Introduction

Gerechte designs are mathematical generalizations of the popular Sudoku number puzzle and extensions of Latin squares.

Definition. A gerechte skeleton is an n × n square grid partitioned into n not necessarily congruent regions each containing n cells.

For example, the skeleton of a Sudoku puzzle is a 9 × 9 grid divided into nine 3 × 3 squares.

Definition. A gerechte design is a gerechte skeleton filled with the symbols 1 · · · n such that every row, column, and region contains each symbol exactly once.

This is a gerechte design which satisfies two different skeletons. One skeleton is designated by color and one by the heavy borders. This is the unique design satisfying this exact set of skeletons, up to relabeling symmetries.

Type the symmetry groups of completed Sudoku puzzles have been well studied [3] for the purpose of counting the total number of distinct Sudoku boards.

Gerechte designs are useful in agricultural experiments, where they help control for irregularities in experimental fields.

Their symmetry groups and groupoids of gerechte designs are used to randomise experimental designs and prevent systematic error. [1]

We present a full characterization of the multiple gerechte properties of 4 × 4 Latin squares and their symmetry groups. We also endeavor to quantify the amount of variability within each set of designs by introducing a measure of cell correlation.

Classification of Multiple Gerechte Designs

There are 24 distinct 4 × 4 Latin squares up to relabeling. Each of these Latin squares satisfies the gerechte property of at least one skeleton with congruent regions.

We classified the 4 × 4 Latin squares based on which of these skeletons they satisfy. We then investigated the position symmetries which act on these squares while preserving their classification. Most, but not all, act indistinguishably from relabeling symmetries.

Pairwise Dependency

Given a set S of Latin squares, let p(S)ab be the probability that cells a and b of a randomly selected design in S contain the same symbol.

For the set of all possible grids containing symbols in 1, ..., n, this probability is 1/n, since the assignment of symbols to cells is completely independent.

Definition. The pairwise dependency P of a set of designs which satisfy the same skeleton G is the sum of the squared deviations of pab for all pairs of cells in G:

\[ P = \sum_{a,b \in G} \left( \frac{1}{n} - \frac{1}{n(n-1)} (np_{ab} - 1)^2 \right) \]

Pairwise dependency measures the amount of variability between designs in a given set.

It is maximal on a set of cardinality 1 because the symbol in every cell is uniquely determined.

It typically decreases with set size and inclusion, but there are exceptions.

Different sets of the same cardinality with isomorphic symmetry groups may not have the same pairwise dependency.

A set of a single 4 × 4 Latin square has pairwise dependency 38.

Symmetry Group Structure

Definition. The symmetry group of a set S of gerechte designs is the set of all invertible permutations of cells or symbols that send one design in S to another design in S. The group operation of the symmetry group is function composition.

A set of a single 4 × 4 Latin square has a symmetry group of order 242. This group is generated by transpositions of cells in the first row and transpositions of cells containing the same symbol.

The sets of designs which satisfy the following sets of skeletons have symmetry groups isomorphic to D3 × S2 × S2 × S4.

This structure arises from the natural symmetries of the square grid, and certain transpositions of columns or other groups of cells that preserve the gerechte designs up to relabeling.

Symmetry Groups of Multiple Gerechte Designs

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Symmetry Group Structure (Continued)

Proposition 1. All elements of the symmetry group of a set S of gerechte designs preserve all values of p(S)ab on the grid.

Every element of the symmetry group is invertible and acts bijectively on S.

The image and preimage of any symmetry acting on S are identical.

P(S) is equal to the image and preimage of any symmetry.

An important corollary of this fact is that if a particular set has a unique value of pab for some cells a, b in the grid, then all position symmetries in the symmetry group of that set must fix both a and b.

Proposition 2. The set of gerechte designs satisfying the skeletons of Set 3 has no transitive symmetry group.

These skeletons have a single pair of cells (a, b) such that p(S)ab = 0.8. By the corollary, all elements of the symmetry group must fix both a and b.

The symbols in cells a and b do not match in only one design, so this design must be fixed by all elements of the symmetry group.

References


[2] Bailey, R. A., J. Kunert, and R. J. Martin. "Some comments on gerechte designs. II; randomization analysis, and other methods that cause it has 5 elements; is not a subset of any other set of designs which share skeletons, and unlike other sets it has no transitive symmetry group.