

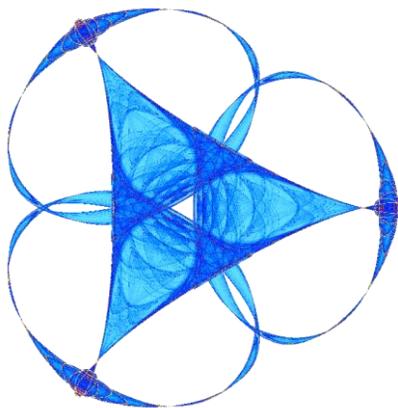
TOWARDS A FORMAL TIE BETWEEN COMBINATORIAL AND CLASSICAL VECTOR  
FIELD DYNAMICS

By

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# TOWARDS A FORMAL TIE BETWEEN COMBINATORIAL AND CLASSICAL VECTOR FIELD DYNAMICS

TOMASZ KACZYNSKI, MARIAN MROZEK, AND THOMAS WANNER

ABSTRACT. Forman's combinatorial vector fields on simplicial complexes are a discrete analogue of classical flows generated by dynamical systems. Over the last decade, many notions from dynamical systems theory have found analogues in this combinatorial setting, such as for example discrete gradient flows and Forman's discrete Morse theory. So far, however, there is no formal tie between the two theories, and it is not immediately clear what the precise relation between the combinatorial and the classical setting is. The goal of the present paper is to establish such a formal tie on the level of the induced dynamics. Following Forman's paper [5], we work with possibly non-gradient combinatorial vector fields on finite simplicial complexes, and construct a flow-like upper semi-continuous acyclic-valued mapping on the underlying topological space whose dynamics is equivalent to the dynamics of Forman's combinatorial vector field on the level of isolated invariant sets and isolating blocks.

## 1. INTRODUCTION

In his seminal work [4, 5], Forman introduced the notion of combinatorial vector fields on finite simplicial complexes. In many aspects, his work was aimed at creating a combinatorial version of concepts from classical dynamical systems theory. For example, Forman's discrete Morse theory [4] is an analogue of classical Morse theory for gradient flows, and his work [5] on combinatorial vector fields and dynamical systems extends the ideas of a combinatorial vector field from the gradient case to the general case. He establishes a number of results which are analogous to Conley's work [2], with the aim of proving the existence of periodic orbits, i.e., recurrent behavior in the combinatorial setting. In particular, he introduces an analogue of the chain recurrent set, proves a combinatorial version

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of Conley's decomposition theorem, as well as discrete analogues of the Morse inequalities. In addition, Forman studies zeta functions on basic or minimal sets.

While Forman's work managed to create combinatorial versions of many aspects of dynamical systems, these results are only informal analogues. To the best of our knowledge, there are no precise ties between combinatorial and classical vector field theory which enable one to go from the combinatorial theory to the classical theory and back, and which therefore facilitate the direct comparison of dynamical objects in either approach. Moreover, Forman does not formally introduce invariant sets, isolated invariant sets, or the concept of Conley index for combinatorial vector fields.

In the present paper we take the first steps towards closing this formal gap. On the one hand, we further complete Forman's combinatorial theory by introducing the notions of invariant set, of isolated invariant set, and of the associated isolating block in the purely discrete setting. This has to be done with extreme care, since in contrast to the classical dynamical systems framework, distinct isolated invariant sets in the combinatorial setting can have the same isolating block. On the other hand, we show that these formal analogues to the classical objects also exhibit equivalent dynamics. More precisely, consider a finite simplicial complex  $\mathcal{X}$  together with a combinatorial vector field  $\mathcal{V}$  in the sense of Forman. Then we define a multivalued discrete dynamical system  $\Phi : \mathbb{Z} \times X \rightrightarrows X$  on the polyhedron  $X$  defined by  $\mathcal{X}$ , together with its multivalued generator  $F : X \rightrightarrows X$  given by  $F(x) = \Phi(1, x)$ , such that the following holds.

- The generator  $F$  is homologous to the identity map on  $X$ , thus making it in principle a multivalued flow.
- There is a correspondence between arbitrary trajectories  $\{x_n\}$  of the discrete dynamical system  $\Phi$ , i.e., of the multivalued map  $F$ , and simplex trajectories  $\{\sigma_n\} \subset \mathcal{X}$  of  $\mathcal{V}$  in the combinatorial framework.
- The isolated invariant sets  $\mathcal{S} \subset \mathcal{X}$  of  $\mathcal{V}$  in the combinatorial setting are in one-to-one correspondence with isolated invariant sets  $S \subset X$  of  $\Phi$ .
- The isolating block associated with an isolated invariant set  $\mathcal{S} \subset \mathcal{X}$  of  $\mathcal{V}$  gives rise to a natural isolating block for the multivalued discrete dynamical system  $\Phi$ .

In other words, the results of the present paper demonstrate that on a dynamical level the combinatorial vector fields of Forman can in fact be viewed as a special case of discrete multivalued dynamical systems. Moreover, our study will further indicate that it should be possible to extend this correspondence to the case of continuous dynamical systems, more precisely, continuous semiflows on the topological space  $X$ .

One of the questions that is not addressed in the current paper, but which is one of the motivations for our work, is the question of reversing the ties between continuous flows and combinatorial vector fields. Given a flow on a finite-dimensional manifold, is it possible to construct a combinatorial vector field on an associated

triangulation or other cell decomposition of the manifold which captures the dynamics of the original flow? For the case of gradient flows, this problem has been studied by experimentalists in such areas as geometric modeling, visualization, and imaging science, yet from a purely numerical point of view. Given a real-valued function on a discrete representation of the manifold, how can one construct a discrete Morse function — or equivalently, a combinatorial Morse vector field — in the sense of Forman which gives rise to the same Morse complex? Algorithms aimed at solving this problem to a certain extent were designed, for example, by King et al. [8] and Robbins et al. [10], and they can be viewed as a compression of information. In contrast, our results are in some sense a decompression of information, as we pass from the combinatorial model to a classical dynamical systems one. While we do not address the inverse problem in the current paper, our hope is that the formal ties established below serve as first steps towards showing that such a study is feasible.

The remainder of this paper is organized as follows. In Section 2 we present our basic notation for simplices and simplicial complexes. Section 3 is devoted to the introduction of combinatorial vector fields on simplicial complexes, their induced discrete flow, as well as the fundamental notion of isolated invariant sets and their associated isolating blocks. Beginning with Section 4 we turn our attention to the case of multivalued maps. We construct a multivalued strongly upper semi-continuous map  $F : X \rightrightarrows X$  which mimics the dynamics of a given combinatorial vector field  $\mathcal{V}$  on a simplicial complex  $\mathcal{X}$ , based on a new cell decomposition of the topological space  $X$  generated by the simplicial complex  $\mathcal{X}$ . Once this has been accomplished, Section 5 shows that orbits of the combinatorial vector field  $\mathcal{V}$  can be associated with orbits of the multivalued map  $F$ , and that isolated invariant sets in the combinatorial setting give rise to corresponding isolating blocks for the multivalued map. Finally, in Section 6 we briefly outline natural continuations of this work.

## 2. PRELIMINARIES

In this section we introduce and review basic notions that will be used throughout the paper. A *partial function* is a triple  $(X, f, Y)$  such that  $X$  and  $Y$  are sets, and  $f \subset X \times Y$  is a relation satisfying the property

$$\text{if } (x, y), (x, y') \in f \text{ then } y = y' .$$

We write  $f : X \dashrightarrow Y$  to indicate that  $(X, f, Y)$  is a partial function. The *domain* of  $f : X \dashrightarrow Y$  is defined as the set

$$\text{dom } f := \{ x \in X \mid \text{there exists } y \in Y \text{ with } (x, y) \in f \} .$$

Given a subset  $A \subset X$ , the *image of  $A$  under  $f$*  is the set

$$f(A) := \{ y \in Y \mid \text{there exists } x \in A \text{ with } (x, y) \in f \} .$$

The *image* of  $f$  is defined as  $\text{im } f := f(X)$ , and the *kernel* of  $f$  is given by  $\ker f := X \setminus \text{dom } f$ . The partial function  $f$  is *injective* if for arbitrary  $x, x' \in \text{dom } f$  the identity

$$f(x) = f(x') \quad \text{implies} \quad x = x' .$$

A partial function  $f : X \dashrightarrow X$  is called *partial self-map*. We call a point  $x \in \text{dom } f$  a *fixed point* of a partial self-map  $f$  if  $f(x) = x$ . The set of all fixed points of a self-map  $f$  is denoted by  $\text{Fix } f$ .

We now turn our attention to simplicial complexes. Recall that the convex hull of  $n + 1$  affinely independent points  $v_0, v_1, \dots, v_n \in \mathbb{R}^d$ , which is abbreviated by the notation  $\langle v_0, v_1, \dots, v_n \rangle$ , is called a *geometric simplex*, or for short *simplex*, of dimension  $n$  in  $\mathbb{R}^d$  spanned by the points  $v_0, v_1, \dots, v_n$ . The points  $v_0, v_1, \dots, v_n$  are called the vertices of  $\langle v_0, v_1, \dots, v_n \rangle$ . Each point  $x \in \langle v_0, v_1, \dots, v_n \rangle$  has a unique representation as the linear combination

$$(1) \quad x = \sum_{i=0}^n t_{v_i}(x) v_i, \quad \text{where} \quad \sum_{i=0}^n t_{v_i}(x) = 1 \quad \text{and} \quad t_{v_i}(x) \geq 0 .$$

The number  $t_{v_i}(x)$  is called the *barycentric coordinate* of  $x$  with respect to the vertex  $v_i$ . Given a simplex  $\sigma = \langle v_0, v_1, \dots, v_n \rangle$ , the associated *cell*  $\overset{\circ}{\sigma}$  consists of those points in  $\sigma$  whose barycentric coordinates are all strictly positive. For each subset  $\{v_{j_0}, v_{j_1}, \dots, v_{j_m}\} \subset \{v_0, v_1, \dots, v_n\}$  the simplex

$$\langle v_{j_0}, v_{j_1}, \dots, v_{j_m} \rangle$$

is called a *face* of  $\langle v_0, v_1, \dots, v_n \rangle$ , and this face is a *proper face* if it is strictly smaller than the simplex itself. An  $(n - 1)$ -dimensional face of an  $n$ -dimensional simplex is called a *facet*. We denote the set of all proper faces of a simplex  $\sigma$  by  $\text{Bd } \sigma$  and call it the *boundary* of the simplex.

We now use simplices as building blocks for associated larger structures. A *simplicial complex* is a finite collection  $\mathcal{X}$  of simplices such that any face of a simplex in  $\mathcal{X}$  is also contained in  $\mathcal{X}$ , and such that the intersection of any two simplices in  $\mathcal{X}$  is either the empty set or a common face. The collection of all  $n$ -dimensional simplices in  $\mathcal{X}$  is denoted by  $\mathcal{X}_n$ . In particular, the set  $\mathcal{X}_0$  consists of all vertices of the simplicial complex  $\mathcal{X}$ . The union of all geometric simplices in  $\mathcal{X}$  is denoted either by  $X$  or by  $|\mathcal{X}|$  and is referred to as the *polyhedron* of  $\mathcal{X}$ .

In combinatorial constructions, it is convenient to think of a simplex  $\sigma \in \mathcal{X}$  as the *abstract simplex*, i.e., as a finite set of vertices  $\{v_0, v_1, \dots, v_n\}$  which is a subset of the set  $\mathcal{X}_0$  of vertices of  $\mathcal{X}$  defined above. Introducing a distinct notation for an abstract simplex would make formulas less readable, so we let the reader deduce from the context whether  $\sigma$  means an abstract or a geometric complex. If we write  $\sigma \subset \mathcal{X}_0$ , we always mean an abstract simplex. Whenever for an arbitrary point  $x \in \mathbb{R}^n$  we write  $x \in \sigma$ , we mean a geometric simplex. If  $v$  is a vertex, writing  $v \in \sigma$  makes the same sense in both cases.

Now let  $\sigma \in \mathcal{X}$  be a simplex in a simplicial complex  $\mathcal{X}$ . Then the *coboundary* of  $\sigma$  is defined as the set of all simplexes in  $\mathcal{X}$  which have  $\sigma$  as a facet. The coboundary is denoted by  $\text{Cbd}_{\mathcal{X}}\sigma$ , or simply  $\text{Cbd}\sigma$  if the underlying simplicial complex  $\mathcal{X}$  is clear from context. For any vertex  $v \in \mathcal{X}_0$  the *star of  $v$*  in  $\mathcal{X}$  is defined as the collection of all simplices in  $\mathcal{X}$  which have  $v$  as a vertex. Finally, there is a well-defined continuous function  $t_v : |\mathcal{X}| \rightarrow [0, 1]$  which assigns to each point  $x \in |\mathcal{X}|$  its barycentric coordinate with respect to the vertex  $v$ , whenever  $x$  belongs to a simplex in the star of  $v$ , and zero otherwise.

Given  $A \subset X = |\mathcal{X}|$ , the topological closure of  $A$  is denoted by  $\text{cl}A$ . The *combinatorial closure*, or simply the *closure* of a subset  $\mathcal{S} \subset \mathcal{X}$  of simplices is denoted by  $\text{Cl}\mathcal{S}$ , and it is defined as the set of faces of all dimensions of simplices in  $\mathcal{S}$ . We say that a collection of simplices  $\mathcal{S}$  is *closed*, if we have  $\text{Cl}\mathcal{S} = \mathcal{S}$ . Equivalently, the combinatorial closure  $\text{Cl}\mathcal{S}$  is the smallest simplicial complex in  $\mathcal{X}$  which contains  $\mathcal{S}$ , and the set  $\mathcal{S}$  is closed if and only if it is a simplicial subcomplex of  $\mathcal{X}$ . For any subset  $\mathcal{S} \subset \mathcal{X}$  we denote the associated subset in the ambient Euclidean space by

$$|\mathcal{S}| = \bigcup_{\sigma \in \mathcal{S}} \overset{\circ}{\sigma} \subset X = |\mathcal{X}|,$$

and usually abbreviate this further as  $S = |\mathcal{S}|$ .

### 3. ISOLATED INVARIANT SETS FOR COMBINATORIAL VECTOR FIELDS

We now turn our attention to combinatorial vector fields on simplicial complexes, which were first introduced by Forman [5]. Since one of our goals is the study of isolated invariant sets in this discrete simplicial setting, Forman's original definition has to be slightly altered, without changing its essential features. We will comment on this change after presenting our definition, as well as the rationale for it.

**Definition 3.1.** *Let  $\mathcal{X}$  denote a simplicial complex. Then an injective partial self-map  $\mathcal{V} : \mathcal{X} \dashrightarrow \mathcal{X}$  is called a combinatorial vector field, or also discrete vector field, on  $\mathcal{X}$  if the following three conditions are satisfied:*

- (i) *For every simplex  $\sigma \in \text{dom}\mathcal{V}$  we either have  $\mathcal{V}(\sigma) = \sigma$ , or  $\sigma$  is a facet of  $\mathcal{V}(\sigma)$ .*
- (ii) *Every simplex in  $\mathcal{X}$  is either in the domain or in the image of  $\mathcal{V}$ , i.e., we have  $\text{dom}\mathcal{V} \cup \text{im}\mathcal{V} = \mathcal{X}$ .*
- (iii) *The only simplices in  $\mathcal{X}$  which are both in the domain and in the image of the combinatorial vector field  $\mathcal{V}$  are fixed points of  $\mathcal{V}$ . In other words, the identity  $\text{dom}\mathcal{V} \cap \text{im}\mathcal{V} = \text{Fix}\mathcal{V}$  holds.*

Combinatorial vector fields on simplicial complexes can easily be understood and visualized using Forman's concept of *arrows*. A pair  $(\tau, \sigma)$  is called an arrow of the discrete vector field  $\mathcal{V}$ , if  $\tau \in \text{dom}\mathcal{V}$  and  $\sigma = \mathcal{V}(\tau) \neq \tau$ . Due to property (i)

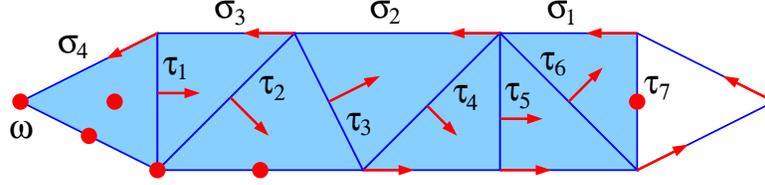


FIGURE 1. A simplicial complex with a sample combinatorial vector field. The arrows of  $\mathcal{V}$  are marked in red, fixed points of  $\mathcal{V}$  are marked by red circles.

above, this implies that the arrow points from an  $n$ -dimensional simplex  $\tau$  in  $\mathcal{X}$  to an  $(n+1)$ -dimensional simplex  $\sigma$  in the coboundary of  $\tau$ . It follows easily from all three of the above properties, that every simplex in  $\mathcal{X}$  is either a fixed point, or it belongs to precisely one arrow of  $\mathcal{V}$ .

The concept of arrows also makes it easy to construct combinatorial vector fields. To see this, let  $\mathcal{X}$  denote a simplicial complex, on which a collection of arrows is given in such a way that every arrow pairs an  $n$ -dimensional simplex in  $\mathcal{X}$  with one of the simplices in its coboundary, and such that every simplex is contained in at most one arrow. If we then define  $\mathcal{V}(\tau) = \sigma$  for every arrow  $(\tau, \sigma)$ , and let  $\mathcal{V}(\omega) = \omega$  for all simplices  $\omega \in \mathcal{X}$  which are not part of any arrow in the collection, then it can easily be seen that (i), (ii), and (iii) are satisfied.

All of these concepts are illustrated in Figure 1. This image shows a two-dimensional simplicial complex in blue, which consists of ten vertices, seventeen edges, and seven triangles. The combinatorial vector field  $\mathcal{V}$  is represented via red arrows, which map  $n$ -dimensional simplices to  $(n+1)$ -dimensional simplices, as well as solid red dots, which correspond to fixed points of  $\mathcal{V}$ . There is a minor difference between Forman's definitions of a combinatorial vector field in [4, 5] and the one given above. The definition given in [4] is somewhat inspired by the definition of a vector field with values in a tangent space, thus  $\mathcal{V}$  may take the value 0 at a simplex which is a rest point of  $\mathcal{V}$ . In [4], the rest points are excluded from the domain of  $\mathcal{V}$ . Thus, any simplex which is not part of an arrow is neither in the domain nor in the image of the map  $\mathcal{V}$ , whereas in our definition it is in both. In other words, rest points in Forman's terminology are just fixed points of  $\mathcal{V}$  in ours, i.e., the red dots in Figure 1.

Our slightly altered point of view will prove to be convenient for discussing orbits in the flow associated with a combinatorial vector field. For this, let  $\mathcal{V} : \mathcal{X} \dashrightarrow \mathcal{X}$  be a discrete vector field. Then the associated *combinatorial multivalued flow* is defined as the multivalued map  $\Pi_{\mathcal{V}} : \mathcal{X} \rightrightarrows \mathcal{X}$  given by

$$(2) \quad \Pi_{\mathcal{V}}(\sigma) := \begin{cases} \text{Cl } \sigma & \text{if } \sigma \in \text{Fix } \mathcal{V}, \\ \text{Cl Bd } \sigma \setminus \{\mathcal{V}^{-1}(\sigma)\} & \text{if } \sigma \in \text{im } \mathcal{V} \setminus \text{Fix}(\mathcal{V}), \\ \{\mathcal{V}(\sigma)\} & \text{if } \sigma \in \text{dom } \mathcal{V} \setminus \text{Fix}(\mathcal{V}). \end{cases}$$

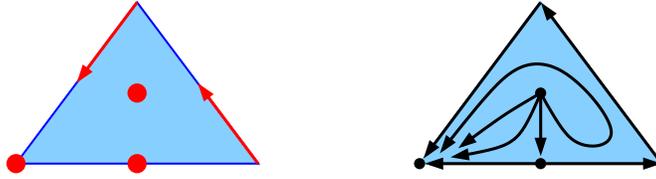


FIGURE 2. The left image shows a simplicial complex with a sample combinatorial vector field. The arrows of  $\mathcal{V}$  are marked in red, fixed points of  $\mathcal{V}$  are marked by red circles. The image on the right depicts a sample time-continuous flow on the underlying set which mimics the dynamics of the combinatorial flow.

A *solution* of the flow  $\Pi_{\mathcal{V}}$ , sometimes also called an *orbit* of the combinatorial vector field  $\mathcal{V}$ , is then a partial function  $\varrho : \mathbb{Z} \dashrightarrow \mathcal{X}$  such that its domain is an interval in  $\mathbb{Z}$ , and such that the inclusion  $\varrho(i+1) \in \Pi_{\mathcal{V}}(\varrho(i))$  holds for all values  $i \in \text{dom } \varrho$  with  $i+1 \in \text{dom } \varrho$ .

Note that this notion of solution generalizes the notion of  $V$ -path in [5]. In the latter case, a solution is a sequence of simplices whose dimensions alternately increase and decrease by one, and there are finite solutions which cannot be continued in forward or in backward time. For the simplicial flow shown in Figure 1, the sequence

$$\tau_1, \mathcal{V}(\tau_1), \tau_2, \mathcal{V}(\tau_2), \dots, \tau_6, \mathcal{V}(\tau_6), \tau_7,$$

is such a maximal finite solution, which consists of seven edges and six triangles. Note that in Forman's situation this orbit cannot be extended in forward time, since the edge  $\tau_7$  is a fixed point. In contrast, in our setting the combinatorial orbit can jump from a fixed point to itself or to its boundary. Thus, it is possible to extend the above orbit through the right vertex of the edge  $\sigma_1$ , as well as the subsequent corner-edge pairs corresponding to  $\sigma_2$ ,  $\sigma_3$ , and  $\sigma_4$ , and finally through an infinite sequence of the repeated vertex  $\omega$ . In other words, in our situation the finite Forman  $V$ -path can be extended to a solution which exists for all positive discrete times. Similarly, this orbit can also be extended in backwards time. Thus, this solution can be extended to a *full solution*  $\varrho : \mathbb{Z} \dashrightarrow \mathcal{X}$ , i.e., a solution with  $\text{dom } \varrho = \mathbb{Z}$ .

At the first glance, the extension of solutions to  $\mathbb{Z}$  seems to be unnecessary, because the simplicial complex  $\mathcal{X}$  is finite anyway. However, it is natural since we are trying to build the bridge to classical Conley theory. In particular allowing  $\mathbb{Z}$  as domain is convenient, because it will allow us to also define dynamical concepts such as heteroclinic connections. Moreover, the rest points in Forman's theory which are  $p$ -dimensional simplexes correspond to stationary points of a flow with a  $p$ -dimensional unstable manifold. In particular, vertices which are rest points correspond to attracting stationary trajectories and top-dimensional simplexes which are rest points correspond to repelling stationary orbits. See also the sample

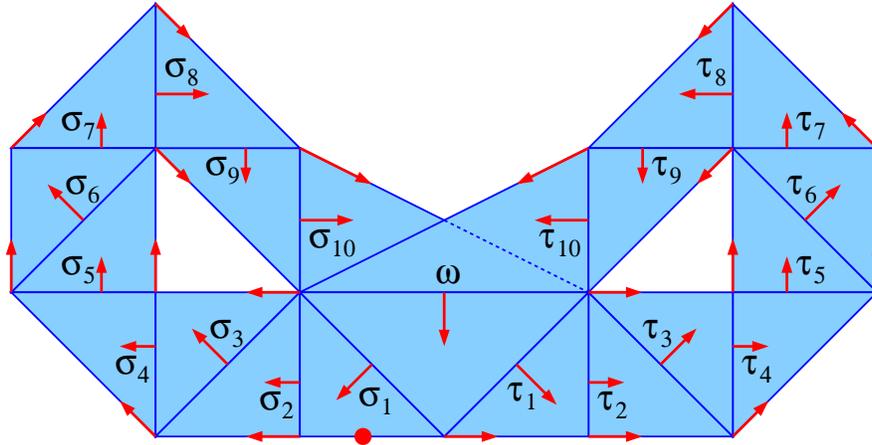


FIGURE 3. A simplicial complex with a sample combinatorial vector field. The arrows of  $\mathcal{V}$  are marked in red, fixed points of  $\mathcal{V}$  are marked by red circles. In this case, the discrete vector field exhibits recurrent, and even chaotic dynamics.

combinatorial vector field in the left image of Figure 2, together with a sample time-continuous flow on the underlying set which mimics the dynamics of the combinatorial flow.

We would also like to point out that using the above definition of the associated flow  $\Pi_{\mathcal{V}}$ , a discrete vector field can exhibit complicated dynamics. This is described in more detail in the following example.

**Example 3.2.** Consider for example the two-dimensional simplicial complex  $\mathcal{X}$  in  $\mathbb{R}^3$  which is sketched in Figure 3, together with a combinatorial vector field  $\mathcal{V}$  shown in red. Despite the planar representation of  $\mathcal{X}$ , notice that the two triangles  $\mathcal{V}(\sigma_{10})$  and  $\mathcal{V}(\tau_{10})$  which have  $\omega$  as their lower edge only intersect in  $\omega$ , i.e., as the dashed blue line indicates, the triangle  $\mathcal{V}(\sigma_{10})$  lies behind  $\mathcal{V}(\tau_{10})$ . One can easily see that both the sequence

$$\omega, \mathcal{V}(\omega), \sigma_1, \mathcal{V}(\sigma_1), \sigma_2, \mathcal{V}(\sigma_2), \dots, \sigma_{10}, \mathcal{V}(\sigma_{10}), \omega$$

and the simplex sequence

$$\omega, \mathcal{V}(\omega), \tau_1, \mathcal{V}(\tau_1), \tau_2, \mathcal{V}(\tau_2), \dots, \tau_{10}, \mathcal{V}(\tau_{10}), \omega$$

are solutions of the flow  $\Pi_{\mathcal{V}}$ . By concatenating these two orbit segments without the trailing  $\omega$ 's, one can easily construct infinitely many different periodic solutions for  $\Pi_{\mathcal{V}}$ . In fact, the shape of the simplicial complex is inspired by the celebrated Lorenz attractor, which of course exhibits chaotic dynamics.  $\diamond$

Using the combinatorial multivalued flow  $\Pi_{\mathcal{V}}$  we now turn our attention to defining the dynamical concepts which lie at the heart of Conley's theory, i.e., we define invariant sets, isolated invariant sets, index pairs, as well as the Conley

index. We begin with the following definition which can be taken almost verbatim from the classical dynamical systems situation.

**Definition 3.3.** *Let  $\mathcal{V}$  denote a combinatorial vector field on a finite simplicial complex  $\mathcal{X}$ . Then a set  $\mathcal{S} \subset \mathcal{X}$  is called invariant for the associated flow  $\Pi_{\mathcal{V}}$ , if for each simplex  $\sigma \in \mathcal{S}$  there exists a full solution  $\varrho : \mathbb{Z} \rightarrow \mathcal{X}$  through  $\sigma$  which lies completely in  $\mathcal{S}$ , i.e., which satisfies both  $\varrho(k) = \sigma$  for some  $k \in \mathbb{Z}$  and  $\varrho(\mathbb{Z}) \subset \mathcal{S}$ .*

Note that in the definition of invariant set, we only require the existence of at least one full solution through every simplex in  $\mathcal{S}$ . In the combinatorial setting, there certainly could be multiple solutions through a simplex in  $\mathcal{S}$ , and some of these certainly could leave the invariant set in forward or backward time. For example, in the situation shown in Figure 2, the set  $\mathcal{S} = \{\omega\}$  is an invariant set, since  $\varrho(k) = \omega$  for all  $k \in \mathbb{Z}$  defines a full solution through  $\sigma$  which lies in  $\mathcal{S}$ . However, one can easily see that there is also a full solution which equals  $\tau_7$  for strictly negative  $k \in \mathbb{Z}$ , then passes through the four edges  $\sigma_1, \dots, \sigma_4$  and the associated four intermediate vertices, and then equals  $\omega$  for  $k \geq 8$ . This latter solution passes through  $\omega$ , but it is not contained in  $\mathcal{S}$  for all times. While at first this merging and splitting of orbits might seem unnatural, it is a necessary consequence of the fact that simplices in  $\mathcal{X}$  of dimension at least one correspond to infinitely many points in the underlying topological space, and therefore possibly to infinitely many different orbits. See also Figure 2.

One of the central insights of Conley was the realization that the study of general invariant sets is ill-suited for a theory which is robust under perturbations. He therefore introduced the stricter notion of isolated invariant set as basic objects of study. In the discrete setting, we propose a new definition of combinatorial isolation, which is motivated by the concept of an isolating block, i.e., it is based on the exclusion of internal tangencies in the theory of flows.

**Definition 3.4.** *Let  $\mathcal{V}$  denote a combinatorial vector field on a finite simplicial complex  $\mathcal{X}$ . Furthermore, let  $\mathcal{S} \subset \mathcal{X}$  denote an invariant set for  $\Pi_{\mathcal{V}}$  in the sense of Definition 3.3, and define the exit set of  $\mathcal{S}$  by*

$$\text{ex } \mathcal{S} := \text{Cl } \mathcal{S} \setminus \mathcal{S}.$$

*Then the invariant set  $\mathcal{S}$  is called an isolated invariant set, if the following two statements hold:*

- (a) *The exit set  $\text{ex } \mathcal{S}$  is closed in the simplicial complex  $\mathcal{X}$ .*
- (b) *There exists no solution  $\varrho : [-1, 1] \cap \mathbb{Z} \rightarrow \mathcal{X}$  of  $\Pi_{\mathcal{V}}$  such that both  $\varrho(-1) \in \mathcal{S}$  and  $\varrho(1) \in \mathcal{S}$  hold, as well as  $\varrho(0) \in \text{ex } \mathcal{S}$ .*

*Note that in (a) we refer to the combinatorial closedness of the set  $\mathcal{S}$ . Finally, the closure  $\text{Cl } \mathcal{S}$  is called an isolating block for the isolated invariant set  $\mathcal{S}$ .*

Note that it does not seem to be possible to define an isolated invariant set for combinatorial flows in complete analogy to the classical dynamical systems definition. In the latter situation, an isolated invariant set is the maximal invariant

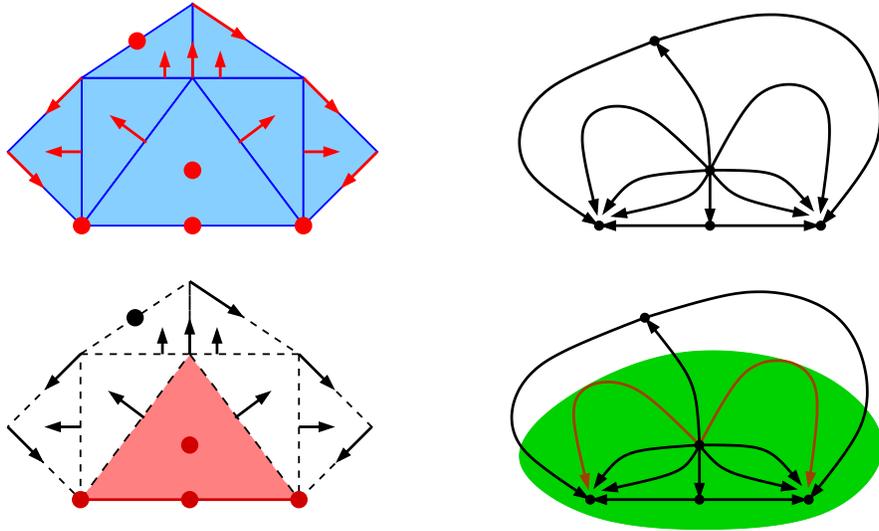


FIGURE 4. The top left image shows a simplicial complex with a sample combinatorial vector field, whose associated continuous flow is shown in the top right image. The lower left indicates an invariant set consisting of two vertices, one edge, and one triangle. This set satisfies part (b) in Definition 3.4, but violates (a). As shown in the lower right image, in the time-continuous case this set should not lead to an isolated invariant set, since there are always internal flow tangencies.

set in a compact isolating neighborhood, which in addition lies in the interior of the isolating neighborhood. In the combinatorial setting, we certainly want to make sure that a simplex which is a fixed point for  $\mathcal{V}$  is an isolated invariant set, such as for example the bottom edge in the left image of Figure 2. However, the smallest closed set in the simplicial complex which contains this edge also contains the lower left corner, which is also a fixed point for  $\mathcal{V}$ . Thus, the maximal invariant set in any closed set containing the bottom edge is actually larger than this one-simplex fixed point. Notice also that the closure of the bottom edge in this example is an isolating block for two different isolated invariant sets.

Since a direct transfer of the notion of isolated invariant set is not possible, Definition 3.4 is based on the notion of an isolating block in continuous-time dynamical systems, which was first introduced in [2, 3]. See also the recent paper [11]. One can easily see that under the above definition, every simplex  $\sigma$  in the simplicial complex  $\mathcal{X}$  which is a fixed point for the combinatorial vector field  $\mathcal{V}$  gives rise to an isolated invariant set  $\mathcal{S} = \{\sigma\}$ . While part (a) in Definition 3.4 is immediate, part (b) follows from the fact that no facet  $\tau$  of  $\sigma$  can satisfy  $\mathcal{V}(\tau) = \sigma$ , and therefore any solution  $\varrho$  of the combinatorial multivalued flow  $\Pi_{\mathcal{V}}$  which satisfies  $\varrho(0) = \tau$  must have  $\varrho(1) \in \text{Bd } \sigma$ , and therefore  $\varrho(1) \notin \mathcal{S}$ .

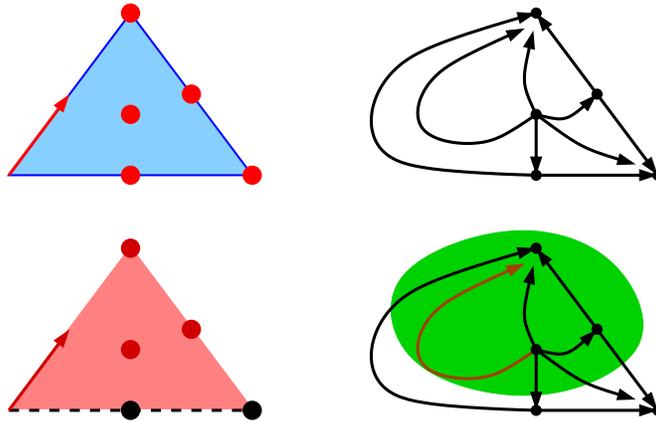


FIGURE 5. The top left image shows a simplicial complex with a sample combinatorial vector field, whose associated continuous flow is shown in the top right image. The lower left indicates an invariant set consisting of one vertex, two edges, and one triangle. This set satisfies part (a) in Definition 3.4, but violates (b). As shown in the lower right image, in the time-continuous case this set should not lead to an isolated invariant set, since there are always internal flow tangencies.

Both parts of Definition 3.4 are necessary if we want to capture the intuitive correspondence between discrete vector fields on a simplicial complex and flows on the underlying topological set. To demonstrate this in the case of condition (a), see the combinatorial vector field shown in the upper left image of Figure 4. In the upper right image we depict the corresponding continuous flow on the underlying topological disk, while in the lower left image an invariant set  $\mathcal{S}$  is indicated in red, which consists of the red fixed point triangle, its bottom edge, as well as its two bottom corners. All of these simplices are fixed points for  $\mathcal{V}$ , which immediately shows invariance. Furthermore, one can readily see that condition (a) is violated, while condition (b) holds. Thus, the invariant set  $\mathcal{S}$  is not isolated in the sense of Definition 3.4. The image in the lower right shows that in fact it should not be considered isolated. The green region in this image depicts a neighborhood which encloses the four equilibrium solutions corresponding to the four simplices in  $\mathcal{S}$ . Due to the nature of the flow, regardless of how this neighborhood is chosen, there will always be orbits which touch the boundary, similar to the two brown orbits in the image. In other words, the largest invariant subset in the green region is not isolated.

Similarly, Figure 5 contains an example which demonstrates the necessity of part (b). The figure shows a combinatorial vector field in the upper left image, and the corresponding continuous flow in the upper right one. The invariant set  $\mathcal{S}$  is again shown in red in the lower left image. It consists of the fixed point triangle,

its upper corner, as well as the two upper edges of the triangle. Clearly the exit set is given by the lower edge and its two vertices, hence it is closed in  $\mathcal{X}$  and (a) holds. However, if we define  $\varrho(-1)$  as the red 2-simplex,  $\varrho(0)$  as the lower left vertex of the triangle, and  $\varrho(1)$  as the non-fixed upper left edge of the triangle, then  $\varrho$  is a solution of the flow  $\Pi_{\mathcal{V}}$  which violates (b). The image in the lower right demonstrates that also in this example, any neighborhood which is a candidate for an isolating neighborhood has to contain internal flow tangencies, so the maximal invariant set in such a neighborhood cannot be isolated.

We now introduce some notation which will prove to be useful in the following. For this, let  $\mathcal{V}$  denote a combinatorial vector field on a simplicial complex  $\mathcal{X}$ . For each simplex  $\sigma \in \mathcal{X}$  we then define

$$(3) \quad \sigma^+ := \begin{cases} \mathcal{V}(\sigma) & \text{if } \sigma \in \text{dom } \mathcal{V} \\ \sigma & \text{otherwise} \end{cases} \quad \text{and} \quad \sigma^- := \begin{cases} \sigma & \text{if } \sigma \in \text{dom } \mathcal{V} \\ \mathcal{V}^{-1}(\sigma) & \text{otherwise} \end{cases}.$$

The simplices  $\sigma^-$  and  $\sigma^+$  can be thought of as the source and the target of the arrow associated with  $\sigma$ , and they will equal  $\sigma$  in the case of a fixed point. For reference, this statement is formalized in the following lemma.

**Lemma 3.5.** *Let  $\mathcal{V}$  denote a combinatorial vector field on a simplicial complex  $\mathcal{X}$ . Then for arbitrary  $\sigma \in \mathcal{X}$  we have*

$$\sigma^- \subset \sigma \subset \sigma^+,$$

*and at least one of the inclusions is an equality. Moreover, both inclusions are equalities if and only if  $\sigma \in \text{Fix } \mathcal{V}$ .*

The next lemma provides us with a simple consequence of Definition 3.4(b) in the context of the simplices  $\sigma^\pm$ . It basically says that if  $\mathcal{S}$  is an isolated invariant set, then every arrow of the combinatorial vector field which has beginning or end simplex in  $\mathcal{S}$  has to lie completely in  $\mathcal{S}$ . This simple observation will be extremely useful later on.

**Lemma 3.6.** *Let  $\mathcal{V}$  denote a combinatorial vector field on a simplicial complex  $\mathcal{X}$ , and let  $\mathcal{S}$  be an isolated invariant set of the combinatorial flow  $\Pi_{\mathcal{V}}$ . Then for every simplex  $\sigma \in \mathcal{X}$  we have  $\sigma^+ \in \mathcal{S}$  if and only if  $\sigma^- \in \mathcal{S}$ . In other words, the simplices  $\sigma$  and  $\sigma^\pm$  either all lie in the isolated invariant set, or they all lie outside  $\mathcal{S}$ .*

**Proof:** If  $\sigma$  is a fixed point of  $\mathcal{V}$ , then  $\sigma = \sigma^+ = \sigma^-$  according to Lemma 3.5 — and the result holds trivially. For the remainder of the proof we therefore assume that  $\sigma \notin \text{Fix}(\mathcal{V})$ . Then Lemma 3.5 implies  $\sigma^- \neq \sigma^+$ , and (3) implies  $\mathcal{V}(\sigma^-) = \sigma^+$ .

Suppose first that the inclusion  $\sigma^- \in \mathcal{S}$  holds. Since  $\mathcal{S}$  is an invariant set, there exists a full solution  $\varrho : \mathbb{Z} \rightarrow \mathcal{S}$  with  $\varrho(0) = \sigma^-$ . Due to the definition of a solution and (2), we then have to have  $\varrho(1) \in \Pi_{\mathcal{V}}(\varrho(0)) = \Pi_{\mathcal{V}}(\sigma^-) = \{\mathcal{V}(\sigma^-)\} = \{\sigma^+\}$ . This finally implies  $\sigma^+ = \varrho(1) \in \mathcal{S}$ .

Now suppose that  $\sigma^+ \in \mathcal{S}$ . Again, the assumed invariance of  $\mathcal{S}$  implies that there exists a full solution  $\varrho : \mathbb{Z} \rightarrow \mathcal{S}$  with  $\varrho(0) = \sigma^+$ . Let  $\tau = \varrho(-1)$  denote the simplex in this solution which precedes  $\sigma^+$ . Then we have the inclusion

$$(4) \quad \sigma^+ = \varrho(0) \in \Pi_{\mathcal{V}}(\varrho(-1)) = \Pi_{\mathcal{V}}(\tau) .$$

In order to prove that  $\sigma^- \in \mathcal{S}$ , we distinguish two cases.

First of all, assume that  $\tau \in \text{dom } \mathcal{V} \setminus \text{Fix}(\mathcal{V})$ . Then  $\Pi_{\mathcal{V}}(\tau) = \{\mathcal{V}(\tau)\}$ , and (4) implies the identity  $\sigma^+ = \mathcal{V}(\tau)$ . Thus, we have  $\sigma^- = \tau \in \mathcal{S}$  according to our construction.

Suppose on the other hand that  $\tau \notin \text{dom } \mathcal{V} \setminus \text{Fix}(\mathcal{V})$ . Then the first two alternatives in definition (2) show that  $\sigma^+$  has to be a proper face of  $\tau$ . (Recall that we assumed earlier  $\sigma \notin \text{Fix}(\mathcal{V})$ , which implies  $\sigma^+ \notin \text{Fix}(\mathcal{V})$ .) Since  $\sigma^-$  is given by  $\mathcal{V}^{-1}(\sigma^+)$ , the simplex  $\sigma^-$  is a proper face of  $\sigma^+$ , hence also of  $\tau$ . But this implies  $\sigma^- \in \Pi_{\mathcal{V}}(\tau)$ , since the images in the first two alternatives of (2) contain with every simplex also all of its faces. If we now define  $\varphi : [-1, 1] \cap \mathbb{Z} \rightarrow \mathcal{X}$  by setting  $\varphi(-1) := \tau \in \mathcal{S}$ ,  $\varphi(0) := \sigma^-$ , and  $\varphi(1) := \sigma^+ \in \mathcal{S}$ , then  $\varphi$  is a solution of  $\Pi_{\mathcal{V}}$ . Since  $\mathcal{S}$  is an isolated invariant set and  $\sigma^- \in \text{Cl } \sigma^+$ , Definition 3.4(b) finally implies  $\sigma^- \in \mathcal{S}$ . This completes the proof of the lemma.  $\square$

We would like to point out that in order for Lemma 3.6 to hold, one only needs to assume that  $\mathcal{S}$  is an invariant set which satisfies Definition 3.4(b). On the other hand, it is not too hard to show that if  $\mathcal{S}$  is an invariant set with combinatorially closed exit set, and if  $\mathcal{S}$  satisfies the conclusion of the above lemma, then Definition 3.4(b) holds. Thus, we get the following result.

**Proposition 3.7.** *An invariant set  $\mathcal{S}$  is an isolated invariant set if it satisfies the following two statements:*

- (a) *The exit set  $\text{ex } \mathcal{S}$  is closed in the simplicial complex  $\mathcal{X}$ .*
- (b) *For every simplex  $\sigma \in \mathcal{X}$  we have  $\sigma^- \in \mathcal{S}$  if and only if  $\sigma^+ \in \mathcal{S}$ .*

This reformulation is somewhat more aligned with the philosophy of combinatorial vector fields, as it avoids the notion of solution and works directly with the arrows of  $\mathcal{V}$ .

#### 4. THE ASSOCIATED TIME-DISCRETE MULTIVALUED DYNAMICAL SYSTEM

While the last two sections were concerned with the beginnings of Conley theory for combinatorial vector fields on simplicial complexes, with this section we turn our attention to establish a link to the classical theory of dynamical systems. This will be accomplished within the framework of multivalued maps. More precisely, let  $X$  denote a topological space. We say that a multivalued map  $F : X \rightrightarrows X$  is *strongly upper semi-continuous*, if for each  $x \in X$  there is an open neighborhood  $V$  of  $x$  such that for every  $y \in V$  the inclusion  $F(y) \subset F(x)$  holds. The multivalued map  $F$  is called *acyclic-valued*, if for every  $x \in X$  the image  $F(x)$  is a non-empty compact subset of  $X$  with the homology of a point.

The dynamical systems generated by such multivalued maps have been studied extensively as a numerical tool for representing the dynamics of single-valued maps. The related concepts of isolated invariant sets, isolating blocks, index pairs, and the Conley index are presented in the next section. The goal of the present section is to construct a strongly upper semi-continuous acyclic-valued map  $F : X \rightrightarrows X$  which is naturally related to a given combinatorial vector field. For this, consider a simplicial complex  $\mathcal{X}$  in  $\mathbb{R}^d$ , and let  $\mathcal{V}$  denote a combinatorial vector field on  $\mathcal{X}$ . Furthermore, consider the underlying topological space  $X := |\mathcal{X}|$ . We are interested in a strongly upper semi-continuous acyclic-valued map  $F : X \rightrightarrows X$  which is related to the combinatorial vector field  $\mathcal{V}$  as follows:

- The multivalued map  $F$  is a flow-type map, in the sense that it is homologous to the identity map on  $X$  within the class of strongly upper semi-continuous acyclic-valued maps.
- We have  $x \notin F(x)$ , unless the point  $x$  is contained in  $\sigma \subset \mathbb{R}^d$  for a fixed point  $\sigma$  of  $\mathcal{V}$ , i.e., unless  $x \in \sigma$  and  $\mathcal{V}(\sigma) = \sigma$ .
- Isolated invariant sets for  $\Pi_{\mathcal{V}}$  are in one-to-one correspondence with isolated invariant sets of  $F$ .
- The Conley indices of isolated invariant sets for  $\Pi_{\mathcal{V}}$  coincide with the Conley indices of their counterparts for  $F$ .

Thus, our goal is to construct a classical multivalued dynamical system which exhibits analogous dynamics to the combinatorial vector field on the simplicial complex  $\mathcal{X}$ .

Our construction of the multivalued map  $F$  is based on a special cellular complex representation of  $X = |\mathcal{X}|$  which is, in a sense, dual to the simplicial representation of it. To describe this representation we first need to introduce some notation.

Consider the set  $E := \{-1, 0, 1\}$  and let  $\text{sgn} : \mathbb{R} \rightarrow E$  denote the standard sign function. Furthermore, consider a parameter

$$(5) \quad 0 \leq \lambda < \frac{1}{d+1},$$

where  $d$  is the dimension of the Euclidean space which contains  $\mathcal{X}$ . With every point  $x \in X$  we associate a function  $\alpha = \text{sign}^\lambda x : \mathcal{X}_0 \rightarrow E$  which is called the  $\lambda$ -signature of  $x$ , and which is defined by

$$(6) \quad \alpha(v) := \text{sgn}(t_v(x) - \lambda) \quad \text{for every vertex } v \in \mathcal{X}_0.$$

Recall that the expression  $t_v(x)$  denotes the barycentric coordinate of the point  $x$  with respect to the vertex  $v$ . Note also that we have  $\alpha(v) \geq 0$  if and only if the barycentric coordinate with respect to  $v$  satisfies  $t_v(x) \geq \lambda$ . If we now consider an arbitrary subset  $A \subset \mathcal{X}_0$  of the vertices of the simplicial complex  $\mathcal{X}$ , then one can easily see that if both  $\lambda > 0$  and  $\alpha|_A \geq 0$  hold, then  $A$  has to be an abstract simplex representing a geometric simplex in  $\mathcal{X}$ . In fact,  $A$  has to be a face of the lowest-dimensional simplex in  $\mathcal{X}$  which contains  $x$ . Even in the case of general  $\lambda$ ,

it will prove to be useful to associate certain simplices to a signature. This is the subject of the following definition.

**Definition 4.1.** *Let  $\mathcal{X}$  denote a simplicial complex in  $\mathbb{R}^d$ , let  $x \in X = |\mathcal{X}|$  be arbitrary, and let  $\alpha = \text{sign}^\lambda x$  be the  $\lambda$ -signature of  $x$  defined in (6). Furthermore, suppose that (5) holds. Then we call an (abstract) simplex  $\sigma \in \mathcal{X}$  a  $\lambda$ -characteristic simplex of  $\alpha$  if both*

$$\alpha|_\sigma \geq 0 \quad \text{and} \quad \alpha^{-1}(\{1\}) \subset \sigma .$$

We denote the family of characteristic simplices of  $\alpha = \text{sign}^\lambda x$  by  $\mathcal{X}^\lambda(\alpha)$ , i.e., we have

$$\mathcal{X}^\lambda(\alpha) := \{ \sigma \in \mathcal{X} \mid \alpha^{-1}(1) \subset \sigma \text{ and } \alpha(v) \geq 0 \text{ for all } v \in \sigma \} .$$

Equivalently, one can easily see that a simplex  $\sigma \in \mathcal{X}$  is in  $\mathcal{X}^\lambda(\alpha)$  if and only if

$$(7) \quad \alpha^{-1}(\{1\}) \subset \sigma \subset \alpha^{-1}(\{0, 1\}) .$$

For any  $\lambda \geq 0$ , the set  $\alpha^{-1}(\{1\})$  is a simplex which is called the *minimal characteristic simplex* of  $\alpha$  and is denoted by  $\sigma_{min}^\lambda(\alpha)$ . If  $\lambda > 0$ , then the set  $\alpha^{-1}(\{0, 1\})$  is also a simplex, which is called the *maximal characteristic simplex* of  $\alpha$  and is denoted by  $\sigma_{max}^\lambda(\alpha)$ . When  $\lambda = 0$ , then the set  $\alpha^{-1}(\{0, 1\})$  is the set of all vertices  $\mathcal{X}_0$  of the simplicial complex, so its maximal characteristic simplex may not exist.

Finally, if rather than emphasizing the signature  $\alpha$  we want to stress the dependence on the point  $x$ , then we call  $\sigma$  a  *$\lambda$ -characteristic simplex* of  $x$ , and the above notions will be abbreviated as  $\mathcal{X}^\lambda(x)$ ,  $\sigma_{min}^\lambda(x)$ , and  $\sigma_{max}^\lambda(x)$ , respectively.

The above definitions can easily be expressed in terms of the barycentric coordinates of  $x$ . For example, the set of all characteristic simplices is equivalently given by

$$(8) \quad \mathcal{X}^\lambda(x) = \{ \sigma \in \mathcal{X} \mid t_v(x) \geq \lambda \text{ for all } v \in \sigma \text{ and } t_v(x) \leq \lambda \text{ for all } v \notin \sigma \} , \quad \text{if } \lambda \geq 0 .$$

Similarly, the minimal characteristic simplex of  $x$  is

$$(9) \quad \sigma_{min}^\lambda(x) = \{ v \in \mathcal{X}_0 \mid t_v(x) > \lambda \} \quad \text{if } \lambda \geq 0 ,$$

while the maximal characteristic simplex is

$$(10) \quad \sigma_{max}^\lambda(x) = \{ v \in \mathcal{X}_0 \mid t_v(x) \geq \lambda \} \quad \text{if } \lambda > 0 .$$

Note that we have to exclude  $\lambda = 0$  for the maximal characteristic simplex.

In order to illustrate the above definition we concentrate on the important case  $\lambda > 0$ . Figure 6 contains a sample simplicial complex in blue with the five vertices  $\langle a \rangle, \langle b \rangle, \dots, \langle e \rangle$ , the six edges  $\langle a, b \rangle, \langle a, c \rangle, \langle b, c \rangle, \langle b, d \rangle, \langle c, d \rangle$ , and  $\langle d, e \rangle$ , and the two triangles  $\langle a, b, c \rangle$  and  $\langle b, c, d \rangle$ . Within the complex, the location of the points  $x$  for which  $t_v(x) = \lambda$  for some vertex  $v$  of  $\mathcal{X}$  is indicated by dashed black lines, as well as the two black dots on the edge  $\langle d, e \rangle$ . Finally, for each of the eight red points  $x_1, \dots, x_8$  the table below the image indicates both the minimal

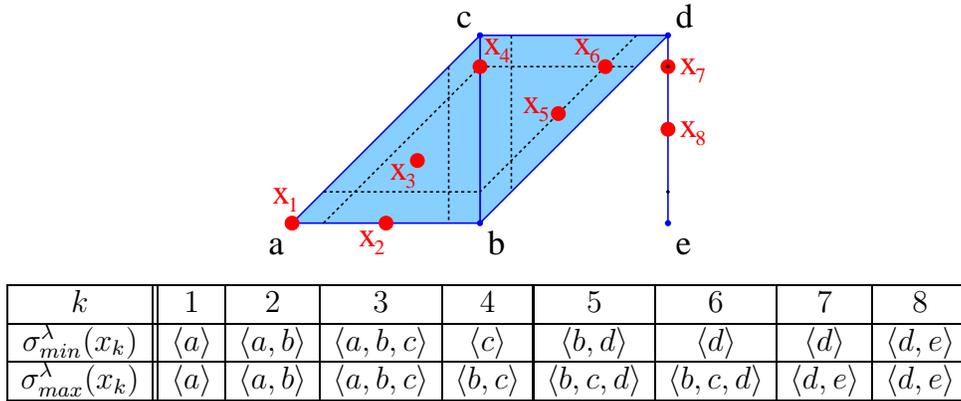


FIGURE 6. The image shows a simplicial complex  $\mathcal{X}$  in blue, together with eight sample points  $x_k \in |\mathcal{X}|$ . The black dashed lines and black dots correspond to points  $x$  for which  $t_v(x) = \lambda$  for some vertex  $v$  of  $\mathcal{X}$ . The table below the image lists both the minimal characteristic simplex  $\sigma_{min}^\lambda(x_k)$  and the maximal characteristic simplex  $\sigma_{max}^\lambda(x_k)$  for each point.

and the maximal characteristic simplex. We would like to point out that away from the dashed black lines and the two black dots, the minimal and maximal characteristic simplices agree, and  $\mathcal{X}^\lambda(x)$  contains exactly one simplex. If  $x$  lies on one of the dashed black lines or black dots, the set  $\mathcal{X}^\lambda(x)$  contains all minimal or maximal characteristic simplices for points  $y$  in a small neighborhood of  $x$ .

The minimal and maximal characteristic simplices of points  $x$  will play a crucial role in the construction of the multivalued map  $F$  associated with a combinatorial vector field  $\mathcal{V}$ . But first we have to collect a few auxiliary results which are needed for that construction.

**Lemma 4.2.** *Assume that  $\lambda$  satisfies (5), let  $0 \leq \varepsilon < \lambda$ , and let  $\mathcal{X}$  denote a simplicial complex in  $\mathbb{R}^d$ . Then  $\sigma_{max}^\lambda(x) \subset \sigma_{min}^\varepsilon(x)$  for any  $x \in X = |\mathcal{X}|$ .*

**Proof:** This lemma is an immediate consequence of the characterizations of minimal and maximal characteristic simplices given in both (9) and (10).  $\square$

In the above lemma, the main reason for admitting the case  $\varepsilon = 0$  is that  $\sigma_{min}^0(x)$  is the smallest simplex of  $\mathcal{X}$  containing  $x$ . As we mentioned earlier, the maximal characteristic simplex may not exist in this case. However, the set  $\mathcal{X}^0(x)$  still has a clear geometric meaning — it is the star of  $x$ , i.e., the set of all simplices which contain  $x$ . In particular, we have  $\mathcal{X}^0(x) \neq \emptyset$ .

The next auxiliary result provides more information of the collections  $\mathcal{X}^\lambda(x)$ . Besides showing that they are nonempty for all possible values of  $\lambda$ , they also establish the fundamental strong upper semi-continuity property.

**Lemma 4.3.** *Assume that  $\lambda$  satisfies (5), and let  $\mathcal{X}$  denote a finite simplicial complex in  $\mathbb{R}^d$ . Then for all  $x \in X = |\mathcal{X}|$  the collection  $\mathcal{X}^\lambda(x)$  from Definition 4.1 is non-empty, i.e., it contains at least one simplex. Moreover, there exists a neighborhood  $U$  of  $x$  such that*

$$\mathcal{X}^\lambda(y) \subset \mathcal{X}^\lambda(x) \quad \text{for all } y \in U .$$

*In other words, the mapping  $x \mapsto \mathcal{X}^\lambda(x)$  is strongly upper semi-continuous.*

**Proof:** In order to verify  $\mathcal{X}^\lambda(x) \neq \emptyset$ , it suffices to show that the minimal characteristic simplex  $\sigma_{min}^\lambda(x)$  contains at least one vertex. Let  $\tau$  denote the smallest simplex in  $\mathcal{X}$  which contains  $x$ , and assume that  $\sigma_{min}^\lambda(x) = \emptyset$ . Then (9) implies  $t_v(x) \leq \lambda$  for all vertices  $v \in \mathcal{X}_0$ , and therefore in particular for all  $v \in \tau$ . If  $k \leq d$  denotes the dimension of  $\tau$ , then one obtains

$$1 = \sum_{v \in \tau} t_v(x) \leq (k+1)\lambda, \quad \text{and therefore} \quad \lambda \geq \frac{1}{k+1} \geq \frac{1}{d+1},$$

which contradicts (5).

In order to prove the second statement of the lemma, suppose that  $x \mapsto \mathcal{X}^\lambda(x)$  is not strongly upper semi-continuous. Then one can find a sequence  $(x_n)$  in  $X$  which converges to  $x$ , as well as a sequence of simplices  $\sigma_n \in \mathcal{X}^\lambda(x_n) \setminus \mathcal{X}^\lambda(x)$ . Since the simplicial complex  $\mathcal{X}$  was assumed to be finite, we can assume without loss of generality, after possibly passing to a subsequence, that the sequence  $(\sigma_n)$  is constant. Let  $\sigma \in \mathcal{X}$  be such that  $\sigma = \sigma_n$  for all  $n \in \mathbb{N}$ . According to (6), we therefore have  $t_v(x_n) \geq \lambda$  for every vertex  $v$  of  $\sigma$ , as well as  $t_v(x_n) \leq \lambda$  for all vertices  $v \notin \sigma$ . Since barycentric coordinates are continuous, the same inequalities hold in the limit, i.e., we have  $t_v(x) \geq \lambda$  for all  $v \in \sigma$ , and  $t_v(x) \leq \lambda$  for all  $v \notin \sigma$ . But this implies  $\sigma \in \mathcal{X}^\lambda(x)$ , contradicting our construction of the sequence  $(\sigma_n)$ . This completes the proof of the lemma.  $\square$

At this point we would like to recall that the main goal of the present section is the construction of a strongly upper semi-continuous multivalued map  $F$  which mimics the dynamics of a given combinatorial vector field  $\mathcal{V}$  on  $\mathcal{X}$ . So far, we have been able to associate to every point  $x \in X = |\mathcal{X}|$  a collection  $\mathcal{X}^\lambda(x)$  which is strongly upper semi-continuous. We now use this association to define the basic building blocks for a new cell decomposition of the topological space  $X$ .

**Definition 4.4.** *Suppose that  $\mathcal{X}$  is a finite simplicial complex in  $\mathbb{R}^d$ , and let*

$$(11) \quad 0 < \lambda < \frac{1}{d+1} .$$

*For any simplex  $\sigma \in \mathcal{X}$  we define the  $\lambda$ -cell generated by  $\sigma$  by*

$$\langle \sigma \rangle_\lambda := \{x \in X \mid \mathcal{X}^\lambda(x) = \{\sigma\}\} ,$$

*i.e., the  $\lambda$ -cell contains all points  $x \in X$  for which  $\sigma = \sigma_{min}^\lambda(x) = \sigma_{max}^\lambda(x)$ .*

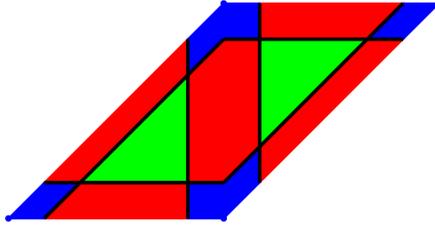


FIGURE 7. The image shows all  $\lambda$ -cells for the simplicial complex from Figure 6. The five  $\lambda$ -cells corresponding to points  $x$  for which  $\mathcal{X}^\lambda(x)$  contains a zero-dimensional simplex  $\sigma$  are shown in blue, the six  $\lambda$ -cells for which  $\sigma$  is one-dimensional are shown in red, and the two  $\lambda$ -cells for which  $\sigma$  is two-dimensional are green. Note that all cells are open sets, and that the points on black lines or dots are not contained in any  $\lambda$ -cell.

The notion of  $\lambda$ -cells is illustrated in Figure 7 for the simplicial complex from Figure 6. In the image,  $\lambda$ -cells corresponding to points  $x$  for which  $\mathcal{X}^\lambda(x)$  contains a zero-, one-, or two-dimensional simplex are shown in blue, red, or green, respectively. Altogether, there are five blue  $\lambda$ -cells, six red, and two green ones. We would like to point out that all cells are open and connected sets, and that the points on black lines or dots are not contained in any  $\lambda$ -cell.

The above-defined  $\lambda$ -cells will be used both for dividing up the domain  $X$  of our multivalued mapping  $F$ , as well as to construct the images  $F(x)$ . This requires a number of auxiliary results, which will be described in the following. To begin with, it is easy to see that a  $\lambda$ -cell can equivalently be described as

$$(12) \quad \langle \sigma \rangle_\lambda = \{x \in X \mid t_v(x) > \lambda \text{ for all } v \in \sigma \text{ and } t_v(x) < \lambda \text{ for all } v \notin \sigma\} .$$

In addition, the following lemma contains useful observations concerning the geometry of  $\lambda$ -cells.

**Lemma 4.5.** *Assume that  $\lambda$  satisfies (11), and let  $\mathcal{X}$  denote a simplicial complex in  $\mathbb{R}^d$ . Then for every simplex  $\sigma \in \mathcal{X}$  the associated  $\lambda$ -cell  $\langle \sigma \rangle_\lambda$  is a non-empty open set in  $X$ , and the intersection of  $\langle \sigma \rangle_\lambda$  with any simplex of  $\mathcal{X}$  is convex. Furthermore, the closure of the  $\lambda$ -cell can be expressed in terms of barycentric coordinates as*

$$(13) \quad \text{cl } \langle \sigma \rangle_\lambda = \{x \in X \mid t_v(x) \geq \lambda \text{ for all } v \in \sigma \text{ and } t_v(x) \leq \lambda \text{ for all } v \notin \sigma\} .$$

**Proof:** Due to (11) and (12) the barycenter of the simplex  $\sigma$  is contained in  $\langle \sigma \rangle_\lambda$ , i.e., the  $\lambda$ -cell is not empty. Moreover, since the barycentric coordinates are continuous functions, the characterization (12) implies that the  $\lambda$ -cell is open.

For the proof of the convexity assertion, it will be convenient to introduce two auxiliary sets. Given any vertex  $v \in \mathcal{X}_0$ , we define

$$A_v := \{x \in X \mid t_v(x) > \lambda\} \quad \text{and} \quad B_v := \{x \in X \mid t_v(x) < \lambda\} .$$

Then the  $\lambda$ -cell  $\langle \sigma \rangle_\lambda$  can be expressed as the intersection

$$(14) \quad \langle \sigma \rangle_\lambda = \bigcap_{v \in \sigma} A_v \cap \bigcap_{v \notin \sigma} B_v .$$

Now let  $\tau \in \mathcal{X}$  denote an arbitrary simplex. Since the restriction of  $t_v$  to  $\tau$  is a linear function, the sets  $A_v \cap \tau$  and  $B_v \cap \tau$  are convex, for every vertex  $v \in \mathcal{X}_0$ . Hence, the intersection  $\langle \sigma \rangle_\lambda \cap \tau$  has to be convex as well.

In order to complete the proof of the lemma, it remains to establish (13). Suppose first that  $x \in \text{cl} \langle \sigma \rangle_\lambda$ . Then there exists a sequence  $(x_n)$  in  $X$  which converges to  $x$ , and such that  $t_v(x_n) > \lambda$  for all  $v \in \sigma$ , as well as  $t_v(x_n) < \lambda$  for all  $v \notin \sigma$ . Due to the continuity of the barycentric coordinates this immediately implies  $t_v(x) \geq \lambda$  for all  $v \in \sigma$ , as well as  $t_v(x) \leq \lambda$  for all  $v \notin \sigma$ , i.e., the right-hand side of (13) contains  $x$ .

Conversely, let  $x$  be contained in the right-hand side of (13), and assume that at least one barycentric coordinate of  $x$  equals  $\lambda$ , since otherwise  $x \in \langle \sigma \rangle_\lambda \subset \text{cl} \langle \sigma \rangle_\lambda$ . Let  $\tau$  denote the smallest simplex in  $\mathcal{X}$  which contains  $x$ , i.e., the vertices of  $\tau$  are precisely the vertices  $v \in \mathcal{X}_0$  for which  $t_v(x) > 0$ . Note that in fact, all barycentric coordinates of  $x$  have to be contained in the open interval  $(0, 1)$  due to (11). We divide the vertices of  $\tau$  into three disjoint sets  $E_1$ ,  $E_2$ , and  $E_3$  in such a way that

$$\begin{aligned} v \in E_1 & \quad \text{if and only if} \quad t_v(x) = \lambda \text{ and } v \in \sigma , \\ v \in E_2 & \quad \text{if and only if} \quad t_v(x) = \lambda \text{ and } v \notin \sigma , \\ v \in E_3 & \quad \text{if and only if} \quad t_v(x) \in (0, 1) \setminus \{\lambda\} . \end{aligned}$$

One can easily see that due to (11) and  $\dim \tau \leq d$  the set  $E_3$  has to contain at least one vertex  $w$ . For  $\delta > 0$ , we now define the point  $x_\delta$  as

$$x_\delta = \sum_{v \in E_1} (\lambda + \delta) v + \sum_{v \in E_2} (\lambda - \delta) v + \sum_{v \in E_3} t_v(x) v + \delta (|E_2| - |E_1|) w ,$$

where  $|E_k|$  denotes the number of elements in  $E_k$ . Then clearly the barycentric coordinates of  $x_\delta$  add up to one. Furthermore, there exists a  $\delta_0 > 0$  such that for all  $0 < \delta < \delta_0$  we have  $t_v(x_\delta) > 0$  for all  $v \in \tau$ , as well as  $t_w(x_\delta) \in (0, 1) \setminus \{\lambda\}$ . For this, one only needs to recall that  $t_w(x)$  lies in the open set  $(0, 1) \setminus \{\lambda\}$  and that  $\lambda > 0$ . In addition, the construction of  $x_\delta$  guarantees that we have  $t_v(x_\delta) > \lambda$  for all  $v \in \sigma$ , as well as  $t_v(x_\delta) < \lambda$  for all  $v \notin \sigma$ . In other words, for all  $0 < \delta < \delta_0$  we have  $x_\delta \in \langle \sigma \rangle_\lambda$ . Since  $x_\delta \rightarrow x$  as  $\delta \rightarrow 0$ , this readily implies that the point  $x$  is contained in  $\text{cl} \langle \sigma \rangle_\lambda$ , i.e., both sides in (13) are in fact equal. This completes the proof of the lemma.  $\square$

The above lemma immediately leads to the following characterization which relates points  $x \in X = |\mathcal{X}|$  to both the simplices in  $\mathcal{X}^\lambda(x)$  and the  $\lambda$ -cells  $\langle \sigma \rangle_\lambda$ .

**Corollary 4.6.** *Assume that  $\lambda$  satisfies (11), and let  $\mathcal{X}$  denote a simplicial complex in  $\mathbb{R}^d$ . Furthermore, let  $\sigma \in \mathcal{X}$  and  $x \in X$ . Then the following statements are equivalent:*

- (i)  $\sigma \in \mathcal{X}^\lambda(x)$ ,
- (ii)  $\sigma_{\min}^\lambda(x) \subset \sigma \subset \sigma_{\max}^\lambda(x)$ ,
- (iii)  $x \in \text{cl} \langle \sigma \rangle_\lambda$ .

**Proof:** Since  $\lambda > 0$ , the statement in (ii) is equivalent to (7) due to the definition of the minimal and maximal characteristic simplices, and it is equivalent to (i) according to Definition 4.1. Finally, the equivalence of (i) and (iii) follows from both (8) and (13).  $\square$

The next lemma demonstrates how  $\lambda$ -cells can be used to construct neighborhoods of points  $x$  in  $X$ .

**Lemma 4.7.** *Assume that  $\lambda$  satisfies (11), and let  $\mathcal{X}$  denote a simplicial complex in  $\mathbb{R}^d$ . Then for any  $x \in X$  the set*

$$\bigcup_{\sigma \in \mathcal{X}^\lambda(x)} \text{cl} \langle \sigma \rangle_\lambda$$

*is a neighborhood of the point  $x$ .*

**Proof:** According to Lemma 4.3 there exists a neighborhood  $U$  of  $x$  such that the inclusion  $\mathcal{X}^\lambda(y) \subset \mathcal{X}^\lambda(x)$  holds for all  $y \in U$ . Now let  $y \in U$  be arbitrary, and let  $\tau \in \mathcal{X}^\lambda(y)$ . Then  $\tau \in \mathcal{X}^\lambda(x)$ , and Corollary 4.6 implies

$$y \in \text{cl} \langle \tau \rangle_\lambda \subset \bigcup_{\sigma \in \mathcal{X}^\lambda(x)} \text{cl} \langle \sigma \rangle_\lambda .$$

Thus, we have  $U \subset \bigcup_{\sigma \in \mathcal{X}^\lambda(x)} \text{cl} \langle \sigma \rangle_\lambda$ , and the claim follows.  $\square$

We would like to point out that the signature provides a decomposition of  $X$  into a union of disjoint connected cells on which the map  $x \mapsto \text{sign}^\lambda x$  is constant. While all of these cells are polyhedral sets, only some are actually  $\lambda$ -cells. In fact, the  $\lambda$ -cells correspond exactly to signatures which are nonzero on all vertices of the simplicial complex. For example, in Figure 7 each connected colored region is a  $\lambda$ -cell, and also a maximal set on which the signature is constant. The remaining points in  $X$ , i.e., the black lines and dots, correspond to signatures which are zero at at least one vertex. By considering regions over which the signature is constant, these black lines and points are decomposed in the following way. The six corners of the two green triangles, together with the two black points on the one-dimensional vertical edge at the right side of the simplicial complex are zero-dimensional cells in the signature decomposition. After removal of these eight points, the black structure breaks up into sixteen one-dimensional connected pieces, and they are the remaining cells. Notice that the dimensions of the cells in the black sub-structure cannot be determined by the number of zeros in the

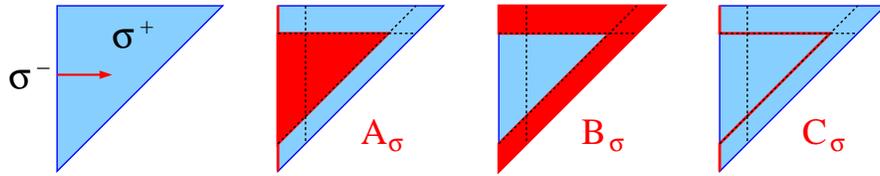


FIGURE 8. Visualization of the three sets  $A_\sigma$ ,  $B_\sigma$ , and  $C_\sigma$  defined in (16), for the case of a two-dimensional simplex  $\sigma \in \mathcal{X} \setminus \text{Fix } \mathcal{V}$ . The first image depicts an arrow of the combinatorial vector field  $\mathcal{V}$ , originating at the one-simplex  $\sigma^-$  and ending at the two-simplex  $\sigma^+ = \sigma$ . The next three figures show the sets  $A_\sigma$ ,  $B_\sigma$ , and  $C_\sigma$ , respectively.

signature alone. They also depend on the dimension of the enclosing simplex. For example, either of the two black points in the vertical edge in Figure 7 corresponds to a signature with exactly one zero, while the corners of the green triangles each correspond to signatures with exactly two zeros.

After these preparations, we are finally in a position to start the construction of the strongly upper semi-continuous map  $F$  which will be associated to a combinatorial vector field. For this, we fix two parameters

$$(15) \quad 0 < \gamma < \varepsilon < \frac{1}{d+1}.$$

Our construction proceeds in several steps. For the remainder of this section, let  $\mathcal{X}$  denote a simplicial complex, let  $\mathcal{V}$  be a combinatorial vector field on  $\mathcal{X}$  in the sense of Definition 3.1, and recall the notation introduced in (3). Then for any simplex  $\sigma \in \mathcal{X}$  we define three subsets of the topological space  $X = |\mathcal{X}|$  by

$$(16) \quad \begin{aligned} A_\sigma &:= \{x \in \sigma^+ \mid t_v(x) \geq \gamma \text{ for all } v \in \sigma^-\} \cup \sigma^-, \\ B_\sigma &:= \{x \in \sigma^+ \mid \text{there exists a } v \in \sigma^- \text{ with } t_v(x) \leq \gamma\}, \text{ and} \\ C_\sigma &:= A_\sigma \cap B_\sigma. \end{aligned}$$

According to our convention from Section 2, the statement  $x \in \sigma^+$  refers to arbitrary points  $x$  in the geometric simplex  $\sigma^+$ , while  $v \in \sigma^-$  only refers to the vertices  $v$  of  $\sigma^-$ . Moreover, note that if  $\sigma$  is a fixed point of  $\mathcal{V}$ , then according to Lemma 3.5 we have  $\sigma^- = \sigma^+$ , i.e., the first set satisfies  $A_\sigma = \sigma$ . Furthermore, one can easily see that we always have

$$A_\sigma = A_{\sigma^-} = A_{\sigma^+}, \quad B_\sigma = B_{\sigma^-} = B_{\sigma^+}, \quad \text{as well as} \quad C_\sigma = C_{\sigma^-} = C_{\sigma^+},$$

since the definitions of these sets only involve references to  $\sigma^\pm$ . This will be useful later on.

The above subsets are in some sense related to the  $\gamma$ -cells associated with  $\mathcal{X}$ , and they are the central building blocks for constructing the images of  $F$ . However, before continuing with this construction we need the following auxiliary result,

which only considers the case of simplices  $\sigma \in \mathcal{X} \setminus \text{Fix } \mathcal{V}$ . According to Lemma 3.5 we then have  $\sigma^- \neq \sigma^+$ . In this case, the three sets are visualized in Figure 8.

**Lemma 4.8.** *Let  $\mathcal{X}$  denote a simplicial complex, let  $\mathcal{V}$  be a combinatorial vector field on  $\mathcal{X}$  in the sense of Definition 3.1, and assume that (15) holds. Then for any simplex  $\sigma \in \mathcal{X} \setminus \text{Fix } \mathcal{V}$  there exists a continuous homotopy  $h : \sigma^+ \times [0, 1] \rightarrow \sigma^+$  with the following properties:*

- (a) *The mapping  $h(\cdot, 0)$  is the identity map on  $\sigma^+$ .*
- (b) *For every  $s \in [0, 1]$  the mapping  $h(\cdot, s)$  is the identity map on  $|\text{Bd } \sigma^+|$ .*
- (c) *For all  $s \in [0, 1]$  the inclusion  $h(A_\sigma, s) \subset A_\sigma$  holds, and  $h(x, 1) \in \sigma^-$  for arbitrary points  $x \in A_\sigma$ .*
- (d) *The restriction of  $h(\cdot, 1)$  to  $B_\sigma$  is a homeomorphism of  $B_\sigma$  onto  $\sigma^+$ .*
- (e) *The restriction of  $h(\cdot, 1)$  to  $C_\sigma$  is a homeomorphism of  $C_\sigma$  onto  $\sigma^-$ .*

As a consequence, all three sets  $A_\sigma$ ,  $B_\sigma$ , and  $C_\sigma$  are contractible.

**Proof:** According to our hypotheses, the simplex  $\sigma^-$  has to be a facet of  $\sigma^+$ . Let  $\sigma^- = \langle v_0, v_1, \dots, v_{k-1} \rangle$ , and let  $v_k$  denote the remaining vertex of  $\sigma^+$ . We view the simplex  $\sigma^+$  as the join  $\sigma^+ = \sigma^- \star \{v_k\}$ , i.e., as the cone with base  $\sigma^-$  and vertex  $v_k$ . Similarly, the set

$$A^+ := \{x \in \sigma^+ \mid t_v(x) \geq \gamma \text{ for all } v \in \sigma^-\}$$

in the definition of  $A_\sigma$  can be written as  $A^+ = A^- \star \{y\}$ , where

$$A^- := \sigma^- \cap \text{cl} \langle \sigma^- \rangle_\gamma,$$

and the point  $y \in \sigma^+$  is the point with barycentric coordinates  $t_{v_i}(y) = \gamma$  for the indices  $i = 0, 1, \dots, k-1$ , as well as  $t_{v_k}(y) = 1 - k\gamma$ . In Figure 8, the point  $y$  is the right-most point of the red region  $A_\sigma$ , and  $A^-$  is the part of the left vertical edge between the two black dashed lines.

It is easy to verify that the set  $A^- \subset \sigma^-$  is also a simplex, which has the same dimension  $k-1$  as  $\sigma^-$ . This implies that  $A^+ \subset \sigma^+$  is a simplex of dimension  $k$ . Moreover, one can readily see that  $A_\sigma = \sigma^- \cup A^+$ , and  $C_\sigma = (\sigma^- \setminus \langle \sigma^- \rangle_\gamma) \cup C^+$ , where we define  $C^+ := \text{bd}(A^- \star \{w\}) \cap \overset{\circ}{\sigma}$ .

We construct the homotopy  $h$  along the intervals  $\{z\} \star \{v_k\}$ , where  $z \in \sigma^-$ . It can easily be verified that if  $z \notin \langle \sigma^- \rangle_\gamma$ , then we have  $(\{z\} \star \{v_k\}) \cap A_\sigma = \{z\}$ . On the other hand, if  $z \in A^-$ , then  $(\{z\} \star \{v_k\}) \cap A_\sigma = \{z\} \star \{x_z\}$  for a uniquely defined point  $x_z \in C^+$ . In the first case, the homotopy  $h$  is the identity map on  $\{z\} \star \{v_k\}$ . In the second case, the homotopy  $h$  contracts the interval  $\{z\} \star \{x_z\}$  to  $\{z\}$ , and expands  $\{x_z\} \star \{v_k\}$  to  $\{z\} \star \{v_k\}$ . An explicit formula for the resulting  $h$ , as well as the verification of its properties (a) through (e) are left as exercise.

The properties (a), (b), and (c) imply that  $\sigma^-$  is a strong deformation retract of  $A_\sigma$ , which therefore has to be contractible due to the contractibility of  $\sigma^-$ . According to property (d), the set  $B_\sigma$  is contractible, and the set  $C_\sigma$  is contractible because of (e).  $\square$

The above-defined sets  $A_\sigma$ ,  $B_\sigma$ , and  $C_\sigma$  are the essential building blocks for defining the images of our multivalued map  $F$ , and they are related to the  $\gamma$ -cells associated with  $\mathcal{X}$ . In order to define itself, we use a piece-wise constant approach, by defining  $F(x)$  in the same way for all  $x$  in certain subsets of  $\mathcal{X}$ . These subsets turn out to be the  $\varepsilon$ -cells of  $\mathcal{X}$  and their boundaries.

As a first step, we define multivalued maps  $F_\sigma : X \rightrightarrows X$  for every simplex  $\sigma \in \mathcal{X}$ . This definition makes use of the sets introduced in (16), and is given by

$$(17) \quad F_\sigma(x) := \begin{cases} \emptyset & \text{if } \sigma \notin \mathcal{X}^\varepsilon(x), \\ A_\sigma & \text{if } \sigma \in \mathcal{X}^\varepsilon(x), \sigma \neq \sigma_{max}^\varepsilon(x)^+, \text{ and } \sigma \neq \sigma_{max}^\varepsilon(x)^-, \\ B_\sigma & \text{if } \sigma = \sigma_{max}^\varepsilon(x)^+ \neq \sigma_{max}^\varepsilon(x)^-, \\ C_\sigma & \text{if } \sigma = \sigma_{max}^\varepsilon(x)^- \neq \sigma_{max}^\varepsilon(x)^+, \\ \sigma & \text{if } \sigma = \sigma_{max}^\varepsilon(x)^- = \sigma_{max}^\varepsilon(x)^+. \end{cases}$$

It can easily be verified that all of the maps  $F_\sigma$  are well-defined, i.e., that the cases are exhaustive and exclusive. Recall in particular that if  $\sigma \in \text{Fix } \mathcal{V}$ , then we have  $A_\sigma = \sigma$ . Moreover, one can easily establish the following lemma, which we state without proof.

**Lemma 4.9.** *Let  $\mathcal{X}$  denote a simplicial complex, let  $\mathcal{V}$  be a combinatorial vector field on  $\mathcal{X}$  in the sense of Definition 3.1, and assume that (15) holds. Then we have*

$$F_\sigma(x) \subset \sigma^+ \quad \text{for all } \sigma \in \mathcal{X} \text{ and } x \in X.$$

After these lengthy preparations we are finally in a position to define the multivalued map  $F : X \rightrightarrows X$  which mimics the dynamics of a given combinatorial vector field  $\mathcal{V}$  on a simplicial complex  $\mathcal{X}$ . For this, let

$$(18) \quad F(x) := \bigcup_{\sigma \in \mathcal{X}} F_\sigma(x) \quad \text{for all } x \in X = |\mathcal{X}|,$$

where  $F_\sigma(x)$  was defined in (17). While the above definition is appealing due to its simplicity, it is actually not necessary to take the union over all simplices  $\sigma \in \mathcal{X}$ . In fact, we have the following result.

**Lemma 4.10.** *Let  $\mathcal{X}$  denote a simplicial complex, let  $\mathcal{V}$  be a combinatorial vector field on  $\mathcal{X}$  in the sense of Definition 3.1, and assume that (15) holds. Furthermore, let  $F : X \rightrightarrows X$  be defined as in (18). If we then define*

$$\mathcal{T}^\varepsilon(x) := \{ \tau \in \mathcal{X}^\varepsilon(x) \setminus \{ \sigma_{max}^\varepsilon(x) \} \mid \tau = \tau^- \text{ and } \tau^+ \notin \text{Cl } \sigma_{max}^\varepsilon(x) \},$$

then the image  $F(x)$  can be expressed alternatively as

$$(19) \quad F(x) = F_{\sigma_{max}^\varepsilon(x)} \cup \bigcup_{\tau \in \mathcal{T}^\varepsilon(x)} F_\tau(x).$$

Furthermore, every  $\tau \in \mathcal{T}^\varepsilon(x)$  automatically satisfies  $\tau \in \text{dom } \mathcal{V} \setminus \text{Fix } \mathcal{V}$ .

**Proof:** According to the first line of the definition of  $F_\sigma(x)$  given in (17), the definition of  $F$  in (18) is equivalent to

$$(20) \quad F(x) = \bigcup_{\sigma \in \mathcal{X}^\varepsilon(x)} F_\sigma(x) \quad \text{for all } x \in X .$$

Due to the characterization of  $\mathcal{X}^\varepsilon(x)$  given in Corollary 4.6, it suffices to prove that the inclusion  $F_\tau(x) \subset F_{\sigma_{max}^\varepsilon(x)}$  holds for all  $\tau \in \mathcal{X}^\varepsilon(x) \setminus \{\sigma_{max}^\varepsilon(x)\}$  for which either

- (i)  $\tau \neq \tau^-$ , or
- (ii)  $\tau = \tau^-$  and  $\tau^+ \in \text{Cl } \sigma_{max}^\varepsilon(x)$

are satisfied. This will be the subject of the remainder of this proof. To simplify notation, we use the abbreviation  $\sigma := \sigma_{max}^\varepsilon(x)$ .

(i) Assume first that  $\tau \in \mathcal{X}^\varepsilon(x) \setminus \{\sigma\}$  satisfies  $\tau \neq \tau^-$ . According to Lemma 3.5 we then have  $\tau = \tau^+$ , and Lemma 4.9 and Corollary 4.6 imply

$$F_\tau(x) \subset \tau^+ = \tau \subset \sigma .$$

If we have  $\sigma \in \text{Fix } \mathcal{V}$ , then  $F_\sigma(x) = \sigma$ , and one trivially obtains  $F_\tau(x) \subset F_\sigma(x)$ .

Now suppose that  $\sigma \in \text{dom } \mathcal{V} \setminus \text{Fix } \mathcal{V}$ . Then we have  $\sigma = \sigma^- \neq \sigma^+$ , and (17) yields  $F_\sigma(x) = C_\sigma$ . Since  $\tau \neq \sigma$ , the simplex  $\tau$  is a proper face of  $\sigma$ , and therefore  $\tau \in \text{Bd } \sigma$ . The definition of  $C_\sigma$  then implies  $\tau \subset C_\sigma$ , which immediately implies  $F_\tau(x) \subset C_\sigma = F_\sigma(x)$ .

Finally, consider the case  $\sigma \in \text{im } \mathcal{V} \setminus \text{Fix } \mathcal{V}$ . Then we have  $\sigma = \sigma^+ \neq \sigma^-$ , and (17) implies  $F_\sigma(x) = B_\sigma$ . Since  $\tau$  is a proper face of  $\sigma = \sigma^+$ , it has to be a face of  $\sigma^-$ . Assume that in fact we have  $\tau = \sigma^-$ . Then  $\tau \in \text{dom } \mathcal{V}$ . Yet, according to  $\tau = \tau^+$  we also have  $\tau \in \text{im } \mathcal{V}$ . This implies  $\tau \in \text{dom } \mathcal{V} \cap \text{im } \mathcal{V} = \text{Fix } \mathcal{V}$ , i.e., we have  $\tau = \tau^-$ , which contradicts (i). Thus, the simplex  $\tau$  has to be a proper face of  $\sigma^-$ , and the definition of  $B_\sigma$  implies  $\tau \subset B_\sigma$ , which yields  $F_\tau(x) \subset B_\sigma = F_\sigma(x)$ .

(ii) We now assume that  $\tau \in \mathcal{X}^\varepsilon(x) \setminus \{\sigma\}$  satisfies both  $\tau = \tau^-$  and  $\tau^+ \in \text{Cl } \sigma$ . Without loss of generality, we can also assume that  $\tau \neq \tau^+$ , since otherwise we can proceed as in the proof of case (i) above. In other words, the simplex  $\tau$  is a proper face of  $\sigma$  as well as the source of an arrow of  $\mathcal{V}$  which points into  $\text{Cl } \sigma$ . Thus, Lemma 4.9 implies

$$F_\tau(x) \subset \tau^+ \subset |\text{Cl } \sigma| \subset \sigma .$$

If we have  $\sigma \in \text{Fix } \mathcal{V}$ , then  $F_\sigma(x) = \sigma$ , and one again obtains  $F_\tau(x) \subset F_\sigma(x)$ .

If on the other hand we assume  $\sigma \in \text{dom } \mathcal{V} \setminus \text{Fix } \mathcal{V}$ , then  $\sigma = \sigma^- \neq \sigma^+$ , as well as  $F_\sigma(x) = C_\sigma$  due to (17). Since  $\sigma \in \text{dom } \mathcal{V}$ , one necessarily has to have  $\tau^+ \neq \sigma$ . This implies  $\tau^+ \subset |\text{Bd } \sigma| \subset C_\sigma = F_\sigma(x)$ , i.e., we have  $F_\tau(x) \subset F_\sigma(x)$ .

Finally, consider the case  $\sigma \in \text{im } \mathcal{V} \setminus \text{Fix } \mathcal{V}$ . Then we have  $\sigma = \sigma^+ \neq \sigma^-$ , and (17) implies  $F_\sigma(x) = B_\sigma$ . If  $\tau^+ = \sigma = \sigma^+$ , then obviously  $\tau = \sigma^-$ . Therefore, another application of (17) implies  $F_\tau(x) = C_\tau = C_{\sigma^-} = C_\sigma \subset B_\sigma = F_\sigma(x)$ .

Assume now that  $\tau^+ \neq \sigma = \sigma^+$ . Then the definition of a discrete vector field implies  $\tau^+ \neq \sigma^-$ . This finally yields  $\tau^+ \subset |\text{Bd } \sigma \setminus \{\sigma^-\}|$ , as well as

$$F_\tau(x) \subset \tau^+ \subset |\text{Bd } \sigma \setminus \{\sigma^-\}| \subset B_\sigma = F_\sigma(x).$$

This completes the proof of the lemma.  $\square$

The somewhat involved definition of the multivalued map  $F$  is illustrated in the following example.

**Example 4.11.** In order to provide more intuition for the definition of the multivalued map  $F$ , consider the diagrams in Figure 9. In this figure, the top left image shows a simplicial complex  $\mathcal{X}$  with a combinatorial vector field  $\mathcal{V}$ . The remaining seven diagrams contain sample images  $F(x)$  of the multivalued map  $F$ . In each case, the point  $x$  is marked by a large black dot and the corresponding image  $F(x)$  is shown in red. The top right image, as well as the images in the second row, correspond to cases where the set  $\mathcal{X}^\varepsilon(x)$  consists of only one simplex. Note that the first image in the second row also contains a second point  $x$ , which leads to the same image  $F(x)$  as the point in the center, even though for the second point the set  $\mathcal{X}^\varepsilon(x)$  contains exactly two simplices. The remaining four images in the third and fourth row are for sets  $\mathcal{X}^\varepsilon(x)$  with three or two simplices. In the figure, the dashed black lines and small black dots delineate  $\varepsilon$ -cells  $\langle \sigma \rangle_\varepsilon$ .  $\diamond$

The following main result of this section establishes the fundamental properties of the multivalued map  $F$ .

**Theorem 4.12.** *Let  $\mathcal{X}$  denote a simplicial complex, let  $\mathcal{V}$  be a combinatorial vector field on  $\mathcal{X}$  in the sense of Definition 3.1, and assume that (15) holds. Furthermore, let  $F : X \rightrightarrows X$  be defined as in (18). Then  $F$  is strongly upper semicontinuous, and its values  $F(x)$  for all  $x \in X$  are non-empty and contractible sets.*

**Proof:** According to Lemma 4.10, the definition of  $F$  in (18) is equivalent to (20). Thus, Lemma 4.3 immediately implies the strong upper semicontinuity of the map  $F$ , as well as  $F(x) \neq \emptyset$  for all  $x \in X$ .

We now turn our attention to the contractibility of the images of  $F$ , and for this we make use of the alternative representation of the image  $F(x)$  in (19), which was derived in Lemma 4.10. According to Lemma 4.8 and the fact that every simplex is contractible, all non-empty values  $F_\tau(x)$  of the map  $F_\tau$  are automatically contractible. Yet, in order to show that this contractibility carries over to  $F(x)$  we need to be more careful, since unions of contractible sets are not contractible in general.

Let  $x \in \mathcal{X}$  be arbitrary, but fixed. If the collection  $\mathcal{T}^\varepsilon(x)$  from Lemma 4.10 is empty, then we have  $F(x) = F_{\sigma_{\max}^\varepsilon(x)}(x)$ , which is clearly contractible. Suppose therefore from now on that  $\mathcal{T}^\varepsilon(x)$  contains at least one element. In order to simplify the notation we use the abbreviation  $\sigma := \sigma_{\max}^\varepsilon(x)$ . We first show that for any  $\tau \in \mathcal{T}^\varepsilon(x)$  the union  $F_\tau(x) \cup F_\sigma(x)$  can be homotopically deformed to the

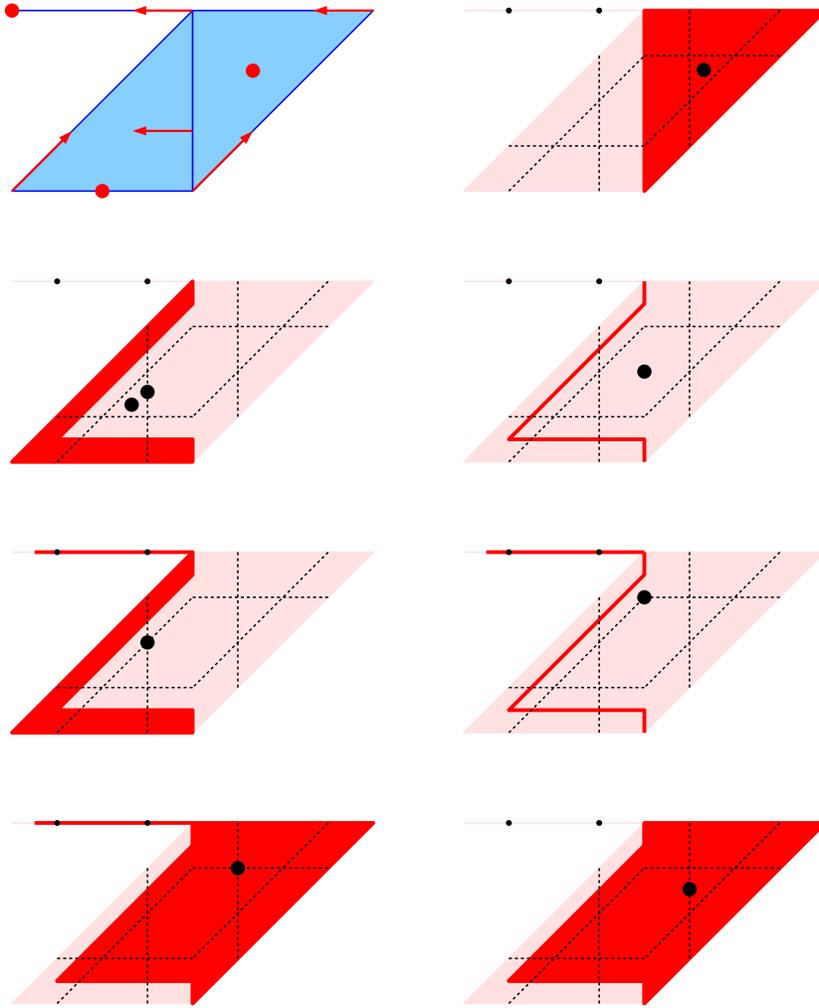


FIGURE 9. The top left image shows a simplicial complex  $\mathcal{X}$  with a combinatorial vector field  $\mathcal{V}$ . The remaining seven images contain sample images  $F(x)$  of the multivalued map  $F$ . In each case, the points  $x$  are marked by large black dots, the corresponding image  $F(x)$  is shown in red. Note that in the second image from the top in the left column, both indicated points  $x$  lead to the same image  $F(x)$ . The dashed black lines and small black dots delineate  $\varepsilon$ -cells  $\langle \sigma \rangle_\varepsilon$ .

simplex  $\sigma$ . This will be accomplished in steps (i), (ii), and (iii) below. After that, in step (iv) we demonstrate that the underlying deformations for different simplices  $\tau$  do not interfere.

(i) Suppose first that  $\sigma \in \text{Fix } \mathcal{V}$ , and let  $\tau \in \mathcal{T}^\varepsilon(x)$  be arbitrary. Then according to the last case in (17) we have  $F_\sigma(x) = \sigma = \sigma^- = \sigma^+$ . Our choice of  $\tau$  implies

both the identity  $\tau = \tau^-$  and the fact that  $\tau^+ \notin \text{Cl } \sigma$ . Thus, definition (17) implies  $F_\tau(x) = A_\tau$ . Note that the intersection  $\sigma \cap A_\tau = \tau$  is convex, and so the union  $F_\sigma(x) \cup F_\tau(x) = \sigma \cup A_\tau$  is a contractible set. Moreover, the contraction homotopy  $h_\tau(\cdot, s)$  of  $A_\tau$  onto  $\tau$ , which is obtained by applying Lemma 4.8 to the simplex  $\tau$ , is the identity on the intersection  $\sigma \cap A_\tau = \tau$ . Therefore, the homotopy can be extended continuously to the union  $\sigma \cup \tau^+$ , by defining it as the identity on the simplex  $\sigma$ . This finally provides a deformation of  $\sigma \cup \tau^+$  onto  $\sigma \cup \tau = \sigma$ .

(ii) In the second case, suppose that  $\sigma \in \text{dom } \mathcal{V} \setminus \text{Fix } \mathcal{V}$ , and let  $\tau \in \mathcal{T}^\varepsilon(x)$  be arbitrary. Then an inspection of (17) shows that  $F_\sigma(x) = C_\sigma$ . Note that  $\sigma = \sigma^-$  is a facet of the simplex  $\sigma^+$ . Due to the definition of  $\mathcal{T}^\varepsilon(x)$ , the simplex  $\tau$  is a proper face of  $\sigma = \sigma^-$ , and therefore also of  $\sigma^+$ . This implies both  $\tau \subset C_\sigma$  and the identity  $F_\tau(x) = A_\tau$ . Since  $\tau^+ \notin \text{Cl } \sigma$ , we have  $A_\tau \cap \sigma = \tau$  and  $C_\sigma \cap A_\tau = \tau$ , and we can conclude as in (i) that  $F_\sigma(x) \cup F_\tau(x) = C_\sigma \cup A_\tau$  is contractible. Moreover, the homotopy equivalences  $h_\sigma(\cdot, s)$  of  $C_\sigma$  with  $\sigma$ , and  $h_\tau(\cdot, s)$  of  $A_\tau$  with  $\tau$ , defined as in Lemma 4.8, are the identity on  $C_\sigma \cap A_\tau = \tau$ , so they can be concatenated to provide a homotopy equivalence between  $C_\sigma \cup A_\tau$  and  $\sigma \cup \tau = \sigma$ .

(iii) Suppose now that  $\sigma \in \text{im } \mathcal{V} \setminus \text{Fix } \mathcal{V}$ , and let  $\tau \in \mathcal{T}^\varepsilon(x)$  be arbitrary. Then  $\sigma^-$  is a facet of  $\sigma = \sigma^+$  and we have  $F_\sigma(x) = B_\sigma$ . If  $\tau = \sigma^-$ , then the choice of  $\tau$  and  $\sigma$  would immediately imply  $\tau^+ = \sigma^+ = \sigma$ , in contrast to  $\tau^+ \notin \text{Cl } \sigma$ . Thus, we have  $\tau \neq \sigma^-$ , and since  $\tau$  is a proper face of  $\sigma$  one obtains  $F_\tau(x) = A_\tau$ . According to  $\tau^+ \notin \text{Cl } \sigma$  the identity  $B_\sigma \cap A_\tau = \tau$  holds, and we can conclude as before that the union  $F_\sigma(x) \cup F_\tau(x) = B_\sigma \cup A_\tau$  is contractible. Moreover, the homotopy expansion  $h_\sigma(\cdot, s)$  of  $B_\sigma$  onto  $\sigma$  and the contraction  $h_\tau(\cdot, s)$  of  $A_\tau$  onto  $\tau$  as guaranteed by Lemma 4.8 are the identity on  $B_\sigma \cap A_\tau = \tau$ , so they can be concatenated to provide a homotopy equivalence between  $B_\sigma \cup A_\tau$  and  $\sigma \cup \tau = \sigma$ .

(iv) It remains to be shown that the deformations constructed for different simplices  $\tau \in \mathcal{T}^\varepsilon(x)$  can be made independently or concatenated, so as to produce a deformation of the image  $F(x)$  onto the simplex  $\sigma$ .

For this, let  $\tau_1, \tau_2 \in \mathcal{T}^\varepsilon(x)$  be two faces of  $\sigma$  with  $\tau_1 \neq \tau_2$ . According to the definition of the set  $\mathcal{T}^\varepsilon(x)$  and parts (i) through (iii) of the present proof, we then have both  $\tau_i \in \text{dom } \mathcal{V}$  and  $F_{\tau_i}(x) = A_{\tau_i}$  for  $i = 1, 2$ .

Since  $\tau_1 \neq \tau_2$ , we have  $\tau_1^+ \neq \tau_2^+$ . Due to (16) this implies  $A_{\tau_i} \cap |\text{Bd } \tau_i^+| = \tau_i$ . It then follows that  $A_{\tau_1} \cap A_{\tau_2} \subset \tau_1 \cap \tau_2$ . If this set is empty, the discussed deformations can be made independently. If it is non-empty, both deformations are the identity on the intersection, so they can be concatenated.  $\square$

We will see in the next section that the dynamics of the above-defined multivalued map  $F$  does indeed mimic the dynamics of the combinatorial vector field  $\mathcal{V}$ . Note, however, that in all our previous illustrations we interpreted the discrete vector field dynamics through induced flows. To make this final connection, we close the present section by showing that on a homological level, the map  $F$  is indeed a flow-type map. For this, recall that for any  $x \in X$  the *simplicial star*

of  $x$  in  $X$  is defined as

$$\text{St}(x) := \{\sigma \in \mathcal{X} \mid x \in \sigma\} \subset \mathcal{X} ,$$

and the *star* of  $x$  is the subset of  $X = |\mathcal{X}|$  given by

$$\text{st}(x) := |\text{St}(x)| \subset X .$$

Note that the star of  $x$  is closed in  $X$ , since simplices are assumed to be closed. We first need the following simple lemma.

**Lemma 4.13.** *Let  $\mathcal{X}$  denote a simplicial complex, let  $\mathcal{V}$  be a combinatorial vector field on  $\mathcal{X}$  in the sense of Definition 3.1, and assume that (15) holds. Furthermore, let  $F : X \rightrightarrows X$  be defined as in (18). Define a new map  $G : X \rightrightarrows X$  by*

$$G(x) := \text{st}(x) \cup F(x) \quad \text{for every } x \in X .$$

*Then  $G$  is strongly upper semicontinuous with non-empty contractible values.*

**Proof:** It follows from [1], applied in the context of simplicial grids, that the map  $x \mapsto \text{st}(x)$  is strongly upper semicontinuous, and one can easily see that it has star-shaped values, which are therefore contractible. Moreover, since the union of two strongly upper semicontinuous maps is again strongly upper semicontinuous, the map  $G$  is as well. Finally, establishing the contractibility of the images  $G(x)$  can be done similar to (i) in the proof of Theorem 4.12. More precisely, let  $x \in X$  be arbitrary and let  $\tau \in \mathcal{X}^\varepsilon(x)$ . Then

$$\tau \subset \sigma_{\max}^\varepsilon(x) \subset \text{st}(x) .$$

The only case in which  $F_\tau(x)$  is not a subset of  $\text{st}(x)$  occurs if  $\tau \subset \text{bd st}(x) \cap \text{dom } \mathcal{V}$  and  $\mathcal{V}(\tau)$  points outside of  $\text{st}(x)$ , i.e., if  $\tau = \tau^-$  and  $\tau^+ \notin \text{St}(x)$ . In this case we have  $F_\tau(x) = A_\tau$ . Since  $\text{st}(x)$  is closed, it then follows that

$$\text{st}(x) \cap F_\tau(x) = \text{st}(x) \cap A_\tau = \tau .$$

Hence, the same homotopy argument as in the proof of Theorem 4.12 can be used to show that  $\text{st}(x) \cup F_\tau(x)$  can be homotopically deformed to  $\text{st}(x)$ . As in part (iv) in the proof of Theorem 4.12 one can show that this deformation can be carried out independently for any  $\tau$  such that  $F_\tau(x)$  is not contained in  $\text{st}(x)$ . This completes the proof of the lemma.  $\square$

After these preparations we now turn our attention to the final result of this section, which shows that on a homological level, the multivalued map  $F$  is of flow type.

**Theorem 4.14.** *Let  $\mathcal{X}$  denote a simplicial complex, let  $\mathcal{V}$  be a combinatorial vector field on  $\mathcal{X}$  in the sense of Definition 3.1, and assume that (15) holds. Furthermore, let  $F : X \rightrightarrows X$  be defined as in (18). Then for the maps induced by homology we have  $H_*(F) = H_*(\text{id})$ , where  $\text{id} : X \rightarrow X$  is the identity map.*

**Proof:** According to Lemma 4.13, both the identity map  $id$  and the multivalued map  $F$  are submaps of  $G$ , i.e., we have

$$x \in G(x) \quad \text{and} \quad F(x) \subset G(x) \quad \text{for all} \quad x \in X .$$

Standard results from the theory of multivalued mappings then imply the identity  $H_*(id) = H_*(G) = H_*(F)$ . For example, this equality can be established by arguments similar to the ones used in the proof of [1, Corollary 4.8].  $\square$

## 5. ISOLATING BLOCKS FOR THE MULTIVALUED MAP

In this section we show that for every isolated invariant set of a given combinatorial vector field  $\mathcal{V}$  on a simplicial complex  $\mathcal{X}$ , we can find an associated isolating block for the multivalued map  $F$  which was constructed in the last section. In the course of this, it will also prove to be illuminating to establish a connection between solutions of the combinatorial flow  $\Pi_{\mathcal{V}}$  and solutions of the discrete dynamical system generated by  $F$ .

We begin by adapting some terminology for discrete multivalued dynamical systems from [6, 7] to the specific framework of the point-to-set map  $F : X \rightrightarrows X$  defined in Section 4. A *solution* of such a map  $F$  is a partial function  $\varphi : \mathbb{Z} \dashrightarrow X$  whose domain is an interval in  $\mathbb{Z}$  and such that  $\varphi(i+1) \in F(\varphi(i))$  for all  $i \in \text{dom } \varphi$  with  $i+1 \in \text{dom } \varphi$ . In the dynamical systems literature, solutions are also often referred to as *trajectories* or *orbits*, and we will also adopt this notation.

Before getting deeper into the notion of isolated invariant sets for multivalued maps, we first have to address the question of whether solutions of the combinatorial flow  $\Pi_{\mathcal{V}}$  give rise to corresponding orbits of the multivalued map  $F$ , and vice versa. Referring back to the construction of the map  $F$  in the last section, one can easily see that the open cells  $\langle \sigma \rangle_{\varepsilon}$  for  $\sigma \in \mathcal{X}$  are in one-to-one correspondence with the simplices of  $\mathcal{X}$ . Recall also that only for the points  $x$  in these open cells the set  $\mathcal{X}^{\varepsilon}(x)$  contains exactly one simplex, which is given by  $\sigma_{max}^{\varepsilon}(x)$ . Therefore the definition of  $F(x)$  for such points  $x$  most closely follows the dynamics of the combinatorial vector field  $\mathcal{V}$ , as it avoids nonempty sets  $\mathcal{T}^{\varepsilon}(x)$ . Yet, as the following lemma makes clear, the analogy is not perfect. Note that this result is formulated for general solutions of  $F$ , whose points do not necessarily need to lie in the open cells  $\langle \sigma \rangle_{\varepsilon}$ .

**Lemma 5.1.** *Let  $\mathcal{X}$  denote a simplicial complex, let  $\mathcal{V}$  be a combinatorial vector field on  $\mathcal{X}$  in the sense of Definition 3.1, and assume that (15) holds. Furthermore, let  $F : X \rightrightarrows X$  be defined as in (18), and let  $\varphi : \mathbb{Z} \dashrightarrow X$  be an orbit of  $F$ . Finally, for all  $k \in \text{dom } \varphi$  define*

$$\sigma_k = \sigma_{max}^{\varepsilon}(\varphi(k)) .$$

*Then the dimensions of the simplices  $\sigma_k$  are monotone decreasing with  $k$ , i.e., we have  $\dim \sigma_{k+1} \leq \dim \sigma_k$  for all  $k \in \text{dom } \varphi$  with  $k+1 \in \text{dom } \varphi$ .*

**Proof:** Let  $x \in |\mathcal{X}|$  be arbitrary, but fixed. In view of Lemma 4.10 it suffices to show that for every  $\sigma \in \{\sigma_{max}^{\varepsilon}(x)\} \cup \mathcal{T}^{\varepsilon}(x)$  and every  $y \in F_{\sigma}(x)$ , the point  $y$

satisfies  $\dim \sigma_{max}^\varepsilon(y) \leq \dim \sigma_{max}^\varepsilon(x)$ , which in turn follows immediately if the point  $y$  is contained in a simplex  $\kappa \in \mathcal{X}$  with  $\dim \kappa \leq \dim \sigma_{max}^\varepsilon(x)$ .

If  $\sigma \in \mathcal{T}^\varepsilon(x)$ , then we automatically have  $\dim \sigma < \dim \sigma_{max}^\varepsilon(x)$ , and the claim follows from  $y \in F_\sigma(x) \subset \sigma^+$ . Assume now that we have  $\sigma = \sigma_{max}^\varepsilon(x)$ , and that in addition  $\sigma = \sigma_{max}^\varepsilon(x)^+$ . Then (17) implies  $F_\sigma(x) = B_\sigma \subset \sigma^+ = \sigma$  or  $F_\sigma(x) = \sigma$ , i.e., in both cases we have  $y \in \sigma$ , and the claim follows again.

It remains to consider the case  $\sigma = \sigma_{max}^\varepsilon(x) = \sigma_{max}^\varepsilon(x)^- \neq \sigma_{max}^\varepsilon(x)^+$ , in which we have  $F_\sigma(x) = C_\sigma$ . This time, it is possible that the smallest simplex which contains  $y \in F_\sigma(x)$  is given by  $\sigma_{max}^\varepsilon(x)^+$ . Note, however, that at least one of the barycentric coordinates of  $y$  with respect to the vertices of  $\sigma_{max}^\varepsilon(x)^+$  is bounded above by  $\gamma < \varepsilon$ . In other words, the  $\varepsilon$ -signature of  $y$  can only be nonnegative at at most  $\dim \sigma_{max}^\varepsilon(x)^+ - 1 = \dim \sigma_{max}^\varepsilon(x)$  vertices, and therefore the simplex  $\sigma_{max}^\varepsilon(y)$  has at most dimension  $\dim \sigma_{max}^\varepsilon(x)$ .  $\square$

The implications of the above lemma are most striking if we assume in addition that the solution  $\varphi$  satisfies  $\varphi(k) \in \langle \sigma_k \rangle_\varepsilon$  for all  $k \in \text{dom } \varphi$ , i.e., that the whole orbit lies in the union of all open  $\varepsilon$ -cells. Suppose further that  $\sigma_k$  is the source of an arrow of  $\mathcal{V}$ . Then the next simplex  $\sigma_{k+1}$  in the sequence cannot be contained in the image of  $\sigma_k$  under the combinatorial flow  $\Pi_{\mathcal{V}}$ , which is given by  $\Pi_{\mathcal{V}}(\sigma_k) = \{\sigma_k^+\}$ , since of course the identity  $\dim \sigma_k^+ = 1 + \dim \sigma_k$  has to hold. Rather, one can easily see that  $\sigma_{k+1} \in \Pi_{\mathcal{V}}(\sigma_k^+)$ .

This discussion shows that even for solutions  $\varphi$  of  $F$  which are contained only in the open  $\varepsilon$ -cells, the sequence of associated simplices  $\sigma_k$  does not generally form a solution of the combinatorial flow  $\Pi_{\mathcal{V}}$ , since for simplices  $\sigma_k \in \text{dom } \mathcal{V} \setminus \text{Fix } \mathcal{V}$  the arrow targets  $\sigma_k^+$  are missing in the sequence. However, if we extend the sequence  $(\sigma_k)_{k \in \text{dom } \varphi}$  by these missing targets, then we do in fact obtain a solution for  $\Pi_{\mathcal{V}}$ . This reduction or extension of the simplex sequence is now defined in more detail.

**Definition 5.2.** *Let  $\mathcal{X}$  denote a simplicial complex, and let  $\mathcal{V}$  be a combinatorial vector field on  $\mathcal{X}$  in the sense of Definition 3.1. Then we define the following two notions:*

- (a) *Let  $\varrho : \mathbb{Z} \rightarrow \mathcal{X}$  denote a full solution of the combinatorial flow  $\Pi_{\mathcal{V}}$ . Then the reduced solution  $\varrho^* : \mathbb{Z} \rightarrow \mathcal{X}$  is obtained from  $\varrho$  by removing  $\varrho(k+1)$  whenever  $\varrho(k+1)$  is the target of an arrow of  $\mathcal{V}$  whose source is  $\varrho(k)$ .*
- (b) *Conversely, let  $\varrho^* : \mathbb{Z} \rightarrow \mathcal{X}$  denote an arbitrary sequence of simplices in  $\mathcal{X}$ . Then its arrowhead extension  $\varrho : \mathbb{Z} \rightarrow \mathcal{X}$  is defined as follows. If  $\varrho^*(k) \in \text{dom } \mathcal{V} \setminus \text{Fix } \mathcal{V}$  and if  $\varrho^*(k+1) \neq \varrho^*(k)^+$ , then we insert  $\varrho^*(k)^+$  between  $\varrho^*(k)$  and  $\varrho^*(k+1)$ . In other words, the arrowhead extension  $\varrho$  is obtained from  $\varrho^*$  by inserting missing targets of arrows.*

Using the above-defined modifications of simplex sequences, we can establish a first correspondence between solutions of the combinatorial flow  $\Pi_{\mathcal{V}}$  and certain, in some sense generic, orbits of the multivalued map  $F$ . This is the subject of the

next result, which for the sake of presentation is only defined for full solutions. It can easily be modified for solutions defined on finite  $\mathbb{Z}$ -intervals, and this is left to the reader.

**Theorem 5.3.** *Let  $\mathcal{X}$  denote a simplicial complex, let  $\mathcal{V}$  be a combinatorial vector field on  $\mathcal{X}$  in the sense of Definition 3.1, and assume that (15) holds. Furthermore, let  $F : X \rightrightarrows X$  be defined as in (18), and let*

$$X^\varepsilon := \bigcup_{\sigma \in \mathcal{X}} \langle \sigma \rangle_\varepsilon \subset X$$

denote the union of all open  $\varepsilon$ -cells of  $X$ . Then the following hold.

- (a) Let  $\varrho : \mathbb{Z} \rightarrow \mathcal{X}$  denote a full solution of the combinatorial flow  $\Pi_{\mathcal{V}}$ . Furthermore, let  $\varrho^* : \mathbb{Z} \rightarrow \mathcal{X}$  denote the reduced solution as in Definition 5.2(a). Then there is a function  $\varphi : \mathbb{Z} \rightarrow X^\varepsilon$  such that for  $k \in \mathbb{Z}$  we have

$$\varphi(k+1) \in F(\varphi(k)) \quad \text{and} \quad \varphi(k) \in \langle \varrho^*(k) \rangle_\varepsilon.$$

In other words,  $\varphi$  is an orbit of  $F$  which follows the dynamics of the combinatorial simplicial solution  $\varrho$  after removing arrowheads.

- (b) Conversely, let  $\varphi : \mathbb{Z} \rightarrow X^\varepsilon$  denote a full solution of  $F$  which is completely contained in  $X^\varepsilon$ . Let  $\varrho^*(k) = \sigma_{\max}^\varepsilon(\varphi(k))$  for  $k \in \mathbb{Z}$ , and let  $\varrho : \mathbb{Z} \rightarrow \mathcal{X}$  denote the arrowhead extension of  $\varrho^*$  as in Definition 5.2(b). Then  $\varrho$  is a solution of the combinatorial flow  $\Pi_{\mathcal{V}}$ .

**Proof:** (a) Note first that all points  $y \in \langle \varrho^*(k) \rangle_\varepsilon$  have the same image  $F(y)$ , since for all of these points we have  $\mathcal{X}^\varepsilon(y) = \{\varrho^*(k)\}$ . Thus, in order to construct a solution of  $F$  we only need to make sure that we can choose  $\varphi(k) = y \in \langle \varrho^*(k) \rangle_\varepsilon$  in such a way that  $y \in F(x)$  for  $x \in \langle \varrho^*(k-1) \rangle_\varepsilon$ .

To begin, let  $\sigma_k = \varrho^*(k)$  and  $\sigma_{k-1} = \varrho^*(k-1)$ . If  $\sigma_{k-1}$  and  $\sigma_k$  were consecutive simplices in the original solution  $\varrho$ , then either  $\sigma_{k-1} \in \text{Fix } \mathcal{V}$  or  $\sigma_{k-1} \in \text{im } \mathcal{V} \setminus \text{Fix } \mathcal{V}$ . In the first case, the inclusion  $\sigma_k \in \Pi_{\mathcal{V}}(\sigma_{k-1}) = \text{Cl } \sigma_{k-1}$  implies that  $\sigma_k$  is a face of  $\sigma_{k-1}$ , and the definition of  $F$  shows that  $F(x) = \sigma_{k-1}$  for every  $x \in \langle \sigma_{k-1} \rangle_\varepsilon$ . Thus, we can pick any  $\varphi(k) \in \sigma_{k-1} \cap \langle \sigma_k \rangle_\varepsilon$ , for example the barycenter of  $\sigma_k$ . If we have  $\sigma_{k-1} \in \text{im } \mathcal{V} \setminus \text{Fix } \mathcal{V}$ , then  $\sigma_k \in \Pi_{\mathcal{V}}(\sigma_{k-1}) = \text{Cl Bd } \sigma_{k-1} \setminus \{\mathcal{V}^{-1}(\sigma_{k-1})\}$ , as well as  $F(x) = B_{\sigma_{k-1}}$  for  $x \in \langle \sigma_{k-1} \rangle_\varepsilon$ . This shows that  $\sigma_k \subset B_{\sigma_{k-1}}$ , and we can pick again  $\varphi(k) \in B_{\sigma_{k-1}} \cap \langle \sigma_k \rangle_\varepsilon$  as the barycenter of  $\sigma_k$ .

Suppose finally that  $\sigma_{k-1}$  and  $\sigma_k$  were not consecutive simplices in the original solution  $\varrho$ . Then  $\sigma_{k-1} \in \text{dom } \mathcal{V} \setminus \text{Fix } \mathcal{V}$  and the arrowhead  $\sigma_{k-1}^+$  was removed between the two simplices. In this case,  $\sigma_k \in \Pi_{\mathcal{V}}(\sigma_{k-1}^+) = \text{Cl Bd } \sigma_{k-1}^+ \setminus \{\sigma_{k-1}\}$ . In addition, the definition of  $F$  shows that  $F(x) = C_{\sigma_{k-1}}$  for  $x \in \langle \sigma_{k-1} \rangle_\varepsilon$ . According to  $C_{\sigma_{k-1}} \cap \langle \sigma_k \rangle_\varepsilon \neq \emptyset$ , and we can choose  $\varphi(k)$  as any point in this intersection.

(b) Let  $\sigma_k = \varrho^*(k)$  and  $\sigma_{k-1} = \varrho^*(k-1)$ . If  $\sigma_{k-1} \in \text{Fix } \mathcal{V}$  or  $\sigma_{k-1} \in \text{im } \mathcal{V} \setminus \text{Fix } \mathcal{V}$ , then the definition of  $F$  in combination with  $\varphi(k-1) \in X^\varepsilon$  implies  $\varphi(k) \in \sigma_{k-1}$ , since we have either  $F(\varphi(k-1)) = \sigma_{k-1}$  or  $F(\varphi(k-1)) = B_{\sigma_{k-1}}$ . Due to Lemma 4.2 the simplex  $\sigma_k$  is a face of the smallest simplex which contains  $\varphi(k)$ , and so  $\sigma_k$

has to be a face of  $\sigma_{k-1}$ . If  $\sigma_{k-1} \in \text{Fix } \mathcal{V}$  this implies  $\sigma_k \in \Pi_{\mathcal{V}}(\sigma_{k-1})$ . If, on the other hand, we have  $\sigma_{k-1} \in \text{im } \mathcal{V} \setminus \text{Fix } \mathcal{V}$ , then the definition of  $B_{\sigma_{k-1}}$  and  $\varepsilon > \gamma$  show that  $\sigma_k \neq \sigma_{k-1}$ , as well as  $\sigma_k \neq \sigma_{k-1}^-$ . Thus,  $\sigma_k \in \text{Cl Bd } \sigma_{k-1} \setminus \{\sigma_{k-1}^-\}$ , and therefore  $\sigma_k \in \Pi_{\mathcal{V}}(\sigma_{k-1})$ .

Finally, suppose that  $\sigma_{k-1} \in \text{dom } \mathcal{V} \setminus \text{Fix } \mathcal{V}$ . Then the definition of the map  $F$  and  $\varphi(k-1) \in X^\varepsilon$  imply  $\varphi(k) \in C_{\sigma_{k-1}}$ , and arguments similar to the ones of the last paragraph show that  $\sigma_k \in \Pi_{\mathcal{V}}(\sigma_{k-1}^+)$ . The result now follows since the arrowhead  $\sigma_{k-1}^+$  had been inserted into the simplex sequence  $\varrho^*$ .  $\square$

While the formulation of the above theorem is quite technical, in practice the correspondence between simplicial solutions and orbits for the multivalued map can easily be obtained. To see this, we take another look at the combinatorial vector field from Example 3.2, see also Figure 3. Consider a solution  $\varrho : \mathbb{Z} \rightarrow \mathcal{X}$  of the combinatorial flow  $\Pi_{\mathcal{V}}$  which contains the segment

$$\dots \rightarrow \omega \xrightarrow{a} \mathcal{V}(\omega) \rightarrow \sigma_1 \xrightarrow{a} \mathcal{V}(\sigma_1) \rightarrow \dots \rightarrow \sigma_{10} \xrightarrow{a} \mathcal{V}(\sigma_{10}) \rightarrow \omega \xrightarrow{a} \mathcal{V}(\omega) \rightarrow \dots ,$$

where we indicate arrows of  $\mathcal{V}$  by the symbol  $\xrightarrow{a}$ . In the reduced simplex sequence  $\varrho^*$  this segment will be replaced by

$$\dots \rightarrow \omega \rightarrow \sigma_1 \rightarrow \dots \rightarrow \sigma_{10} \rightarrow \omega \rightarrow \dots ,$$

and the corresponding sequence of points in  $X^\varepsilon$  is taken from the associated  $\varepsilon$ -cells. In other words, arrows of  $\mathcal{V}$  have to be considered as a ‘‘unit’’, and each point in the solution of  $F$  corresponds to one of these units.

Theorem 5.3 provides a straightforward correspondence of orbits in the discrete simplicial and the multivalued setting, as long as the solution of  $F$  stays in the open  $\varepsilon$ -cells  $X^\varepsilon$ . It is natural to wonder whether arbitrary solutions  $\varphi : \mathbb{Z} \rightarrow X$  of the multivalued map  $F$  give rise to corresponding solutions of the combinatorial flow  $\Pi_{\mathcal{V}}$ . The positive answer is given in the following result.

**Theorem 5.4.** *Let  $\mathcal{X}$  denote a simplicial complex, let  $\mathcal{V}$  be a combinatorial vector field on  $\mathcal{X}$  in the sense of Definition 3.1, and assume that (15) holds. Furthermore, let  $F : X \rightrightarrows X$  be defined as in (18), and let  $\varphi : \mathbb{Z} \rightarrow X$  denote an arbitrary full solution of the multivalued map  $F$ . Finally, let  $\varrho^*(k) = \sigma_{\max}^\varepsilon(\varphi(k))$  for  $k \in \mathbb{Z}$ , and extend this sequence of simplices in the following way:*

- (1) *For every  $k \in \mathbb{Z}$  for which  $\varphi(k) \notin \varrho^*(k-1)^+$ , choose a face  $\tau \subset \varrho^*(k-1)$  such that  $\varphi(k) \in \text{Cl } \tau^+ \setminus \{\tau\}$ , and then insert  $\tau$  between  $\varrho^*(k-1)$  and  $\varrho^*(k)$ .*
- (2) *Let  $\varrho : \mathbb{Z} \rightarrow \mathcal{X}$  denote the arrowhead extension of the sequence created in (1), according to Definition 5.2(b).*

*Then the simplex sequence  $\varrho : \mathbb{Z} \rightarrow \mathcal{X}$  is a solution of the combinatorial flow  $\Pi_{\mathcal{V}}$ .*

**Proof:** For arbitrary  $k \in \mathbb{Z}$  we use the abbreviations  $\sigma_k = \varrho^*(k)$  and  $x_k = \varphi(k)$ . Assume now that  $k \in \mathbb{Z}$  is a fixed index with  $x_k \in \sigma_{k-1}^+$ . Then no simplex is inserted between  $\sigma_{k-1}$  and  $\sigma_k$  in step (1), and according to Lemma 4.10 we necessarily have  $x_k \in F_{\sigma_{k-1}}(x_{k-1})$ . Thus, it is possible to proceed as in the proof

of Theorem 5.3(b) to show that together with the potential extension in step (2), the extended sequence  $\varrho$  is indeed a solution of the combinatorial flow  $\Pi_{\mathcal{V}}$  between the two simplices  $\sigma_{k-1}$  and  $\sigma_k$ .

In order to complete the proof we now assume  $x_k \notin \sigma_{k-1}^+$  for some  $k \in \mathbb{Z}$ . Then Lemma 4.10 shows that  $x_k \in A_{\tau} \setminus \tau$  for a face  $\tau$  of  $\sigma_{k-1}$  as in step (1). If we further suppose  $\sigma_{k-1} \in \text{Fix } \mathcal{V}$  or  $\sigma_{k-1} \in \text{im } \mathcal{V} \setminus \text{Fix } \mathcal{V}$ , the steps (1) and (2) lead to the possible extensions

$$\sigma_{k-1} \rightarrow \sigma_k, \quad \sigma_{k-1} \rightarrow \tau \xrightarrow{a} \sigma_k, \quad \text{or} \quad \sigma_{k-1} \rightarrow \tau \xrightarrow{a} \tau^+ \rightarrow \sigma_k,$$

depending on whether  $\sigma_k = \tau$ ,  $\sigma_k = \tau^+$ , or  $\sigma_k \in \text{Cl } \tau^+ \setminus \{\tau, \tau^+\}$ , respectively. All three sequences can easily be verified as valid solutions of the combinatorial flow  $\Pi_{\mathcal{V}}$ . On the other hand, if we have  $\sigma_{k-1} \in \text{dom } \mathcal{V} \setminus \text{Fix } \mathcal{V}$ , then steps (1) and (2) lead to the extensions

$$\sigma_{k-1} \xrightarrow{a} \sigma_{k-1}^+ \rightarrow \sigma_k, \quad \sigma_{k-1} \xrightarrow{a} \sigma_{k-1}^+ \rightarrow \tau \xrightarrow{a} \sigma_k, \quad \sigma_{k-1} \xrightarrow{a} \sigma_{k-1}^+ \rightarrow \tau \xrightarrow{a} \tau^+ \rightarrow \sigma_k,$$

for  $\sigma_k = \tau$ ,  $\sigma_k = \tau^+$ , or  $\sigma_k \in \text{Cl } \tau^+ \setminus \{\tau, \tau^+\}$ , respectively. Since also these sequences are solutions of  $\Pi_{\mathcal{V}}$ , this completes the proof of the theorem.  $\square$

At first glance the construction of the simplicial orbit in Theorem 5.4 seems to be somewhat convoluted. However, we believe that the above approach does in fact reflect the correct underlying dynamics. To illustrate this, consider the simplicial complex  $\mathcal{X}$  and the combinatorial vector field  $\mathcal{V}$  shown in the left image of Figure 10. The image on the right contains a point  $x_{k-1} = \varphi(k-1)$ , together with the next iterate  $x_k = \varphi(k) \in F(x_{k-1})$  of some solution  $\varphi$  for the multivalued map  $F$ , while the image  $F(x_{k-1})$  is shown in red as in Figure 9. Due to Theorem 5.4 the transition  $x_{k-1} \rightarrow x_k$  in the multivalued discrete dynamical systems setting is represented by the solution segment  $\sigma_{k-1} \xrightarrow{a} \sigma_{k-1}^+ \rightarrow \tau \xrightarrow{a} \tau^+ \rightarrow \sigma_k$  in the simplicial setting. Since the original point  $x_{k-1}$  is actually contained in the open cell  $\overset{\circ}{\sigma}_{k-1}$ , the continuous flow indicated by the arrows of the combinatorial vector field  $\mathcal{V}$  should drive the point  $x_{k-1}$  into the interior of  $\sigma_{k-1}^+$ , and from there through  $\tau$  and  $\tau^+$  towards the final image  $x_k$  which lies in the  $\varepsilon$ -cell associated with  $\sigma_k$ , i.e., with the rest point on the upper left corner of the simplicial complex. This is exactly reflected by the above orbit segment.

After having established the orbit correspondence between the combinatorial vector field  $\mathcal{V}$  and the multivalued map  $F$ , we now turn our attention to the question of isolating blocks. A set  $S_F \subset X$  is called *invariant* under the map  $F$ , if for each point  $x \in S_F$  there exists a *full solution*  $\varphi : \mathbb{Z} \rightarrow S_F$  of the map  $F$  which passes through  $x$ , i.e., for which  $\varphi(0) = x$ . Let  $N$  be a compact subset of  $X$ . Then the set  $N$  is called an *isolating neighborhood* for the map  $F$ , if its *invariant part*, which is defined by

$$\text{inv } N := \{x \in N \mid \text{there exists a full solution } \varphi : \mathbb{Z} \rightarrow N \text{ through } x\},$$



complex  $\mathcal{X}$ , and let  $\mathcal{S}$  be an isolated invariant set for the combinatorial flow  $\Pi_{\mathcal{V}}$  as in Definition 3.4. Now consider the multivalued map  $F : X \rightrightarrows X$  on  $X = |\mathcal{X}|$  which was defined in (18). We are interested in identifying an associated isolating neighborhood  $N$  for  $F$  which gives rise to equivalent dynamics in the sense of the Conley index. After all our preparations, the set  $N$  can easily be defined. For this, choose parameters

$$(23) \quad 0 < \delta < \gamma < \varepsilon < \frac{1}{d+1} .$$

Recall that in Section 4 the parameters  $\gamma$  and  $\varepsilon$  were used to construct a cell decomposition of  $X$ , as well as the multivalued map  $F$ . More precisely, the parameter  $\varepsilon$  was used for building suitable regions in the domain of  $F$ , while  $\gamma$  was used to construct the images of  $F$ .

The new parameter  $\delta$  in (23) will be used to construct the isolating neighborhood  $N$  for the multivalued map  $F$ . For this, let  $\mathcal{S}$  be an isolated invariant set for  $\Pi_{\mathcal{V}}$  in the sense of Definition 3.4. Then we define

$$(24) \quad N := \bigcup_{\sigma \in \mathcal{S}} \text{cl} \langle \sigma \rangle_{\delta} .$$

In the remainder of this section, we show that this set  $N$  is indeed an isolating block. Before presenting our main theorem, we need to establish two auxiliary results.

**Lemma 5.5.** *Let  $\mathcal{X}$  denote a simplicial complex, let  $\mathcal{V}$  be a combinatorial vector field on  $\mathcal{X}$  in the sense of Definition 3.1, and assume that (23) holds. Let  $\mathcal{S}$  be an isolated invariant set for  $\Pi_{\mathcal{V}}$  in the sense of Definition 3.4, and let  $N \subset X = |\mathcal{X}|$  be defined as in (24). Finally, let  $F : X \rightrightarrows X$  be defined as in (18). Then for any point  $x \in \text{bd } N$  we have*

$$\mathcal{X}^{\delta}(x) \cap \mathcal{S} \neq \emptyset \quad \text{and} \quad \mathcal{X}^{\delta}(x) \setminus \mathcal{S} \neq \emptyset .$$

**Proof:** Let  $x \in \text{bd } N$  be an arbitrary point. Since the set  $N$  is closed, we have to have  $x \in N$ . According to (24), there exists a simplex  $\sigma \in \mathcal{S}$  with  $x \in \text{cl} \langle \sigma \rangle_{\delta}$ . Corollary 4.6 then implies  $\sigma \in \mathcal{X}^{\delta}(x)$ , and therefore we have  $\sigma \in \mathcal{X}^{\delta}(x) \cap \mathcal{S} \neq \emptyset$ . This establishes the first inequality.

In order to prove the second inequality, note that every neighborhood of  $x$  contains a point which is not contained in  $N$ . At the same time, Lemma 4.7 shows that the set

$$\bigcup_{\sigma \in \mathcal{X}^{\delta}(x)} \text{cl} \langle \sigma \rangle_{\delta}$$

is a neighborhood of  $x$ . Thus, there exists a point  $y \notin N$  and a simplex  $\sigma \in \mathcal{X}^{\delta}(x)$  such that  $y \in \text{cl} \langle \sigma \rangle_{\delta}$ . According to our definition of  $N$  given in (24) this implies  $\sigma \notin \mathcal{S}$ . Hence, the inequality  $\sigma \in \mathcal{X}^{\delta}(x) \setminus \mathcal{S} \neq \emptyset$  has to be satisfied.  $\square$

The second auxiliary result is of a technical nature. However, it will be used in two different contexts in the proof of our main result below.

**Lemma 5.6.** *Let  $\mathcal{X}$  denote a simplicial complex, let  $\mathcal{V}$  be a combinatorial vector field on  $\mathcal{X}$ , assume that (23) holds, and let  $F_\sigma : X \rightrightarrows X$  be defined as in (17). Finally, suppose that  $\sigma \in \mathcal{X}$  and  $\tau \in \mathcal{X}$  are two simplices such that*

$$(F_\sigma(x) \setminus \sigma) \cap \text{cl} \langle \tau \rangle_\delta \neq \emptyset .$$

*Then the inclusion  $\sigma \subset \tau$  has to hold.*

**Proof:** Due to the assumption of the lemma, and the fact that  $F_\sigma(x) \subset \sigma^+$  according to Lemma 4.9, we must have  $\sigma \neq \sigma^+$ . This immediately implies

$$(25) \quad \sigma = \sigma^- \neq \sigma^+ .$$

Now let  $y \in (F_\sigma(x) \setminus \sigma) \cap \text{cl} \langle \tau \rangle_\delta$ . The inclusion  $y \in F_\sigma(x)$  implies  $\sigma \in \mathcal{X}^\varepsilon(x)$ . In addition, (25) implies that either  $F_\sigma(x) = A_\sigma$  or  $F_\sigma(x) = C_\sigma \subset A_\sigma$ . On the other hand, according to  $y \in \text{cl} \langle \tau \rangle_\delta$  Lemma 4.5 implies that the inequalities

$$(26) \quad t_v(y) \geq \delta \quad \text{for all } v \in \tau \quad \text{and} \quad t_v(y) \leq \delta \quad \text{for all } v \notin \tau$$

are satisfied. Because of  $y \in F_\sigma(x) \subset A_\sigma$  and  $y \notin \sigma = \sigma^-$ , the definition of  $A_\sigma$  in (16) implies

$$(27) \quad t_v(y) \geq \gamma \quad \text{for all } v \in \sigma^- = \sigma .$$

Now let  $w \in \sigma$  be an arbitrary vertex, and assume that  $w \notin \tau$ . Then (26) implies the estimate  $t_w(y) \leq \delta$ , while (27) implies  $t_w(y) \geq \gamma > \delta$  due to (23), which is a contradiction. Therefore, our assumption  $w \notin \tau$  was wrong, and  $w \in \tau$  has to be satisfied. This finally implies  $\sigma \subset \tau$ .  $\square$

The following result is the main result of the present section. It shows that the set  $N$  is indeed an isolating block for  $F$ .

**Theorem 5.7.** *Let  $\mathcal{X}$  denote a simplicial complex, let  $\mathcal{V}$  be a combinatorial vector field on  $\mathcal{X}$  in the sense of Definition 3.1, and assume that (23) holds. Let  $\mathcal{S}$  be an isolated invariant set for  $\Pi_\mathcal{V}$  in the sense of Definition 3.4, and let  $N \subset X = |\mathcal{X}|$  be defined as in (24). Finally, let  $F : X \rightrightarrows X$  be defined as in (18). Then the set  $N$  is an isolating block for  $F$ .*

**Proof:** Assume that  $N$  is not an isolating block, i.e., suppose that the implication (22) does not hold. Then there exist points  $x \in \text{bd} N$  and  $y, z \in N$  such that  $x \in F(z)$  and  $y \in F(x)$ . Let  $\tau$  denote the smallest simplex which contains  $x$ , i.e., define

$$\tau := \sigma_{\min}^0(x) = \{v \in \mathcal{X}_0 \mid t_v(x) > 0\} .$$

Due to Lemma 5.5 we can find two simplices  $\sigma_0 \in \mathcal{X}^\delta(x) \setminus \mathcal{S}$  and  $\sigma_1 \in \mathcal{X}^\delta(x) \cap \mathcal{S}$ , and Lemma 4.2 and Corollary 4.6 imply

$$(28) \quad \sigma_{\min}^\delta(x) \subset \sigma_k \subset \sigma_{\max}^\delta(x) \subset \sigma_{\min}^0(x) = \tau \quad \text{for } k = 0, 1 .$$

Finally, choose the simplex  $\omega \in \mathcal{X}$  such that  $x \in F_\omega(z)$ , and the simplex  $\xi \in \mathcal{X}$  such that  $y \in F_\xi(x)$ . The definition of the map  $F_\sigma$  then implies both  $\omega \in \mathcal{X}^\varepsilon(z)$

and  $\xi \in \mathcal{X}^\varepsilon(x)$ . Now Lemma 4.9 implies  $x \in F_\omega(z) \subset \omega^+$ , and since  $\tau$  is the smallest simplex containing  $x$ , we further get

$$(29) \quad \tau \subset \omega^+ .$$

Finally, since  $z \in N$ , there exists a simplex  $\chi \in \mathcal{S}$  such that  $z \in \text{cl}\langle\chi\rangle_\delta$ , and Lemma 4.2 and Corollary 4.6 imply both  $\chi \in \mathcal{X}^\delta(z)$  and  $\sigma_{max}^\varepsilon(z) \subset \sigma_{min}^\delta(z) \subset \chi$ , and together with  $\omega \in \mathcal{X}^\varepsilon(z)$  this yields

$$(30) \quad \omega \subset \chi .$$

We now distinguish the two cases (i)  $\tau \notin \mathcal{S}$  and (ii)  $\tau \in \mathcal{S}$  to arrive at a contradiction.

(i) We begin by assuming that  $\tau \notin \mathcal{S}$ , and further divide our first case by distinguishing the two subcases  $x \in \omega$  and  $x \notin \omega$ .

As our first subcase, suppose that  $x \in \omega$ . Then the definition of  $\tau$  and (30) imply  $\tau \subset \omega \subset \chi$ . Together with (28) this implies the inclusions  $\sigma_1 \subset \tau \subset \chi$ , as well as  $\sigma_1 \in \mathcal{S}$ ,  $\tau \notin \mathcal{S}$ , and  $\chi \in \mathcal{S}$ . Since  $\tau$  is not an element of  $\mathcal{S}$ , but a face of a simplex in  $\mathcal{S}$ , it has to be an element of the closed exit set  $\text{ex}\mathcal{S}$ . But then according to Definition 3.4, the simplex  $\sigma_1 \subset \tau$  has to be in the exit set as well, which contradicts  $\sigma_1 \in \mathcal{S}$ .

Suppose now that  $x \notin \omega$ . We have already seen that  $x \in F_\omega(z)$ , and according to  $\sigma_1 \in \mathcal{X}^\delta(x)$  we also have  $x \in \text{cl}\langle\sigma_1\rangle_\delta$ . This immediately implies the inclusion

$$x \in (F_\omega(z) \setminus \omega) \cap \text{cl}\langle\sigma_1\rangle_\delta .$$

An application of Lemma 5.6 then implies  $\omega \subset \sigma_1$ . Together with (28) and (29) this yields  $\omega \subset \sigma_1 \subset \tau \subset \omega^+$ . Due to our choices we have  $\sigma_1 \in \mathcal{S}$  and  $\tau \notin \mathcal{S}$ . Since the dimensions of the two simplices  $\omega$  and  $\omega^+$  differ by at most one, the inclusion  $\omega \subset \sigma_1 \subset \tau \subset \omega^+$  implies both  $\omega = \sigma_1$  and  $\tau = \omega^+$ , i.e., we have  $\omega \in \mathcal{S}$  and  $\omega^+ \notin \mathcal{S}$ . This contradicts Lemma 3.6.

(ii) For our second case we assume that  $\tau \in \mathcal{S}$ . According to Lemma 4.2 and (28) we obtain

$$\sigma_{max}^\varepsilon(x) \subset \sigma_{min}^\delta(x) \subset \sigma_0 \subset \sigma_{max}^\delta(x) \subset \tau \quad \text{with} \quad \sigma_0 \notin \mathcal{S} \quad \text{and} \quad \tau \in \mathcal{S} .$$

Thus, the simplex  $\sigma_0$  has to be contained in the exit set  $\text{ex}\mathcal{S}$ , and since  $\mathcal{S}$  is an isolated invariant set, all faces of  $\sigma_0$  have to be in the exit set as well. In particular, due to  $\sigma_{max}^\varepsilon(x) \subset \sigma_0$  all simplices in  $\mathcal{X}^\varepsilon(x)$  are elements of the exit set of  $\mathcal{S}$ . Now recall that earlier we have chosen  $\xi \in \mathcal{X}^\varepsilon(x)$  such that  $y \in F_\xi(x)$ . Together with Lemma 3.6 the above discussion then implies

$$(31) \quad \xi \subset \tau , \quad \text{as well as} \quad \xi \notin \mathcal{S} \quad \text{and} \quad \xi^\pm \notin \mathcal{S} .$$

Since  $y \in N$ , there exists a simplex  $\kappa \in \mathcal{S}$  such that  $y \in \text{cl}\langle\kappa\rangle_\delta$ , and therefore we also have  $\kappa \in \mathcal{X}^\delta(y)$ . Due to  $F_\xi(x) \subset \xi^+$ , we further obtain  $y \in \xi^+$ , i.e., the simplex  $\xi^+$  contains  $y$ . According to its definition,  $\sigma_{min}^0(y)$  is the smallest simplex

which contains  $y$ , and so an application of Lemma 4.2 implies with  $\kappa \in \mathcal{X}^\delta(y)$  the inclusions

$$\kappa \subset \sigma_{max}^\delta(y) \subset \sigma_{min}^0(y) \subset \xi^+ .$$

This gives

$$(32) \quad \kappa \subset \xi^+ , \quad \text{as well as} \quad \kappa \in \mathcal{S} \quad \text{and} \quad \kappa^\pm \in \mathcal{S} .$$

To complete the proof, we distinguish the cases  $y \in \xi$  and  $y \notin \xi$ .

Suppose first that the inclusion  $y \in \xi$  is satisfied. Then arguing as above we must have  $\kappa \subset \sigma_{min}^0(y) \subset \xi$ . This immediately implies together with (31) the inclusions  $\kappa \subset \xi \subset \tau$ , as well as  $\kappa \in \mathcal{S}$ ,  $\xi \notin \mathcal{S}$ , and  $\tau \in \mathcal{S}$ . As in the first subcase of case (i), this contradicts the closedness of the exit set  $\text{ex } \mathcal{S}$ .

Suppose finally that  $y \notin \xi$ . We have already seen that  $y \in F_\xi(x)$  and  $y \in \text{cl } \langle \kappa \rangle_\delta$ , and this implies

$$y \in (F_\xi(x) \setminus \xi) \cap \text{cl } \langle \kappa \rangle_\delta .$$

An application of Lemma 5.6 then implies  $\xi \subset \kappa$ . Together with (31) and (32) this gives the inclusions  $\xi \subset \kappa \subset \xi^+$ , as well as  $\xi \notin \mathcal{S}$ ,  $\kappa \in \mathcal{S}$ , and  $\xi^+ \notin \mathcal{S}$ . Since the dimensions of  $\xi$  and  $\xi^+$  differ by at most one, this leads to a contradiction as there can be no simplex  $\kappa$  which is a proper face of  $\xi^+$ , and which has  $\xi$  as a proper face. This completes the proof of the theorem.  $\square$

## 6. FUTURE WORK

As we mentioned in the introduction, this paper serves as a first step towards establishing formal ties between Forman's combinatorial vector fields and classical dynamical systems theory. These ties are based on equivalent dynamics, and above we have concentrated on the case of isolated invariant sets and associated isolating blocks. Further extensions of these results are in preparation. In particular, we work on extending the ties established in this paper to the case of the Conley index and showing that the combinatorial multivalued dynamical system generated by  $F$  admits a minimal Morse decomposition which corresponds to the Morse decomposition of the combinatorial vector field  $\mathcal{V}$ .

Beyond these natural extensions, it would be interesting to extend our formal correspondence also to the case of continuous-time dynamical systems. In other words, it would be useful to construct a continuous semiflow on the space  $X = |\mathcal{X}|$  with the same one-to-one correspondence of Morse decompositions. Ultimately, however, the question of reversing the ties between continuous flows and combinatorial vector fields is of utmost importance, i.e., continuing the work which was initiated in [8, 10]. Moreover, since the latter paper is formulated for the framework of cubical grids, it would also be interesting to extend our results to complexes other than simplicial ones.

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