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Ezio Marchi* and Ruth Martinez†

Abstract: The quasi-polymatrix which are almost polymatrix games in n-person non-cooperative theory are studied in this paper. The real computation of all the equilibria are obtained by suitable convex combination of the extreme points of a problem lineal convex associate.

Keywords: Non-cooperative theory, quasi-polymatrix game, equilibria point, mixed strategy, convex polyhedron.

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1 Introduction

In this paper we study the structure of all the equilibrium points of the game quasi-polymatrix, which are obtained by considering an appropriate problem of convex polyhedrons one gets in a natural way as the one presented by Tucker for other games. From a mathematical point of view, the characterization of all the equilibria in a quasi-polymatrix game, given in this paper, it is similar to the characterization of all the equilibria in polymatrix games given by Quintas [1968] and Jansen 1981].

Yanosvshaya [1968] and Marchi 1968] introduced the polymatrix games. These games involve a class of non-cooperative multiperson games in normal form. The equilibria of them were studied by them as a linear complementary problem. In the same context Howson [1972] has obtained the existence of equilibria using the technique of Lemke-Howson [1964]. Yanosvshaya [1968] also proved that every non degenrate polymatrix game has odd number of equilibrium points. The exact meaning of the non degenerations is very technical.

The presentation of the paper is as follows: in the second section we introduce notations and definitions. In the third section we give a characterization of all the equilibrium strategies for quasi-polymatrix games which coincide with a polymatrix games associated to it. Finally, we present a numerical example.

2 Notations and Definitions

Definition 1 *A quasi-polymatrix game is a n-person non cooperative with a normal form game:* $\Gamma = \{\Sigma_i, A_i, i \in N\}$, *where* N *is the finite set* $\{1, 2, \ldots, n\}$, *with* $n \geq 2$.

$$\Sigma_i = \{\sigma_i^1, \ldots, \sigma_i^{m_i}\}$$

denote the strategy set of player $i \in N$. *If player* i *chooses a pure strategy* $\sigma_i^{r_i} \in \Sigma_i$ *and player* j *choose a pure strategy* $\sigma_j^{r_j} \in \Sigma_j$, *it is possible to assign a partial payoff* $a^{rj}(\sigma_i^{r_i}, \sigma_j^{r_j})$ *such that for any choice of pure strategies* $\{\sigma_i, \ldots, \sigma_n\}$ *by the n players, the payoff to player* i *is given by:*

$$A_i(\sigma_1, \ldots, \sigma_i, \ldots, \sigma_n) = \sum_{j \neq i} a^{ij}(\sigma_i^{r_i}, \sigma_j^{r_j}) + D_i(\sigma_{N-\{i\}}) \tag{1}$$

where $\sigma_{N-\{i\}} = (\sigma_1^{r_1}, \ldots, \sigma_{i-1}^{r_i - 1}, \sigma_{i+1}^{r_i+1}, \ldots, \sigma_n^{r_n})$ *and* D_i *is a suitable function.*

Definition 2 *A mixed strategy for player* i *is a probability distribution* x_i *over the pure strategies. That is, it is a vector:*

$$x_i = (x_i(\sigma_i^1), \ldots, x_i(\sigma_i^{m_i}) = (x_i^1, \ldots, x_i^{m_i})$$

where x_i^S *is the probability of* i *playing this strategy* $\sigma_i^S \in \Sigma_i$. *Let* $\widetilde{\Sigma}_i$ *be the set of mixed strategies for player* i:

$$\widetilde{\Sigma}_i = \{x_i : e_i x_i^T = 1 \quad and \quad x_i \geq 0\}$$

where e_i *is the vector of dimension* m_i *consisting of 1 in each component.*

The inequality $x_i \geq 0$ means inequality between the respective elements of the vectors. Here it is denoted by $(.)^T$ the transposed of the corresponding vector or matrix.

Definition 3 *Let $x = x_1, \ldots, x_n \in \prod_{i \in N}$ be a n-tuple of mixed strategies of de n players. The expected payoff to player i is:*

$$E_i(\mathbf{x}) = \sum_{j \neq i} e^{ij}(x_i, x_j) + H_i(x_{N-\{i\}}) \tag{2}$$

where the e's are the expected values of the a's and H's the expected values of the D's.

Definition 4 *Let $S(x_j)$ be the support of the mixed strategy x_j, that is:*

$$S(x_j) = \{\sigma_j^p \in \Sigma_j : x_j(\sigma_j^p) > 0\}$$

3 Characterization of the all equilibrium points for quasi-polymatrix games

We present the following know result:

Theorem 1 *The point $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)$ is an equilibrium point if and only if*

$$\left\{ \begin{array}{rll} \sum\limits_{j \neq i} e^{ij}(\sigma_i, \bar{x}_j) - \lambda & = 0 & \forall \sigma_i, \sigma_i \in S(\bar{x}_i) \\ \sum\limits_{j \neq i} e^{ij}(\sigma_i, \bar{x}_j) - \lambda & \leq 0 & \forall \sigma_i, \sigma_i \in \Sigma_i - S(\bar{x}_i) \\ \bar{x}_i(\sigma_i) & > 0 & \sigma_i \in S(\bar{x}_i) \\ \sum\limits_{j=1}^{m_i} \bar{x}_i(\sigma_i^j) & = 1 & \end{array} \right. \tag{3}$$

where $\lambda_i = E_i(\bar{x}) - H_i(\bar{x}_{N-\{i\}})$ for $i = 1, \ldots, n$

We will consider a further system of inequalities namely

$$\left\{ \begin{array}{ll} \sum\limits_{j \neq i} e^{ij}(\sigma_i, x_j) - \lambda_i & \leq 0 \\ x_i(\sigma_i) & \geq 0 \\ \sum\limits_{j=1}^{m_i} x_i(\sigma_i^j) & = 1 \end{array} \right. \tag{4}$$

$\forall i \in N$ and $\forall \sigma_i \in \Sigma_i$
Since such a system is linear the set of solutions is a convex polyhedron Q. We have the followin result:

Theorem 2 *Consider the extreme points x^k of the polyhedron Q. Each equilibrium point can be written in the following way*

$$x_\mu = \sum_k \mu_k x^k$$

with $\mu_k \geq 0$ and $\sum_k \mu = 1$. If for some $k : \mu_k > 0$ the the extreme point of the polyhedron x^k is also an equilibrium point.

Proof. The first part is clear. This is concerned with the fact that each equilibrium point it belongs to Q. Now for the second part take a point $x^k = (x_i^k, \dots, x_n^k, \lambda_1^k, \dots, \lambda_n^k)$ and assume that such a point were not an equilibrium point.

Then there exist and $i \in N$ such that:

$$\sum_{j \neq i} e^{ij}(\sigma_i, x_j^k) - \lambda_i^k < 0$$

for some $\sigma_i \in S(x_i^k)$. Furthermore for other $\bar{k}$ it holds true

$$\sum_{j \neq i} e^{ij}(\sigma_i, x_j^{\bar{x}} \leq 0$$

for every $\sigma_i \in \sum_i$. Now adding up over k

$$\sum_k \mu_k \sum_{j \neq i} e^{ij}(\sigma_i, x_j^k) - \sum_k \mu_k \lambda_i^k = \sum_{j \neq i} e^{ij}(\sigma_i, x_\mu) - \lambda_i < 0$$

for some $\sigma_i \in S(x_i^k) \subseteq \bigcup_k S(x_\mu)$ and then x_μ would not be an equilibrium point which is impossible.

With regard to this theorem; we would like to point out that the extreme points of a polyhedron was

Further we have the following result. We remind the reader that a completely mixed game is a game that has all the equilibrium strategies $\bar{x}$ with the property $x_i(\sigma_i) > 0$, $\forall i \in N, \forall \sigma_i$

Theorem 3 *The set of equilibrium points of any polymatrix game completely mixed is convex.*

Proof. Consider two completely mixed equilibria $\bar{x}$ and $\tilde{x}$ then they fulfill

$$\sum_{j \neq i} e^{ij}(\sigma_i, \bar{x}_j) = \lambda_i \quad \forall i \in N, \forall \sigma_i$$

and

$$\sum_{j \neq i} e^{ij}(\sigma_i, \tilde{x}_j) = \tilde{\lambda}_i \quad \forall i \in N, \forall \sigma_i$$

then for any $\mu \in [0, 1]$. From here it follows

$$\sum_{j \neq i} e^{ij}(\sigma_i, \mu\bar{x}_j + (1-\mu)\tilde{x}_j) = \mu\lambda_i + (1-\mu)\tilde{\lambda}_i \quad \forall i \in N, \, \forall \sigma_i$$

which says that point $\mu\bar{x} + (1-\mu)\tilde{x}$ is also a completely mixed equilibrium point.

Corollary 1 *Any completely mixed polymatrix game has a unique equilibrium point.*

Proof. By Theorem 5 and Theorem 9 of Chin, Parthasarathy and Raghavan [1974] the result follows easily. Now, we characterize the extreme points of the polyhedron Q. By theorem 2 we know that there is at least, an extreme point which is an equilibrium point.

Theorem 4 *Consider an extreme point* $\bar{y} = (\bar{x}_1, \ldots, \bar{x}_n, \bar{\lambda})$ *of* Q*,where* $\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_n)$*, then there exists a submatrix* $A^{ij}, j \neq i, i, j \in N$ *such that the square matrix*

$$\begin{pmatrix} 0 & A^{12} & A^{13} & \cdots & A^{1n} & -1^T & 0 & 0 & \cdots & 0 \\ A^{21} & 0 & A^{23} & \cdots & A^{2n} & 0 & -1^T & 0 & \cdots & 0 \\ A^{31} & A^{32} & 0 & \cdots & A^{3n} & 0 & 0 & -1^T & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ A^{n1} & A^{n2} & A^{n3} & \cdots & 0 & 0 & 0 & 0 & \cdots & -1^T \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \tag{5}$$

where the vector $\mathbf{1} = (1, \ldots, 1)$ is a row vector and $\mathbf{0}$ is the matrix with all the entries 0 with the respective dimension. This is a non singular matrix.

Proof. Let $\bar{y} = (\bar{x}_1, \ldots, \bar{x}_n, \bar{\lambda})$ where $\bar{\lambda} = (\lambda_1, \ldots, \lambda_n)$ be an extreme solution of the polyhedron convex Q given by the inequalities (4), then there exists a $(\sigma_i, \ldots, \sigma_n)$, such that $\forall i \in N$

$$\sum_{s=i}^{n} a_{\sigma_i} \bar{x}_s(\sigma_s) = \bar{\lambda}_i \tag{6}$$

where here a_{σ_i} stand for the respective rows in the matrices obtained from the system (4).

We, reorder the rows, in order that (4) is satisfied by

$$\sigma_j = 1, \ldots, s_j \qquad j = 1, \ldots, n$$

These rows are linearly independent. Therefore, we have

$$\sum_{s=1}^{n} a_{\sigma_i} \bar{x}_s(\sigma_s) \leq \bar{\lambda}_i \tag{7}$$

for

$$\sigma_j = s_j + 1, \ldots, m_j \qquad m_j = |\Sigma_j| \qquad j = 1, \ldots, n$$

From $\bar{x}_j(\sigma_j) \geq 0$, for all σ_j and $\sum_{\sigma_i \in \Sigma_i} \bar{x}(\sigma_i) = 1$ then for some $\sigma_i, \bar{x}_i(\sigma_i) > 0$, for all i belonging to N. We, recorder the columns in such a way that

$$\begin{cases} \bar{x}_j(\sigma_j) > 0 & \sigma_j = 1, \ldots, r_j \\ \bar{x}_j(\sigma_j) = 0 & \sigma_j = r_j + 1, \ldots, m_j \\ \sum_{\sigma_j=1}^{r_j} \sum_{j \neq 1} A^{ij}(\sigma_i, \sigma_j) \bar{x}_j(\sigma_j) = \lambda_i & \end{cases} \tag{8}$$

for $\sigma_i = 1, \ldots, r_i, i \in N$ has an unique solution. In this system each component must appear.

Assume the contrary that is, to say there would exist two different solutions which we write as

$$y' = (y'_1, \ldots, y'_n, \lambda') \quad , \quad \lambda' = (\lambda'_1, \ldots, \lambda'_n)$$

and

$$y'' = (y''_1, \ldots, y''_n, \lambda'') \quad , \quad \lambda'' = (\lambda''_1, \ldots, \lambda''_n)$$

Let us define

$$x'_1(\sigma_i) = \bar{x}_i(\sigma_i) + \epsilon(y'_i(\sigma_1) - y''_i(\sigma_i)) \qquad i = 1, \ldots, n$$

and

$$x''_1(\sigma_i) = \bar{x}_i(\sigma_i) + \epsilon(y''_i(\sigma_1) - y'_i(\sigma_i)) \qquad i = 1, \ldots, n$$

with $\epsilon > 0$. From the fact that $x_i(\sigma_i) > 0, \sigma_i = 1, \ldots, r_i$, then there exist and $\epsilon > 0$ and sufficiently small such that $x'_i(\sigma_i) \geq 0$ and $x''_i(\sigma_i) \geq 0, i = 1, \ldots, n$

On the other hand since y' and y'' are solutions of the system (8) we have that

$$\sum_{\sigma_j=1}^{r_j} A_i(\sigma_i, \sigma_j) x'_j(\sigma_j) = \bar{\lambda}_i + \epsilon(\lambda'_i - \lambda''_i) \quad , \quad \sigma_i = 1, \ldots, s_i \quad , \quad i = 1, \ldots, n$$

In a similar form for y''. Moreover

$$\sum_{\sigma_j=1}^{r_j} A_i(\sigma_i, \sigma_j) x'_j(\sigma_j) = \sum_{\sigma_j=1}^{r_j} \bar{x}_j(\sigma_j) + \epsilon \sum_{\sigma_j=1}^{r_j} A_i(\sigma_i, \sigma_j)(y'_i(\sigma_i) - y''_i(\sigma_i)) \quad ,$$
$$\sigma_i = s_1 + 1, \ldots, m_i \quad , \quad i = 1, \ldots, n$$

Similarly for y''. On the other hand, let us choose ϵ so small enough such that

$$\delta_i = \epsilon(\lambda'_i - \lambda''_i) \quad , \quad i = 1, \ldots, n$$

and x'_i and x''_i be solutions of the system (8). Then we have

$$\sum_{i \neq j} e^{ij}(\sigma_i, x'_j) - (\bar{\lambda}_i + \delta_i) \leq 0$$

and

$$\sum_{i \neq j} e^{ij}(\sigma_i, x''_j) - (\bar{\lambda}_i + \delta_i) \leq 0$$

for $\sigma_i = s_i + 1, \ldots, m_i$. Therefore the solutions

$$\alpha = (x'_1, \ldots, x'_n, \bar{\lambda}_1 + \delta_1, \ldots, \bar{\lambda}_n + \delta_n) \beta = (x''_1, \ldots, x''_n, \bar{\lambda}_1 + \delta_1, \ldots, \bar{\lambda}_n + \delta_n)$$

are points satisfying (3). But

$$\bar{y} = \frac{1}{2}(\alpha + \beta)$$

which is impossible since $\bar{y}$ was an extreme solution. Therefore we obtain that the system (8) has only a solution.

We point out that the matrix given in the previous theorem, is indeed the scheme of Tucker mentioned by Howson in [1972]. The only difference is that the first column of the scheme of Tucker in the paper of Howson is deleted and it is considered out of the system.

By the way we would like to point out that we have the general theory of almost polymatrix games in a different form as that presented by Quintas [1989].

Having this strong result, we are able to compute extreme points by using the Cramer's rule with the system (4) and the vector $(0, 0, \ldots, 0, 1)$. Another way is to apply the technique of Marchi and Quintas [1987].

4 Example

Having this strong result, we are able to compute actual extreme points by using Cramer's rule with the system (4) and suitable vectors.

In this section, we considerer a general example of three players.

The system to be solved for the computation of extreme points is given by

$$\begin{pmatrix} 0 & A^{12} & A^{13} & -1^T & 0 & 0 \\ A^{21} & 0 & A^{23} & 0 & -1^T & 0 \\ A^{31} & A^{32} & A^{33} & 0 & 0 & -1^T \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \tag{9}$$

Following the theorem arranging in an adequate way the rows and columns the x_i is solution of a subsystem (9). Now will present a numerical example which is the same as that given by Quintas [1989].Consider the 3-person polymatrix game defined by:

$$A^{ij}(\sigma_i, \sigma_j) = \begin{cases} 1 & \text{if } \sigma_i = \sigma_j \\ 0 & \text{otherwise} \end{cases}$$

$\Sigma_i = \{1, 2\}, i = 1, 2$

The system (9) now becomes

$$\begin{pmatrix} 0 & \mathrm{I} & \mathrm{I} & -1^T & 0 & 0 \\ \mathrm{I} & 0 & \mathrm{I} & 0 & -1^T & 0 \\ \mathrm{I} & \mathrm{I} & 0 & 0 & 0 & -1^T \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

where x $= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, y $= \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, z $= \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, $1 = \begin{pmatrix} 1 & 1 \end{pmatrix}$ and 0 is the matrix with all the entries 0 with the respective dimension.

From here we have the equations

$$\begin{array}{rclcrcl} y_1 + z_1 - \lambda_1 & = & 0 & \qquad & y_2 + z_2 - \lambda_1 & = & 0 \\ x_1 + z_1 - \lambda_2 & = & 0 & & x_2 + z_2 - \lambda_2 & = & 0 \\ x_1 + y_1 - \lambda_3 & = & 0 & & x_2 + y_2 - \lambda_3 & = & 0 \end{array}$$

which give us

$$2 - 2\lambda_1 = 0 \qquad 2 - 2\lambda_2 = 0 \qquad 2 - 2\lambda_3 = 0$$

in consequence $\lambda_1 = \lambda_2 = \lambda_3$ and $x_1 = x_2 = y_1 = y_2 = z_1 = z_2 = \frac{1}{2}$ In a similar way it is possible to see and compute that the strategies $\{(1, 0), (1, 0), (1, 0)\}$ and $\{(0, 1), (0, 1), (0, 1)\}$ are equilibria payoff 2 for every one.

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