ the reachability Gramian $M(s,0) = P(s) > 0$ satisfies the differential Lyapunov equation (6.16b) with $P(0) = 0$. The controllability Gramian

$$N(s,t) := \int_t^s \Phi(t,\tau)B(\tau)B(\tau)'\Phi(t,\tau)'d\tau,$$

is necessarily also nonsingular for all $t < s$ ($t, s \in [0,1]$) and if, we similarly set $Q(t) = N(1,t)$, then $Q(t)$ satisfies (6.16a) with $Q(1) = 0$.

However, as suggested in the previous section, we need to consider solutions $P(\cdot)$, $Q(\cdot)$ of these two differential Lyapunov equations (6.16) that satisfy boundary conditions that are coupled through (6.17). In general, $P(t)$ and $Q(t)$ do not need to be sign definite, but in order for

$$\Sigma(t)^{-1} = P(t)^{-1} + Q(t)^{-1}. \tag{6.19}$$

to qualify as a covariance of the controlled process (6.3), $P(t)$ and $Q(t)$ need to be invertible. This condition is also sufficient and $\Sigma(t)$ satisfies the corresponding differential Lyapunov equation for the covariance of the controlled process

$$\dot{\Sigma}(t) = A_Q(t)\Sigma(t) + \Sigma(t)A_Q(t)' + B(t)B(t)' \tag{6.20}$$

with

$$A_Q(t) := (A(t) - B(t)B(t)'Q(t)^{-1}). \tag{6.21}$$

The following proposition shows the existence and uniqueness of an admissible pair $(P_-(t), Q_-(t))$ of solutions to (6.16)-(6.17) that are invertible on $[0,1]$. Interestingly, there is always a second solution $(P_+(t), Q_+(t))$ to the nonlinear problem (6.16)-(6.17) which is not admissible as it fails to be invertible on $[0,1]$. See Appendix A.3 for the proof.

Proposition 6 *Consider $\Sigma_0, \Sigma_1 > 0$ and a controllable pair $(A(t), B(t))$. The system of differential Lyapunov equations* (6.16) *has two sets of solutions $(P_\pm(\cdot), Q_\pm(\cdot))$ over $[0,1]$ that simultaneously satisfy the coupling boundary conditions* (6.17)*. These two solutions are specified by*

$$\begin{aligned} Q_\pm(0) &= N(1,0)^{1/2} S_0^{1/2} \left(S_0 + \frac{1}{2} I \pm \left(S_0^{1/2} S_1 S_0^{1/2} + \frac{1}{4} I \right)^{1/2} \right)^{-1} S_0^{1/2} N(1,0)^{1/2}, \\ P_\pm(0) &= \left(\Sigma_0^{-1} - Q_\pm(0)^{-1} \right)^{-1} \end{aligned}$$

and the two differential equations (6.16)*, where*

$$\begin{aligned} S_0 &= N(1,0)^{-1/2} \Sigma_0 N(1,0)^{-1/2}, \\ S_1 &= N(1,0)^{-1/2} \Phi(0,1) \Sigma_1 \Phi(0,1) N(1,0)^{-1/2}. \end{aligned}$$

The two pairs $(P_\pm(t), Q_\pm(t))$ with subscript $-$ and $+$, respectively, are distinguished by the following:

i) $Q_-(t)$ and $P_-(t)$ are both nonsingular on $[0,1]$, whereas

ii) $Q_+(t)$ and $P_+(t)$ become singular for some $t \in [0,1]$, possibly not for the same value of t.

Remark 13 *We have numerically observed that the iteration*

$$\begin{aligned}
&P(0)\\
&\ \downarrow\\
&P(1) &&= \Phi(1,0)P(0)\Phi(1,0)' + M(1,0)\\
&\ \downarrow\\
&Q(1) &&= (\Sigma_1^{-1} - P(1)^{-1})^{-1}\\
&\ \downarrow\\
&Q(0) &&= \Phi(0,1)(Q(1) + M(1,0))\Phi(0,1)'\\
&\ \downarrow\\
&P(0) &&= (\Sigma_0^{-1} - Q(0)^{-1})^{-1}
\end{aligned}$$

using (6.17)*, converges to* $Q_-(0)$*,* $P_-(0)$*,* $Q_-(1)$*,* $P_-(1)$*, starting from a generic choice for* $Q(0)$*. The choice with a "$-$" is the one that generates the Schrödinger bridge as explained below. It is interesting to compare this property with similar properties of iterations that lead to solutions of Schrödinger systems in [86, 91] (see also Chapter 4).*

Remark 14 *Besides the expression in the proposition, another equivalent closed form formula for* $Q_\pm(0)$ *is*

$$\begin{aligned}
Q_\pm(0) \;=\; & \Sigma_0^{1/2}\Bigg(\frac{1}{2}I + \Sigma_0^{1/2}\Phi(1,0)'M(1,0)^{-1}\Phi(1,0)\Sigma_0^{1/2} \pm \\
& (\frac{1}{4}I + \Sigma_0^{1/2}\Phi(1,0)'M(1,0)^{-1}\Sigma_1 M(1,0)^{-1}\Phi(1,0)\Sigma_0^{1/2})^{1/2}\Bigg)^{-1}\Sigma_0^{1/2}
\end{aligned}$$

Remark 15 *Interestingly, the solution* $\Pi_+(t) = Q_+(t)^{-1}$ *of the Riccati equation* (6.7) *corresponding to the choice "+" in* $Q_\pm$ *has a* finite escape time.

We are now in a position to state the full solution to Problem 11.

Theorem 16 *Assuming that the pair* $(A(t), B(t))$ *is controllable and that* $\Sigma_0, \Sigma_1 > 0$*, Problem 11 has a unique optimal solution*

$$u^\star(t) = -B(t)'Q_-(t)^{-1}x(t) \tag{6.22}$$

where $Q_-(\cdot)$ (together with a corresponding matrix function $P_-(\cdot)$) solves to the pair of coupled Lyapunov differential equations in Proposition 6.

Proof Since Proposition 6 has established existence and uniqueness of nonsingular solutions $(P_-(\cdot), Q_-(\cdot))$ to the system (6.16), the result now follows from Corollary 12.

Thus, the controlled process with optimal control (6.10) and $\Pi(t) = Q_-(t)^{-1}$, i.e.,

$$dx^* = (A(t) - B(t)B(t)'Q_-(t)^{-1})x^*(t)dt + B(t)dw(t) \tag{6.23}$$

steers the beginning density ρ_0 to the final one, ρ_1, with the least cost. It turns out that this controlled stochastic differential equation specifies the random evolution which is closest to the prior in the sense of relative entropy among those with the two given marginal distributions. This will be explained in Section 6.1.3.

Remark 17 *The variant of Problem 11 where the two marginals have a non-zero mean is of great practical significance. The formulae for the optimal control easily extend to this case as follows. Assuming that the Gaussian marginals ρ_0 and ρ_1 have mean m_0 and m_1, respectively, a deterministic term is needed in* (6.23) *for the bridge to satisfy the means. The controlled process becomes*

$$dx^* = (A(t) - B(t)B(t)'Q_-(t)^{-1})x^*(t)dt + B(t)B(t)'m(t)dt + B(t)dw(t) \tag{6.24}$$

where

$$m(t) = \hat{\Phi}(0,t)'\hat{M}(1,0)^{-1}(m_1 - \hat{\Phi}(1,0)m_0)$$

and $\hat{\Phi}(t,s)$, $\hat{M}(t,s)$ satisfy

$$\frac{\partial \hat{\Phi}(t,s)}{\partial t} = (A(t) - B(t)B(t)'Q_-(t)^{-1})\hat{\Phi}(t,s), \quad \hat{\Phi}(t,t) = I$$

and

$$\hat{M}(t,s) = \int_s^t \hat{\Phi}(t,\tau)B(t)B(t)'\hat{\Phi}(t,\tau)'d\tau.$$

It is easy to verify that (6.24) *meets the condition on the two marginal distributions. To see* (6.24) *is in fact optimal, observe that Problem 11 is equivalent to minimizing* *the augmented cost functional*

$$\tilde{J}(u) = \mathbb{E}\{\int_0^1 u(t)'u(t)\,dt + x(1)'Q_-(1)^{-1}x(1) - 2m(1)'x(1) - x(0)'Q_-(0)^{-1}x(0) + 2m(0)'x(0)\}.$$

over $\mathcal{U}$*. On the other hand, we have*

$$\begin{aligned}\tilde{J}(u) &= \mathbb{E}\{\int_0^1 u(t)'u(t)\,dt + \int_0^1 d(x(t)'Q_-(t)^{-1}x(t) - 2m(t)'x(t))\}\\ &= \mathbb{E}\{\int_0^1 \|u(t) + B(t)'Q_-(t)^{-1}x(t) - B(t)'m(t)\|^2 dt\\ &\quad + \int_0^1 [\mathrm{trace}\left(Q_-(t)^{-1}B(t)B(t)'\right) - m(t)'B(t)B(t)'m(t)]dt\},\end{aligned}$$

and thus

$$u(t) = -B(t)'Q_-(t)^{-1}x(t) + B(t)'m(t)$$

is indeed the optimal control strategy.

6.1.3 Minimum relative entropy interpretation of optimal control

In Chapter 5 we have showed that SBP can be recast as an optimal control problem. In particular, Problem 11 is equivalent to an SBP. Therefore, the controlled process under optimal control strategy has the property of minimizing its relative entropy with respect to the prior process (6.6). Below we present a direct proof of this property.

For the purposes of this section we denote by $\Omega = C([0,1],\mathbb{R}^n)$ the space of continuous, n-dimensional sample paths of a linear diffusion as in (6.6) and by $\mathcal{P}$ the induced probability measure on Ω. By disintegration of measure [42], one can describe $\mathcal{P}$ as a mixture of measures pinned at the two ends of the interval $[0,1]$, that is,

$$\mathcal{P}(\cdot) = \int_{\mathbb{R}^n\times\mathbb{R}^n} \mathcal{P}^{x_0x_1}(\cdot)\mathcal{P}_{01}(dx_0dx_1)$$

where $\mathcal{P}^{x_0x_1}(\cdot)$ is the conditional probability and $\mathcal{P}_{01}(\cdot)$ is the joint probability of $(x(0)=x_0, x(1)=x_1)$. The two end-point joint measure $\mathcal{P}_{01}(\cdot)$, which is Gaussian, has a (zero-mean) probability density function $g_{S_{01}}(x_0,x_1)$ with covariance

$$S_{01} = \begin{bmatrix} S_0 & S_0\Phi(1,0)' \\ \Phi(1,0)S_0 & S_1 \end{bmatrix} \tag{6.25}$$

where

$$S_0 = \mathbb{E}\{x_0 x_0'\}$$
$$S_t = \Phi(t,0)S_0\Phi(t,0)' + \int_0^t \Phi(t,\tau)B(\tau)B(\tau)'\Phi(t,\tau)'d\tau.$$

The probability law induced by the controlled process under optimal control, minimizes the relative entropy

$$H(\tilde{\mathcal{P}}, \mathcal{P}) := \int_\Omega \log\left(\frac{d\tilde{\mathcal{P}}}{d\mathcal{P}}\right) d\tilde{\mathcal{P}}$$

among those probability distributions on Ω that have the prescribed marginals. Evidently, this is an abstract problem on an infinite-dimensional space. However, since

$$\tilde{\mathcal{P}}(\cdot) = \int \tilde{\mathcal{P}}^{x_0 x_1}(\cdot)\tilde{\mathcal{P}}_{01}(dx_0 dx_1),$$

the relative entropy can be readily written as the sum of two nonnegative terms, the relative entropy between the two end-point joint measures

$$\int \log\left(\frac{d\tilde{\mathcal{P}}_{01}}{d\mathcal{P}_{01}}\right) d\tilde{\mathcal{P}}_{01}$$

and

$$\int \log\left(\frac{d\tilde{\mathcal{P}}^{x_0 x_1}(\cdot)}{d\mathcal{P}^{x_0 x_1}(\cdot)}\right) d\tilde{\mathcal{P}}.$$

The second term becomes zero (and therefore minimal) when the conditional probability $\tilde{\mathcal{P}}^{x_0 x_1}(\cdot)$ is taken to be the same as $\mathcal{P}^{x_0 x_1}(\cdot)$. Thus, the solution is in the same *reciprocal class* [69] as the prior evolution and, as already observed by Schrödinger [4] in a simpler context, the problem reduces to the problem of minimizing relative entropy of the joint initial-final distribution among those that have the prescribed marginals.

Below we show that the probability law induced by (6.23) is indeed the minimizer by verifying directly that the densities between the two are identical when conditioned at the two end points, i.e., they share the same bridges, and that the end-point joint marginal for (6.23) is indeed closest to the corresponding joint marginal for the prior (6.6).

In order to show that these two linear systems share the same bridges, we need the following lemma which is based on [46].

Lemma 18 *The probability law of the SDE* (6.6)*, when conditioned on* $x(0) = x_0$, $x(1) = x_1$*, for any* x_0, x_1*, reduces to the probability law induced by the SDE*

$$dx = (A - BB'R(t)^{-1})xdt + BB'R(t)^{-1}\Phi(t,1)x_1dt + Bdw$$

where $R(t)$ *satisfies*

$$\dot{R}(t) = AR(t) + R(t)A' - BB'$$

with $R(1) = 0$*.*

The stochastic process specified by this conditioning, or the latter SDE, will be referred to as the *pinned bridge associated to* (6.6). Thus, in order to establish that the probability laws of (6.23) and (6.6) conditioned on $x(0) = x_0$, $x(1) = x_1$ are identical, it suffices to show that they have the same pinned bridges for any x_0, x_1. All these are stated in the following theorem, and the proof is in Appendix A.4.

Theorem 19 *The probability law induced by* (6.23) *represents the minimum of the relative entropy with respect to the law of* (6.6) *over all probability laws on* Ω *that have Gaussian marginals with zero mean and covariances* Σ_0 *and* Σ_1*, respectively, at the two end-points of the interval* $[0,1]$*.*

6.2 Optimal mass transport: $\epsilon = 0$

We now consider the OMT-wpd problem for the special case where the marginals are Gaussian distributions. The OMT-wpd solution corresponds to the zero-noise limit of the Schrödinger bridges, which is of course a consequence of Theorem 10. Therefore, by taking the zero-noise limit, we obtain explicit expressions for the optimal control strategy.

Consider the reference evolution

$$dx(t) = A(t)x(t)dt + \sqrt{\epsilon}B(t)dw(t) \tag{6.26}$$

and the two marginals

$$\rho_0(x) = (2\pi)^{-n/2} \det(\Sigma_0)^{-1/2} \exp\left[-\frac{1}{2}(x-m_0)'\Sigma_0^{-1}(x-m_0)\right], \tag{6.27a}$$

$$\rho_1(x) = (2\pi)^{-n/2} \det(\Sigma_1)^{-1/2} \exp\left[-\frac{1}{2}(x-m_1)'\Sigma_1^{-1}(x-m_1)\right], \tag{6.27b}$$

where, as usual, the system with matrices $(A(t), B(t))$ is controllable. As in Section 6.1, we derived a closed-form expression for the Schrödinger bridge, namely,

$$dx(t) = (A(t) - B(t)B(t)'\Pi_\epsilon(t))x(t)dt + B(t)B(t)'m(t)dt + \sqrt{\epsilon}B(t)dw(t) \tag{6.28}$$

with $\Pi_\epsilon(t)$ satisfying the matrix Riccati equation

$$\dot{\Pi}_\epsilon(t) + A(t)'\Pi_\epsilon(t) + \Pi_\epsilon(t)A(t) - \Pi_\epsilon(t)B(t)B(t)'\Pi_\epsilon(t) = 0, \tag{6.29}$$

$$\Pi_\epsilon(0)=\Sigma_0^{-1/2}[\frac{\epsilon}{2}I + \Sigma_0^{1/2}\Phi_{10}'M_{10}^{-1}\Phi_{10}\Sigma_0^{1/2} - (\frac{\epsilon^2}{4}I + \Sigma_0^{1/2}\Phi_{10}'M_{10}^{-1}\Sigma_1 M_{10}^{-1}\Phi_{10}\Sigma_0^{1/2})^{1/2}]\Sigma_0^{-1/2} \tag{6.30}$$

and

$$m(t) = \hat{\Phi}(1,t)'\hat{M}(1,0)^{-1}(m_1 - \hat{\Phi}(1,0)m_0), \tag{6.31}$$

where $\hat{\Phi}(t,s), \hat{M}(t,s)$ satisfy

$$\frac{\partial\hat{\Phi}(t,s)}{\partial t} = (A(t) - B(t)B(t)'\Pi_\epsilon(t))\hat{\Phi}(t,s), \quad \hat{\Phi}(t,t) = I$$

and

$$\hat{M}(t,s) = \int_s^t \hat{\Phi}(t,\tau)B(\tau)B(\tau)'\hat{\Phi}(t,\tau)'d\tau.$$

The probability law $\mathcal{P}^\epsilon(\cdot)$ induced on path space by the stochastic process in (6.28) is indeed the solution of the SBP with prior corresponding to the law induced by the stochastic process in (6.26) and marginals (6.27).

Next we consider the zero-noise limit by letting ϵ go to 0. By taking $\epsilon = 0$ in (6.30) we obtain

$$\Pi_0(0) = \Sigma_0^{-1/2}[\Sigma_0^{1/2}\Phi_{10}'M_{10}^{-1}\Phi_{10}\Sigma_0^{1/2} - (\Sigma_0^{1/2}\Phi_{10}'M_{10}^{-1}\Sigma_1 M_{10}^{-1}\Phi_{10}\Sigma_0^{1/2})^{1/2}]\Sigma_0^{-1/2}, \tag{6.32}$$

and the corresponding limiting process

$$dx(t) = (A(t) - B(t)B(t)'\Pi_0(t))x(t)dt + B(t)B(t)'m(t)dt, \quad x(0) \sim (m_0, \Sigma_0) \quad (6.33)$$

with $\Pi_0(t), m(t)$ satisfying (6.29), (6.31) and (6.32). In fact $\Pi_0(t)$ has the explicit expression

$$\begin{aligned}\Pi_0(t) = -M(t,0)^{-1} - M(t,0)^{-1}\Phi(t,0)\Big[&\Phi_{10}' M_{10}^{-1}\Phi_{10} - \Phi(t,0)'M(t,0)^{-1}\Phi(t,0) \\ &-\Sigma_0^{-1/2}(\Sigma_0^{1/2}\Phi_{10}' M_{10}^{-1}\Sigma_1 M_{10}^{-1}\Phi_{10}\Sigma_0^{1/2})^{1/2}\Sigma_0^{-1/2}\Big]^{-1}\Phi(t,0)'M(t,0)^{-1}. \quad (6.34)\end{aligned}$$

As indicated earlier, Theorem 10 already implies that (6.33) yields an optimal solution to the OMT-wpd (5.14). Here we give an alternative proof by completion of squares.

Proposition 7 *Given Gaussian marginal distributions as in* (6.27)*, the optimal control law is*

$$u(t,x) = -B(t)'\Pi_0(t)x + B(t)'m(t), \qquad (6.35)$$

with Π_0 *in* (6.29) *and* m *in* (6.31)*.*

Proof We show first that u in (6.35) is a feasible control by proving that the corresponding probability density function ρ satisfies the boundary condition (6.27), and second, that this control u is the optimal one.

The controlled process (6.33) is linear with gaussian initial condition, hence $x(t)$ is a gaussian process. We claim that density of $x(t)$ is

$$\rho(t,x) = (2\pi)^{-n/2}\det(\Sigma(t))^{-1/2}\exp\left[-\frac{1}{2}(x-n(t))'\Sigma(t)^{-1}(x-n(t))\right]$$

where

$$n(t) = \hat{\Phi}(t,0)m_0 + \int_0^t \hat{\Phi}(t,\tau)B(\tau)B(\tau)'m(\tau)d\tau$$

and

$$\begin{aligned}\Sigma(t) =& M(t,0)\Phi(0,t)'\Sigma_0^{-1/2}\Big[-\Sigma_0^{1/2}\Phi_{10}' M_{10}^{-1}\Phi_{10}\Sigma_0^{1/2} + (\Sigma_0^{1/2}\Phi_{10}' M_{10}^{-1}\Sigma_1 M_{10}^{-1}\Phi_{10}\Sigma_0^{1/2})^{1/2} \\ &+ \Sigma_0^{1/2}\Phi(t,0)'M(t,0)^{-1}\Phi(t,0)\Sigma_0^{1/2}\Big]^2\Sigma_0^{-1/2}\Phi(0,t)M(t,0)\end{aligned}$$

for $t \in (0, 1]$. It is obvious that $\mathbb{E}\{x(t)\} = n(t)$ and it is also immediate that

$$\lim_{t \to 0} \Sigma(t) = \Sigma_0.$$

Straightforward but lengthy computations show that $\Sigma(t)$ satisfies the Lyapunov differential equation

$$\dot{\Sigma}(t) = (A(t) - B(t)B(t)'\Pi_0(t))\Sigma(t) + \Sigma(t)(A(t) - B(t)B(t)'\Pi_0(t))'.$$

Hence, $\Sigma(t)$ is the covariance of $x(t)$. Now, observing that

$$\begin{aligned} n(1) &= \hat{\Phi}(1,0)m_0 + \int_0^1 \hat{\Phi}(1,\tau)B(\tau)B(\tau)'m(\tau)d\tau \\ &= \hat{\Phi}(1,0)m_0 + \int_0^1 \hat{\Phi}(1,\tau)B(\tau)B(\tau)'\hat{\Phi}(1,\tau)'d\tau \hat{M}(1,0)^{-1}(m_1 - \hat{\Phi}(1,0)m_0) \\ &= m_1 \end{aligned}$$

and

$$\begin{aligned} \Sigma(1) &= M(1,0)\Phi(0,1)'\Sigma_0^{-1/2}\left[(\Sigma_0^{1/2}\Phi_{10}'M_{10}^{-1}\Sigma_1 M_{10}^{-1}\Phi_{10}\Sigma_0^{1/2})^{1/2}\right]^2\Sigma_0^{-1/2}\Phi(0,1)M(1,0) \\ &= \Sigma_1, \end{aligned}$$

allows us to conclude that ρ satisfies $\rho(1, x) = \rho_1(x)$.

For the second part, consider the OMT-wpd (5.14) with the augmented cost functional

$$\begin{aligned} J(u) &= \mathbb{E}\{\int_0^1 \frac{1}{2}\|u(t)\|^2 dt + \frac{1}{2}x(1)'\Pi_0(1)x(1) \\ &\quad -\frac{1}{2}x(0)'\Pi_0(0)x(0) - m(1)'x(1) + m(0)'x(0)\}. \end{aligned}$$

This doesn't change the minimizer because the extra terms are constant under the fixed boundary distributions. Since

$$\begin{aligned} J(u) &= \mathbb{E}\{\int_0^1 \frac{1}{2}\|u(t)\|^2 dt + \frac{1}{2}d(x(t)'\Pi_0(t)x(t)) - d(m(t)'x(t))\} \\ &= \mathbb{E}\{\int_0^1 \frac{1}{2}\|u(t) + B(t)'\Pi_0(t)x(t) - B(t)'m(t)\|^2 dt\} + \int_0^1 \frac{1}{2}m(t)'B(t)B(t)'m(t)dt, \end{aligned}$$

it is easy to see that u in (6.35) achieves the minimum of $J(u)$.

6.3 Different input and noise channels

In this section we formulate the control problem to optimally steer a stochastic linear system with different input and noise channels from an initial Gaussian distribution to a final target Gaussian distribution. In parallel, we formulate the problem to maintain a stationary Gaussian state distribution by constant state feedback for time-invariant dynamics.

The mean value of the state-vector is effected only by a deterministic mean value for the input process. Thus, throughout this section and without loss of generality we assume that all processes have zero-mean and we only focus on our ability to assign the state-covariance in those two instances.

6.3.1 Finite-horizon optimal steering

Consider the controlled evolution

$$dx^u(t) = A(t)x^u(t)dt + B(t)u(t)dt + B_1(t)dw(t) \tag{6.36}$$

where $A(\cdot)$, $B(\cdot)$ and $B_1(\cdot)$ are continuous matrix functions of t taking values in $\mathbb{R}^{n\times n}$, $\mathbb{R}^{n\times m}$ and $\mathbb{R}^{n\times p}$, respectively. The goal is to steer the system from initial distribution (6.4) at $t = 0$ to terminal distribution (6.5) at $t = 1$ with minimum effort.

As in Section 6.1, denote by $\mathcal{U}$ the family of *adapted*, *finite-energy* control functions such that (6.36) has a strong solution and $x^u(1)$ is distributed according to (6.5). Therefore, $\mathcal{U}$ represents the class of admissible control inputs. The existence of such control inputs will be established in the following section, i.e., that $\mathcal{U}$ is not empty. At present, assuming this to be the case, we formulate the following:

Problem 20 *Determine* $u^* := \operatorname{argmin}_{u\in\mathcal{U}} J(u)$.

With the same completion of the squares argument used in Section 6.1.1 we obtain a sufficient conditions in Proposition 8 and show that a *control-theoretic view* of the SBP [63] carries through in this more general setting.

Proposition 8 *Let $\{\Pi(t) \mid 0 \le t \le 1\}$ be a solution of the matrix Riccati equation*

$$\dot{\Pi}(t) = -A(t)'\Pi(t) - \Pi(t)A(t) + \Pi(t)B(t)B(t)'\Pi(t). \tag{6.37}$$

Define the feedback control law

$$u(x,t) := -B(t)'\Pi(t)x \tag{6.38}$$

and let $x^u = x^*$ be the Gauss-Markov process

$$dx^*(t) = (A(t) - B(t)B(t)'\Pi(t))\, x^*(t)dt + B_1(t)dw(t), \quad x^*(0) = x_0 \;\; a.s. \tag{6.39}$$

If $x^*(1)$ has probability density ρ_1, then $u(x^*(t),t) = u^*(t)$, i.e., it is the solution to Problem 20.

Below we recast Proposition 8 in the form of a Schrödinger system.

Let $\Sigma(t) := \mathbb{E}\{x^*(t)x^*(t)'\}$ be the state covariance of (6.39) and assume that the conditions of the proposition hold. Then

$$\dot{\Sigma}(t) = (A(t) - B(t)B(t)'\Pi(t))\,\Sigma(t) + \Sigma(t)\,(A(t) - B(t)B(t)'\Pi(t))' + B_1(t)B_1(t)' \tag{6.40}$$

holds together with the two boundary conditions

$$\Sigma(0) = \Sigma_0, \quad \Sigma(1) = \Sigma_1. \tag{6.41}$$

Further, since $\Sigma_0 > 0$, $\Sigma(t)$ is positive definite on $[0,1]$. Now define

$$\mathrm{H}(t) := \Sigma(t)^{-1} - \Pi(t).$$

A direct calculation using (6.40) and (6.37) leads to (6.42b) below. We have therefore derived a *nonlinear* generalized Schrödinger system

$$\begin{aligned}
\dot{\Pi} &= -A'\Pi - \Pi A + \Pi BB'\Pi && (6.42a)\\
\dot{\mathrm{H}} &= -A'\mathrm{H} - \mathrm{H}A - \mathrm{H}BB'\mathrm{H} + (\Pi + \mathrm{H})\,(BB' - B_1B_1')\,(\Pi + \mathrm{H}) && (6.42b)\\
\Sigma_0^{-1} &= \Pi(0) + \mathrm{H}(0) && (6.42c)\\
\Sigma_1^{-1} &= \Pi(1) + \mathrm{H}(1). && (6.42d)
\end{aligned}$$

Indeed, in contrast to the case when $B = B_1$ (see Section 6.1), the two Riccati equations in (6.42) are coupled not only through their boundary values (6.42c)-(6.42d) but also in a nonlinear manner through their dynamics in (6.42b). Clearly, the case $\Pi(t) \equiv 0$ corresponds to the situation where the uncontrolled evolution already satisfies the boundary marginals and, in that case, $\mathrm{H}(t)^{-1}$ is simply the prior state covariance. We summarize our conclusion in the following proposition.

Proposition 9 *Assume that* $\{(\Pi(t), \mathrm{H}(t)) \mid 0 \leq t \leq 1\}$ *satisfy* (6.42a)-(6.42d). *Then the feedback control law* (6.38) *is the solution to Problem 20 and the corresponding optimal evolution is given by* (6.39).

The existence and uniqueness of solutions for the Schrödinger system is quite challenging already in the classical case where the two dynamical equations are uncoupled. It is therefore hardly surprising that at present we don't know how to prove existence of solutions for (6.42a)-(6.42d) A direct proof of existence of solutions for (6.42) would in particular imply *feasibility* of Problem 20, i.e., that $\mathcal{U}$ is nonempty and that there exists a minimizer. At present we do not have a proof that a minimizer exists. However, in Section 6.3.3 we establish that the set of admissible controls $\mathcal{U}$ is not empty and in Section 6.3.4 we provide an approach that allows constructing suboptimal controls incurring cost that is arbitrarily close to $\inf_{u \in \mathcal{U}} J(u)$.

6.3.2 Infinite-horizon optimal steering

Suppose now that A, B and B_1 do not depend on time and that the pair (A, B) is controllable. We seek a constant state feedback law $u(t) = -Kx(t)$ to maintain a stationary state-covariance $\Sigma > 0$ for (6.36). In particular, we are interested in one that minimizes the expected input power (energy rate)

$$J_{\text{power}}(u) := \mathbb{E}\{u'u\} \tag{6.43}$$

and thus we are led to the following problem[2].

Problem 21 *Determine u^* that minimizes $J_{\text{power}}(u)$ over all $u(t) = -Kx(t)$ such*

[2] An equivalent problem is to minimize $\lim_{T\to\infty} \frac{1}{T_f}\mathbb{E}\left\{\int_0^{T_f} u(t)'u(t)dt\right\}$ for a given terminal state covariance as $T_f \to \infty$.

that

$$dx(t) = (A - BK)x(t)dt + B_1 dw(t) \tag{6.44}$$

admits

$$\rho(x) = (2\pi)^{-n/2} \det(\Sigma)^{-1/2} \exp\left(-\frac{1}{2}x'\Sigma^{-1}x\right) \tag{6.45}$$

as the invariant probability density.

Interestingly, the above problem may not have a solution in general since not all values for Σ can be maintained by state feedback. In fact, Theorem 24 in Section 6.3.3, provides conditions that ensure Σ is admissible as a stationary state covariance for a suitable input. Moreover, as it will be apparent from what follows, even when the problem is feasible, i.e., there exist controls which maintain Σ, an optimal control may fail to exist. The relation between this problem and Jan Willems' classical work on the Algebraic Riccati Equation (ARE) [24] is provided after Proposition 10 below.

Let us start by observing that the problem admits the following finite-dimensional reformulation. Let $\mathcal{K}$ be the set of all $m \times n$ matrices K such that the corresponding feedback matrix $A - BK$ is Hurwitz. Since

$$\mathbb{E}\{u'u\} = \mathbb{E}\{x'K'Kx\} = \operatorname{trace}(K\Sigma K'),$$

Problem 21 reduces to finding a $m \times n$ matrix $K^* \in \mathcal{K}$ which minimizes the criterion

$$J(K) = \operatorname{trace}(K\Sigma K') \tag{6.46}$$

subject to the constraint

$$(A - BK)\Sigma + \Sigma(A' - K'B') + B_1B_1' = 0. \tag{6.47}$$

Now, consider the Lagrangian function with symmetric multiplier Π

$$\mathcal{L}(K, \Pi) = \operatorname{trace}(K\Sigma K') + \operatorname{trace}(\Pi((A - BK)\Sigma + \Sigma(A' - K'B') + B_1B_1'))$$

which is a simple quadratic form in the unknown K. Observe that $\mathcal{K}$ is *open*, hence a minimum point may fail to exist. Nevertheless, at any point $K \in \mathcal{K}$ we can take a directional derivative in any direction $\delta K \in \mathbb{R}^{m \times n}$ to obtain

$$\delta\mathcal{L}(K,\Pi;\delta K) = \operatorname{trace}\left((\Sigma K' + K\Sigma - \Sigma\Pi B - B'\Pi\Sigma)\,\delta K\right).$$

Setting $\delta\mathcal{L}(K,\Pi;\delta K) = 0$ for all variations, which is a sufficient condition for optimality, we get the form

$$K^* = B'\Pi. \tag{6.48}$$

To compute K^*, we calculate the multiplier Π as a maximum point of the dual functional

$$\begin{aligned} G(\Pi) &= \mathcal{L}(K^*,\Pi) \\ &= \operatorname{trace}\left((A'\Pi + \Pi A - \Pi BB'\Pi)\,\Sigma + \Pi B_1 B_1'\right). \end{aligned} \tag{6.49}$$

The unconstrained maximization of the concave functional G over symmetric $n \times n$ matrices produces matrices Π^* which satisfy (6.47), namely

$$(A - BB'\Pi^*)\Sigma + \Sigma(A' - \Pi^* BB') + B_1 B_1' = 0. \tag{6.50}$$

There is no guarantee, however, that $K^* = B'\Pi^*$ is in $\mathcal{K}$, namely that $A - BB'\Pi^*$ is Hurwitz. Nevertheless, since (6.50) is satisfied, the spectrum of $A - BB'\Pi^*$ lies in the *closed* left half-plane. Thus, our analysis leads to the following result.

Proposition 10 *Assume that there exists a symmetric matrix* Π *such that* $A - BB'\Pi$ *is a Hurwitz matrix and*

$$(A - BB'\Pi)\Sigma + \Sigma(A - BB'\Pi)' + B_1 B_1' = 0 \tag{6.51}$$

holds. Then

$$u^*(t) = -B'\Pi x(t) \tag{6.52}$$

is the solution to Problem 21.

We now draw a connection to some classical results due to Jan Willems [24]. In our setting, minimizing (6.43) is equivalent to minimizing

$$J_{\text{power}}(u) + \mathbb{E}\{x'Qx\} \tag{6.53}$$

for an arbitrary symmetric matrix Q since the portion

$$\mathbb{E}\{x'Qx\} = \text{trace}\{Q\Sigma\}$$

is independent of the choice of K. On the other hand, minimization of (6.53) for specific Q, but without the constraint that $\mathbb{E}\{xx'\} = \Sigma$, was studied by Willems [24] and is intimately related to the *maximal* solution of the Algebraic Riccati Equation (ARE)

$$A'\Pi + \Pi A - \Pi BB'\Pi + Q = 0. \tag{6.54}$$

Under the assumption that the Hamiltonian matrix

$$H = \begin{bmatrix} A & -BB' \\ -Q & -A' \end{bmatrix}$$

has no pure imaginary eigenvalues, Willems' result states that $A - BB'\Pi$ is Hurwitz and that (6.52) is the optimal solution.

Thus, starting from a symmetric matrix Π as in Proposition 10, we can define Q using

$$Q = -A'\Pi - \Pi A + \Pi BB'\Pi.$$

Since, by Willems' results, (6.54) has at most one "stabilizing" solution Π, the matrix in the proposition coincides with the maximal solution to (6.54). Therefore, if our original problem has a solution, this same solution can be recovered by solving for the maximal solution of a corresponding ARE, for a particular choice of Q. Interestingly, neither Π nor Q, which correspond to an optimal control law and satisfy (6.51), are unique, whereas K is.

6.3.3 Controllability of state statistics

We now return to the "controllability" question of whether there exist admissible controls to steer the controlled evolution

$$dx(t) = Ax(t)dt + Bu(t)dt + B_1 dw(t) \quad x(0) = x_0 \text{ a.s.} \tag{6.55}$$

to a target Gaussian distribution at the end of a finite interval [0, 1], or, for the stationary case, whether a stationary Gaussian distribution can be achieved by constant state feedback. From now on, we assume that $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $B_1 \in \mathbb{R}^{n\times p}$, are time-invariant and that (A, B) is controllable. In view of the earlier analysis, we search over controls that are linear functions of the state, i.e.,

$$u(t) = -K(t)x(t), \quad \text{for } t \in [0, 1], \tag{6.56}$$

and where K is constant and $A - BK$ Hurwitz for the stationary case.

We first consider finite horizon case. We assume that $\mathbb{E}\{x_0\} = 0$ while $\mathbb{E}\{x_0 x_0'\} = \Sigma_0$. The state covariance

$$\Sigma(t) := \mathbb{E}\{x(t)x(t)'\}$$

of (6.36) with input as in (6.56) satisfies the differential Lyapunov equation

$$\dot{\Sigma}(t) = (A - BK(t))\Sigma(t) + \Sigma(t)(A - BK(t))' + B_1B_1' \tag{6.57}$$

and $\Sigma(0) = \Sigma_0$. Regardless of the choice of $K(t)$, (6.57) specifies dynamics that leave invariant the cone of positive semi-definite symmetric matrices

$$\mathcal{S}_n^+ := \{\Sigma \mid \Sigma \in \mathbb{R}^{n\times n},\ \Sigma = \Sigma' \geq 0\}.$$

To see this, note that the solution to (6.57) is of the form

$$\Sigma(t) = \hat{\Phi}(t, 0)\Sigma_0\hat{\Phi}(t, 0)' + \int_0^t \hat{\Phi}(t, \tau)B_1B_1'\hat{\Phi}(t, \tau)'d\tau$$

where $\hat{\Phi}(t, 0)$ satisfies

$$\frac{\partial\hat{\Phi}(t, 0)}{\partial t} = (A - BK(t))\hat{\Phi}(t, 0)$$

and $\hat{\Phi}(0, 0) = I$, i.e., $\hat{\Phi}(t, 0)$ is the state-transition matrix of the system $\dot{x}(t) = (A - BK(t))x(t)$.

Assuming $\Sigma_0 > 0$, it follows that $\Sigma(t) > 0$ for all t and finite $K(\cdot)$. Our interest is in our ability to specify $\Sigma(1)$ via a suitable choice of $K(t)$. To this end, we define

$$U(t) := -\Sigma(t)K(t)',$$

we observe that $U(t)$ and $K(t)$ are in bijective correspondence provided that $\Sigma(t) > 0$, and we now consider the differential Lyapunov system

$$\dot{\Sigma}(t) = A\Sigma(t) + \Sigma(t)A' + BU(t)' + U(t)B'. \tag{6.58}$$

This should be compared with the linear system

$$\dot{x}(t) = Ax(t) + Bu(t). \tag{6.59}$$

Reachability/controllability of a differential system such as (6.59) (or (6.58)), is the property that with suitable bounded control input $u(t)$ (or $U(t)$), the solution can be driven to any finite value. Interestingly, if either (6.59) or (6.58) is controllable, so is the other. But, more importantly, when (6.58) is controllable, the control authority allowed is such that steering from one value for the covariance to another can be done by remaining within the non-negative cone. This is stated as our first theorem below. See [92] for the proof.

Theorem 22 *The Lyapunov system* (6.58) *is controllable iff* (A, B) *is a controllable pair. Furthermore, if* (6.58) *is controllable, then for any two positive definite matrices* Σ_0 *and* Σ_1 *and an arbitrary* $Q \geq 0$*, there is a smooth input* $U(\cdot)$ *so that the solution of the (forced) differential equation*

$$\dot{\Sigma}(t) = A\Sigma(t) + \Sigma(t)A' + BU(t)' + U(t)B' + Q \tag{6.60}$$

satisfies the boundary conditions $\Sigma(0) = \Sigma_0$ *and* $\Sigma(1) = \Sigma_1$ *and* $\Sigma(t) > 0$ *for all* $t \in [0, 1]$.

Remark 23 *It is interesting to observe that, in the case when* $B = B_1$*, steering the state-covariance via state-feedback is equivalent to modeling the evolution of state-covariances as due to an external input process. Specifically, given the Gauss-Markov model*

$$dx(t) = Ax(t)dt + Bdy(t)$$

and a path of state-covariances $\{\Sigma(t) \mid t \in [0, 1]\}$ *that satisfies* (6.60) *for some* $U(t)$*, the claim is that there is a suitable process* $y(t)$ *that can account for the time-evolution* *of this state-covariance. Indeed, starting from the Gauss-Markov process*

$$\begin{aligned} d\xi(t) &= (A - BK(t))\xi(t)dt + Bdw(t) \\ dy(t) &= -K(t)\xi(t)dt + dw(t), \end{aligned} \tag{6.61}$$

with $\mathbb{E}\{\xi(0)\xi(0)'\} = \Sigma_0$ *and*

$$K(t) = -U(t)'\Sigma(t)^{-1},$$

we observe that

$$d\xi(t) = A\xi(t)dt + Bdy(t).$$

Therefore $\xi(t)$ *and* $x(t)$ *share the same statistics. In the converse direction, the state covariance of* (6.61) *satisfies* (6.60).

We now consider the problem to maintain the state process of a dynamical system at an equilibrium distribution with a specified state-covariance Σ via static state-feedback

$$u(t) = -Kx(t). \tag{6.62}$$

Due to linearity, the distribution will then be Gaussian. However, depending on the value of Σ this may not always be possible. The precise characterization of admissible stationary state-covariances is provided in Theorem 24 given below.

Let $\mathcal{S}_n$ denote the linear vector space of symmetric matrices of dimension n and note that the map

$$\mathfrak{g}_B \ : \ \mathcal{S}_n \to \mathcal{S}_n \ : \ Y \mapsto \Pi_{\mathcal{R}(B)^\perp} Y \Pi_{\mathcal{R}(B)^\perp} \tag{6.63}$$

is self-adjoint. Hence, the orthogonal complement of its range is precisely its null space, which according to the lemma in Appendix A.5, is also the range of

$$\mathfrak{f}_B \ : \ \mathbb{R}^{n\times m} \to \mathcal{S}_n \ : \ X \mapsto BX' + XB'. \tag{6.64}$$

Assuming that $A - BK$ is a Hurwitz matrix, which is necessary for the state process $\{x(t) \mid t \in [0,\infty)\}$ to be stationary, the (stationary) state-covariance $\Sigma =$

$\mathbb{E}\{x(t)x(t)'\}$ satisfies the algebraic Lyapunov equation

$$(A - BK)\Sigma + \Sigma(A - BK)' = -B_1B_1'. \tag{6.65}$$

Thus, the equation

$$A\Sigma + \Sigma A' + B_1B_1' + BX' + XB' = 0 \tag{6.66a}$$

can be solved for X,

which in particular can be taken to be $X = -\Sigma K'$. The solvability of (6.66a) is obviously a necessary condition for Σ to qualify as a stationary state-covariance attained via feedback. Alternatively, (6.66a) is equivalent to the statement that

$$A\Sigma + \Sigma A' + B_1B_1' \in \mathcal{R}(\mathfrak{f}_B). \tag{6.66b}$$

The latter can be expressed as a rank condition [93, Proposition 1] in the form

$$\operatorname{rank}\begin{bmatrix} A\Sigma + \Sigma A' + B_1B_1' & B \\ B & 0 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix}. \tag{6.66c}$$

In view of Lemma 37 (Appendix A.5), (6.66b) is equivalent to

$$A\Sigma + \Sigma A' + B_1B_1' \in \mathcal{N}(\mathfrak{g}_B). \tag{6.66d}$$

Therefore, the conditions (6.66a)-(6.66d), which are all equivalent, are necessary for the existence of a state-feedback gain K that ensures $\Sigma > 0$ to be the stationary state covariance of (6.36).

Conversely, if $\Sigma > 0$ satisfies (6.66) and X is the corresponding solution to (6.66a), then (6.65) holds with $K = -X'\Sigma^{-1}$. Provided $A - BK$ is a Hurwitz matrix, Σ is an admissible stationary covariance. The property of $A - BK$ being Hurwitz can be guaranteed when $(A - BK,\, B_1)$ is a controllable pair. In turn, controllability of $(A - BK,\, B_1)$ is guaranteed when $\mathcal{R}(B) \subseteq \mathcal{R}(B_1)$. Thus, we have established the following.

Theorem 24 *Consider the Gauss-Markov model* (6.36) *and assume that* $\mathcal{R}(B) \subseteq \mathcal{R}(B_1)$. *A positive-definite matrix* Σ *can be assigned as the stationary state covariance* *via a suitable choice of state-feedback if and only if* Σ *satisfies any of the equivalent statements* (6.66a)-(6.66d).

Interest in (6.66d) was raised in [10] where it was shown to characterize state-covariances that can be maintained by state-feedback. On the other hand, conditions (6.66a)-(6.66c) were obtained in [93, 94], for the special case when $B = B_1$, as being necessary and sufficient for a positive-definite matrix to materialize as the state covariance of the system driven by a stationary stochastic process (not-necessarily white). It should be noted that in [93], the state matrix A was assumed to be already Hurwitz so as to ensure stationarity of the state process. However, if the input is generated via feedback as above, A does not need to be Hurwitz whereas, only $A - BK$ needs to be.

Remark 25 *We now turn to the question of which positive definite matrices materialize as state covariances of the Gauss-Markov model*

$$dx(t) = Ax(t) + Bdy(t), \tag{6.67}$$

with (A, B) *controllable and* A *Hurwitz, when driven by some stationary stochastic process* $y(t)$*. The characterization of admissible state covariances was obtained in [93] and amounts to the condition that*

$$A\Sigma + \Sigma A' \in \mathcal{R}(\mathfrak{f}_B)$$

which coincides with the condition that Σ *can be assigned as in Theorem 24 by state-feedback. A feedback system can be implemented, separate from* (6.67)*, to generate a suitable input process to give rise to* Σ *as the state covariance of* (6.67)*. Specifically, let* X *be a solution of*

$$A\Sigma + \Sigma A' + BX' + XB' = 0, \tag{6.68}$$

and

$$\begin{aligned} d\xi(t) &= (A - BK)\xi(t)dt + Bdw(t) \\ dy(t) &= -K\xi(t)dt + dw(t) \end{aligned}$$

with

$$K = \frac{1}{2}B'\Sigma^{-1} - X'\Sigma^{-1}. \tag{6.69}$$

Trivially,

$$d\xi(t) = A\xi(t)dt + Bdy(t),$$

and therefore, $\xi(t)$ *shares the same stationary statistics with* $x(t)$*. But if* $S = \mathbb{E}\{\xi(t)\xi(t)'\}$*,*

$$(A - BK)S + S(A - BK)' + BB' = 0,$$

which, in view of (6.68)-(6.69)*, is satisfied by* $S = \Sigma$*.*

6.3.4 Numerical computation of optimal control

Having established feasibility for the problem to steer the state-covariance to a given value at the end of a time interval, it is of interest to design efficient methods to compute the optimal controls of Section 6.3.1. As an alternative to solving the generalized Schrödinger system (6.42), we formulate the optimization as a semidefinite program (SDP), and likewise for the infinite-horizon problem.

We are interested in computing an optimal feedback gain $K(t)$ so that the control signal $u(t) = -K(t)x(t)$ steers (6.36) from Σ_0 at $t = 0$ to Σ_1 at $t = 1$. The expected control energy functional

$$\begin{aligned} J(u) &:= \mathbb{E}\left\{\int_0^1 u(t)'u(t)dt\right\} \\ &= \int_0^1 \text{trace}(K(t)\Sigma(t)K(t)')dt \end{aligned} \tag{6.70}$$

needs to be optimized over $K(t)$ so that (6.57) holds as well as the boundary conditions

$$\Sigma(0) = \Sigma_0, \text{ and } \Sigma(1) = \Sigma_1. \tag{6.71a}$$

If instead we sought to optimize over $U(t) := -\Sigma(t)K(t)'$ and $\Sigma(t)$, the functional (6.70) becomes

$$J = \int_0^1 \text{trace}(U(t)'\Sigma(t)^{-1}U(t))dt$$

which is jointly convex in $U(t)$ and $\Sigma(t)$, while (6.57) is replaced by

$$\dot{\Sigma}(t) = A\Sigma(t) + \Sigma(t)A' + BU(t)' + U(t)B' + B_1B_1' \tag{6.71b}$$

which is now linear in both. Thus, finally, the optimization can be written as an SDP to minimize

$$\int_0^1 \text{trace}(Y(t))dt \tag{6.71c}$$

subject to (6.71a)-(6.71b) and

$$\begin{bmatrix} Y(t) & U(t)' \\ U(t) & \Sigma(t) \end{bmatrix} \geq 0. \tag{6.71d}$$

This can be solved numerically after discretization in time and a corresponding (sub-optimal) gain recovered as $K(t) = -U(t)'\Sigma(t)^{-1}$.

For the infinite horizon case, as noted earlier, a positive definite matrix Σ is admissible as a stationary state covariance provided (6.66a) holds for some X and $A + BX'\Sigma^{-1}$ is a Hurwitz matrix. The condition $\mathcal{R}(B) \subseteq \mathcal{R}(B_1)$ is a sufficient condition for the latter to be true always, but it may be true even if $\mathcal{R}(B) \subseteq \mathcal{R}(B_1)$ fails. Either way, the expected input power (energy rate) is

$$\begin{aligned} \mathbb{E}\{u'u\} &= \text{trace}(K\Sigma K') \\ &= \text{trace}(X'\Sigma^{-1}X), \end{aligned} \tag{6.72}$$

expressed in either K, or X. Thus, assuming that $\mathcal{R}(B) \subseteq \mathcal{R}(B_1)$ holds, and in case (6.66a) has multiple solutions, the optimal constant feedback gain K can be obtained by solving the convex optimization problem

$$\min \left\{ \text{trace}(K\Sigma K') \mid \quad (6.66a) \text{ holds} \right\}. \tag{6.73}$$

Remark 26 *In case $\mathcal{R}(B) \nsubseteq \mathcal{R}(B_1)$, the condition that $A - BK$ be Hurwitz needs to be verified separately. If this fails, we cannot guarantee that Σ is an admissible stationary state-covariance that can be maintained with constant state-feedback. However, it is always possible to maintain a state-covariance that is arbitrarily close. To see this, consider the control*

$$K_\epsilon = K + \frac{1}{2}\epsilon B'\Sigma^{-1}$$

for $\epsilon > 0$. Then, from (6.65),

$$\begin{aligned}(A - BK_\epsilon)\Sigma + \Sigma(A - BK_\epsilon)' &= -\epsilon BB' - B_1B_1' \\ &\leq -\epsilon BB'.\end{aligned}$$

The fact that $A - BK_\epsilon$ is Hurwitz is obvious. If now Σ_ϵ is the solution to

$$(A - BK_\epsilon)\Sigma_\epsilon + \Sigma_\epsilon(A - BK_\epsilon)' = -B_1B_1'$$

the difference $\Delta = \Sigma - \Sigma_\epsilon \geq 0$ and satisfies

$$(A - BK_\epsilon)\Delta + \Delta(A - BK_\epsilon)' = -\epsilon BB',$$

and hence is of the order of ϵ.

6.4 Covariance control with state penalty

In this section, we consider the optimal steering problem for the dynamics (6.1) with the more general cost function

$$\mathbb{E}\left\{\int_0^1 [\|u(t)\|^2 + x(t)'S(t)x(t)]dt\right\}, \tag{6.74}$$

where $S(t)$ is positive semi-definite for all $t \in [0, 1]$.

For the case $\epsilon > 0$, the solution can be obtained by solving a Schrödinger system. For the case $\epsilon = 0$, we can compute the solution by taking the zero-noise limit of the previous case.

For simplicity, we present the solution for $\epsilon = 1$. It corresponds to a Markov evolution with losses. Nevertheless, the Markov kernel $q(s, x, t, y)$ is a positive everywhere, continuous function. By Theorem 5, the solution exists and is unique. Moreover, it is not difficult to see that the factors $\varphi(t, x)$ and $\hat{\varphi}(t, x)$ (see Chapter 4) have the form

$$\begin{aligned}\varphi(t, x) &= c(t)\exp\{-\frac{1}{2}x'\Pi(t)x\} \\ \hat{\varphi}(t, x) &= \hat{c}(t)\exp\{-\frac{1}{2}x'\mathrm{H}(t)x\}.\end{aligned}$$

By substituting the above into the Schrödinger system (4.13) and separating variables, we arrive at the following two coupled Riccati equations with split boundary conditions

$$-\dot{\Pi}(t) = A'\Pi(t) + \Pi(t)A - \Pi(t)BB'\Pi(t) + S(t) \tag{6.75a}$$

$$-\dot{\mathrm{H}}(t) = A'\mathrm{H}(t) + \mathrm{H}(t)A + \mathrm{H}(t)BB'\mathrm{H}(t) - S(t) \tag{6.75b}$$

with

$$\Sigma_0^{-1} = \Pi(0) + \mathrm{H}(0) \quad \text{and} \quad \Sigma_1^{-1} = \Pi(1) + \mathrm{H}(1) \tag{6.75c}$$

and

$$\begin{aligned} c(t) &= \exp\{\frac{1}{2}\int_0^t \mathrm{trace}(BB'\Pi(\tau))d\tau\} \\ \hat{c}(t) &= \exp\left\{-\int_0^t \mathrm{trace}\left[A(\tau) + \frac{1}{2}BB'\mathrm{H}(\tau)\right]d\tau\right\}. \end{aligned}$$

Thus, the problem boils down to finding a pair $(\Pi(t), \mathrm{H}(t))$ satisfying (6.75). Such a pair always exists due to the existence of the solution to the SBP. For the case when $S(t) \equiv 0$, it has been shown in Section 6.1 that the solution of this system has an explicit form. For general $S(\cdot)$, we can compute the optimal solution as described below using an SDP formulation [81].

The goal is to compute a feedback gain $K(t)$ so that the control signal $u(t) = -K(t)x(t)$ steers (6.1) from the initial state-covariance Σ_0 at $t = 0$ to the final state-covariance Σ_1 at $t = 1$. Namely, we need to minimize

$$\begin{aligned} J &= \mathbb{E}\left\{\int_0^1 \left[\|u\|^2 + x(t)'S(t)x(t)\right]dt\right\} \qquad (6.76) \\ &= \int_0^1 \left[\mathrm{trace}(K(t)\Sigma(t)K(t)') + \mathrm{trace}(S(t)\Sigma(t))\right]dt \end{aligned}$$

subject to the corresponding differential Lyapunov equation for the state covariance

$$\dot{\Sigma}(t) = (A - BK)\Sigma(t) + \Sigma(t)(A - BK)' + BB' \tag{6.77}$$

satisfying the boundary conditions

$$\Sigma(0) = \Sigma_0, \quad \text{and} \quad \Sigma(1) = \Sigma_1. \tag{6.78a}$$

If we replace $K(t)$ by $U(t) := -\Sigma(t)K(t)'$, then

$$J = \int_0^1 [\text{trace}(U(t)'\Sigma(t)^{-1}U(t)) + \text{trace}(S(t)\Sigma(t))]dt$$

becomes *jointly convex* in $U(t)$ and $\Sigma(t)$. On the other hand, the Lyapunov equation (6.77) becomes

$$\dot{\Sigma}(t) = A\Sigma(t) + \Sigma(t)A' + BU(t)' + U(t)B' + BB' \tag{6.78b}$$

and is now *linear* in both U and Σ. Thus, our optimization problem reduces to a SDP to minimize

$$\int_0^1 [\text{trace}(Y(t)) + \text{trace}(S(t)\Sigma(t))]dt \tag{6.78c}$$

subject to (6.78a-6.78b) and

$$\begin{bmatrix} Y(t) & U(t)' \\ U(t) & \Sigma(t) \end{bmatrix} \geq 0. \tag{6.78d}$$

After discretization in time, (6.78a-6.78d) can be solved numerically and a (suboptimal) gain can be recovered as

$$K(t) = -U(t)'\Sigma(t)^{-1}.$$

6.5 Covariance control through output feedback

In certain cases, precise measurements of the states are not available. The measurements might be corrupted with noise. For instance, we can consider the system

$$\begin{aligned} dx(t) &= Ax(t)dt + Bu(t)dt + B_1dw(t) \\ dy(t) &= Cx(t)dt + dv(t), \end{aligned}$$

where $v(t)$ represents the measurement noise.

In the standard linear quadratic gaussian control theory, one can obtain the best estimation of the state using a Kalman filter and then design the controller based on the estimated state. Here we take a similar approach. It turns out that, as expected, our ability to steer the distribution of the state vector, as compared to what is possible by noise free state feedback, is only limited by an extra inequality of admissible state-covariances to exceed the error covariance of a corresponding Kalman filter; see [95] for additional details.

6.6 Examples

We present three examples to illustrate the results. All the examples are related to inertial particle systems on the phase space. In the first example, we study the effect of reducing stochastic disturbance (see Section 6.1 and 6.2). In the second example, we consider the case when the control input and disturbance enter the system through different channels (see Section 6.3). In the third example, we consider similar dynamics to the first example but add an extra penalty on the state (see Section 6.4).

6.6.1 Schrödinger bridges to OMT

Consider a large collection of inertial particles

$$\begin{aligned} dx(t) &= v(t)dt \\ dv(t) &= u(t)dt + \sqrt{\epsilon}dw(t). \end{aligned}$$

moving in a 1-dimension configuration space (i.e., for each particle, the position $x(t) \in \mathbb{R}$). The position x and velocity v of particles are assumed to be jointly normally distributed in the 2-dimensional phase space ($(x, v) \in \mathbb{R}^2$) with mean and variance

$$m_0 = \begin{bmatrix} -5 \\ -5 \end{bmatrix}, \text{ and } \Sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

at $t = 0$. We seek to steer the particles to a new joint Gaussian distribution with mean and variance

$$m_1 = \begin{bmatrix} 5 \\ 5 \end{bmatrix}, \text{ and } \Sigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

at $t = 1$. The problem to steer the particles provides also a natural way to interpolate these two end-point marginals by providing a flow of one-time marginals at intermediary points $t \in [0, 1]$. In particular, we are interested in the behavior of trajectories when the random forcing is negligible compared to the "deterministic" drift.

Figure 6.1a depicts the flow of the one-time marginals of the Schrödinger bridge with $\epsilon = 9$. The transparent tube represents the 3σ region

$$(\xi(t)' - m_t')\Sigma_t^{-1}(\xi(t) - m_t) \leq 9, \quad \xi(t) = \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}$$

and the curves with different color stand for typical sample paths of the Schrödinger bridge. Similarly, Figures 6.1b and 6.1c depict the corresponding flows for $\epsilon = 4$ and $\epsilon = 0.01$, respectively. The interpolating flow in the absence of stochastic disturbance ($\epsilon = 0$), i.e., for the optimal transport with prior, is depicted in Figure 6.1d; the sample paths are now smooth as compared to the corresponding sample paths with stochastic disturbance. As $\epsilon \searrow 0$, the paths converge to those corresponding to optimal transport and $\epsilon = 0$.

6.6.2 Different input and disturbance channels

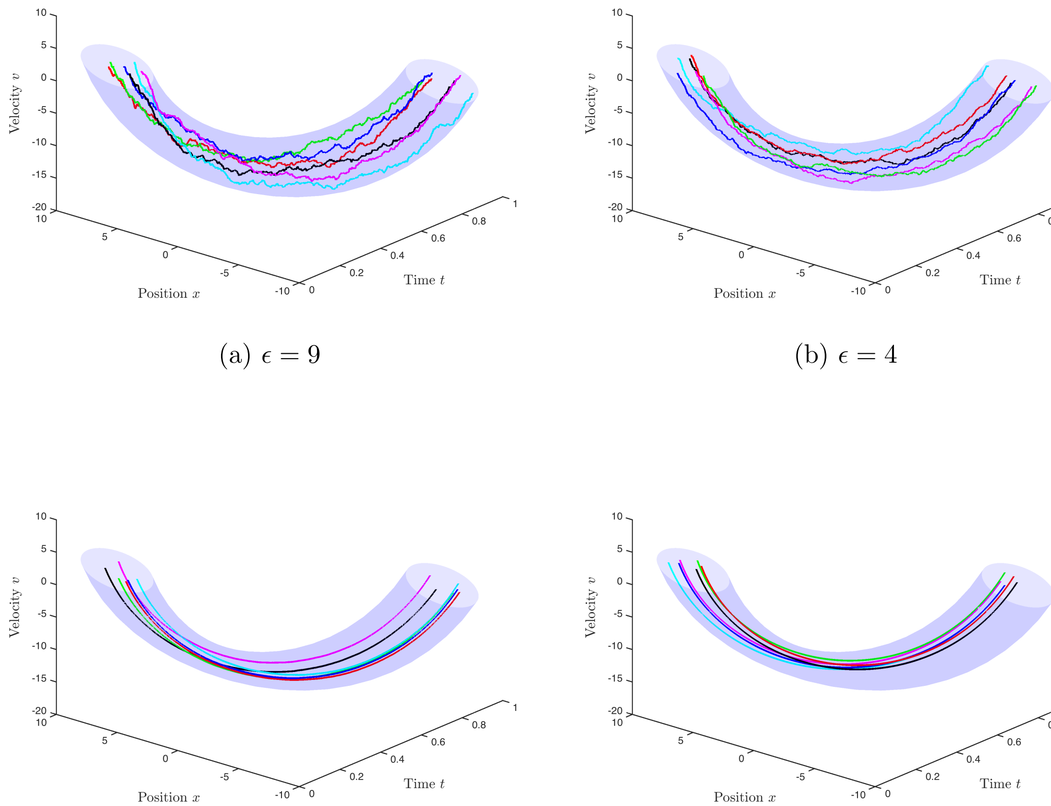
Consider particles that are modeled by

$$\begin{aligned} dx(t) &= v(t)dt + dw(t) \\ dv(t) &= u(t)dt. \end{aligned}$$

Here, $u(t)$ is the control input (forcing) at our disposal, $x(t)$ represents position, $v(t)$ is the velocity (integral of acceleration due to input forcing), while $w(t)$ represents random displacement due to impulsive accelerations. The purpose of the example is to highlight a case where the control is handicapped compared to the effect of noise. Indeed, the displacement $w(t)$ is directly affecting the position while the control effort needs to be integrated before it impacts the position of the particles.

Another interesting aspect of this example is that $\mathcal{R}(B) \not\subseteq \mathcal{R}(B_1)$ since $B = [0,\ 1]'$ while $B_1 = [1,\ 0]'$. If we choose

$$\Sigma_1 = \begin{bmatrix} 1 & -1/2 \\ -1/2 & 1/2 \end{bmatrix} \tag{6.79}$$



(a) $\epsilon = 9$

(b) $\epsilon = 4$

(c) $\epsilon = 0.01$

(d) $\epsilon = 0$

Figure 6.1: Interpolation with different ϵ

as a candidate stationary state-covariance, it can be seen that (6.66a) has a unique solution X giving rise to $K = [1,\ 1]$ and a stable feedback since $A - BK$ is Hurwitz.

We wish to steer the spread of the particles from an initial Gaussian distribution with $\Sigma_0 = 2I$ at $t = 0$ to the terminal marginal Σ_1 at $t = 1$, and from there on, since Σ_1 is an admissible stationary state-covariance, to maintain with constant state-feedback control.

Figure 6.2 displays typical sample paths in phase space as functions of time. These are the result of using the optimal feedback strategy derived following (6.71c) over the time interval $[0,\ 1]$. The optimal feedback gains $K(t) = [k_1(t),\ k_2(t)]$ are shown in Figure 6.3 as functions of time over the interval $[0, 1]$, where the state-covariance transitions to the chosen admissible steady-state value Σ_1. The corresponding cost is $J(u) = 9.38$. Past the point in time $t = 1$, the state-covariance of the closed-loop sys-

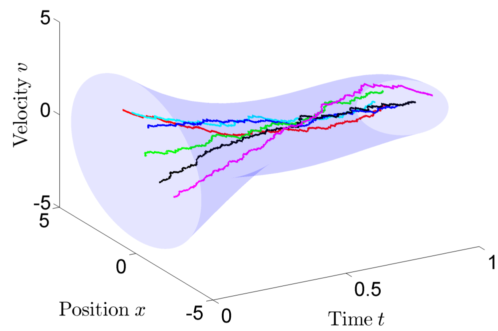


Figure 6.2: Finite-interval steering in phase space

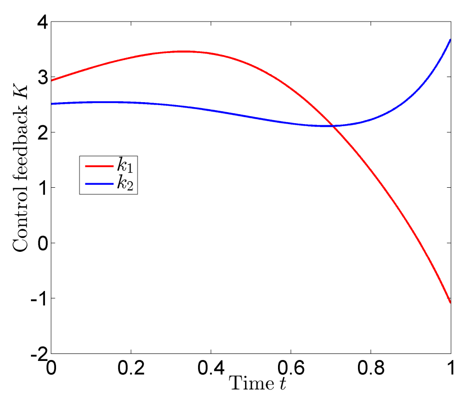


Figure 6.3: Optimal feedback gains in finite-interval steering

tem is maintained at this stationary value in (6.79). Figure 6.4 shows representative sample paths in phase space under the now constant state feedback gain $K = [1,\ 1]$ over the time window $[1,\ 5]$. Finally, Figure 6.5 displays the corresponding control action for each trajectory over the complete time interval $[0,\ 5]$, which consists of the "transient" interval $[0,\ 1]$ to the target (stationary) distribution and the "stationary" interval $[1,\ 5]$.

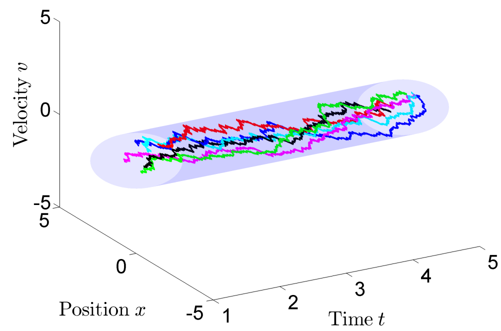


Figure 6.4: Steady state trajectories in phase space

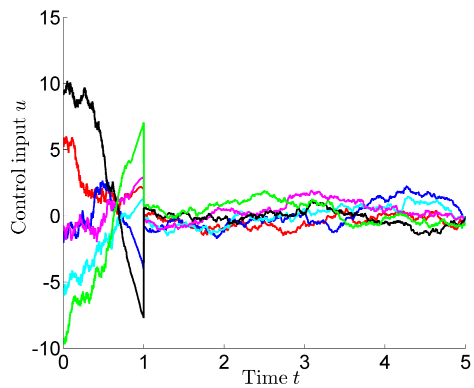


Figure 6.5: Control inputs

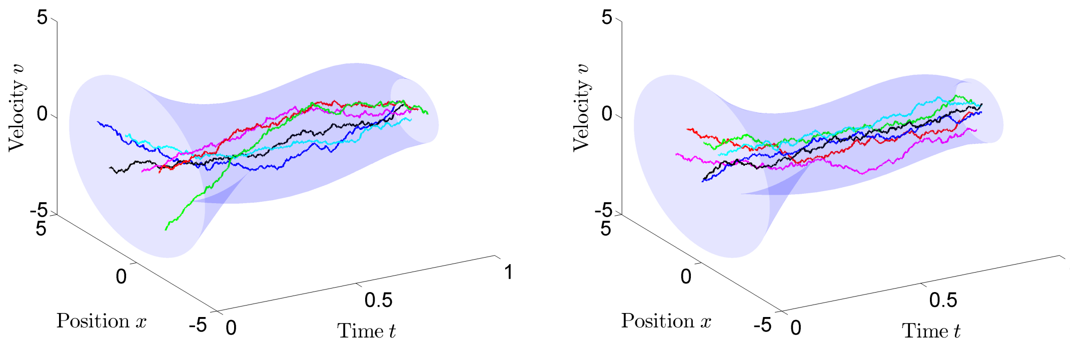
6.6.3 State penalty

We consider again inertial particles modeled by

$$\begin{aligned} dx(t) &= v(t)dt \\ dv(t) &= u(t)dt + dw(t). \end{aligned}$$

We wish to steer the spread of the particles from an initial Gaussian distribution with $\Sigma_0 = 2I$ at $t = 0$ to the terminal marginal $\Sigma_1 = 1/4I$ in a optimal way such that the cost function (6.76) is minimized.

Figure 6.6a displays typical sample paths $\{(x(t), v(t)) \mid t \in [0, 1]\}$ in phase space, as a function of time, that are attained using the optimal feedback strategy derived following (6.78c) and $S = I$. The feedback gains $K(t) = [k_1(t), k_2(t)]$ are shown in Figure 6.7 as a function of time. Figure 6.8 shows the corresponding control action for each trajectory.



(a) $S = I$ (b) $S = 10I$

Figure 6.6: State trajectories

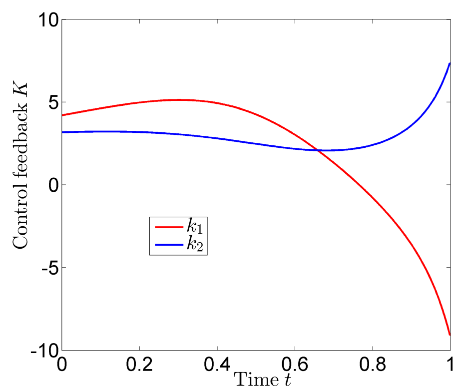


Figure 6.7: Feedback gains

For comparison, Figure 6.6b displays typical sample paths when optimal control is used and $S = 10I$. As expected, $\Sigma(\cdot)$ shrinks faster as we increase the state penalty S which is consistent with the reference evolution loosing probability mass at a higher rate at places where the state penalty $U(x)$ is large.

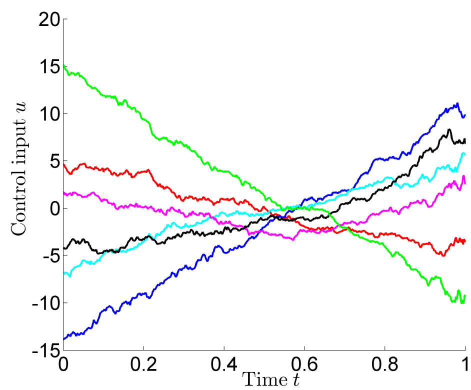


Figure 6.8: Control inputs

Chapter 7

Robust transport over networks

In this chapter, we generalize our framework of density steering from Euclidean spaces to graphs. On a graph, the mass on the nodes may represent products or other resources, and the goal is to reallocate their distributions. Our method is based on the Schrödinger bridge problem (SBP) with the Ruelle-Bowen (RB) measure as the prior and it leads to a robust way of transporting mass on networks.

7.1 Markov chain

A Markov chain is a stochastic process on a discrete space with the property that the next state of the process depends only on the current state. We denote by $\mathcal{X}$ the underlying discrete space. We are interested in the case when the Markov chain has finite states, that is, $\mathcal{X}$ has a finite number of elements. Without loss of generality, we consider the finite state space

$$\mathcal{X} = \{1, \ldots, n\}. \tag{7.1}$$

A Markov chain is defined as a sequence of random variables $\{X_1, X_2, \ldots, X_t, \ldots\}$ taking values in $\mathcal{X}$, and the sequence fulfills the condition

$$\mathrm{Prob}(X_{t+1} = x \mid X_1 = x_1, X_2 = x_2, \ldots, X_t = x_t) = \mathrm{Prob}(X_{t+1} = x \mid X_t = x_t)$$

for all t. Let

$$m_{t,ij} = \mathrm{Prob}(X_{t+1} = j \mid X_t = i),$$

then the matrix $M_t = [m_{t,ij}]_{i,j=1}^n$ is a stochastic matrix, i.e., the sum of each row of M_t is 1. The stochastic matrix determines the propagation of the probability distribution of the associated Markov chain. Let $\mu_t \in \mathbb{R}^n$ be the probability distribution of X_t, then it satisfies

$$\mu_{t+1} = M_t' \mu_t.$$

One can also view a Markov chain as a random walk on a graph with nodes in $\mathcal{X}$. In this case, the stochastic matrix M_t describes the probability distribution of jumps of the random walker at the t-th step.

When M_t is independent of t, we call the Markov chain time-homogenous. Under some proper assumptions like the Markov chain is irreducible and aperiodic [96], the distribution μ_t converges to the invariant measure, which is the left eigenvector μ of M associated with the eigenvalue 1, i.e.,

$$\mu = M' \mu.$$

Sometimes there is probability for the random walker to disappear. This pertains to the Markov chain with "killing" or "creation". The evolution matrix M in this case might not be a stochastic matrix; its row sum might be different to 1.

7.2 Schrödinger bridges on graphs

We discuss a generalization of the discrete Schrödinger bridge problem (SBP) considered in [86, 91], where the "prior" is not necessarily a probability law and mass is not necessarily preserved during the evolution. Consider a finite state space $\mathcal{X}$ over a time-indexing set

$$\mathcal{T} = \{0, 1, \ldots, N\}.$$

Our goal is to determine a probability distribution P on the space of paths $\mathcal{X}^{N+1}$ in such a way that it matches the specified marginal distributions $\nu_0(\cdot)$ and $\nu_N(\cdot)$ and the resulting random evolution is the closest to the "prior" in a suitable sense.

The prior law is induced by the time-homogenous Markovian evolution

$$\mu_{t+1}(x_{t+1}) = \sum_{x_t \in \mathcal{X}} \mu_t(x_t) m_{x_t x_{t+1}} \tag{7.2}$$

for nonnegative distributions $\mu_t(\cdot)$ over $\mathcal{X}$ with $t \in \mathcal{T}$, as is explained in what follows.

Throughout, we assume that $m_{ij} \geq 0$ for all indices $i, j \in \mathcal{X}$ and for simplicity, for the most part, that the matrix

$$M = [m_{ij}]_{i,j=1}^{n}$$

does not depend on t. In this case, we will often assume that all entries of M^N are positive. The rows of the transition matrix M do not necessarily sum up to one, meaning that the "total transported mass" is not necessarily preserved. This is the case, in particular, for a Markov chain with "creation" and "killing". It also occurs when M simply encodes the topological structure of a directed network with m_{ij} being zero or one, depending on whether a certain transition is allowed. The evolution (7.2), together with the measure $\mu_0(\cdot)$, which we assume positive on $\mathcal{X}$, i.e.,

$$\mu_0(x) > 0 \text{ for all } x \in \mathcal{X}, \tag{7.3}$$

induces a measure $\mathfrak{M}$ on $\mathcal{X}^{N+1}$ as follows. It assigns to a path $x = (x_0, x_1, \ldots, x_N) \in \mathcal{X}^{N+1}$ the value

$$\mathfrak{M}(x_0, x_1, \ldots, x_N) = \mu_0(x_0) m_{x_0 x_1} \cdots m_{x_{N-1} x_N}, \tag{7.4}$$

and gives rise to a flow of the one-time marginal

$$\mu_t(x_t) = \sum_{x_{\ell \neq t}} \mathfrak{M}(x_0, x_1, \ldots, x_N), \quad t \in \mathcal{T}.$$

The "prior" distribution $\mathfrak{M}$ on the space of paths may be at odds with a pair of specified marginals ν_0 and ν_N in that one or possibly both,

$$\mu_0(x_0) \neq \nu_0(x_0), \quad \mu_N(x_N) \neq \nu_N(x_N).$$

We denote by $\mathcal{P}(\nu_0, \nu_N)$ the family of probability distributions on $\mathcal{X}^{N+1}$ having the prescribed marginals. We seek a distribution in this set which is the closest to the prior $\mathfrak{M}$ in a suitable entropic sense.

Recall that, for P and Q probability distributions, the *Kullback-Leibler distance* (divergence, relative entropy) $H(P, Q)$ is nonnegative and equal to zero if and only if $P = Q$ [97]. This can be extended to positive measures that are not probability distributions.

Naturally, while the value of $H(P, Q)$ may turn out negative due to the mismatch between scalings, the relative entropy is always jointly convex. We view the prior $\mathfrak{M}$ (specified by M and μ_0) in a similar manner, and consider the SBP:

Problem 27 *Determine*

$$\mathfrak{M}[\nu_0, \nu_N] = \operatorname{argmin}\{H(P, \mathfrak{M}) \mid P \in \mathcal{P}(\nu_0, \nu_N)\}. \tag{7.5}$$

In terms of notation, we denote by $\mathfrak{M}[\nu_0, \nu_N]$ the solution to Problem 27 with prior measure $\mathfrak{M}$ and marginals ν_0, ν_N. Provided all entries of M^N are positive, the problem has a solution, which is unique due to the strict convexity of the relative entropy function. This is stated next.

Theorem 28 *Assume that M^N has all positive elements. Then there exist nonnegative functions $\varphi(\cdot)$ and $\hat{\varphi}(\cdot)$ on $[0, N] \times \mathcal{X}$ satisfying, for $t \in [0, N-1]$, the system*

$$\begin{aligned} \varphi(t, i) &= \sum_j m_{ij} \varphi(t+1, j), && (7.6a) \\ \hat{\varphi}(t+1, j) &= \sum_i m_{ij} \hat{\varphi}(t, i) && (7.6b) \end{aligned}$$

with the boundary conditions

$$\begin{aligned} \varphi(0, x_0)\hat{\varphi}(0, x_0) &= \nu_0(x_0) && (7.6c) \\ \varphi(N, x_N)\hat{\varphi}(N, x_N) &= \nu_N(x_N), && (7.6d) \end{aligned}$$

for all $x_0, x_N \in \mathcal{X}$. Moreover, the solution $\mathfrak{M}[\nu_0, \nu_N]$ to Problem 27 is unique and obtained by

$$\mathfrak{M}[\nu_0, \nu_N](x_0, \ldots, x_N) = \nu_0(x_0)\pi_{x_0 x_1}(0) \cdots \pi_{x_{N-1} x_N}(N-1),$$

where[1]

$$\pi_{ij}(t) := m_{ij} \frac{\varphi(t+1, j)}{\varphi(t, i)}. \tag{7.7}$$

[1]Here we use the convention that $0/0 = 0$.

Equation (7.7) *specifies* one-step transition probabilities *that are well defined.*

The factors φ and $\hat{\varphi}$ are unique up to multiplication of φ by a positive constant and division of $\hat{\varphi}$ by the same constant. The statement of the theorem is analogous to results for the classical Schrödinger system (7.6) of diffusions that have been established by Fortet, Beurling, Jamison and Föllmer [38–41]. The requirement for M^N to have positive entries can be slightly relaxed and replaced by a suitable condition that guarantees existence of solution for the particular ν_0 and ν_N. The case when M is time varying can also be readily established along the lines of [91, Theorem 4.1] and [86, Theorem 2].

Finally, to simplify the notation, let $\varphi(t)$ and $\hat{\varphi}(t)$ denote the column vectors with components $\varphi(t,i)$ and $\hat{\varphi}(t,i)$, respectively, with $i \in \mathcal{X}$. In matrix form, (7.6a), (7.6b) and (7.7) read

$$\varphi(t) = M\varphi(t+1), \quad \hat{\varphi}(t+1) = M'\hat{\varphi}(t), \tag{7.8a}$$

and

$$\Pi(t) = [\pi_{ij}(t)] = \operatorname{diag}(\varphi(t))^{-1} M \operatorname{diag}(\varphi(t+1)). \tag{7.8b}$$

7.2.1 Time-homogeneous bridges

We now consider the SBP when the marginals are idential, namely, $\nu_0 = \nu_N = \nu$. In particular, we are interested in the case when the solution of the SBP corresponds to a time-homogeneous Markov evolution. We begin by stating the celebrated Perron-Frobenous theorem (see [98]), which will be needed later.

Theorem 29 (Perron-Frobenius) *Let $A = (a_{ij})$ be an $n \times n$ matrix with nonnegative elements. Suppose there exists N such that A^N has only positive elements. Let λ_A be the spectral radius of A, then*

i) $\lambda_A > 0$ is an eigenvalue of A;

ii) λ_A is a simple eigenvalue;

iii) there exists an eigenvector v corresponding to λ_A with strictly positive entries;

iv) v is the only non-negative eigenvector of A;

v) let $B = [b_{ij}]$ be a $n \times n$ matrix with nonnegative elements. If $a_{ij} \leq b_{ij}$, $\forall i, j \leq n$ and $A \neq B$, then $\lambda_A < \lambda_B$.

Since M^N has only positive elements by assumption, we conclude, by the above Perron-Frobenous theorem, that M has a unique positive eigenvalue λ_M and it is equal to the spectral radius. Let ϕ and $\hat{\phi}$ be the corresponding right and left eigenvectors, then both of them have only positive components. If we normalize ϕ and $\hat{\phi}$ so that

$$\sum_{x \in \mathcal{X}} \phi(x)\hat{\phi}(x) = 1,$$

then

$$\bar{\nu}(x) = \phi(x)\hat{\phi}(x) \tag{7.9}$$

is a probability vector. Furthermore, $\bar{\nu}$ has the following property.

Theorem 30 *Let M be a nonnegative matrix such that M^N has only positive elements, and $\mathfrak{M}$ the measure on $\mathcal{X}^{N+1}$ given by (7.2) with μ_0 satisfying (7.3). Then the solution to the SBP*

$$\mathfrak{M}[\bar{\nu}, \bar{\nu}] = \operatorname{argmin}\{H(P, \mathfrak{M}) | P \in \mathcal{P}(\bar{\nu}, \bar{\nu})\}, \tag{7.10}$$

where $\bar{\nu}$ is as in (7.9), has the time-invariant transition matrix

$$\bar{\Pi} = \lambda_M^{-1} \operatorname{diag}(\phi)^{-1} M \operatorname{diag}(\phi) \tag{7.11}$$

and invariant measure $\bar{\nu}$.

Proof Since ϕ and $\hat{\phi}$ are right and left eigenvectors of M associated with the eigenvalue λ_M, the nonnegative functions φ and $\hat{\varphi}$ defined by

$$\varphi(t, x) = \lambda_M^t \phi(x), \quad \hat{\varphi}(t, x) = \lambda_M^{-t} \hat{\phi}(x)$$

satisfy the Schrödinger system (7.6). By Theorem 28, the solution $\mathfrak{M}[\bar{\nu}, \bar{\nu}]$ of the SBP (7.10) has the transition matrix (see (7.7))

$$\begin{aligned} \bar{\Pi} &= \operatorname{diag}(\varphi(0))^{-1} M \operatorname{diag}(\varphi(1)) \\ &= \lambda_M^{-1} \operatorname{diag}(\phi)^{-1} M \operatorname{diag}(\phi), \end{aligned}$$

which is exactly (7.11). Moreover, since

$$\bar{\Pi}'\bar{\nu} = \lambda_M^{-1}\operatorname{diag}(\phi)M'\hat{\phi} = \bar{\nu},$$

it follows that $\bar{\nu}$ is the corresponding invariant measure.

We refer to this special Schrödinger bridge as the time-homogenous bridge associated with M. As we shall see in the next section, when M is the adjacency matrix of a strongly connected, directed graph, the associated time-homogenous bridge turns out to be the Ruelle-Bowen measure $\mathfrak{M}_{\mathrm{RB}}$ [12, Section III]. This probability measure has a number of beautiful properties. In particular, it gives the same probability to paths of the same length between any two given nodes.

7.3 Ruelle Bowen random walks

In this section we explain the Ruelle-Bowen random walk [11] and some of its properties. We follow closely Delvenne and Libert [12]. The RB random walk amounts to a Markovian evolution on a directed graph that assigns equal probabilities to all paths of equal length between any two nodes. The motivation of [12] was to assign a natural invariant probability to nodes based on relations that are encoded by a graph topology, and thereby determine a *centrality measure*, akin to Google Page ranking, yet more robust and discriminating [12]. Our motivation is quite different. The RB random walk provides a uniform distribution on paths. Therefore, it represents a natural distribution to serve as prior in the SBP in order to achieve a maximum spreading of the mass transported over the available paths. In this section, besides reviewing basics on the RB random walk, we show that the RB distribution is itself a solution to a SBP.

We consider a strongly connected aperiodic, directed graph

$$\mathcal{G} = (\mathcal{X}, \mathcal{E})$$

with nodes in $\mathcal{X}$. The idea in Google Page rank is based on a random walk where a jump takes place from one node to any of its neighbors with equal probability. The alternative proposed in [12] is an *entropy ranking*, based on the stationary distribution of the RB random walk [11, 99]. The transition mechanism is such that it induces a uniform distribution on *paths* of equal length joining any two nodes. This distribution is characterized as the one maximizing the *entropy rate* [97] for the random walker. Let us briefly recall the relevant concept. The Shannon entropy for paths of length t is at most

$$\log |\{\text{paths of length } t\}|.$$

Here $|\{\cdot\}|$ denotes the cardinality of a set. Hence, the entropy rate is bounded by the *topological entropy rate*

$$H_{\mathcal{G}} = \lim_{t\to\infty} \sup(\log |\{\text{paths of length } t\}|/t).$$

Notice that $H_{\mathcal{G}}$ only depends on the graph $\mathcal{G}$ and not on the probability distribution on paths. More specifically, if A denotes the adjacency matrix of the graph, then the number of paths of length t is the sum of all the entries of A^t. Thus, it follows that $H_{\mathcal{G}}$ is the logarithm of the spectral radius of A, namely,

$$H_{\mathcal{G}} = \log(\lambda_A). \tag{7.12}$$

We next construct the Ruelle-Bowen random walk. Since A is the adjacency matrix of a strongly connected aperiodic graph, it satisfies that A^N has only positive elements for some $N > 0$. By Perron-Frobenius Theorem, the spectral radius λ_A is an eigenvalue of A, and the associated left and right eigenvectors[2] u and v, i.e.,

$$A'u = \lambda_A u, \quad Av = \lambda_A v \tag{7.13}$$

have only positive components. We further normalize u and v so that

$$\langle u, v\rangle := \sum_{i\in\mathcal{X}} u_i v_i = 1.$$

As in Section 7.2.1, it is readily seen that their componentwise multiplication

$$\mu_{\mathrm{RB}}(i) = u_i v_i \tag{7.14}$$

[2] We are now following the notation in [12] for ease of comparison. Hence we use u and v rather than $\hat{\phi}$ and ϕ.

defines a probability distribution that is invariant under the transition matrix

$$R = [r_{ij}], \quad r_{ij} = \frac{v_j}{\lambda_A v_i} a_{ij}, \tag{7.15}$$

namely,

$$R' \mu_{\mathrm{RB}} = \mu_{\mathrm{RB}}. \tag{7.16}$$

The transition matrix R in (7.15) together with the stationary measure μ_{RB} in (7.14), define the *Ruelle-Bowen path measure*

$$\mathfrak{M}_{\mathrm{RB}}(x_0, x_1, \ldots, x_N) := \mu_{\mathrm{RB}}(x_0) r_{x_0 x_1} \cdots r_{x_{N-1} x_N}. \tag{7.17}$$

Proposition 11 *The RB measure* $\mathfrak{M}_{\mathrm{RB}}$ (7.17) *assigns probability* $\lambda_A^{-t} u_i v_j$ *to any path of length* t *from node* i *to node* j.

Proof Starting from the stationary distribution (7.14), and in view of (7.15), the probability of a path ij is

$$u_i v_i \left(\frac{1}{\lambda_A} v_i^{-1} v_j \right) = \frac{1}{\lambda_A} u_i v_j,$$

assuming that node j is accessible from node i in one step. Likewise, the probability of the path ijk is

$$u_i v_i \left(\frac{1}{\lambda_A} v_i^{-1} v_j \right) \left(\frac{1}{\lambda_A} v_j^{-1} v_k \right) = \frac{1}{\lambda_A^2} u_i v_k$$

independent of the intermediate state j, and so on. Thus, the claim follows.

Thus, the Ruelle-Bowen measure $\mathfrak{M}_{\mathrm{RB}}$ has the striking property that it induces a uniform probability measure on paths of equal length between any two given nodes. We quote from [12] "Since the number of paths of length t is of the order of λ_A^t (up to a factor) the distribution on paths of fixed length is uniform up to a factor (which does not depend on t). Hence the Shannon entropy of paths of length t grows as $t \log \lambda_A$, up to an additive constant. The entropy rate of this distribution is thus $\log \lambda_A$ which is optimal" by the expression for $H_{\mathcal{G}}$ in (7.12).

The construction of the RB measure is obviously a special case of the measure $\bar{\nu}$ in (7.9) in Section 7.2.1 when M is the adjacency matrix A of a graph. Therefore, the RB measure is the solution of the particular SBP when the "prior" transition mechanism is given by the adjacency matrix! This observation is apparently new and beautifully links the topological entropy rate to a maximum entropy problem on path space. This is summarized as follows.

Proposition 12 *Let A be the adjacency matrix of a strongly connected graph aperiodic $\mathcal{G}$, $\mathfrak{M}$ the nonnegative measure on $\mathcal{X}^{N+1}$ given by* (7.2) *with $M = A$ and μ_0 satisfying* (7.3)*. Then, the Ruelle-Bowen measure $\mathfrak{M}_{\mathrm{RB}}$* (7.17) *solves the SBP* (7.5) *with marginals $\nu_0 = \nu_N = \mu_{\mathrm{RB}}$.*

7.4 Robust transport on graphs

Once again we consider a strongly connected aperiodic, directed graph $\mathcal{G}$ with n vertices and seek to transport a unit mass from initial distribution ν_0 to terminal distribution ν_N in at most N steps. We identify node 1 as a *source* and node n as a *sink*. The task is formalized by setting an initial marginal distribution $\nu_0(x) = \delta_{1x}(x)$ the Kronecker's delta. Similarly, the final distribution is $\nu_N(x) = \delta_{nx}(x)$. We seek a transportation plan which is *robust* and avoids *congestion* as much as the topology of the graph permits. This latter feature of the transportation plan will be achieved in this section indirectly, without explicitly bringing into the picture the capacity of each edge. With these two key specifications in mind, we intend to control the flux so that the initial mass *spreads as much as possible* on the feasible paths joining vertices 1 and n in N steps before reconvening at time N in vertex n. We shall achieve this by constructing a suitable, possibly time-varying, Markovian transition mechanism. As we want to allow for the possibility that all or part of the mass reaches node n at some time less than N, we always include a loop in node n so that our adjacency matrix A always has $a_{nn} = 1$. As explained in Section 7.3 that $\mathfrak{M}_{\mathrm{RB}}$ gives equal probability to paths joining two specific vertices, it is natural to use it as a prior in the SBP with marginals δ_{1x}, δ_{nx} so as to achieve the spread of the probability mass on the feasible paths joining the source with the sink. Thus, we consider the following.

Problem 31 *Determine*

$$\mathfrak{M}_{\mathrm{RB}}[\delta_{1x}, \delta_{nx}] = \operatorname{argmin}\{H(P, \mathfrak{M}_{\mathrm{RB}}) | P \in \mathcal{P}(\delta_{1x}, \delta_{nx})\}.$$

By Theorem 28, the optimal, time varying transition matrix $\Pi(t)$ of the above problem is given, recalling the notation in (7.8), by

$$\Pi(t) = \mathrm{diag}(\varphi(t))^{-1} R\, \mathrm{diag}(\varphi(t+1)), \tag{7.18}$$

where

$$\varphi(t) = R\varphi(t+1),\ \hat{\varphi}(t+1) = R^T\hat{\varphi}(t),$$

with the boundary conditions

$$\varphi(0,x)\hat{\varphi}(0,x) = \delta_{1x}(x),\ \varphi(N,x)\hat{\varphi}(N,x) = \delta_{nx}(x) \tag{7.19}$$

for all $x \in \mathcal{X}$. In view of (7.15), if we define

$$\varphi_v(t) := \lambda_A^{-t}\, \mathrm{diag}(v)\varphi(t), \quad \hat{\varphi}_v(t) := \lambda_A^t\, \mathrm{diag}(v)^{-1}\hat{\varphi}(t),$$

then we have

$$\varphi_v(t) = A\varphi_v(t+1),\ \hat{\varphi}_v(t+1) = A'\hat{\varphi}_v(t),\ t = 0, \ldots, N-1.$$

Moreover,

$$\varphi_v(t,x)\hat{\varphi}_v(t,x) = \varphi(t,x)\hat{\varphi}(t,x),\ t = 0, \ldots, N-1,\ x \in \mathcal{X}.$$

Here, again, A is the adjacency matrix of $\mathcal{G}$ and v is a right eigenvector corresponding to the spectral radius λ_A.

The above analysis provides another interesting way to express $\mathfrak{M}_{\mathrm{RB}}[\delta_{1x}, \delta_{nx}]$; it also solves the SBP with the same marginals δ_{1x} and δ_{nx} but with a different prior transition matrix A, the adjacency matrix. Thus, we can replace the two-step procedure by a single bridge problem. This is summarized in the following proposition.

Proposition 13 *Let A be the adjacency matrix of a strongly connected aperiodic graph $\mathcal{G}$, $\mathfrak{M}$ the nonnegative measure on $\mathcal{X}^{N+1}$ given by* (7.2) *with $M = A$ and μ_0 satisfying* (7.3)*. Then, the solution $\mathfrak{M}_{\mathrm{RB}}[\delta_{1x}, \delta_{nx}]$ of Problem 31 also solves the*

$$\min\{H(P, \mathfrak{M}) | P \in \mathcal{P}(\delta_{1x}, \delta_{nx})\}. \tag{7.20}$$

The iterative algorithm of [86, Section III] can now be based on (7.20) to efficiently compute the transition matrix of the optimal robust transport plan $\mathfrak{M}_{\rm RB}[\delta_{1x}, \delta_{nx}]$, or equivalently, $\mathfrak{M}[\delta_{1x}, \delta_{nx}]$.

Remark 32 *Observe that if A^N has also zero elements, the robust transport described in this section may still be feasible provided there is at least one path of length N joining node 1 with node n, i.e., $(A^N)_{1n} > 0$.*

As we discussed in the beginning of this section, the intuition to use $\mathfrak{M}_{\rm RB}$ as a prior is to achieve a uniform spreading of the probability on all the feasible paths connecting the source and the sink. It turns out that this is indeed the case; the solution $\mathfrak{M}_{\rm RB}[\delta_{1x}, \delta_{nx}]$ of Problem 31 assigns equal probability to all the feasible paths of lengths N joining the source 1 with the sink n. To see this, by (7.18), the probability of the optimal transport plan $\mathfrak{M}_{\rm RB}[\delta_{1x}, \delta_{nx}]$ assigns on path $x = (x_0,\, x_1, \ldots, x_N)$ is

$$\begin{aligned}\mathfrak{M}_{\rm RB}[\delta_{1x}, \delta_{nx}](x) \;&=\; \delta_{1x}(x_0)\prod_{t=1}^{N-1} r_{x_t x_{t+1}} \frac{\varphi(t+1, x_{t+1})}{\varphi(t, x_t)} \\ &=\; \delta_{1x}(x_0)\frac{\varphi_v(N, x_N)}{\varphi_v(0, x_0)} \prod_{t=1}^{N-1} a_{x_t x_{t+1}}.\end{aligned}$$

Indeed, $\prod_{t=1}^{N-1} a_{x_t x_{t+1}} = 1$ for a feasible path and 0 otherwise. Moreover, $\delta_{1x}(x_0)\frac{\varphi_v(N,x_N)}{\varphi_v(0,x_0)}$ depends only on the boundary points x_0, x_N. We conclude that $\mathfrak{M}_{\rm RB}[\delta_{1x}, \delta_{nx}]$ assigns equal probability to all the feasible paths. Finally, there are $(A^N)_{1n}$ feasible paths of length N connecting nodes 1 and n. Thus we have established the following result.

Proposition 14 *$\mathfrak{M}_{\rm RB}[\delta_{1x}, \delta_{nx}]$ assigns probability $1/(A^N)_{1n}$ to each of the feasible paths of length N connecting node 1 with node n.*

7.5 OMT on graphs and its regularization

We consider the graph $\mathcal{G} = (\mathcal{X}, \mathcal{E})$ and seek to transport a given probability distribution ν_0 to a specified destination ν_N, in N steps, by suitably distributing the probability mass along paths $(x_0, \ldots, x_N)$. We consider that there is a nontrivial cost U_{ij} for traversing an edge (i,j) at any given time[3] and relate the classical Monge-Kantorovich problem to a stochastic regularization that can be cast as a SBP.

We first consider minimizing the transportation cost

$$\mathcal{U}(P) := \sum_{\{(x_0,\ldots,x_N)\}} \nu_0(x_0) \prod_{t=0}^{N-1} p_{x_t x_{t+1}}(t) \left(\sum_{t=0}^{N-1} U_{x_t x_{t+1}} \right).$$

over a transportation plan specified by a measure

$$P(x_0, \ldots, x_N) = \nu_0(x_0) \prod_{t=0}^{N-1} p_{x_t x_{t+1}}(t), \tag{7.21}$$

so as to be consistent with an end-point specified marginal

$$\nu_N(x_N) = \sum_{x_0,\ldots,x_{N-1}} P(x_0, \ldots, x_N). \tag{7.22}$$

Problem 33 *Determine transition probabilities $p_{x_t x_{t+1}}$ to minimize $\mathcal{U}(P)$ subject to* (7.22)*.*

This a discrete version of the Monge-Kantorovich Optimal Mass Transport (OMT) problem. One approach (e.g., see [100]) is to first identify the least costly path(s) $(x_0, x_1^*, \ldots, x_{N-1}^*, x_N)$ from any starting node $x_0 \in \mathcal{X}$ to any ending node x_N, along with the corresponding end-point cost for a unit mass[4],

$$C_{x_0 x_N} = \min_{x_1^*,\ldots,x_{N-1}^*} \left(U_{x_0 x_1^*} + \ldots + U_{x_{N-1}^* x_N} \right).$$

This is a *combinatorial problem* but can also be cast as a linear program [101]. Having a solution to this first problem, the OMT problem can then be recast as the linear

[3]For simplicity, we assume that this cost does not vary with time.
[4]We assume a self loop for each node with zero cost, i.e., $U_{xx} = 0$ for each $x \in \mathcal{X}$.

program

$$\min_{q}\left\{\sum_{x_0,x_N} q_{x_0,x_N} C_{x_0 x_N} \mid q_{x_0,x_N} \geq 0, \sum_{x_0} q_{x_0,x_N} = \nu_N(x_N), \sum_{x_N} q_{x_0,x_N} = \nu_0(x_0)\right\}. \tag{7.23}$$

The solution to (7.23) is the transport plan q_{x_0,x_N} which dictates the portion of mass that is to be sent from x_0 to x_N along the corresponding least costly path $(x_0, x_1^*, \ldots, x_{N-1}^*, x_N)$. Alternatively, the MK problem can be directly cast as a linear program in as many variables as there are edges [101].

An apparent shortcoming of the OMT formalism is the "rigidity" of the transportation to utilize only paths with minimal cost from starting to ending node. Herein, instead, we seek a mechanism that allows the usage of additional paths as a way to provide robustness and reduce congestion. To this end, as an alternative to the OMT problem, we propose to consider as our cost the Helmholtz free energy

$$\mathcal{F} := \mathcal{U} - T_{\text{eff}}\mathcal{S} \tag{7.24}$$

where T_{eff} is an effective "temperature" and $\mathcal{S}$ is the entropy of the transport plan

$$\mathcal{S} = -\sum_{x_0,\ldots,x_N} P(x_0,\ldots,x_N)\log\left(P(x_0,\ldots,x_N)\right)$$

The temperature T_{eff} serves the role of a regularization parameter, with higher temperature corresponding to a larger spread of the distribution along alternative paths. In the other direction, when $T_{\text{eff}} \to 0$, we recover the solution to the optimal mass transport (cf. [42, 49, 51, 87]).

The roots of our formalism can be traced to thermodynamics and the connection to SBP can be seen via the well-known correspondence between free energy and the relative entropy between corresponding distributions. In more detail, let $\mathcal{G} = (\mathcal{X}, \mathcal{E})$ by a graph with n nodes and adjacency matrix $A = [a_{ij}]_{i,j=1}^n$, and let P in (7.21) a probability law on N-step paths between nodes. Define $B = [b_{ij}]_{i,j=1}^n$ with

$$b_{ij} = a_{ij}\exp\left(-\frac{1}{T_{\text{eff}}}U_{ij}\right), \tag{7.25}$$

when $(i,j) \in \mathcal{E}$ and zero otherwise, and the nonnegative measure

$$\mathfrak{M}^U(x_0, x_1, \ldots, x_N) = b_{x_0 x_1} \cdots b_{x_{N-1} x_N}$$

By direct substitution, it can be seen that

$$\mathcal{F}(P) = T_{\text{eff}}\, H(P, \mathfrak{M}^U). \tag{7.26}$$

Thus, we are led to the following SBP.

Problem 34 *Determine*

$$\mathfrak{M}^U[\nu_0, \nu_N] = \operatorname{argmin}\{H(P, \mathfrak{M}^U) \mid P \in \mathcal{P}(\nu_0, \nu_N)\}.$$

It is evident that a judicious selection of U_{ij}'s (or time-varying $U_{ij}(t)$'s more generally) and T_{eff}, can influence the choice and spread of paths in effecting the transport task. This can be done in accordance with the capacity or cost of transporting along various edges. Moreover, the U_{ij}'s can be used to reduce congestion by imposing added cost on using critical nodes.

Besides the added flexibility in dictating preferences in the choice of paths, the regularized transport problem is precisely a SBP between specified marginals ν_0 and ν_N with respect to a prior specified by B. Thus, the tools and ideas that we saw earlier are applicable verbatim. The significance of the new formulation, as compared to the OMT problem, stems from the fact that the SBP is *computationally more attractive* as its solution can be computed iteratively as the fixed point of a map [51, 87].

In a similar manner as in Section 7.4, where we showed the equivalence of Problems 31 and that in (7.20), Problem 34 is equivalent to Problem 35 given below. To this end, we apply the Perron-Frobenius theorem. Now u^B and v^B are the left and right eigenvectors of the matrix B with positive entries. They correspond to the spectral radius λ_B of B, i.e.,

$$B' u^B = \lambda_B u^B, \quad B v^B = \lambda_B v^B.$$

They are normalized so that $\langle u^B, v^B \rangle = \sum_i u_i^B v_i^B = 1$. Then the vector μ_{RB}^U with entries $u_i^B v_i^B$ represents a probability distribution which is invariant under the transition matrix

$$R^U = [r^U_{ij}] = \lambda_B^{-1} \operatorname{diag}(v^B)^{-1} B \operatorname{diag}(v^B),$$

with entries

$$r^U_{ij} = \frac{1}{\lambda_B}\frac{v^B_j}{v^B_i} \exp\left(-\frac{1}{T_{\text{eff}}} U_{ij}\right).$$

Thus, the stationary measure μ^U_{RB} defines a *weighted Ruelle-Bowen path-space measure*

$$\mathfrak{M}^U_{\text{RB}}(x_0, x_1, \ldots, x_N) := \mu^U_{\text{RB}}(x_0) r^U_{x_0x_1} \cdots r^U_{x_{N-1}x_N}, \tag{7.27}$$

and we arrive at the problem below.

Problem 35 *Determine*

$$\mathfrak{M}^U_{\text{RB}}[\nu_0, \nu_N] = \operatorname{argmin}\{H(P, \mathfrak{M}^U_{\text{RB}}) \mid P \in \mathcal{P}(\nu_0, \nu_N)\}.$$

The solution $\mathfrak{M}^U_{\text{RB}}[\nu_0, \nu_N]$ to Problem 35 is equal to the solution $\mathfrak{M}^U[\nu_0, \nu_N]$ to Problem 34. They have the same pinned bridges [46, 73]as the weighted RB path space measure $\mathfrak{M}^U_{\text{RB}}$. The nature of these pinned bridges become evident from the probability assigned to individual paths which is given next.

Proposition 15 *The measure* $\mathfrak{M}^U_{\text{RB}}$ (7.27) *assigns probability*

$$\lambda_B^{-t} \exp\left(\frac{1}{T_{\text{eff}}} \sum_{\ell=1}^{t-1} U_{x_\ell x_{\ell+1}}\right) u^B_i v^B_j$$

to any path of length t from node i to j.

Clearly, the path space measure $\mathfrak{M}^U_{\text{RB}}$ is no longer uniform on paths of equal length, and the pinned bridges now are

$$\text{Prob}(x_1, \ldots, x_{n-1} | x_0, x_N) \sim \exp\left(\frac{1}{T_{\text{eff}}} \sum_{\ell=1}^{N-1} U_{x_\ell x_{\ell+1}}\right).$$

This represents a *Boltzmann distribution* on the path space (cf. [12, Section IV]) and

$\mathfrak{M}_{\text{RB}}^U$ minimizes *free energy* (often referred to as topological pressure) giving a value $\mathcal{F}(\mathfrak{M}_{\text{RB}}^U) = T_{\text{eff}} \log \lambda_B$.

Remark 36 *This framework can be applied to study transport problem over graph which is not strongly connected. One can add fake links to the graph to make it strongly connected. However, to make the flow on those fake edges negligible, each fake edge comes with a large cost U_{ij}.*

7.6 Examples

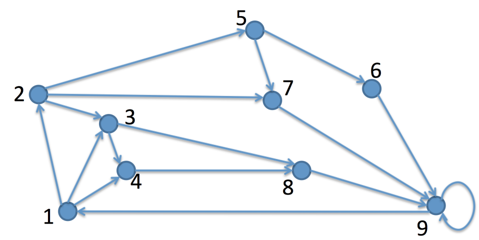


Figure 7.1: Network topology

We present a simple academic example to illustrate our method. Consider the graph in Figure 7.1 with the following adjacency matrix

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

We seek to transport a unit mass from node 1 to node 9 in $N = 3$ and 4 steps. We add a self loop at node 9, i.e., $a_{99} = 1$, to allow for transport paths with different step sizes.

The shortest path from node 1 to 9 is of length 3 and there are three such paths, which are $1-2-7-9$, $1-3-8-9$, and $1-4-8-9$. If we want to transport the mass with minimum number of steps, we may end up using one of these three paths. This is not so robust. On the other hand, if we apply the Schrödinger bridge framework with the RB measure $\mathfrak{M}_{\rm RB}$ as the prior, then we get a transport plan with equal probabilities using all these three paths. The evolution of mass distribution is given by

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/3 & 1/3 & 1/3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/3 & 2/3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

where the four rows of the matrix show the mass distribution at time step $t = 0, 1, 2, 3$ respectively. As we can see, the mass spreads out first and then goes to node 9. When we allow for more steps $N = 4$, the mass spreads even more before reassembling at node 9, as shown below

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4/7 & 2/7 & 1/7 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/7 & 1/7 & 2/7 & 0 & 1/7 & 2/7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/7 & 1/7 & 2/7 & 3/7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Now we change the graph by adding a cost on the edge (7, 9). In particular, we consider the weighted adjacency matrix

$$B = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

When $N = 3$ steps is allowed to transport a unit mass from node 1 to node 9, the evolution of mass distribution for the optimal transport plan is given by

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/5 & 2/5 & 2/5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/5 & 4/5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The mass travels through paths $1-2-7-9$, $1-3-8-9$ and $1-4-8-9$, but unlike the unweighted case, the transport plan doesn't take equal probability for these three paths Since we added a cost on the edge (7, 9), the probability that the mass takes this path becomes smaller. The plan does, however, assign equal probability to the two minimum cost paths $1-3-8-9$ and $1-4-8-9$ in agreement with Theorem 14. Similar phenomenon appears when we allow for more steps $N=4$, with mass evolution as shown below

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 1/3 & 1/6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/6 & 1/6 & 1/4 & 0 & 1/12 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/6 & 1/12 & 1/3 & 5/12 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Next we consider the case where the underlying graph is not strongly connected. In particular, we delete several links in Figure 7.1 to make it not strongly connected and consider the graph in Figure 7.2. Again we want to transport a unit mass from

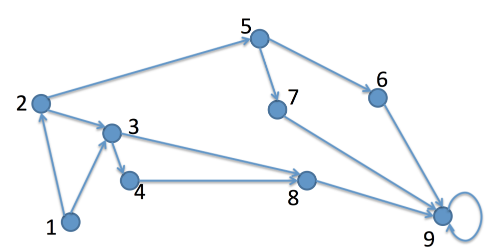


Figure 7.2: Network topology

node 1 to node 9. In order to do this, we add an artificial energy U_0 to each non existing link as discussed in Section 7.5. We display the results for $N=4$ steps.

When we take $U_0 = 2$, the evolution of mass is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.0415 & 0.4079 & 0.3416 & 0.0326 & 0.0462 & 0.0326 & 0.0326 & 0.0326 & 0.0326 \\ 0.0270 & 0.0349 & 0.1740 & 0.1477 & 0.2330 & 0.0603 & 0.0603 & 0.1614 & 0.1014 \\ 0.0116 & 0.0152 & 0.0199 & 0.0242 & 0.0163 & 0.1709 & 0.1709 & 0.2641 & 0.3069 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

We can see that there are quite a portion of mass traveling along non existing edges. If we increase the value to $U_0 = 8$, then the mass evolution becomes

$$\begin{bmatrix} 1.0000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.0001 & 0.5995 & 0.4000 & 0.0001 & 0.0001 & 0.0001 & 0.0001 & 0.0001 & 0.0001 \\ 0.0000 & 0.0000 & 0.2000 & 0.1999 & 0.3994 & 0.0002 & 0.0002 & 0.1999 & 0.0004 \\ 0.0000 & 0.0000 & 0.0000 & 0.0001 & 0.0000 & 0.1999 & 0.1999 & 0.3995 & 0.2007 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.0000 \end{bmatrix}.$$

The portion of mass traveling along non existing edges is negligible. Eventually, all the mass would be transported along feasible paths and the mass evolution becomes

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3/5 & 2/5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/5 & 1/5 & 2/5 & 0 & 0 & 1/5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/5 & 1/5 & 2/5 & 1/5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Chapter 8

Conclusion and discussion

The theme of this work has been the modeling and control of a collection of identical and indistinguishable dynamical systems. Equivalently, the setting may be seen as modeling and controlling a single system with stochastic uncertainty. There are two complementing viewpoints and goals. First, for the modeling problem, to determine the most likely path that the dynamical systems have followed between known empirical marginal distributions and, second, for the control problem, how to effect the transition of the collection between specified marginals with minimal cost. Evidently, the latter is an optimal control problem whereas the former relates to the latter via Girsanov's theory for the case of quadratic cost. The framework should be contrasted with standard optimal control aimed at driving the state of a dynamical system from one value to another with minimum cost. In the present setting, initial and final states are thought to be uncertain and only required to abide by specified marginal distributions.

We make contact with the theory of optimal mass transport (OMT) and the theory of the Schrödinger bridge problem (SBP). Historically, OMT (Monge 1781) and SBP (Schrödinger 1931/32) represent the first formulation of a deterministic and a stochastic density steering problem, respectively. In this work we have considered various natural generalizations of OMT and SBP which, in particular, include the problem to transport inertial particles, particles with killing, between end-point distributions in continuous space as well as particles transported between discrete end-point distributions on a graph. In the case of the SBP the cost can be conveniently expressed as an entropy functional, which encompasses the setting of discrete and continuous spaces within the same framework. There are important differences in the absence of stochastic excitation (OMT) as well as in when control and stochastic excitation influence the process along different directions in state space, and in those cases there is a dichotomy between the control-theoretic viewpoint and a probabilistic one.

The special case of linear dynamics and Gaussian marginals has been dealt in detail with the optimal control given in closed form based on Riccati equations. For the general case of arbitrary marginals it is shown that OMT and the SBP are closely related and that the solution to OMT can be obtained as a limiting case. For the general case, we focused on algorithmic aspects in addressing numerically such problems. In particular, it is shown that the geometry of positive cones and that of the Hilbert metric play an important role.

At the other end, over discrete spaces, stochastic transport relates to Markov chains and the classical Sinkhorn iteration. In particular, we have proposed a novel approach to design robust transportation plans on graphs. It relies on the SBP for measures on paths of the given network. Taking as prior measure the Ruelle-Bowen random walker, the solution naturally tends to spread the mass on all available routes joining the source and the sink. Hence, the resulting transport appears robust against links/nodes failure. This approach can be adapted to weighted graphs to effectively compromise between robustness and cost. Indeed, we exhibit a *robust* transportation plan which assigns *maximum probability* to *minimum cost paths* and therefore appears attractive when compared with OMT approaches. The transport plan is computed by solving a SBP, for which an efficient iterative algorithm is available. In addition, the concepts like entropy, free energy, and temperature we used in this chapter may open a door to study robustness of graphs. In fact, this may impact the current trends and interests of the controls community, leading to the development of useful models for power flow and the interaction of subsystems through communication and transportation networks.

There is a wide range of possible applications as well as extensions of the theory that lay ahead. The problem of combining estimation and control in this new, non-classical, setting is important for applications to sensing and control of micro-systems, especially when thermal noise is a limiting factor. In particular, we expect that the theory of the SBP will be especially important in the control of thermodynamic systems. Ideas from OMT and the SBP are increasing pertinent in control of networks and geometric notions that characterize connectivity of graphs can be based on the ease of solving relevant transport problems. This direction may provide new insights and directions in the theory of networks. Finally, the inverse problem of identifying flows that are consistent with empirical marginals, may be used to identify potential that is responsible for density flows and corresponding physical laws.

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Appendix A

Appendix

A.1 Proof of Proposition 4

The velocity field associated with $u(t, x) = B(t)'\nabla\psi(t, x)$ is

$$v(t, x) = A(t)x + B(t)B(t)'\nabla\psi(t, x),$$

which is well-defined almost everywhere (as it will be shown below that ψ is indeed differentiable almost everywhere). Since we already know from previous discussion that T_t in (5.28b) gives the trajectories associated with the optimal transportation plan, it suffices to show

$$v(t, \cdot) \circ T_t = dT_t/dt,$$

that is, $v(t, x)$ is the velocity field associated with the trajectories $(T_t)_{0\leq t\leq 1}$. We next prove $v(t, \cdot) \circ T_t = dT_t/dt$.

For $0 < t < 1$, formula (5.35) can be rewritten as

$$g(x) = \sup_y \left\{x'M(t, 0)^{-1}\Phi(t, 0)y - f(y)\right\},$$

with

$$\begin{aligned} g(x) &= \frac{1}{2}x'M(t, 0)^{-1}x - \psi(t, x) \\ f(y) &= \frac{1}{2}y'\Phi(t, 0)'M(t, 0)^{-1}\Phi(t, 0)y + \psi(0, y). \end{aligned}$$

The function

$$\begin{aligned} f(y) &= \frac{1}{2}y'\Phi(t,0)'M(t,0)^{-1}\Phi(t,0)y + \psi(0,y) \\ &= \frac{1}{2}y'\left[\Phi(t,0)'M(t,0)^{-1}\Phi(t,0) - \Phi_{10}'M_{10}^{-1}\Phi_{10}\right]y + \phi(M_{10}^{-1/2}\Phi_{10}y) \end{aligned}$$

is uniformly convex since ϕ is convex and the matrix

$$\begin{aligned} \Phi(t,0)'M(t,0)^{-1}\Phi(t,0) - \Phi_{10}'M_{10}^{-1}\Phi_{10} = &\left(\int_0^t \Phi(0,\tau)B(\tau)B(\tau)'\Phi(0,\tau)'d\tau\right)^{-1} \\ &- \left(\int_0^1 \Phi(0,\tau)B(\tau)B(\tau)'\Phi(0,\tau)'d\tau\right)^{-1} \end{aligned}$$

is positive definite. Hence, f, g, ψ are differentiable almost everywhere, and from a similar argument to the case of Legendre transform, we obtain

$$\nabla g \circ (M(t,0)\Phi(0,t)'\nabla f(x)) = M(t,0)^{-1}\Phi(t,0)x$$

for all $x \in \mathbb{R}^n$. It follows

$$\begin{aligned} &(M(t,0)^{-1} - \nabla\psi(t,\cdot)) \circ \left(M(t,0)\Phi(0,t)'\left[\Phi(t,0)'M(t,0)^{-1}\Phi(t,0)x + \nabla\psi(0,x)\right]\right) \\ =&M(t,0)^{-1}\Phi(t,0)x. \end{aligned}$$

After some cancellations it yields

$$\nabla\psi(t,\cdot)\circ\Phi(t,0)x + \nabla\psi(t,\cdot)\circ M(t,0)\Phi(0,t)'\nabla\psi(0,x) - \Phi(0,t)'\nabla\psi(0,x) = 0.$$

On the other hand, since

$$T(x) = M_{10}^{-1/2}\nabla\phi(M_{10}^{-1/2}\Phi_{10}x) = M_{10}\Phi_{01}'\nabla\psi(0,x) + \Phi_{10}x,$$

we have

$$\begin{aligned} T_t(x) &= \Phi(t,1)M(1,t)M_{10}^{-1}\Phi_{10}x + M(t,0)\Phi(1,t)'M_{10}^{-1}T(x) \\ &= \Phi(t,0)x + M(t,0)\Phi(0,t)'\nabla\psi(0,x), \end{aligned}$$

from which it follows

$$\frac{dT_t(x)}{dt} = A(t)\Phi(t,0)x + A(t)M(t,0)\Phi(0,t)'\nabla\psi(0,x) + B(t)B(t)'\Phi(0,t)'\nabla\psi(0,x).$$

Therefore,

$$\begin{aligned}
& v(t,\cdot)\circ T_t(x) - \frac{dT_t(x)}{dt} \\
=& [A(t) + B(t)B(t)'\nabla\psi(t,\cdot)]\circ[\Phi(t,0)x + M(t,0)\Phi(0,t)'\nabla\psi(0,x)] \\
& - [A(t)\Phi(t,0)x + A(t)M(t,0)\Phi(0,t)'\nabla\psi(0,x) + B(t)B(t)'\Phi(0,t)'\nabla\psi(0,x)] \\
=& B(t)B(t)'\left\{\nabla\psi(t,\cdot)\circ\Phi(t,0)x + \nabla\psi(t,\cdot)\circ M(t,0)\Phi(0,t)'\nabla\psi(0,x) - \Phi(0,t)'\nabla\psi(0,x)\right\} \\
=& 0,
\end{aligned}$$

which completes the proof.

A.2 Proof of Theorem 10

The Markov kernel of (5.39) is

$$q^\epsilon(s,x,t,y) = (2\pi\epsilon)^{-n/2}|M(t,s)|^{-1/2}\exp\left(-\frac{1}{2\epsilon}(y-\Phi(t,s)x)'M(t,s)^{-1}(y-\Phi(t,s)x)\right).$$

Comparing this and the Brownian kernel $q^{B,\epsilon}$ we obtain

$$q^\epsilon(s,x,t,y) = (t-s)^{n/2}|M(t,s)|^{-1/2}q^{B,\epsilon}(s, M(t,s)^{-1/2}\Phi(t,s)x, t, M(t,s)^{-1/2}y).$$

Now define two new marginal distributions $\hat\rho_0$ and $\hat\rho_1$ through the coordinates transformation C in (5.24),

$$\begin{aligned}
\hat\rho_0(x) &= |M_{10}|^{1/2}|\Phi_{10}|^{-1}\rho_0(\Phi_{10}^{-1}M_{10}^{1/2}x), \\
\hat\rho_1(x) &= |M_{10}|^{1/2}\rho_1(M_{10}^{1/2}x).
\end{aligned}$$

Let $(\hat\varphi_0, \varphi_1)$ be a pair that solves the Schrödinger bridge problem with kernel q^ϵ and marginals ρ_0, ρ_1, and define $(\hat\varphi_0^B, \varphi_1^B)$ as

$$\hat\varphi_0(x) = |\Phi_{10}|\hat\varphi_0^B(M_{10}^{-1/2}\Phi_{10}x), \tag{A.1a}$$

$$\varphi_1(x) = |M_{10}|^{-1/2}\varphi_1^B(M_{10}^{-1/2}x), \tag{A.1b}$$

then the pair $(\hat{\varphi}_0^B, \varphi_1^B)$ solves the Schrödinger bridge problem with kernel $q^{B,\epsilon}$ and marginals $\hat{\rho}_0, \hat{\rho}_1$. To verify this, we need only to show that the joint distribution

$$\mathcal{P}_{01}^{B,\epsilon}(E) = \int_E q^{B,\epsilon}(0,x,1,y)\hat{\varphi}_0^B(x)\varphi_1^B(y)dxdy$$

matches the marginals $\hat{\rho}_0, \hat{\rho}_1$. This follows from

$$\begin{aligned}
&\int_{\mathbb{R}^n} q^{B,\epsilon}(0,x,1,y)\hat{\varphi}_0^B(x)\varphi_1^B(y)dy \\
&= \int_{\mathbb{R}^n} q^{B,\epsilon}(0,x,1,M_{10}^{-1/2}y)\hat{\varphi}_0^B(x)\varphi_1^B(M_{10}^{-1/2}y)d(M_{10}^{-1/2}y) \\
&= |M_{10}|^{1/2}|\Phi_{10}|^{-1}\int_{\mathbb{R}^n} q^{\epsilon}(0,\Phi_{10}^{-1}M_{10}^{1/2}x,1,y)\hat{\varphi}_0(\Phi_{10}^{-1}M_{10}^{1/2}x)\varphi_1(y)dy \\
&= |M_{10}|^{1/2}|\Phi_{10}|^{-1}\rho_0(\Phi_{10}^{-1}M_{10}^{1/2}x) = \hat{\rho}_0(x),
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^n} q^{B,\epsilon}(0,x,1,y)\hat{\varphi}_0^B(x)\varphi_1^B(y)dx \\
&= \int_{\mathbb{R}^n} q^{B,\epsilon}(0,M_{10}^{-1/2}\Phi_{10}x,1,y)\hat{\varphi}_0^B(M_{10}^{-1/2}\Phi_{10}x)\varphi_1^B(y)d(M_{10}^{-1/2}\Phi_{10}x) \\
&= |M_{10}|^{1/2}\int_{\mathbb{R}^n} q^{\epsilon}(0,x,1,M_{10}^{1/2}y)\hat{\varphi}_0(x)\varphi_1(M_{10}^{1/2}y)dx \\
&= |M_{10}|^{1/2}\rho_1(M_{10}^{1/2}y) = \hat{\rho}_1(y).
\end{aligned}$$

Compare $\mathcal{P}_{01}^{B,\epsilon}$ with $\mathcal{P}_{01}^{\epsilon}$ it is not difficult to find out that $\mathcal{P}_{01}^{B,\epsilon}$ is a push-forward of $\mathcal{P}_{01}^{\epsilon}$, that is,

$$\mathcal{P}_{01}^{B,\epsilon} = C_\sharp\mathcal{P}_{01}^{\epsilon}.$$

On the other hand, let π^B be the solution to classical OMT (3.3) with marginals $\hat{\rho}_0, \hat{\rho}_1$, then

$$\pi^B = C_\sharp\pi.$$

Now since $\mathcal{P}_{01}^{B,\epsilon}$ weakly converge to π^B from Theorem 9, we conclude that $\mathcal{P}_{01}^{\epsilon}$ weakly converge to π as ϵ goes to 0.

We next show $\mathcal{P}_t^{\epsilon}$ weakly converges to μ_t as ϵ goes to 0 for all t. The corresponding path space measure μ can be expressed as

$$\mu(\cdot) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \delta_{\gamma^{xy}}(\cdot)\ \pi(dxdy),$$

where γ^{xy} is the minimum energy path (5.22) connecting x, y, and $\delta_{\gamma^{xy}}$ is the Dirac measure concentrated on γ^{xy}. Similarly, the Schrödinger bridge $\mathcal{P}^\epsilon$ can be decomposed [42] as

$$\mathcal{P}^\epsilon(\cdot) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \mathcal{Q}^{xy,\epsilon}(\cdot)\ \mathcal{P}_{01}^\epsilon(dxdy),$$

where $\mathcal{Q}^{xy,\epsilon}$ is the pinned bridge [46] (a generalization of Brownian bridge) associated with (5.39) conditioned on $x^\epsilon(0) = x$ and $x^\epsilon(1) = y$, and it has the stochastic differential equation representation

$$\begin{aligned} dx^\epsilon(t) &= (A(t) - B(t)B(t)'\Phi(1,t)'M(1,t)^{-1}\Phi(1,t))x^\epsilon(t)dt \\ &\quad + B(t)B(t)'\Phi(1,t)'M(1,t)^{-1}ydt + \sqrt{\epsilon}B(t)dw(t) \end{aligned} \tag{A.2}$$

with initial value $x^\epsilon(0) = x$. As ϵ goes to zero, $\mathcal{Q}^{xy,\epsilon}$ tends to concentrate on the solution of

$$\begin{aligned} dx^0(t) &= (A(t) - B(t)B(t)'\Phi(1,t)'M(1,t)^{-1}\Phi(1,t))x^0(t)dt \\ &\quad + B(t)B(t)'\Phi(1,t)'M(1,t)^{-1}ydt,\ x^0(0) = x, \end{aligned}$$

which is γ^{xy}. The linear stochastic differential equation (A.2) represents a Gaussian process. It has the following explicit expression

$$x^\epsilon(t) = \gamma^{xy}(t) + \sqrt{\epsilon}\int_0^t \tilde{\Phi}(t,\tau)B(\tau)dw(\tau),\ 0 \le t < 1, \tag{A.3}$$

where $\tilde{\Phi}$ is the transition matrix of the dynamics (A.2), and $x^\epsilon(t)$ converges to y almost surely as t goes to 1, see [46]. From (A.3) it is easy to see that the autocovariance of $x^\epsilon(\cdot)$ depends linearly on ϵ and therefore goes to 0 as $\epsilon \to 0$. Combining this and the fact $x^\epsilon(\cdot)$ is a Gaussian process we conclude that the set of processes $x^\epsilon(\cdot)$ is tight [31, Theorem 7.3] and their finite dimensional distributions converge weakly to those of $x^0(\cdot)$. Hence, $\mathcal{Q}^{xy,\epsilon}$ converges weakly to $\delta_{\gamma^{xy}}$ [31, Theorem 7.1] as ϵ goes to 0.

We finally claim that $\mathcal{P}_t^\epsilon$ weakly converges to μ_t as ϵ goes to 0 for each t. To see this, choose a bounded, uniformly continuous[1] function h and define

$$g^\epsilon(x,y) := \langle \mathcal{Q}_t^{xy,\epsilon}, h\rangle,$$

$$\begin{aligned} g(x,y) &:= \langle \delta_{\gamma^{xy},t}, h\rangle \\ &= h(\gamma^{xy}(t)), \end{aligned}$$

where $\langle\cdot,\cdot\rangle$ denotes the integration of the function against the measure. From (5.22) it is immediate that g is a bounded continuous functions of x, y. Since $\mathcal{Q}_t^{xy,\epsilon}$ is a Gaussian distribution with mean $\gamma^{xy}(t)$ and covariance which is independent of x, y and tends to zero as $\epsilon \to 0$ based on (A.3), $g^\epsilon \to g$ uniformly as $\epsilon \to 0$. It follows

$$\begin{aligned} \langle \mathcal{P}_t^\epsilon, h\rangle - \langle \mu_t, h\rangle &= \langle \mathcal{P}_{01}^\epsilon, g^\epsilon\rangle - \langle \pi, g\rangle \\ &= (\langle \mathcal{P}_{01}^\epsilon, g\rangle - \langle \pi, g\rangle) + \langle \mathcal{P}_{01}^\epsilon, g^\epsilon - g\rangle. \end{aligned}$$

Both summands tend to zero as $\epsilon \to 0$, the first due to weak convergence of $\mathcal{P}_{01}^\epsilon$ to π and the second due to the uniform convergence of g^ϵ to g. This completes the proof.

A.3 Proof of Proposition 6

Apply the time-varying change of coordinates

$$\xi(t) = N(1,0)^{-1/2}\Phi(0,t)x(t).$$

Then, in this new coordinates the dynamical system (6.6) becomes

$$d\xi(t) = \underbrace{N(1,0)^{-1/2}\Phi(0,t)B(t)}_{B_{\text{new}}(t)}\, dw(t).$$

We will prove the statement in this new set of coordinates for the state, where the state matrix $A_{\text{new}} = 0$ and the state equation is simply $d\xi(t) = B_{\text{new}}(t)dw(t)$, and then revert back to the original set of coordinates at the end. Accordingly,

$$\dot{P}_{\text{new}}(t) = B_{\text{new}}(t)B_{\text{new}}(t)',$$

[1]To guarantee weak convergence, it suffices to have bounded, uniform continuous test functions by Portmanteau Theorem [31].

$$\dot{Q}_{\text{new}}(t) = -B_{\text{new}}(t)B_{\text{new}}(t)',$$

along with $M_{\text{new}}(1,0) = N_{\text{new}}(1,0) = I$ and

$$\Sigma_{0,\text{new}} = N(1,0)^{-1/2}\Sigma_0 N(1,0)^{-1/2}, \tag{A.4a}$$

while

$$\Sigma_{1,\text{new}} = N(1,0)^{-1/2}\Phi(0,1)\Sigma_1\Phi(0,1)'N(1,0)^{-1/2}. \tag{A.4b}$$

The relation between $Q_{\text{new}}(t)$ and $Q(t)$ is given by

$$Q_{\text{new}}(t) = N(1,0)^{-1/2}\Phi(0,t)Q(t)\Phi(0,t)'N(1,0)^{-1/2}.$$

This can be seen by taking the derivative of both sides

$$\begin{aligned}
\dot{Q}_{\text{new}}(t) &= -N(1,0)^{-1/2}\Phi(0,t)A(t)Q(t)\Phi(0,t)'N(1,0)^{-1/2} \\
&\quad -N(1,0)^{-1/2}\Phi(0,t)Q(t)A(t)'\Phi(0,t)'N(1,0)^{-1/2} \\
&\quad +N(1,0)^{-1/2}\Phi(0,t)\dot{Q}(t)\Phi(0,t)'N(1,0)^{-1/2} \\
&= -N(1,0)^{-1/2}\Phi(0,t)B(t)B(t)'\Phi(0,t)'N(1,0)^{-1/2} \\
&= -B_{\text{new}}(t)B_{\text{new}}(t)'.
\end{aligned}$$

In the next paragraph, for notational convenience, we drop the subscript "new" and prove the statement assuming that $A(t) = 0$ as well as $N(1,0) = I$. We will return to the notation that distinguishes the two sets of coordinates with the subscript "new" and relate back to the original ones at the end of the proof.

Since $A(t) = 0$, then $\Phi(t,x) = I$ for all $s,t \in [0,1]$. Further, $M(1,0) = N(1,0) = I$. Thus,

$$\begin{aligned}
P(1) &= P(0) + I \\
Q(1) &= Q(0) - I.
\end{aligned}$$

Substituting in (6.17), we obtain that

$$\begin{aligned}
Q(0)^{-1} + P(0)^{-1} &= \Sigma_0^{-1} \\
(Q(0) - I)^{-1} + (P(0) + I)^{-1} &= \Sigma_1^{-1}.
\end{aligned}$$

Solving the first for $P(0)$ as a function of $Q(0)$ and substituting in the second, we have

$$\begin{aligned}
\Sigma_1^{-1} &= ((\Sigma_0^{-1} - Q(0)^{-1})^{-1} + I)^{-1} + (Q(0) - I)^{-1} \\
&= ((\Sigma_0^{-1} - Q(0)^{-1})^{-1} + I)^{-1}(Q(0) + (\Sigma_0^{-1} - Q(0)^{-1})^{-1}))(Q(0) - I)^{-1} \\
&= ((\Sigma_0^{-1} - Q(0)^{-1})^{-1} + I)^{-1}(\Sigma_0^{-1} - Q(0)^{-1})^{-1}\Sigma_0^{-1}Q(0)(Q(0) - I)^{-1} \\
&= (\Sigma_0^{-1} + I - Q(0)^{-1})^{-1}\Sigma_0^{-1}(I - Q(0)^{-1})^{-1}.
\end{aligned}$$

After inverting both terms, simple algebra leads to

$$(I - Q(0)^{-1})\Sigma_0(I - Q(0)^{-1}) + (I - Q(0)^{-1}) = \Sigma_1.$$

This is a quadratic expression and has two Hermitian solutions

$$I - Q(0)^{-1} = \Sigma_0^{-1/2}\left(-\frac{1}{2}I \mp \left(\Sigma_0^{1/2}\Sigma_1\Sigma_0^{1/2} + \frac{1}{4}I\right)^{1/2}\right)\Sigma_0^{-1/2}. \tag{A.5}$$

This gives that

$$Q(0) = \Sigma_0^{1/2}\left(\Sigma_0 + \frac{1}{2}I \pm \left(\Sigma_0^{1/2}\Sigma_1\Sigma_0^{1/2} + \frac{1}{4}I\right)^{1/2}\right)^{-1}\Sigma_0^{1/2}.$$

To see that i) holds evaluate (in these simplified coordinates where there is no drift and $M(1,0) = I$)

$$\begin{aligned}
Q_-(t)^{-1} &= (Q_-(0) - M(t,0))^{-1} \\
&= -M(t,0)^{-1} - M(t,0)^{-1}(Q_-(0)^{-1} - M(t,0)^{-1})^{-1}M(t,0)^{-1} \\
&= -M(t,0)^{-1} - M(t,0)^{-1}\Sigma_0^{1/2}\left(\Sigma_0 + \frac{1}{2}I\right. \\
&\qquad \left. - \left(\Sigma_0^{1/2}\Sigma_1\Sigma_0^{1/2} + \frac{1}{4}I\right)^{1/2} - \Sigma_0^{1/2}M(t,0)^{-1}\Sigma_0^{1/2}\right)^{-1}\Sigma_0^{1/2}M(t,0)^{-1}
\end{aligned}$$

for $t > 0$. For $t \in (0,1]$, the expression in parenthesis

$$\Sigma_0 + \frac{1}{2}I - \left(\Sigma_0^{1/2}\Sigma_1\Sigma_0^{1/2} + \frac{1}{4}I\right)^{1/2} - \Sigma_0^{1/2}M(t,0)^{-1}\Sigma_0^{1/2}$$

is clearly maximal when $t = 1$. However, for $t = 1$ when $M(1,0) = I$, this expression is seen to be

$$\frac{1}{2}I - \left(\Sigma_0^{1/2}\Sigma_1\Sigma_0^{1/2} + \frac{1}{4}I\right)^{1/2} < 0.$$

Therefore, the expression in parenthesis is never singular and we deduce that $Q_-(t)^{-1}$ remains bounded for all $t \in (0,1]$, i.e., $Q_-(t)$ remains non-singular. For $t = 0$, $Q(0)^{-1}$ is seen to be finite from (A.5). The argument for $P_-(t)$ is similar. Regarding ii), it suffices to notice that $0 < Q_+(0) < I$ while $Q_+(1) = Q_+(0) - I < 0$. The statement ii) follows by continuity of $Q_+(t)$, and similarly for $P_+(t)$.

We now revert back to the set of coordinates where the drift is not necessarily zero and where $N(1,0)$ may not be the identity. We see that

$$\begin{aligned} Q_\pm(0) &= N(1,0)^{1/2}(Q_\pm(0))_{\text{new}}N(1,0)^{1/2} \\ &= N(1,0)^{1/2}\Sigma_{0,\text{new}}^{1/2}\left(\Sigma_{0,\text{new}} + \frac{1}{2}I \pm \left(\Sigma_{0,\text{new}}^{1/2}\Sigma_{1,\text{new}}\Sigma_{0,\text{new}}^{1/2} + \frac{1}{4}I\right)^{1/2}\right)^{-1} \\ &\quad \Sigma_{0,\text{new}}^{1/2}N(1,0)^{1/2} \end{aligned}$$

where $\Sigma_{0,\text{new}}, \Sigma_{1,\text{new}}$ as in (A.4a)-(A.4b), which for compactness of notation in the statement of the proposition we rename S_0 and S_1, respectively.

A.4 Proof of Theorem 19

We show that i) the joint distribution between the two end-points of $[0,1]$ for (6.23) is the minimizer of the relative entropy, with respect to the corresponding two-endpoint joint distribution of (6.6), over distributions that satisfy the endpoint constraint that the marginals are Gaussian with specified covariances, and ii) the probability laws of these two SDEs on sample paths, conditioned on $x(0) = x_0$, $x(1) = x_1$ for any x_0, x_1, are identical by showing that they have the same pinned processes. We use the notation

$$g_S(x) := (2\pi)^{-n/2}\det(S)^{-1/2}\exp\left[-\frac{1}{2}x'S^{-1}x\right],$$

to denote the standard Gaussian probability density function with mean zero and covariance S.

We start with i). In general, the relative entropy between two Gaussian distributions $g_S(x)$ and $g_\Sigma(x)$ is

$$\begin{aligned}\int_{\mathbb{R}^n} g_\Sigma(x)\log\left(\frac{g_\Sigma}{g_S}\right)dx &= \int_{\mathbb{R}^n} g_\Sigma \log\left(\frac{\det(S)^{1/2}}{\det(\Sigma)^{1/2}}\right)dx + \frac{1}{2}\int_{\mathbb{R}^n} g_\Sigma(x)(x'S^{-1}x - x'\Sigma x)dx\\ &= \frac{1}{2}\log(\det(S)) - \frac{1}{2}\log(\det(\Sigma)) + \frac{1}{2}\operatorname{trace}(S^{-1}\Sigma) - \frac{1}{2}\operatorname{trace}(I).\end{aligned}$$

If p_Σ is a probability density function, not necessarily Gaussian, having covariance Σ, then

$$\begin{aligned}\int_{\mathbb{R}^n} p_\Sigma(x)\log\left(\frac{p_\Sigma}{g_S}\right)dx &= \int_{\mathbb{R}^n} p_\Sigma(x)\log\left(\frac{p_\Sigma}{g_S}\frac{g_\Sigma}{g_\Sigma}\right)dx\\ &= \int_{\mathbb{R}^n} p_\Sigma(x)\log\left(\frac{p_\Sigma}{g_\Sigma}\right)dx + \int_{\mathbb{R}^n} p_\Sigma(x)\log\left(\frac{g_\Sigma}{g_S}\right)dx \qquad \text{(A.6)}\end{aligned}$$

where we multiplied and divided by g_Σ and then partitioned accordingly. We observe that

$$\int_{\mathbb{R}^n} p_\Sigma(x)\log\left(\frac{g_\Sigma}{g_S}\right)dx = \int_{\mathbb{R}^n} g_\Sigma(x)\log\left(\frac{g_\Sigma}{g_S}\right)dx.$$

since $\log\left(\frac{g_\Sigma}{g_S}\right)$ is a quadratic form in x. Thus, the minimizer of relative entropy to g_S among probability density functions with covariance Σ is Gaussian since the first term in (A.6) is positive unless $p_\Sigma = g_\Sigma$, in which case it is zero.

We consider two-point joint Gaussian distributions with covariances S_{01} as in (6.25) with $S_0 = \Sigma_0$, and

$$\Sigma_{01} := \begin{bmatrix}\Sigma_0 & Y'\\ Y & \Sigma_1\end{bmatrix}$$

and evaluate Y that minimizes the relative entropy. To this end we focus on

$$\operatorname{trace}(S_{01}^{-1}\Sigma_{01}) - \log\det(\Sigma_{01}). \qquad \text{(A.7)}$$

Since

$$S_{01} = \begin{bmatrix} I\\ \Phi(1,0)\end{bmatrix}\Sigma_0\begin{bmatrix} I, & \Phi(1,0)'\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & M(1,0)\end{bmatrix},$$

it follows that

$$S_{01}^{-1} = \begin{bmatrix}\Sigma_0^{-1} + \Phi' M^{-1}\Phi & -\Phi' M^{-1}\\ -M^{-1}\Phi & M^{-1}\end{bmatrix},$$

where we simplified notation by setting $\Phi := \Phi(1,0)$ and $M := M(1,0)$. Then, the expression in (A.7) becomes

$$\begin{aligned}&\operatorname{trace}\big((\Sigma_0^{-1} + \Phi' M^{-1}\Phi)\Sigma_0 - \Phi' M^{-1} Y - Y' M^{-1}\Phi + M^{-1}\Sigma_1\big)\\ &- \log\det(\Sigma_0) - \log\det(\Sigma_1 - Y\Sigma_0^{-1}Y').\end{aligned}$$

Retaining only the terms that involve Y leads us to seek a maximizing choice for Y in

$$f(Y) := \log\det(\Sigma_1 - Y\Sigma_0^{-1}Y') + 2\operatorname{trace}(\Phi' M^{-1} Y).$$

Equating the differential of this last expression as a function of Y to zero gives

$$-2\Sigma_0^{-1}Y'(\Sigma_T - Y\Sigma_0^{-1}Y')^{-1} + 2\Phi' M^{-1} = 0 \tag{A.8}$$

To see this, denote by Δ a small perturbation of Y and retain the linear terms in Δ in

$$\begin{aligned}&f(Y+\Delta) - f(Y)\\ =\ &\log\det(I - (\Sigma_1 - Y\Sigma_0^{-1}Y')^{-1}(\Delta\Sigma_0^{-1}Y' + Y\Sigma_0^{-1}\Delta')) + 2\operatorname{trace}(\Phi' M^{-1}\Delta)\\ \simeq\ &-\operatorname{trace}((\Sigma_1 - Y\Sigma_0^{-1}Y')^{-1}(\Delta\Sigma_0^{-1}Y' + Y\Sigma_0^{-1}\Delta')) + 2\operatorname{trace}(\Phi' M^{-1}\Delta)\\ =\ &-2\operatorname{trace}(\Sigma_0^{-1}Y'(\Sigma_1 - Y\Sigma_0^{-1}Y')^{-1}\Delta) + 2\operatorname{trace}(\Phi' M^{-1}\Delta)\end{aligned}$$

Let now

$$\Sigma_{01} = \begin{bmatrix}\Sigma_0 & \Sigma_0\Phi_{Q_-}(1,0)' \\ \Phi_{Q_-}(1,0)\Sigma_0 & \Sigma_1\end{bmatrix}$$

where $\Phi_{Q_-}(1,0)$ is the state-transition matrix of $A_{Q_-}(t)$, i.e., it satisfies

$$\begin{aligned}\frac{\partial}{\partial t}\Phi_{Q_-}(t,s) &= A_{Q_-}(t)\Phi_{Q_-}(t,s), \text{ and}\\ -\frac{\partial}{\partial s}\Phi_{Q_-}(t,s) &= \Phi_{Q_-}(t,s)A_{Q_-}(s),\end{aligned}$$

with $\Phi_{Q_-}(s,s) = I$. We need to show that Σ_{01} here is the solution of the relative entropy minimization problem above. By concavity of $f(Y)$, it suffices to show that

$Y = \Phi_{Q_-}(1,0)\Sigma_0$ satisfies the first-order condition (A.8), that is,

$$\begin{aligned}\Phi_{Q_-}(1,0)'(\Sigma_1 - \Phi_{Q_-}(1,0)\Sigma_0\Phi_{Q_-}(1,0)')^{-1} &= \Phi(1,0)'M(1,0)^{-1} \\ &= \Phi(1,0)'(S_1 - \Phi(1,0)S_0\Phi(1,0)')^{-1},\end{aligned}$$

where S_t is as in (6.25) with $S_0 = \Sigma_0$. By taking inverse of both sides we obtain an equivalent formula

$$\Sigma_1\Phi_{Q_-}(0,1)' - \Phi_{Q_-}(1,0)\Sigma_0 = S_1\Phi(0,1)' - \Phi(1,0)\Sigma_0. \tag{A.9}$$

We claim

$$\Sigma_t\Phi_{Q_-}(0,t)' - \Phi_{Q_-}(t,0)\Sigma_0 = S_t\Phi(0,t)' - \Phi(t,0)\Sigma_0,$$

then (A.9) follows by taking $t = 1$. We now prove our claim. For convenience, denote

$$\begin{aligned}F_1(t) &= \Sigma_t\Phi_{Q_-}(0,t)' - \Phi_{Q_-}(t,0)\Sigma_0 \\ F_2(t) &= S_t\Phi(0,t)' - \Phi(t,0)\Sigma_0 \\ F_3(t) &= Q_-(t)(\Phi_{Q_-}(0,t)' - \Phi(0,t)').\end{aligned}$$

We will show that $F_1(t) = F_2(t) = F_3(t)$. First we show $F_2(t) = F_3(t)$. Since $F_2(0) = F_3(0) = 0$, we only need to show that they satisfy the same differential equation. To this end, compare

$$\begin{aligned}\dot F_2(t) &= \dot S_t\Phi(0,t)' - S_tA'\Phi(0,t)' - A\Phi(t,0)\Sigma_0 \\ &= (AS_t + S_tA' + BB')\Phi(0,t)' - S_tA'\Phi(0,t)' - A\Phi(t,0)\Sigma_0 \\ &= AF_2(t) + BB'\Phi(0,t)',\end{aligned}$$

with

$$\begin{aligned}\dot F_3(t) &= \dot Q_-(t)(\Phi_{Q_-}(0,t)' - \Phi(0,t)') + Q_-(t)(-A_{Q_-}(t)'\Phi_{Q_-}(0,t)' + A'\Phi(0,t)') \\ &= (AQ_-(t) + Q_-(t)A' - BB')(\Phi_{Q_-}(0,t)' - \Phi(0,t)') \\ &\quad -Q_-(t)A'(\Phi_{Q_-}(0,t)' - \Phi(0,t)') + BB'\Phi_{Q_-}(0,t)' \\ &= AF_3(t) + BB'\Phi(0,t)',\end{aligned}$$

which proves the claim $F_2(t) = F_3(t)$. We next show that $F_1(t) = F_3(t)$. Let

$$\begin{aligned} H(t) &= Q_-(t)^{-1}(F_3(t) - F_1(t)) \\ &= -(Q_-(t)^{-1} - \Sigma_t^{-1})\Sigma_t \Phi_{Q_-}(0,t)' + Q_-(t)^{-1}\Phi_{Q_-}(t,0)\Sigma_0 - \Phi(0,t)' \\ &= P(t)^{-1}\Sigma_t \Phi_{Q_-}(0,t)' + Q_-(t)^{-1}\Phi_{Q_-}(t,0)\Sigma_0 - \Phi(0,t)', \end{aligned}$$

then

$$\begin{aligned} \dot{H}(t) &= \dot{P}(t)^{-1}\Sigma_t \Phi_{Q_-}(0,t)' + P(t)^{-1}\dot{\Sigma}_t \Phi_{Q_-}(0,t)' - \\ &\quad P(t)^{-1}\Sigma_t A_{Q_-}(t)'\Phi_{Q_-}(0,t)' + \dot{Q}_-(t)^{-1}\Phi_{Q_-}(t,0)\Sigma_0 \\ &\quad + Q_-(t)^{-1}A_{Q_-}(t)\Phi_{Q_-}(t,0)\Sigma_0 + A'\Phi(0,t)' \\ &= -A'H(t). \end{aligned}$$

Since $H(0) = Q_-(0)^{-1}(F_3(0) - F_1(0)) = 0$, it follows that $H(t) = 0$ for all t, and therefore, $F_1(t) = F_3(t)$. This completes the proof of the first part.

We now prove ii). According to Lemma 18, the pinned process corresponding to (6.6) satisfies

$$dx = (A - BB'R_1(t)^{-1})xdt + BB'R_1(t)^{-1}\Phi(t,1)x_1 dt + Bdw \tag{A.10}$$

where $R_1(t)$ satisfies

$$\dot{R}_1(t) = AR_1(t) + R_1(t)A' - BB'$$

with $R_1(1) = 0$, while the pinned process corresponding to (6.23) satisfies

$$dx = (A_{Q_-}(t) - BB'R_2(t)^{-1})xdt + BB'R_2(t)^{-1}\Phi_{Q_-}(t,1)x_1 dt + Bdw \tag{A.11}$$

where $R_2(t)$ satisfies

$$\dot{R}_2(t) = A_{Q_-}(t)R_2(t) + R_2(t)A_{Q_-}(t)' - BB'$$

with $R_2(1) = 0$. We next show (A.10) and (A.11) are identical. It suffices to prove that

$$A - BB'R_1(t)^{-1} = A_{Q_-}(t) - BB'R_2(t)^{-1} \tag{A.12}$$

and

$$R_1(t)^{-1}\Phi(t,1) = R_2(t)^{-1}\Phi_{Q_-}(t,1). \tag{A.13}$$

Equation (A.12) is equivalent to

$$R_1(t)^{-1} = R_2(t)^{-1} + Q_-(t)^{-1}.$$

Multiply $R_1(t)$ and $R_2(t)$ on both sides to obtain

$$R_2(t) = R_1(t) + R_1(t)Q_-(t)^{-1}R_2(t).$$

Now let

$$J(t) = R_1(t) + R_1(t)Q_-(t)^{-1}R_2(t) - R_2(t).$$

Then

$$\begin{aligned} \dot{J}(t) &= \dot{R}_1(t) + \dot{R}_1(t)Q_-(t)^{-1}R_2(t) + R_1(t)\dot{Q}_-(t)^{-1}R_2(t) + R_1(t)Q_-(t)^{-1}\dot{R}_2(t) - \dot{R}_2(t) \\ &= AJ(t) + J(t)A_{Q_-}(t)'. \end{aligned}$$

Since

$$J(1) = R_1(1) + R_1(1)Q_-(1)^{-1}R_2(1) - R_2(1) = 0,$$

it follows that $J(t) = 0$. This completes the proof of (A.12). Equation (A.13) is equivalent to

$$\Phi(1,t)R_1(t) = \Phi_{Q_-}(1,t)R_2(t).$$

Let

$$K(t) = \Phi(1,t)R_1(t) - \Phi_{Q_-}(1,t)R_2(t),$$

and then

$$\begin{aligned} \dot{K}(t) &= -\Phi(1,t)AR_1(t) + \Phi(1,t)\dot{R}_1(t) + \Phi_{Q_-}(1,t)A_{Q_-}(t)R_2(t) - \Phi_{Q_-}(1,t)\dot{R}_2(t) \\ &= K(t)(A' - R_1(t)^{-1}BB'). \end{aligned}$$

Since

$$K(1) = \Phi(1,1)R_1(1) - \Phi_{Q_-}(1,1)R_2(1) = 0,$$

it follows that $K(t) = 0$ as well for all t. This completes the proof.

A.5 Lemma on $\mathfrak{f}_B$ and $\mathfrak{g}_B$

Lemma 37 *Consider the maps* $\mathfrak{f}_B$ *and* $\mathfrak{g}_B$ *defined in* (6.63)-(6.64)*. The range of* $\mathfrak{f}_B$ *coincides with the null space of* $\mathfrak{g}_B$*, that is,*

$$\mathcal{R}(\mathfrak{f}_B) = \mathcal{N}(\mathfrak{g}_B).$$

Proof It is immediate that

$$\mathcal{R}(\mathfrak{f}_B) \subseteq \mathcal{N}(\mathfrak{g}_B).$$

To show equality it suffices to show that $(\mathcal{R}(\mathfrak{f}_B))^\perp \subseteq \mathcal{N}(\mathfrak{g}_B)^\perp$. To this end, consider

$$M \in \mathcal{S}_n \cap (\mathcal{R}(\mathfrak{f}_B))^\perp .$$

Then

$$\operatorname{trace}(M(BX + X'B')) = 0$$

for all $X \in \mathbb{R}^{m\times n}$. Equivalently, for $Z = MB \in \mathbb{R}^{n\times m}$, $\operatorname{trace}(ZX) + \operatorname{trace}(X'Z') = 0$ for all X. Thus, $\operatorname{trace}(ZX) = 0$ for all X and hence $Z = 0$. Since $MB = Z = 0$, then $M\Pi_{\mathcal{R}(B)} = 0$ or, equivalently, $M\Pi_{\mathcal{R}(B)^\perp} = M$. Therefore $\Pi_{\mathcal{R}(B)^\perp} M \Pi_{\mathcal{R}(B)^\perp} = M$, i.e., $M \in (\mathcal{R}(\mathfrak{g}_B))$. Therefore,

$$(\mathcal{R}(\mathfrak{f}_B))^\perp \subseteq (\mathcal{R}(\mathfrak{g}_B)) = \mathcal{N}(\mathfrak{g}_B)^\perp$$

since $\mathfrak{g}_B$ is self-adjoint, which completes the proof.