Convergence in law of the centered maximum of the mollified Gaussian free field in two dimensions

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Javier E. Acosta

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Dedication

To my beloved mother.

Abstract

Consider a family of centered Gaussian fields on the d-dimensional unit box, whose covariance decreases logarithmically in the distance between points. In Part I of this thesis, we prove tightness of the centered maximum of the Gaussian fields and provide exponentially decaying bounds on the right and left tails.

Part II is devoted to the study of a specific and fundamental example of a log-correlated Gaussian field in two dimensions, namely, the mollified Gaussian free field (MGFF). The MGFF is a random field obtained by suitably mollifying the covariance of the continuum Gaussian free field, which is a generalized random field defined on measures in the unit square. We prove that the centered maximum of the MGFF converges in law as the mollifier approaches the Dirac delta function. We moreover show that this limit law does not depend on the specific mollifier that is employed, and give a representation for it.

Our approach in both Part I and II is similar to the approach employed by Bramson, Ding and Zeitouni in their papers on the centered maximum of the discrete Gaussian free field.

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Chapter 1

Introduction

The standard Brownian bridge is usually defined as the stochastic process obtained by conditioning the one-dimensional Brownian motion to have value 0 at time 1.

An equivalent definition of the Brownian bridge is as follows. Let $I = [0,1]$ and denote by $C_0^\infty(I)$ the space of test functions supported in the interior of I. For a given $h \in C_0^\infty(I)$, consider the Poisson problem of finding $f \in C_0^\infty(I)$ such that

$$-\Delta f = h \qquad \text{on } I, \tag{1.0.1}$$

where Δ denotes the one-dimensional Laplace operator. An integral representation of the solution of (1.0.1) is given by means of the *Green function* $G : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ defined by

$$G(x,y) = \begin{cases} x(1-y) & \text{if } 0 \le x \le y \le 1, \\ y(1-x) & \text{if } 0 \le y < x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to check that, for any $h \in C_0^\infty(I)$, the function

$$f(x) := \int_{\mathbb{R}} G(x,y)h(y)dy \tag{1.0.2}$$

satisfies (1.0.1). The Brownian bridge can then be defined as the real-valued Gaussian process B_t, indexed by $t \in [0,1]$, such that, for all $s,t \in [0,1]$,

$$\mathbb{E}[B_t] = 0 \qquad \text{and} \qquad Cov(B_s, B_t) = G(s,t). \tag{1.0.3}$$

(Recall that the law of a Gaussian process is uniquely determined by the point-wise expectation and the covariance structure of the process.)

Random fields can be viewed as generalizations of stochastic processes to higher dimensions. For a stochastic process $(X_t)_{t\in T}$, the index set T is typically a subset of the real line, while, for a random field, the index set T is arbitrary. A *Gaussian field* is a random field for which the collection of random variables $(X_t)_{t\in T}$ is jointly Gaussian (see Chapter 2 for a precise definition of Gaussian fields).

We are interested in the *continuum Gaussian free field* (CGFF), which can be viewed as a generalization of the Brownian bridge to higher dimensions. To define it, we now let Δ denote the d-dimensional Laplace operator and let $G : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty]$ be the Green function for the analogous Poisson problem (1.0.1) on the d-dimensional unit box $I = [0, 1]^d$. We recall that the Green function has the following representation in terms of Brownian motion (see [1, Chapter 3]): Let $(W_t)_{t\geq 0}$ denote a d-dimensional Brownian motion and let $\mathbb{E}^x$ denote the expectation with respect to the probability law of $(W_t)_{t\geq 0}$ with starting point $x \in \mathbb{R}^d$. Then, for all $x, y \in \mathbb{R}^d$,

$$G(x, y) = \Phi(\|x - y\|) - \mathbb{E}^x[\Phi(\|W_\tau - y\|)], \tag{1.0.4}$$

where $\tau = \inf\{t \geq 0 : W_t \notin [0, 1]^d\}$, $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^d$ and

$$\Phi(r) = \begin{cases} -\frac{1}{2\pi}\log(r) & \text{if } d = 2, \\ \frac{1}{d(d-2)\omega(d)} r^{2-d} & \text{if } d \geq 3, \end{cases}$$

for all $r \geq 0$, where $\omega(d)$ denotes the volume of the d-dimensional unit ball. (Note that $G(x, y) = 0$ if x or y belongs to $\mathbb{R}^d \setminus (0, 1)^d$.)

For $d \geq 2$, the Green function diverges on the diagonal (i.e., $G(x, x) = \infty$ for all x in $(0, 1)^d$), hence, it can not be used to define the covariance structure of a Gaussian field with index set $[0, 1]^d$. Consider instead the collection $\mathcal{M}$ of probability measures μ on $\mathbb{R}^d$ such that

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} G(x, y)d\mu(x)d\mu(y) < \infty,$$

and, analogously to (1.0.3), define the CGFF as the Gaussian field $(X(\mu) : \mu \in \mathcal{M})$ such that, for all $\mu, \nu \in \mathcal{M}$,

$$\mathbb{E}[X(\mu)] = 0 \quad \text{and} \quad Cov(X(\mu), X(\nu)) = \iint_{\mathbb{R}^d\times\mathbb{R}^d} G(x, y)d\mu(x)d\nu(y).$$

This construction of the CGFF relies on the Green function being positive semi-definite. We mention [2] as an excellent survey on the different ways to construct the CGFF.

We will be concerned here with the CGFF when $d = 2$. There are two main reasons for this. First, the two-dimensional Green function $G(x, y)$ in (1.0.4) is logarithmic in the distance between x and y, so the CGFF fits naturally in the context of the *log-correlated Gaussian fields*, which we define below and are the subject of Part I of this thesis. The logarithmic covariance of the CGFF, when $d = 2$, produces a rich structure that is absent in the CGFF when $d \geq 3$: log-correlated Gaussian fields are conformally invariant (and they therefore have applications in mathematical physics, see [2]) and are related to additive cascade models and branching random walks (see [3]). The second reason for focusing on $d = 2$ is given by the recent advances in the study of the two-dimensional *discrete Gaussian free field* (DGFF), which we will define below. The techniques developed in [4] for the DGFF can be extended and applied in the continuum, as we will see throughout Part II.

A fundamental object of study for any random field is its supremum (or maximum, when it is attained). In the case of the Brownian bridge, it is possible to compute explicitly the distribution of the maximum using the reflection principle of Brownian motion. For the CGFF, the problem of characterizing the distribution of the maximum is considerably more complicated and the bulk of this thesis is devoted to this problem.

Instead of working with $\mathcal{M}$ as the index set, we will restrict ourselves to measures in $\mathcal{M}$ induced by test functions, as follows. Let θ be a fixed test function on $\mathbb{R}^2$ with $\int_{\mathbb{R}^2} \theta(u)du = 1$. For a constant $R > 0$ and $x \in [0, 1]^2$, define the measure $\rho_{R,x}$ by

$$\rho_{R,x}(A) = \int_A R^2 \theta(R(x - u))du \tag{1.0.5}$$

for all Borel sets $A \subseteq \mathbb{R}^2$. (The exponent 2 in the scaling factor R^2 is needed for $\rho_{R,x}$ to be a probability measure.) Note that $\rho_{R,x}$ is (roughly) centered at x and it converges weakly to the Dirac delta at x as $R \to \infty$. The field $(X(\rho_{R,x}) : R > 0, x \in [0, 1]^2)$ is the *mollified Gaussian free field* (MGFF) associated with θ.

This thesis is devoted to the study of $\Theta^*_R := \max_{x\in[0,1]^2} X(\rho_{R,x})$, the maximum of the MGFF. In Part II, we prove that Θ^*_R, after being appropriately centered, converges in law as $R \to \infty$. Furthermore, up to a shift in the centering, the limit law does not depend on the test function θ. (See Theorem 6.1.1 for the precise statement of this result.)

As mentioned before, this work is motivated by recent advances for the discrete Gaussian free field, which can be defined in analogy to (1.0.1) and (1.0.3), as follows. Consider the discrete two-dimensional lattice $V_N = \{0, 1, \ldots, N-1\}^2 \subset \mathbb{Z}^2$ of size $N \times N$, and denote by ∂V_N the points in $\mathbb{Z}^2 \setminus V_N$ that have neighbors in V_N. Let $h : V_N \cup \partial V_N \to \mathbb{R}$ be a given function such that $h\,|_{\partial V_N} = 0$. The analog to the problem (1.0.1) in this setting is to find a function $f : V_N \cup \partial V_N \to \mathbb{R}$ such that $f\,|_{\partial V_N} = 0$ and

$$-\Delta f = h \qquad \text{on } V_N, \tag{1.0.6}$$

where Δ is the discrete Laplace operator defined by

$$(\Delta f)(x) = -f(x) + \frac{1}{4} \sum_e f(x+e)$$

for all $x \in V_N$, and the sum is over the unit vectors $e \in \{(\pm 1, 0), (0, \pm 1)\}$.

When we view h and f as vectors in $\mathbb{R}^{V_N}$, the discrete Laplace operator is the matrix $\Delta(x, y) = I(x, y) - P(x, y)$ in $\mathbb{R}^{V_N \times V_N}$, where $I(x, y) = 1_{\{x=y\}}$ is the identity matrix and

$$P(x, y) = \begin{cases} \frac{1}{4} & \text{if } x - y \in \{(\pm 1, 0), (0, \pm 1)\}, \\ 0 & \text{otherwise,} \end{cases}$$

is the one-step transition probability matrix of the simple symmetric random walk on V_N. The discrete Green function is then defined, analogously to (1.0.2), as the matrix G_N in $\mathbb{R}^{V_N \times V_N}$, so that

$$f(x) := \sum_{y \in V_N} G_N(x, y) h(y) \qquad \text{for all } x \in V_N$$

satisfies (1.0.6). The DGFF is then defined, in analogy with (1.0.3), as the Gaussian field $(\eta_{N,x} : x \in V_N)$ such that, for all $x, y \in V_N$,

$$\mathbb{E}[\eta_{N,x}] = 0 \qquad \text{and} \qquad Cov(\eta_{N,x}, \eta_{N,y}) = G_N(x, y).$$

In [4], Bramson, Ding and Zeitouni established the convergence in law of the centered maximum of the DGFF, $\eta_N^* := \max_{x \in V_N} \eta_{N,x}$, as $N \to \infty$. Theorem 6.1.1 in Part II of this thesis can be viewed as a continuum analog of Theorem 1.1 in [4].

The MGFF is a two-dimensional example of the family of log-correlated Gaussian fields. We say that a (point-wise zero-mean) Gaussian field $Y = (Y_{\epsilon,x} : \epsilon > 0, x \in [0,1]^d)$ is log-correlated if, together with additional technical conditions (see (3.1.2)),

$$Cov(Y_{\epsilon,x}, Y_{\epsilon,y}) = \begin{cases} -\log \|x-y\| + O(1) & \text{if } \|x-y\| \geq \epsilon, \\ -\log(\epsilon) + O(1) & \text{if } \|x-y\| < \epsilon, \end{cases} \tag{1.0.7}$$

where the order 1 term depends on the family Y and the constant $\epsilon > 0$ can be viewed as a parameter that truncates the variance when x and y are close together.

In Part I, we investigate the question of tightness as $\epsilon \to 0$ of the maximum $Y_\epsilon^* := \max_{x\in[0,1]^d} Y_{\epsilon,x}$, after it is appropriately centered. We show that tightness holds and prove that the left and right tails of the centered maximum exhibit exponential decay. (See Theorem 3.1.1 for a precise statement of this result.)

Tightness is a weaker result than the convergence that we will be establish for the MGFF, but this weaker result is partially offset by the generality of the Gaussian field Y satisfying (1.0.7). The proof of the tightness result in Part I is relatively short compared with that of the convergence result in Part II, but the question of tightness of the centered maximum of the DGFF was an open conjecture for a number of years before being resolved in [5], where the expectation of the maximum of the DGFF was computed up to an order 1 term, from which tightness followed.

Main techniques

Slepian's Lemma

An important tool employed throughout this thesis is *Slepian's Lemma*, which we state in Theorem 2.2.1. The intuition behind Slepian's Lemma is simple: if two centered (that is, point-wise zero-mean) Gaussian fields on the same index set have the same point-wise variance, then the more "disordered" field (i.e., the field with smaller covariance) has the greater supremum. More precisely, if two centered fields X and Y satisfy $Var(X_t) = Var(Y_t)$ and $Cov(X_s, X_t) \leq Cov(Y_s, Y_t)$ for all s, t in the index set, then the supremum of X stochastically dominates the supremum of Y. This result was established by Slepian in [6].

Slepian's Lemma points to the following strategy when studying Gaussian fields: if one desires to obtain tight bounds on the distribution of the supremum of a field, one should compare it to other (carefully chosen) fields with known bounds. This idea will be employed throughout this thesis.

Since the convergence of the centered maximum of the DGFF was established in [4], we would like to compare the MGFF to the DGFF for the convergence result in Part II. However, an immediate difficulty arises: the index of the DGFF is discrete, whereas the index set of the MGFF is the continuum square $[0,1]^2$. At first sight, then, Slepian's Lemma is difficult to apply in this situation.

We surmount this difficulty as follows. First, Slepian's Lemma can still be applied directly to obtain lower bounds on the right tail of the MGFF, since its global maximum is greater than the maximum over a finite subset of $[0,1]^2$. We can therefore compare this maximum with the maximum of the DGFF and obtain lower bounds.

Second, we extend the DGFF to a continuum field in order to compare it to the MGFF using Slepian's Lemma, so that the maximum of this extension stochastically dominates the maximum of the MGFF, while not being much greater than the maximum of the DGFF. This is accomplished by adding a Brownian sheet to each point in the square lattice V_N. The covariance structure of the Brownian sheet allows us to accomplish our goal and to find the correct upper bounds of the right tail of the MGFF. (Brownian sheet is the analog of Brownian motion when the time index set is multidimensional. See Section 4.1 for a definition of the Brownian sheet.)

The tightness result in Part I will be proved by employing an analogous technique to the one described above. We will compare a general log-correlated Gaussian field with the *continuous modified branching random walk* (CMBRW), which is a continuous-time extension of the *modified branching random walk* (MBRW), a field that was introduced Bramson and Zeitouni in [5]. (See Section 4.1 for a definition of the CMBRW.)

Coarse and fine fields of the MGFF

As we will explain in detail in Section 7.1, the MGFF can be decomposed in the following fashion. Let $Q \subseteq [0,1]^2$ be a closed sub-square and denote by $\mathcal{F}$ the sigma-algebra generated by $\{X(\mu) : \mu \in \mathcal{M}$ and is supported on $\partial Q\}$. Following the nomenclature in [4], the Gaussian fields

$$(\mathbb{E}[X(\rho_{R,x}) \mid \mathcal{F}] : \mathrm{supp}(\rho_{R,x}) \subseteq Q)$$

and

$$(X(\rho_{R,x}) - \mathbb{E}[X(\rho_{R,x}) \mid \mathcal{F}] : \mathrm{supp}(\rho_{R,x}) \subseteq Q)$$

are called the *coarse* and *fine fields*, respectively. (Here, $\mathrm{supp}(\rho_{R,x})$ denotes the support of the measure $\rho_{R,x}$ that was defined in (1.0.5).) By basic properties of Gaussian random variables, the coarse and fine fields of the MGFF are independent of each other.

The fine field is independent of any $X(\rho_{R,y})$ with $\rho_{R,y}$ supported on $[0,1]^2 \setminus Q$. This establishes a (spatial) Markov property of the MGFF, which is analogous to the Markov property of Brownian bridge $(B_t)_{t\geq 0}$: if $0 < t_1 < t < t_2$, then $B_t - \mathbb{E}[B_t \mid B_{t_1}, B_{t_2}]$ is independent of B_r for all $r \notin (t_1, t_2)$.

The fine field is also a re-scaled copy of the MGFF. More precisely, suppose for simplicity that $Q = [0,S]^2$, where $0 < S < 1$. Then,

$$(X(\rho_{R,x}) - \mathbb{E}[X(\rho_{R,x}) \mid \mathcal{F}] : \mathrm{supp}(\rho_{R,x}) \subseteq Q) \overset{law}{=} (X(\rho_{x/S,R/S}) : \mathrm{supp}(\rho_{R,x}) \subseteq Q),$$

where the right hand side is again the MGFF on the unit square. This establishes a self-similarity property of the MGFF, which is analogous to the self-similarity property of Brownian bridge $(B_t)_{t\geq 0}$: if $0 < t_1 < t < t_2$, then $B_t - \mathbb{E}[B_t \mid B_{t_1}, B_{t_2}]$ is again a Brownian bridge on $[t_1, t_2]$.

We can decompose the MGFF by partitioning the unit square into a number of disjoint sub-squares of the same size. The fine fields on each sub-square are then i.i.d. copies of the MGFF, which produces a tree-like structure. This decomposition will be used repeatedly in Chapters 8, 9 and 10.

Outline of the thesis

The following outline summarizes each section of the thesis.

Part I, which consists of Chapters 2, 3, 4 and 5, establishes the tightness for the centered maximum of a family of log-correlated Gaussian fields. The main result of Part I is Theorem 3.1.1.

In Chapter 2, we present some introductory material on the theory of general Gaussian fields. In Section 2.1, we define Gaussian fields and state conditions for their existence and almost sure continuity. In Section 2.2, we state the three main tools for the study of the supremum of general Gaussian fields: Slepian's Lemma (Theorem 2.2.1), Borell's Inequality (Theorem 2.2.2) and Fernique's Majorizing Criterion (Theorem 2.2.3).

In Chapter 3, we define the log-correlated Gaussian fields. In Section 3.1, we state our main result for tightness and also give upper bounds for the left and right tails of the maximum, as well as discussing the main idea of the proof. In Section 3.2, we provide motivation for the study of log-correlated Gaussian fields and outline the recent related literature.

In Chapter 4, we introduce the CMBRW and recall the definition of the Brownian sheet. We state some of the properties of these fields and provide proofs of these properties.

In Chapter 5, we compare an arbitrary log-correlated Gaussian field to the CMBRW. In Section 5.1, we employ the CMBRW in order to obtain an upper bound for the right tail of the maximum of the log-correlated field. In Section 5.2, we compare the log-correlated field directly with the CMBRW by restricting it to a discrete index set. This allows us to compute an upper bound on the left tail of the maximum of the log-correlated field.

Part II, which consists of Chapters 6, 7, 8, 9 and 10, is devoted to the proof of convergence of the centered maximum of the MGFF. In Chapter 6, we formally introduce the MGFF. In Sections 6.1 and 6.2, we state the main result of this part and explain the main ideas behind its proof. In Section 6.3, we provide background and related work for this result.

In Chapter 7, we state and prove preliminary results and properties that will be used in the remaining chapters. In Section 7.1, we prove the spatial Markov and the self-similarity properties of the MGFF. As explained before, these two properties imply that the MGFF has a useful tree-like structure, which will be exploited in the remaining chapters. In Section 7.2, we establish continuity of the MGFF and, using Theorem 3.1.1 from Part I, we establish tightness of the centered maximum. In Sections 7.3, 7.4 and 7.5, we recall the definitions of the DGFF, the Brownian sheet and the CMBRW, respectively, and we state and prove some of their properties.

Chapter 8 is the first step in the proof of Theorem 6.1.1. The main result of this chapter is Proposition 8.0.1, which gives precise asymptotics for the right tail of the MGFF and an asymptotic distribution for the point on the unit square where the MGFF is maximized.

Chapter 9 employs the above self-similarity and Markov properties to decompose the MGFF into the coarse and fine fields, as mentioned before. We do this by partitioning the unit square, for a large integer $K > 0$, into (roughly) $K \times K$ disjoint sub-squares of side length (roughly) $1/K$. The main result of Chapter 9 is Proposition 9.1.2, which shows that the global maximum Θ_R^* is approximated, as $R \to \infty$ and then $K \to \infty$, by the maximum of the MGFF among the points that maximize the fine fields on each sub-square.

In Chapter 10, we construct a coupling of the MGFF using the results in the previous chapters. This coupling allows us to prove that the law of the centered maximum is Cauchy with respect to the Lévy metric, from which Theorem 6.1.1 follows quickly. Lastly, we use Theorem 6.1.1 to prove Theorem 6.1.2, thus obtaining a representation for the limit law.

Parts I and II are self-contained. A variation of Part I has appeared as [7] (DOI: 10.1214/EJP.v19-3170). In it, tightness of the centered maximum of log-correlated fields was established and this result was then applied to the two-dimensional MGFF. Part II consists of a separate article that will be submitted in the future. The thesis has been structured so that both parts can be read independently. With this in mind, the auxiliary fields (DGFF, Brownian sheet and CMBRW) are introduced and explained in both parts.

Part I

Tightness of the centered maximum of log-correlated Gaussian fields

Chapter 2

Gaussian fields

In this introductory chapter, we recall the definition of Gaussian random variables and Gaussian fields, and state some properties of these fields that will be used in later chapters. The following definitions can be found in [8, Chapter 1].

2.1 Existence and continuity

A real-valued random variable X with finite mean $x_0 = \mathbb{E}[X]$ and finite variance $\sigma^2 = \mathbb{E}[(X - x_0)^2] > 0$ is said to be *Gaussian* if it has a probability density of the form

$$\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x - x_0)^2}{2\sigma^2}\right) \qquad \text{for all } x \in \mathbb{R}.$$

We also consider deterministic constants to be (degenerate) Gaussian random variables.

A *random field* is a collection of random variables $X = (X_t : t \in T)$ defined over the same probability space, where T is an arbitrary index set. We say that this collection is a *Gaussian field* if, for any finite subset $K \subseteq T$ and any $\alpha = (\alpha_k)_{k \in K} \in \mathbb{R}^K$, the random variable $\sum_{k \in K} \alpha_k X_k$ is Gaussian. If T is finite, we say that X is a *Gaussian vector*.

Given a Gaussian field $(X_t : t \in T)$, we define the *mean function* $m : T \to \mathbb{R}$ and the *covariance function* $V : T \times T \to \mathbb{R}$ by

$$m(t) = \mathbb{E}[X_t] \qquad \text{and} \qquad V(s,t) = Cov(X_s, X_t)$$

for all $s, t \in T$. The covariance function V is positive semi-definite: letting $K \subseteq T$ be finite and $\alpha = (\alpha_k)_{k \in K} \in \mathbb{R}^K$, we have

$$\sum_{j,k \in K} \alpha_j V(j,k) \alpha_k = Var\left(\sum_{k \in K} \alpha_k X_k\right) \geq 0.$$

Suppose, on the other hand, that we are given a function $m : T \to \mathbb{R}$ and a positive semi-definite function $V : T \times T \to \mathbb{R}$. Then, there exists a Gaussian field with index set T, mean function m and covariance function V.

The proof of this statement is straightforward if T is finite and, for all $s, t \in T$, $m(t) = 0$ and $V(s,t) = 1_{\{s=t\}}$: In this case, we can exhibit the multivariate probability density

$$\frac{1}{\sqrt{(2\pi)^{|T|}}} \exp\left(-\frac{1}{2}\|x\|^2\right) \qquad \text{for all } x \in \mathbb{R}^T, \tag{2.1.1}$$

where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^T$ and $|T|$ denotes the cardinality of T. A Gaussian vector $(X_t : t \in T)$ with probability density as in (2.1.1) is called a *standard Gaussian vector.*

If T is finite and V is an arbitrary positive semi-definite matrix, the existence statement is a simple exercise in linear algebra: Since V is positive semi-definite, it can be written as $V = RR'$ for some matrix $R \in \mathbb{R}^{T \times T}$ (where R' denotes the transpose of R). If $Z = (Z_t : t \in T)$ is a standard Gaussian vector, then the field $X = (X_t : t \in T)$ defined by

$$X_t = m(t) + \sum_{s \in T} R(t,s) Z_s$$

is a Gaussian vector with mean function m and covariance function V.

When T is infinite, the existence of the Gaussian field follows from the existence in the finite case and Kolmogorov's Extension Theorem (see [9, Appendix I]), which we state below.

Denote by $\mathcal{B}$ the Borel sigma-algebra on $\mathbb{R}$ and, for any set T, denote by $\mathcal{B}^T$ the product sigma-algebra on $\mathbb{R}^T$ generated by finite-dimensional Borel sets. Also, for any two sets S and T with $S \subseteq T$, let $\pi_{T,S}$ be the canonical projection from $\mathbb{R}^T$ onto $\mathbb{R}^S$.

Theorem 2.1.1 (Kolmogorov's Extension Theorem)**.** *Let T be an arbitrary set. For each finite subset $K \subseteq T$, let ν_K be a probability measure on $(\mathbb{R}^K, \mathcal{B}^K)$ and suppose that* *the probability measures ν_K satisfy the following consistency condition: For any finite sets K and L with $K \subseteq L \subseteq T$, and for all $A \in \mathcal{B}^K$,*

$$\nu_L(\pi_{L,K}^{-1}(A)) = \nu_K(A).$$

Then, there exists a unique probability measure ν on $(\mathbb{R}^T, \mathcal{B}^T)$ such that, for any finite subset $K \subseteq T$ and all $A \in \mathcal{B}^K$,

$$\nu(\pi_{T,K}^{-1}(A)) = \nu_K(A).$$

As a consequence of Kolmogorov's Extension Theorem, the mean and covariance functions m and V uniquely determine the law that the Gaussian field X induces on $(\mathbb{R}^T, \mathcal{B}^T)$.

Suppose now that X is a Gaussian field defined on $T \subseteq \mathbb{R}^d$. Using the metric from $\mathbb{R}^d$, we can study the continuity, or lack thereof, of X.

We recall the notion of a *modification* of a random field. If $X = (X_t : t \in T)$ and $Y = (Y_t :\in T)$ are two random fields defined on the same index set T, we say that X is a modification of Y (and vice versa) if $\mathbb{P}(X_t = Y_t) = 1$ for all $t \in T$. In particular, X and Y induce the same law on $(\mathbb{R}^T, \mathcal{B}^T)$. The following theorem (see [10, Theorem 1.4.17]) provides sufficient conditions for the existence of continuous modifications when the index set is the d-dimensional unit box $[0, 1]^d$.

Theorem 2.1.2 (Kolmogorov's Continuity Criterion)**.** *Let X be a random field with index set $[0, 1]^d$ for some integer $d \geq 1$. Suppose that there exist constants $\alpha, \beta, C \in (0, \infty)$ such that*

$$\mathbb{E}[(X_s - X_t)^\alpha] \leq C\|s - t\|^{d+\beta}$$

for all $s, t \in [0, 1]^d$, where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^d$. Then, there exists a continuous modification of X.

For a Gaussian field X with index set $[0, 1]^d$, a sufficient condition for the existence of a continuous modification is the existence of constants constants $\gamma, C \in (0, \infty)$ such that, for all $s, t \in [0, 1]^d$,

$$\mathbb{E}[(X_s - X_t)^2] \leq C\|s - t\|^\gamma.$$

This statement follows from Kolmogorov's Continuity Criterion and basic properties of the moments of Gaussian random variables.

2.2 Supremum of a Gaussian field

We now state three of the most fundamental results in the theory of Gaussian fields. These results are stated for centered (i.e., point-wise zero-mean) Gaussian fields.

Slepian's Lemma (see [8, Corollary 2.4]), which was mentioned in the introduction, provides sufficient conditions for the supremum of one field to stochastically dominate another.

Theorem 2.2.1 (Slepian's Lemma)**.** *Let X and Y be almost surely bounded, centered Gaussian fields on T, such that*

$$Var(X_t) = Var(Y_t)$$

and

$$Cov(X_s, X_t) \leq Cov(Y_s, Y_t)$$

for all $s, t \in T$. Then, for all $x \in \mathbb{R}$,

$$\mathbb{P}\left(\sup_{t \in T} X_t > x\right) \geq \mathbb{P}\left(\sup_{t \in T} Y_t > x\right).$$

The next result, known as Borell's Inequality, informs us that the right tail of the supremum of a Gaussian field decays like the right tail of a Gaussian random variable with variance equal to the supremum of the variance of the field. This important result was proved by Borell in [11] in a highly abstract setting by employing isoperimetric inequalities. The following statement can be found in [8, Theorem 2.1].

Theorem 2.2.2 (Borell's Inequality)**.** *Let X be an almost surely bounded, centered Gaussian field on T. Then,*

$$\mathbb{E}\left[\sup_{t \in T} X_t\right] < \infty$$

and, for all $x > 0$,

$$\mathbb{P}\left(\sup_{t \in T} X_t - \mathbb{E}\left[\sup_{t \in T} X_t\right] > x\right) \leq \exp\left(-\frac{x^2}{2\sigma_T^2}\right),$$

where

$$\sigma_T^2 := \sup_{t \in T} Var(X_t).$$

Note that Borell's Inequality requires knowing (an upper bound of) the expected supremum of the field. Fernique's Majorizing Criterion (see [8, Theorem 4.1]) provides such an upper bound.

If X is a Gaussian field on T, then

$$\rho(s,t) := \sqrt{\mathbb{E}[(X_s - X_t)^2]}$$

is a pseudo-metric on T. It is called the *canonical pseudo-metric* on T for the Gaussian field X. The next theorem relates the canonical pseudo-metric with the supremum of X.

Theorem 2.2.3 (Fernique's Majorizing Criterion)**.** *Let X be a centered Gaussian field on T. For $\epsilon > 0$, let $B(t,\epsilon) := \{s \in T : \rho(t,s) < \epsilon\}$ denote the ϵ-ball under the canonical pseudo-metric ρ. Assume that T is totally bounded under ρ. Then,*

$$\mathbb{E}\left[\sup_{t\in T} X_t\right] \leq C \inf_{\mu} \sup_{t\in T} \int_0^{\infty} \sqrt{-\log(\mu(B(t,\epsilon)))}d\epsilon, \tag{2.2.1}$$

where $C > 1$ is an absolute constant and the infimum is taken over all probability measures μ on T.

The probability measure μ is said to be a "majorizing measure" if the right hand side of (2.2.1) is finite. These measures were introduced by Fernique in [12]. Note that the existence of a majorizing measure is a sufficient condition for the finiteness of the expected supremum. In [13], Talagrand showed that the existence of a majorizing measure is also a necessary condition.

Chapter 3

Log-correlated Gaussian fields

3.1 Setup and main result

Let $\left\{\left(Y_\epsilon^x : x \in [0,1]^d\right)\right\}_{\epsilon>0}$ be a family of centered Gaussian fields indexed by the d-dimensional unit box $[0,1]^d$, where d is any positive integer. Suppose that the family satisfies, for some constant $0 < C_Y < \infty$ and all $x, y \in [0,1]^d$, $\epsilon > 0$,

$$|Cov\,(Y_\epsilon^x, Y_\epsilon^y) + \log\left(\max\{\epsilon, \|x-y\|\}\right)| \leq C_Y \tag{3.1.1}$$

and

$$\mathbb{E}\left[(Y_\epsilon^x - Y_\epsilon^y)^2\right] \leq C_Y \epsilon^{-1} \|x-y\| \qquad \text{if} \qquad \|x-y\| \leq \epsilon, \tag{3.1.2}$$

where $\|\cdot\|$ is Euclidean distance. Display (3.1.1) implies that the covariance is logarithmic for distant points and that the variance is nearly constant. The second condition is imposed so that the field does not vary too much for close points. Display (3.1.2), basic relations between the moments of Gaussian random variables and Kolmogorov's Continuity Criterion (see [10, Theorem 1.4.17]) imply that the fields have continuous modifications.

When $d = 2$, an example of a field satisfying (3.1.1) and (3.1.2) is the bulk of the mollified Gaussian free field (MGFF), which will be the object of our attention in Part II.

Set $m_\epsilon = m_{\epsilon,d} = \sqrt{2d}\log(1/\epsilon) - \frac{3/2}{\sqrt{2d}}\log\log(1/\epsilon)$. The main result of this part is the following.

Theorem 3.1.1. *There exist constants $0 < c, C < \infty$ (depending on C_Y and d) and a small $\epsilon_0 > 0$ (depending on C_Y and d), such that, for all $\epsilon \in (0, \epsilon_0]$ and all $\lambda \geq 0$,*

$$\mathbb{P}\left(\left|\max_{x\in[0,1]^d} Y_\epsilon^x - m_\epsilon\right| \geq \lambda\right) \leq Ce^{-c\lambda}. \tag{3.1.3}$$

Theorem 3.1.1 implies, in particular, that $\left\{\max_{x\in[0,1]^d} Y_\epsilon^x - m_\epsilon : \epsilon \in (0, \epsilon_0]\right\}$ is tight and that, for all $\epsilon \in (0, \epsilon_0]$,

$$\left|\mathbb{E}\left[\max_{x\in[0,1]^d} Y_\epsilon^x\right] - m_\epsilon\right| \leq C$$

for some constant C depending on C_Y and d.

The main idea of the proof of Theorem 3.1.1 is to use Slepian's Lemma (see [8, Corollary 2.4]) to compare the maximum of the field Y_ϵ with the maximum of the continuous modified branching random walk (CMBRW), which is a continuous-time version of the modified branching random walk (MBRW), a field introduced by Bramson and Zeitouni in [5]. Since Slepian's Lemma only allows comparison of fields with the same index set, we will add an appropriately chosen independent continuous field to the CMBRW. Adding an independent continuous field to the CMBRW does not change the maximum much, provided the continuous field is small and smooth enough. These auxiliary fields are defined in detail in Section 4.1. After defining the fields, we compare the right and left tails in Sections 5.1 and 5.2.

A comment on constants: c will always denote a small positive constant and C will always denote a large positive constant. Both constants are allowed to change from line to line. The dependence of the constants will be explicit or will be clear from the context. The phrase "absolute constant" will refer to fixed numbers that are independent of everything.

3.2 Related work

Our approach is motivated by recent advances in the study of the two-dimensional discrete Gaussian free field (DGFF). In [5], Bramson and Zeitouni computed the expected maximum of the DGFF up to an order 1 error and concluded tightness of the centered maximum. In [14], Ding obtained bounds on the right and left tail of the centered maximum of the DGFF. Later on, in [4], Bramson, Ding and Zeitouni proved convergence in distribution of the centered maximum. The approach of this line of research is to use first and second moment methods, together with decomposition properties of the DGFF, to obtain good estimates on tail events. Previous work on the DGFF includes [15], where Bolthausen, Deuschel and Giacomin obtained asymptotics for the maximum of the DGFF, and [16], where Daviaud studied the extreme points of the DGFF. On the other hand, previous work on the continuous Gaussian free field (CGFF) includes [17], where Hu, Miller, and Peres studied the Hausdorff dimension of the "thick points" of the MGFF, which are closely related to the work of Daviaud. We also mention [18] for a nice discussion of Gaussian fields induced by Markov processes, and [2] for a survey on the CGFF.

As mentioned before, our approach consists on extending the CMBRW by Brownian sheet, so that it is possible to compare the extended field with scaled log-correlated continuous fields. Log-correlated Gaussian fields are subject of current interest (see [19], [20], [21]). In particular, in [20], Madaule proved convergence for stationary centered Gaussian fields $\big(Z_\epsilon(x) : x \in [0,1]^d\big)$ whose covariance satisfies

$$Cov(Z_\epsilon(0), Z_\epsilon(x)) = \int_0^{\log(1/\epsilon)} k(e^r x)dr,$$

where the fixed kernel $k : \mathbb{R}^d \to \mathbb{R}$ is of class C^1, vanishes outside $[-1,1]^d$, and satisfies $k(0) = 1$. Theorem 3.1.1 has weaker conditions on the covariance structure, and consequently, only tightness is achieved.

In [21], the authors proved the so called "Freezing Theorem for GFF in planar domains" for a sequence of Gaussian fields approximating the continuous GFF by cutting-off white noise, so that the covariance kernel is proportional to the function $G_t : [0,1]^2 \times [0,1]^2 \to \mathbb{R}$ given by

$$G_t(x,y) = \int_{e^{-t}}^{\infty} p_{\partial[0,1]^2}(r,x,y)dr,$$

where $p_{\partial[0,1]^2}(r,x,y)$ is the transition probability density of a Brownian motion killed at $\partial[0,1]^2$.

Chapter 4

Auxiliary fields

4.1 Definition and properties of the auxiliary fields

In this section, we rigorously define the fields we mentioned in Chapter 3. A number of properties of these fields will be stated; the proofs of these properties will be given at the end of the chapter.

In order to define these fields, it will be notationally more convenient to use $[0,1)^d$ instead of $[0,1]^d$ as the index set. This will not affect the proof of Theorem 3.1.1 because the supremum of Y_ϵ over $[0,1)^d$ is the same, due to continuity, as the maximum over $[0,1]^d$.

Continuous modified branching random walk

We first divide $[0,1)^d$ into boxes of side length $\epsilon > 0$. Let $V_\epsilon = \left(\epsilon\mathbb{Z}^d\right) \cap [0,1)^d$ and, for $v = (v_i)_{1\le i\le d} \in V_\epsilon$, let $\square_\epsilon^v = (\prod_{1\le i\le d}[v_i, v_i+\epsilon)) \cap [0,1)^d$. Moreover, if $x \in \square_\epsilon^v$, let $[x] := v$. The set V_ϵ is, of course, a discretized version of $[0,1)^d$.

We now define the *continuous modified branching random walk* (CMBRW) as the centered Gaussian field $\{\xi_\epsilon^v(t) : v \in V_\epsilon, 0 \le t \le \log(1/\epsilon)\}$ with covariance structure

$$Cov(\xi_\epsilon^v(t), \xi_\epsilon^u(s)) = \int_0^{\min\{t,s\}} \prod_{1\le i\le d} (1 - e^r\,|v_i - u_i|)_+ dr \tag{4.1.1}$$

for all $0 \le t, s \le \log(1/\epsilon)$ and $v, u \in V_\epsilon$, where $(\cdot)_+ = \max\{\cdot, 0\}$. For simplicity, write $\xi_\epsilon^v = \xi_\epsilon^v(\log(1/\epsilon))$.

Note that, for each point $v \in V_\epsilon$, the process $(\xi_\epsilon^v(t))_{t\geq 0}$ is a standard Brownian motion. Moreover, for each pair $v, u \in V_\epsilon$, the Brownian motions are correlated until $t = -\log \|v-u\|_\infty$, at which time their increments become independent. The end time is $t = \log(1/\epsilon)$, because, for the "usual" d-ary branching Brownian motion, it takes $\log(1/\epsilon)$ units of time to generate $|V_\epsilon|$ particles (see the proof of Proposition 4.2.3 for a definition of the usual d-ary branching Brownian motion).

It will be proved (see Proposition 4.2.1) that the CMBRW exists and that it satisfies

$$Var(\xi_\epsilon^v) = \log(1/\epsilon) \tag{4.1.2}$$

and, for $v \neq u$ (so that $\|v-u\|_\infty \geq \epsilon$),

$$-\log \|v-u\|_\infty - C \leq Cov(\xi_\epsilon^v, \xi_\epsilon^u) \leq -\log \|v-u\|_\infty \tag{4.1.3}$$

for some constant C depending on d. The CMBRW also satisfies (see Proposition 4.2.2)

$$\mathbb{P}\left(\max_{v\in V_\epsilon} \xi_\epsilon^v \geq m_\epsilon\right) \geq c > 0, \tag{4.1.4}$$

where c is a constant depending only on d. It will also be proved (see Proposition 4.2.3) that there exist constants $0 < c, C < \infty$ (depending on d) such that

$$\mathbb{P}\left(\max_{v\in A} \xi_\epsilon^v \geq m_\epsilon + z\right) \leq C\left(\epsilon^d |A|\right)^{1/2} e^{-cz} \tag{4.1.5}$$

for all $A \subseteq V_\epsilon$, $z \in \mathbb{R}$ and $\epsilon > 0$ small enough, where $|A|$ is the cardinality of A.

Brownian sheet

As mentioned before, we will need an additional continuous Gaussian field. For $x = (x_i)_{1\leq i\leq d} \in \mathbb{R}_+^d$, let ψ^x denote the centered standard Brownian sheet. Recall that it satisfies, for all $x, y \in \mathbb{R}_+^d$,

$$\mathbb{E}\left[\psi^x \psi^y\right] = \prod_{1\leq i\leq d} \min\left\{x_i, y_i\right\}.$$

Define a new field $\left(\psi_\epsilon^x : x \in [0,1)^d\right)$, depending on a parameter $p \geq 1$, as follows. For $v \in V_\epsilon$, let l be the linear map from $\Box_\epsilon^v$ onto $[p, 2p)^d$ sending v to $(p)_{1\leq i\leq d} = (p, p, \ldots, p)$. Set

$$(\psi_\epsilon^x : x \in \Box_\epsilon^v) \stackrel{d}{:=} \left(\psi^{l(x)} : x \in \Box_\epsilon^v\right) = \left(\psi^{l(x)} : l(x) \in [p, 2p)^d\right) \tag{4.1.6}$$

for each $v \in V_\epsilon$, and choose ψ_ϵ^x and ψ_ϵ^y to be independent if $[x] \neq [y]$. Note that the collection of fields $\{(\psi_\epsilon^x : x \in \Box_\epsilon^v)\}_{v \in V_\epsilon}$ consists of i.i.d. copies of Brownian sheet on $[p, 2p)^d$. Using the covariance structure of the Brownian sheet, it is not hard to see that

$$p^d \leq Var\left(\psi_\epsilon^x\right) \leq (2p)^d \tag{4.1.7}$$

for all $x \in [0,1)^d$, and that (see Proposition 4.2.5)

$$p^d \epsilon^{-1} \left\|x - y\right\|_1 \leq \mathbb{E}\left[\left(\psi_\epsilon^x - \psi_\epsilon^y\right)^2\right] \leq (2p)^d \epsilon^{-1} \left\|x - y\right\|_1 \tag{4.1.8}$$

for all $[x] = [y]$. Note that p can be chosen as large as desired.

To understand the motivation behind the previous definitions, we invite the reader to compare the bounds (3.1.1) and (3.1.2) with (4.1.3) and (4.1.8), respectively. These bounds will be used in Chapter 5.

4.2 Proofs of Chapter 4

We prove here the claims made in Chapter 4.

Proposition 4.2.1. *The CMBRW, defined by display* (4.1.1)*, exists and satisfies*

$$Var(\xi_\epsilon^v(t)) = t$$

for all $0 \leq t \leq \log(1/\epsilon)$ *and all* $v \in V_\epsilon$*, and*

$$t - C \leq Cov(\xi_\epsilon^v(t), \xi_\epsilon^w(t)) \leq t$$

for all $0 \leq t \leq -\log\left\|v - w\right\|_\infty$ *and all* $v, w \in V_\epsilon$*, where* C *is a constant depending on the dimension.*

Proof. We show that the mapping $(V_\epsilon \times [0, \log(1/\epsilon)])^2 \to \mathbb{R}$ given by

$$((v,t),(u,s)) \mapsto \int_0^{\min\{t,s\}} \prod_{1 \leq i \leq d} (1 - e^r \left|v_i - u_i\right|)_+ dr$$

is positive semi-definite. Note first that

$$\prod_{1 \leq i \leq d} (1 - e^r \left|v_i - u_i\right|)_+ = \int_{\mathbb{R}^d} 1_{A(v,r)}(z) 1_{A(u,r)}(z) dz,$$

where dz is d-dimensional Lebesgue measure and $A(v, r)$ is the d-dimensional box of side length 1, centered at $e^r v$. Let $\{(v^\alpha, t^\alpha)\}_\alpha$ be any finite subset of $V_\epsilon \times [0, \log(1/\epsilon)]$, and let $\{c_\alpha\}_\alpha$ be any finite set of real numbers. Then, applying the previous display, we obtain

$$\sum_{\alpha,\beta} c_\alpha c_\beta \int_0^{\min\{t^\alpha, t^\beta\}} \prod_{1\le i\le d} (1 - e^r \left|v_i^\alpha - v_i^\beta\right|)_+ dr$$
$$= \int_0^\infty \int_{\mathbb{R}^d} \sum_{\alpha,\beta} c_\alpha c_\beta 1_{[0,t^\alpha]}(r) 1_{[0,t^\beta]}(r) 1_{A(v^\alpha, r)}(z) 1_{A(v^\beta, r)}(z)\, dz\, dr$$
$$= \int_0^\infty \int_{\mathbb{R}^d} \left(\sum_\alpha c_\alpha 1_{[0,t^\alpha]}(r) 1_{A(v^\alpha, r)}(z) \right)^2 dz\, dr \ge 0,$$

as desired. This shows that the CMBRW exists.

For any $v \in V_\epsilon$ and $t \le \log(1/\epsilon)$,

$$Var(\xi_\epsilon^v(t)) = \int_0^t \prod_{1\le i\le d} (1)\, dr = t.$$

Moreover, if $v \ne w$,

$$\prod_{1\le i\le d} (1 - e^r |v_i - w_i|)_+ \begin{cases} > 0 & \text{if } r < -\log \|v - w\|_\infty \\ = 0 & \text{if } r \ge -\log \|v - w\|_\infty \end{cases}.$$

Therefore, if $t < -\log \|v - w\|_\infty$,

$$t \ge Cov(\xi_\epsilon^v(t), \xi_\epsilon^w(t)) \ge \int_0^t \prod_{1\le i\le d} (1 - e^r |v_i - w_i|)\, dr \ge \int_0^t (1 - e^r \|v - w\|_\infty)^d\, dr,$$

where the last inequality follows because $1 - e^r |v_i - w_i| \ge 1 - e^r \|v - w\|_\infty$ for all $i \in \{1, 2, \ldots, d\}$. Expanding and integrating, we obtain that the previous display is

$$\ge t + \sum_{k=1}^d \binom{d}{k} (-1)^k \|v - w\|_\infty^k \left(\frac{e^{kt} - 1}{k} \right) \ge t - \sum_{k=1}^d \binom{d}{k} \|v - w\|_\infty^k \left(e^{kt} + 1 \right). \quad (4.2.1)$$

But, since $\|v - w\|_\infty \le 1$ and $t < -\log \|v - w\|_\infty$, we have

$$\|v - w\|_\infty^k \left(e^{kt} + 1 \right) \le \left(\|v - w\|_\infty e^{-\log \|v - w\|_\infty} \right)^k + 1 \le 2.$$

Therefore, display (4.2.1) is

$$\geq t - 2\sum_{k=1}^{d}\binom{d}{k} \geq t - C$$

for some constant $C < \infty$ depending on d only. Similarly, if $t \geq -\log\|v-w\|_\infty$,

$$-\log\|v-w\|_\infty \geq Cov(\xi_\epsilon^v(t), \xi_\epsilon^w(t)) \geq -\log\|v-w\|_\infty - C.$$

$\square$

Proposition 4.2.2. *Let $(\xi_\epsilon^v : v \in V_\epsilon)$ be the CMBRW and let m_ϵ be the number defined in the line preceding Theorem 3.1.1. Then, there exists a constant $c > 0$ (depending on the dimension) such that*

$$\mathbb{P}\left(\max_{v\in V_\epsilon}\xi_\epsilon^v \geq m_\epsilon\right) \geq c.$$

Proof. We use a second moment method. Let $T = T_\epsilon = \log(1/\epsilon)$ and let

$$\begin{aligned} A_v &= \left\{\xi_\epsilon^v \geq m_\epsilon, \xi_\epsilon^v(t) \leq \frac{m_\epsilon}{T}t + 1 \text{ for all } 0 \leq t \leq T\right\}, \\ Z &= \sum_{v\in V_\epsilon} 1_{A_v}. \end{aligned}$$

Note that

$$\mathbb{P}\left(\max_{v\in V_\epsilon}\xi_\epsilon^v \geq m_\epsilon\right) \geq \mathbb{P}(Z>0) \geq \frac{(\mathbb{E}[Z])^2}{\mathbb{E}[Z^2]}, \tag{4.2.2}$$

where the second inequality follows from Cauchy-Schwarz. We first compute a lower bound for $\mathbb{E}[Z]$. Note that

$$\mathbb{E}[Z] = \epsilon^{-d}\mathbb{P}(A_v).$$

Let $\bar{\xi}_\epsilon^v(t) = \xi_\epsilon^v(t) - \frac{m_\epsilon}{T}t$. Define a probability measure $\mathbb{Q}$ by

$$\frac{d\mathbb{P}}{d\mathbb{Q}} = \exp\left(-\frac{m_\epsilon}{T}\bar{\xi}_\epsilon^v(T) - \frac{m_\epsilon^2}{2T}\right).$$

Girsanov's Theorem (see [10, Theorem 6.8.8]) implies that $\bar{\xi}_\epsilon^v(t)$ is Brownian motion under $\mathbb{Q}$. Note that

$$\begin{aligned} \mathbb{P}(A_v) &= \int_{A_v} \exp\left(-\frac{m_\epsilon}{T}\bar{\xi}_\epsilon^v(T) - \frac{m_\epsilon^2}{2T}\right) d\mathbb{Q} \geq \exp\left(-\frac{m_\epsilon}{T} - \frac{m_\epsilon^2}{2T}\right)\mathbb{Q}(A_v) \\ &\geq ce^{-\sqrt{2d}}\epsilon^d T^{3/2}\mathbb{Q}(A_v) \end{aligned}$$

for some absolute constant $c > 0$. It follows easily from the reflection principle (see [1, Theorem 2.19]) that $\mathbb{Q}(A_v) = \mathbb{Q}(\bar{\xi}_\epsilon^v \geq 0, \bar{\xi}_\epsilon^v(t) \leq 1 \text{ for all } 0 \leq t \leq T) \geq cT^{-3/2}$ for some absolute constant $c > 0$. Combining the three previous displays, we obtain

$$\mathbb{E}\left[Z\right] \geq c \tag{4.2.3}$$

for some constant $c > 0$, depending on the dimension d.

We now compute an upper bound for $\mathbb{E}\left[Z^2\right]$. Note that

$$\mathbb{E}\left[Z^2\right] = \sum_{v,w \in V_\epsilon} \mathbb{P}\left(A_v \cap A_w\right) = \sum_{v,w \in V_\epsilon} \mathbb{P}\left(\bar{\xi}_\epsilon^v, \bar{\xi}_\epsilon^w \geq 0, \bar{\xi}_\epsilon^v(t), \bar{\xi}_\epsilon^w(t) \leq 1 \text{ for all } 0 \leq t \leq T\right). \tag{4.2.4}$$

Both $\xi_\epsilon^v(\cdot), \xi_\epsilon^w(\cdot)$ are Brownian motions, which have independent increments starting at time $s = s_{v,w} = -\log\left(\max\left\{\epsilon, \|v-w\|_\infty\right\}\right)$. Therefore,

$$\begin{aligned}
&\mathbb{P}\left(A_v \cap A_w\right) \\
&\quad\leq \sum_{-\infty < x,y \leq 1} p(x)p(y)\times \\
&\qquad\quad \mathbb{P}\left(\bar{\xi}_\epsilon^v(t), \bar{\xi}_\epsilon^w(t) \leq 1 \text{ for all } t \in [0,s], \bar{\xi}_\epsilon^v(s) \in [x-1,x], \bar{\xi}_\epsilon^w(s) \in [y-1,y]\right) \\
&\quad\leq \sum_{-\infty < y \leq x \leq 1} 2p(x)p(y)\times \\
&\qquad\quad \mathbb{P}\left(\bar{\xi}_\epsilon^v(t), \bar{\xi}_\epsilon^w(t) \leq 1 \text{ for all } t \in [0,s], \bar{\xi}_\epsilon^v(s) \in [x-1,x], \bar{\xi}_\epsilon^w(s) \in [y-1,y]\right),
\end{aligned} \tag{4.2.5}$$

where

$$p(x) = \sup_{z \in [x-1,x]} \times \mathbb{P}\left(\bar{\xi}_\epsilon^v(t) \leq 1 - z \text{ for all } t \in [0, T-s], \bar{\xi}_\epsilon^v(T-s) \geq -z\right).$$

Assume $0 < s < T$. Applying Girsanov's Theorem and the reflection principle, we obtain

$$p(x) \leq C \exp\left(\frac{m_\epsilon}{T}x - \frac{m_\epsilon^2}{2T^2}(T-s)\right)\frac{(1-x)}{(T-s)^{3/2}}$$

for some constant C. Therefore, from (4.2.5) and the previous display,

$$\begin{aligned}
&\mathbb{P}\left(A_v \cap A_w\right) \\
&\quad \leq \sum_{-\infty<y\leq x\leq 1} Cp(x)^2 \times \\
&\qquad\quad \mathbb{P}\left(\bar{\xi}_\epsilon^v(t), \bar{\xi}_\epsilon^w(t) \leq 1 \text{ for all } t \in [0,s], \bar{\xi}_\epsilon^v(s) \in [x-1,x], \bar{\xi}_\epsilon^w(s) \in [y-1,y]\right) \\
&\quad \leq \sum_{-\infty<x\leq 1} Cp(x)^2 \mathbb{P}\left(\bar{\xi}_\epsilon^v(t) \leq 1 \text{ for all } t \in [0,s], \bar{\xi}_\epsilon^v(s) \in [x-1,x]\right).
\end{aligned}$$

Applying Girsanov's Theorem and the reflection principle again,

$$\begin{aligned}
\mathbb{P}\left(A_v \cap A_w\right) &\leq C \sum_{-\infty<x\leq 1} p(x)^2 \exp\left(-\frac{m_\epsilon}{T}x - \frac{m_\epsilon^2}{2T^2}s\right)\frac{(1-x)}{s^{3/2}} \\
&\leq C\frac{1}{(T-s)^3 s^{3/2}} \exp\left(-\frac{m_\epsilon^2}{2T^2}(2T-s)\right)
\end{aligned} \tag{4.2.6}$$

for some constant C.

Consider now the case $s=0$. Then, the independence between $\xi_\epsilon^v(\cdot)$ and $\xi_\epsilon^w(\cdot)$ implies

$$\begin{aligned}
\mathbb{P}\left(A_v \cap A_w\right) = \mathbb{P}\left(A_v\right)^2 &= \mathbb{P}\left(\bar{\xi}_\epsilon^v(t) \leq 1 \text{ for all } t \in [0,T], \bar{\xi}_\epsilon^v(T) \geq 0\right)^2 \\
&\leq C\frac{1}{T^3}\exp\left(-\frac{m_\epsilon^2}{T}\right),
\end{aligned} \tag{4.2.7}$$

where the last bound follows from Girsanov's Theorem and the reflection principle. In the case $s=T$,

$$\mathbb{P}\left(A_v \cap A_w\right) \leq \mathbb{P}\left(A_v\right) \leq C\frac{1}{T^{3/2}}\exp\left(-\frac{m_\epsilon^2}{2T}\right). \tag{4.2.8}$$

In consequence, for any pair $v, w \in V_\epsilon$, displays (4.2.6), (4.2.7) and (4.2.8) imply

$$\mathbb{P}\left(A_v \cap A_w\right) \leq C\frac{1}{((T-s)\vee 1)^3 (s \vee 1)^{3/2}} \exp\left(-\frac{m_\epsilon^2}{2T^2}(2T-s)\right),$$

where $(\cdot \vee \cdot) = \max\{\cdot,\cdot\}$. For any fixed $v \in V_\epsilon$, there are $O(e^{(d-1)(T-s)})$ points w such that $-\log \|v-w\|_\infty = s$. Therefore, from (4.2.4) and the previous display, we obtain

$$\begin{aligned}
\mathbb{E}\left[Z^2\right] &\le C \sum_{0\le s\le T} |V_\epsilon|\, e^{(d-1)(T-s)} \frac{1}{\left((T-s)\vee 1\right)^3 (s\vee 1)^{3/2}} \exp\left(-\frac{m_\epsilon^2}{2T^2}(2T-s)\right) \\
&\le C + C \sum_{0<s<T} |V_\epsilon|\, e^{(d-1)(T-s)} \frac{\exp\left(-\frac{m_\epsilon^2}{2T^2}(2T-s)\right)}{(T-s)^3\, s^{3/2}} \\
&= C + C \sum_{0<s<T} e^{dT} e^{(d-1)(T-s)} \frac{\exp\left(-\frac{m_\epsilon^2}{2T^2}(2T-s)\right)}{(T-s)^3\, s^{3/2}}.
\end{aligned}$$

But

$$\begin{aligned}
&\sum_{0<s<T} e^{dT} e^{(d-1)(T-s)} \frac{\exp\left(-\frac{m_\epsilon^2}{2T^2}(2T-s)\right)}{(T-s)^3\, s^{3/2}} \\
&\quad \le \sum_{0<s<T} e^{d(2T-s)} \frac{\exp\left(\left(-d+\frac{3\log T}{2T}\right)(2T-s)\right)}{(T-s)^3\, s^{3/2}} = \sum_{0<s<T} \frac{\exp\left(\frac{3}{2}\frac{\log T}{T}(2T-s)\right)}{(T-s)^3 s^{3/2}} \\
&\quad \le C \sum_{0<s<T/2} \frac{1}{s^{3/2}} + \sum_{T/2\le s<T} \frac{\exp\left(\frac{3}{2}\frac{\log T}{T}(T-s)\right)}{(T-s)^3} \frac{T^{3/2}}{s^{3/2}} \\
&\quad \le C + C \sum_{0<s\le T/2} \frac{T^{3s/2T}}{s^3} \le C < \infty,
\end{aligned}$$

because the last expression is (eventually) decreasing in T. Proposition 4.2.2 follows from the previous display, (4.2.2) and (4.2.3). $\square$

Proposition 4.2.3. *Let $(\xi_\epsilon^v : v \in V_\epsilon)$ be the CMBRW and let m_ϵ be the number defined in the line preceding Theorem 3.1.1. Then, there exist constants $0 < c, C < \infty$ (depending on the dimension d) such that*

$$\mathbb{P}\left(\max_{v\in A} \xi_\epsilon^v \ge m_\epsilon + z\right) \le C\left(\epsilon^d |A|\right)^{1/2} e^{-cz}$$

for all $A \subseteq V_\epsilon$, $z \in \mathbb{R}$ and $\epsilon > 0$ small enough.

Proof. We define the d-ary *branching Brownian motion* (BBM) as follows. Let $\epsilon = 2^{-n}$ for some $n \in \mathbb{N}$. At each time $T_k = k\log 2; k = 0, 1, \ldots, n$, we partition $[0,1)^d$ into 2^{kd} disjoint boxes of side length 2^{-k}. For a pair $v, w \in V_\epsilon$, denote by $l(v,w)$ the first time that v, w lie in different boxes of the partition. With this notation, define the BBM as the Gaussian field $(\eta_\epsilon^v(t) : v \in V_\epsilon, t \in [0, T_n])$ with

$$Cov(\eta_\epsilon^v(t), \eta_\epsilon^w(s)) = \min\{t, s, l(v, w)\}.$$

For simplicity, let $T = T_n$ and $\eta_\epsilon^v = \eta_\epsilon^v(T)$. It is not hard to show that such a field exists. Note that our BBM can be interpreted as a branching Brownian motion that splits every $\log 2$ units of time into 2^d independent Brownian motions. Following the argument given in [22, Lemma 3.7], one can show that there exists C (depending on the dimension) such that

$$\mathbb{P}\left(\max_{v \in A} \xi_\epsilon^v \geq m_\epsilon + \lambda\right) \leq C\mathbb{P}\left(\max_{v \in A} \eta_{\epsilon/C}^v \geq m_\epsilon + \lambda\right)$$

for all $A \subseteq V_\epsilon \subset V_{\epsilon/C}$ and all $\lambda \in \mathbb{R}$. Therefore, it is enough to prove Proposition 4.2.3 for the BBM. We do so by following very closely the proof in [4, Lemma 3.8].

We will use the following estimate, which is proved in [4, Lemma 3.6]: let W_s be standard Brownian motion under $\mathbb{P}$ and fix a large constant C_1. Then, if

$$\mu_{q,r}^*(x) = \mathbb{P}\left(W_q \in dx, W_s \leq r + C_1(\min\{s, q-s\})^{1/20} \text{ for all } 0 \leq s \leq q\right)/dx,$$

we have

$$\mu_{q,r}^*(x) \leq C_2 r(r-x)/q^{3/2} \tag{4.2.9}$$

for all $x \leq r$, where C_2 depends on C_1.

We next define the event

$$G(\lambda) = \left\{\exists t \leq T, v \in V_\epsilon : \eta_\epsilon^v(t) - \frac{m_\epsilon}{T}t - 10\log(\min\{t, T-t\})_+ \geq \lambda\right\}$$

and we prove the following claim.

Claim 4.2.4. There exists a constant $C > 0$ (depending on d) such that

$$\mathbb{P}(G(\lambda)) \leq C\lambda e^{-\sqrt{2d}\lambda}$$

for all $\lambda \geq 1$.

Proof. Following the proof of [4, Lemma 3.7], we define $\psi_t = \lambda + 10\log(\min\{t, T-t\})_+$ and $\chi_{T_k}(x) = \mathbb{P}\left(\eta_\epsilon^v(t) - \frac{m_\epsilon}{T}t \leq \psi_t \text{ for all } t \leq T_k, \eta_\epsilon^v(T_k) - \frac{m_\epsilon}{T}T_k \in dx\right)/dx$. Then, by decomposing based on the first time such that $\eta_\epsilon^v(t) - \frac{m_\epsilon}{T}t \geq \psi_t$, we obtain that

$$\mathbb{P}\left(G(\lambda)\right) \leq \sum_{k=1}^{n} 2^{dk} \int_{-\infty}^{\psi_{T_k}} \chi_{T_k}(x) \mathbb{P}\left(\max_{s \leq \log 2} \eta_\epsilon^v(s) \geq \psi_{T_k} - x - C\right) dx,$$

where C is an absolute constant. Display (4.2.9) and Girsanov's Theorem imply that

$$\chi_{T_k}(x) \leq C 2^{-dk} e^{-x\left(\sqrt{2d} - O(\log T/T)\right)} \psi_{T_k}(\psi_{T_k} - x),$$

where C depends on d. On the other hand,

$$\mathbb{P}\left(\max_{s \leq \log 2} \eta_\epsilon^v(t) \geq \psi_{T_k} - x - C\right) \leq C e^{-\left(\psi_{T_k} - x - C\right)^2 / 2\log 2}$$

for some absolute constant C. Therefore, by the three previous displays, we obtain

$$\mathbb{P}\left(G(\lambda)\right) \leq C \sum_{k=1}^{n} \psi_{T_k} \int_{-\infty}^{\psi_{T_k}} e^{-x\left(\sqrt{2d} - O(\log T/T)\right)} (\psi_{T_k} - x) e^{-\left(\psi_{T_k} - x - C\right)^2 / 2\log 2} dx.$$

The change of variables $u = \psi_{T_k} - x$ yields

$$\begin{aligned}
\mathbb{P}\left(G(\lambda)\right) &\leq C \sum_{k=1}^{n} \psi_{T_k} e^{-\sqrt{2d}\psi_{T_k}} \\
&= C \sum_{k=1}^{n} \left(\lambda + 10 \log\left(\min\left\{T_k, T - T_k\right\} \vee 1\right)\right) e^{-\sqrt{2d}(\lambda + 10\log(\min\{T_k, T - T_k\} \vee 1))} \\
&= C \sum_{k=1}^{n} \frac{\left(\lambda + 10 \log\left(\min\left\{T_k, T - T_k\right\} \vee 1\right)\right)}{\left(\min\left\{T_k, T - T_k\right\} \vee 1\right)^{10}} e^{-\sqrt{2d}\lambda} \leq C\lambda e^{-\sqrt{2d}\lambda},
\end{aligned}$$

where $(\cdot \vee \cdot) = \max\{\cdot, \cdot\}$, and the convergence of the last sum is due the exponent 10 in the denominator (with room to spare). $\square$

We now finish the proof of Proposition 4.2.3. Fix $A \subset V_\epsilon$ and $z \in \mathbb{R}$. For $z + \left(|V_\epsilon| / |A|\right)^{1/4} \geq 1$, let $\lambda = z + \left(|V_\epsilon| / |A|\right)^{1/4}$, and continuing with the notation of Claim 4.2.4, we let

$$F_v = \left\{\eta_\epsilon^v(t) \leq \frac{m_\epsilon}{T} t + \psi_t \text{ for all } 0 \leq t \leq T, \eta_\epsilon^v \geq m_\epsilon + z\right\},$$

where $v \in V_\epsilon$. We now compute

$$\begin{aligned}
\mathbb{P}\left(F_v(\lambda)\right) &= \int_z^{\psi_T} \frac{d\mathbb{P}}{d\mathbb{Q}}(x + m_\epsilon) \chi_T(x) dx \\
&\leq C \int_z^{\psi_T} 2^{-dn} e^{-x\left(\sqrt{2d} - O(\log T/T)\right)} \psi_T\left(\psi_T - x\right) dx \\
&\leq C 2^{-dn} \psi_T e^{-\sqrt{2d}\psi_T} \int_0^{\psi_T - z} e^u u \, du \leq C 2^{-dn} \psi_T e^{-\sqrt{2d}z}\left(\psi_T - z\right).
\end{aligned}$$

Recalling that $\psi_T = \lambda = z + (|V_\epsilon| / |A|)^{1/4}$, we obtain

$$\begin{aligned}\mathbb{P}\left(F_v(\lambda)\right) &\le C 2^{-dn} \left(z + (|V_\epsilon| / |A|)^{1/4}\right) (|V_\epsilon| / |A|)^{1/4} e^{-\sqrt{2d} z} \\ &\le C 2^{-dn} (|V_\epsilon| / |A|)^{1/2} e^{-cz}.\end{aligned}$$

Adding the previous display for $v \in A$ and using Claim 4.2.4, we obtain

$$\begin{aligned}\mathbb{P}\left(\max_{v \in A} \eta_\epsilon^v \ge m_\epsilon + z\right) &\le C \left(\epsilon^d |A|\right)^{1/2} e^{-cz} + C\left(z + (|V_\epsilon| / |A|)^{1/4}\right) e^{-\sqrt{2d}\left(z + (|V_\epsilon|/|A|)^{1/4}\right)} \\ &\le C \left(\epsilon^d |A|\right)^{1/2} e^{-cz}\end{aligned}$$

for some $0 < c, C < \infty$ (depending on d only), as desired. The previous computation was made under the assumption $z + (|V_\epsilon| / |A|)^{1/4} \ge 1$. Assume now $(|V_\epsilon| / |A|)^{1/4} - 1 \le -z$. In this case,

$$\left(\epsilon^d |A|\right)^{1/2} e^{-cz} \ge c \left(\epsilon^d |A|\right)^{1/2} e^{c\left(\epsilon^d |A|\right)^{-1/4}}.$$

But $\inf_{0<x<1} x^{1/2} e^{cx^{-1/4}} \ge c > 0$, where c depends only d. Therefore, in this case, Proposition 4.2.3 holds trivially by adjusting the constant C. $\square$

Proposition 4.2.5. *Let $(\psi_\epsilon^x : x \in \square_\epsilon^v)$ be the Brownian sheet defined in* (4.1.6)*. Then, for all $x, y \in \square_\epsilon^v$,*

$$p^d \epsilon^{-1} \|x - y\|_1 \le \mathbb{E}\left[(\psi_\epsilon^x - \psi_\epsilon^y)^2\right] \le (2p)^d \epsilon^{-1} \|x - y\|_1 .$$

Proof. By (4.1.6),

$$\mathbb{E}\left[(\psi_\epsilon^x - \psi_\epsilon^y)^2\right] = \mathbb{E}\left[\left(\psi^{l(x)} - \psi^{l(y)}\right)^2\right], \tag{4.2.10}$$

where l is the linear map from $\square_\epsilon^v$ onto $[p, 2p)^d$ sending v to $(p)_{1 \le i \le d} = (p, p, \ldots, p)$. Call $l(x) = x'$ and $l(y) = y'$. Note that

$$\begin{aligned}\mathbb{E}\left[\left(\psi^{x'} - \psi^{y'}\right)^2\right] &= \left(\prod_{1 \le i \le d} x_i' - \prod_{1 \le i \le d} \min\left\{x_i', y_i'\right\}\right) + \left(\prod_{1 \le i \le d} y_i' - \prod_{1 \le i \le d} \min\left\{x_i', y_i'\right\}\right) \\ &=: A + B. \end{aligned}\tag{4.2.11}$$

Consider the first term, A. Adding and subtracting the intermediate terms

$$\left(\prod_{j \le i \le d} x_i'\right) \left(\prod_{1 \le i \le j-1} \min\left\{x_i', y_i'\right\}\right)$$

for $j = 2, \ldots, d$, we obtain

$$
\begin{aligned}
A = \sum_{1 \le j \le d} \Bigg(& \left(\prod_{j \le i \le d} x'_i \right) \left(\prod_{1 \le i \le j-1} \min\left\{x'_i, y'_i\right\} \right) \\
& - \left(\prod_{j+1 \le i \le d} x'_i \right) \left(\prod_{1 \le i \le j} \min\left\{x'_i, y'_i\right\} \right) \Bigg) \\
= \sum_{1 \le j \le d} & \left(\prod_{j+1 \le i \le d} x'_i \right) \left(\prod_{1 \le i \le j-1} \min\left\{x'_i, y'_i\right\} \right) \left(x'_j - \min\left\{x'_j, y'_j\right\}\right).
\end{aligned}
$$

Since both x' and y' belong to $[p, 2p)^d$, we obtain

$$
p^{d-1} \sum_{1 \le j \le d} \left(x'_j - \min\left\{x'_j, y'_j\right\}\right) \le A \le (2p)^{d-1} \sum_{1 \le j \le d} \left(x'_j - \min\left\{x'_j, y'_j\right\}\right).
$$

An analogous expression holds for B. Then,

$$
\begin{aligned}
p^{d-1} \sum_{1 \le j \le d} \left(x'_j + y'_j - 2\min\left\{x'_j, y'_j\right\}\right) &\le A + B \\
&\le (2p)^{d-1} \sum_{1 \le j \le d} \left(x'_j + y'_j - 2\min\left\{x'_j, y'_j\right\}\right),
\end{aligned}
$$

so, from the previous display, (4.2.10) and (4.2.11)

$$
p^{d-1} \left\|x' - y'\right\|_1 \le \mathbb{E}\left[\left(\psi^x_\epsilon - \psi^y_\epsilon\right)^2\right] \le (2p)^{d-1} \left\|x' - y'\right\|_1.
$$

But, from the definition of x' and y', we see that $\|x' - y'\|_1 = \epsilon^{-1} p \|x - y\|_1$. Therefore,

$$
p^d \epsilon^{-1} \|x - y\|_1 \le \mathbb{E}\left[\left(\psi^x_\epsilon - \psi^y_\epsilon\right)^2\right] \le (2p)^d \epsilon^{-1} \|x - y\|_1,
$$

as desired. $\square$

Chapter 5

Comparison to CMBRW

We now proceed to the comparison of the right and left tail of the maximum of the field Y_ϵ (which was defined in Chapter 3 and satisfies (3.1.1) and (3.1.2)) and the maximum of an appropriate combination of the fields ξ_ϵ and ψ_ϵ (which will be specified in Sections 5.1 and 5.2). Note that we will only use Brownian sheet when comparing the right tail; for the left tail, we will compare directly the CMBRW with the field Y_ϵ on a discrete index set.

5.1 The right tail of (3.1.3)

Proposition 5.1.1. *For $\epsilon > 0$, let $(\xi_\epsilon^v : v \in V_\epsilon)$ and $\left(\psi_\epsilon^x : x \in [0,1)^d\right)$ be independent fields, defined as in* (4.1.1) *and* (4.1.6)*, respectively. Then, there exist $\delta > 0$ small enough and p large enough (depending on C_Y and d) such that, for all $\epsilon > 0$ small enough (depending on C_Y and d),*

$$\mathbb{P}\left(\sup_{x\in[0,1)^d} Y_{\delta\epsilon}^{\delta x} \geq \lambda\right) \leq \mathbb{P}\left(\sup_{x\in[0,1)^d} a(x)\xi_\epsilon^{[x]} + \psi_\epsilon^x \geq \lambda\right)$$

for all $\lambda \in \mathbb{R}$, where $a(x) := \sqrt{\left(Var(Y_{\delta\epsilon}^{\delta x}) - Var(\psi_\epsilon^x)\right) / Var(\xi_\epsilon^{[x]})}$.

Proof. We first make sure that $Var(Y_{\delta\epsilon}^{\delta x}) - Var(\psi_\epsilon^x) \geq 0$, so that $a(x)$ is well defined. Note that (3.1.1) and (4.1.7) imply

$$Var(Y_{\delta\epsilon}^{\delta x}) - Var(\psi_\epsilon^x) \geq \log(1/\epsilon) + \log(1/\delta) - C_Y - (2p)^d \geq 0$$

for all $\epsilon > 0$ small enough (depending on C_Y, d and p). As we will see in this proof, p depends only on C_Y and d, so $a(x)$ is well defined for all $\epsilon > 0$ small enough, depending only on C_Y and d.

We now check the hypotheses of Slepian's Lemma (see [8, Corollary 2.4]). The variances of the fields $Y_{\delta\epsilon}^{\delta x}$ and $a(x)\xi_\epsilon^{[x]} + \psi_\epsilon^x$ are equal by the definition of $a(x)$. We first choose p so that $a(x) \leq 1$. Note that (3.1.1) and (4.1.7) imply

$$a(x)^2 = \frac{Var(Y_{\delta\epsilon}^{\delta x}) - Var(\psi_\epsilon^x)}{Var(\xi_\epsilon^{[x]})} \leq \frac{\log(1/\epsilon) + \log(1/\delta) + C_Y - p^d}{\log(1/\epsilon)},$$

so, by choosing p large enough (depending on C_Y, d and δ), we obtain $a(x) \leq 1$, for all x.

We now compare the covariance for points $x \neq y$, for which we distinguish two cases:

1. $[x] = [y]$ (that is, $\square_\epsilon^{[x]} = \square_\epsilon^{[y]}$). In this case, (3.1.2) and (4.1.8) imply

$$\mathbb{E}\left[\left(Y_{\delta\epsilon}^{\delta x} - Y_{\delta\epsilon}^{\delta y}\right)^2\right] \leq C_Y(\delta\epsilon)^{-1}\|\delta x - \delta y\| \leq p^d\epsilon^{-1}\|x-y\|_1 \leq \mathbb{E}\left[(\psi_\epsilon^x - \psi_\epsilon^y)^2\right]$$
$$\leq \mathbb{E}\left[\left(a(x)\xi_\epsilon^{[x]} + \psi_\epsilon^x - a(y)\xi_\epsilon^{[y]} - \psi_\epsilon^y\right)^2\right]$$

for p large enough (depending on C_Y and d). The last inequality is due to the independence between ξ_ϵ and ψ_ϵ.

2. $[x] \neq [y]$. In this case, we can apply (4.1.3) and the independence between ξ_ϵ, $\psi_\epsilon^{[x]}$ and $\psi_\epsilon^{[y]}$ to obtain

$$\begin{aligned} Cov(a(x)\xi_\epsilon^{[x]} + \psi_\epsilon^x, a(y)\xi_\epsilon^{[y]} + \psi_\epsilon^y) &\leq a(x)a(y)Cov(\xi_\epsilon^{[x]}, \xi_\epsilon^{[y]}) \\ &\leq a(x)a(y)\left(-\log\|[x]-[y]\| + C\right). \end{aligned}$$

But $a(x)a(y) \leq 1$, so

$$Cov(a(x)\xi_\epsilon^{[x]} + \psi_\epsilon^x, a(y)\xi_\epsilon^{[y]} + \psi_\epsilon^y) \leq -\log\|[x]-[y]\| + C.$$

Note that $-\log\|[x]-[y]\| \leq -\log\left(\max\{\epsilon, \|x-y\|\}\right) + C$. Applying (3.1.1), we obtain

$$-\log\left(\max\{\epsilon, \|x-y\|\}\right) + C \leq -\log\left(\max\{\delta\epsilon, \|\delta x - \delta y\|\}\right) - C_Y \leq Cov(Y_{\delta\epsilon}^{\delta x}, Y_{\delta\epsilon}^{\delta y})$$

for some $\delta > 0$ small enough (depending on C_Y). Proposition 5.1.1 follows now from Slepian's Lemma. $\square$

Proposition 5.1.1 provides an upper bound for the right tail of the supremum of $Y_{\delta\epsilon}$ taken over the δ-box $\delta[0,1)^d$. The same proof works for any δ-box. Therefore, a union bound implies

$$\mathbb{P}\left(\sup_{x\in[0,1)^d} Y^x_{\delta\epsilon} \geq \lambda\right) \leq \left(\frac{1}{\delta}\right)^d \mathbb{P}\left(\sup_{x\in[0,1)^d} a(x)\xi^{[x]}_\epsilon + \psi^x_\epsilon \geq \lambda\right) \tag{5.1.1}$$

for all $\lambda \in \mathbb{R}$.

We now provide an upper bound for the probability on the right hand side of the previous display. We first prove an upper bound on the supremum of the Brownian sheet.

Lemma 5.1.2. *There exist constants* $0 < c, C < \infty$ *(depending on* p *and* d*) such that*

$$\sup_{v\in V_\epsilon} \mathbb{P}\left(\sup_{x\in\square^v_\epsilon} \psi^x_\epsilon \geq \lambda\right) \leq Ce^{-c\lambda^2}$$

for all $\lambda \geq 0, \epsilon > 0$.

Proof. Let $v \in V_\epsilon$. Fernique's Majorizing Criterion (see [8, Theorem 4.1]) implies that

$$\mathbb{E}\left[\sup_{x\in\square^v_\epsilon} \psi^x_\epsilon\right] \leq C \sup_{x\in\square^v_\epsilon} \int_0^\infty \sqrt{-\log\left(\mu(B(x,r))\right)}dr$$

for some absolute constant C, where μ is the normalized d-dimensional Lebesgue measure on $\square^v_\epsilon$ and $B(x,r) = \left\{y \in \square^v_\epsilon : \mathbb{E}\left[(\psi^x_\epsilon - \psi^y_\epsilon)^2\right] \leq r^2\right\}$. But (4.1.8) implies

$$B(x,r) \supseteq \left\{y \in \square^v_\epsilon : (2p)^d \epsilon^{-1} \left\|y - x\right\|_1 \leq r^2\right\}.$$

Therefore, $\mu\left(B(x,r)\right) \geq cr^{2d}$ for some constant $c > 0$ depending on p and d. Applying the previous display and Fernique's Majorizing Criterion, we obtain

$$\mathbb{E}\left[\sup_{x\in\square^v_\epsilon} \psi^x_\epsilon\right] \leq C \int_0^\infty \sqrt{-\log\left(cr^{2d}\right)}dr \leq C < \infty,$$

where C depends on p and d. Borell's Inequality (see [8, Theorem 2.1]) and (4.1.7) imply

$$\mathbb{P}\left(\sup_{x\in\square^v_\epsilon} \psi^x_\epsilon \geq C + \lambda\right) \leq e^{-\lambda^2/(2(2p)^d)},$$

where C is the constant obtained in the previous display. Lemma 5.1.2 now follows from a change of variables. $\square$

Proposition 5.1.3. *Let p and δ be as in Proposition 5.1.1. There exist constants $0 < c, C < \infty$ (depending on C_Y and d) such that*

$$\mathbb{P}\left(\sup_{x\in[0,1)^d} a(x)\xi_\epsilon^{[x]} + \psi_\epsilon^x \geq \lambda + m_\epsilon\right) \leq Ce^{-c\lambda}$$

for all $\lambda \geq 0$ and all $\epsilon > 0$ small enough (depending on C_Y and d).

Proof. By letting $\psi_\epsilon^{*,[x]} = \sup_{y\in\Box_\epsilon^{[x]}} \psi_\epsilon^y$, we have

$$\sup_{x\in[0,1)^d} a(x)\xi_\epsilon^{[x]} + \psi_\epsilon^x \leq \max_{x\in[0,1)^d} a(x)\xi_\epsilon^{[x]} + \psi_\epsilon^{*,[x]}.$$

The previous display implies

$$\sup_{x\in[0,1)^d} a(x)\xi_\epsilon^{[x]} + \psi_\epsilon^x \geq m_\epsilon + \lambda \implies \sup_{x\in[0,1)^d} a(x)\xi_\epsilon^{[x]} + \psi_\epsilon^{*,[x]} \geq m_\epsilon + \lambda.$$

We now compute an upper bound for the right hand side of the previous display. Define the random sets $\Gamma_y = \{v \in V_\epsilon : \psi_\epsilon^{*,v} \in [y-1, y)\}$ for $y \geq 1$, and $\Gamma_0 = \{v \in V_\epsilon : \psi_\epsilon^{*,v} \leq 0\}$. Note that

$$\mathbb{P}\left(\sup_{x\in[0,1)^d} a(x)\xi_\epsilon^{[x]} + \psi_\epsilon^{*,[x]} \geq m_\epsilon + \lambda\right) \leq \sum_{y\geq 0} \mathbb{P}\left(\sup_{x:[x]\in\Gamma_y} a(x)\xi_\epsilon^{[x]} \geq m_\epsilon + \lambda - y\right).$$

By the definition of $a(x)$ and the choice of p and δ in Proposition 5.1.1,

$$a(x)^2 = \frac{Var(Y_{\delta\epsilon}^{\delta x}) - Var(\psi_\epsilon^x)}{Var(\xi_\epsilon^{[x]})} \leq 1,$$

and by (3.1.1) and (4.1.7),

$$a(x)^2 \geq \frac{\log(1/\epsilon) + \log(1/\delta) - C_Y - (2p)^d}{\log(1/\epsilon)} \geq \frac{1}{2}$$

for $\epsilon > 0$ small enough (depending on δ, p, C_Y and d, all of which ultimately depend on C_Y and d). Therefore, the last three displays imply

$$\mathbb{P}\left(\sup_{x\in[0,1)^d} a(x)\xi_\epsilon^{[x]} + \psi_\epsilon^{*,[x]} \geq m_\epsilon + \lambda\right) \leq \sum_{y\geq 0} \mathbb{P}\left(\max_{v\in\Gamma_y} \xi_\epsilon^v \geq m_\epsilon + \lambda - 2y\right). \tag{5.1.2}$$

But $\mathbb{P}\left(\max_{v\in\Gamma_y}\xi_\epsilon^v \geq m_\epsilon+\lambda-2y\right) = \mathbb{E}\left[\mathbb{P}\left(\max_{v\in\Gamma_y}\xi_\epsilon^v \geq m_\epsilon+\lambda-2y \mid \Gamma_y\right)\right]$. Since ψ_ϵ and ξ_ϵ are independent, from (4.1.5) we obtain

$$\mathbb{P}\left(\max_{v\in\Gamma_y}\xi_\epsilon^v \geq m_\epsilon+\lambda-2y \mid \Gamma_y\right) \leq C\left(\epsilon^d\,|\Gamma_y|\right)^{1/2} e^{-c(\lambda-2y)}.$$

Then,

$$\mathbb{P}\left(\max_{v\in\Gamma_y}\xi_\epsilon^v \geq m_\epsilon+\lambda-2y\right) \leq Ce^{-c(\lambda-2y)}\left(\mathbb{E}\left[\epsilon^d\,|\Gamma_y|\right]\right)^{1/2}. \tag{5.1.3}$$

But, by Lemma 5.1.2, $\mathbb{E}\left[|\Gamma_y|\right] = \sum_{v\in V_\epsilon}\mathbb{P}\left(\psi_\epsilon^{*,v}\in[y-1,y)\right) \leq C\epsilon^{-d}e^{-cy^2}$. For $y=0$, we simply use $|\Gamma_0|\leq\epsilon^{-d}$. Therefore, from displays (5.1.2) and (5.1.3), we obtain

$$\mathbb{P}\left(\sup_{x\in[0,1)^d} a(x)\xi_\epsilon^{[x]}+\psi_\epsilon^{*,[x]} \geq m_\epsilon+\lambda\right) \leq Ce^{-c\lambda}$$

for some constants $0<c,C<\infty$ (depending on C_Y and d). □

Proof of Theorem 3.1.1, (3.1.3)*, the right tail.* Display (5.1.1) and Proposition 5.1.3 imply that there exist constants $0<c,C<\infty$ (depending on C_Y and d) such that, for all $\epsilon>0$ small enough (depending on C_Y and d),

$$\mathbb{P}\left(\max_{x\in[0,1)^d} Y_{\delta\epsilon}^x \geq m_\epsilon+\lambda\right) \leq \left(\frac{1}{\delta}\right)^2 \mathbb{P}\left(\max_{x\in[0,1)^d} a(x)\xi_\epsilon^{[x]}+\psi_\epsilon^x \geq \lambda+m_\epsilon\right) \leq Ce^{-c\lambda}.$$

It is easy to see from the definition that $m_{\delta\epsilon}\leq m_\epsilon+C'$ for some C' depending on δ and d. Therefore,

$$\mathbb{P}\left(\max_{x\in[0,1)^d} Y_{\delta\epsilon}^x \geq m_{\delta\epsilon}+\lambda-C'\right) \leq Ce^{-c\lambda}.$$

The upper bound (3.1.3) for the right tail follows by adjusting the constants. □

5.2 The left tail of (3.1.3)

In this section we prove the upper bound (3.1.3) for the left tail. As previously mentioned, we can reduce the set under maximization to a discrete set. More precisely, if $\{D_\epsilon : \epsilon>0\}$ is any collection of subsets of $[0,1)^d$, then

$$\mathbb{P}\left(\sup_{x\in[0,1)^d} Y_\epsilon^x \leq m_\epsilon-\lambda\right) \leq \mathbb{P}\left(\sup_{x\in D_\epsilon} Y_\epsilon^x \leq m_\epsilon-\lambda\right). \tag{5.2.1}$$

If we select D_ϵ appropriately, we can perform a comparison with the CMBRW using Slepian's Lemma.

Proposition 5.2.1. *There exist $\delta, \rho > 0$ small enough (depending on C_Y and d) such that*

$$\mathbb{P}\left(\max_{u \in V_{\epsilon/\rho}} Y^u_{\delta\epsilon} \leq \lambda\right) \leq \mathbb{P}\left(\max_{u \in V_\epsilon \cap \rho[0,1)^d} b(u)\xi^u_\epsilon \leq \lambda\right)$$

for all $\epsilon > 0$ and all $\lambda \in \mathbb{R}$, where $b(u) := \sqrt{Var(Y^u_{\delta\epsilon})/Var(\xi^u_\epsilon)}$ for $u \in V_\epsilon \cap \rho[0,1)^d$.

Proof. Note that (3.1.1) and (4.1.2) imply that $b(u) \geq \frac{\log(1/\epsilon)+\log(1/\delta)-C_Y}{\log(1/\epsilon)}$, which is greater than 1 for $\delta > 0$ small enough (depending on C_Y).

Let $u, v \in V_{\epsilon/\rho}$, with $u \neq v$. Then, for $0 < \delta, \rho \leq 1$, we have $\|u - v\| \geq \epsilon/\rho \geq \delta\epsilon$. Display (3.1.1) therefore implies

$$Cov(Y^u_{\delta\epsilon}, Y^v_{\delta\epsilon}) \leq -\log\|u - v\| + C_Y.$$

Choose $\rho > 0$ small enough (depending on C_Y and d) so that

$$-\log\|u - v\| + C_Y \leq -\log\|\rho u - \rho v\| - C \leq Cov(\xi^{\rho u}_\epsilon, \xi^{\rho v}_\epsilon) \leq Cov(b(\rho u)\xi^{\rho u}_\epsilon, b(\rho v)\xi^{\rho v}_\epsilon),$$

where the second to last bound follows from (4.1.3). All the hypotheses of Slepian's Lemma are satisfied, so

$$\mathbb{P}\left(\max_{u \in V_{\epsilon/\rho}} Y^u_{\delta\epsilon} \leq \lambda\right) \leq \mathbb{P}\left(\max_{u \in V_{\epsilon/\rho}} b(\rho u)\xi^{\rho u}_\epsilon \leq \lambda\right)$$

for all $\lambda \in \mathbb{R}$. Proposition 5.2.1 follows by observing that $\rho V_{\epsilon/\rho} = V_\epsilon \cap \rho[0,1]^d$. $\square$

Proposition 5.2.2. *Let $\rho > 0$ and $\{b(u) : u \in V_\epsilon \cap \rho[0,1]^d\}$ be as in Proposition 5.2.1. Then,*

$$\mathbb{P}\left(\max_{u \in V_\epsilon \cap \rho[0,1]^d} b(u)\xi^u_\epsilon \leq m_\epsilon - \lambda\right) \leq \mathbb{P}\left(\max_{u \in V_\epsilon \cap \rho[0,1]^d} \xi^u_\epsilon \leq m_\epsilon - \lambda/2\right)$$

for all $\lambda \geq 0$ and all $\epsilon > 0$ small enough (depending on C_Y).

Proof. It follows from the definition of $b(u)$ and the choices made in Proposition 5.2.1 that, for all u,

$$1 \leq b(u) = \sqrt{Var(Y^u_{\delta\epsilon})/Var(\xi^u_\epsilon)} \leq \sqrt{\frac{\log(1/\epsilon) + \log(1/\delta) + C_Y}{\log(1/\epsilon)}} \leq 2$$

for small enough $\epsilon > 0$ (depending on C_Y and δ, which itself depends on C_Y). Let ν be the (a.s. well-defined) point that maximizes ξ^u_ϵ, for $u \in V_\epsilon \cap \rho[0,1]^d$. Then, the previous display implies

$$b(\nu)\xi^\nu_\epsilon \leq m_\epsilon - \lambda \implies \xi^\nu_\epsilon \leq m_\epsilon/b(\nu) - \lambda/b(\nu) \leq m_\epsilon - \lambda/2.$$

□

Our task is now to find an upper bound for the probability on the right hand side of Proposition 5.2.2.

Proposition 5.2.3. *There exist constants* $0 < c, C < \infty$ *(depending on* ρ *and* d*) such that*

$$\mathbb{P}\left(\max_{v \in V_\epsilon \cap \rho[0,1]^d} \xi_\epsilon^v \leq m_\epsilon - \lambda\right) \leq Ce^{-c\lambda}$$

for all $\lambda \geq 0$ *and all* $\epsilon > 0$ *small enough.*

Proof. We distinguish three cases for λ.

- $\lambda \in [0, 2/\rho]$. In this case, the proposition is trivially true by simply adjusting the constants c and C so that $Ce^{-c\lambda} \geq 1$.

- $\lambda \in (2/\rho, \sqrt{1/\epsilon}]$. Let $n := \left\lfloor \frac{\rho\lambda}{2} \right\rfloor$ and let $\left\{B^i : i = 1, \ldots, n\right\}$ be a collection of boxes of side length λ^{-1} inside $\rho[0,1)^d$, such that the distance between any pair of boxes is at least λ^{-1}. Set $B_\epsilon^i = B^i \cap V_\epsilon$. We claim that the field

$$\left(\xi_\epsilon^v - \xi_\epsilon^v(\log \lambda) : v \in B_\epsilon^i\right)$$

is a copy of $(\xi_{\lambda\epsilon}^v : v \in V_{\lambda\epsilon})$, and that the fields $\left\{\left(\xi_\epsilon^v - \xi_\epsilon^v(\log \lambda) : v \in B_\epsilon^i\right)\right\}_{1 \leq i \leq n}$ are independent. Indeed, if $v, u \in B_\epsilon^i$, then (4.1.1) implies

$$\begin{aligned} Cov(\xi_\epsilon^v - \xi_\epsilon^v(\log \lambda), \xi_\epsilon^u - \xi_\epsilon^u(\log \lambda)) &= \int_{\log(\lambda)}^{\log(1/\epsilon)} \prod_{1 \leq j \leq d} \left(1 - e^r \left|v_j - u_j\right|\right)_+ dr \\ &= \int_0^{-\log(\lambda\epsilon)} \prod_{1 \leq j \leq d} \left(1 - e^r \left|\lambda v_j - \lambda u_j\right|\right)_+ dr, \end{aligned} \tag{5.2.2}$$

and the set $\lambda B_\epsilon^i = \left\{\lambda v : v \in B_\epsilon^i\right\}$ coincides with $V_{\lambda\epsilon}$ after a translation. This shows that $\left(\xi_\epsilon^v - \xi_\epsilon^v(\log \lambda) : v \in B_\epsilon^i\right) \stackrel{d}{=} (\xi_{\lambda\epsilon}^v : v \in V_{\lambda\epsilon})$. Moreover, from (5.2.2), it is easy to see that $\|v - u\| \geq \lambda^{-1}$ (which is true for points v, u in different boxes B_ϵ^i, by construction) implies

$$Cov(\xi_\epsilon^v - \xi_\epsilon^v(\log \lambda), \xi_\epsilon^u - \xi_\epsilon^u(\log \lambda)) = 0,$$

as desired.

Therefore, independence of the fields $\left\{\left(\xi_\epsilon^v - \xi_\epsilon^v(\log\lambda) : v \in B_\epsilon^i\right)\right\}_{1\le i\le n}$ and (4.1.4) imply

$$\mathbb{P}\left(\max_{v\in\cup_i B_\epsilon^i} (\xi_\epsilon^v - \xi_\epsilon^v(\log\lambda)) \le m_{\lambda\epsilon}\right) \le e^{-cn}$$

for some constant $c > 0$ depending on d. But $n \ge c\lambda$ for some constant $c > 0$ depending on ρ. Therefore,

$$\mathbb{P}\left(\max_{v\in\cup_i B_\epsilon^i} (\xi_\epsilon^v - \xi_\epsilon^v(\log\lambda)) \le m_{\lambda\epsilon}\right) \le e^{-c\lambda}$$

for some constant $c > 0$ depending on both d and ρ. By letting

$$\nu = \arg\max\left\{\xi_\epsilon^v - \xi_\epsilon^v(\log\lambda) : v \in \bigcup_i B_\epsilon^i\right\},$$

the previous display implies

$$\begin{aligned}
&\mathbb{P}\left(\max_{v\in V_\epsilon\cap\rho[0,1)^d} \xi_\epsilon^v \le m_\epsilon - \lambda\right)\\
&\quad\le \mathbb{P}\left(\xi_\epsilon^\nu \le m_\epsilon - \lambda\right)\\
&\quad\le \mathbb{P}\left(\xi_\epsilon^\nu(\log\lambda) \le m_\epsilon - m_{\lambda\epsilon} - \lambda\right) + \mathbb{P}\left(\xi_\epsilon^\nu - \xi_\epsilon^\nu(\log\lambda) \le m_{\lambda\epsilon}\right)\\
&\quad\le \mathbb{P}\left(\xi_\epsilon^\nu(\log\lambda) \le m_\epsilon - m_{\lambda\epsilon} - \lambda\right) + e^{-c\lambda}.
\end{aligned}$$

Moreover, it is clear from (4.1.1) that the fields

$$(\xi_\epsilon^v - \xi_\epsilon^v(\log\lambda) : v \in V_\epsilon) \qquad \text{and} \qquad (\xi_\epsilon^v(\log\lambda) : v \in V_\epsilon)$$

are independent. Hence, ν is independent from $\xi_\epsilon^{(\cdot)}(\log\lambda)$, and $\xi_\epsilon^\nu(\log\lambda)$ is therefore a Gaussian random variable with mean zero and variance $\log\lambda$. But

$$m_\epsilon - m_{\lambda\epsilon} \le \sqrt{2d}\log\lambda.$$

Therefore, the last two displays imply

$$\mathbb{P}\left(\max_{v\in V_\epsilon\cap\rho[0,1)^d} \xi_\epsilon^v \le m_\epsilon - \lambda\right) \le Ce^{-c\frac{(\lambda-\sqrt{2d}\log\lambda)^2}{\log\lambda}} + e^{-c\lambda} \le Ce^{-c\lambda},$$

proving Proposition 5.2.3 in the case $\lambda \in [2/\rho, \sqrt{1/\epsilon}]$.

- $\lambda \in (\sqrt{1/\epsilon}, \infty)$. In this case, we have

$$\mathbb{P}\left(\max_{v \in V_\epsilon \cap \rho[0,1)^d} \xi_\epsilon^v \leq m_\epsilon - \lambda\right) \leq \mathbb{P}\left(\xi_\epsilon^v \leq m_\epsilon - \lambda\right) \leq Ce^{-c\frac{(\lambda - m_\epsilon)^2}{\log(1/\epsilon)}} \leq Ce^{-c\lambda}$$

(where v is any point), which implies Proposition 5.2.3 in this case.

$\square$

Using Propositions 5.2.1, 5.2.2 and 5.2.3, we are now ready to finish the proof of Theorem 3.1.1.

Proof of Theorem 3.1.1, (3.1.3)*, the left tail.* Propositions 5.2.1, 5.2.2 and 5.2.3 imply the existence of constants $0 < \delta, \rho, c, C < \infty$, depending on C_Y and d, such that

$$\mathbb{P}\left(\max_{u \in V_{\epsilon/\rho}} Y_{\delta\epsilon}^u \leq m_\epsilon - \lambda\right) \leq Ce^{-c\lambda}$$

for all $\lambda \geq 0$ and all $\epsilon > 0$ small enough (depending on C_Y). But $m_{\delta\epsilon} \leq m_\epsilon + C'$, where C' depends on δ and d. Therefore,

$$\mathbb{P}\left(\max_{u \in V_{\epsilon/\rho}} Y_{\delta\epsilon}^u \leq m_{\delta\epsilon} - \lambda - C'\right) \leq Ce^{-c\lambda}.$$

The bound (3.1.3) for the left tail follows by adjusting the constants. $\square$

Part II

Convergence in law of the centered maximum of the mollified Gaussian free field in two dimensions

Chapter 6

The mollified Gaussian free field

6.1 Setup and main result

We begin by defining the *continuum Gaussian free field* (CGFF). Let $(W_t)_{t\geq 0}$ be a two-dimensional Brownian motion and let $\mathbb{E}^x$ denote the expectation with respect to the law of $(W_t)_{t\geq 0}$ with starting point $x \in \mathbb{R}^2$. Define the *Green function* $G_I : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, \infty]$ associated with $I = [0,1]^2$ by

$$G_I(x,y) = -\log\|x-y\| + \mathbb{E}^x[\log\| W_{\tau_I} - y\|], \tag{6.1.1}$$

where $\tau_I = \inf\{t \geq 0 : W_t \notin I\}$ and $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^2$. The Green function G_I is symmetric and positive semi-definite. (Note that the function G_I differs from the Green function G defined in (1.0.4) by a multiplicative factor; this change is made here to simplify the notation.)

It is tempting to define a centered Gaussian field $(X(x) : x \in I)$ with covariance structure given by $Cov(X(x), Y(y)) = G_I(x,y)$. Unfortunately, such a field does not exist because $G_I(x,x) = \infty$ for all $x \in (0,1)^2$. To overcome this difficulty, we follow a standard procedure (see, for example, [18]) and introduce a generalized field, which is a function on measures instead of points in I. Denote by $\mathcal{M}$ the collection of probability measures μ on $\mathbb{R}^2$ such that

$$\iint_{\mathbb{R}^2\times\mathbb{R}^2} G_I(x,y)d\mu(x)d\mu(y) < \infty, \tag{6.1.2}$$

and let $(X(\mu) : \mu \in \mathcal{M})$ be the centered Gaussian field with covariance structure

$$Cov(X(\mu), X(\nu)) = \iint_{\mathbb{R}^2 \times \mathbb{R}^2} G_I(x, y) d\mu(x) d\nu(y)$$

for all $\mu, \nu \in \mathcal{M}$. The field X is known as the continuum Gaussian free field (CGFF). It is not difficult to show that $\sup_{\mu \in \mathcal{M}} X(\mu) = \infty$ almost surely.

We next introduce a family of fields $\{(\Theta_{r,x} : x \in I)\}_{r>0}$ consisting of mollified versions of the CGFF that has the advantages that (i) for each $r > 0$, the field $\Theta_{r,x}$ has a version which is continuous in $x \in I$ (see Remark 7.2.2) and (ii) the family of fields $\Theta_{r,(\cdot)}$ approximates the CGFF X as $r \to \infty$.

A *mollifier* is a compactly supported C^∞ function $\theta : \mathbb{R}^2 \to [0, \infty)$ such that $\int_{\mathbb{R}^2} \theta(x) dx = 1$. For a fixed mollifier $\theta(\cdot)$, define the family of mollifiers $\theta_r(\cdot) = e^{2r} \theta(e^r(\cdot))$ for each $r > 0$. The family $\{\theta_r\}_{r>0}$ induces a collection of probability measures $\{\rho_{r,x} : r > 0, x \in I\}$, with

$$\rho_{r,x}(D) := \int_D \theta_r(x - u) du$$

for all Borel sets $D \subset \mathbb{R}^2$. Note that $\rho_{r,x}$ converges weakly to the Dirac delta function at x as $r \to \infty$. We define the *mollified Gaussian free field* (MGFF) with respect to $\theta(\cdot)$ as the centered Gaussian field $(\Theta_{r,x} : r > 0, x \in I)$ given by

$$\Theta_{r,x} = X(\rho_{r,x}), \tag{6.1.3}$$

where X is the CGFF. As mentioned before, we will show later (see Remark 7.2.2) that, for each $r > 0$, the MGFF $\Theta_{r,x}$ has a version which is continuous in $x \in I$, so it is possible to define, for any probability measure μ, a family of random variables $\Theta_{r,\mu}$ given by

$$\Theta_{r,\mu} = \int_{\mathbb{R}^2} \Theta_{r,x} d\mu(x).$$

Note that, for all $\mu, \nu \in \mathcal{M}$,

$$Cov(\Theta_{r,\mu}, \Theta_{r,\nu}) = \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \left(\iint_{\mathbb{R}^2 \times \mathbb{R}^2} G_I(x + u, y + v) \theta_r(u) \theta_r(v) du dv \right) d\mu(x) d\nu(y), \tag{6.1.4}$$

so the covariance structure of $(\Theta_{r,\mu} : \mu \in \mathcal{M})$ is simply a mollified version of the covariance structure of the CGFF. The previous display implies that, for all $\mu, \nu \in \mathcal{M}$, we have $Cov(\Theta_{r,\mu}, \Theta_{r,\nu}) \to Cov(X(\mu), X(\nu))$ as $r \to \infty$, so, in this sense, the CGFF is the limit of the MGFF as $r \to \infty$. Note also that a simple convexity argument implies

$$\Theta_r^* := \sup_{\mu \in \mathcal{M}} \Theta_{r,\mu} = \max_{x \in I} \Theta_{r,x}$$

almost surely for every $r > 0$.

The main result of this part is

Theorem 6.1.1. *Let $m_r := 2r - \frac{3}{4}\log(r)$. There exists a constant β_θ (depending only on the mollifier θ) and a probability law μ^* on $\mathbb{R}$ (not depending on θ) such that*

$$\Theta_r^* - m_r + \beta_\theta \to \mu^*$$

in law as $r \to \infty$.

Since the MGFF approximates the CGFF as $r \to \infty$, the law μ^* can be viewed informally as a centered version of $\sup_{\mu \in \mathcal{M}} X(\mu)$. Additionally, we obtain the following representation of μ^*.

Theorem 6.1.2. *There exists a random variable $Z > 0$ such that*

$$\mu^*((\infty, x]) = \mathbb{E}\left[\exp(-Ze^{-2x})\right]. \tag{6.1.5}$$

Thus, the limit law can be viewed as a randomly shifted Gumbel distribution. The random variable Z is the limit of a sequence of random variables with known distribution (see Remark 10.4.2).

6.2 Main ideas of the proof of Theorem 6.1.1

The outline of the proof of Theorem 6.1.1 follows that of the analogous result in [4] for the discrete Gaussian free field (DGFF), with the proof here being organized as follows. The first step is to prove the existence of a constant α_θ^* (depending on θ) and probability density ζ on I (not depending on θ) such that, for any closed square $A \subseteq I$,

$$z^{-1}e^{2z}\mathbb{P}\left(\max_{x \in A} \Theta_{r,x} - m_r \geq z\right) \to \alpha_\theta^* \int_A \zeta(x)dx \tag{6.2.1}$$

when first $r \to \infty$ and then $z \to \infty$; this is carried out in Chapter 8. We obtain in display (8.0.1) an explicit formula for ζ in terms of a Brownian motion killed at ∂I.

For the second step of the proof, for $\delta > 0$ small and $k \geq 1$ large, we divide $[\delta, 1-\delta]^2$ into a collection of adjacent sub-squares $\{S_{k,i}\}_i$ of side length e^{-k}. As will be explained in detail in Chapter 7, the MGFF can be decomposed into two independent centered Gaussian fields $\Theta^c_{r,(\cdot)}$ and $\Theta^f_{r,(\cdot)}$ by

$$\Theta_{r,x} = \Theta^c_{r,x} + \Theta^f_{r,x},$$

where $\Theta^c_{r,x}$ is the expected value of $\Theta_{r,x}$ conditioned on the values of the CGFF on the boundary of the sub-square $S_{k,i}$ containing x. We call $\Theta^c_{r,(\cdot)}$ the *coarse field* and $\Theta^f_{r,(\cdot)}$ the *fine field*. On each sub-square $S_{k,i}$, the fine field is a copy of the MGFF $\Theta_{r-k,(\cdot)}$, and the fine fields on different sub-squares are independent. We will show in Chapter 9 that the maximum of the MGFF can be approximated by restricting our attention to the points on each sub-square that maximize the corresponding fine field.

We then prove via a coupling argument (based in the above decomposition) that the sequence $\Theta^*_r - m_r + \beta_\theta$ is Cauchy with respect to the Lévy distance, where β_θ is a constant that depends only on θ. At the same time, and using that the probability density ζ mentioned before does not depend on the mollifier, we show that the limit law does not depend on θ. The coupling argument is described in detail in Chapter 10.

Throughout Part II, we will use two techniques repeatedly. The first technique consists of omitting the points near the boundary of the squares; that is, for $\delta > 0$ small, we will consider the maximum of the MGFF restricted to sets such as $I_\delta = \{x \in I : dist(x, \partial I) \geq \delta\}$. These restricted maxima will approximate the global maximum as $\delta \to 0$. The reason for omitting points near the boundary is that the covariance $Cov(\Theta_{r,x}, \Theta_{r,y})$ of the MGFF, for points x and y in I_δ, is logarithmic in $\|x - y\|$ (see (7.2.1)). Since we will be able to control the covariance on I_δ, we will be able to employ techniques from the general theory of Gaussian fields, such as Borell's Inequality, Slepian's Lemma and Fernique's Majorizing Criterion (see [8, Theorem 2.1], [8, Corollary 2.4] and [8, Theorem 4.1], respectively).

The second technique consists of comparing the maximum of continuum fields with the maximum of discrete fields, by using Slepian's Lemma. For Slepian's Lemma, we will need to employ fields with the same index set, which is accomplished by adding small continuum fields to the corresponding discrete fields.

In each chapter, we first present the propositions and lemmas and their relationship, and then present the proofs. A number of the proofs follow closely the proofs of the analogous results in [4]; it will only be necessary to make minor modifications to adapt them to the continuum case. Other proofs follow the general outline of the proofs presented in [4], but require special techniques in the context of the continuum. In both cases, we will mention the analogous results in [4] and the corresponding modifications.

A comment on constants: c will always denote a small positive constant and C will always denote a large positive constant, which will be allowed to change from line to line. Any dependence on other parameters will be made explicit by introducing subscripts. The phrase "absolute constant" will refer to fixed numbers that are independent of everything.

6.3 Related work

As mentioned earlier, the discrete analog of the MGFF is the discrete Gaussian free field (DGFF), which we will define in Chapter 7. We denote it here by $(\eta_{N,v} : v \in V_N)$, where V_N is a square lattice of size $N \times N$.

The DGFF has been studied extensively. In [15], Bolthausen, Deuschel and Giacomin showed that

$$\lim_{N\to\infty} \max_{v\in V_N} \frac{\eta_{N,v}}{\log N} = 2\sqrt{\frac{2}{\pi}}$$

in probability. In [16], Daviaud studied the number of extreme points of the DGFF and showed that, for all $0 < b < 1$,

$$\lim_{N\to\infty} \frac{\log(\mathcal{H}_N(b))}{\log N} = 2(1-b^2)$$

in probability, where $\mathcal{H}_N(b) = \#\left\{v \in V_N : \eta_{N,v} \geq 2\sqrt{\frac{2}{\pi}}b\log N\right\}$. In [5], Bramson and Zeitouni showed that the sequence of centered maxima $\{\max_{v\in V_N} \eta_{N,v} - m_N\}$ is tight (where $m_N = \sqrt{\frac{2}{\pi}}(2\log N - \frac{3}{4}\log\log N)$), and later on, Bramson, Ding and Zeitouni then established in [4] the convergence in law of $\max_{v\in V_N}(\eta_{N,v} - m_N)$ as $N \to \infty$. More recently, in [23], Biskup and Louidor studied the behavior of local maxima of the DGFF. Setting

$$\zeta_{N,r}(A \times B) = \sum_{v\in V_N} \mathbf{1}_{\{v/N\in A\}}\mathbf{1}_{\{\eta_{N,v}-m_N\in B\}}\mathbf{1}_{\{\eta_{N,v}=\max_{\|u-v\|_1\leq r}\eta_{N,u}\}}$$

for all Borel sets $A \subseteq [0,1]^2$ and $B \subseteq \mathbb{R}$, they showed that there exists an a.s. finite random measure $Z(dx)$ on $[0,1]^2$ that is a.s. strictly positive on open sets and such that, for every sequence r_N with $r_N \to \infty$ and $r_N/N \to 0$,

$$\zeta_{N,r_N} \to PPP(Z(dx) \otimes e^{-\sqrt{2\pi}y} dy)$$

in law as $N \to \infty$, where $PPP(Z(dx) \otimes e^{-\sqrt{2\pi}y} dy)$ is a Cox process with random intensity $Z(dx) \otimes e^{-\sqrt{2\pi}y} dy$.

A nice survey on the CGFF can be found in [2]. In [17], Hu, Miller and Peres studied the so called "thick points" of the MGFF. More specifically, they studied the MGFF $\Theta_{r,x}$ with respect to $\theta(w) = \mathbf{1}_{D(0,1)}(w)/\pi$ and referred to $x \in I$ as an *a-thick point* if

$$\lim_{r \to \infty} \frac{\Theta_{r,x}}{r} = \sqrt{a/2}.$$

(Note that the collection of a-thick points is a random set.) They showed that, for any $a \in [0,2]$, the Hausdorff dimension of the set of a-thick points is $2-a$ a.s.

We also mention [21], where Madaule, Rhodes and Vargas study a sequence of Gaussian fields that approximate the CGFF as follows. Let W_t be a Brownian motion and, for $x, y \in I$, let

$$p_{\partial I}(t,x,y) = \mathbb{P}^x(W_t \in dy, \tau_I > t)/dy,$$

where $\mathbb{P}^x$ means that the Brownian motion is started at x and $\tau_I = \inf\{t \geq 0 : W_t \notin I\}$. Consider the sequence of centered Gaussian fields $\{(X_n(x) : x \in I)\}_{n \geq 1}$ with covariance $Cov(X_n(x), X_n(y)) = \pi \int_{e^{-n}}^{\infty} p_{\partial I}(t,x,y)dt$ (where the pre-factor π is chosen so that the covariance is logarithmic in $\|x-y\|$). Madaule, Rhodes and Vargas proved the so called "Freezing Theorem for GFF", which states that, for any $\gamma > 2$, the sequence of random measures

$$\left\{ n^{3\gamma/4} e^{t_n(\gamma/\sqrt{2}-\sqrt{2})^2} M_n^{\gamma}(dx) \right\}_{n \geq 1},$$

where

$$M_n^{\gamma}(dx) = e^{\gamma X_n(x) - \gamma^2 n/2} dx,$$

converges in law towards a purely atomic random measure. Note that, for all $x, y \in I$, the covariance $\pi \int_{e^{-n}}^{\infty} p_{\partial I}(t,x,y)dt \to G_I(x,y)$ as $n \to \infty$, so this gives another way of approximating G_I. In this thesis, we approximate G_I by mollifying it.

More recently, in [24], Ding, Roy and Zeitouni proved convergence in law of the centered maximum for a more general class of discrete Gaussian fields, in any dimension. For $d \geq 1$, they considered log-correlated Gaussian field $(\varphi_{N,v} : v \in V_N)$ (where $V_N = \{0, 1, \ldots, N-1\}^d$), and they showed the convergence in law of $\max_{v \in V_N}(\varphi_{N,v} - m_{N,d})$ (where $m_{N,d} = \sqrt{2d} \log N - \frac{3/2}{\sqrt{2d}} \log\log N$) under the assumption that the covariance structure of φ_N converges off-diagonal in a macroscopic level, and it converges in finite scale around the diagonal (for the precise statements, see assumptions (A.2) and (A.3) in [24]).

Chapter 7

Preliminaries

In this chapter we present general results of the Gaussian free field and state a few lemmas that will be useful in the sequel. The proofs are deferred to the end of the chapter.

7.1 Markov property and self-similarity of the MGFF

Recall that $\mathcal{M}$ denotes the set of probability measures μ on $\mathbb{R}^2$ satisfying (6.1.2). For any closed set $K \subseteq I$, denote by $\sigma(K)$ the sigma-algebra generated by the collection $\{X(\mu) : \mu \in \mathcal{M}, \mathrm{supp}(\mu) \subseteq K\}$. When conditioning, we will abbreviate by using K instead of $\sigma(K)$. The CGFF satisfies the following Markov property (see [18, Theorem 1.2.1]): let $Q \subseteq I$ be a closed square, and let $\mu, \nu \in \mathcal{M}$ have support on Q and $\overline{Q^c}$, respectively. Then,

$$X(\mu) - \mathbb{E}[X(\mu) \mid \partial Q] \text{ is independent of } X(\nu). \tag{7.1.1}$$

Remark 7.1.1. Note that $supp(\mu) \subseteq Q$ is required for (7.1.1) to be true. In the case of the MGFF, we need $x + supp(\theta_r) \subseteq Q$. In other words, the Markov property is true for the MGFF only if x is away from the boundary of Q by a distance of order at least e^{-r}.

The following lemma provides a self-similarity property for the MGFF. This important property will be used repeatedly in the Chapters 8, 9 and 10. The proof uses basic properties of the Green function.

Lemma 7.1.2. *Fix $r > 0$ and let $Q \subseteq I$ be a sub-square with side length e^{-q}, where $q \in [0, r)$. Then,*

$$(\Theta_{r,x} - \mathbb{E}[\Theta_{r,x} \mid \partial Q] : x + \mathrm{supp}(\theta_r) \subseteq Q) \overset{law}{=} \left(\Theta_{r-q,h(x)} : x + \mathrm{supp}(\theta_r) \subseteq Q\right), \quad (7.1.2)$$

where $h : Q \to I$ is the map that stretches Q linearly onto I.

7.2 Estimates on the MGFF, continuity and tightness

In this section, we present some rough estimates on the covariance structure of the MGFF and on the tail probabilities of its maximum. We also prove continuity of the MGFF and tightness of the centered maximum.

For $\delta > 0$, let $I_\delta := \{x \in I : dist(x, \partial I) \geq \delta\}$, and recall that $\theta = \theta(\cdot)$ is a fixed mollifier.

The following lemma follows from basic bounds by employing the logarithmic nature of the two-dimensional Green function. Note that (7.2.1) is the continuum analog of (7.3.2). On the other hand, (7.2.2) provides an upper bound on the intrinsic (i.e., L^2) distance of the MGFF for close points. This upper bound will be used in later proofs (e.g., whenever Fernique's Majorizing Criterion is employed).

Lemma 7.2.1. *Fix $\delta > 0$. There exists a constant $C_{\delta,\theta} \in (0, \infty)$ and a positive number $r_{\delta,\theta}$ such that, for all $x, y \in I_\delta$ and all $r \geq r_{\delta,\theta}$,*

$$\left|Cov(\Theta_{r,x}, \Theta_{r,y}) + \log(\max\left\{e^{-r}, \|x - y\|\right\})\right| \leq C_{\delta,\theta}. \quad (7.2.1)$$

Moreover, there exists a constant $C_\theta \in (0, \infty)$ such that, for all r large enough (depending on θ), and all $x, y \in I$,

$$\mathbb{E}\left[(\Theta_{r,x} - \Theta_{r,y})^2\right] \leq C_\theta e^r \|x - y\|. \quad (7.2.2)$$

(Note that the last bound is of interest when $\|x - y\| \leq e^{-r}$.)

Remark 7.2.2. Note that (7.2.2) and Kolmogorov's Continuity Criterion (see [10, Theorem 1.4.17]) imply that the MGFF $\Theta_{r,x}$ has a version which is continuous in x. From now on, we assume we are working with this version. In particular, $\Theta_r^* = \max_{x \in I} \Theta_{r,x}$ is attained.

We next prove that the MGFF has a unique maximum point. This property will be useful in Chapter 10, where we employ the maxima of the fine fields. The main tools used in its proof are the Markov property and the (only) lemma in [25], which provides sufficient conditions for the maximum of a Gaussian field to have a continuous distribution.

Lemma 7.2.3. *Fix $r \geq 0$. There exists almost surely a unique random point $\chi \in I$ that maximizes the MGFF $\Theta_{r,(\cdot)}$ over I.*

Remark 7.2.4. Lemma 7.2.3 is also true if we replace I by any closed sub-square of I.

A consequence of Lemmas 7.2.1, 7.2.3 and [7, Theorem 1.1] is the tightness of the centered MGFF.

Lemma 7.2.5. *There exists $r_\theta > 0$ large enough (depending on θ) such that the collection $\{\Theta_r^* - m_r : r \geq r_\theta\}$ is tight.*

The following two lemmas are a continuum version of Lemma 7.3.2. Their proofs use (7.3.3) and (7.3.4) together with comparison arguments employing Slepian's Lemma and Fernique's Majorizing Criterion. As mentioned before, small squares of Brownian sheet will be added to the DGFF in order to perform the comparison with the MGFF.

Lemma 7.2.6. *There exist constants $c_\theta, C_\theta \in (0,\infty)$ and a positive constant r_θ such that, for all $r \geq r_\theta$,*

$$c_\theta \lambda e^{-2\lambda} \leq \mathbb{P}\left(\Theta_r^* - m_r \geq \lambda\right) \leq C_\theta \lambda e^{-2\lambda} e^{-c_\theta \lambda^2/r}, \tag{7.2.3}$$

where the lower bound holds for $\lambda \in [1, \sqrt{r}]$ and the upper bound holds for all $\lambda \geq 1$.

Lemma 7.2.7. *There exist a constant $C_\theta \in (0,\infty)$ and a positive constant r_θ such that, for all closed sub-squares $A \subseteq I$, all $r \geq r_\theta$ and all $\lambda \in \mathbb{R}$,*

$$\mathbb{P}\left(\max_{x\in A} \Theta_{r,x} - m_r \geq \lambda\right) \leq C_\theta |A|^{1/2} \max\{\lambda, 1\} e^{-2\lambda}, \tag{7.2.4}$$

where $|A|$ is the Lebesgue measure of A.

For the next lemma, we let $K = \left[\delta, \delta + e^{-k}\right]^2$, where $k > 0$, and we let $K_\delta = \left\{x \in K : dist(x, \partial K) \geq \delta e^{-k}\right\}$. Then, by denoting $\Phi_{r,x} = \mathbb{E}\left[\Theta_{r,x} \mid \partial K\right]$, we have the following estimates.

Lemma 7.2.8. *There exists a constant $C_{\delta,\theta} \in (0,\infty)$ such that, for all $x, y, y' \in K_\delta$,*

$$\left|Cov(\Phi_{r,x}, \Phi_{r,y}) - Cov(\Phi_{r,x}, \Phi_{r,y'})\right| \le C_{\delta,\theta} \|y - y'\| e^k. \tag{7.2.5}$$

Additionally, there exist constants $c_{\delta,\theta}, C_{\delta,\theta} \in (0,\infty)$ such that, for all $x, y \in K_\delta$,

$$c_{\delta,\theta}\|x-y\|^2 e^{2k} \le \mathbb{E}\left[(\Phi_{r,x} - \Phi_{r,y})^2\right] \le C_{\delta,\theta}\|x-y\|^2 e^{2k}. \tag{7.2.6}$$

Estimate (7.2.5) follows from basic properties of the Green function, similarly to the proof of (7.2.1). On the other hand, (7.2.6) follows from analogous estimates on the DGFF (see [4, Lemma 3.10]).

7.3 Discrete Gaussian free field

In this section we review the two-dimensional discrete Gaussian free field (DGFF) and state some of its properties. The lemmas in this section will not be proved, as they are immediate consequence of lemmas proved in [4].

For $r \ge 1$, let $V_r = I \cap (e^{-r}\mathbb{Z}^2)$, and, for $x \in V_r$, let $\mathbb{E}^x$ denote the expectation with respect to the law of a simple symmetric random walk $(S_i)_{i\ge 0}$ on $e^{-r}\mathbb{Z}^2$ started at x. The discrete Green function $G_r : V_r \times V_r \to [0,\infty)$ associated with V_r is defined by

$$G_r(x,y) = \frac{\pi}{2}\mathbb{E}^x\left[\sum_{0\le i<\tau_r} \mathbf{1}_{\{S_i = y\}}\right], \tag{7.3.1}$$

where $\tau_r := \min\{i \ge 0 : S_i \notin V_r\}$ and the pre-factor $\frac{\pi}{2}$ is chosen so that G_r is logarithmic (see (7.3.2)). For $r \ge 1$, the DGFF on V_r is defined as the centered Gaussian field $(\eta_{r,v} : v \in V_r)$ with covariance structure

$$Cov(\eta_{r,v}, \eta_{r,w}) = G_r(v,w)$$

for all $v, w \in V_r$. Let $V_r^\delta = \{v \in V_r : dist(v, \partial I) \ge \delta\}$. Then, from [4, Lemma 3.1], it follows easily that the Green function G_r satisfies:

Lemma 7.3.1. *There exists a constant $C_\delta \in (0,\infty)$ such that, for all $v, w \in V_r^\delta$,*

$$|G_r(x,y) + \log(\max\{e^{-r}, \|x-y\|\})| \le C_\delta. \tag{7.3.2}$$

Additionally, from [22, Theorem 1.4] and [4, Lemma 3.8], it follows that the DGFF satisfies the following estimates:

Lemma 7.3.2. *There exist absolute constants $c, C \in (0,\infty)$ such that, for all $r \geq 1$,*

$$c\lambda e^{-2\lambda} \leq \mathbb{P}\left(\max_{v\in V_r} \eta_{r,v} - m_r \geq \lambda\right) \leq C\lambda e^{-2\lambda} e^{-c\lambda^2/r}, \tag{7.3.3}$$

where the lower bound holds for $\lambda \in [1, \sqrt{r}]$ and the upper bound holds for all $\lambda \geq 1$. Additionally, for any closed sub-square $A \subseteq I$,

$$\mathbb{P}\left(\max_{v\in V_r\cap A} \eta_{r,v} - m_r \geq \lambda\right) \leq C|A|^{1/2} \max\{\lambda, 1\} e^{-2\lambda} \tag{7.3.4}$$

for all $\lambda \in \mathbb{R}$, where $|A|$ is the Lebesgue measure of A.

7.4 Brownian sheet

We will need the Brownian sheet, which is defined as the centered Gaussian field $\left(\phi_x : x \in [0,\infty)^2\right)$ with covariance structure

$$Cov(\phi_x, \phi_y) = \min\{x_1, y_1\} \min\{x_2, y_2\} \tag{7.4.1}$$

for all $x, y \in [0,\infty)^2$. It is not difficult to show that, for any $p \geq 1$, the Brownian sheet satisfies the following properties: for all $x, y \in [p, p+1)^2$,

$$p\|x-y\|_1 \leq \mathbb{E}\left[(\phi_x - \phi_y)^2\right] \leq (p+1)\|x-y\|_1, \tag{7.4.2}$$

$$p^2 \leq Cov(\phi_x, \phi_y) \leq (p+1)^2 \tag{7.4.3}$$

and, for some constants $c_p, C_p \in (0,\infty)$ and all $\lambda \geq 0$,

$$\mathbb{P}\left(\sup_{x\in[p,p+1)^2} \phi_x \geq \lambda\right) \leq C_p e^{-c_p\lambda^2}. \tag{7.4.4}$$

Remark 7.4.1. A proof of (7.4.4) can be found in [7, Lemma 2.2]. Displays (7.4.2) and (7.4.3) follow easily from the definition; hence, we omit their proofs.

7.5 Continuous modified branching random walk

In this section we define the continuous modified branching random walk (CMBRW), which is a continuum version of the modified branching random walk (MBRW), a field introduced by Bramson and Zeitouni in [5]. We let $(\xi_x(t) : x \in \mathbb{R}^2, t \geq 0)$ be the centered Gaussian field with covariance structure

$$Cov(\xi_x(t), \xi_y(s)) = \int_0^{min\{t,s\}} \Big(1 - e^h|x_1 - y_1|\Big)_+ \Big(1 - e^h|x_2 - y_2|\Big)_+ dh, \qquad (7.5.1)$$

where $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$ and $t, s \geq 0$, and $(\cdot)_+ = \max\{\cdot, 0\}$. Note that an alternative expression for the covariance of the CMBRW is

$$Cov(\xi_x(t), \xi_y(s)) = \int_0^{min\{t,s\}} Area(B(h, x) \cap B(h, y))dh$$

where $B(h, x)$ is a box with side length 1 centered at $e^h x$. Points that are far apart will produce processes $\xi_x(t)$ and $\xi_y(t)$ with low correlation, while points that are close will produce processes with high correlation. Note that, for a fixed point x, the process $(\xi_x(t))_{t\geq 0}$ is a standard Brownian motion. A proof that the right hand side of (7.5.1) is positive semi-definite can be found in [7, Proposition 4.1]. The structure of the CMBRW satisfies the following properties.

Lemma 7.5.1. *For a pair of different points $x, y \in \mathbb{R}^2$, let $s_{x,y} := -\log(\|x - y\|_\infty)$. Then, for all $t \in [0, s_{x,y}]$,*

$$t - 2 \leq Cov(\xi_x(t), \xi_y(t)) \leq t. \qquad (7.5.2)$$

Moreover, for $t \geq s_{x,y}$,

$$Cov(\xi_x(t), \xi_y(t)) = Cov(\xi_x(s_{x,y}), \xi_y(s_{x,y})). \qquad (7.5.3)$$

Additionally, the CMBRW satisfies the following estimates on the right tail distribution of the maximum over $V_r = I \cap (e^{-r}\mathbb{Z}^2)$.

Lemma 7.5.2. *There exist constants $c, C \in (0, \infty)$ such that, for all $\lambda \in [1, \sqrt{r}]$,*

$$c\lambda e^{-2\lambda} \leq \mathbb{P}\left(\max_{v \in V_r} \xi_v(r) \geq m_r + \lambda\right) \leq C\lambda e^{-2\lambda}. \qquad (7.5.4)$$

Moreover, for any sub-square $A \subseteq I$, and all r large enough (depending on $|A|$),

$$c|A|\lambda e^{-2\lambda} \leq \mathbb{P}\left(\max_{v \in A \cap V_r} \xi_v(r) \geq m_r + \lambda\right) \tag{7.5.5}$$

for all $\lambda \in [1, \sqrt{r}]$, where $|A|$ is the Lebesgue measure of A.

Remark 7.5.3. The proof of (7.5.4) is given in [22, Lemma 3.7]. We will prove (7.5.5), whose proof employs (7.5.4) and the stationarity of the CMBRW.

We now provide the proofs of Lemmas 7.1.2, 7.2.1, 7.2.3, 7.2.6, 7.2.7, 7.2.8, 7.5.1 and 7.5.2.

7.6 Proof of Lemma 7.1.2

Proof of (7.1.2). We choose $Q = [0, e^{-q}]^2$ in order to keep the notation simple, but we note that the proof for an arbitrary Q is analogous. Since the fields are Gaussian and centered, it is enough to show that the covariance structures are equal. Let $x, y \in Q$ be such that $x + \mathrm{supp}(\theta_r), y + \mathrm{supp}(\theta_r) \subseteq Q$, and note that

$$\begin{aligned} &Cov\left(\Theta_{r,x} - \mathbb{E}\left[\Theta_{r,x} \mid \partial Q\right], \Theta_{r,y} - \mathbb{E}\left[\Theta_{r,y} \mid \partial Q\right]\right) \\ &= Cov\left(\Theta_{r,x} - \mathbb{E}\left[\Theta_{r,x} \mid \partial Q\right], \Theta_{r,y}\right). \end{aligned}$$

It follows from the definition in (6.1.3) and [18, Theorem 1.2.2] that the previous display is

$$= Cov(X(\rho_{r,x}) - X(\beta_{\rho_{r,x}}), X(\rho_{r,y})), \tag{7.6.1}$$

where $\beta_{\rho_{r,x}}$ is the harmonic measure on ∂Q of a Brownian motion with initial distribution $\rho_{r,x}$. That is, for all Borel sets $E \subseteq \partial Q$,

$$\beta_{\rho_{r,x}}(E) = \int \mathbb{P}^u(W_{\tau_Q} \in E)\theta_r(x-u)du,$$

where W_t is a Brownian motion started at u and $\tau_Q = \inf\{t \geq 0 : W_t \notin Q\}$. Then, (7.6.1) is

$$= \iint \left(G_I(u,v) - \mathbb{E}^u\left[G_I\left(W_{\tau_Q}, v\right)\right]\right)\theta_r(x-u)\theta_r(y-v)dudv, \tag{7.6.2}$$

where W_t is a Brownian motion started at u and $\tau_Q = \inf\{t \geq 0 : W_t \notin Q\}$. On the other hand,

$$\begin{aligned}
&Cov(\Theta_{r-q,e^q x}, \Theta_{r-q,e^q y}) \\
&\quad = \iint G_I(u,v)\Theta_{r-q}(e^q x - u)\Theta_{r-q}(e^q y - v)dudv \\
&\quad = \iint G_I(e^q u, e^q v)\Theta_r(x-u)\Theta_r(y-v)dudv, \qquad (7.6.3)
\end{aligned}$$

where the previous display follows from the change of variables $(u,v) \mapsto e^q(u,v)$. Therefore, the covariance structures (7.6.2) and (7.6.3) are equal if

$$G_I(u,v) - \mathbb{E}^u\left[G_I\left(W_{\tau_Q}, v\right)\right] = G_I(e^q u, e^q v).$$

We now prove the previous display. From (6.1.1), we obtain

$$\begin{aligned}
&G_I(u,v) - \mathbb{E}^u\left[G_I(W_{\tau_Q}, v)\right] \\
&\quad = -\log\|u-v\| + \mathbb{E}^u[\log\|W_{\tau_I} - v\|] \\
&\qquad - \mathbb{E}^u\left[-\log\|W_{\tau_Q} - v\| + \mathbb{E}^{W_{\tau_Q}}[\log\|W_{\tau_I} - v\|]\right] \\
&\quad = -\log\|u-v\| + \mathbb{E}^u[\log\|W_{\tau_Q} - v\|],
\end{aligned}$$

where the last equality follows because, by the Markov property of Brownian motion, $\mathbb{E}^u[\mathbb{E}^{W_{\tau_Q}}[\log\|W_{\tau_I} - v\|]] = \mathbb{E}^u[\log\|W_{\tau_I} - v\|]$. Note that, by elementary geometry,

$$\mathbb{E}^u[\log\|W_{\tau_Q} - v\|] = \mathbb{E}^{e^q u}[\log(\|W_{\tau_I} - e^q v\|/e^q)].$$

The last two displays imply

$$\begin{aligned}
G_I(u,v) - \mathbb{E}^u\left[G_I(W_{\tau_Q}, v)\right] &= -\log\|u-v\| + \mathbb{E}^{e^q u}[\log(\|W_{\tau_I} - e^q v\|/e^q)] \\
= -\log\|e^q u - e^q v\| + \mathbb{E}^{e^q u}[\log\|W_{\tau_I} - e^q v\|] &= G_I(e^q u, e^q v),
\end{aligned}$$

as desired. $\square$

7.7 Proof of Lemma 7.2.1

Proof of (7.2.1). From (6.1.1),

$$\begin{aligned}
&\left|Cov(\Theta_{r,x}, \Theta_{r,y}) + \iint \log\|x-u-y+v\|\theta_r(u)\theta_r(v)dudv\right| \\
&\quad \leq \iint \mathbb{E}^{x-u}\left[|\log\|y-v-W_{\tau_I}\||\right]\theta_r(u)\theta_r(v)dudv.
\end{aligned}$$

We choose $r_{\delta,\theta}$ so that, for all $r \geq r_{\delta,\theta}$, we have $(y + \text{supp}(\theta_r)) \subseteq I_{\delta/2}$. Therefore, the previous display is

$$\leq -\log(\delta/2).$$

Thus, in order to prove (7.2.1), it is enough to prove

$$\left| \iint \log \|x-u-y+v\| \theta_r(u)\theta_r(v) du dv - \log\left(\max\left\{e^{-r}, \|x-y\|\right\}\right) \right| \leq C_\theta.$$

We choose a large constant $C'_\theta \in (0,\infty)$ such that, if $\|x-y\| \geq C'_\theta e^{-r}$, then $\|x-y\|/C'_\theta \leq \|x-u-y+v\| \leq C'_\theta \|x-y\|$ for all $u, v \in \text{supp}(\theta_r)$. It then follows that, if $\|x-y\| \geq C'_\theta e^{-r}$,

$$\left| \iint \log \|x-u-y+v\| \theta_r(u)\theta_r(v) du dv - \log \|x-y\| \right| \leq \log C'_\theta =: C_\theta.$$

On the other hand, when $\|x-y\| \leq C_\theta e^{-r}$,

$$\begin{aligned} &\left| \iint \left(\log\left(\|x-u-y+v\|\right) + n\right) \theta_r(u)\theta_r(v) du dv \right| \\ &\qquad \leq \max_{\|z\| \leq C_\theta} \iint \left|\log \|z + (-u+v)\|\right| \theta(u)\theta(v) du dv \leq C_\theta, \end{aligned}$$

as desired. Display (7.2.1) follows by adjusting the constants and using basic properties of log. □

Proof of (7.2.2). Since

$$\begin{aligned} &\mathbb{E}\left[(\Theta_{r,x} - \Theta_{r,y})^2\right] \\ &\qquad \leq \left|Var\left(\Theta_{r,x}\right) - Cov\left(\Theta_{r,x}, \Theta_{r,y}\right)\right| + \left|Var\left(\Theta_{r,y}\right) - Cov\left(\Theta_{r,x}, \Theta_{r,y}\right)\right|, \end{aligned}$$

it is enough to bound $|Var(\Theta_{r,x}) - Cov(\Theta_{r,x}, \Theta_{r,y})|$ (because the bound for the other term is analogous). Note that

$$\begin{aligned} &|Var(\Theta_{r,x}) - Cov(\Theta_{r,x}, \Theta_{r,y})| \\ &\quad = \left| \iint \left(G_I(x-u, x-v) - G_I(x-u, y-v)\right) \theta_r(u)\theta_r(v) du dv \right| \\ &\quad \leq \iint \left|\log \|u-v\| - \log \|x-y-u+v\|\right| \theta_r(u)\theta_r(v) du dv \\ &\qquad + \iint \mathbb{E}^{x-u}\left[\left|\log \|x-v-W_{\tau_I}\| - \log \|y-u-W_{\tau_I}\|\right|\right] \theta_r(u)\theta_r(v) du dv, \end{aligned}$$

where W_t is a Brownian motion started at $x-u$ and $\tau_I = \inf\{t \geq 0 : W_t \notin I\}$. But

$$|\log\|u-v\| - \log\|x-y-u+w\|| \leq \|x-y\| \left(\frac{1}{\|u-v\|} + \frac{1}{\|x-y-u+v\|} \right).$$

For r large enough (depending on δ and θ), we have $\|x-v-W_{\tau_I}\| \geq \delta/2$ for all $v \in \mathrm{supp}(\theta_r)$. Therefore,

$$\begin{aligned} &|\log\|x-v-W_{\tau_I}\| - \log\|y-v-W_{\tau_I}\|| \\ &\quad \leq \|x-y\| \left(\frac{1}{\|x-v-W_{\tau_I}\|} + \frac{1}{\|y-v-W_{\tau_I}\|} \right) \leq C_\delta \|x-y\|. \end{aligned}$$

The last three displays imply

$$\begin{aligned} &|Var(\Theta_{r,x}) - Cov(\Theta_{r,x}, \Theta_{r,y})| \\ &\quad \leq \|x-y\| \iint \left(\frac{1}{\|u-v\|} + \frac{1}{\|x-y+u-v\|} \right) \theta_r(u)\theta_r(v) du dv + C_\delta \|x-y\| \\ &\quad \leq 2\|x-y\| \sup_{\|z\|\leq C} \iint \frac{1}{\|z+u-v\|} \theta_r(u)\theta_r(v) du dv + C_\delta \|x-y\| \\ &\quad \leq C_\theta e^r \|x-y\| + C_\delta \|x-y\| \leq C_{\delta,\theta} e^r \|x-y\|. \end{aligned}$$

Therefore,

$$\mathbb{E}[(\Theta_{r,x} - \Theta_{r,y})^2] \leq C_{\delta,\theta} e^r \|x-y\|.$$

We now improve the previous display so that the constant term does not depend on δ. Let Q_0 be the square of side length e^{-1}, which is concentric with I. Then, from the previous display, we obtain that there exists a constant C_θ (depending only on θ) such that

$$\mathbb{E}[(\Theta_{r,x} - \Theta_{r,y})^2] \leq C_\theta e^r \|x-y\|$$

for all $x, y \in Q_0$. Note that

$$\mathbb{E}[(\Theta_{r,x} - \mathbb{E}[\Theta_{r,x} \mid \partial Q_0] - \Theta_{r,y} + \mathbb{E}[\Theta_{r,y} \mid \partial Q_0])^2] \leq \mathbb{E}[(\Theta_{r,x} - \Theta_{r,y})^2].$$

Therefore, the last two displays and (7.1.2) imply

$$\mathbb{E}[(\Theta_{r-1,h(x)} - \Theta_{r-1,h(y)})^2] \leq C_\theta e^r \|x-y\| = C_\theta e^{n-1} \|h(x) - h(y)\|,$$

where $h : Q_0 \to I$ is the linear map that stretches Q_0 onto I. This finishes the proof of (7.2.2). □

7.8 Proof of Lemma 7.2.3

Proof. Let $H > 1$ be an arbitrary integer. Divide I into H^2 (closed) sub-squares of side length $1/H$. We show that, for any two non-adjacent sub-squares S_1, S_2, the probability of the event

$$\max_{x \in S_1} \Theta_{r,x} = \max_{x \in S_2} \Theta_{r,x}$$

is 0. Augment S_2 by considering the closed set $S_2' = \{x \in I : \text{ there exists } y \in S_2 \text{ such that } x - y \in \text{supp}(\theta_r)\}$. Let $\mathcal{F}$ denote the sigma-algebra generated by $\{X(\mu) : \mu \in \mathcal{M} \text{ and } \text{supp}(\mu) \subseteq S_2'\}$.

By a similar argument to that in the proof of Lemma 7.1.2, it is possible to conclude that the field

$$(\Theta_{r,x} - \mathbb{E}[\Theta_{r,x} \mid \mathcal{F}] : x \in S_1)$$

has variance structure

$$Var(\Theta_{r,x} - \mathbb{E}[\Theta_{r,x} \mid \mathcal{F}]) = \iint G_{S_2}(x-u, x-v)\theta_r(u)\theta_r(v)dudv$$

where

$$G_{S_2}(u,v) = -\log \|u - v\| + \mathbb{E}^u[\log \|W_\tau - v\|],$$

W_t is a Brownian motion and $\tau = \inf\{t \geq 0 : W_t \in S_2 \cup I^c\}$. Therefore, since G_{S_2} is strictly positive outside of $S_2 \cup I^c$, for any $x \in S_1$,

$$Var(\Theta_{r,x} \mid \mathcal{F}) > 0$$

almost surely. The previous display implies, by using the (only) lemma in [25], that $\max_{x \in S_1} \Theta_{r,x}$, conditional on $\mathcal{F}$, has a continuous distribution. In particular,

$$\mathbb{P}\left(\max_{x \in S_1} \Theta_{r,x} = \max_{x \in S_2} \Theta_{r,x} \mid \mathcal{F}\right) = 0$$

since $\max_{x \in S_2} \Theta_{r,x}$ is a constant conditional on $\mathcal{F}$. By taking the expectation,

$$\mathbb{P}\left(\max_{x \in S_1} \Theta_{r,x} = \max_{x \in S_2} \Theta_{r,x}\right) = 0.$$

Since the previous display can be proved for any two non-adjacent squares, we obtain that if the MGFF has two global maxima x and y, then they are within distance $2\sqrt{2}/H$ of each other almost surely. Sinc H is arbitrary, we can take the union over all $H > 1$ and we obtain that the maximum is unique almost surely. $\square$

7.9 Proof of Lemma 7.2.5

Proof. Let Q be the closed square centered at $(\frac{1}{2}, \frac{1}{2}) \in \mathbb{R}^2$, with side length e^{-1}. From Lemma 7.2.1 and [7, Theorem 1.1], there exist constants $c_\theta, C_\theta, r_\theta > 0$, depending only on the mollifier θ, such that, for all $r \geq r_\theta$,

$$\mathbb{P}\left(\left|\max_{x\in Q} \Theta_{r,x} - m_r\right| \geq \lambda\right) \leq C_\theta e^{-c_\theta \lambda} \tag{7.9.1}$$

for all $\lambda \geq 0$. (Note that the side length e^{-1} of Q is arbitrary: it can be changed to any other side length less than 1 and (7.9.1) still holds by adjusting the constants c_θ, C_θ.) From this, it immediately follows that

$$\mathbb{P}\left(\Theta_r^* - m_r \leq -\lambda\right) \leq C_\theta e^{-c_\theta \lambda} \tag{7.9.2}$$

for all $\lambda \geq 0$.

For $x \in Q$, we decompose

$$\Theta_{r,x} = \Phi_{r,x} + \mathbb{E}[\Theta_{r,x} \mid \partial Q];$$

due to Lemma 7.1.2,

$$(\Phi_{r,x} : x \in Q) \overset{law}{=} (\Theta_{r-1,h(x)} : x \in Q),$$

where $h : Q \to I$ is the map that stretches Q linearly onto I.

Let χ be the (random) point that maximizes of $\Phi_{r,(\cdot)}$ over Q, which is unique due to Lemma 7.2.3. Then,

$$\{\Phi_{r,\chi} - m_r \geq \lambda, \mathbb{E}[\Theta_{r,\chi} \mid \partial Q] \geq 0\} \subseteq \{\max_{x\in Q} \Theta_{r,x} - m_r \geq \lambda\}.$$

Since $\mathbb{E}[\Theta_{r,\chi} \mid \partial Q]$ is symmetric around 0, this display, (7.1.1) and (7.9.1) together imply that

$$\mathbb{P}(\Theta_{r-1}^* - m_r \geq \lambda) \leq 2C_\theta e^{-c_\theta \lambda}.$$

Since $m_r = m_{r-1} + O(1)$, the previous display implies that, for all $\lambda \geq 0$,

$$\mathbb{P}(\Theta_{r-1}^* - m_{r-1} \geq \lambda) \leq C_\theta e^{-c_\theta \lambda} \tag{7.9.3}$$

for some constants $c_\theta, C_\theta > 0$. Displays (7.9.2) and (7.9.3) together imply Lemma 7.2.5. $\square$

7.10 Proof of Lemma 7.2.6

The proof of (7.2.3) is based on a comparison argument using Slepian's Lemma. Recall that $(\eta_{r,v})_{v\in V_r}$ is DGFF on $V_r = I \cap (e^{-r}\mathbb{Z}^2)$. We are comparing a continuum field (the MGFF) with a discrete field (the DGFF). In order to perform the comparison, we will add small patches of Brownian sheet to the DGFF. We will need the following notation.

For $r \geq 1$ and $v = (v_1, v_2) \in V_r$, let $\square_{r,v} = [v_1, v_1+e^{-r}) \times [v_2, v_2+e^{-r}) \cap I$ and, for $x \in \square_{r,v}$, set $[x] = v$. The notation $[\cdot]$ depends on r, but we keep this dependence implicit. Let p and q be large constants that will be chosen later (see (7.10.3), (7.10.4), Claim 7.10.1 and (7.10.9)). Denote by Q the square of side length e^{-q} that is concentric with I, and let $h : Q \to [1/4, 3/4]^2$ be the map that stretches Q linearly onto $[1/4, 3/4]^2 \subset I$.

For the proof of the upper bound of (7.2.3), we will compare the fields

$$(\Theta_{r,x} : x \in Q) \text{ and } \left(a_r(x)\eta_{r,h([x])} + \psi_{r,x} : x \in Q\right),$$

where $a_r(x) \geq 0$ is defined so that

$$Var(\Theta_{r,x}) = Var(a_r(x)\eta_{r,h([x])} + \psi_{r,x}) \tag{7.10.1}$$

for all $x \in Q$, and $(\psi_{r,x} : x \in Q)$ is defined as follows. For each $v \in V_r$, let P_v be the linear map from $\square_{r,v}$ onto $[p, p+1)^2$ sending v to (p, p). Recall from (7.4.1) that $(\phi_x : x \in [p, p+1)^2)$ is a two-dimensional Brownian sheet. For each $v \in V_r$, let

$$(\psi_{r,x} : x \in \square_{r,v}) := (\phi_{P_v(x)} : x \in \square_{r,v}).$$

Note that the collection $\{(\psi_{r,x} : x \in \square_{r,v})\}_{v\in V_r}$ consists of $O(e^{2r})$ identical (not independent) copies of a scaled Brownian sheet. On the other hand, for the proof of the lower bound of (7.2.3), we will compare the fields

$$(\eta_{r,v} : v \in V_r \cap Q) \text{ and } \left(b_r(v)\Theta_{r+p,h(v)} : v \in V_r \cap Q\right),$$

where $b_r(v) \geq 0$ is defined so that

$$Var(\eta_{r,v}) = Var(b_r(v)\Theta_{r+p,h(v)}) \tag{7.10.2}$$

for all $v \in V_r \cap Q$.

Note that, by applying (7.2.1), (7.3.2) and (7.4.3) on (7.10.1) and (7.10.2), we obtain

$$\frac{1}{4} \leq \frac{r - C_\theta - (p+1)^2}{r+C} \leq \frac{Var(\Theta_{r,x}) - Var(\psi_{r,x})}{Var(\eta_{r,h([x])})} = a_r^2(x) \leq \frac{r + C_\theta - p^2}{r - C} \leq 1 \quad (7.10.3)$$

and

$$b_r(v)^2 = \frac{Var(\eta_{r,v})}{Var(\Theta_{r+p,h(v)})} \leq \frac{r+C}{r+p-C_\theta} \leq 1 \qquad (7.10.4)$$

for $p = p_\theta$ large enough and for all r large enough, depending on p and θ. (Here, C is an absolute constant because $h([x]) \in [1/4, 3/4]^2$.)

We first prove the upper bound of (7.2.3). We begin by proving the following claim.

Claim 7.10.1. There exist $p = p_\theta$ and $q = q_\theta$ large enough such that, for all r large enough (depending on θ),

$$Cov(a_r(x)\eta_{r,h([x])} + \psi_{r,x}, a_n(y)\eta_{r,h([y])} + \psi_{r,y}) \leq Cov(\Theta_{r,x}, \Theta_{r,y})$$

for all $x, y \in Q$.

Proof of Claim 7.10.1. We distinguish two cases:

- Case 1: $[x] = [y]$. In this case, (7.2.2) and (7.4.2) imply

  $$\mathbb{E}[(\Theta_{r,x} - \Theta_{r,y})^2] \leq C_\theta e^r \|x - y\| \leq p e^r \|x-y\|_1 \leq \mathbb{E}[(\psi_{r,x} - \psi_{r,y})^2]$$

  for $p = p_\theta$ large enough.

- Case 2: $[x] \neq [y]$. Using the upper bound of (7.10.3), together with (7.2.1), (7.3.2), (7.4.3) and the definition of $h(\cdot)$,

  $$\begin{aligned} &Cov(a_r(x)\eta_{r,h([x])} + \psi_{r,x}, a_n(y)\eta_{r,h([y])} + \psi_{r,y}) \\ &\quad \leq -q - \log \|[x] - [y]\| + (p+1)^2 + C \\ &\quad \leq -\log(\max\{e^{-r}, \|x-y\|\}) - C_\theta \leq Cov(\Theta_{r,x}, \Theta_{r,y}) \end{aligned}$$

  for $q = q_{p,\theta}$ large enough.

$\square$

Claim 7.10.1 and Slepian's Lemma (see [8, Corollary 2.4]) imply

$$\mathbb{P}\left(\max_{x\in Q}\Theta_{r,x}\geq\lambda\right)\leq\mathbb{P}\left(\sup_{x\in Q}a_r(x)\eta_{r,h([x])}+\psi_{r,x}\geq\lambda\right) \tag{7.10.5}$$

for all $\lambda\in\mathbb{R}$. We can now prove (7.2.3).

Proof of (7.2.3)*, the upper bound.* Let $\psi^*=\sup_{x\in Q}\psi_{r,x}$. Then, by applying (7.10.5) and decomposing according to the value of ψ^*,

$$\begin{aligned}\mathbb{P}\left(\max_{x\in Q}\Theta_{r,x}\geq m_r+\lambda\right)&\leq\mathbb{P}\left(\sup_{x\in Q}a_r(x)\eta_{r,h([x])}+\psi^*\geq m_r+\lambda\right)\\&\leq\sum_{z\geq 0}\mathbb{P}\left(\sup_{x\in Q}a_r(x)\eta_{r,h([x])}\geq m_r+\lambda-z\right)\left(\mathbb{P}\left(z\leq\psi^*<z+1\right)+1_{z=0}\right),\end{aligned} \tag{7.10.6}$$

where the $1_{z=0}$ term accounts for the probability that $\psi^*<0$. By applying (7.10.3), we obtain that the previous display is

$$\begin{aligned}&\leq\sum_{z\geq 0}\mathbb{P}\left(\sup_{x\in Q}\eta_{r,h([x])}\geq m_r+\lambda-2z\right)\left(\mathbb{P}\left(z\leq\psi^*<z+1\right)+1_{z=0}\right)\\&\leq\sum_{z\geq 0}\mathbb{P}\left(\max_{v\in V_r}\eta_{r,v}\geq m_r+\lambda-2z\right)\left(\mathbb{P}\left(z\leq\psi^*<z+1\right)+1_{z=0}\right)\\&\leq\sum_{0\leq z<\lambda/2}\mathbb{P}\left(\max_{v\in V_r}\eta_{r,v}\geq m_r+\lambda-2z\right)\left(\mathbb{P}\left(z\leq\psi^*<z+1\right)+1_{z=0}\right)\\&\quad+\mathbb{P}(\psi^*>\tfrac{\lambda}{2}).\end{aligned}$$

We now apply (7.3.3) and (7.4.4) to obtain that the previous display is

$$\begin{aligned}&\leq\sum_{0\leq z<\lambda/2}(\lambda-2z)e^{-2(\lambda-2z)}e^{-c(\lambda-2z)^2/r}C_pe^{-c_pz^2}+C_pe^{-c_p\lambda^2}\\&\leq C_p\lambda e^{-2\lambda}e^{-c_p\lambda^2/r}.\end{aligned}$$

(recall that the constant C_p is allowed to change from line to line). Thus, from (7.10.6) and the previous display,

$$\mathbb{P}\left(\max_{x\in Q}\Theta_{r,x}-m_r\geq\lambda\right)\leq C_p\lambda e^{-2\lambda}e^{-c_p\lambda^2/r}. \tag{7.10.7}$$

By letting, for $x \in Q$, $\Phi_x = \mathbb{E}[\Theta_{r,x} \mid \partial Q]$ and $\Psi_x = \Theta_{r,x} - \Phi_x$, we see from (7.1.2) that

$$(\Psi_x : x \in Q) \overset{law}{=} \left(\Theta_{r-q,g(x)} : x \in Q\right), \tag{7.10.8}$$

where g is the map that stretches Q linearly onto I. Let χ be the (random) point that maximizes Ψ_x over $x \in Q$ (which exists by Lemma 7.2.3). Using that Φ and Ψ are independent, we obtain from the previous display

$$\begin{aligned} \mathbb{P}\left(\max_{x\in Q} \Theta_{r,x} \geq m_r + \lambda\right) &\geq \mathbb{P}\left(\Psi_\chi \geq m_r + \lambda\right) \mathbb{P}\left(\Phi_\chi \geq 0\right) \\ &= \mathbb{P}\left(\Theta^*_{r-q} \geq m_r + \lambda\right)/2. \end{aligned}$$

From (7.10.7) and the previous display,

$$\mathbb{P}\left(\Theta^*_{r-q} \geq m_r + \lambda\right) \leq C_p \lambda e^{-2\lambda} e^{-c_p \lambda^2/r}.$$

Since $m_r = m_{r-q} + O(q)$, the upper bound in (7.2.3) follows by adjusting the constants (all of which depend on θ). $\square$

Proof of (7.2.3), *the lower bound.* The proof of (7.2.3) is based on applying Slepian's Lemma together with [22, Theorem 1.4]. Let $v, w \in V_r \cap Q$, and note that $h(v)$ and $h(w)$ belong to $[1/4, 3/4]^2$. Then, from (7.2.1), (7.3.2) and (7.10.4),

$$\begin{aligned} Cov(b_r(v)\Theta_{r+p,h(v)}, b_r(w)\Theta_{r+p,h(w)}) &\leq -q - \log\|v-w\| + C_\theta \\ &\leq -\log\|v-w\| - C \leq Cov(\eta_{r,v}, \eta_{r,w}) \end{aligned} \tag{7.10.9}$$

for $q = q_\theta$ large enough. The previous display and Slepian's Lemma imply

$$\mathbb{P}\left(\max_{v\in V_r\cap Q} \eta_{r,v} \geq m_r + \lambda\right) \leq \mathbb{P}\left(\max_{v\in V_r\cap Q} b_r(v)\Theta_{r+p,h(v)} \geq m_r + \lambda\right)$$

for all $\lambda \geq 1$. Using (7.10.4), we obtain that the previous display is

$$\leq \mathbb{P}\left(\max_{v\in V_r\cap Q} \Theta_{r+p,v} \geq m_r + \lambda\right) \leq \mathbb{P}\left(\Theta^*_{r+p} \geq m_r + \lambda\right)$$

for all $\lambda \geq 1$. From the proof of [22, Theorem 1.4], we obtain that there exists a constant $c_q \in (0, \infty)$ such that, for all $\lambda \in [1, \sqrt{r}]$,

$$\mathbb{P}\left(\max_{v\in V_r\cap Q} \eta_{r,v} \geq m_r + \lambda\right) \geq c_q \lambda e^{-2\lambda}.$$

Therefore, from the last three displays,

$$\mathbb{P}\left(\Theta^*_{r+p} \geq m_r + \lambda\right) \geq c_q \lambda e^{-2\lambda}$$

for all $\lambda \in [1, \sqrt{r}]$. Since $m_{r+p} = m_r + O(p)$, the lower bound in (7.2.3) follows by adjusting the constants (all of which depend on θ). $\square$

7.11 Proof of Lemma 7.2.7

Proof of (7.2.4). Using the same arguments that were used in the proof of the upper bound of (7.2.3), we see from (7.10.6) (and the display that follows it) that

$$\mathbb{P}\left(\max_{x\in A} \Theta_{r,x} \geq m_r + \lambda\right) \leq \sum_{z\geq 0} \mathbb{P}\left(\max_{h(A)\cap V_r} \eta_{r,v} \geq m_r + \lambda - 2z\right) C_p e^{-c_p z^2}$$

for all $\lambda \in \mathbb{R}$ and any closed sub-square $A \subseteq Q$. Applying (7.3.4), we obtain that the previous display is

$$\leq \sum_{z\geq 0} |h(A)|^{1/2} \max\{\lambda - 2z, 1\} e^{-2(\lambda-2z)} C_p e^{-c_p z^2}. \tag{7.11.1}$$

Assume $\lambda \geq 1$. Then, (7.11.1) is

$$\leq C_p |h(A)|^{1/2} \left(\sum_{0\leq z<\lambda/2} (\lambda - 2z) e^{-2(\lambda-2z)} e^{-c_p z^2} + \sum_{z\geq \lambda/2} e^{-2(\lambda-2z)} e^{-c_p z^2} \right)$$
$$\leq C_p |h(A)|^{1/2} \lambda e^{-2\lambda} \leq C_p e^q |A|^{1/2} \lambda e^{-2\lambda}.$$

Assume now $\lambda < 1$. Then, (7.11.1) is

$$\leq C_p e^q |A|^{1/2} e^{-2\lambda} \sum_{z\geq 0} e^{4z} e^{-c_p z^2} = C_p e^q |A|^{1/2} e^{-2\lambda}.$$

Therefore, the last two displays imply

$$\mathbb{P}\left(\max_{x\in A} \Theta_{r,x} \geq m_r + \lambda\right) \leq C_{p,q} |A|^{1/2} \max\{\lambda, 1\} e^{-2\lambda}.$$

By using the same argument as in (7.10.8) and the paragraph that follows it, we obtain that

$$\mathbb{P}\left(\max_{x\in g(A)} \Theta_{r-q,x} \geq m_r + \lambda\right) \leq C_{p,q} |g(A)|^{1/2} \max\{\lambda, 1\} e^{-2\lambda}$$

where $g : Q \to I$ is the concentric map that stretches Q linearly onto I. Since $m_r = m_{r-q} + O(q)$, (7.2.4) follows from the previous display by adjusting the constant $C_{p,q}$ (which depends on θ). $\square$

7.12 Proof of Lemma 7.2.8

Proof of (7.2.5). For $y \in K_\delta$, let $\beta_{r,y}$ be the harmonic measure on ∂K of a Brownian motion with initial distribution $\rho_{r,y}$. Then,

$$\begin{aligned} Cov(\Phi_{r,x}, \Phi_{r,y}) &= Cov(\mathbb{E}[\Theta_{r,x} \mid \partial K], \Theta_{r,y}) = Cov(X(\beta_{r,x}), X(\rho_{r,y})) \\ &= \int_{\partial K} \int_I G_I(u, y+v)\theta_r(v)dv\, \beta_{r,x}(du). \end{aligned}$$

Therefore,

$$\begin{aligned} &\left|Cov(\Phi_{r,x}, \Phi_{r,y}) - Cov(\Phi_{r,x}, \Phi_{r,y'})\right| \\ &\quad \leq \int_I \int_{\partial K} \left|G_I(u, y+v) - G_I(u, y'+v)\right| \theta_r(v)dv\, \beta_{r,x}(du). \end{aligned}$$

But

$$\begin{aligned} G_I(u, y+v) - G_I(u, y'+v) &\leq \left|\log \|u-y-v\| - \log \|u-y'-v\|\right| \\ &+ \mathbb{E}^u\left[\left|\log \|W_{\tau_I} - y - v\| - \log \|W_{\tau_I} - y' - v\|\right|\right] \leq C_\delta e^k, \end{aligned}$$

where the last inequality follows from the the same argument as in the proof of (7.2.1). $\square$

Proof of (7.2.6). For $x \in K$, let $\psi_{r,x} = \Theta_{r,x} - \Phi_{r,x}$. Recall that the fields $(\Phi_{r,x} : x \in K)$ and $(\psi_{r,x} : x \in K)$ are independent. Therefore, for all $x, y \in K_\delta$,

$$\mathbb{E}\left[(\Phi_{r,x} - \Phi_{r,y})^2\right] = \mathbb{E}\left[(\Theta_{r,x} - \Theta_{r,y})^2\right] - \mathbb{E}\left[(\psi_{r,x} - \psi_{r,y})^2\right].$$

Applying the self-similarity property (7.1.2) to the previous display,

$$\mathbb{E}\left[(\Phi_{r,x} - \Phi_{r,y})^2\right] = \mathbb{E}\left[(\Theta_{r,x} - \Theta_{r,y})^2\right] - \mathbb{E}\left[(\Theta_{r-k,e^k x} - \Theta_{r-k,e^k y})^2\right]. \tag{7.12.1}$$

From (6.1.1) and (6.1.4),

$$\mathbb{E}\left[(\Theta_{r,x} - \Theta_{r,y})^2\right] =$$

$$\iint \left(G_I(x+u,x+v)+G_I(y+u,y+v)-2G_I(x+u,y+v)\right)\theta_r(u)\theta_r(v)dudv$$

$$= \iint 2\log\left(\frac{\|x+u-y-v\|}{\|u-v\|}\right)\theta_r(u)\theta_r(v)dudv + \tag{7.12.2}$$

$$\iiint_{\partial I}(\log\|x+u-z\|-\log\|y+u-z\|)(p_I(x+v,z)-p_I(y+v,z))dz\,\theta_r(u)\theta_r(v)dudv,$$

where $p_I(x+u,\cdot)$ is the density (with respect to the Lebesgue measure) of the harmonic measure on ∂I of a Brownian motion started at $x+u$. We study the second term of the previous display. Note first that there exist constants $c_\delta, C_\delta \in (0,\infty)$ such that

$$c_\delta \le \frac{\int_{\partial I}(\log\|x+u-z\|-\log\|y+u-z\|)(p_I(x+v,z)-p_I(y+v,z))dz}{-\|x-y\|^2} \le C_\delta.$$

The lower bound on the previous display follows from log and p_I being Lipschitz on I_δ (where the Lipschitz constant depends on δ). The upper bound follows from [4, Display (3.31)] and [4, Display (3.32)], by increasing the size of the lattice to infinity. From the last two displays,

$$c_\delta \le \frac{\mathbb{E}\left[(\Theta_{r,x}-\Theta_{r,y})^2\right] - \iint 2\log\left(\frac{\|x+u-y-v\|}{\|u-v\|}\right)\theta_r(u)\theta_r(v)dudv}{-\|x-y\|^2} \le C_\delta.$$

Therefore, from (7.12.1), and applying the previous display on $\mathbb{E}[(\Theta_{r,x}-\Theta_{r,y})^2]$ and on $\mathbb{E}[(\Theta_{r-k,e^kx}-\Theta_{r-k,e^ky})^2]$,

$$c_\delta e^{2k}\|x-y\|^2 \le \mathbb{E}\left[(\Phi_{r,x}-\Phi_{r,y})^2\right] \le C_\delta e^{2k}\|x-y\|^2,$$

as desired. □

7.13 Proof of Lemma 7.5.1

Proof of (7.5.2). It is clear that $Cov(\xi_x(t),\xi_y(t)) \le t$. In order to prove the lower bound, note that, for $t \le s_{x,y} = -\log(\|x-y\|_\infty)$, we obtain from (7.5.1)

$$\begin{aligned} Cov(\xi_x(t),\xi_y(t)) &= t-(e^t-1)\|x-y\|_1+\frac{1}{2}(e^{2t}-1)|x_1-y_1||x_2-y_2| \\ &\ge t-e^t\|x-y\|_1 \ge t-e^{s_{x,y}}\|x-y\|_1 = t-\frac{\|x-y\|_1}{\|x-y\|_\infty} \ge t-2. \end{aligned}$$

□

Proof of (7.5.3). Note that, for $h \geq s_{x,y}$, we have

$$(1 - e^h|x_1 - y_1|)_+(1 - e^h|x_2 - y_2|)_+ = 0.$$

Therefore, if $t \geq s_{x,y}$,

$$\begin{aligned} &Cov(\xi_x(t), \xi_y(t)) - Cov(\xi_x(s_{x,y}), \xi_y(s_{x,y})) \\ &\quad = \int_{s_{x,y}}^{t} (1 - e^h|x_1 - y_1|)_+(1 - e^h|x_2 - y_2|)_+ dh = 0. \end{aligned}$$

□

7.14 Proof of Lemma 7.5.2

Proof of (7.5.5). Let $A \subseteq I$ be a sub-square. Note that, for r large enough (depending on $|A|$), $A \cap V_r$ is a sub-lattice of V_r, and we can cover V_r with $O(1/|A|)$ copies of $A \cap V_r$. We denote these copies by $\{A_i\}$. Since $\xi_v(r)$ is translation-invariant, the probability

$$\mathbb{P}\left(\max_{v \in A_i} \xi_v(r) \geq m_r + \lambda\right)$$

is the same for each copy A_i. Therefore, a simple union bound implies

$$\mathbb{P}(\max_{v \in V_r} \xi_v(r) \geq m_r + \lambda) \leq O(1/|A|)\mathbb{P}(\max_{v \in A \cap V_r} \xi_v(r) \geq m_r + \lambda).$$

Hence, (7.5.4) implies

$$c|A|\lambda e^{-2\lambda} \leq \mathbb{P}\left(\max_{v \in A \cap V_r} \xi_v(r) \geq m_r + \lambda\right),$$

as desired. □

Chapter 8

Asymptotics on the tail probability

The goal of this chapter is to prove the following strengthening of Lemma 7.2.6, which will be needed in order to prove Theorem 6.1.1.

Proposition 8.0.1. *There exists a constant $\alpha_\theta^* > 0$ (depending on θ) and a continuous function $\zeta : (0,1)^2 \to (0,\infty)$ (not depending on θ) with $\int_{(0,1)^2} \zeta(x)dx = 1$, such that, for any closed square $A \subset (0,1)^2$,*

$$\lim_{z\to\infty} \limsup_{r\to\infty} \left| z^{-1} e^{2z} \mathbb{P}\left(\max_{x\in A} \Theta_{r,x} \geq m_r + z \right) - \alpha_\theta^* \int_A \zeta(x)dx \right| = 0.$$

Remark 8.0.2. As we will see in the proof of Proposition 8.1.7, the function ζ can be obtained from (8.6.11) by taking the limit there as $\delta \to 0$, obtaining

$$\zeta(x) := e^{2S(x)} / \int_I e^{2S(y)} dy, \tag{8.0.1}$$

where $S(x) = \mathbb{E}^x[\log \|x - W_\tau\|]$ and $\tau = \inf\{t \geq 0 : W_t \notin I\}$.

We will present first an outline of the propositions and lemmas needed for the proof of Proposition 8.0.1, with their proofs being deferred until the end of the chapter.

8.1 Main ideas of the proof of Proposition 8.0.1

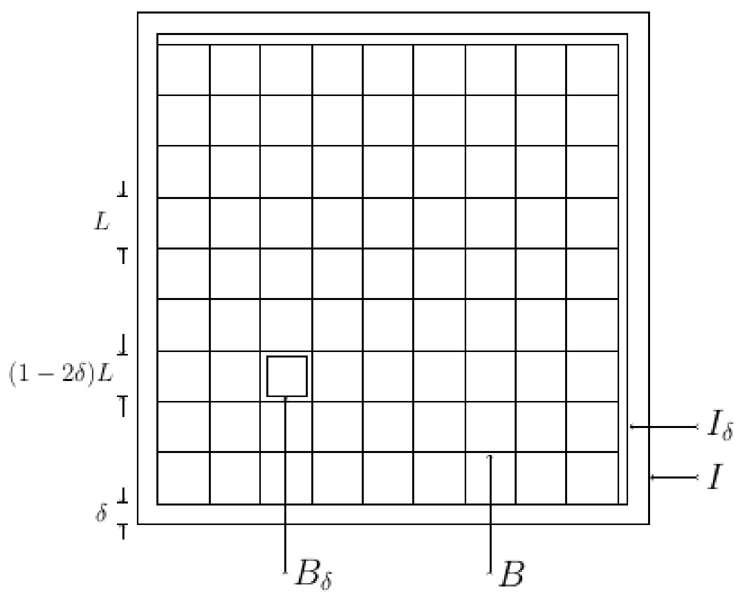
Because of Lemma 7.2.7, it is unlikely that the maximum of the MGFF occurs on a given subset of I whose area is small. Therefore, we will restrict our attention to subsets of I that cover most of the area. We proceed to define such a set, $\tilde{I}$. At the same time, we define two collections, $\mathcal{B}$ and $\tilde{\mathcal{B}}$, of small disjoint (except for the boundary) squares inside the unit square. These squares will allow us to decompose the MGFF in a tree-like fashion, using (7.1.1) and (7.1.2).

Let us fix a small $\delta > 0$. We define two functions of z, $l = l(z)$ and $\tilde{l} = \tilde{l}(z)$, such that both l and $l - \tilde{l}$ increase to infinity as $z \to \infty$. For the proofs of this chapter, we will employ the following restrictions on l and $\tilde{l}$:

$$l \geq e^{z^{1/20}} \text{ and } l - \tilde{l} \leq \log\log(z). \tag{8.1.1}$$

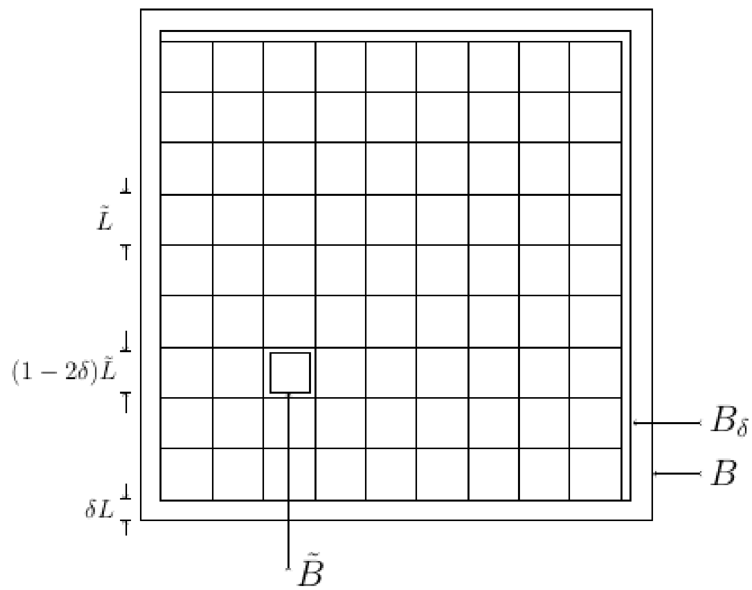
We next divide I into two collections of sub-squares: B-boxes and $\tilde{B}$-boxes.

Figure 8.1: B-boxes



- The first collection of sub-squares is defined as follows. Set $L = e^{-r+l}$. Consider the square grid consisting of $(\lfloor(1 - 2\delta)/L\rfloor)^2$ adjacent closed squares with side length L, placed inside I_δ so that the left-bottom corner of the grid and the left-bottom corner of I_δ coincide. The side length e^{-r+l} is chosen so that it decreases exponentially as $r \to \infty$, together with the support of the mollified of the MGFF. Note that the grid covers all of I_δ with the possible exception of a band of width strictly less than L at the upper and right sides of I_δ. Let $\mathcal{B}$ denote the collection of all $(\lfloor(1-2\delta)/L\rfloor)^2$ squares in the grid. For each $B \in \mathcal{B}$, let $B_\delta = \{x \in B : dist(x, \partial B) \geq \delta L\}$. We call $\mathcal{B}$ the collection of *B-boxes* (see Figure 8.1 above).

Figure 8.2: $\tilde{B}$-boxes



- The second collection of sub-squares is defined as follows. Set $\tilde{L} = e^{-r+\tilde{l}}$. Similarly to the previous definition, for each $B \in \mathcal{B}$, consider the square sub-grid consisting of $(\lfloor(1-2\delta)e^{l-\tilde{l}}\rfloor)^2$ adjacent closed sub-squares with side length $\tilde{L}$, placed inside B_δ so that the left-bottom corner of the grid and the left-bottom corner of B_δ coincide. Note that the grid covers all of B_δ with the possible exception of a band of width strictly less than $\tilde{L}$ at the upper and right sides of B_δ. For each $B \in \mathcal{B}$, let $\mathcal{L}_B$ be the collection of sub-squares in the sub-grid. Also, let $\tilde{\mathcal{B}}_B$ denote the collection of squares that are concentric with those in $\mathcal{L}_B$, but with side length $(1-2\delta)\tilde{L}$. Let $\tilde{\mathcal{B}} = \bigcup_{B\in\mathcal{B}} \tilde{\mathcal{B}}_B$. We call $\tilde{\mathcal{B}}$ the collection of *$\tilde{B}$-boxes* (see Figure 8.2 above).

- Let $\tilde{I} = \bigcup_{B\in\mathcal{B}} \bigcup_{\tilde{B}\in\tilde{\mathcal{B}}} \tilde{B}$. For $x \in \tilde{I}$, let B_x be the B-box containing x, $\tilde{B}_x$ be the

$\tilde{B}$-box containing x, and let $\tilde{x}$ be the center of $\tilde{B}_x$. Also, let $\Xi = \left\{\tilde{x} : x \in \tilde{I}\right\}$ be the collection of center points of the $\tilde{B}$-boxes.

The bottom-left positioning of the grids described above is arbitrary; we choose this positioning for concreteness. Note that the cardinality of Ξ is $O(e^{2r-2\tilde{l}})$, and the number of $\tilde{B}$-boxes inside each B-box does not depend on r. Also, note that $\tilde{I}$ depends on δ, and, for some absolute constant $C > 0$, the area of $I \backslash \tilde{I}$, is at most $C\delta$.

We follow the same proof outline as in the proof of [4, Proposition 4.1]. Proposition 8.0.1 easily follows from Lemma 7.2.7 and the following proposition.

Proposition 8.1.1. *For all $\delta > 0$ small enough, there exist a constant $\alpha_\theta^\delta > 0$ (depending on δ and θ), a continuous function $\zeta_\delta : [\delta, 1-\delta]^2 \to (0,\infty)$, with $\int_{[\delta,1-\delta]^2} \zeta_\delta(x)dx = 1$, and a continuous function $\zeta : (0,1)^2 \to (0,\infty)$, with $\zeta_\delta(x) \to \zeta(x)$ uniformly in x on closed sets as $\delta \to 0$ (with neither ζ nor ζ_δ depending on θ), such that, for any closed square $A \subseteq [\delta, 1-\delta]^2$,*

$$\lim_{z\to\infty} \limsup_{r\to\infty} \left| z^{-1} e^{2z} \mathbb{P}\left(\max_{x \in \tilde{I} \cap A} \Theta_{r,x} \geq m_r + z \right) - \alpha_\theta^\delta \int_A \zeta_\delta(x) dx \right| = 0.$$

In order to prove Proposition 8.1.1, we first replace Θ_r by a simpler field $\tilde{\Theta}_r$ as follows. For $x \in \tilde{I}$, and using the notation of Chapter 7, we decompose $\Theta_{r,x} = \Theta^c_{r,x} + \Theta^f_{r,x}$, where

$$\Theta^c_{r,x} = \mathbb{E}\left[\Theta_{r,x} \mid \partial B_x\right].$$

Following [4], we will refer to the fields $\Theta^c_{r,x}$ and $\Theta^f_{r,x}$ as the coarse and fine field, respectively. These fields depend on z via $l = l(z)$ and $\tilde{l} = \tilde{l}(z)$, but we keep this dependence implicit. Note that (7.1.2) implies

$$\left(\Theta^f_{r,x} : x \in B\right) \overset{law}{=} \left(\Theta_{l,h(x)} : x \in B\right), \tag{8.1.2}$$

where h is the map that stretches B linearly onto I. We define the approximating field

$$\tilde{\Theta}_{r,x} = \Theta^c_{r,\tilde{x}} + \Theta^f_{r,x},$$

which can be viewed as a "semi-discretized" version of Θ_r. The fields Θ_r and $\tilde{\Theta}_r$ are related by:

Proposition 8.1.2. *There exists $\epsilon_z \to_{z\to\infty} 0$ such that, for all closed squares with non-empty interior $A \subseteq [\delta, 1-\delta]^2$,*

$$\limsup_{z\to\infty} \limsup_{r\to\infty} \frac{\mathbb{P}\left(\max_{x\in\tilde{I}\cap A} \Theta_{r,x} \geq m_r + z\right)}{\mathbb{P}\left(\max_{x\in\tilde{I}\cap A} \tilde{\Theta}_{r,x} \geq m_r + z - \epsilon_z\right)} \leq 1 \tag{8.1.3}$$

and

$$\liminf_{z\to\infty} \liminf_{r\to\infty} \frac{\mathbb{P}\left(\max_{x\in\tilde{I}\cap A} \Theta_{r,x} \geq m_r + z\right)}{\mathbb{P}\left(\max_{x\in\tilde{I}\cap A} \tilde{\Theta}_{r,x} \geq m_r + z + \epsilon_z\right)} \geq 1. \tag{8.1.4}$$

Assuming Proposition 8.1.2, Proposition 8.1.1 follows from:

Proposition 8.1.3. *For all $\delta > 0$ small enough, there exist a constant $\alpha_\theta^\delta > 0$ (depending on δ and θ), a continuous function $\zeta_\delta : [\delta, 1-\delta]^2 \to (0,\infty)$, with $\int_{[\delta,1-\delta]^2} \zeta_\delta(x)dx = 1$, and a continuous function $\zeta : (0,1)^2 \to (0,\infty)$, with $\zeta_\delta(x) \to \zeta(x)$ uniformly in x on closed sets as $\delta \to 0$ (with neither ζ nor ζ_δ depending on θ), such that, for any closed square $A \subseteq [\delta, 1-\delta]^2$,*

$$\lim_{z\to\infty} \limsup_{r\to\infty} \left| z^{-1} e^{2z} \mathbb{P}\left(\max_{x\in\tilde{I}\cap A} \tilde{\Theta}_{r,x} \geq m_r + z\right) - \alpha_\theta^\delta \int_A \zeta_\delta(x)dx \right| = 0.$$

In order to prove Proposition 8.1.3, we first need some preparation. Recall that the continuous modified branching random walk (CMBRW) is the centered Gaussian field $(\xi_x(t) : x \in I, t \geq 0)$ with covariance structure defined in (7.5.1). For $v \in \Xi$, let $T_v = Var(\Theta^c_{r,v})$. Fix $R = C_{\delta,\theta}(l - \tilde{l})$ large enough (which will be chosen in Proposition 8.1.4). For $v \in \Xi$, define the Brownian motion $\xi^{\mathrm{lw}}_{r,v}(t)$ by setting

$$\xi^{\mathrm{lw}}_{r,v}(t) = \begin{cases} \sqrt{\gamma_v}\, W(t/\gamma_v) & \text{for } t \in [0, \gamma_v] \\ \sqrt{\gamma_v}\, W(1) + \xi_v(t - \gamma_v) & \text{for } t \in [\gamma_v, T_v] \end{cases},$$

where $\gamma_v = T_v - r + l + R$, and $(W(t) : t \in [0,1])$ is an independent Brownian motion, and define the Brownian motion $\xi^{\mathrm{up}}_{r,v}(t)$ by setting

$$\xi^{\mathrm{up}}_{r,v}(t) = \begin{cases} \xi_v(t + \gamma_v) - \xi_v(\gamma_v) & \text{for } t \in [0, T_v - \gamma_v] \\ \xi_v(T_v) - \xi_v(\gamma_v) + W_v(t - T_v + \gamma_v) & \text{for } t \in [T_v - \gamma_v, T_v] \end{cases},$$

where $\{W_v(t) : v \in \Xi\}$ is a family of independent Brownian motions. To simplify some notation, we let $\xi_{r,v} = \xi_v(T_v)$, $\xi^{\mathrm{up}}_{r,v} = \xi^{\mathrm{up}}_{r,v}(T_v)$ and $\xi^{\mathrm{lw}}_{r,v} = \xi^{\mathrm{lw}}_{r,v}(T_v)$. The fields $\xi_{r,(\cdot)}$, $\xi^{\mathrm{up}}_{r,(\cdot)}$, $\xi^{\mathrm{lw}}_{r,(\cdot)}$ and $\Theta^c_{r,(\cdot)}$ are related by:

Proposition 8.1.4. *There exists $R = C_{\delta,\theta}(l - \tilde{l})$ such that, for all r large enough (depending on θ, δ and z) and all $v, w \in \Xi$,*

$$Cov(\xi^{\mathrm{up}}_{r,v}, \xi^{\mathrm{up}}_{r,w}) \leq Cov(\xi_{r,v}, \xi_{r,w}) \leq Cov(\xi^{\mathrm{lw}}_{r,v}, \xi^{\mathrm{lw}}_{r,w}) \tag{8.1.5}$$

and

$$Cov(\xi^{\mathrm{up}}_{r,v}, \xi^{\mathrm{up}}_{r,w}) \leq Cov(\Theta^{c}_{r,v}, \Theta^{c}_{r,w}) \leq Cov(\xi^{\mathrm{lw}}_{r,v}, \xi^{\mathrm{lw}}_{r,w}). \tag{8.1.6}$$

Moreover, if $v = w$, equality holds in (8.1.5) *and in* (8.1.6).

This last proposition and Slepian's Lemma imply that, for any closed square $A \subseteq [\delta, 1-\delta]^2$, both $\max_{x \in \tilde{I} \cap A}(\xi_{r,\tilde{x}} + \Theta^{f}_{r,x})$ and $\max_{x \in \tilde{I} \cap A} \tilde{\Theta}_{r,x}$ (i) are dominated stochastically by $\max_{x \in \tilde{I} \cap A}(\xi^{\mathrm{up}}_{r,\tilde{x}} + \Theta^{f}_{r,x})$, and (ii) dominate stochastically $\max_{x \in \tilde{I} \cap A}(\xi^{\mathrm{lw}}_{r,\tilde{x}} + \Theta^{f}_{r,x})$. Therefore, Proposition 8.1.3 follows from

$$\lim_{z \to \infty} \limsup_{r \to \infty} \left| \frac{\mathbb{P}\left(\max_{x \in \tilde{I} \cap A}(\xi^{\mathrm{lw}}_{r,\tilde{x}} + \Theta^{f}_{r,x}) \geq m_r + z\right)}{\mathbb{P}\left(\max_{x \in \tilde{I} \cap A}(\xi^{\mathrm{up}}_{r,\tilde{x}} + \Theta^{f}_{r,x}) \geq m_r + z\right)} - 1 \right| = 0, \tag{8.1.7}$$

together with Proposition 8.1.4 and the following proposition.

Proposition 8.1.5. *For all $\delta > 0$ small enough, there exist a constant $\alpha^{\delta}_{\theta} > 0$ (depending on δ and θ), a continuous function $\zeta_\delta : [\delta, 1-\delta]^2 \to (0, \infty)$, with $\int_{[\delta,1-\delta]^2} \zeta_\delta(x)dx = 1$, and a continuous function $\zeta : (0,1)^2 \to (0, \infty)$, with $\zeta_\delta(x) \to \zeta(x)$ uniformly in x on closed sets as $\delta \to 0$ (with neither ζ nor ζ_δ depending on θ), such that, for any closed square $A \subseteq [\delta, 1-\delta]^2$,*

$$\lim_{z \to \infty} \limsup_{r \to \infty} \left| z^{-1} e^{2z} \mathbb{P}\left(\max_{x \in \tilde{I} \cap A}(\xi_{r,\tilde{x}} + \Theta^{f}_{r,x}) \geq m_r + z\right) - \alpha^{\delta}_{\theta} \int_A \zeta_\delta(x)dx \right| = 0.$$

In order to prove (8.1.7) and Proposition 8.1.5, we define the following sets and random variables. For $v \in \Xi$, let

$$E_{r,v}(z) = \left\{ \xi_{r,v}(t) \leq \frac{m_r}{r} t + z \text{ for all } t \in [0, T_v], \xi_{r,v} + \max_{x \in \tilde{B}_v} \Theta^{f}_{r,x} \geq m_r + z \right\}$$

and, for $A \subseteq [\delta, 1-\delta]^2$, let

$$\Lambda_{r,A}(z) = \sum_{v \in \Xi \cap A} \mathbf{1}_{E_{r,v}(z)}.$$

Similarly, define $E^{\mathrm{up}}_{r,v}(z)$, $\Lambda^{\mathrm{up}}_{r,A}(z)$ and $E^{\mathrm{lw}}_{r,v}(z)$, $\Lambda^{\mathrm{lw}}_{r,A}(z)$ by replacing $\xi_{r,v}$ by $\xi^{\mathrm{up}}_{r,v}$ and $\xi^{\mathrm{lw}}_{r,v}$, respectively. The previous definitions are employed in:

Proposition 8.1.6. *For any closed sub-square with nonempty interior $A \subseteq [\delta, 1-\delta]^2$,*

$$\lim_{z\to\infty} \limsup_{r\to\infty} \left| \frac{\mathbb{P}\left(\max_{x\in\tilde{I}\cap A}(\xi^{\mathrm{up}}_{r,\tilde{x}} + \Theta^f_{r,x}) \geq m_r + z\right)}{\mathbb{E}\Lambda^{\mathrm{up}}_{r,A}(z)} - 1 \right| = 0$$

and

$$\lim_{z\to\infty} \limsup_{r\to\infty} \left| \frac{\mathbb{P}\left(\max_{x\in\tilde{I}\cap A}(\xi^{\mathrm{lw}}_{r,\tilde{x}} + \Theta^f_{r,x}) \geq m_r + z\right)}{\mathbb{E}\Lambda^{\mathrm{lw}}_{r,A}(z)} - 1 \right| = 0.$$

Since $\mathbb{E}\Lambda_{r,A}(z) = \mathbb{E}\Lambda^{\mathrm{up}}_{r,A}(z) = \mathbb{E}\Lambda^{\mathrm{lw}}_{r,A}(z)$, Proposition 8.1.6 implies (8.1.7). Additionally, Proposition 8.1.5 follows from Proposition 8.1.4, Proposition 8.1.6 and the following proposition.

Proposition 8.1.7. *For all $\delta > 0$ small enough, there exist a constant $\alpha^\delta_\theta > 0$ (depending on δ and θ), a continuous function $\zeta_\delta : [\delta, 1-\delta]^2 \to (0,\infty)$, with $\int_{[\delta,1-\delta]^2} \zeta_\delta(x)dx = 1$, and a continuous function $\zeta : (0,1)^2 \to (0,\infty)$, with $\zeta_\delta(x) \to \zeta(x)$ uniformly in x on closed sets as $\delta \to 0$ (with neither ζ nor ζ_δ depending on θ), such that, for any closed square $A \subseteq [\delta, 1-\delta]^2$,*

$$\lim_{z\to\infty} \limsup_{r\to\infty} \left| z^{-1} e^{2z} \mathbb{E}\Lambda_{r,A}(z) - \alpha^\delta_\theta \int_A \zeta_\delta(x)dx \right| = 0.$$

The rest of Chapter 8 is devoted to proving Propositions 8.1.2, 8.1.4, 8.1.6 and 8.1.7. We conclude this section by outlining how these propositions are employed and comparing them to how similar results are employed in [4].

Proposition 8.1.2 has its analog in [4, Lemma 4.5]; however, the proof in our setting requires extensive use of Slepian's Lemma, together with the technique mentioned in Chapter 6 of adding small patches of Brownian sheet to the field $\tilde{\Theta}$. Proposition 8.1.6 has its analog in [4, Proposition 4.8], although the proof in our setting employs properties of the CBRW and estimates on the covariance structure of the MGFF. (See Lemmas 8.2.1 and 8.2.3 below.) Proposition 8.1.6 establishes an asymptotic equivalence between the tail event of the maximum and a first order approximation that employs the events $E_{n,v}(z)$. The proof of Proposition 8.1.6 uses a fairly standard first and second moment method to find tight bounds on the tail event (see Lemma 8.5.3 and (8.5.16)); we follow almost verbatim the argument in the proof of [4, Proposition 4.8]. The above Proposition 8.1.7 also has its analog in [4, Proposition 4.12], although, in its proof here, we obtain the limit $\mathbb{E}[\Lambda_{r,A}(z)]$ as $r \to \infty$, with $z > 0$ fixed (see (8.6.9)). When subsequently taking the limit of this expectation as $z \to \infty$, we will explicitly employ the term ζ_δ from the proposition (see (8.6.11)).

8.2 Preliminary lemmas for the proofs of Chapter 8

We will need the following estimates. Lemma 8.2.1 follows from the self-similarity property of the MGFF, while Lemma 8.2.3 follows from tight bounds on the Green functions of the square and a half-plane.

Lemma 8.2.1. *There exists a constant $C_{\delta,\theta} \in (0, \infty)$ such that, for all $z > 0$, all r large enough (depending on z), and all $x, y \in \tilde{I}$ in the same B-box,*

$$\left|Cov(\Theta^c_{r,x}, \Theta^c_{r,y}) - r + l\right| \leq C_{\delta,\theta}. \tag{8.2.1}$$

Moreover, suppose that x, y, y' belong to the same B-box. Then,

$$\left|Cov(\Theta^c_{r,x}, \Theta^c_{r,y}) - Cov(\Theta^c_{r,x}, \Theta^c_{r,y'})\right| \leq C_{\delta,\theta} e^{r-l} \|y - y'\|. \tag{8.2.2}$$

Proof of Lemma 8.2.1. The proof of (8.2.2) is a straightforward application of (7.2.5). We now prove (8.2.1). Note that (7.1.1) and (7.1.2) imply

$$Cov(\Theta^c_{r,x}, \Theta^c_{r,y}) = Cov(\Theta_{r,x}, \Theta_{r,y}) - Cov(\Theta_{l,h(x)}, \Theta_{l,h(y)}),$$

where h is the map that stretches B linearly onto I. Applying (7.2.1) to both terms in the right hand side of the previous display,

$$\left|Cov(\Theta^c_{r,x}, \Theta^c_{r,y}) + \log \max\{e^{-r}, \|x - y\|\} - \log \max\{e^{-l}, \|h(x) - h(y)\|\}\right| \leq C_{\delta,\theta}.$$

From the definition of a B-box, we obtain that $\|h(x) - h(y)\| = e^{r-l}\|x - y\|$. Then, the previous two displays imply

$$\left|Cov(\Theta^c_{r,x}, \Theta^c_{r,y}) - r + l\right| \leq C_{\delta,\theta},$$

as desired. $\square$

Remark 8.2.2. Note that (8.2.1) implies that $T_v := Var(\Theta^c_{r,v})$ satisfies

$$|T_v - r + l| \leq C_{\delta,\theta} \tag{8.2.3}$$

for all $v \in \Xi$.

We will also need the following extension of Lemma 7.2.8.

Lemma 8.2.3. *There exist constants $c_{\delta,\theta}, C_{\delta,\theta} \in (0,\infty)$ such that, for all r and l large enough (depending on δ and θ), and for all $x, y \in \tilde{I}$ in different B-boxes,*

$$\mathbb{E}[(\Theta^c_{r,x} - \Theta^c_{r,y})^2] \geq c_{\delta,\theta} - C_{\delta,\theta}\|x-y\|^2 \tag{8.2.4}$$

and

$$\mathbb{E}[(\Theta^c_{r,x} - \Theta^c_{r,y})^2] \geq 2\log\|e^{r-l}(x-y)\| - C_{\delta,\theta}. \tag{8.2.5}$$

Remark 8.2.4. Display (8.2.4) is of interest when x and y are close, and (8.2.5) is of interest when x and y are far apart.

Proof of Lemma 8.2.3. Note that, by the definition of a B-box, $dist(x, \partial B_x) \geq \delta e^{-r+l}$. Therefore, (7.1.1) implies that the fine fields restricted to $\tilde{I}$ on different B-boxes are independent. Thus,

$$\mathbb{E}[(\Theta^c_{r,x} - \Theta^c_{r,y})^2] = \mathbb{E}[(\Theta_{r,x} - \Theta_{r,y})^2] - \mathbb{E}[(\Theta^f_{r,x})^2] - \mathbb{E}[(\Theta^f_{r,y})^2].$$

From (7.12.2),

$$\begin{aligned}
\mathbb{E}[(\Theta_{r,x} - \Theta_{r,y})^2] &\geq \iint 2\log\left(\frac{\|x+u-y-v\|}{\|u-v\|}\right)\theta_r(u)\theta_r(v)dudv - C_{\delta,\theta}\|x-y\|^2 \\
&\geq 2\log\|x-y\| - C_{\delta,\theta}e^{-r} - 2\iint \log\|u-v\|\theta_r(u)\theta_r(v)dudv - C_{\delta,\theta}\|x-y\|^2 \\
&= 2\log\|x-y\| - C_{\delta,\theta}e^{-r} + 2r - 2\iint \log\|u-v\|\theta(u)\theta(v)dudv - C_{\delta,\theta}\|x-y\|^2.
\end{aligned}$$

On the other hand, by (6.1.1) and (7.1.2),

$$\begin{aligned}
\mathbb{E}[(\Theta^f_{r,x})^2] &= \iint (-\log\|u-v\| + \mathbb{E}^{x'+u}[\log\|x'+v-W_{\tau_I}\|])\,\theta_l(u)\theta_l(v)dudv \\
&= l - \iint \log\|u-v\|\theta(u)\theta(v)dudv - C_{\delta,\theta}e^{-l} + \mathbb{E}^{x'}[\log\|x'-W_{\tau_I}\|],
\end{aligned}$$

where x' is the image of x after we map B onto I. By putting together the last two displays, we obtain

$$\begin{aligned}
\mathbb{E}[(\Theta_{r,x} - \Theta_{r,y})^2] \geq{}& 2\log\|e^{r-l}(x-y)\| - C_{\delta,\theta}e^{-r} + C_{\delta,\theta}e^{-l} - C_{\delta,\theta}\|x-y\|^2 \\
&- \mathbb{E}^{x'}[\log\|x'-W_{\tau_I}\|] - \mathbb{E}^{y'}[\log\|y'-W_{\tau_I}\|].
\end{aligned}$$

Display (8.2.5) follows by observing that $x' \in I_\delta$, so $\|x' - W_{\tau_I}\| \geq c_{\delta,\theta}$.

Display (8.2.4) requires a finer analysis. Note that

$$\|x - y\| \geq dist(x, \partial B) + dist(y, \partial B').$$

Therefore, recalling that L denotes the side-length of the B-boxes, we obtain

$$\|x - y\|/L \geq dist(x', \partial I) + dist(y', \partial I),$$

which implies

$$2\log(e^{r-l}(x-y)) \geq \log(2\,dist(x', \partial I)) + \log(2\,dist(y', \partial I)).$$

The last two displays imply that to prove (8.2.4), it is enough to show there exists $c_{\delta,\theta} \in (0,\infty)$ such that, for any $a \in I_\delta$,

$$\log(2\,dist(a, \partial I)) - \mathbb{E}^a[\log\|a - W_{\tau_I}\|] \geq c_{\delta,\theta}.$$

Since the last expression is continuous in a, it is enough to show that, for any a in the interior of I,

$$\log(2\,dist(a, \partial I)) > \mathbb{E}^a[\log\|a - W_{\tau_I}\|].$$

Let M be the line that contains the side of I that is closest to a, and let $\tau_M = \min\{t \geq 0 : W_t \in M\}$. Note that W_{τ_M} has a Cauchy distribution on M. Therefore, from [26, 4.295-7], we deduce that

$$\mathbb{E}^a[\log\|a - W_{\tau_M}\|] = \log(2\,dist(a, \partial I)). \tag{8.2.6}$$

It is therefore enough to show

$$\mathbb{E}^a[\log\|a - W_{\tau_M}\|] > \mathbb{E}^a[\log\|a - W_{\tau_I}\|].$$

We will do so by using the Green functions G_I (of the unit square) and G_M (of the half-space delimited by M). Since they do not exist along the diagonal, we consider first a small disk D around a. Then,

$$\begin{aligned}
&\int_{w\in D} \mathbb{E}^a[\log\|w - W_{\tau_M}\|] - \mathbb{E}^a[\log\|w - W_{\tau_I}\|]\,dw \\
&\quad = \int_{w\in D} (G_M(a,w) - G_I(a,w))dw = \mathbb{E}^a\left[\int_{\tau_I}^{\tau_M} 1_D(W_t)dt\right] \\
&\quad = \int_{z\in\partial I}\int_{w\in D} G_M(z,w)\,dw\,p_I(a,z)\,dz,
\end{aligned}$$

where $p_I(a,\cdot)$ is the harmonic measure on ∂I of a Brownian motion started at a. By continuity in a of the previous expression, we obtain

$$\mathbb{E}^a[\log \|a - W_{\tau_M}\|] - \mathbb{E}^a[\log \|a - W_{\tau_I}\|] = \int_{z\in\partial I} G_M(z,a)p_I(a,z)dz,$$

which is strictly positive. This completes the proof of (8.2.4). □

8.3 Proof of Proposition 8.1.2

We will first prove the upper bound (8.1.3) by employing Slepian's Lemma. Since $\tilde{\Theta}_r$ has a discretized part, we will add patches of Brownian sheet, similarly to what we did in the proof of (7.2.3).

Let $P : \tilde{I} \to [p, p+1]^2$ be the map that translates and stretches each $\tilde{B} \in \tilde{\mathcal{B}}$ onto $[p, p+1]^2$, where $p > 0$ is a large constant, independent of l and $\tilde{l}$, that will chosen later (see Claim 8.3.1). For each $\tilde{B}$, define the field

$$\left(\psi_{z,x} : x \in \tilde{B}\right) \overset{law}{:=} \left(e^{(\tilde{l}-l)/2}\phi_{P(x)} : x \in \tilde{B}\right), \tag{8.3.1}$$

where ϕ is the standard Brownian sheet defined in (7.4.1), and assume the collection of fields $\left\{\left(\psi_{z,x} : x \in \tilde{B}\right)\right\}_{\tilde{B}\in\tilde{\mathcal{B}}}$ is independent. We will compare the fields

$$\left(\Theta_{r,x} : x \in \tilde{I}\right) \text{ and } \left(a'_r(x)\Theta^c_{r,\tilde{x}} + \Theta^f_{r,x} + \psi_{z,x} : x \in \tilde{I}\right),$$

where $a'_r(x) \geq 0$ is defined so that

$$Var(\Theta_{r,x}) = Var(a'_r(x)\Theta^c_{r,\tilde{x}} + \Theta^f_{r,x} + \psi_{z,x}) \tag{8.3.2}$$

for all $x \in \tilde{I}$.

We first prove

Claim 8.3.1. For all p large enough (depending on δ and θ), all $z > 0$ and all r large enough (depending on z),

$$Cov(a'_r(x)\Theta^c_{r,\tilde{x}} + \Theta^f_{r,x} + \psi_{z,x}, a'_r(y)\Theta^c_{r,\tilde{y}} + \Theta^f_{r,y} + \psi_{z,y}) \leq Cov(\Theta_{r,x}, \Theta_{r,y})$$

for all $x, y \in \tilde{I}$.

Proof of Claim 8.3.1. Note that it is enough to prove

$$Cov(a'_r(x)\Theta^c_{r,\tilde{x}} + \psi_{z,x}, a'_r(y)\Theta^c_{r,\tilde{y}} + \psi_{z,y}) \le Cov(\Theta^c_{r,x}, \Theta^c_{r,y}).$$

We distinguish three cases:

- Case 1: x and y belong to the same $\tilde{B}$ box. In this case, (7.4.2), (8.2.2) and (8.3.1) imply

$$\begin{aligned}\mathbb{E}\left[\left(\Theta^c_{r,x} - \Theta^c_{r,y}\right)^2\right] &\le C_{\delta,\theta}e^{r-l}\|x-y\| \le pe^{r-l}\|x-y\|_1 \le \mathbb{E}\left[(\psi_{z,x} - \psi_{z,y})^2\right] \\ &\le \mathbb{E}[(a'_r(x)\Theta^c_{r,\tilde{x}} + \psi_{z,x} - a'_r(y)\Theta^c_{r,\tilde{y}} - \psi_{z,y})^2]\end{aligned}$$

  for $p = p_{\delta,\theta}$ large enough, where the last inequality follows from the independence between $\psi_{z,(\cdot)}$ and $\Theta^c_{r,(\cdot)}$.

- Case 2: x and y belong to different $\tilde{B}$-boxes but to the same B-box. Then,

$$Cov(a'_r(x)\Theta^c_{r,\tilde{x}} + \psi_{z,x}, a'_r(y)\Theta^c_{r,\tilde{y}} + \psi_{z,y}) = a'_r(x)a'_r(y)Cov(\Theta_{r,\tilde{x}}, \Theta_{r,\tilde{y}}).$$

  It is therefore enough to show

$$a'_r(x)a'_r(y) \le \frac{Cov(\Theta^c_{r,x}, \Theta^c_{r,y})}{Cov(\Theta^c_{r,\tilde{x}}, \Theta^c_{r,\tilde{y}})}.$$

  We do this by showing that the right hand side is closer to 1 than the left hand side. Note that (8.3.2) implies

$$\begin{aligned}1 - a'_r(x)^2 &= 1 - \frac{Var(\Theta^c_{r,x}) - Var(\psi_{z,x})}{Var(\Theta^c_{r,\tilde{x}})} \\ &= \frac{Var(\Theta^c_{r,\tilde{x}}) - Var(\Theta^c_{r,x}) + Var(\psi_{z,x})}{Var(\Theta^c_{r,\tilde{x}})}.\end{aligned} \tag{8.3.3}$$

  Then, displays (7.4.3) (8.2.1), (8.2.2) and (8.3.1) imply

$$1 - a'_r(x)^2 \ge \frac{-C_{\delta,\theta}e^{r-l}\|x-\tilde{x}\| + p^2e^{\tilde{l}-l}}{r-l+C_{\delta,\theta}} \ge \frac{(p^2 - C_{\delta,\theta})e^{\tilde{l}-l}}{r-l+C_{\delta,\theta}}. \tag{8.3.4}$$

  On the other hand, displays (8.2.1) and (8.2.2) imply

$$\begin{aligned}\left|\frac{Cov(\Theta^c_{r,x}, \Theta^c_{r,y})}{Cov(\Theta^c_{r,\tilde{x}}, \Theta^c_{r,\tilde{y}})} - 1\right| &= \frac{\left|Cov(\Theta^c_{r,x}, \Theta^c_{r,y}) - Cov(\Theta^c_{r,\tilde{x}}, \Theta^c_{n,\tilde{y}})\right|}{Cov(\Theta^c_{r,\tilde{x}}, \Theta^c_{n,\tilde{y}})} \\ &\le \frac{C_{\delta,\theta}e^{\tilde{l}-l}}{r-l-C_{\delta,\theta}}.\end{aligned}$$

The previous two displays imply that for $p = p_{\delta,\theta}$ large enough,

$$1 - a_r'(x)a_r'(y) \geq \min\{1 - a_r'(x)^2, 1 - a_r'(y)^2\} \geq \left| \frac{Cov(\Theta^c_{r,x}, \Theta^c_{r,y})}{Cov(\Theta^c_{r,\tilde{x}}, \Theta^c_{r,\tilde{y}})} - 1 \right|,$$

as desired.

- Case 3: x and y belong to different B-boxes. Then, since the fine fields are independent,

  $$Cov(\Theta^c_{r,x}, \Theta^c_{r,y}) = Cov(\Theta_{r,x}, \Theta_{r,y}).$$

  Therefore, it is enough to prove

  $$a_r'(x)a_r'(y) \leq \frac{Cov(\Theta_{r,x}, \Theta_{r,y})}{Cov(\Theta_{r,\tilde{x}}, \Theta_{r,\tilde{y}})}.$$

  We proved in Case 2 that

  $$1 - a_r'(x)a_r'(y) \geq \frac{(p^2 - C_{\delta,\theta})e^{\tilde{l}-l}}{r - l + C_{\delta,\theta}}. \tag{8.3.5}$$

  We therefore study

  $$\left| \frac{Cov(\Theta_{r,x}, \Theta_{r,y})}{Cov(\Theta_{r,\tilde{x}}, \Theta_{r,\tilde{y}})} - 1 \right| = \frac{|Cov(\Theta_{r,x}, \Theta_{r,y}) - Cov(\Theta_{r,\tilde{x}}, \Theta_{r,\tilde{y}})|}{Cov(\Theta_{r,\tilde{x}}, \Theta_{r,\tilde{y}})}.$$

  By (7.2.1), the denominator of the previous display is bounded below by $-\log \|\tilde{x} - \tilde{y}\| - C_{\delta,\theta}$. On the other hand, the numerator is not greater than

  $$|Cov(\Theta_{r,x}, \Theta_{r,y}) - Cov(\Theta_{r,x}, \Theta_{r,\tilde{y}})| + |Cov(\Theta_{r,x}, \Theta_{r,\tilde{y}}) - Cov(\Theta_{r,\tilde{x}}, \Theta_{r,\tilde{y}})|.$$

  By (6.1.4),

  $$\begin{aligned} &|Cov(\Theta_{r,x}, \Theta_{r,y}) - Cov(\Theta_{r,x}, \Theta_{r,\tilde{y}})| \\ &\quad \leq \iint |G_I(x+u, y+v) - G_I(x+u, \tilde{y}+v)| \, \theta_r(u)\theta_r(v)dudv. \end{aligned}$$

  By (6.1.1), we can bound

  $$\begin{aligned} &|G_I(x+u, y+v) - G_I(x+u, \tilde{y}+v)| \\ &\quad \leq \left| \log \frac{\|x+u-y-v\|}{\|x+u-\tilde{y}-v\|} \right| + \mathbb{E}^{x+u} \left[\left| \log \frac{\|W_{\tau_I} - y - v\|}{\|W_{\tau_I} - \tilde{y} - v\|} \right| \right] \\ &\quad \leq C_\delta \frac{\|y - \tilde{y}\|}{\|x - y\|} + C_\delta \|y - \tilde{y}\| \leq C_\delta \frac{\|y - \tilde{y}\|}{\|x - y\|}. \end{aligned}$$

Combining the last four displays, we obtain

$$\left|\frac{Cov(\Theta_{r,x},\Theta_{r,y})}{Cov(\Theta_{r,\tilde{x}},\Theta_{r,\tilde{y}})}-1\right|\leq C_\delta\frac{\|x-\tilde{x}\|+\|y-\tilde{y}\|}{-\|x-y\|(\log(\|x-y\|)+C_{\delta,\theta})}.$$

But, since x and y belong to different B-boxes, we have $\|x-y\|\geq\delta e^{-r+l}$. Therefore, the previous display is

$$\leq C_\delta\frac{e^{\tilde{l}-l}}{r-l-C_{\delta,\theta}}.$$

The previous display and (8.3.5) imply

$$1-a_r'(x)a_r'(y)\geq\left|\frac{Cov(\Theta_{r,x},\Theta_{r,y})}{Cov(\Theta_{r,\tilde{x}},\Theta_{r,\tilde{y}})}-1\right|$$

for $p=p_{\delta,\theta}$ large enough, completing the proof of Claim 8.3.1.

□

Slepian's Lemma and Claim 8.3.1 imply that, for any closed square $A\subseteq[\delta,1-\delta]^2$,

$$\mathbb{P}\left(\max_{x\in\tilde{I}\cap A}\Theta_{r,x}\geq m_r+\lambda\right)\leq\mathbb{P}\left(\max_{x\in\tilde{I}\cap A}\left(a_r'(x)\Theta^c_{r,\tilde{x}}+\Theta^f_{r,x}+\psi_{z,x}\right)\geq m_r+\lambda\right)\quad(8.3.6)$$

for all $\lambda\in\mathbb{R}$. In order to control the right hand side of (8.3.6), we will need:

Claim 8.3.2. Let $a_r'(x)$ be chosen as in Claim 8.3.1. Then, there exists a constant $C_\theta\in(0,\infty)$ such that, for all r and z large enough (depending on δ and θ),

$$\mathbb{P}\left(\max_{x\in\tilde{I}\cap A}(a_r'(x)\Theta^c_{r,\tilde{x}}+\Theta^f_{r,x})\geq m_r+\lambda\right)\leq C_\theta|A|^{1/2}\max\{\lambda,1\}e^{-2\lambda}\quad(8.3.7)$$

for all $\lambda\in\mathbb{R}$ and any closed square $A\subseteq[\delta,1-\delta]^2$. Moreover, for a fixed closed square $A\subseteq[\delta,1-\delta]^2$, there exists a constant $c_{\delta,\theta}\in(0,\infty)$ such that, for all r large enough (depending on δ,θ and $|A|$),

$$\mathbb{P}\left(\max_{x\in\tilde{I}\cap A}\left(a_r'(x)\Theta^c_{r,\tilde{x}}+\Theta^f_{r,x}\right)\geq m_r+\lambda\right)\geq c_{\delta,\theta}|A|\lambda e^{-2\lambda}\quad(8.3.8)$$

for all $\lambda\geq1$.

Proof of Claim 8.3.2, Display (8.3.7). The proof of (8.3.7) is analogous to that of (7.2.4). All we need to check is the analog of Lemma 7.2.1 for the field $(a'_r(x)\Theta^c_{r,\tilde{x}} + \Theta^f_{r,x} : x \in \tilde{I})$. Suppose first that x and y belong to the same B-box. Then, displays (8.2.1), (8.3.4), together with (7.1.2) and (7.2.1), imply

$$\begin{aligned}
&Cov(a'_r(x)\Theta^c_{r,\tilde{x}} + \Theta^f_{r,x}, a'_r(y)\Theta^c_{r,\tilde{y}} + \Theta^f_{r,x}) \\
&\quad = a'_r(x)a'_r(y)Cov(\Theta^c_{r,\tilde{x}}, \Theta^c_{r,\tilde{y}}) + Cov(\Theta^f_{r,x}, \Theta^f_{r,y}) \\
&\quad = (1 - \frac{O(1)e^{\tilde{l}-l}}{r-l+O(1)})(r-l+O(1)) - \log(\max\{e^{r-l}\|x-y\|, e^{-l}\}) + O(1) \\
&\quad = -\log(\max\{\|x-y\|, e^{-r}\}) + O(1),
\end{aligned}$$

where the $O(1)$ terms depend on δ and θ.

Suppose next that x and y belong to different B-boxes. Then,

$$\begin{aligned}
&Cov(a'_r(x)\Theta^c_{r,\tilde{x}} + \Theta^f_{r,x}, a'_r(y)\Theta^c_{r,\tilde{y}} + \Theta^f_{r,x}) = a'_r(x)a'_r(y)Cov(\Theta_{r,\tilde{x}}, \Theta_{r,\tilde{y}}) \\
&\quad = (1 - \frac{O(1)e^{\tilde{l}-l}}{r-l+O(1)})(-\log\|\tilde{x}-\tilde{y}\|) + O(1)) = -\log\|\tilde{x}-\tilde{y}\| + O(1),
\end{aligned}$$

where in the last equality we have used that $\|x-y\| \geq \delta e^{-r+l}$. Applying basic properties of log and that $\|\tilde{x}-\tilde{x}\| \leq C_\delta e^{-r+l}$, we obtain $|\log(\|x-y\|/\|\tilde{x}-\tilde{y}\|)| \leq C_\delta e^{\tilde{l}-l}$. Therefore, the previous display is

$$= -\log\|x-y\| + O(1),$$

where the $O(1)$ term depends on δ and θ. This proves the analog of (7.2.1).

We next prove the analog of (7.2.2). For points $\|x-y\| \leq e^{-r}$, we have $\tilde{x} = \tilde{y}$, so, by applying (7.2.2),

$$\begin{aligned}
&\mathbb{E}[(a'_r(x)\Theta^c_{r,\tilde{x}} + \Theta^f_{r,x} - a'_r(y)\Theta^c_{r,\tilde{y}} - \Theta^f_{r,y})^2] \\
&\quad = (a'_r(x) - a'_r(y))^2 Var(\Theta^c_{r,\tilde{x}}) + \mathbb{E}[(\Theta^f_{r,x} - \Theta^f_{r,y})^2] \\
&\quad \leq |a'_r(x)^2 - a'_r(y)^2| Var(\Theta^c_{r,\tilde{x}}) + C_\theta e^r\|x-y\|.
\end{aligned}$$

But, by applying (8.2.2) to (8.3.3), we deduce that

$$|a'_r(x)^2 - a'_r(y)^2| Var(\Theta^c_{r,\tilde{x}}) \leq C_{\delta,\theta} e^{r-l}\|x-y\| + |Var(\psi_{z,x}) - Var(\psi_{z,y})|.$$

From the definition of $\psi_{z,x}$ in (8.3.1), we can deduce that the previous display is

$$\leq C_{\delta,\theta} e^{r-l}\|x-y\| \leq e^r\|x-y\|$$

for z large enough (depending on δ and θ). Combining the last three displays, we obtain the analog of (7.2.2) for the field $(a'_r(x)\Theta^c_{r,\tilde{x}} + \Theta^f_{r,x} : x \in \tilde{I})$. Therefore, the analog of Lemma 7.2.1 holds, as desired. $\square$

Proof of Claim 8.3.2, (8.3.8). Let q be a large constant that we will choose below. We will compare the fields

$$\left(a'_r(x)\Theta^c_{r,\tilde{x}} + \Theta^f_{r,x} : x \in A \cap V_{r-2q}\right) \text{ and } (b(x)\xi_{xe^{-q}}(r-q) : x \in A \cap V_{r-2q})$$

where $b_r(x) \geq 0$ is defined so that

$$Var(a'_r(x)\Theta^c_{r,x} + \Theta^f_{r,x}) = Var(b_r(x)\xi_{xe^{-q}}(r-q)).$$

It follows from (8.3.2) and the definition of the CMBRW that, for q large enough (depending on δ and θ), we have $b_r(x) \geq 1$ for all $x \in A \cap V_{r-2q}$.

We compare the covariance of the previous fields. From the proof of (8.3.7), we obtain that, for different points $x, y \in A \cap V_{r-2q}$,

$$Cov(a'_r(x)\Theta^c_{r,\tilde{x}} + \Theta^f_{r,x}, a'_r(y)\Theta^c_{r,\tilde{y}} + \Theta^f_{r,y}) \leq -\log\|x-y\| + C_{\delta,\theta}.$$

Then, from (7.5.2) and (7.5.3), we obtain that, for q large enough (depending on δ and θ), the previous display is

$$\leq -\log\|xe^{-q} - ye^{-q}\| - 2 \leq Cov(\xi_{xe^{-q}}(r-q), \xi_{ye^{-q}}(r-q)).$$

Therefore, by using Slepian's Lemma, we obtain

$$\begin{aligned}
&\mathbb{P}\left(\max_{x\in A\cap V_{r-2q}} a'_r(x)\Theta^c_{r,\tilde{x}} + \Theta^f_{r,x} \geq m_r + \lambda\right) \\
&\quad \geq \mathbb{P}\left(\max_{x\in A\cap V_{r-2q}} b(x)\xi_{xe^{-q}}(r-q) \geq m_r + \lambda\right) \\
&\quad \geq \mathbb{P}\left(\max_{x\in(e^{-q}A)\cap V_{r-q}} b(xe^q)\xi_x(r-q) \geq m_r + \lambda\right).
\end{aligned}$$

Since $b(xe^q) \geq 1$, we obtain that this is

$$\geq \mathbb{P}\left(\max_{x\in(e^{-q}A)\cap V_{r-q}} \xi_x(r-q) \geq m_r + \lambda\right).$$

But $m_r = m_{r-q} + O(q)$, so the previous display is

$$\geq \mathbb{P}\left(\max_{x \in (e^{-q}A) \cap V_{r-q}} \xi_x(r-q) \geq m_{r-q} + \lambda + O(q) \right).$$

By (7.5.5), the previous display is greater than $c_{\delta,\theta}|A|\lambda e^{-2\lambda}$, as desired. $\square$

We are now ready to prove Proposition 8.1.2.

Proof of (8.1.3). For $v \in \Xi$, let $\psi^*_{z,v} = \sup_{x \in \tilde{B}_v} \psi_{z,x}$. Define also, for integers $y \geq 1$ and for a closed square $A \subseteq [\delta, 1-\delta]^2$, the random sets $\Gamma_A(y) = \{x \in \tilde{I} \cap A : \psi^*_{z,\tilde{x}} \in [y-1, y)\}$. Then,

$$\begin{aligned} &\mathbb{P}\left(\max_{x \in \tilde{I} \cap A} \left(a_r(x)\Theta^c_{r,\tilde{x}} + \Theta^f_{r,x} + \psi^*_{z,\tilde{x}} \right) \geq m_r + z \right) \\ &\quad \leq \mathbb{P}\left(\max_{x \in \tilde{I} \cap A} \left(a'_r(x)\Theta^c_{r,\tilde{x}} + \Theta^f_{r,x} \right) \geq m_r + z \right) \\ &\qquad + \sum_{y \geq 1} \mathbb{P}\left(\max_{x \in \Gamma_A(y)} \left(a'_r(x)\Theta^c_{r,\tilde{x}} + \Theta^f_{r,x} \right) \geq m_r + z - y \right). \end{aligned} \tag{8.3.9}$$

The independence of $\Theta_{r,(\cdot)}$ and the random sets $\Gamma_A(y)$, together with (8.3.7) implies that the previous display is

$$\leq \mathbb{P}\left(\max_{x \in \tilde{I} \cap A} \left(a'_r(x)\Theta^c_{r,\tilde{x}} + \Theta^f_{r,x} \right) \geq m_r + z \right) + C_\theta z e^{-2z} \sum_{y \geq 1} \mathbb{E}[|\Gamma_A(y)|]^{1/2} e^{2y}$$

But (7.4.4) and (8.3.1) imply that $\mathbb{P}\left(\psi^*_{z,v} \geq \lambda\right) \leq C_p \exp(-c_p e^{l-\tilde{l}} \lambda^2)$. Therefore, the previous display is

$$\begin{aligned} &\leq \mathbb{P}\left(\max_{x \in \tilde{I} \cap A} \left(a'_r(x)\Theta^c_{r,\tilde{x}} + \Theta^f_{r,x} \right) \geq m_r + z \right) \\ &\quad + C_\theta |A|^{1/2} z e^{-2z} \sum_{y \geq 1} C_p \exp(-c_p e^{l-\tilde{l}} y^2 + 2y). \end{aligned} \tag{8.3.10}$$

Dividing this by $\mathbb{P}\left(\max_{x \in \tilde{I} \cap A} \left(a'_r(x)\Theta^c_{r,\tilde{x}} + \Theta^f_{r,x} \right) \geq m_r + z\right)$ and using both (8.3.6) and (8.3.8), we obtain

$$\begin{aligned} &\frac{\mathbb{P}\left(\max_{x \in \tilde{I} \cap A} \Theta_{r,x} \geq m_r + z\right)}{\mathbb{P}\left(\max_{x \in \tilde{I} \cap A} \left(a'_r(x)\Theta^c_{r,\tilde{x}} + \Theta^f_{r,x} \right) \geq m_r + z\right)} \\ &\quad \leq 1 + C_{\delta,\theta,A} \sum_{y \geq 1} C_p \exp(-c_p e^{l-\tilde{l}} y^2 + 2y). \end{aligned}$$

The right hand side of the previous display converges to 1 as $z \to \infty$, because $l - \tilde{l} \to \infty$ as $z \to \infty$. Therefore,

$$\limsup_{z\to\infty} \limsup_{r\to\infty} \frac{\mathbb{P}\left(\max_{x\in\tilde{I}\cap A} \Theta_{r,x} \geq m_r + z\right)}{\mathbb{P}\left(\max_{x\in\tilde{I}\cap A} \left(a'_r(x)\Theta^c_{r,\tilde{x}} + \Theta^f_{r,x}\right) \geq m_r + z\right)} \leq 1. \tag{8.3.11}$$

In order to finish the proof of (8.1.3), we must choose $\epsilon_z \to 0$ such that

$$\limsup_{z\to\infty} \limsup_{r\to\infty} \frac{\mathbb{P}\left(\max_{x\in\tilde{I}\cap A} \left(a'_r(x)\Theta^c_{r,\tilde{x}} + \Theta^f_{r,x}\right) \geq m_r + z\right)}{\mathbb{P}\left(\max_{x\in\tilde{I}\cap A} \tilde{\Theta}_{r,x} \geq m_r + z - \epsilon_z\right)} \leq 1. \tag{8.3.12}$$

Note that

$$\begin{aligned}
&\mathbb{P}\left(\max_{x\in\tilde{I}\cap A} \left(a'_r(x)\Theta^c_{r,\tilde{x}} + \Theta^f_{r,x}\right) \geq m_r + z\right) \\
&\quad \leq \mathbb{P}\left(\max_{x\in\tilde{I}\cap A} \left(\Theta^c_{r,\tilde{x}} + \Theta^f_{r,x}\right) \geq m_r + z - \epsilon_z\right) \\
&\qquad + \mathbb{P}\left(\min_{x\in\tilde{I}\cap A} \left(1 - a'_r(x)\right)\Theta^c_{r,\tilde{x}} \leq -\epsilon_z\right).
\end{aligned}$$

Dividing by the left hand side of the previous display and using (8.3.8), we obtain

$$\begin{aligned}
1 \leq\ & \frac{\mathbb{P}\left(\max_{x\in\tilde{I}\cap A} \left(\Theta^c_{r,\tilde{x}} + \Theta^f_{r,x}\right) \geq m_r + z - \epsilon_z\right)}{\mathbb{P}\left(\max_{x\in\tilde{I}\cap A} \left(a'_r(x)\Theta^c_{r,\tilde{x}} + \Theta^f_{r,x}\right) \geq m_r + z\right)} \\
&+ \frac{\mathbb{P}\left(\min_{x\in\tilde{I}\cap A} \left(1 - a'_r(x)\right)\Theta^c_{r,\tilde{x}} \leq -\epsilon_z\right)}{c_{\delta,\theta,A} z e^{-2z}}.
\end{aligned}$$

It is therefore enough to show that the second term in the right hand side of the previous display goes to 0 as $r \to \infty$. Applying (7.4.3), (8.2.1), (8.2.2) and (8.3.1) to (8.3.3) and using that $p = p_{\delta,\theta}$ depends only on δ and θ, we obtain

$$1 - a'_r(x)^2 \leq \frac{C_{\delta,\theta} e^{\tilde{l}-l}}{r - l - C_{\delta,\theta}}.$$

Since $0 \leq a'_r(x) \leq 1$, it follows that

$$1 - a'_r(x) \leq \frac{C_{\delta,\theta} e^{\tilde{l}-l}}{r - l - C_{\delta,\theta}}.$$

Therefore,

$$\mathbb{P}\left(\min_{x\in\tilde{I}\cap A}(1-a_r'(x))\Theta^c_{r,\tilde{x}}\le -\epsilon_z\right)\le \mathbb{P}\left(\min_{v\in\Xi\cap A}\Theta^c_{r,v}\le -c_{\delta,\theta}(r-l-C_{\delta,\theta})e^{l-\tilde{l}}\epsilon_z\right) \tag{8.3.13}$$

$$\le \sum_{v\in\Xi\cap A}\mathbb{P}\left(\Theta^c_{r,v}\le -c_{\delta,\theta}(r-l-C_{\delta,\theta})e^{l-\tilde{l}}\epsilon_z\right).$$

From (8.2.3) and using $|\Xi\cap A|\le O(|A|e^{2r-2\tilde{l}})$, this is

$$\le C|A|e^{2r-2\tilde{l}}\exp\left(-c_{\delta,\theta}\frac{(r-l-C_{\delta,\theta})^2}{(r-l+C_{\delta,\theta})}e^{2l-2\tilde{l}}\epsilon_z^2\right).$$

Therefore, if we choose ϵ_z so that $c_{\delta,\theta}e^{l-\tilde{l}}\epsilon_z=\sqrt{3}$, we obtain that the previous display goes to 0 as $r\to\infty$, as desired. Note that, since $l-\tilde{l}\to\infty$, then $\epsilon_z\to 0$ as $z\to\infty$. This concludes the proof of (8.1.3). $\square$

We will now prove the second half of Proposition 8.1.2, the lower bound (8.1.4). Let $(\psi_{z,v}:v\in\Xi)$ be a collection of centered Gaussian variables with variance $pe^{\tilde{l}-l}$, where $p>0$ is a large constant that will be chosen later (see (8.3.14)). We will compare the fields

$$\left(\tilde{\Theta}_{r,x}:x\in\tilde{I}\right)\ \text{ and }\ \left(b_r(x)\Theta^c_{r,x}+\Theta^f_{r,x}+\psi_{z,\tilde{x}}:x\in\tilde{I}\right),$$

where $b_r(x)$ is defined so that

$$Var(\tilde{\Theta}_{r,x})=Var(b_r(x)\Theta^c_{r,x}+\Theta^f_{r,x}+\psi_{z,\tilde{x}})$$

for all $x\in\tilde{I}$.

Proof of (8.1.4). A proof analogous to that of Claim 8.3.1 shows that, for p large enough (depending on δ and θ),

$$Cov(b_r(x)\Theta^c_{r,\tilde{x}}+\psi_{z,\tilde{x}},b_r(y)\Theta^c_{r,\tilde{y}}+\psi_{z,\tilde{y}})\le Cov(\tilde{\Theta}_{r,x},\tilde{\Theta}_{r,y}). \tag{8.3.14}$$

for all $x,y\in\tilde{I}\cap A$. Therefore, Slepian's Lemma implies

$$\mathbb{P}\left(\max_{x\in\tilde{I}\cap A}\tilde{\Theta}_{r,x}\ge\lambda\right)\le\mathbb{P}\left(\max_{x\in\tilde{I}\cap A}(b_r(x)\Theta^c_{r,x}+\Theta^f_{r,x}+\psi_{z,\tilde{x}})\ge\lambda\right)$$

for all $\lambda \in \mathbb{R}$. As in the proof of (8.1.3), using the argument starting in (8.3.9) and ending in (8.3.11), we obtain

$$\limsup_{z\to\infty}\limsup_{r\to\infty}\frac{\mathbb{P}\left(\max_{x\in\tilde{I}\cap A}\tilde{\Theta}_{r,x}\geq m_r+z\right)}{\mathbb{P}\left(\max_{x\in\tilde{I}\cap A}(b_r(x)\Theta^c_{r,x}+\Theta^f_{r,x})\geq m_r+z\right)}\leq 1.$$

We therefore show, for some $\epsilon_z \to_{z\to\infty} 0$,

$$\limsup_{z\to\infty}\limsup_{r\to\infty}\frac{\mathbb{P}\left(\max_{x\in\tilde{I}\cap A}(b_r(x)\Theta^c_{r,x}+\Theta^f_{r,x})\geq m_r+z\right)}{\mathbb{P}\left(\max_{x\in\tilde{I}\cap A}\Theta_{r,x}\geq m_r+z-\epsilon_z\right)}\leq 1.$$

As in the proof of (8.1.3), using the argument starting in (8.3.12) and ending in (8.3.13), we deduce that it is enough to show that

$$\mathbb{P}\left(\max_{x\in\tilde{I}\cap A}\Theta^c_{r,x}\geq c_{\delta,\theta}(r-l-C_{\delta,\theta})e^{l-\tilde{l}}\epsilon_z\right)$$

goes to 0 as $n\to\infty$. But the previous display is

$$\leq \sum_{v\in\Xi\cap A}\mathbb{P}\left(\max_{x\in\tilde{B}_v}\Theta^c_{r,x}\geq c_{\delta,\theta}(r-l-C_{\delta,\theta})e^{l-\tilde{l}}\epsilon_z\right). \tag{8.3.15}$$

Fernique's Majorizing Criterion (see [8, Theorem 4.1]) and (8.2.2) imply

$$\mathbb{E}\left[\max_{x\in\tilde{B}_v}\Theta^c_{r,x}\right]\leq C\int_0^\infty\sqrt{-\log\left(c_{\delta,\theta}r^4e^{2l-2\tilde{l}}\right)}dr=C_{\delta,\theta}e^{(l-\tilde{l})/2}.$$

Therefore, Borell's Inequality (see [8, Theorem 2.1]) and (8.2.3) imply

$$\begin{aligned}\mathbb{P}&\left(\max_{x\in\tilde{B}_v}\Theta^c_{r,x}\geq c_{\delta,\theta}(r-l-C_{\delta,\theta})e^{l-\tilde{l}}\epsilon_z\right)\\&=\mathbb{P}\left(\max_{x\in\tilde{B}_v}\Theta^c_{r,x}\geq c_{\delta,\theta}(r-l-C_{\delta,\theta})e^{l-\tilde{l}}\epsilon_z-C_{\delta,\theta}e^{(l-\tilde{l})/2}+C_{\delta,\theta}e^{(l-\tilde{l})/2}\right)\\&\leq C\exp\left(-(c_{\delta,\theta}(r-l-C_{\delta,\theta})e^{l-\tilde{l}}\epsilon_z-C_{\delta,\theta}e^{(l-\tilde{l})/2})^2/(r-l+C_{\delta,\theta})\right).\end{aligned}$$

By choosing $\epsilon_z\to 0$ so that $c_{\delta,\theta}e^{l-\tilde{l}}\epsilon_z=\sqrt{3}$, the previous display is

$$\leq C_{\delta,\theta}e^{-3r}f(z)$$

for some function $f(z)$ of z. This display, combined with (8.3.15), implies

$$\begin{aligned}\mathbb{P}&\left(\max_{x\in\tilde{I}\cap A}\Theta^c_{r,x}\geq c_{\delta,\theta}(r-l-C_{\delta,\theta})e^{l-\tilde{l}}\epsilon_z\right)\leq C_{\delta,\theta}|\Xi\cap A|e^{-3r}f(z)\\&\leq C_{\delta,\theta}|A|e^{2r-2\tilde{l}}e^{-3r}f(z),\end{aligned}$$

which goes to 0 as $r\to\infty$. This concludes the proof of (8.1.4). $\square$

8.4 Proof of Proposition 8.1.4

The proof of the first inequality (8.1.5) only involves rough estimates on the covariance of the CMBRW, and it is therefore simpler than the proof of (8.1.6), which requires Lemma 8.2.3.

Proof of (8.1.5). By the definition of $\xi^{\mathrm{up}}_{r,(\cdot)}$,

$$
\begin{aligned}
Cov(\xi^{\mathrm{up}}_{r,v}, \xi^{\mathrm{up}}_{r,w}) &= Cov(\xi_v(T_v) - \xi_v(\gamma_v), \xi_w(T_w) - \xi_w(\gamma_w)) \\
&= Cov(\xi_v(T_v), \xi_v(T_w)) - Cov(\xi_v(\gamma_v), \xi_w(\gamma_w)) \leq Cov(\xi_{r,v}, \xi_{r,w}),
\end{aligned}
$$

proving the lower bound of (8.1.5). For the proof of the upper bound, assume without losing generality that $T_v \leq T_w$. Note that for $R = C_{\delta,\theta}(l - \tilde{l})$ and r large enough (depending on δ, θ and z), we have $r - l - R \leq T_v$. Then,

$$
\int_{r-l-R}^{T_v} (1 - e^r|v_1 - w_1|)_+ (1 - e^r|v_2 - w_2|)_+ dr \leq T_v - r + l + R = \gamma_v \leq \sqrt{\gamma_v \gamma_w}.
$$

Therefore, from the definition of $\xi^{\mathrm{lw}}_{r,(\cdot)}$, we obtain

$$
\begin{aligned}
Cov(\xi^{\mathrm{lw}}_{r,v}, \xi^{\mathrm{lw}}_{r,w}) &= \sqrt{\gamma_v, \gamma_w} + Cov(\xi_v(r - l - R), \xi_w(r - l - R)) \\
&\geq Cov(\xi_v(T_v), \xi_w(T_w)),
\end{aligned}
$$

as desired. □

Proof of (8.1.6)*, the lower bound.* We distinguish two cases:

- Case 1: v and w belong to the same B-box. Then, by (8.2.1),

  $$Cov(\Theta^c_{r,v}, \Theta^c_{r,w}) \geq r - l - C_{\delta,\theta}.$$

  On the other hand, from (7.5.1) and (7.5.2),

  $$
  \begin{aligned}
  &Cov(\xi^{\mathrm{up}}_{r,v}, \xi^{\mathrm{up}}_{r,w}) \\
  &\quad = Cov(\xi_{r,v} - \xi_v(\gamma_v) + W_v(\gamma_v), \xi_{r,w} - \xi_w(\gamma_w) + W_w(\gamma_w)) \\
  &\quad = Cov(\xi_{r,v}, \xi_{r,w}) - Cov(\xi_v(\gamma_v), \xi_w(\gamma_w)) \leq r - l - R + C_{\delta,\theta} \\
  &\quad \leq r - l - C_{\delta,\theta}.
  \end{aligned}
  $$

  for $R = C_{\delta,\theta}(l - \tilde{l})$ large enough.

- Case 2: v and w belong to different B-boxes. In this case, (7.2.1) implies

$$Cov(\Theta^c_{r,v}, \Theta^c_{r,w}) = Cov(\Theta_{r,v}, \Theta_{r,w}) \geq -\log \|v-w\| - C_{\delta,\theta}.$$

  On the other hand, if we let $s = -\log \|x-y\|_\infty$,

$$\begin{aligned} Cov(\xi^{\text{up}}_{r,v}, \xi^{\text{up}}_{r,w}) &= Cov(\xi_v(s), \xi_w(s)) - Cov(\xi_v(\gamma_v), \xi_w(\gamma_w)) \\ &\leq s - R + C_{\delta,\theta} \leq -\log \|v-w\| - C_{\delta,\theta} \end{aligned}$$

  for $R = C_{\delta,\theta}(l-\tilde{l})$ large enough.

□

Proof of (8.1.6)*, the upper bound.* Note that, by definition,

$$\begin{aligned} \mathbb{E}\left[\left(\xi^{\text{lw}}_{r,v} - \xi^{\text{lw}}_{r,w}\right)^2\right] = \mathbb{E}&\left[(\xi_v(r-l-R) - \xi_w(r-l-R))^2\right] \\ &+ \left(\sqrt{T_v - r + l + R} - \sqrt{T_w - r + l + R}\right)^2. \end{aligned}$$

Let $s = \min\{r-l-R, -\log \|v-w\|_\infty\}$. Then, by (7.5.1),

$$\begin{aligned} &\mathbb{E}\left[(\xi_v(r-l-R) - \xi_w(r-l-R))^2\right] \\ &\quad = 2(r-l-R) - 2\int_0^s (1-e^r|v_1-w_1|)(1-e^r|v_2-w_2|)dr \\ &\quad \leq 2(r-l-R-s) + 2\|v-w\|_1 e^s. \end{aligned}$$

On the other hand, using (8.2.3),

$$\left(\sqrt{T_v - r + l + R} - \sqrt{T_w - r + l + R}\right)^2 \leq C_{\delta,\theta}(T_v - T_w)^2/R.$$

Combining the last three displays, we obtain:

$$\mathbb{E}\left[\left(\xi^{\text{lw}}_{r,v} - \xi^{\text{lw}}_{r,w}\right)^2\right] \leq 2(r-l-R-s) + 2\|v-w\|_1 e^s + C_{\delta,\theta}(T_v-T_w)^2/R. \tag{8.4.1}$$

We now distinguish four cases, according to the positions of v and w:

- Case 1: v and w belong to the same B-box. Then, $\|v-w\|_\infty \leq L$, so $s = r-l-R$. Therefore, (7.2.5) and (8.4.1) imply

$$\mathbb{E}\left[\left(\xi^{\text{lw}}_{r,v} - \xi^{\text{lw}}_{r,w}\right)^2\right] \leq C\|v-w\|e^{r-l-R} + C_{\delta,\theta}\|v-w\|^2 e^{2r-2l}/R.$$

On the other hand, by (7.2.6),

$$\mathbb{E}\left[\left(\Theta^c_{r,v} - \Theta^c_{r,w}\right)^2\right] \geq c_{\delta,\theta}\|v-w\|^2 e^{2r-2l}$$

Therefore, a sufficient condition for

$$\mathbb{E}\left[\left(\xi^{\mathrm{lw}}_{r,v} - \xi^{\mathrm{lw}}_{r,w}\right)^2\right] \leq \mathbb{E}\left[\left(\Theta^c_{r,v} - \Theta^c_{r,w}\right)^2\right]$$

is

$$C\|v-w\|e^{r-l}e^{-R} \leq c_{\delta,\theta}\|v-w\|^2 e^{2r-2l}.$$

Since v and w belong to Ξ, their distance is at least $e^{-r+\tilde{l}}$. Therefore, a sufficient condition for the previous inequality is

$$R = C_{\delta,\theta}(l-\tilde{l}).$$

The previous choice of R depends only on z, δ and θ, as desired.

- Case 2: v and w belong to different B-boxes, but $\|v-w\| \leq e^{-r+l+R/2}$. Then, $s = r-l-R$ and we obtain from (8.4.1)

$$\mathbb{E}\left[\left(\xi^{\mathrm{lw}}_{r,v} - \xi^{\mathrm{lw}}_{r,w}\right)^2\right] \leq Ce^{-R/2} + C_{\delta,\theta}/R$$

which is smaller than

$$c_{\delta,\theta} \leq \mathbb{E}[(\Theta^c_{r,v} - \Theta^c_{r,w})^2]$$

for R large enough, with the last display following from (8.2.4).

- Case 3: v and w satisfy $e^{-r+l+R/2} \leq \|v-w\| \leq e^{-r+l+R}$. We still have $s = r-l-R$, so (8.2.3) and (8.4.1) imply

$$\mathbb{E}\left[\left(\xi^{\mathrm{lw}}_{r,v} - \xi^{\mathrm{lw}}_{r,w}\right)^2\right] \leq C_{\delta,\theta}.$$

On the other hand, (8.2.5) implies that, for R large enough,

$$\mathbb{E}[(\Theta_{r,v} - \Theta_{r,w})^2] \geq R/2 \geq C_{\delta,\theta},$$

as desired.

- Case 4: $\|v-w\| \geq e^{-r+l+R}$. In this case, $s = -\log \|v-w\|_\infty$, so

$$Cov(\xi^{\mathrm{lw}}_{r,v}, \xi^{\mathrm{lw}}_{r,w}) \geq \sqrt{\gamma_v \gamma_w} + \int_0^s (1 - e^r|v_1 - w_1|)(1 - e^r|v_2 - w_2|)dr$$
$$\geq R - C_{\delta,\theta} - \log \|v-w\|_\infty.$$

On the other hand, since v and w belong to different B-boxes,

$$Cov(\Theta^c_{r,v}, \Theta^c_{r,w}) = Cov(\Theta_{r,v}, \Theta_{r,w}),$$

and (7.2.1) implies

$$Cov(\Theta_{r,v}, \Theta_{r,w}) \leq -\log \|v-w\| + C_{\delta,\theta}.$$

Therefore, the desired inequality follows from the previous three displays by taking R large enough (with R depending on δ and θ).

□

8.5 Proof of Proposition 8.1.6

In order to prove Proposition 8.1.6, we will first prove a series of properties of the events $E_{r,v}(z)$ and the random variables $\Lambda_{r,A}(z)$ defined in Section 8.1. We will need the following estimate, whose proof is almost verbatim that of [4, Lemma 4.9].

Lemma 8.5.1. *There exist constants $c_{\delta,\theta}, C_{\delta,\theta} \in (0,\infty)$ and $r_0 = r_{\delta,\theta,z} \geq 1$ such that, for any $\tilde{B}_1 \neq \tilde{B}_2$, any $\lambda_1, \lambda_2 \geq 1$ and any $r \geq r_0$,*

$$\mathbb{P}\left(\max_{x\in\tilde{B}_1} \Theta^f_{r,x} \geq l\frac{m_r}{r} + \lambda_1, \max_{x\in\tilde{B}_2} \Theta^f_{r,x} \geq l\frac{m_r}{r} + \lambda_2\right)$$
$$\leq C_{\delta,\theta} e^{C_{\delta,\theta}(l-\tilde{l})} l^{-3} (\lambda_1 + \log l)(\lambda_2 + \log l) e^{-2(\lambda_1+\lambda_2)} e^{-c_{\delta,\theta}(\lambda_1^2+\lambda_2^2)/l}. \tag{8.5.1}$$

Moreover, for $\lambda \geq -\log l + 1$,

$$\mathbb{P}\left(\max_{x\in\tilde{B}_1} \Theta^f_{r,x} \geq l\frac{m_r}{r} + \lambda\right) \leq C_{\delta,\theta} e^{C_{\delta,\theta}(l-\tilde{l})} l^{-3/2} (\lambda + \log l) e^{-2\lambda} e^{-c_{\delta,\theta}\lambda^2/l}. \tag{8.5.2}$$

Proof of Lemma 8.5.1. We give a proof of (8.5.1); the proof of (8.5.2) is similar, but simpler. For $i = 1, 2$, let $\hat{B}_i$ be a box of side length $(1-\delta)e^{-r+\tilde{l}}$ that is concentric with $\tilde{B}_i$, and note that $\tilde{B}_i \subset \hat{B}_i$. For $x \in \tilde{B}_i$, let

$$\Phi_{r,x} = \mathbb{E}[\Theta^f_{r,x} \mid \partial \hat{B}_i] \text{ and } \Psi_{x,r} = \Theta^f_{r,x} - \Phi_{r,x}.$$

Then, from (7.2.6),

$$\mathbb{E}\left[(\Phi_{r,x} - \Phi_{r,y})^2\right] \leq C_{\delta,\theta}\|x-y\|^2 e^{2r-2\tilde{l}}.$$

An application of Fernique's Majorizing Criterion and the previous display imply that, for $i = 1, 2$,

$$\mathbb{E}[\Phi^{(i)}_r] \leq C_{\delta,\theta},$$

where $\Phi^{(i)}_r = \max_{x\in\tilde{B}_i} \Phi_{r,x}$. Furthermore, by the same argument as in the proof of (8.2.1), we see that

$$\max_{x\in\tilde{B}_1\cup\tilde{B}_2} Var(\Phi_{r,x}) \leq l - \tilde{l} + C_{\delta,\theta}.$$

Therefore, the last two displays and Borell's Inequality imply that there exist constants $c_{\delta,\theta}, C_{\delta,\theta} \in (0,\infty)$ such that, for $i = 1, 2$,

$$\mathbb{P}(\Phi^{(i)}_r \geq \lambda - 1) \leq C_{\delta,\theta} e^{-c_{\delta,\theta}\lambda^2/(l-\tilde{l})} \tag{8.5.3}$$

for all $\lambda \geq 0$. We can now find the upper bound (8.5.1) as follows

$$\begin{aligned}
&\mathbb{P}\left(\max_{x\in\tilde{B}_1}\Theta^f_{r,x} \geq l\frac{m_r}{r} + \lambda_1, \max_{x\in\tilde{B}_2}\Theta^f_{r,x} \geq l\frac{m_r}{r} + \lambda_2\right) \\
&\quad\leq C\int_0^\infty \max_{i=1,2}\mathbb{P}\left(\Phi^{(i)}_r \geq \lambda - 1\right)\mathbb{P}\left(\max_{x\in\hat{B}_1}\Psi_{r,x} \geq l\frac{m_r}{r} + \lambda_1 - \lambda\right) \\
&\qquad\times\mathbb{P}\left(\max_{x\in\hat{B}_2}\Psi_{r,x} \geq l\frac{m_r}{r} + \lambda_2 - \lambda\right)d\lambda \\
&\quad= C\int_0^\infty \max_{i=1,2}\mathbb{P}\left(\Phi^{(i)}_r \geq \lambda - 1\right) \\
&\qquad\times\mathbb{P}\left(\max_{x\in\hat{B}_1}\Psi_{r,x} \geq m_l + \tfrac{3}{4}\log(l) + \lambda_1 - \lambda + O(l\log(r)/r)\right) \\
&\qquad\times\mathbb{P}\left(\max_{x\in\hat{B}_2}\Psi_{r,x} \geq m_l + \tfrac{3}{4}\log(l) + \lambda_2 - \lambda + O(l\log(r)/r)\right)d\lambda,
\end{aligned}$$

where the last equality follows because $l\,m_r/r = m_l + \frac{3}{4}\log(l) + O(l\log(r)/r)$. By (8.5.3), and by applying (7.1.2) and (7.2.3) to the MGFF $\Psi_{r,x}$ in the boxes $\hat{B}_i$, we obtain that the previous display is

$$\begin{aligned}
&\leq C_{\delta,\theta}\int_0^\infty e^{-c_{\delta,\theta}\lambda^2/(l-\tilde{l})} l^{-3}(\lambda_1+\log l)(\lambda_2+\log l) \\
&\quad\times e^{-2(\lambda_1+\lambda_2-2\lambda)} e^{-c_{\delta,\theta}((\lambda_1-\lambda)^2+(\lambda_2-\lambda)^2)/l} d\lambda \\
&\leq C_{\delta,\theta} e^{C(l-\tilde{l})} l^{-3}(\lambda_1+\log l)(\lambda_2+\log l) e^{-2(\lambda_1+\lambda_2)} e^{-c_{\delta,\theta}(\lambda_1^2+\lambda_2^2)/l},
\end{aligned}$$

as desired. □

In order to study $E_{r,v}(z)$ and $\Lambda_{r,A}(z)$, we define the following auxiliary sets. For $v\in\Xi$, let

$$\begin{aligned}
D_{r,v}(z) = \{\xi_{r,v}(t) \leq{}& \frac{m_r}{r}t + z + z^{1/20} \\
&+ 10\log_+(\min\{t, T_v - t\}) \text{ for all } t\in[0,T_v]\}, \\
F_{r,v}(z) = D_{r,v} \cap{}& \{\xi_{r,v} + \max_{x\in\tilde{B}_v}\Theta^f_{r,x} \geq m_r + z\}
\end{aligned}$$

and

$$G_r(z) = \bigcup_{v\in\Xi} D_{r,v}(z)^c,$$

where $(\cdot)^c$ means complement. We also define, for $A\subseteq[\delta,1-\delta]^2$, the random variables

$$\Gamma_{r,A}(z) = \sum_{v\in\Xi\cap A} \mathbf{1}_{F_{r,v}(z)}.$$

Similarly, we define $D^{\mathrm{up}}_{r,v}(z)$, $F^{\mathrm{up}}_{r,v}(z)$, $G^{\mathrm{up}}_r(z)$, $\Gamma^{\mathrm{up}}_{r,A}(z)$, $D^{\mathrm{lw}}_{r,v}(z)$, $F^{\mathrm{lw}}_{r,v}(z)$, $G^{\mathrm{lw}}_n(z)$ and $\Gamma^{\mathrm{lw}}_{r,A}(z)$. An application of Slepian's Lemma and [4, Lemma 3.7] imply that there exist constants $c_{\delta,\theta}, C_{\delta,\theta}, \gamma\in(0,\infty)$ such that, for all r large enough,

$$\mathbb{P}\left(G^{\mathrm{up}}_r(z)\right) \leq C_{\delta,\theta} z e^{-2(z+z^\gamma)} e^{-c_{\delta,\theta}z^2/r}, \tag{8.5.4}$$

where the argument to prove the previous display is analogous to that of the proof of [4, Display (4.50)].

Using Lemma 8.5.1 and (8.5.4), we can show that $\Lambda_{r,A}(z)$ and $\Gamma_{r,A}(z)$ have asymptotically the same expectation, which is an important property necessary to prove Proposition 8.1.6. The following proof is almost verbatim that of [4, Lemma 4.10].

Lemma 8.5.2. *For any closed sub-square with nonempty interior $A \subseteq [\delta, 1-\delta]^2$,*

$$\lim_{z\to\infty} \limsup_{r\to\infty} \left| \frac{\mathbb{E}\Lambda_{r,A}(z)}{\mathbb{E}\Gamma_{r,A}(z)} - 1 \right| = 0.$$

Proof of Lemma 8.5.2. For $v \in \Xi \cap A$ and $t \in [0, T_v]$, let $\bar{\xi}_{r,v}(t) = \xi_{r,v}(t) - \frac{m_r}{r}t$. Define the probability measure $\mathbb{Q}$ by

$$\frac{d\mathbb{P}}{d\mathbb{Q}} = \exp\left(-\frac{m_r}{r}\bar{\xi}_{r,v}(T_v) - \frac{m_r^2}{2r^2}T_v\right),$$

and note that, by Girsanov's Theorem, the process $\bar{\xi}_{r,v}(t)$ is a standard Brownian motion under $\mathbb{Q}$. Define the probability densities $\mu_v = \mu_{r,v,z}$ and $\hat{\mu}_v = \hat{\mu}_{r,v,z}$ on $\mathbb{R}$ by

$$\mu_v(y) = \mathbb{Q}\left(\bar{\xi}_{r,v}(T_v) \in dy, \bar{\xi}_{r,v}(t) \leq z \text{ for all } t \in [0, T_v]\right)/dy$$

and

$$\begin{aligned}\hat{\mu}_v(y) = \mathbb{Q}(&\bar{\xi}_{r,v}(T_v) \in dy, \bar{\xi}_{r,v}(t) \leq z + z^{1/20} \\ &+ 10\log_+(\min\{t, T_v - t\}) \text{ for all } t \in [0, T_v])/dy.\end{aligned}$$

Then, conditioning on the value of $\bar{\xi}_{r,v}(T_v)$, we obtain

$$\begin{aligned}
&\mathbb{P}\left(F_{r,v}(z) \backslash E_{r,v}(z)\right) \\
&= \int_0^{z+z^{1/20}} \exp\left(-\frac{m_r}{r}y - \frac{m_r^2}{2r^2}T_v\right)\hat{\mu}_v(y) \\
&\quad \times \mathbb{P}\left(\max_{x\in\tilde{B}_v} \Theta^f_{r,x} \geq \frac{m_r}{r}(r - T_v) + z - y\right) dy \\
&\quad + \int_{-\infty}^0 \exp\left(-\frac{m_r}{r}y - \frac{m_r^2}{2r^2}T_v\right)(\hat{\mu}_v(y) - \mu_v(y)) \\
&\quad \times \mathbb{P}\left(\max_{x\in\tilde{B}_v} \Theta^f_{r,x} \geq \frac{m_r}{r}(r - T_v) + z - y\right) dy \\
&=: I_1 + I_2.
\end{aligned}$$

We first bound I_1. By (8.2.3),

$$\mathbb{P}\left(\max_{x\in\tilde{B}_v} \Theta^f_{r,x} \geq \frac{m_r}{r}(r - T_v) + z - y\right) \leq \mathbb{P}\left(\max_{x\in\tilde{B}_v} \Theta^f_{r,x} \geq \frac{m_r}{r}l + z - y - C_{\delta,\theta}\right).$$

Applying (8.5.2), we obtain that the previous display is

$$\leq C_{\delta,\theta}e^{C_{\delta,\theta}(l-\tilde{l})}l^{-3/2}(z - y + \log l)e^{-2(z-y)}e^{-c_{\delta,\theta}(z-y)^2/l}.$$

Therefore,

$$
\begin{aligned}
I_1 &\leq C_{\delta,\theta} e^{C_{\delta,\theta}(l-\tilde{l})} l^{-3/2} \\
&\times \int_0^{z+z^{1/20}} \exp\left(-\frac{m_r}{r} y - \frac{m_r^2}{2r^2} T_v\right) \hat{\mu}_v(y)(z-y+\log l) e^{-2(z-y)} e^{-c_{\delta,\theta}(z-y)^2/l} dy \\
&\leq C_{\delta,\theta} e^{C_{\delta,\theta}(l-\tilde{l})} l^{-3/2} \\
&\times \int_0^{z+z^{1/20}} e^{-(2-O(\frac{\log r}{r}))y} e^{-2T_v} T_v^{3/2} \hat{\mu}_v(y)(z-y+\log l) e^{-2(z-y)} e^{-c_{\delta,\theta}(z-y)^2/l} dy.
\end{aligned}
$$

But, by [4, Lemma 3.6], we know that $\hat{\mu}_v(y) \leq Cz(z+z^{1/20}-y)T_v^{-3/2}$. Therefore, the previous display is

$$
\begin{aligned}
&\leq C_{\delta,\theta} e^{C_{\delta,\theta}(l-\tilde{l})} l^{-3/2} z e^{-2z} e^{-2T_v} \\
&\quad \times \int_0^{z+z^{1/20}} e^{O(\frac{\log r}{r})y} (z+z^{1/20}-y)(z-y+\log l) e^{-c_{\delta,\theta}(z-y)^2/l} dy \\
&\leq C_{\delta,\theta} e^{C_{\delta,\theta}(l-\tilde{l})} l^{-3/2} z^3 e^{-2z} e^{-2r+2l}(z+\log l). \qquad (8.5.5)
\end{aligned}
$$

On the other hand, [4, Lemma 3.6] implies that there exists $\delta_z \to 0$ as $z \to \infty$ such that

$$
(\hat{\mu}_v(y) - \mu_v(y)) \leq \delta_z \mu_v(y)
$$

for all $y \leq 0$. Therefore,

$$
I_2 \leq \delta_z \mathbb{P}(E_{r,v}(z)). \qquad (8.5.6)
$$

Together, (8.5.5) and (8.5.6) imply

$$
\mathbb{P}\left(F_{r,v}(z) \backslash E_{r,v}(z)\right) \leq C_{\delta,\theta} e^{C_{\delta,\theta}(l-\tilde{l})} l^{-3/2} z^3 e^{-2z} e^{-2r}(z+\log l) + \delta_z \mathbb{P}(E_{r,v}(z)).
$$

Adding the previous display over all $v \in \Xi \cap A$, and using that the cardinality of $\Xi \cap A$ is of the order of $|A|e^{2r-2\tilde{l}}$ (where $|A|$ is the Lebesgue measure of A), we obtain

$$
\mathbb{E}[\Gamma_{r,A}(z)] - \mathbb{E}[\Lambda_{r,A}(z)] \leq C_{\delta,\theta} e^{C_{\delta,\theta}(l-\tilde{l})} l^{-3/2} z^3 e^{-2z}(z+\log l)|A| + \delta_z \mathbb{E}[\Lambda_{r,A}(z)].
$$

Dividing the previous display by $\mathbb{E}[\Gamma_{r,A}(z)]$, we obtain

$$
1 - (1+\delta_z)\frac{\mathbb{E}[\Lambda_{r,A}(z)]}{\mathbb{E}[\Gamma_{r,A}(z)]} \leq C_{\delta,\theta} e^{C_{\delta,\theta}(l-\tilde{l})} l^{-3/2} z^3 e^{-2z}(z+\log l)|A|/\mathbb{E}[\Gamma_{r,A}(z)]. \qquad (8.5.7)
$$

In order to finish this upper bound, we show that there exists a constant $c_{\delta,\theta} \in (0,\infty)$ such that $\mathbb{E}\left[\Gamma_{r,A}(z)\right] \geq c_{\delta,\theta}|A|ze^{-2z}$, and we argue as follows. First, set $A = I_\delta$. A straightforward decomposition shows that

$$\mathbb{P}\left(\max_{x\in\tilde{I}}(\xi^{\mathrm{up}}_{r,\tilde{x}} + \Theta^f_{r,x}) \geq m_r + z\right) \leq \mathbb{P}\left(G^{\mathrm{up}}_r(z)\right) + \mathbb{E}[\Gamma^{\mathrm{up}}_{r,I_\delta}(z)].$$

Applying Lemma 7.2.6, Propositions 8.1.2 and 8.1.4 and display (8.5.4) to the previous display, we obtain, for some small absolute constant $\gamma > 0$,

$$c_{\delta,\theta}ze^{-2z} \leq C_{\delta,\theta}ze^{-2(z+z^\gamma)} + \mathbb{E}[\Gamma^{\mathrm{up}}_{r,I_\delta}(z)].$$

So, for z large enough (depending on δ and θ), we obtain

$$c_{\delta,\theta}ze^{-2z} \leq \mathbb{E}[\Gamma^{\mathrm{up}}_{r,I_\delta}(z)] = \mathbb{E}[\Gamma_{r,I_\delta}(z)].$$

Consider two B-boxes B_1 and B_2. Since the CMBRW $\xi_{r,v}$ is translation invariant and the fine field $\Theta^f_{r,x}$ is identical for different B-boxes, it follows that

$$\Gamma_{r,B_1}(z) \overset{law}{=} \Gamma_{r,B_2}(z).$$

This implies that, for any B-box, $\mathbb{E}[\Gamma_{r,B}(z)] = \mathbb{E}[\Gamma_{r,I_\delta}(z)]/|\mathcal{B}|$, where $\mathcal{B}$ is the collection of all B-boxes. Therefore,

$$\mathbb{E}[\Gamma_{r,A}(z)] \geq \frac{\#\{B\in\mathcal{B} : B\subset A\}}{|\mathcal{B}|}\mathbb{E}[\Gamma_{r,I_\delta}(z)] \geq |A|c_{\delta,\theta}ze^{-2z}. \tag{8.5.8}$$

Applying the last inequality in (8.5.7), we obtain

$$1-(1+\delta_z)\frac{\mathbb{E}[\Lambda_{r,A}(z)]}{\mathbb{E}[\Gamma_{r,A}(z)]} \leq C_{\delta,\theta}e^{C_{\delta,\theta}(l-\tilde{l})}l^{-3/2}z^2(z+\log l).$$

A pair of sufficient conditions for the right hand side of the previous display to converge to 0 as $z\to\infty$ are given by (8.1.1). Therefore, using the trivial inequality $\Lambda_{r,A}(z) \leq \Gamma_{r,A}(z)$, the previous display implies

$$\lim_{z\to\infty}\limsup_{r\to\infty}\left|1-\frac{\mathbb{E}[\Lambda_{r,A}(z)]}{\mathbb{E}[\Gamma_{r,A}(z)]}\right| = 0,$$

as desired. □

We now study the second moment of $\Lambda^{\mathrm{lw}}_{r,A}(z)$ and show that it is asymptotically equal to its first moment. Together, Lemmas 8.5.2 and 8.5.3 are the essential tools employed in the first and second moment method that we use to prove Proposition 8.1.6. The proof of Lemma 8.5.3 is almost verbatim that of [4, Lemma 4.11].

Lemma 8.5.3. *For any closed sub-square with nonempty interior $A \subseteq [\delta, 1-\delta]^2$,*

$$\lim_{z\to\infty} \limsup_{r\to\infty} \left| \frac{\mathbb{E}\left[(\Lambda^{\mathrm{lw}}_{r,A}(z))^2\right]}{\mathbb{E}\left[\Lambda^{\mathrm{lw}}_{r,A}(z)\right]} - 1 \right| = 0.$$

Proof of Lemma 8.5.3. Note that

$$\frac{\mathbb{E}\left[(\Lambda^{\mathrm{lw}}_{r,A}(z))^2\right]}{\mathbb{E}\left[\Lambda^{\mathrm{lw}}_{r,A}(z)\right]} - 1 = \frac{\sum_{v,w\in\Xi\cap A, v\neq w} \mathbb{P}(E^{\mathrm{lw}}_{r,v}(z) \cap E^{\mathrm{lw}}_{r,w}(z))}{\mathbb{E}[\Lambda^{\mathrm{lw}}_{r,A}(z)]}. \tag{8.5.9}$$

We will find an upper bound for the right hand side of (8.5.9). We know from Lemma 8.5.2 and (8.5.8) that

$$\limsup_{z\to\infty} \limsup_{r\to\infty} \frac{|A|ze^{-2z}}{\mathbb{E}[\Lambda^{\mathrm{lw}}_{r,A}(z)]} \le C_{\delta,\theta}. \tag{8.5.10}$$

Therefore, it suffices to find an upper bound for $\mathbb{P}(E^{\mathrm{lw}}_{r,v}(z) \cap E^{\mathrm{lw}}_{r,w}(z))$ for $v \neq w$. Let $\bar{\xi}^{\mathrm{lw}}_{r,v}(t) = \xi^{\mathrm{lw}}_{r,v}(t) - \frac{m_r}{r} t$ and $s = -\log \|v-w\|_\infty + R + C_{\delta,\theta}$. Note that $(\bar{\xi}^{\mathrm{lw}}_{r,v}(t) - \bar{\xi}^{\mathrm{lw}}_{r,v}(s))$ and $(\bar{\xi}^{\mathrm{lw}}_{r,w}(t) - \bar{\xi}^{\mathrm{lw}}_{r,w}(s))$ are independent for $t \ge s$. We distinguish three cases: (i) $s \le z/3$, (ii) $z/3 \le s \le r - l - z/3$ and (iii) $z/3 \ge r - l - z/3$.

- Case (i): $s \le z/3$. Note that, in this case, v and w belong to different B-boxes. Therefore,

$$\begin{aligned} &\mathbb{P}(E^{\mathrm{lw}}_{r,v}(z) \cap E^{\mathrm{lw}}_{r,w}(z)) \\ &\quad \le \sum_{-\infty < x,y \le z} p_v(x) p_w(y) \\ &\quad \times \mathbb{P}(\bar{\xi}^{\mathrm{lw}}_{r,v}(t) \le z \text{ for all } t \in [0,s], \bar{\xi}^{\mathrm{lw}}_{r,v}(s) \in [x-1,x], \bar{\xi}^{\mathrm{lw}}_{r,v}(s) \in [y-1,y]), \end{aligned} \tag{8.5.11}$$

  where we define $p_v(x)$ as the supremum over $\gamma \in [x-1,x]$ of

$$\mathbb{P}(\bar{\xi}^{\mathrm{lw}}_{r,v}(t) \le z - \gamma \text{ for all } t \in [0, T_v - s], \bar{\xi}^{\mathrm{lw}}_{r,v}(T_v - s) + \Theta^{f,*}_{r,v} \ge l\frac{m_r}{r} + z - \gamma),$$

and $\Theta^{f,*}_{r,v} := \max_{v'\in\tilde{B}_v} \Theta^{f}_{r,v'}$. By decomposing according to the value of $\bar{\xi}^{\mathrm{lw}}_{r,v}(T_v - s)$, we obtain

$$p_v(x) \le \sum_{x'\le z} p'_v(x')\times$$
$$\mathbb{P}(\bar{\xi}^{\mathrm{lw}}_{r,v}(t) \le z - x \text{ for all } t \in [0, T_v - s], \bar{\xi}^{\mathrm{lw}}_{r,v}(T_v - s) \in [x' - x - 1, x' - x]),$$

where

$$p'_v(x') = \mathbb{P}(\Theta^{f,*}_{r,v} \ge l\frac{m_r}{r} + z - x').$$

Applying Girsanov's Theorem and the reflection principle, we obtain

$$p_v(x) \le \sum_{x'\le z} e^{-2(x'-x)-\frac{1}{2}(m_r/r)^2(T_v-s)}(z-x)(z-x')p'_v(x')/(T_v-s)^{3/2}.$$

By (8.5.2), this display is

$$\begin{aligned}
&\le C_{\delta,\theta}\sum_{x'\le z} e^{-2(x'-x)}\frac{e^{-\frac{1}{2}(m_r/r)^2(T_v-s)}}{(T_v-s)^{3/2}}(z-x)(z-x')e^{C_{\delta,\theta}(l-\tilde{l})}l^{-3/2}\\
&\quad\times (z-x'+\log l)e^{-2(z-x')}e^{-c_{\delta,\theta}(z-x')^2/l}\\
&\le C_{\delta,\theta}e^{2x}e^{-\frac{1}{2}(m_r/r)^2(T_v-s)}(z-x)e^{C_{\delta,\theta}(l-\tilde{l})}e^{-2z}/(T_v-s)^{3/2} \qquad (8.5.12)\\
&\le C_{\delta,\theta}e^{2x-2(T_v-s)}(z-x)e^{C_{\delta,\theta}(l-\tilde{l})}e^{-2z}.
\end{aligned}$$

Using (8.5.11), we obtain

$$\begin{aligned}
&\mathbb{P}(E^{\mathrm{lw}}_{r,v}(z)\cap E^{\mathrm{lw}}_{r,w}(z))\\
&\quad\le \sum_{-\infty<x\le z} Cp_v(x)^2\mathbb{P}(\bar{\xi}^{\mathrm{lw}}_{r,v}(t)\le z \text{ for all } t\in[0,s], \bar{\xi}^{\mathrm{lw}}_{r,v}(s)\in[x-1,x]). \qquad (8.5.13)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\mathbb{P}(\bar{\xi}^{\mathrm{lw}}_{r,v}(t)\le z \text{ for all } t\in[0,s], \bar{\xi}^{\mathrm{lw}}_{r,v}(s)\in[x-1,x]) \le \mathbb{P}(\bar{\xi}^{\mathrm{lw}}_{r,v}(s)\in[x-1,x])\\
&\quad\le C_{\delta,\theta}e^{-\frac{m_r}{r}x-\frac{1}{2}(\frac{m_r}{r})^2 s}e^{-\frac{1}{2}x^2/s}/\sqrt{s} \le C_{\delta,\theta}e^{-2x-2s}se^{-\frac{1}{2}x^2/s}.
\end{aligned}$$

The last three displays and (8.2.3) imply

$$\begin{aligned}
&\mathbb{P}(E^{\mathrm{lw}}_{r,v}(z) \cap E^{\mathrm{lw}}_{r,w}(z)) \\
&\quad \leq C_{\delta,\theta} \sum_{-\infty<x\leq z} e^{-2x-2s} s e^{x^2/2s} e^{4x} e^{-4(r-l-s)} (z-x) e^{C_{\delta,\theta}(l-\tilde{l})} e^{-4z} \\
&\quad \leq C_{\delta,\theta} e^{-4r+4l} e^{2s} s e^{C_{\delta,\theta}(l-\tilde{l})} e^{-4z} \sum_{-\infty<x\leq z} (z-x)^2 e^{2x} e^{-x^2/2s} \\
&\quad \leq C_{\delta,\theta} e^{-4r+4l} e^{5s} z^2 e^{C_{\delta,\theta}(l-\tilde{l})} e^{-4z}.
\end{aligned}$$

For a fixed point $v \in \Xi$, we sum over all points w such that $s \leq z/3$, and obtain

$$\begin{aligned}
&\sum_{w:s\leq z/3} \mathbb{P}(E^{\mathrm{lw}}_{r,v}(z) \cap E^{\mathrm{lw}}_{r,w}(z)) \\
&\quad \leq C_{\delta,\theta} e^{-2r+2l} z^2 e^{-4z} e^{C_{\delta,\theta}(l-\tilde{l})} \sum_{-\infty<s<z/3} e^{3s} \\
&\quad \leq C_{\delta,\theta} e^{-2r+2l} z^2 e^{-3z} e^{C_{\delta,\theta}(l-\tilde{l})}.
\end{aligned}$$

Summing over all points $v \in \Xi \cap A$, it follows that

$$\sum_{\substack{v,w\in\Xi\cap A:\\ s\leq z/3}} \mathbb{P}(E^{\mathrm{lw}}_{r,v}(z) \cap E^{\mathrm{lw}}_{r,w}(z)) \leq C_{\delta,\theta} |A| z^2 e^{-3z} e^{C_{\delta,\theta}(l-\tilde{l})}.$$

Consequently, from (8.5.10) and the previous display,

$$\frac{\sum_{\substack{v,w\in\Xi\cap A:\\ s\leq z/3}} \mathbb{P}(E^{\mathrm{lw}}_{r,v}(z) \cap E^{\mathrm{lw}}_{r,w}(z))}{\mathbb{E}[\Lambda^{\mathrm{lw}}_{r,A}(z)]} \to_{z\to\infty} 0$$

uniformly in r, as desired for (8.5.9).

- Case (ii): $z/3 \leq s \leq r - l - z/3$. Note that v and w still belong to different B-boxes. By the same argument as in the previous case, we obtain (8.5.13). Using Girsanov's Theorem and the reflection principle, we obtain the bound

$$\begin{aligned}
&\mathbb{P}\left(\bar{\xi}^{\mathrm{lw}}_{r,v}(t) \leq z \text{ for all } t \in [0,s], \bar{\xi}^{\mathrm{lw}}_{r,v}(s) \in [x-1,x]\right) \\
&\quad \leq C_{\delta,\theta} e^{-2x-2s} z(z-x) \frac{\exp\left(\frac{3\log(r)}{2r} s\right)}{s^{3/2}}.
\end{aligned}$$

Then, from the previous display, (8.5.12) and (8.5.13), we obtain

$$\begin{aligned}&\mathbb{P}(E^{\mathrm{lw}}_{r,v}(z)\cap E^{\mathrm{lw}}_{r,w}(z))\\&\quad\le C_{\delta,\theta}\sum_{x\le z}e^{2x}e^{-4r}e^{4l}e^{2s}(z-x)^3ze^{-4z}e^{C_{\delta,\theta}(l-\tilde{l})}\frac{(r-l)^{3/2}}{s^{3/2}(r-l-s)^{3/2}},\end{aligned}$$

where, in the last inequality, we have used that $e^{\frac{3\log r}{2r}(r-l)}\le(r-l)^{3/2}$ and $e^{\frac{3\log r}{2r}(r-l-s)}\le(r-l-s)^{3/2}$. Fix $v\in\Xi$. Summing over all $w\in\Xi$ such that $z/3\le s\le r-l-z/3$, we obtain

$$\begin{aligned}&\sum_{w:z/3\le s\le r-l-z/3}\mathbb{P}(E^{\mathrm{lw}}_{r,v}(z)\cap E^{\mathrm{lw}}_{r,w}(z))\\&\le C_{\delta,\theta}e^{-2r+2l}ze^{-2z}e^{C_{\delta,\theta}(l-\tilde{l})}(r-l)^{3/2}\sum_{z/3\le s\le r-l-z/3}s^{-3/2}(r-l-s)^{-3/2}\\&\le C_{\delta,\theta}e^{-2r+2l}ze^{-2z}e^{C_{\delta,\theta}(l-\tilde{l})}z^{-1/2}.\end{aligned}$$

Adding over all $v\in\Xi\cap A$, we obtain:

$$\sum_{\substack{w,v\in\Xi\cap A:\\ z/3\le s\le r-l-z/3}}\mathbb{P}(E^{\mathrm{lw}}_{r,v}(z)\cap E^{\mathrm{lw}}_{r,w}(z))\le C_{\delta,\theta}|A|z^{1/2}e^{-2z}e^{C_{\delta,\theta}(l-\tilde{l})}.$$

From (8.5.10) and the previous display, it follows

$$\frac{\sum_{\substack{v,w\in\Xi\cap A:\\ z/3\le s\le r-l-z/3}}\mathbb{P}(E^{\mathrm{lw}}_{r,v}(z)\cap E^{\mathrm{lw}}_{r,w}(z))}{\mathbb{E}[\Lambda^{\mathrm{lw}}_{r,A}(z)]}\to_{z\to\infty}0$$

uniformly in r, as desired for (8.5.9).

- Case (iii): $s\ge r-l-z/3$. We employ the decomposition

$$\begin{aligned}&\mathbb{P}(E^{\mathrm{lw}}_{r,v}(z)\cap E^{\mathrm{lw}}_{r,w}(z))\\&\le\sum_{-\infty<x',y'\le z}\mathbb{P}(\Theta^{*,f}_{r,v}\ge l\frac{m_r}{r}+z-x',\Theta^{*,f}_{r,w}\ge l\frac{m_r}{r}+z-y')\times\\&\mathbb{P}(\bar{\xi}^{\mathrm{lw}}_{r,v}(t),\bar{\xi}^{\mathrm{lw}}_{r,w}(t)\le z\text{ for all }t\le T_v,T_w,\bar{\xi}_{r,v}\in[x'-1,x'],\bar{\xi}_{r,w}\in[y'-1,y'])\end{aligned}\tag{8.5.14}$$

The second term in the right hand side of the previous display can be decomposed further by

$$\begin{aligned}
&\mathbb{P}(\bar{\xi}^{\mathrm{lw}}_{r,v}(t), \bar{\xi}^{\mathrm{lw}}_{r,w}(t) \le z \text{ for all } t \le T_v, T_w,\ \bar{\xi}_{r,v} \in [x'-1, x'], \bar{\xi}_{r,w} \in [y'-1, y'])\\
&\le \sum_{-\infty < y \le x \le z} q_v(x) q_w(y)\\
&\times \mathbb{P}(\bar{\xi}^{\mathrm{lw}}_{r,v}(t), \bar{\xi}^{\mathrm{lw}}_{r,w}(t) \le z \text{ for all } t \le s, \bar{\xi}^{\mathrm{lw}}_{r,v}(s) \in [x-1, x], \bar{\xi}^{\mathrm{lw}}_{r,w}(s) \in [y-1, y]),
\end{aligned}$$

where

$$q_v(x) = \mathbb{P}(\bar{\xi}^{\mathrm{lw}}_{r,v}(t) \le z - x \text{ for all } t \in [0, T_v - s], \bar{\xi}^{\mathrm{lw}}_{r,v}(T_v - s) \in [x' - x - 1, x' - x]).$$

Since $q_v(x) \le \mathbb{P}(\bar{\xi}^{\mathrm{lw}}_{r,v}(T_v - s) \in [x' - x - 1, x' - x])$, we obtain

$$q_v(x) \le C_{\delta,\theta} e^{-\frac{m_r^2}{2r^2}(r-l-s)} e^{-\frac{m_r}{r}(x'-x)} \frac{e^{-(x'-x)^2/2(r-l-s)}}{\sqrt{r-l-s}}.$$

The last three displays imply

$$\begin{aligned}
&\mathbb{P}(\bar{\xi}^{\mathrm{lw}}_{r,v}(t), \bar{\xi}^{\mathrm{lw}}_{r,w}(t) \le z \text{ for all } t \le T_v, T_w,\ \bar{\xi}_{r,v} \in [x'-1, x'], \bar{\xi}_{r,w} \in [y'-1, y'])\\
&\quad \le C_{\delta,\theta} \sum_{-\infty < x \le z} \mathbb{P}(\bar{\xi}^{\mathrm{lw}}_{r,v}(t) \le z \text{ for all } t \in [0, s], \bar{\xi}^{\mathrm{lw}}_{r,v}(s) \in [x-1, x])\\
&\qquad \times e^{-\frac{m_r^2}{r^2}(r-l-s)} e^{-\frac{m_r}{r}(x'+y'-2x)} (r-l-s)^{-1} e^{-\frac{(x'-x)^2 + (y'-y)^2}{2(r-l-s)}}.
\end{aligned}$$

Applying Girsanov's Theorem and the reflection principle, the previous display is

$$\begin{aligned}
&\le C_{\delta,\theta} e^{-\frac{m_r}{r}(x'+y')} e^{-\frac{m_r^2}{2r^2}(2r-2l-s)} s^{-3/2} (r-l-s)^{-1}\\
&\quad \times \sum_{-\infty < x \le z} (z-x) e^{\frac{m_r}{r}x} e^{-\frac{(x'-x)^2 + (y'-x)^2}{2(r-l-s)}}.
\end{aligned} \tag{8.5.15}$$

On the other hand, the first term in the right hand side of (8.5.14) can be bounded using (8.5.1). Therefore, from (8.5.1), (8.5.14) and (8.5.15), we obtain

$$
\begin{aligned}
&\mathbb{P}(E^{\mathrm{lw}}_{r,v}(z) \cap E^{\mathrm{lw}}_{r,w}(z)) \\
&\le C_{\delta,\theta} e^{-\frac{m_r^2}{2r^2}(2r-2l-s)} \frac{e^{C_{\delta,\theta}(l-\tilde{l})} z e^{-4z}}{(r-l-s)s^{3/2}l^3} \\
&\quad \times \sum_{x \le z} e^{2x}(z-x) \left(\sum_{x' \le z} (z-x'+\log l) e^{-\frac{(x'-x)^2}{2(r-l-s)}} \right)^2 \\
&\le C_{\delta,\theta} e^{-\frac{m_r^2}{2r^2}(2r-2l-s)} \frac{e^{C_{\delta,\theta}(l-\tilde{l})} z e^{-4z}}{s^{3/2}l^3} \sum_{x \le z} e^{2x}(z-x+\log l+\sqrt{r-l-s})^3 \\
&\le C_{\delta,\theta} e^{-\frac{m_r^2}{2n^2}(2n-2l-s)} \frac{e^{C_{\delta,\theta}(l-\tilde{l})} z e^{-2z}}{s^{3/2}l^3} (\log l+\sqrt{r-l-s})^3 \\
&\le C_{\delta,\theta} e^{-4r+4l+2s}(r-l-s)^{3/2} e^{C_{\delta,\theta}(l-\tilde{l})} z e^{-2z} l^{-3} (\log l+\sqrt{r-l-s})^3 \\
&\le C_{\delta,\theta} e^{-4r+4l+2s}(r-l-s)^{3} e^{C_{\delta,\theta}(l-\tilde{l})} z e^{-2z} l^{-3} (\log l)^3.
\end{aligned}
$$

For fixed $v \in \Xi$, we sum the previous expression over all w such that $s \ge r-l-z/3$, and obtain

$$
\begin{aligned}
&\sum_{w: s \ge r-l-z/3} \mathbb{P}(E^{\mathrm{lw}}_{r,v}(z) \cap E^{\mathrm{lw}}_{r,w}(z)) \\
&\qquad \le C_{\delta,\theta} e^{-2r+2l} e^{C_{\delta,\theta}(l-\tilde{l})} z e^{-2z} l^{-3} (\log l)^3 \sum_{s=r-l-z/3}^{r-l+C_{\delta,\theta}(l-\tilde{l})} (r-l-s)^3,
\end{aligned}
$$

where the upper bound of this last sum comes from the definition of s. The previous display is

$$
\le C_{\delta,\theta} e^{-2r+2l} e^{C_{\delta,\theta}(l-\tilde{l})} z e^{-2z} l^{-3} (\log l)^3 z^4.
$$

Summing the previous expression over all $v \in \Xi \cap A$, we obtain

$$
\sum_{\substack{v,w \in \Xi \cap A: \\ s \ge r-l-z/3}} \mathbb{P}(E^{\mathrm{lw}}_{r,v}(z) \cap E^{\mathrm{lw}}_{r,w}(z)) \le C_{\delta,\theta} |A| e^{C_{\delta,\theta}(l-\tilde{l})} z e^{-2z} l^{-3} (\log l)^3 z^4.
$$

From (8.5.10) and the previous display, we obtain

$$
\frac{\sum_{\substack{v,w \in \Xi \cap A: \\ s \ge r-l-z/3}} \mathbb{P}(E^{\mathrm{lw}}_{r,v}(z) \cap E^{\mathrm{lw}}_{r,w}(z))}{\mathbb{E}[\Lambda^{\mathrm{lw}}_{r,A}(z)]} \le C_{\delta,\theta} e^{C_{\delta,\theta}(l-\tilde{l})} l^{-3} (\log l)^3 z^4 \to 0
$$

as $z \to \infty$, because $l \geq e^{z^{1/20}}$. This concludes the proof of Lemma 8.5.3.

□

We now have all the ingredients to prove Proposition 8.1.6.

Proof of Proposition 8.1.6. Note that

$$\frac{\mathbb{P}(\max_{x \in \tilde{I} \cap A} \xi^{\mathrm{up}}_{r,\tilde{x}} + \Theta^{f}_{r,x} \geq m_r + z)}{\mathbb{E}[\Lambda^{\mathrm{up}}_{r,A}(z)]} \leq \frac{\mathbb{P}(G^{\mathrm{up}}_r(z)) + \mathbb{E}[\Gamma^{\mathrm{up}}_{r,A}(z)]}{\mathbb{E}[\Lambda^{\mathrm{up}}_{r,A}(z)]}.$$

It follows from (8.5.4) and (8.5.10) that, for some small absolute constant $\gamma > 0$, the previous display is

$$\leq C_{\delta,\theta} e^{-2z^{\gamma}} + \frac{\mathbb{E}[\Gamma^{\mathrm{up}}_{r,A}(z)]}{\mathbb{E}[\Lambda^{\mathrm{up}}_{r,A}(z)]};$$

using Lemma 8.5.2, we conclude that

$$\limsup_{z \to \infty} \limsup_{r \to \infty} \frac{\mathbb{P}(\max_{x \in \tilde{I} \cap A} \xi^{\mathrm{up}}_{r,\tilde{x}} + \Theta^{f}_{r,x} \geq m_r + z)}{\mathbb{E}[\Lambda^{\mathrm{up}}_{r,A}(z)]} \leq 1.$$

On the other hand,

$$\mathbb{P}\left(\max_{x \in \tilde{I} \cap A} \xi^{\mathrm{lw}}_{r,\tilde{x}} + \Theta^{f}_{r,x} \geq m_r + z\right) \geq \mathbb{P}\left(\bigcup_{v \in \Xi} E^{\mathrm{lw}}_{r,A}(z)\right) \geq \frac{(\mathbb{E}[\Lambda^{\mathrm{lw}}_{r,A}(z)])^2}{\mathbb{E}[(\Lambda^{\mathrm{lw}}_{r,A}(z))^2]}. \tag{8.5.16}$$

Therefore, from Lemma 8.5.3,

$$\liminf_{z \to \infty} \liminf_{r \to \infty} \frac{\mathbb{P}(\max_{x \in \tilde{I} \cap A} \xi^{\mathrm{lw}}_{r,\tilde{x}} + \Theta^{f}_{r,x} \geq m_r + z)}{\mathbb{E}[\Lambda^{\mathrm{lw}}_{r,A}(z)]} \geq 1.$$

Proposition 8.1.6 follows by observing $\mathbb{E}[\Lambda^{\mathrm{lw}}_{r,A}(z)] = \mathbb{E}[\Lambda^{\mathrm{up}}_{r,A}(z)]$ and

$$\mathbb{P}\left(\max_{x \in \tilde{I} \cap A} \xi^{\mathrm{lw}}_{r,\tilde{x}} + \Theta^{f}_{r,x} \geq m_r + z\right) \leq \mathbb{P}\left(\max_{x \in \tilde{I} \cap A} \xi^{\mathrm{up}}_{r,\tilde{x}} + \Theta^{f}_{r,x} \geq m_r + z\right),$$

for r large enough (depending on δ, θ and z). (This last inequality follows from Proposition 8.1.4 and Slepian's Lemma.) □

8.6 Proof of Proposition 8.1.7

We first study, for a fixed $z > 0$, the convergence of $\mathbb{E}[\Lambda_{r,A}(z)]$ as $r \to \infty$. Recall that

$$\Lambda_{r,A}(z) = \sum_{v \in \Xi \cap A} \mathbf{1}_{E_{r,v}(z)}$$

and that

$$E_{r,v}(z) = \left\{ \bar{\xi}_{r,v}(t) \le z \text{ for all } [0, T_v], \bar{\xi}_{r,v} + \Theta^{f,*}_{r,v} \ge \frac{m_r}{r}(r - T_v) + z \right\}, \tag{8.6.1}$$

where $\bar{\xi}_{r,v}(t) = \xi_{r,v}(t) - \frac{m_r}{r} t$ and $\Theta^{f,*}_{r,v} = \max_{y \in \tilde{B}_v} \Theta^f_{r,y}$. We will show that the limit of $\mathbb{E}[\Lambda_{r,A}(z)]$ as $r \to \infty$ is of the form

$$z e^{-2z} \sum_{w \in \Phi} \int_A \Upsilon^{w,l}(x) dx, \tag{8.6.2}$$

where $\Upsilon^{w,l}(x)$ is defined in (8.6.8), and Φ is defined shortly.

Note that the distribution of $\Theta^{f,*}_{r,v}$ depends only on $l, \tilde{l}$ and the position of v in B_v. We will make the previous sentence more precise by using the following definitions. For each $B \in \mathcal{B}$, let $h : B \to I$ be the linear function that stretches B onto I. For $v \in \Xi$, denote $h(v)$ by $\hat{v}$. Then, by (7.1.2),

$$\max_{y \in \tilde{B}_v} \Theta^f_{r,y} \overset{law}{=} \max_{y \in h(\tilde{B}_v)} \Theta_{l,y} =: \Theta^*_{l,\hat{v}}.$$

Let $\Phi = \{\hat{v} : v \in \Xi\}$, and note that it has cardinality $O(e^{2(l - \tilde{l})})$. Note also that $(\Theta^*_{l,w} : w \in \Phi)$ does not depend on r.

In order to study $T_v = Var(\Theta^c_{r,v})$, we define the function $S : (0,1)^2 \to \mathbb{R}$ by

$$S(w) = \mathbb{E}^w[\log \|w - W_\tau\|], \tag{8.6.3}$$

where $\tau = \inf\{t \ge 0 : W_t \notin I\}$. For $r \ge 1$, define similarly

$$S_r(w) = \iint \mathbb{E}^{w+y}[\log \|w + x - W_\tau\|] \theta_r(x) \theta_r(y) dx dy. \tag{8.6.4}$$

With the previous notation, we can improve the bound (8.2.3) by the following lemma.

Lemma 8.6.1. *There exists a constant $C_{\delta,\theta} \in (0, \infty)$ such that, for each $z > 0$ and all r large enough (depending on z),*

$$|T_v - r + l + S_l(\hat{v}) - S(v)| \le C_{\delta,\theta} e^{-r},$$

for all $v \in \Xi$.

Proof of Lemma 8.6.1. By the definition of $\Theta^c_{r,v}$ and the harmonicity of the Green function,

$$T_v = \iiint_{\partial B_v} G_I(v+x,u)p_{\partial B_v}(v+y,u)du\,\theta_r(x)\theta_r(y)dxdy,$$

where $p_{\partial B_v}(v+y,\cdot)$ is the harmonic density on ∂B_v of a Brownian motion started at $v+y$. Applying (6.1.1) to the previous display, we obtain $T_v = J_v + S_r(v)$, where

$$J_v = -\iiint_{\partial B_v} \log\|v+x-u\|p_{\partial B_v}(v+y,u)du\,\theta_r(x)\theta_r(y)\,dxdy.$$

By changing variables,

$$\begin{aligned} J_v &= r-l-\iiint_{\partial I} \log\|\hat{v}+x-u\|p_{\partial I}(\hat{v}+y,u)\,\theta_l(x)\theta_l(y)dxdy \\ &= r-l-S_l(\hat{v}). \end{aligned}$$

Therefore,

$$T_v = J_v + S_r(v) = r-l-S_l(\hat{v})+S_r(v).$$

It is not difficult to check that there exists a constant $C_{\delta,\theta}\in(0,\infty)$ so that

$$|\log\|v+x-w\| - \log\|v-w\|| \le C_{\delta,\theta}e^{-r} \tag{8.6.5}$$

for all $x\in\operatorname{supp}(\theta_r)$, all $w\in\partial I$ and all $v\in I_\delta$. We also know that, if we denote by $p_{\partial I}(v,\cdot)$ the harmonic measure on ∂I with starting point v, then there exists a constant $C_{\delta,\theta}\in(0,\infty)$ such that

$$|p_{\partial I}(v,w) - p_{\partial I}(v+y,w)| \le C_{\delta,\theta}e^{-r} \tag{8.6.6}$$

for all $y\in\operatorname{supp}(\theta_r)$, all $w\in\partial I$ and all $v\in I_\delta$. The last three displays and (8.6.4) imply the existence of $C_{\delta,\theta}\in(0,\infty)$ such that

$$|T_v - r + l + S_l(\hat{v}) - S(v)| \le C_{\delta,\theta}e^{-r},$$

as desired. □

For each $w\in\Phi$ and $A\subseteq[\delta,1-\delta]^2$, define the auxiliary quantity

$$\Upsilon^{w,l}_{r,A} = \sum_{v\in\Xi\cap A:\hat{v}=w} \frac{2e^{2S_l(w)-2S(v)}}{\sqrt{2\pi}e^{2r-2l}} \int_0^\infty e^{2\lambda}\lambda\mathbb{P}(\Theta^*_{l,w} \ge 2(l+S_l(w)-S(v))+\lambda)d\lambda,$$

where the convergence of this integral is guaranteed by (8.5.2). (Recall that $\Theta^*_{l,w} \overset{law}{=} \max_{y\in\tilde{B}_v}\Theta^f_{r,y}$, for some $v\in\Xi$.) Using Lemma 8.6.1, we show the following lemma.

Lemma 8.6.2. *For any closed sub-square with nonempty interior $A \subseteq [\delta, 1-\delta]^2$ and any $z > 0$,*

$$\lim_{r\to\infty} \frac{ze^{-2z}\sum_{w\in\Phi}\Upsilon^{w,l}_{r,A}}{\mathbb{E}[\Lambda_{r,A}(z)]} = 1.$$

Proof of Lemma 8.6.2. By applying Girsanov's Theorem and the reflection principle, we obtain from (8.6.1)

$$\begin{aligned}\mathbb{P}(E_{r,v}(z)) = & \frac{e^{-\frac{m_r}{r}z - \frac{m_r^2}{2r^2}T_v}}{\sqrt{2\pi}T_v^{3/2}} \\ & \times \int_0^\infty e^{\frac{m_r}{r}\lambda}\left[\int_{\lambda-z}^{\lambda+z} \rho e^{-\rho^2/2T_v} d\rho\right] \mathbb{P}(\Theta^*_{l,\hat{v}} \geq \frac{m_r}{r}(r-T_v)+\lambda) d\lambda.\end{aligned}$$

Note that

$$e^{-\frac{m_r}{r}z}/e^{-2z} \to_{r\to\infty} 1.$$

Additionally, Lemma 8.6.1 implies

$$\frac{e^{-\frac{m_r^2}{2r^2}T_v}}{T_v^{-3/2}} \Bigg/ \frac{e^{2S_l(\hat{v})-2S(v)}}{e^{2r-2l}} \to_{r\to\infty} 1,$$

uniformly in $v \in \Xi$. The last three displays imply that $\mathbb{E}\left[\Lambda_{r,A}(z)\right]$ is asymptotically equivalent to

$$\begin{aligned}& \sum_{v\in\Xi\cap A} \frac{e^{-2z}e^{2S_l(\hat{v})-2S(v)}}{\sqrt{2\pi}e^{2r-2l}} \\ & \times \int_0^\infty e^{\frac{m_r}{r}\lambda}\left[\int_{\lambda-z}^{\lambda+z} \rho e^{-\rho^2/2T_v} d\rho\right] \mathbb{P}(\Theta^*_{l,\hat{v}} \geq \frac{m_r}{r}(r-T_v)+\lambda) d\lambda \qquad (8.6.7)\end{aligned}$$

as $r \to \infty$.

We next study the inner integral in (8.6.7). Note that

$$\begin{aligned}\left|\int_{\lambda-z}^{\lambda+z} \rho e^{-\rho^2/2T_v} d\rho - 2z\lambda\right| &\leq \int_{\lambda-z}^{\lambda+z} |\rho| \left|e^{-\rho^2/2T_v} - 1\right| d\rho \\ &\leq \int_{\lambda-z}^{\lambda+z} \rho^2(\lambda+z) d\rho / T_v \leq \frac{2z(\lambda+z)^3}{T_v}.\end{aligned}$$

Displays (8.2.3) and (8.5.2) imply

$$\int_0^\infty e^{\frac{m_r}{r}\lambda}\left[\frac{2z(\lambda+z)^3}{T_v}\right] \mathbb{P}(\Theta^*_{l,\hat{v}} \geq \frac{m_r}{r}(r-T_v)+\lambda) d\lambda \to_{r\to\infty} 0,$$

uniformly in v. Therefore, by plugging the last two displays in (8.6.7), we obtain that $\mathbb{E}[\Lambda_{r,A}(z)]$ is asymptotically equivalent to

$$\sum_{v\in\Xi\cap A}\frac{e^{-2z}e^{2S_l(\hat{v})-2S(v)}}{\sqrt{2\pi}e^{2r-2l}}\int_0^\infty e^{\frac{m_r}{r}\lambda}2z\lambda\mathbb{P}(\Theta^*_{l,\hat{v}}\geq\frac{m_r}{r}(r-T_v)+\lambda)d\lambda$$

as $r\to\infty$. But $|e^{\frac{m_r}{r}\lambda}-e^{2\lambda}|\leq\lambda(2-\frac{m_r}{r})e^{2\lambda}$. Thus, similar reasoning shows that $\mathbb{E}[\Lambda_{r,A}(z)]$ is asymptotically equivalent to

$$\sum_{v\in\Xi\cap A}\frac{e^{-2z}e^{2S_l(\hat{v})-2S(v)}}{\sqrt{2\pi}e^{2r-2l}}\int_0^\infty e^{2\lambda}2z\lambda\mathbb{P}(\Theta^*_{l,\hat{v}}\geq\frac{m_r}{r}(r-T_v)+\lambda)d\lambda$$

as $r\to\infty$.

Note next that Lemma 8.6.1 implies

$$\frac{m_r}{r}(r-T_v)=2(l+S_l(\hat{v})-S(v))+O(l\log r/r)$$

A change of variables therefore implies

$$\begin{aligned}&\int_0^\infty e^{2\lambda}\lambda\mathbb{P}(\Theta^*_{l,\hat{v}}\geq\frac{m_r}{r}(r-T_v)+\lambda)d\lambda\\&=\int_{O(\frac{l\log r}{r})}^\infty e^{2\lambda+O(\frac{l\log r}{r})}(\lambda+O\left(\frac{l\log r}{r}\right))\mathbb{P}(\Theta^*_{l,\hat{v}}\geq 2(l+S_l(\hat{v})-S(v))+\lambda)d\lambda.\end{aligned}$$

Another application of (8.5.2) shows that $\mathbb{E}[\Lambda_{r,A}(z)]$ is asymptotically equivalent to

$$\begin{aligned}&\sum_{v\in\Xi\cap A}\frac{2ze^{-2z}e^{2S_l(\hat{v})-2S(v)}}{\sqrt{2\pi}e^{2r-2l}}\int_0^\infty e^{2\lambda}\lambda\mathbb{P}(\Theta^*_{l,\hat{v}}\geq 2(l-S_l(\hat{v})-S(v))+\lambda)d\lambda\\&=ze^{-2z}\sum_{w\in\Phi}\Upsilon^{w,l}_{r,A},\end{aligned}$$

as $r\to\infty$, as desired. This finishes the proof of Lemma 8.6.2. $\square$

Note that $\Upsilon^{w,l}_{r,A}$ is a Riemann sum. Therefore, by defining

$$\Upsilon^{w,l}(x)=\frac{2e^{2S_l(w)}}{\sqrt{2\pi}e^{2S(x)}}\int_0^\infty e^{2\lambda}\lambda\mathbb{P}(\Theta^*_{l,w}\geq 2(l+S_l(w)-S(x))+\lambda)d\lambda \tag{8.6.8}$$

and using that $\Upsilon^{w,l}(x)$ is continuous in x, we obtain

$$\Upsilon^{w,l}_{r,A}\to_{r\to\infty}\int_A\Upsilon^{w,l}(x)dx.$$

From the previous display and Lemma 8.6.2, we obtain, for any $z > 0$,

$$\mathbb{E}[\Lambda_{r,A}(z)] \to ze^{-2z} \sum_{w \in \Phi} \int_A \Upsilon^{w,l}(x)dx \tag{8.6.9}$$

as $r \to \infty$, as asserted in (8.6.2).

We next study the asymptotic behavior of the right hand side of (8.6.9) as $z \to \infty$. We define a simplified version of $\Upsilon^{w,l}$, which we denote by $\tilde{\Upsilon}^{w,l}$, as follows:

$$\tilde{\Upsilon}^{w,l}(x) = \frac{2e^{2S(x)}}{\sqrt{2\pi}e^{2S_l(w)}} \int_0^\infty e^{2\lambda} \lambda \mathbb{P}(\Theta^*_{l,w} \geq 2l + \lambda) d\lambda. \tag{8.6.10}$$

The functions $\Upsilon^{w,l}(x)$ and $\tilde{\Upsilon}^{w,l}(x)$ are related as follows.

Lemma 8.6.3. *For any closed sub-square with nonempty interior $A \subseteq [\delta, 1-\delta]^2$,*

$$\lim_{z \to \infty} \frac{\sum_{w \in \Phi} \int_A \tilde{\Upsilon}^{w,l}(x)dx}{\sum_{w \in \Phi} \int_A \Upsilon^{w,l}(x)dx} = 1$$

Proof of Lemma 8.6.3. Using that $S_l(w)$ and $S(x)$ are both $O(1)$ (depending on δ and θ), we obtain by a change of variables

$$\Upsilon^{w,l}(x) = \frac{2e^{2S(x)}}{\sqrt{2\pi}e^{2S_l(w)}} \int_{O(1)}^\infty e^{2\lambda}(\lambda + O(1)) \mathbb{P}(\Theta^*_{l,w} \geq 2l + \lambda) d\lambda.$$

Therefore, for some $C_{\delta,\theta} \in (0, \infty)$,

$$\begin{aligned} &|\tilde{\Upsilon}^{w,l}(x) - \Upsilon^{w,l}(x)| \\ &\quad \leq C_{\delta,\theta} \frac{2e^{2S(x)}}{\sqrt{2\pi}e^{2S_l(w)}} \left(\mathbb{P}(\Theta^*_{l,w} \geq 2l) + \int_0^\infty e^{2\lambda} \mathbb{P}(\Theta^*_{l,w} \geq 2l + \lambda) d\lambda \right). \end{aligned}$$

Using that $c_{\delta,\theta} \leq S(x), S_l(w) \leq C_{\delta,\theta}$, and applying (8.5.2), we obtain that the right hand side of the previous display is

$$\leq C_{\delta,\theta} \log(z)^{C_{\delta,\theta}} l^{-1/2}.$$

Since $|\Phi| \leq C_\delta e^{2(l-\tilde{l})} = C_\delta \log(z)^{C_{\delta,\theta}}$,

$$\left| \sum_{w \in \Phi} \int_A \tilde{\Upsilon}^{w,l}(x)dx - \sum_{w \in \Phi} \int_A \Upsilon^{w,l}(x)dx \right| \leq C_{\delta,\theta} |A| \log(z)^{C_{\delta,\theta}} l^{-1/2}.$$

Thus,

$$\left|\frac{\sum_{w\in\Phi}\int_A \tilde{\Upsilon}^{w,l}(x)dx}{\sum_{w\in\Phi}\int_A \Upsilon^{w,l}(x)dx}-1\right| \le \frac{C_{\delta,\theta}|A|\log(z)^{C_{\delta,\theta}}l^{-1/2}}{\sum_{w\in\Phi}\int_A \Upsilon^{w,l}(x)dx}.$$

From (8.5.10) and (8.6.9), we obtain that the denominator in the last expression is greater than $c_{\delta,\theta}|A|$. Therefore, the previous display is

$$\le C_{\delta,\theta}\log(z)^{C_{\delta,\theta}}l^{-1/2} \to 0$$

as $z\to\infty$, as desired, which finishes the proof of Lemma 8.6.3. □

We now prove the last proposition of Chapter 8.

Proof of Proposition 8.1.7. We first define the function ζ_δ in Proposition 8.1.7. Note that, by (8.6.10),

$$e^{-2S(x)}\sum_{w\in\Phi}\tilde{\Upsilon}^{w,l}(x)$$

depends only on δ, l, $\tilde{l}$ and θ. We therefore define

$$\zeta_\delta(x) = e^{2S(x)}/\int_{I_\delta} e^{2S(y)}dy, \tag{8.6.11}$$

where, we recall, $S(x) := \mathbb{E}^x[\log\|x-W_\tau\|]$, as in (8.6.3). Note that ζ_δ converges uniformly on compact sets as $\delta\to 0$. Additionally, we define

$$\alpha_\delta^{l,\tilde{l}} = \sum_{w\in\Phi}\tilde{\Upsilon}^{w,l}(x)/\zeta_\delta(x).$$

Note that $\alpha_\delta^{l,\tilde{l}}$ depends only on δ, l, $\tilde{l}$ and θ (but not on x). Lemmas 8.6.2 and 8.6.3 imply

$$\lim_{z\to\infty}\lim_{r\to\infty}\frac{\mathbb{E}[\Lambda_{r,A}(z)]}{\alpha_\delta^{l,\tilde{l}}ze^{-2z}} = \int_A \zeta_\delta(x)dx \tag{8.6.12}$$

for all closed squares $A\subseteq[\delta,1-\delta]^2$. Therefore, in order to prove Proposition 8.1.7, it is enough to prove

$$\lim_{z_1,z_2\to\infty}\limsup_{r\to\infty}|z_1^{-1}e^{2z_1}\mathbb{E}[\Lambda_{r,I_\delta}(z_1)]-z_2^{-1}e^{2z_2}\mathbb{E}[\Lambda_{r,I_\delta}(z_2)|=0.$$

For $z_1 < z_2$, let $\tilde{l} = e^{z_2^{1/20}}$ and $l - \tilde{l} = \log\log(z_1)$. Note that (8.5.10) and (8.6.12) imply that $\alpha_\delta^{l,\tilde{l}} \geq c_{\delta,\theta}$. Therefore,

$$\begin{aligned}
&\left|z_1^{-1}e^{2z_1}\mathbb{E}[\Lambda_{r,I_\delta}(z_1)] - z_2^{-1}e^{2z_2}\mathbb{E}[\Lambda_{r,I_\delta}(z_2)]\right| \\
&\quad \leq C_{\delta,\theta}\left|\frac{\mathbb{E}[\Lambda_{r,I_\delta}(z_1)]}{\alpha_\delta^{l,\tilde{l}} z_1 e^{-2z_1}} - \frac{\mathbb{E}[\Lambda_{r,I_\delta}(z_2)]}{\alpha_\delta^{l,\tilde{l}} z_2 e^{-2z_2}}\right|.
\end{aligned}$$

An application of (8.6.12) implies that the previous display converges to 0 as first $r \to \infty$ and then $z_1, z_2 \to \infty$, as desired. □

Chapter 9

Relationship with the fine field maxima

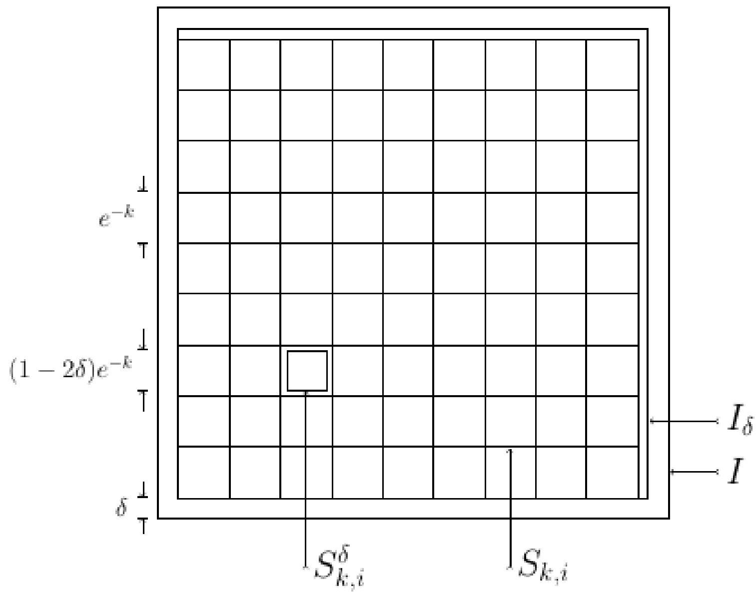
In this chapter, we study of the relationship between the maxima of the fine field and the maximum of the MGFF. We will show that the global maximum of the MGFF can be approximated by the maximum of the MGFF restricted to the set of maxima of the field fields. We defer all proofs to the end of the chapter.

9.1 The coarse and fine fields

We modify here the definition of the fine field used in Chapter 8: the side length of the sub-squares will be a fixed value e^{-k}, with $k > 0$ not depending on r (as opposed to the fine field in Chapter 8, where the sub-squares had side length e^{-r+l}, depending on r).

Let $k > 0$ and consider the square grid consisting of $K := \lfloor (1-2\delta)^2 e^{2k} \rfloor$ adjacent squares of side length e^{-k}, placed inside $I_\delta = [\delta, 1-\delta]^2$ so that the left-bottom corner of the grid coincides with the left-bottom corner of I_δ. Denote the collection of squares in this grid by $\{S_{k,i} : 1 \leq i \leq K\}$. Also, let $S_{k,i}^\delta = \{x \in S_{k,i} : dist(x, \partial S_{k,i}) \geq \delta e^{-k}\}$ and $S_k^\delta = \bigcup_{i \leq K} S_{k,i}^\delta$ (see Figure 9.1 below).

Figure 9.1: $S_{k,i}$ and $S^{\delta}_{k,i}$ squares



For $x \in S^{\delta}_{k,i}$, define the coarse field $\Theta^c_{r,x}$ and fine field $\Theta^f_{r,x}$ by

$$\Theta^c_{r,x} = \mathbb{E}[\Theta_{r,x} \mid \partial S_{k,i}] \tag{9.1.1}$$

and

$$\Theta^f_{r,x} = \Theta_{r,x} - \Theta^c_{r,x}. \tag{9.1.2}$$

Note that both fields depend on k, but we will keep this dependence implicit. Also, let

$$\Theta^{*,\delta}_r := \max_{x \in S^{\delta}_k} \Theta_{r,x}.$$

We will prove the following relationship between Θ^*_r and $\Theta^{*,\delta}_r$. Its proof is a straightforward application of [7, Theorem 3.7] and (7.2.4).

Proposition 9.1.1. *The maximum Θ^*_r of the MGFF satisfies:*

$$\lim_{\delta\to 0} \limsup_{k\to\infty} \limsup_{r\to\infty} \mathbb{P}\left(\Theta^{*,\delta}_r \neq \Theta^*_r\right) = 0 \tag{9.1.3}$$

Because of Proposition 9.1.1, we will study $\Theta_r^{*,\delta}$. For $1 \leq i \leq K$, choose $z(i) = z(i, r, k, \delta)$ so that

$$\max_{x \in S_{k,i}^{\delta}} \Theta_{r,x}^{f} = \Theta_{r,z(i)}^{f}$$

and $\bar{z} = \bar{z}(r, k, \delta)$ so that

$$\max_{1 \leq i \leq K} \Theta_{r,z(i)} = \Theta_{r,\bar{z}}.$$

The main result of this chapter is:

Proposition 9.1.2. *For any fixed $\epsilon > 0$ and small enough $\delta > 0$,*

$$\lim_{k \to \infty} \limsup_{r \to \infty} \mathbb{P}(\Theta_r^{*,\delta} \geq \Theta_{r,\bar{z}} + \epsilon) = 0. \tag{9.1.4}$$

Additionally, there exists a function $g : [0, \infty) \to [0, \infty)$, with $g(k) \to \infty$ as $k \to \infty$, such that

$$\lim_{k \to \infty} \limsup_{r \to \infty} \mathbb{P}(\Theta_{r,\bar{z}}^{f} \leq m_{r-k} + g(k)) = 0. \tag{9.1.5}$$

The proofs of Propositions 9.1.1 and 9.1.2 follow the same outline as the proofs of [4, Proposition 5.1] and [4, Proposition 5.2]. However, in the present setting, we will employ Slepian's Lemma and Fernique's Majorizing Criterion to compare the maxima of continuum and discrete fields, as in the comparisons in the previous chapters (see the proofs of Lemmas 9.3.1, 9.3.2, 9.3.3 and 9.3.4). Proposition 9.1.1 allows us to restrict the index set of the MGFF so I_δ, so that the covariance of the MGFF becomes logarithmic in the distance between points (see Lemma 7.2.1). Proposition 9.1.2 will be instrumental in the proof of Theorem 6.1.1; it allows us to replace the maximum of the MGFF over I_δ by the maximum over the set of points that maximize the fine field.

9.2 Proof of Proposition 9.1.1

Proof of (9.1.3). Let $\epsilon > 0$. From [7, Theorem 3.7], we know that $\{\Theta_r^* - m_r\}_r$ is tight for r large enough. Thus, we can choose $M = M_\epsilon \in (0, \infty)$ such that

$$\limsup_{r \to \infty} \mathbb{P}(\Theta_r^* - m_r \leq -M) \leq \epsilon.$$

Applying the previous display and (7.2.4),

$$\begin{aligned}
&\limsup_{\delta\to 0}\limsup_{k\to\infty}\limsup_{r\to\infty}\mathbb{P}(\max_{x\in I\setminus S_k^\delta}\Theta_{r,x}=\Theta_r^*)\\
&\le \limsup_{\delta\to 0}\limsup_{k\to\infty}\limsup_{r\to\infty}\left(\mathbb{P}(\Theta_r^*-m_r\le -M)+\mathbb{P}(\max_{x\in I\setminus S_k^\delta}\Theta_{r,x}-m_r\ge -M)\right)\\
&\le \epsilon+\limsup_{\delta\to 0} C_\theta e^{2M}\delta^{1/2}=\epsilon.
\end{aligned}$$

Since $\epsilon>0$ is arbitrary, display (9.1.3) follows. □

9.3 Proof of Proposition 9.1.2

Note that

$$\left\{\Theta_r^{*,\delta}\ge\Theta_{r,\bar z}+\epsilon\right\}\subseteq\left\{\Theta_r^{*,\delta}\ge\Theta_{r,z(i^*)}+\epsilon\right\} \tag{9.3.1}$$

where i^* is the index of the square containing $\arg\max_{x\in S_k^\delta}\Theta_{r,x}$. Fix two large constants $C_1,C_2\in(0,\infty)$. The right hand side of the previous display is contained in the union of the following four events:

$$\begin{aligned}
\mathcal{A}_1&=\left\{\Theta_{r,x}<m_r-C_1 \text{ for all } x\in S_k^\delta\right\}\\
\mathcal{A}_2&=\left\{\exists x,y\in S_k^\delta \text{ such that } \|x-y\|\le e^{-r+k} \text{ and } \Theta_{r,x}^c-\Theta_{r,y}^c\ge\epsilon\right\}\\
\mathcal{A}_3&=\Big\{\exists x,y\in S_k^\delta \text{ such that } e^{-r+k}\le\|x-y\|\le e^{-k} \text{ and}\\
&\qquad \Theta_{r,x}\ge m_r-C_1,\Theta_{r,y}\ge m_r-2C_1-C_2\}\\
\mathcal{A}_4&=\left\{\exists i \text{ and } x,y\in S_{k,i}^\delta \text{ such that } \Theta_{r,x}+\Theta_{r,x}^c-\Theta_{r,y}^c\ge m_r+C_2\right\}
\end{aligned}$$

In Lemmas 9.3.1, 9.3.2, 9.3.3 and 9.3.4, we will obtain an upper bound on the probability of the events $\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3$ and $\mathcal{A}_4$, respectively. The proofs of the lemmas will be deferred to the next section.

Lemma 9.3.1. *For any small enough $\delta>0$,*

$$\lim_{C_1\to\infty}\limsup_{k\to\infty}\limsup_{r\to\infty}\mathbb{P}\left(\Theta_r^{*,\delta}<m_r-C_1\right)=0$$

Lemma 9.3.2. *Let $T=\left\{(x,y)\in(S_k^\delta)^2:\|x-y\|\le e^{-r+k}\right\}$. Then, for any fixed $\epsilon>0$,*

$$\lim_{r\to\infty}\mathbb{P}\left(\max_{(x,y)\in T}(\Theta_{r,x}^c-\Theta_{r,y}^c)\ge\epsilon\right)=0.$$

Lemma 9.3.3. *Let* $R = R_{r,k} = \{(x,y) \in (S_k^\delta)^2 : e^{-r+k} \leq \|x-y\| \leq e^{-k}\}$*. Then,*

$$\lim_{k\to\infty} \limsup_{r\to\infty} \mathbb{P}\left(\max_{(x,y)\in R_{n,k}} (\Theta_{r,x} + \Theta_{r,y}) \geq 2m_r - \lambda\right) = 0$$

for any $\lambda \geq 0$*.*

Lemma 9.3.4. *There exists* $C_{\delta,\theta} \in (0,\infty)$ *such that, for all* λ *large enough,*

$$\mathbb{P}\left(\max_{1\leq i\leq K} \max_{x,y\in S_{k,i}^\delta} (\Theta_{r,x} + \Theta_{r,x}^c - \Theta_{r,y}^c) \geq m_r + \lambda\right) \leq C_{\delta,\theta}/\lambda.$$

Using Lemmas 9.3.1, 9.3.2, 9.3.3 and 9.3.4, we prove both (9.1.4) and (9.1.5) of Proposition 9.1.2.

Proof of (9.1.4). As we remarked earlier (in the paragraph following (9.3.1)), for any pair of constants $C_1, C_2 \in (0,\infty)$ the following holds:

$$\mathbb{P}(\Theta_r^{*,\delta} \geq \Theta_{r,\bar{z}} + \epsilon) \leq \mathbb{P}(\mathcal{A}_1) + \mathbb{P}(\mathcal{A}_2) + \mathbb{P}(\mathcal{A}_3) + \mathbb{P}(\mathcal{A}_4)$$

Applying Lemmas 9.3.1, 9.3.2, 9.3.3 and 9.3.4, we obtain

$$\begin{aligned}&\limsup_{k\to\infty} \limsup_{r\to\infty} \mathbb{P}(\Theta_r^{*,\delta} \geq \Theta_{r,\bar{z}} + \epsilon)\\ &\quad\leq \limsup_{C_1,C_2\to\infty} \limsup_{k\to\infty} \limsup_{r\to\infty} (\mathbb{P}(\mathcal{A}_1) + \mathbb{P}(\mathcal{A}_2) + \mathbb{P}(\mathcal{A}_3) + \mathbb{P}(\mathcal{A}_4)) = 0,\end{aligned}$$

as desired. □

We now prove the second claim of Proposition 9.1.2.

Proof of (9.1.5). Let $\epsilon > 0$. We know from Lemma 9.3.1 that

$$\lim_{\lambda\to\infty} \limsup_{k\to\infty} \limsup_{r\to\infty} \mathbb{P}\left(\Theta_r^{*,\delta} < m_r - \lambda\right) = 0.$$

The previous display and (9.1.4) imply

$$\lim_{k\to\infty} \liminf_{r\to\infty} \mathbb{P}\left(\Theta_{r,\bar{z}} \geq m_r - \epsilon \log k\right) = 1.$$

Thus, display (9.1.5) follows from the previous display and

$$\lim_{k\to\infty} \limsup_{r\to\infty} \mathbb{P}\left(\max_{1\leq i\leq K} \Theta_{r,z(i)}^c \geq 2k - \epsilon \log k - g(k)\right) = 0. \tag{9.3.2}$$

Set $g(k) = \gamma \log k$ and $\gamma' = \gamma + \epsilon$, and decompose

$$\mathbb{P}\left(\max_{1\le i\le K} \Theta^c_{r,z(i)} \ge 2k - \gamma' \log k\right) \le \sum_{1\le i\le K} \mathbb{P}\left(\Theta^c_{r,z(i)} \ge 2k - \gamma' \log k\right).$$

Since $z(i)$ is independent of the coarse field $\Theta^c_{r,(\cdot)}$, the variance of $\Theta^c_{r,z(i)}$ is $k + O(1)$ (where the order 1 term depends on δ, θ). Therefore, the previous display is at most

$$C_{\delta,\theta} \exp\left(2k - \frac{(2k-\gamma'\log k)^2}{2k - C_{\delta,\theta}}\right) / \sqrt{k - C_{\delta,\theta}} \le C_{\delta,\theta} k^{2\gamma' - 1/2}.$$

So, any choice of γ' with $\gamma' < 1/4$ is enough to show (9.3.2), which finishes the proof of (9.1.5). $\square$

9.4 Proofs of Lemmas 9.3.1, 9.3.2, 9.3.3 and 9.3.4

Proof of Lemma 9.3.1. From the proofs of [7, Proposition 2.4] and [7, Proposition 2.5], we obtain that, for p, q large enough (depending on δ and θ),

$$\mathbb{P}\left(\max_{x\in V_{r-q}\cap S^\delta_k} \Theta_{r+p,x} < m_{r+p} - \lambda\right) \le \mathbb{P}\left(\max_{x\in V_r \cap e^{-q} S^\delta_k} \xi_x(r) < m_{r+p} - \lambda/2\right)$$

for all $\lambda \ge 0$, where $V_r = I \cap (e^{-r}\mathbb{Z}^2)$. Recall that S^δ_k is the union of $(1-2\delta)^2 e^{2k}$ squares of side length $(1-2\delta)e^{-k}$, so its total area is $(1-2\delta)^4$. Therefore, it is possible to map one-to-one all the points in $V_r \cap e^{-q} S^\delta_k$ to the set $V_r \cap e^{-q}[0,(1-2\delta)^2]^2$, which does not depend on k. Moreover, this can be done such that every pairwise distance between points decreases. Since the CMBRW covariance $Cov(\xi_x(r), \xi_y(r))$ decreases with the Euclidean distance between x and y, a straightforward application of Slepian's Lemma implies

$$\mathbb{P}\left(\max_{x\in V_{r-q}\cap S^\delta_k} \Theta_{r+p,x} < m_{r+p} - \lambda\right) \le \mathbb{P}\left(\max_{x\in V_r \cap e^{-q}[0,1-2\delta]^2} \xi_x(r) < m_{r+p} - \lambda/2\right).$$

Note that the right hand side does not depend on k. Therefore, [7, Proposition 2.6] implies

$$\mathbb{P}(\Theta^{*,\delta}_r < m_r - \lambda) \le C_{\delta,\theta} e^{-c_{\delta,\theta}\lambda} \tag{9.4.1}$$

for $\lambda \ge 0$. This finishes the proof of Lemma 9.3.1. $\square$

Proof of Lemma 9.3.2. Note that, if r is large enough (depending on δ and k), and if $(x, y) \in T$, then x and y belong to the same sub-square $S^{\delta}_{k,i}$. Therefore, it is enough to prove Lemma 9.3.2 when T is replaced by $T_i := \{(x, y) \in (S^{\delta}_{k,i})^2 : \|x - y\| \leq e^{-r+k}\}$.

For a fixed i, let $Z \subset S^{\delta}_{k,i}$ be a collection of $O(e^{2r-4k})$ points such that every point $x \in S^{\delta}_{k,i}$ is within distance e^{-r+k} of some point $z \in Z$. Note that if $(x, y) \in T_i$ and $(\Theta^c_{r,x} - \Theta^c_{r,y}) \geq \epsilon$, then there exists $z \in Z$ within distance $2e^{-r+k}$ of x and y, and such that either $|\Theta^c_{r,x} - \Theta^c_{r,z}| \geq \epsilon/2$ or $|\Theta^c_{r,y} - \Theta^c_{r,z}| \geq \epsilon/2$. Therefore, in order to prove Lemma 9.3.2, it is enough to show

$$\lim_{r\to\infty} \mathbb{P}\left(\max_{z\in Z} \max_{\|x-z\|\leq 2e^{-r+k}} |\Theta^c_{r,x} - \Theta^c_{r,z}| \geq \epsilon/2\right) = 0.$$

We will employ Fernique's Majorizing Criterion and Borell's Inequality to prove the previous display. Let x, x' be within distance $2e^{-r+k}$ of z. Then, from (7.2.6),

$$\mathbb{E}[(\Theta^c_{r,x} - \Theta^c_{r,x'})^2] \leq C_{\delta,\theta} \|x - x'\|^2 e^{2k}.$$

Therefore, for a fixed x', the Lebesgue measure of the set $\{x : \mathbb{E}[(\Theta^c_{r,x} - \Theta^c_{r,x'})^2] \leq \rho^2\}$ is at least $c_{\delta,\theta}\rho^2 e^{-2k}$. On the other hand, the Lebesgue measure of the set $\{x : \|x - z\| \leq 2e^{-r+k}\}$ is at most $4e^{-2r+2k}$. An application of Fernique's Majorizing Criterion implies

$$\mathbb{E}\left[\max_{\|x-z\|\leq 2e^{-r+k}} \Theta^c_{r,x} - \Theta^c_{r,z}\right] \leq C_{\delta,\theta} e^{-r+2k}.$$

Therefore, by Borell's Inequality and (7.2.6),

$$\mathbb{P}\left(\max_{\|x-z\|\leq 2e^{-r+k}} |\Theta^c_{r,x} - \Theta^c_{r,z}| \geq C_{\delta,\theta} e^{-r+2k} + \lambda\right) \leq 2\exp(-c_{\delta,\theta}\lambda^2 e^{2r-4k}).$$

By letting $\lambda = \epsilon/4$ in the previous display, we obtain, for r large enough (depending on ϵ, θ, δ and k),

$$\mathbb{P}\left(\max_{\|x-z\|\leq 2e^{-r+k}} |\Theta^c_{r,x} - \Theta^c_{r,z}| \geq \epsilon/2\right) \leq 2\exp(-c_{\delta,\theta}\epsilon^2 e^{2r-4k}).$$

By a union bound, we obtain

$$\mathbb{P}\left(\max_{z\in Z} \max_{\|x-z\|\leq 2e^{-r+k}} |\Theta^c_{r,x} - \Theta^c_{r,z}| \geq \epsilon/2\right) \leq C\exp(2r - 4k - c_{\delta,\theta}\epsilon^2 e^{2r-4k}) \to 0$$

as $r \to \infty$, as desired. $\square$

Proof of Lemma 9.3.3. Recall that $V_r = I \cap e^{-r}\mathbb{Z}^2$. From the proof of [22, Theorem 1.1], we obtain

$$\lim_{k\to\infty} \limsup_{r\to\infty} \mathbb{P}\left(\max_{(v,w)\in V_r : e^{-r+k}\|v-w\|\le e^{-k}} (\eta_{r,v} + \eta_{r,w}) \ge 2m_r - \lambda \right) = 0 \tag{9.4.2}$$

for any $\lambda \ge 0$. We will use the previous display to prove Lemma 9.3.3 by employing Slepian's Lemma.

Let p, q be large numbers that will be chosen later (see (9.4.3),(9.4.4) and (9.4.5)) and let $\psi^{(1)}$ and $\psi^{(2)}$ be independent copies of a Brownian sheet defined on $[0, e^{-r-q}]^2$ with covariance structure

$$Cov(\psi^{(1)}_x, \psi^{(1)}_y) = (e^{r+q}\min\{x_1, y_1\} + p)(e^{r+q}\min\{x_2, y_2\} + p)$$

for all $x = (x_1, x_2), y = (y_1, y_2) \in [0, e^{-r-q}]^2$. Define Q and Q_0 as the squares with the same center as I and with side length e^{-q-1} and e^{-1} respectively. Let g be the linear map that stretches Q onto Q_0. Also, for $x \in I$, define the point $[x] = ([x]_1, [x]_2)$ to be the closest point to x in V_{r+q} such that $[x]_1 \le x_1$ and $[x]_2 \le x_2$.

We will compare, with index set $U_q := \{(x,y) \in Q^2 : e^{-r+k-q} \le \|x-y\| \le e^{-k-q}\}$, the fields

$$\Theta^d_{r+q,x,y} := \Theta_{r+q,x} + \Theta_{r+q,y}$$

and

$$\eta^d_{r+q,x,y} := a_r(x,y)(\eta_{r+q,g([x])} + \eta_{r+q,g([y])}) + \psi^{(1)}_{x-[x]} + \psi^{(2)}_{y-[y]},$$

where $a_r(x,y) \ge 0$ is defined so that $Var(\Theta^d_{r+q,x,y}) = Var(\eta^d_{r+q,x,y})$ for all $(x,y) \in U_q$. We will show that the maximum of the second field of the previous display stochastically dominates that of the first field. Note that (7.2.1), (7.3.2) and (7.4.3) imply

$$\begin{aligned} &\frac{2(r+q) - 2\log\|x-y\| - 2(p+1)^2 - C_{\delta,\theta}}{2r - 2\log\|[x]-[y]\| + C_{\delta,\theta}} \\ &\qquad \le a_r(x,y)^2 \le \frac{2(r+q) - 2\log\|x-y\| - 2p^2 + C_{\delta,\theta}}{2r - 2\log\|[x]-[y]\| - C_{\delta,\theta}}. \end{aligned}$$

(Note that $[x] \ne [y]$ because $(x,y) \in U_q$.) From the previous display, by choosing

$$p^2 \ge q + C_{\delta,\theta}, \tag{9.4.3}$$

we obtain $a_r(x,y) \le 1$. Moreover, for r large enough (depending on p, q, δ and θ), we obtain $a_r(x,y) \ge 1/2$.

We now compare the covariance structure of the fields $\Theta^d_{r+q,x,y}$ and $\eta^d_{r+q,x,y}$ by distinguishing three cases:

- Case 1: $[x] = [x']$ and $[y] = [y']$. Then, from (7.2.2),

$$\mathbb{E}[(\Theta_{r+q,x} + \Theta_{r+q,y} - \Theta_{r+q,x'} - \Theta_{r+q,y'})^2] \leq C_\theta(\|x - x'\| + \|y - y'\|)e^{r+q}.$$

From (7.4.2), we conclude that this display is at most

$$p(\|x - x'\|_1 + \|y - y'\|_1)e^{r+q} \leq \mathbb{E}[(\psi^{(1)}_{x-[x]} + \psi^{(2)}_{y-[y]} - \psi^{(1)}_{x'-[x']} - \psi^{(2)}_{y-[y']})^2],$$

provided

$$p \geq C_{\delta,\theta}. \tag{9.4.4}$$

Combining the last three displays, we obtain

$$\mathbb{E}[(\Theta^d_{r+q,x,y} - \Theta^d_{r+q,x',y'})^2] \leq \mathbb{E}[(\eta^d_{r+q,x,y} - \eta^d_{r+q,x',y'})^2],$$

as desired.

- Case 2: $[x] \neq [x']$ and $[y] = [y']$. Since $a_r(x, y) \leq 1$ and $dist(Q_0, \partial I) > 0$, displays (7.3.2) and (7.4.3) imply

$$\begin{aligned} &Cov(\eta^d_{r+q,x,y}, \eta^d_{r+q,x',y'}) \\ &\quad \leq -\log(\|[x] - [x']\|\|[x] - [y']\|\|[y] - [x']\|) + r - 2q + 2(p + 1)^2 + C. \end{aligned}$$

On the other hand, from (7.2.1),

$$\begin{aligned} &Cov(\Theta^d_{r+q,x,y}, \Theta^d_{r+q,x',y'}) \\ &\quad \geq -\log(\max\{e^{-r-q}, \|x - x'\|\}\|x - y'\|\|y - x'\|) + r + q - C_{\delta,\theta} \end{aligned}$$

Therefore, the condition

$$3q/2 \geq (p + 1)^2 + C_{\delta,\theta} \tag{9.4.5}$$

implies

$$Cov(\Theta^d_{r+q,x,y}, \Theta^d_{r+q,x',y'}) \geq Cov(\eta^d_{r+q,x,y}, \eta^d_{r+q,x',y'}),$$

as desired.

- Case 3: $[x] \neq [x']$ and $[y] \neq [y']$. Then, using (7.2.1), (7.3.2) and (7.4.3) again, we obtain

$$\begin{aligned} &Cov(\eta^d_{r+q,x,y}, \eta^d_{r+q,x',y'}) \\ &\quad \leq -\log(\|[x]-[x']\|\|[x]-[y']\|\|[y]-[x']\|\|[y]-[y']\|) \\ &\quad - 4q + 2(p+1)^2 + C_\delta \\ &\quad \leq -\log(\|x-x'\|\|x-y'\|\|y-x'\|\|y-y'\|) - C_{\delta,\theta} \\ &\quad \leq Cov(\Theta^d_{r+q,x,y}, \Theta^d_{r+q,x',y'}) \end{aligned}$$

  under the condition $2q \geq (p+1)^2 + C_{\delta,\theta}$ (which is ensured by (9.4.5)).

Conditions (9.4.3), (9.4.4) and (9.4.5) can be simultaneously satisfied by choosing $q = p^2 - C_{\delta,\theta}$ and p large enough so that $p \geq C_{\delta,\theta}$ and $3p^2 - 2(p+1)^2 \geq 5C_{\delta,\theta}$. Cases 1, 2 and 3 imply

$$Cov(\eta^d_{r+q,x,y}, \eta^d_{r+q,x',y'}) \leq Cov(\Theta^d_{r+q,x,y}, \Theta^d_{r+q,x',y'})$$

for all $(x,y),(x',y') \in U_q$. The previous display and an application of Slepian's Lemma imply

$$\mathbb{P}\left(\max_{(x,y)\in U_q} \Theta^d_{r+q,x,y} \geq 2m_{r+q} - \lambda\right) \leq \mathbb{P}\left(\max_{(x,y)\in U_q} \eta^d_{r+q,x,y} \geq 2m_{r+q} - \lambda\right)$$

for all $\lambda \geq 0$. Next, define $\psi^* = \sup_{(x,y)\in U_q}(\psi^{(1)}_{x-[x]} + \psi^{(2)}_{y-[y]})$. An application of Fernique's Majorizing Criterion and Borell's Inequality yields $\mathbb{P}(\psi^* \geq w) \leq C_{\delta,\theta}\exp(-c_{\delta,\theta}w^2)$ for all $w \geq 0$. Therefore, by conditioning on the value of ψ^*, we obtain that the previous display is at most

$$\sum_{w\geq 0} \mathbb{P}\left(\max_{(x,y)\in U_q} a_r(x,y)(\eta_{r+q,g([x])} + \eta_{r+q,g([y])}) \geq 2m_{r+q} - \lambda - w\right) C_{\delta,\theta}e^{-c_{\delta,\theta}w^2}.$$

But, since $1/2 \leq a_r(x,y) \leq 1$, the previous display is at most

$$\sum_{w\geq 0} \mathbb{P}\left(\max_{(x,y)\in U_q} (\eta_{r+q,g([x])} + \eta_{r+q,g([y])}) \geq 2m_{r+q} - 2\lambda - 2w\right) C_{\delta,\theta}e^{-c_{\delta,\theta}w^2}.$$

An application of (9.4.2) and the Lebesgue dominated convergence for infinite sums imply that the previous display converges to 0 as $r \to \infty$ and then $k \to \infty$. Therefore, the last three displays imply

$$\limsup_{k\to\infty} \limsup_{r\to\infty} \mathbb{P}\left(\max_{(x,y)\in U_q} \Theta^d_{r+q,x,y} \geq 2m_{r+q} - \lambda\right) = 0 \tag{9.4.6}$$

for any $\lambda \geq 0$. Let Q_δ be the square with side length $(1-2\delta)e^{-q-1}$ that is concentric with Q. Let h denote the map that stretches Q linearly onto I. Then, from (7.1.2),

$$\begin{aligned}&(\Theta^d_{r+q,x,y} - \mathbb{E}[\Theta^d_{r+q,x,y} \mid \partial Q] : (x,y) \in Q_\delta^2 \text{ with } e^{-r+k-q} \leq \|x-y\| e^{-k-q})\\ &\quad \overset{law}{=} (\Theta_{r-1,h(x)} + \Theta_{r-1,h(y)} : (x,y) \in Q_\delta^2 \text{ with } e^{-r+k-q} \leq \|x-y\| e^{-k-q}).\end{aligned}$$

Let $(\bar{x},\bar{y})$ be the point that maximizes $\Theta^d_{r+q,x,y} - \mathbb{E}[\Theta^d_{r+q,x,y} \mid \partial Q]$ over $\{(x,y) \in Q_\delta^2 \text{ with } e^{-r+k-q} \leq \|x-y\| e^{-k-q})\}$ (which exists by an argument similar to that of the proof of Lemma 7.2.3). Since $(\bar{x},\bar{y})$ is independent of $\mathbb{E}[\Theta^d_{r+q,x,y} \mid \partial Q]$ and $\mathbb{E}[\Theta^d_{r+q,x,y} \mid \partial Q]$ is centered,

$$\begin{aligned}&\mathbb{P}\left(\max_{(x,y)\in U_q} \Theta^d_{r+q,x,y} \geq 2m_{r+q} - \lambda\right)\\ &\quad \geq \mathbb{P}\left(\Theta^d_{r+q,\bar{x},\bar{y}} - \mathbb{E}[\Theta^d_{r+q,\bar{x},\bar{y}} \mid \partial Q] \geq m_{r+q} - \lambda, \mathbb{E}[\Theta^d_{r+q,\bar{x},\bar{y}} \mid \partial Q] \geq 0\right)\\ &\quad = \mathbb{P}\left(\max_{x,y \in I_\delta : e^{-r+k+1} \leq \|x-y\| \leq e^{-k+1}} \Theta_{r-1,x} + \Theta_{r-1,y} \geq 2m_{n+q} - \lambda\right)/2.\end{aligned}$$

Lemma 9.3.3 follows from (9.4.6) and the previous display, by observing that $m_{r+q} - m_{r-1} = O(q)$ (which depends only on δ and θ). □

Proof of Lemma 9.3.4. Define $\Theta^d_{r,x,y} = \Theta_{r,x} + \Theta^c_{r,x} - \Theta^c_{r,y}$ for all $(x,y) \in U_k := \bigcup_i (S^\delta_{k,i})^2$. Note that, for all $(x,y),(x',y') \in U_k$,

$$\begin{aligned}&\mathbb{E}[(\Theta^d_{r,x,y} - \Theta^d_{r,x',y'})^2]\\ &\quad = \mathbb{E}[(\Theta_{r,x} - \Theta_{r,x'})^2] + \mathbb{E}[(\Theta^c_{r,x} - \Theta^c_{r,x'} - \Theta^c_{r,y} + \Theta^c_{r,y'})^2]\\ &\qquad + 2\mathbb{E}[(\Theta^c_{r,x} - \Theta^c_{r,x'})(\Theta^c_{r,x} - \Theta^c_{r,x'} - \Theta^c_{r,y} + \Theta^c_{r,y'})^2].\end{aligned} \tag{9.4.7}$$

We will define a field $(Z^d_{x,y} : (x,y) \in U_k)$ such that

$$\mathbb{E}[(\Theta^d_{r,x,y} - \Theta^d_{r,x',y'})^2] \leq \mathbb{E}[(Z^d_{x,y} - Z^d_{x',y'})^2] \tag{9.4.8}$$

for all $(x,y),(x',y') \in U_k$. Let $p = p_{\delta,\theta} \in (0,\infty)$ be a constant that will be chosen later. For each $S^\delta_{k,i}$, let $h : S^\delta_{k,i} \to [p,p+1]^2$ be the map that stretches $S^\delta_{k,i}$ linearly onto $[p,p+1]^2$. For $j = 1,2$, define two i.i.d. centered Gaussian fields $(Z^j_x : x \in S^\delta_k)$ by

$$Cov(Z^j_x, Z^j_{x'}) = \begin{cases} Cov(\phi_{h(x)}, \phi_{h(x')}) & \text{for } x, x' \text{ in the same } S^\delta_{k,i},\\ 0 & \text{otherwise,}\end{cases}$$

where ϕ is the standard Brownian sheet. We first will show that there exists a constant $p = p_{\delta,\theta} \in (0, \infty)$ such that

$$\begin{aligned} &\mathbb{E}[(\Theta^c_{r,x} - \Theta^c_{r,x'} - \Theta^c_{r,y} + \Theta^c_{r,y'})^2] \\ &+ 2\mathbb{E}[(\Theta^c_{r,x} - \Theta^c_{r,x'})(\Theta^c_{r,x} - \Theta^c_{r,x'} - \Theta^c_{r,y} + \Theta^c_{r,y'})^2] \\ &\quad \le \mathbb{E}[(Z^1_x - Z^1_{x'})^2] + \mathbb{E}[(Z^2_y - Z^2_{y'})^2] \end{aligned} \tag{9.4.9}$$

for all $(x, y), (x', y') \in U_k$. We distinguish two cases:

- Case 1: x and x' belong to the same square $S^\delta_{k,i}$. In this case, by applying (7.2.6), we obtain that the left hand side of (9.4.9) is

  $$\le C_{\delta,\theta} e^{2k}(\|x - x'\|^2 + \|y - y'\|^2).$$

  On the other hand, by the definition of Z^j_x and (7.4.2), the right hand side of (9.4.9) is

  $$\ge pe^k(\|x - x'\|_1 + \|y - y'\|_1) \ge C_{\delta,\theta} e^{2k}(\|x - x'\|^2 + \|y - y'\|^2)$$

  for p large enough (depending on δ and θ).

- Case 2: x and x' belong to different squares $S^\delta_{k,i}$. In this case, since the fine fields are independent for different squares $S^\delta_{k,i}$, we obtain

  $$\begin{aligned} &\mathbb{E}[\Theta^c_{r,x}(\Theta^c_{r,x} - \Theta^c_{r,x'} - \Theta^c_{r,y} + \Theta^c_{r,y'})] \\ &\quad = Cov(\Theta_{r,x}, \Theta_{r,y'} - \Theta_{r,x'}) + Cov(\Theta^c_{r,x}, \Theta^c_{r,x} - \Theta^c_{r,y}) \\ &\quad \le Cov(\Theta_{r,x}, \Theta_{r,y'} - \Theta_{r,x'}) + C_{\delta,\theta}, \end{aligned}$$

  where in the last inequality we have used (8.2.1). But $\|x - x'\|, \|x - y'\| \ge c_\delta e^{-k}$, so (7.2.1) implies that the previous display is

  $$\le C_{\delta,\theta} e^k \|x' - y'\| + C_{\delta,\theta} \le C_{\delta,\theta},$$

  where the last inequality follows because x', y' belong to the same $S^\delta_{k,i}$ square. Repeating the same argument for the rest of the terms in the left hand side of (9.4.9), we obtain that the left hand side of (9.4.9) is bounded above by $C_{\delta,\theta}$. On the other hand, the right hand side of (9.4.9) is at least $4p^2$. Therefore, by choosing p large enough (depending on δ and θ), we obtain (9.4.9).

We now define $Z^d_{x,y} := \Theta_{r,x} + Z^1_x + Z^2_y$ for all $(x,y) \in U_k$. Then, (9.4.7) and (9.4.9) imply (9.4.8), as desired. An application of the Sudakov-Fernique Inequality (see [8, Theorem 2.9]) implies

$$\mathbb{E}\left[\max_{(x,y)\in U_k} Z^d_{x,y}\right] \geq \mathbb{E}\left[\max_{(x,y)\in U_k} (\Theta_{r,x} + \Theta^c_{r,x} - \Theta^c_{r,y})\right]. \tag{9.4.10}$$

Let $S^\delta_{k,i(x)}$ be the square $S^\delta_{k,i}$ containing x, and define

$$\bar{Z}_x := \max_{x',y' \in S^\delta_{k,i(x)}} (Z^1_{x'} + Z^2_{y'})$$

for all $x \in S^\delta_k$. Consider, for $z \geq 1$, the collection of random sets $\Gamma(z) := \{x \in S^\delta_k : \bar{Z}_x \in [z-1, z)\}$. Then,

$$\mathbb{P}\left(\max_{(x,y)\in U_k} Z^d_{x,y} \geq m_r + \lambda\right) \leq \mathbb{P}(\Theta^{*,\delta}_r \geq m_r + \lambda) + \sum_{z\geq 1} \mathbb{P}\left(\max_{x\in\Gamma(z)} \Theta_{r,x} \geq m_r + \lambda - z\right).$$

Applying (7.2.4), we obtain that the sum in the previous display is at most

$$C_{\delta,\theta}\lambda e^{-2\lambda} \sum_{z\geq 1} \mathbb{E}[|\Gamma(z)|]^{1/2} e^{2z}.$$

We claim that there exist constants $c_{\delta,\theta}, C_{\delta,\theta} \in (0,\infty)$ such that, for all $x \in S^\delta_k$,

$$\mathbb{P}(\bar{Z}_x \geq z) \leq C_{\delta,\theta} e^{-c_{\delta,\theta} z^2} \tag{9.4.11}$$

for all $z \geq 0$. Let us assume (9.4.11) momentarily. Then, from the previous two displays, we obtain

$$\mathbb{P}\left(\max_{(x,y)\in U_k} Z^d_{x,y} \geq m_r + \lambda\right) \leq \mathbb{P}(\Theta^{*,\delta}_r \geq m_r + \lambda) + C_{\delta,\theta}\lambda e^{-2\lambda}.$$

Integrating the previous display over $\lambda \geq 0$, and applying (9.4.10), we obtain that

$$\mathbb{E}\left[\max_{(x,y)\in U_k} (\Theta_{r,x} + \Theta^c_{r,x} - \Theta^c_{r-y})\right] \leq m_r + C_{\delta,\theta}.$$

On the other hand, it follows by definition that

$$\max_{(x,y)\in U_k} (\Theta_{r,x} + \Theta^c_{r,x} - \Theta^c_{r,y}) \geq \Theta^{*,\delta}_r$$

a.s.. Furthermore, [7, Theorem 1.1] implies that $\mathbb{E}[(\Theta_r^{*,\delta} - m_r)_+] \leq C_{\delta,\theta}$ and (9.4.1) implies $\mathbb{E}[(\Theta_r^{*,\delta} - m_r)_-] \leq C_{\delta,\theta}$. Therefore, a first moment argument applied to

$$\left| \max_{(x,y)\in U_k} (\Theta_{r,x} + \Theta_{r,x}^c - \Theta_{r,y}^c) - m_r \right|$$

implies Lemma 9.3.4.

We now prove (9.4.11). Let $x, x', y, y' \in S_{k,i(x)}^\delta$. Note that (7.4.2) implies

$$\begin{aligned} \mathbb{E}[(Z_x^1 + Z_y^2 - Z_{x'}^1 - Z_{y'}^2)^2] &= \mathbb{E}[(Z_x^1 - Z_{x'}^1)^2] + \mathbb{E}[(Z_y^2 - Z_{y'}^2)^2] \\ &\leq (p+1)e^k(\|x - x'\| + \|y - y'\|). \end{aligned}$$

An application of Fernique's Majorizing Criterion implies $\mathbb{E}[\bar{Z}_x] \leq C_{\delta,\theta}$. And, since $Var(Z_x^1 + Z_y^2) \leq C_{\delta,\theta}$, an application of Borell's Inequality implies (9.4.11). This finishes the proof of Lemma 9.3.4. $\square$

Chapter 10

Coupling construction

In this chapter we construct a coupling of the coarse field and fine fields based on the results of the preceding chapters. The proofs of the results presented here are deferred until the end of the chapter.

10.1 Definition of the coupling and proof of Theorem 6.1.1

Note that Proposition 8.0.1 produces an asymptotic approximation for the right tail of the MGFF, while Propositions 9.1.1 and 9.1.2 state that the global maximum of the MGFF can be approximated by first taking the maximum of the MGFF among the points maximizing the fine fields.

We will employ the collection of squares $\{S_{k,i} : 1 \leq i \leq K\}$ (and their union S_k^δ), the fine field $\Theta^f_{r,x}$ and coarse field $\Theta^c_{r,x}$, which were all defined at the beginning of Chapter 9. We first find the limiting behavior of the coarse field $\Theta^c_{r,(\cdot)}$ as $r \to \infty$.

Lemma 10.1.1. *Fix a small enough $\delta > 0$ and a large enough $k > 0$. Then, there exists a centered Gaussian field $(Z_k^\delta(x) : x \in S_k^\delta)$ with covariance structure*

$$Cov(Z_k^\delta(x), Z_k^\delta(y)) := h_k^\delta(x, y),$$

with $h_k^\delta : S_k^\delta \times S_k^\delta \to \mathbb{R}$ defined by

$$h_k^\delta(x,y) = \begin{cases} \mathbb{E}^x[\log \|W_\tau - y\|] - \mathbb{E}^x[\log \|W_{\tau_i} - y\|] & \text{for } x, y \text{ in the same } S_{k,i}^\delta, \\ \mathbb{E}^x[\log \|W_\tau - y\|] - \log \|x - y\| & \text{otherwise,} \end{cases}$$

for all $x, y \in S_k^\delta$, where $\tau := \inf\{t \geq 0 : W_t \notin I\}$, $\tau_i := \inf\{t \geq 0 : W_t \notin S_{k,i}\}$ and W_t is a two-dimensional Brownian motion started at x. Moreover,

$$\sup_{x,y \in S_k^\delta} |h_k^\delta(x,y) - Cov(\Theta_{r,x}^c, \Theta_{r,y}^c)| \to_{r\to\infty} 0. \tag{10.1.1}$$

We then proceed to define the coupling construction mentioned before that will be used to prove Theorem 6.1.1. It is immediate that Proposition 8.0.1 can be re-stated as follows.

Proposition 10.1.2. *There exists a constant β_θ (depending only on θ) and a continuous probability density $\zeta : (0,1)^2 \to (0,\infty)$ (not depending on θ) such that, for any closed square $A \subseteq I$,*

$$\lim_{\lambda\to\infty} \limsup_{r\to\infty} \left| \lambda^{-1} e^{2\lambda} \mathbb{P}\left(\max_{x\in A} \Theta_{r,x} - m_r + \beta_\theta \geq \lambda \right) - \int_A \zeta(x)dx \right| = 0.$$

We construct a random variable $G_k^{*,\delta}$ which will be coupled with $\Theta_r^* - m_r + \beta_\theta$. We first need a series of definitions. Set $b_\delta = \int_{[\delta,1-\delta]^2} \zeta(x)dx$ and $\zeta^\delta(x) = \zeta(x)/b_\delta$. Let $z^\delta \in I_\delta$ be a random point with probability density ζ^δ on I_δ. That is,

$$\mathbb{P}(z^\delta \in E) = \int_E \zeta^\delta(x)dx \tag{10.1.2}$$

for all Borel sets $E \subseteq [\delta, 1-\delta]^2$. Let also p_k^δ be an independent Bernoulli random variable with

$$\mathbb{P}(p_k^\delta = 1) = b_\delta g(k) e^{-2g(k)}, \tag{10.1.3}$$

where $g(k) = \gamma \log(k)$ is defined as in the proof of (9.1.5). Additionally, let Y_k be an independent random variable satisfying, for all $\lambda \geq 0$,

$$\mathbb{P}(Y_k \geq \lambda) = \frac{g(k)+\lambda}{g(k)} e^{-2\lambda}. \tag{10.1.4}$$

Using z^δ, p_k^δ and Y_k, we define the following collections of random variables. We let $\{Y_{k,i} : 1 \leq i \leq K\}$ and $\{p_{k,i}^\delta : 1 \leq i \leq K\}$ be i.i.d. copies of Y_k and p_k^δ, respectively. Additionally, we set $\{z_{k,i}^\delta : 1 \leq i \leq K\}$ to be an independent collection of random points, where, for each i, the random point $z_{k,i}^\delta$ has a density which is a scaled version of ζ^δ on $S_{k,i}^\delta$.

Using all the previously defined random variables and points, we define

$$G_k^{*,\delta} = \max_{\{i:p_{k,i}^\delta=1\}} (g(k) + Y_{k,i} + Z_k^\delta(z_{k,i}^\delta) - 2k). \tag{10.1.5}$$

Note that the law of $G_k^{*,\delta}$ does not depend on r or θ.

Proposition 10.1.3. *Let $\mu_{r,\theta}$ denote the law of $\Theta_r^* - m_r + \beta_\theta$ and let ν_k^δ denote the law of $G_k^{*,\delta}$. Then,*

$$\lim_{\delta\to 0} \limsup_{k\to\infty} \limsup_{r\to\infty} d(\mu_{r,\theta}, \nu_k^\delta) = 0,$$

where $d(\cdot,\cdot)$ is the Lévy distance.

Remark 10.1.4. Recall that the Lévy distance between any two probability measures μ, ν on $\mathbb{R}^d$, for $d \geq 1$, is defined as

$$d(\mu,\nu) = \inf\{\delta > 0 : \mu(B) \leq \nu(B^\delta) + \delta \text{ for all open sets } B \subseteq \mathbb{R}^d\},$$

where $B^\delta := \{y : \|x - y\| < \delta \text{ for some } x \in B\}$. For two random variables X, Y, we also denote by $d(X,Y)$ the Lévy distance between the law of X and the law of Y.

Theorem 6.1.1 follows easily from Proposition 10.1.3:

Proof of Theorem 6.1.1, assuming Proposition 10.1.3. Note that, if $\epsilon > 0$, then there exists $k(\epsilon,\theta), \delta(\epsilon,\theta)$ such that, for all $k \geq k(\epsilon,\theta)$ and $0 < \delta \leq \delta(\epsilon,\theta)$,

$$\limsup_{r\to\infty} d(\mu_{r,\theta}, \nu_k^\delta) < \epsilon.$$

Therefore, the sequence $\mu_{r,\theta}$ is Cauchy and it converges to a limit μ_θ^*. Applying Proposition 10.1.3 again, and using that ν_k^δ does not depend on θ, we obtain that the limit $\mu_\theta^* = \mu^*$ does not depend on θ. □

10.2 Proof of Lemma 10.1.1

The existence of the Gaussian field Z_k^δ is a consequence of (10.1.1), because a uniform limit of positive semi-definite functions is a positive semi-definite function. We proceed to prove (10.1.1).

Proof of (10.1.1). Note that, if x and y belong to different $S_{k,i}^{\delta}$, then $Cov(\Theta_{r,x}^c, \Theta_{r,y}^c)$ $= Cov(\Theta_{r,x}, \Theta_{r,y})$. Then, by (6.1.1),

$$\begin{aligned}
&Cov(\Theta_{r,x}, \Theta_{r,y}) \\
&\quad = \iint \mathbb{E}^{x+u}[\log \|W_\tau - y - v\|] - \log \|x + u - y - v\| \, \theta_r(u)\theta_r(v)dudv \\
&\quad = \mathbb{E}^{x}[\log \|W_\tau - y\|] - \log \|x - y\| + O(e^{-r}),
\end{aligned}$$

where, in the previous display, the $O(e^{-r})$ term depends on δ and θ. Similarly, if x and y belong to the same $S_{k,i}^{\delta}$, then

$$\begin{aligned}
&Cov(\Theta_{r,x}^c, \Theta_{r,y}^c) = Cov(\Theta_{r,x}, \Theta_{r,y}) - Cov(\Theta_{r,x}^f, \Theta_{r,x}^f) \\
&= \iint \mathbb{E}^{x+u}[\log \|W_\tau - y - v\|] - \mathbb{E}^{x+u}[\log \|W_{\tau_i} - y - v\|] \, \theta_r(u)\theta_r(v)dudv \\
&= \mathbb{E}^{x}[\log \|W_\tau - y\|] - \mathbb{E}^{x}[\log \|W_{\tau_i} - y\|] + O(e^{-r}),
\end{aligned}$$

where the $O(e^{-r})$ term depends on δ and θ, and the last equality follows from (8.6.5) and (8.6.6). Display (10.1.1) follows from the last two displays. □

10.3 Proof of Proposition 10.1.3

We first prove the following three lemmas. Recall from (10.1.2), (10.1.3) and (10.1.4) the definitions of z^{δ}, p_k^{δ} and Y_k, respectively. The proof of the following lemma is almost verbatim that of [4, Lemma 6.2].

Lemma 10.3.1. *For $j \geq 0$, there exist numbers $\alpha_{r,k}^{\delta}(j) = \alpha(j, r, k, \delta, \theta)$ satisfying*

$$\mathbb{P}(p_k^{\delta} = 1)\mathbb{P}(Y_k \geq j) = \mathbb{P}\left(\max_{x \in I_\delta} \Theta_{r-k,x} - m_{r-k} + \beta_\theta \geq \alpha_{r,k}^{\delta}(j)\right).$$

Furthermore,

$$\sup_{j \geq 0} \limsup_{r \to \infty} |\alpha_{r,k}^{\delta}(j) - g(k) - j| \to 0 \tag{10.3.1}$$

as $k \to \infty$, where $g(k) = \gamma \log(k)$ is defined as in the proof of (9.1.5).

Proof of Lemma 10.3.1. We first prove that $\max_{x \in I_\delta} \Theta_{r-k,x}$ has a continuous distribution. Note that $Var(\Theta_{r-k,x})$ is continuous in $x \in I_\delta$ and

$$\min_{x \in I_\delta} Var(\Theta_{x,r-k}) > 0.$$

Therefore, the (only) lemma of [25] implies that $\max_{x\in I_\delta}\Theta_{r-k,x}$ has a continuous distribution. Thus, there exist numbers $\alpha^\delta_{r,k}(j)$ such that

$$\begin{aligned}&\mathbb{P}\left(\max_{x\in I_\delta}\Theta_{r-k,x}-m_{r-k}+\beta_\theta\geq\alpha^\delta_{r,k}(j)\right)\\&\qquad=b_\delta(g(k)+j)e^{-2(g(k)+j)}=\mathbb{P}(p^\delta_k=1)\mathbb{P}(Y_k\geq j)\end{aligned}$$

for all $j\geq 0$. Note that $\alpha^\delta_{r,k}(0)\leq\alpha^\delta_{r,k}(1)\leq\ldots$, and $\alpha^\delta_{r,k}(0)\to\infty$ as $k\to\infty$. Therefore, applying Proposition 10.1.2, we obtain

$$\sup_{j\geq 0}\limsup_{r\to\infty}\left|\frac{(g(k)+j)e^{2\alpha^\delta_{r,k}(j)}}{\alpha^\delta_{r,k}(j)e^{2(g(k)+j)}}-1\right|\to 0$$

as $k\to\infty$. By the continuity of log, this implies

$$\sup_{j\geq 0}\limsup_{r\to\infty}\left|\log\left(\frac{g(k)+j}{\alpha^\delta_{r,j}(j)}\right)+2(\alpha^\delta_{r,k}(j)-2g(k)-2j)\right|\to 0$$

as $k\to\infty$. The previous display and basic properties of log imply

$$\sup_{j\geq 0}\limsup_{r\to\infty}\left|\alpha^\delta_{r,k}(j)-g(k)-j\right|\to 0$$

as $k\to\infty$, as desired. □

Note that Proposition 10.1.2 implies the existence of a constant $C\in(0,\infty)$ such that

$$\limsup_{r\to\infty}\mathbb{P}\left(\max_{x\in I_\delta}\Theta_{r-k,x}-m_{r-k}+\beta_\theta\geq 2k\right)\leq Ce^{-3k}\tag{10.3.2}$$

for all $k\geq 0$. We next use the finite sequence $\alpha^\delta_{r,k}(j), j=1,2,\ldots,2k$, to construct the following coupling. Define $z^{*,\delta}_{r-k}$ to be the point that maximizes $\Theta_{r-k,x}$ over $x\in I_\delta$. The proof of the following lemma is almost verbatim that of [4, Proposition 6.3].

Lemma 10.3.2. *Let $\bar{p}^\delta_{r,k}$ to be the indicator function of the event $\{\max_{x\in I_\delta}\Theta_{r-k,x}-m_{r-k}+\beta_\theta\geq\alpha^\delta_{r,k}(0)\}$. Then, we can construct $(\bar{p}^\delta_{r,k},\max_{x\in I_\delta}\Theta_{r-k,x},z^{*,\delta}_{r-k})$ and $(p^\delta_k,Y_k,z^\delta)$ on the same probability space, such that $\bar{p}^\delta_{r,k}=p^\delta_k$ almost surely, and such that, on the event $\{\max_{x\in I_\delta}\Theta_{r-k,x}-m_{r-k}+\beta_\theta\leq\alpha^\delta_{r,k}(2k)\}$,*

$$p^\delta_k\left|g(k)+Y_k-\max_{x\in I_\delta}\Theta_{r-k}+m_{r-k}-\beta_\theta\right|+\left|z^\delta-z^{*,\delta}_{r-k}\right|\leq\epsilon_{r,k}\tag{10.3.3}$$

almost surely, where $\epsilon_{r,k} > 0$ *are deterministic numbers satisfying*

$$\lim_{k\to\infty} \limsup_{r\to\infty} \epsilon_{r,k} = 0. \tag{10.3.4}$$

Proof of Lemma 10.3.2. Define a piece-wise linear function $L : [\alpha^{\delta}_{r,k}(0), \alpha^{\delta}_{r,k}(2k)] \to [g(k), g(k) + 2k]$, such that, for $j = 0, 1, \ldots, 2k$, $L(\alpha^{\delta}_{r,k}(j)) = g(k) + j$. Lemma 10.3.1 then implies that, on the event $\{\alpha^{\delta}_{r,k}(0) \le \max_{x\in I_\delta} \Theta_{r-k,x} - m_{r-k} + \beta_\theta \le \alpha^{\delta}_{r,k}(2k)\}$,

$$\left| L\left(\max_{x\in I_\delta} \Theta_{r-k,x} - m_{r-k} + \beta_\theta\right) - \max_{x\in I_\delta} \Theta_{r-k,x} + m_{r-k} - \beta_\theta \right| \le \epsilon_{r,k},$$

where $\epsilon_{r,k} > 0$ satisfies (10.3.4). Therefore, it is enough to prove display (10.3.3) with $\max_{x\in I_\delta} \Theta_{r-k,x} - m_{r-k} + \beta_\theta$ replaced by $L(\max_{x\in I_\delta} \Theta_{r-k,x} - m_{r-k} + \beta_\theta)$. Next, define, for each $j = 0, 1, \ldots, 2k - 1$, the probability measures μ^j_g and μ^j_c on $[j, j + 1) \times I_\delta$ by

$$\begin{aligned}\mu^j_g&([j, j + y) \times A)\\ &= \frac{\mathbb{P}(L(\max_{x\in I_\delta} \Theta_{r-k,x} - m_{r-k} + \beta_\theta) - g(k) \in [j, j + y), z^{*,\delta}_{r-k} \in A)}{\mathbb{P}(L(\max_{x\in I_\delta} \Theta_{r-k,x} - m_{r-k} + \beta_\theta) - g(k) \in [j, j + 1))}\end{aligned} \tag{10.3.5}$$

and

$$\mu^j_c([j, j + y) \times A) = \frac{\mathbb{P}(Y_k \in [j, j + y), z^\delta \in A)}{\mathbb{P}(Y_k \in [j, j + 1))}$$

for all $y \in [0, 1)$ and all closed squares $A \subseteq I_\delta$. We claim

$$\lim_{k\to\infty} \limsup_{r\to\infty} \max_{0\le j\le 2k-1} d(\mu^j_g, \mu^j_c) = 0. \tag{10.3.6}$$

Assuming (10.3.6), we can finish the proof of Lemma 10.3.2 as follows. We note that μ^j_c has a positive density that is uniformly bounded below by a positive number not depending on j or k. Therefore, (10.3.6) and [27, Theorem 1.2] imply the existence of a coupling satisfying the analog of (10.3.3) but restricted to the interval $[j, j + 1)$. Lemma 10.3.2 follows by combining the couplings for different j.

We now prove (10.3.6). We first study the denominator of $\mu^j_g([j, j + y) \times A)$ in (10.3.5). By definition of L and Lemma 10.3.1,

$$\mathbb{P}\left(L\left(\max_{x\in I_\delta} \Theta_{r-k,x} - m_{r-k} + \beta_\theta\right) - g(k) \in [j, j + 1)\right) = \mathbb{P}(p^\delta_k = 1)\mathbb{P}(Y_k \in [j, j + 1)). \tag{10.3.7}$$

Setting $y' = y(\alpha^\delta_{r,k}(j+1) - \alpha^\delta_{r,k}(j))$, the numerator of $\mu^j_g([j, j+y) \times A)$ in (10.3.5) is

$$\begin{aligned} &= \mathbb{P}\left(\max_{x\in I_\delta} \Theta_{r-k,x} - m_{r-k} + \beta_\theta \geq \alpha^\delta_{r,k}(j), z^{*,\delta}_{r-k} \in A\right) \\ &\quad - \mathbb{P}\left(\max_{x\in I_\delta} \Theta_{r-k,x} - m_{r-k} + \beta_\theta \geq \alpha^\delta_{r,k}(j) + y', z^{*,\delta}_{r-k} \in A\right). \end{aligned} \tag{10.3.8}$$

We claim

$$\lim_{\lambda\to\infty} \limsup_{r\to\infty} \left| \lambda^{-1} e^{2\lambda} \mathbb{P}\left(\max_{x\in I_\delta} \Theta_{r-k,x} - m_{r-k} + \beta_\theta \geq \lambda, z^{*,\delta}_{r-k} \in A\right) - \int_A \zeta(x)dx \right| = 0. \tag{10.3.9}$$

Assuming (10.3.9), we can prove (10.3.6). We apply the definitions (10.1.3) and (10.1.4) to (10.3.7), and displays (10.3.1) and (10.3.9) to (10.3.8), obtaining

$$\lim_{k\to\infty} \limsup_{r\to\infty} \max_{0\leq j\leq 2k-1} \left| \mu^j_g([j, j+y) \times A) - \frac{1-e^{-2y}}{1-e^{-2}} \int_A \zeta^\delta(x)dx \right| = 0.$$

On the other hand, by applying (10.1.2), (10.1.4) and (10.1.3) to μ^j_c, we obtain from the previous display that (10.3.6) holds.

We now prove (10.3.9). Note that the event $\{\max_{x\in I_\delta} \Theta_{r-k,x} - m_{r-k} + \beta_\theta \geq \lambda, z^{*,\delta}_{r-k} \in A\}$ is contained in the event $\{\max_{x\in A} \Theta_{r-k} - m_{r-k} + \beta_\theta \geq \lambda\}$. Therefore, (10.3.9) follows from

$$\begin{aligned} \lim_{\lambda\to\infty} \limsup_{r\to\infty} \lambda^{-1} e^{2\lambda} \mathbb{P}\Bigg(&\max_{x\in A} \Theta_{r-k,x} - m_{r-k} + \beta_\theta \geq \lambda, \\ &\max_{x\in \bar{A}^c} \Theta_{r-k,x} - m_{r-k} + \beta_\theta \geq \lambda\Bigg) = 0. \end{aligned}$$

The previous display follows easily from the inclusion-exclusion principle and Proposition 10.1.2 applied to the sets A, $\bar{A}^c$ and $A \cup \bar{A}^c = I$. This finishes the proof of Lemma 10.3.2. □

We also need the following lemma, which has its analog in [4, Lemma 6.4]. Recall from Chapter 9 that we denote by $z(i) = z(i, r, k, \delta)$ the point that maximizes $\Theta^f_{r,x}$ for $x \in S^\delta_{k,i}$.

Lemma 10.3.3. *Let $\{z'_i : 1 \leq i \leq K\}$ denote a family of independent points, chosen so that z'_i is measurable with respect to the sigma-algebra generated by $\{\Theta^f_{r,x} : x \in S^\delta_{k,i}\}$ and so that $e^k \|z'_i - z(i)\| \leq \epsilon_{r,k}$, where $\epsilon_{r,k} > 0$ satisfies (10.3.4). Then,*

$$\lim_{k\to\infty} \limsup_{r\to\infty} d\left(\max_{1\leq i\leq K}(\Theta^f_{r,z(i)} + \Theta^c_{r,z(i)}), \max_{1\leq i\leq K}(\Theta^f_{r,z(i)} + \Theta^c_{r,z'_i})\right) = 0. \tag{10.3.10}$$

Proof of Lemma 10.3.3. From Proposition 9.1.2, we conclude that, for any $\epsilon > 0$,

$$\lim_{k\to\infty} \limsup_{r\to\infty} \mathbb{P}\left(\left|\Theta_r^{*,\delta} - \max_{1\le i\le K}(\Theta^f_{r,z(i)} + \Theta^c_{r,z(i)})\right| > \epsilon\right) = 0.$$

It is therefore enough to show (10.3.10) with $\max_{1\le i\le K}(\Theta^f_{r,z(i)} + \Theta^c_{r,z(i)})$ replaced by $\Theta_r^{*,\delta}$. Note that

$$\begin{aligned}\mathbb{P}&\left(\left|\Theta_r^{*,\delta} - \max_{1\le i\le K}(\Theta^f_{r,z(i)} + \Theta^c_{r,z_i'})\right| > \epsilon\right)\\ &\le \mathbb{P}\left(\Theta_r^{*,\delta} - \max_{1\le i\le K}(\Theta^f_{r,z(i)} + \Theta^c_{r,z_i'}) > \epsilon\right) + \mathbb{P}\left(-\Theta_r^{*,\delta} + \max_{1\le i\le K}(\Theta^f_{r,z(i)} + \Theta^c_{r,z_i'}\right) > \epsilon).\end{aligned} \tag{10.3.11}$$

The first term in the right hand side of the previous display is bounded by an argument analogous to that of the proof of Proposition 9.1.2. We now bound the second term. By a first-moment bound, we obtain

$$\mathbb{P}\left(-\Theta_r^{*,\delta} + \max_{1\le i\le K}(\Theta^f_{r,z(i)} + \Theta^c_{r,z_i'}) > \epsilon\right) \le \epsilon^{-1}\mathbb{E}[\zeta^{*,\delta}_{r,k} - \Theta_r^{*,\delta}],$$

where $\zeta^*_{r,k} = \max\{\Theta_{r,x} + \Theta^c_{r,y} - \Theta^c_{r,x} : x \in S_k^\delta, \|x-y\| \le e^{-k}\epsilon_{r,k}\}$. Divide S_k^δ into $O(e^{2k}\epsilon_{r,k}^{-2})$ adjacent sub-squares with side length $e^{-k}\epsilon_{r,k}$. Denote the collection of centers of these sub-squares by $G_{r,k}$, and, for $g \in G_{r,k}$, denote by $\Box^g_{r,k}$ the sub-square centered at g.

For each $g \in G_{r,k}$, define an independent centered Gaussian field on $\Box^g_{r,k}$ as follows. Let $r = \epsilon_{r,k}$ and $h = \frac{1}{2}e^k/(e^r - 1)$. Choose $z^*_{r,k}$ so that

$$(Z^x_{r,k} : x \in \Box^g_{r,k}) \stackrel{law}{=} (\xi_{hx}(r) : x \in \Box^g_{r,k}),$$

where ξ is the CMBRW. From (7.5.1), it follows that the field $Z_{r,k}$ satisfies, for all $x, y \in \Box^g_{r,k}$,

$$Var(Z^x_{r,k}) = \epsilon_{r,k}$$

and

$$ce^k\|x-y\| \le \mathbb{E}[(Z^x_{r,k} - Z^y_{r,k})^2] \le Ce^k\|x-y\|,$$

for some absolute constants $c, C \in (0,\infty)$. Let $p > 0$ be a large constant that will be chosen later. We consider two independent copies of the field $pZ^x_{r,k}$ for $x \in \Box^g_{r,k}$, and denote them by $(Z^{(1),x}_{r,k} : x \in \Box^g_{r,k})$ and $(Z^{(2),x}_{r,k} : x \in \Box^g_{r,k})$.

We now follow the same line of reasoning as in the proof of Lemma 9.3.4. Let $T = \{(x,y) : x \in S_k^\delta, \|x-y\| \leq e^{-k}\epsilon_{r,k}\}$, and let, for all $(x,y) \in T$,

$$\zeta_{r,k}^{x,y} := \Theta_{r,x} + \Theta_{r,y}^c - \Theta_{r,x}^c \text{ and } \psi_{r,k}^{x,y} := \Theta_{r,x} + Z_{r,k}^{(2),y} + Z_{r,k}^{(1),x}$$

for some $p > 0$ large enough that will be chosen later. We will show

$$\mathbb{E}[(\zeta_{r,k}^{x,y} - \zeta_{r,k}^{x',y'})^2] \leq \mathbb{E}[(\psi_{r,k}^{x,y} - \psi_{r,k}^{x',y'})^2]$$

for all $(x,y),(x',y') \in T$. By expanding the squares in the previous inequality, we obtain that it is equivalent to the following inequality:

$$\begin{aligned} &\mathbb{E}[(\Theta_{r,x}^c - \Theta_{r,x'}^c - \Theta_{r,y}^c + \Theta_{r,y'}^c)^2] \\ &+ 2\mathbb{E}[(\Theta_{r,x}^c - \Theta_{r,x'}^c)(\Theta_{r,x}^c - \Theta_{r,x'}^c - \Theta_{r,y}^c + \Theta_{r,y'}^c)] \\ &\quad \leq \mathbb{E}[(Z_{r,k}^{(1),x} - Z_{r,k}^{(1),x'})^2] + \mathbb{E}[(Z_{r,k}^{(2),y} - Z_{r,k}^{(2),y'})^2]. \end{aligned} \tag{10.3.12}$$

In order to prove (10.3.12), we distinguish three cases:

- Case 1: x and x' belong to the same sub-square $\square_{r,k}^g$. Then, (7.2.6) implies that the left hand side of (10.3.12) is less than $C_{\delta,\theta}e^{2k}(\|x-x'\|^2 + \|y-y'\|^2)$. On the other hand, $\mathbb{E}[(Z_{r,k}^{(1),x} - Z_{r,k}^{(1),x'})^2] \geq cpe^k\|x-x'\|$. Additionally, $\mathbb{E}[(Z_{r,k}^{(1),y} - Z_{r,k}^{(1),y'})^2]$ is greater than $cpe^k\|y-y'\|$ (if y, y' belong to the same sub-square) or greater than $2p^2\epsilon_{r,k}$ (due to independence if y, y' belong to different sub-squares). In either case, by choosing $p = p_{\delta,\theta}$ large enough, we obtain (10.3.12).

- Case 2: x and x' belong to different sub-squares $\square_{r,k}^g$, but the same square $S_{k,i}^\delta$. In this case, (7.2.5) implies that the left hand side of (10.3.12) is less than $C_{\delta,\theta}e^k(\|x-y\| + \|x'-y'\|) \leq C_{\delta,\theta}\epsilon_{r,k}$. On the other hand, we know that $\mathbb{E}[(Z_{r,k}^{(1),x} - Z_{r,k}^{(1),x'})^2] \geq p^2\epsilon_{r,k}$. Therefore, (10.3.12) is satisfied for $p = p_{\delta,\theta}$ large enough.

- Case 3: x and x' belong to different squares $S_{k,i}^\delta$. Then, from similar reasoning to the proof of (7.2.2), we obtain that the left hand side of (10.3.12) is less than $C_{\delta,\theta}\epsilon_{r,k}$. As in Case 2, by choosing $p = p_{\delta,\theta}$ we obtain (10.3.12).

From (10.3.12) and Slepian's Lemma, we obtain

$$\mathbb{E}\left[\max_{(x,y)\in T} \zeta_{r,k}^{x,y}\right] \leq \mathbb{E}\left[\max_{(x,y)\in T} \psi_{r,k}^{x,y}\right] \tag{10.3.13}$$

The same reasoning that was used to obtain (8.3.10) implies

$$\mathbb{P}\left(\max_{(x,y)\in T}\psi^{x,y}_{r,k}\geq m_r+\lambda\right)\leq \mathbb{P}(\Theta^{*,\delta}_r\geq m_r+\lambda)+C_{\delta,\theta}\max\{\lambda,1\}e^{-2\lambda}\sqrt{\epsilon_{r,k}}$$

for all $\lambda\in\mathbb{R}$. By applying the previous display when $\lambda\geq-\log(\epsilon_{r,k})/8$ and applying (9.4.1) for $\lambda<-\log(\epsilon_{r,k})/8$, we obtain

$$\mathbb{E}\left[\max_{(x,y)\in T}\psi^{x,y}_{r,k}\right]\leq \mathbb{E}[\Theta^{*,\delta}_r]+C_{\delta,\theta}\epsilon^{c_{\delta,\theta}}_{r,k}.$$

Therefore, from (10.3.13), we obtain

$$\mathbb{E}\left[\max_{(x,y)\in T}\zeta^{x,y}_{r,k}\right]-\mathbb{E}[\Theta^{*,\delta}_r]\leq C_{\delta,\theta}\epsilon^{c_{\delta,\theta}}_{r,k}\to 0$$

as $r\to\infty$ and then $k\to\infty$. This finishes the proof of Lemma 10.3.3. $\square$

Using Lemmas 10.3.1, 10.3.2 and 10.3.3, we can now prove Proposition 10.1.3. We follow almost verbatim the proof of [4, Theorem 2.4].

Proof of Proposition 10.1.3. Let $\epsilon>0$. For r and k large, let $z(i)$ be the point that maximizes the fine field $\Theta^f_{r,x}$ over $S^\delta_{k,i}$. Define

$$\bar{\Theta}^*_{r,k}=\max_{\{i:\Theta^f_{r,z(i)}-m_{r-k}+\beta_\theta>g(k)\}}(\Theta^f_{r,z(i)}+\Theta^c_{r,z(i)}),$$

where $g(k)=\gamma\log(k)$ is defined as in (9.1.5). Applying Propositions 9.1.1 and 9.1.2, we obtain that, for small enough $\delta=\delta(\epsilon)>0$,

$$\limsup_{k\to\infty}\limsup_{r\to\infty}\mathbb{P}(\Theta^*_r>\bar{\Theta}^*_{r,k}+\epsilon)\leq\epsilon.$$

Since $\Theta^*_r\geq\bar{\Theta}^*_{r,k}$ almost surely, we obtain

$$\limsup_{k\to\infty}\limsup_{r\to\infty}d(\mu_{r,\theta},\bar{\nu}_{r,k,\theta})\leq\epsilon, \tag{10.3.14}$$

where $\bar{\nu}_{r,k,\theta}$ is the law of $\bar{\Theta}^*_{r,k}-m_r+\beta_\theta$. Next, let

$$\bar{p}^\delta_{r,k,i}:=1_{\{\Theta^f_{r,z(i)}-m_{r-k}+\beta_\theta\geq\alpha^\delta_{r,k}(0)\}},$$

where $\alpha^\delta_{r,k}(0)$ is chosen as in Lemma 10.3.1, and let $\{(p^\delta_{k,i},Y_{k,i},z^\delta_{k,i})\}_{i\leq K}$ be an i.i.d. sequence of random vectors, with each $(p^\delta_{k,i},Y_{k,i},z^\delta_{k,i})$ coupled to $(\bar{p}^\delta_{r,k,i},\Theta^f_{r,z(i)}-m_{r-k}+$

$\beta_\theta, z(i))$ as in Lemma 10.3.2. From (10.3.3), we obtain $\|z(i) - z^\delta_{k,i}\| \le e^{-k}\epsilon_{r,k}$, where $\epsilon_{r,k} > 0$ satisfies (10.3.4). It follows from Lemmas 10.3.2 and 10.3.3 and display (10.3.2) that

$$\limsup_{k\to\infty} \limsup_{r\to\infty} d(\bar{\nu}_{r,k,\theta}, \nu'_{r,k,\theta}) \le \epsilon, \tag{10.3.15}$$

where $\nu'_{r,k,\theta}$ is the law of $\max_{\{i:p^\delta_{k,i}=1\}}(g(k) + Y_{k,i} + \Theta^c_{r,z^\delta_{k,i}} + m_{r-k} - m_r)$. Additionally, from Lemma 10.1.1, we obtain

$$\lim_{r\to\infty} d(\nu'_{r,k,\theta}, \nu^\delta_k) = 0,$$

where ν^δ_k is the law of $G^{*,\delta}_k$, which was defined in (10.1.5). Together with (10.3.14) and (10.3.15), this implies

$$\limsup_{k\to\infty} \limsup_{r\to\infty} d(\mu_{r,\theta}, \nu^\delta_k) \le 2\epsilon$$

for all $\delta = \delta(\epsilon) > 0$ small enough, which finishes the proof of Proposition 10.1.3. □

10.4 Proof of Theorem 6.1.2

We now prove Theorem 6.1.2. Recall the definitions of Z^δ_k in Lemma 10.1.1 and $z^\delta_{k,i}$ in the paragraph before (10.1.5). Let $Z^\delta_{k,i} = Z^\delta_k(z^\delta_{k,i})$. We need the following lemma, which is analogous to [4, Lemma 6.5].

Lemma 10.4.1. *There exists $\gamma > 0$ so that*

$$\lim_{k\to\infty} \mathbb{P}\left(\max_{1\le i\le K} Z^\delta_{k,i} \ge 2k - \gamma \log k\right) = 0.$$

Proof. It follows from Lemma (10.1.1) that, conditionally on the collection $\{z^\delta_{k,i}\}$, the Gaussian random variables $Z^\delta_{k,i}$ have mean zero and variance bounded above by $\sigma^2_{k,\delta} := k + C_\delta$. Therefore,

$$\begin{aligned}\mathbb{P}\left(\max_{1\le i\le K} Z^\delta_{k,i} \ge 2k - \gamma \log k\right) &\le C_\delta e^{2k} \max_{1\le i\le K} \mathbb{P}(Z^\delta_{k,i} \ge 2k - \gamma \log k)\\ &\le C_\delta k^{2\gamma - 1} \to 0\end{aligned}$$

as $k \to \infty$, for $\gamma < 1/2$. □

The proof of Theorem 6.1.2 is almost verbatim that of [4, Proof of Theorem 2.5].

Proof of Theorem 6.1.2. We will construct random variables $\bar{Z}_k^\delta \geq 0$ such that

$$\lim_{\delta\to 0}\limsup_{k\to\infty}\left|\mu^*((-\infty,x]) - \mathbb{E}[\exp(-\bar{Z}_k^\delta e^{-2x})]\right| = 0 \tag{10.4.1}$$

for all $x \in \mathbb{R}$. Theorem 6.1.2 follows from (10.4.1): The expectation term in (10.4.1) is, after a change of variables, the Laplace transform of $\bar{Z}_k^\delta$. By the continuity theorem of the Laplace transform (see [28, Theorem XIII.1.2]), there exists a random variable $Z \geq 0$ such that (6.1.5) holds and Z is the limit in distribution of $\bar{Z}_k^\delta$ as $k \to \infty$ and then $\delta \to 0$. Moreover, it is immediate from (6.1.5) that $Z > 0$ almost surely.

We now prove (10.4.1). Let γ be as in the proof of Lemma 10.4.1 and let $0 < \gamma' < \gamma$. Set $g(k) = \gamma' \log k$, where $g(k)$ is the function used to define $G_k^{*,\delta}$ in (10.1.5).Let $\mathcal{F}^c$ denote the sigma-algebra generated by the random variables $\{Z_{k,i}^\delta\}_i$. Then, for any $x \in \mathbb{R}$,

$$\begin{aligned}\mathbb{P}(G_k^{*,\delta} \leq x) &= \mathbb{E}\left[\mathbb{P}\left(\max_{\{i:p_{k,i}^\delta=1\}}(g(k)+Y_{k,i}+Z_{k,i}^\delta-2k) \leq x \mid \mathcal{F}^c, \{p_{k,i}^\delta\}_i\right)\right] \\ &= \mathbb{E}\left[\prod_{1\leq i\leq K}(1-\mathbb{P}(p_{k,i}^\delta=1)\mathbb{P}\left(Y_{k,i}+g(k) \geq x - \bar{Z}_{k,i}^\delta \mid \mathcal{F}^c)\right)\right], \end{aligned} \tag{10.4.2}$$

where $\bar{Z}_{k,i}^\delta = Z_{k,i}^\delta - 2k$. But, from Lemma 10.4.1 and the choice of γ', there exists a function $h(k) \to \infty$ such that the event $D_k = \{-\max_i \bar{Z}_{k,i}^\delta - g(k) \geq h(k)\}$ satisfies $\mathbb{P}(D_k) \to 1$ as $k \to \infty$. Therefore, from the definitions of $p_{k,i}^\delta$ and $Y_{k,i}$, there exists a large $k_x > 0$ (depending on x) such that, for all $k \geq k_x$, on the event D_k,

$$P_{k,i} := \mathbb{P}(p_{k,i}^\delta = 1)\mathbb{P}(Y_{k,i}+g(k) \geq x - \bar{Z}_{k,i}^\delta \mid \mathcal{F}^c) = b_\delta(x-\bar{Z}_{k,i}^\delta)e^{-2(x-\bar{Z}_{k,i}^\delta)} < \epsilon_{k,x}$$

for all $i = 1, 2, \ldots, K$, where $\epsilon_{k,x} > 0$ satisfies

$$\lim_{k\to\infty}\epsilon_{k,x} = 0.$$

This implies, on the event D_k,

$$\exp(-(1+\epsilon_{k,x})P_{k,i}) \leq 1 - P_{k,i} \leq \exp(-P_{k,i}).$$

Let $\tilde{P}_{k,i} := -\bar{Z}_{k,i}^\delta e^{2\bar{Z}_{k,i}^\delta}$. On the event D_k,

$$(1-\epsilon_{k,x,\delta}) \leq \frac{P_{k,i}}{\tilde{P}_{k,i}e^{-2x}} \leq 1,$$

where $\epsilon_{k,x,\delta} > 0$ satisfies $\lim_{\delta\to 0} \limsup_{k\to\infty} \epsilon_{k,x,\delta} = 0$. The previous two displays imply

$$\exp(-(1+\epsilon_{k,x,\delta})\tilde{P}_{k,i}) \leq 1 - P_{k,i} \leq \exp(-(1-\epsilon_{k,x,\delta})\tilde{P}_{k,i})$$

on the event D_k, and hence

$$\exp(-(1+\epsilon_{k,x,\delta})\tilde{Z}_k^\delta e^{-2x}) \leq \prod_{i=1}^{K}(1-P_{k,i}) \leq \exp(-(1-\epsilon_{k,x,\delta})\tilde{Z}_k^\delta e^{-2x}), \tag{10.4.3}$$

where

$$\tilde{Z}_k^\delta := \sum_{i=1}^{K} \tilde{P}_{k,i} = -\sum_{i=1}^{K} \bar{Z}_{k,i}^\delta e^{2\bar{Z}_{k,i}^\delta}.$$

Since $\lim_{\delta\to 0} \limsup_{k\to\infty} \epsilon_{k,x,\delta} = 0$ and $\lim_{k\to\infty} \mathbb{P}(D_k) \to 1$, display (10.4.3) implies

$$\lim_{\delta\to 0} \limsup_{k\to\infty} \left| \prod_{i=1}^{K}(1-P_{k,i}) - \exp(-\tilde{Z}_k^\delta e^{-2x}) \right| \to 0 \tag{10.4.4}$$

in probability. Moreover, $\tilde{Z}_k^\delta \geq 0$ on the event D_k, and so (10.4.4) still holds if we replace $\tilde{Z}_k^\delta$ with $\bar{Z}_k^\delta := \max(\tilde{Z}_k^\delta, 0)$. Additionally, since $\exp(-\bar{Z}_k^\delta e^{-2x}) \leq 1$ almost surely, we obtain from (10.4.4)

$$\lim_{\delta\to 0} \limsup_{k\to\infty} \left| \mathbb{P}(G_k^{*,\delta} \leq x) - \mathbb{E}[\exp(-\bar{Z}_k^\delta e^{-2x})] \right| = 0.$$

Display (10.4.1) follows from this and Proposition 10.1.3. □

Remark 10.4.2. The random variable Z in Theorem 6.1.2 was defined as the limit in distribution of the random variables

$$\bar{Z}_k^\delta = \max\left(-\sum_{i=1}^{K} \bar{Z}_{k,i}^\delta e^{2\bar{Z}_{k,i}^\delta}, 0\right),$$

where $\bar{Z}_{k,i}^\delta = Z_{k,i}^\delta - 2k$. It is therefore possible to compute the distribution of $\bar{Z}_k^\delta$ from the distributions of $Z_{k,i}^\delta$. In order to compute the distribution of $Z_{k,i}^\delta$, recall that

$$Z_{k,i}^\delta = Z_k^\delta(z_{k,i}^\delta),$$

where Z_k^δ was defined in Lemma 10.1.1 and $z_{k,i}^\delta$ was defined in the paragraph before (10.1.5).

Since the field $(Z_k^\delta(x) : x \in S_k^\delta)$ is independent of the random variables $\{z_{k,i}^\delta : 1 \leq i \leq K\}$, the random variables $\{Z_{k,i}^\delta : 1 \leq i \leq K\}$ are jointly Gaussian conditional on the sigma-algebra $\mathcal{G}$ generated by $\{z_{k,i}^\delta : 1 \leq i \leq K\}$. Moreover, their conditional covariance is given by

$$\mathbb{E}[Z_{k,i}^\delta Z_{k,j}^\delta \mid \mathcal{G}] = h_k^\delta(z_{k,i}^\delta, z_{k,j}^\delta),$$

for all $1 \leq i, j \leq K$, where h_k^δ was explicitly defined in Lemma 10.1.1. The (unconditional) covariance between $Z_{k,i}^\delta$ and $Z_{k,j}^\delta$ can therefore be computed explicitly by integrating the right hand side of the previous display and using that the distribution of the (independent) random points $z_{k,i}^\delta$ and $z_{k,j}^\delta$ are known explicitly. (Recall that these points are scaled versions of z^δ, whose distribution was defined in (10.1.2) using the function ζ in (8.0.1).)

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