EXACT CONTROLLABILITY FOR STRING WITH ATTACHED MASSES

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Abstract: We consider the problem of boundary control for a one dimensional wave equation with N interior point masses. We assume the control is at the left end, and the string is fixed the right end. Singularities in waves are "smoothed" out to one order as they cross a point mass. We show that the reachable set for a L^2 control equals $(L^2 \times H^{-1}) \oplus (H^1 \times L^2) \oplus ... \oplus (H^N \times H^{N-1})$ plus compatibility and boundary conditions. The control problem is reduced to a moment problem, which is then solved using the theory of exponential divided differences in tandem with unique shape and velocity controllability results.

1 Introduction

There has been much interest in so called "hybrid systems" in which the dynamics of elastic systems and possibly rigid structures are related through some form of coupling. The study of controllability and stabilization of such structures has made in a number of works, see [16] and [10] and references therein, also [13]. Networks of strings with attached masses have also been studied by many of authors in the context of inverse problems, see for instance [11] and references therein. The controllability of a string with a single attached mass was considered in [14], [8], [9]. The controllability of a series of Euler-Bernoulli beams with interior attached masses was considered in [18].

In this paper, we consider the controllability of a string with N attached masses. In particular we consider the wave equation on the interval $[0, \ell]$ with N masses $M_j > 0$ attached at the points $a_j,\ j = 1, \ldots, N$, where $0 = a_0 < a_1 < ... < a_N < a_{N+1} = \ell$. This is modeled by

System A_N:

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + q(x)u &= 0,\ t \in (-\infty, T),\ x \in (0, \ell) - \{a_j\}_{j=1}^N, \\
u(x,t) &= 0,\ t \le 0, \\
u(a_j^-, t) &= u(a_j^+, t),\ j = 1, \ldots, N,
\end{aligned}$$

$$
\begin{aligned}
M_j u_{tt}(a_j,t) &= u_x(a_j^+,t) - u_x(a_j^-,t),\\
u(0,t) &= f(t),\\
u(\ell,t) &= 0.
\end{aligned}
$$

We will assume that $f \in L^2(0,T)$ for some $T > 0$, and for convenience we assume throughout that $q \in C^\infty[0,\ell]$. It is known that there exists a unique solution $u^f(x,t)$ to System A_N. We will prove the following regularity result, which shows that the attached masses will mollify transmitted waves. This effect has already been noted in the single mass case in [14].

Proposition 1 *For any $T > 0$, and for $i = 0, 1, 2$,*

$$u^f(x,t) \in C^i([0,T], H^{-i}(0,a_1) \oplus H^{1-i}(a_1,a_2) \oplus ...H^{N-i}(a_N,\ell)).$$

To discuss the reachable set for System A_N, we must first discuss compatibility conditions satisfied by u^f at the masses and at $x = \ell$. These include

$$u^f(a_j^-,t) = u^f(a_j^+,t),\ j \geq 2,\ \text{and}\ u_x^f(a_j^-,t) = u_x^f(a_j^+,t) - M_j[u_{xx}^f(a_j^+,t) - q(a_j)u^f(a_j^+,t)],\ j \geq 3, \tag{1.1}$$

and also compatibility conditions on higher order derivatives provided $u^f(x,t)$ is sufficiently regular. In addition, the wave equation in tandem with $u(\ell^-,t) = 0$ impose higher order conditions on $u(x,t)$ at $x = \ell$. For instance, for $N \geq 3$ we have

$$u_{xx}(\ell^-,t) = 0,\ \forall t > 0. \tag{1.2}$$

Thus for a fixed T, a function ϕ satisfying $\phi(x) = u(x,T)$ must satisfy

$$\phi(a_j^-) = \phi(a_j^+), j \geq 2,\ \phi'(a_j^-) = \phi'(a_j^+) - M_j[\phi_{xx}(a_j^+) - q(a)\phi(a_j^+)]\ j \geq 3,\ \phi(\ell^-,t) = 0,$$

and higher order conditions as necessary. In what follows, we will say that a function $\phi(x)$ satisfies the *Condition C* if it satisfies and all compatibility conditions and boundary conditions at $x = \ell$ that are applicable. Furthermore, similar arguments show that u_t must also satisfy the same compatibility conditions and boundary conditions provided u is sufficiently regular, for instance

$$u_t^f(a_j^-,t) = u_t^f(a_j^+,t),\ j = 3, ..., N,\ \text{and}$$

$$u_{xt}^f(a_j^-,t) = u_{xt}^f(a_j^+,t) - M_j[u_{xxt}^f(a_j^+,t) - q(a)u_t^f(a_j^+,t)],\ j = 4, ..., N,$$

and so on. In addition,the wave equation imposes higher order conditions on $u_t(x,t)$ at $x = \ell$. For instance, for $N \geq 4$ we have

$$u_{xxt}(\ell^-,t) = 0,\ \forall t > 0.$$

We will say that a function $\psi(x)$ satisfies the *Condition C'* if

$$\psi(a_j^-) = \phi(a_j^+), j = 3, ..., N,\ \psi'(a_j^-) = \psi'(a_j^+) - M_j[\psi_{xx}(a_j^+) - q(a)\psi(a^+)]\ j = 4, ..., N,$$

and higher order conditions as necessary, and also satisfies the appropriate boundary conditions at $x = \ell$.

We define the space W_0 by

$$W_0 := \{\phi \in L^2(0,a_1) \times H^1(a_1,a_2) \times ...H^j(a_N,\ell) : \ \phi \text{ satisfies Condition } \ \text{C} \ \}.$$

Similarly,

$$W_{-1} := \{\phi \in H^{-1}(0,a_1) \times L^2(a_1,a_2) \times ...H^{j-1}(a_N,\ell) : \ \phi \text{ satisfies Condition } \ \text{C'} \ \}.$$

We thus have the following

Theorem 1 *For any* $T > 0$,

$$\{(u^f(\cdot,T), u_t^f(\cdot,T) : f \in L^2(0,T)\} \subset W_0 \times W_{-1}.$$

We can now state our main result.

Theorem 2 *Let* $T > 2\ell$ *for* $N \geq 2$, *and* $T \geq 2\ell$ *for* $N = 1$. *Then for any* $(u_0,u_1) \in W_0 \times W_{-1}$, *there exists a control* $f \in L^2(0,T)$ *such that*

$$u(x,T) = u_0(x), \ u_t(x,T) = u_1(x), \ x \in (0,\ell).$$

Furthermore,

$$\|f\|^2_{L^2(0,T)} \asymp \|u_0\|^2_W + \|u_1\|^2_{W_{-1}}.$$

We now give some ideas of the proof of this result. The spectral problem associated to System A_N is

$$\begin{aligned} -u''(x) + q(x)u(x) &= \lambda^2 u(x), \ x \in (0,\ell) - \cup_{j=1}^N a_j, \\ u(0) &= u(\ell) = 0, \\ u(a_j^-) &= u(a_j^+), \\ u'(a_j^+) &= u'(a_j^-) - M_j\lambda^2 u(a_j^+), \ j = 1,\dots,N. \end{aligned}$$

Let the lengths of the subintervals $[a_j, a_{j+1}]$ be given by $\{\ell_j\}_0^N$ We show that the set of all eigenfrequencies $\{\lambda_n\}$ equals the union:

$$\cup_{j=0}^N \{\frac{\pi n}{\ell_j} + O(1/n)\}.$$

We use this to construct an associated family of exponential divided differences, see [3], [4], [5]. Studying the associated generating function and using results developed in [19], [1], [2], we show that the associated family of exponential divided differences, with $(N-1)$ terms deleted, forms a Riesz basis of $L^2(0,2\ell)$. Arguments using Riesz sequences and Riesz bases are then used to solve the moment problem associated to the exact control problem. Key ingredients here are the following unique shape controllability and unique velocity controllability results, which are of independent interest, that hold for $T \leq \ell$ and which allows us to characterize the solvable set in the moment problem as $W_0 \times W_{-1}$. Let

$$W_i^T := \{\phi \in W_i : \phi(t) = 0, \ \forall t \geq T\} \, i = -1, 0.$$

Theorem 3

A) Suppose $T \le \ell$. For any $\phi \in W_0^T$, there exists a unique $f \in L^2(0,T)$ such that $u^f(x,T) = \phi(x)$. Furthermore,

$$\|u^f(\cdot,T)\|_{W_0} \asymp \|f\|_{L^2(0,T)}.$$

B) For $T > \ell$,

$$\{u^f(\cdot,T) : f \in L^2(0,T)\} = W_0.$$

C) Suppose $T \le \ell$. For any $\phi \in W_{-1}^T$, there exists a unique $f \in L^2(0,T)$ such that $u_t^f(x,T) = \phi(x)$. Furthermore,

$$\|u_t^f(\cdot,T)\|_{W_{-1}} \asymp \|f\|_{L^2(0,T)}.$$

D) For $T > \ell$,

$$\{u_t^f(\cdot,T) : f \in L^2(0,T)\} = W_{-1}.$$

In [14] (also see [9]), Hansen and Zuazua prove the conclusion of Theorem 2 for $N = 1$. Their method of proof involves using the theory of characteristics for the constant coefficient wave equation to prove an observability estimate. They then indicate how their results can be extended to the case of variable coefficients. It is not clear that the methods in their paper can easily be extended to our setting. Hansen and Zuazua also study controllability using spectral analysis in the special case where $q = 0$ and $a_1 = \ell/2$, and they are able to given a rather precise spectral characterization of the reachable set. However, as we observe in this paper, it might not be easy to extend this analysis to a more general case. In particular, in the case $q = 0$ and $a_1 = \ell/2$ and $N = 1$, the associated unit norm eigenfunctions have the asymptotics $|\phi_j'(0)| \asymp j$, whereas we shall show that this is not generally the case.

This paper is organized as follows. In section 2 first we prove the regularity results, and give a representation of the solution u that effectively models the propagation, transmission, and reflection of waves. We then discuss the various compatibility conditions that arise. In Section 3, we prove the shape and velocity controllability results. In Section 4, we study the spectral theory associated to System A_N, and prove some Riesz basis results for the associated family of divided differences of exponentials. Then in Section 5, we prove the main result.

2 Existence and uniqueness

We begin with some notation.

Fix $T > 0$. Define $H^j(a,b)$ to be the set of functions in $L^2[a,b]$ whose weak derivatives up to order j are in $L^2[a,b]$. We set the following notation

$$C_*^j = \{f \in C^j(-\infty,T) : f(t) = 0 \text{ if } t \le 0\},$$

$$L_*^2 = \{f \in L^2(-\infty,T) : f(t) = 0 \text{ if } t \le 0\},$$

$$H_*^n = \{f \in H^n(-\infty,T) : f(t) = 0 \text{ if } t \le 0\}.$$

Note that $f \in H_*^n$ implies $f^{(j)}(0) = 0$ for $j = 0,\dots,n-1$, where $f^{(j)}$ denotes the jth derivative. Finally,

$$H_*^{-1} = \{f \in H^{-1}(-\infty,T) : f|_{(-\infty,0)} = 0 \text{ as a distribution. }\}.$$

2.1 Case of single mass

It is instructive to consider first the case of a single mass.

System A

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + q(x)u &= 0,\ t \in (0,T),\ x \in (0,\ell) - \{a\}, && (2.3)\\
u(x,t) &= 0,\ t \le 0, && (2.4)\\
u(a^-,t) &= u(a^+,t), && (2.5)\\
Mu_{tt}(a,t) &= u_x(a^+,t) - u_x(a^-,t), && (2.6)\\
u(0,t) &= f(t), && (2.7)\\
u(\ell,t) &= 0. && (2.8)\\
& && (2.9)
\end{aligned}$$

Following [14], we define a weak solution to System A, using the method of transposition. To discuss the weak solution of System A, it is convenient to parametrize the position of the mass separately from the position of the string. Given $(p,r) \in L^1(0,T;L^2(0,\ell)\oplus\mathbb{R})$, the adjoint system to System A is

$$\begin{aligned}
\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} + q(x)\phi &= p(x,t),\ t \in (0,T),\ x \in (0,a)\cup(a,\ell),\\
\phi(0,t) = \phi(\ell,t) &= 0,\ j = 1,\dots,N,\\
\phi(a^-,t) = \phi(a^+,t) &= z(t),\ t > 0, j = 1,\dots,N,\\
Mz''(t) + \phi_x(a^-,t) - \phi_x(a^+,t) &= r(t),\ t > 0,\\
\phi(x,T) = \phi_t(x,T) &= 0,\ x \in \Omega,\\
z(T) = z'(T) &= 0,
\end{aligned}$$

with the position of the mass given by $z(t)$. This system has a unique solution $(\phi,z) \in C(0,T;H^1(0,a)\oplus H^1(a,\ell)\oplus\mathbb{R})$ such that $(\phi_t,z') \in C([0,T];L^2(0,\ell)\oplus\mathbb{R})$.

In what follows, we let $\tilde h(t)$ be the position of mass in System A. Then $(u,\tilde h) \in L^\infty(0,T;L^2(\Omega)\oplus\mathbb{R})$ is defined to be a weak solution to System A if and only if the following equation holds for all (p,r):

$$\int_0^T \int_0^\ell up\ dxdt\ + \int_0^T \tilde h r\ dt\ = \int_0^T f(t)\phi_x(0^+,t)\ dt. \tag{2.10}$$

Proposition 2 *For $f \in L^2(0,T)$, there exists a unique weak solution $(u(x,t),\tilde h(t))$ to System A such that*

$$(u,\tilde h) \in L^\infty([0,T],L^2(\Omega)\oplus\mathbb{R}). \tag{2.11}$$

Furthermore,

$$\|(u,\tilde h)\|_{L^\infty(0,T;L^2(0,\ell)\oplus\mathbb{R})} \le C\|f\|_{L^2(0,T)}. \tag{2.12}$$

This result is basically well known. For q constant on each interval, the result is proven in [14], where also indications are given on how to extend the proof to the variable coefficient case. The goal of this section is to develop a representation of the solution u, and to prove various properties of u. One important consequence of this representation of u is the following improvement on Proposition 2.

Theorem 4 *For $f \in L^2(0,T)$, there exists a unique weak solution (u,h) to System A such that $h(t-a) = u(a^+,t)$ and*

$$u \in C([0,T], L^2(0,a) \oplus H^1(a,\ell)). \tag{2.13}$$

This result is essentially proven in [14], but the representation of u in this paper is different. In the next subsection we will easily extend the proof of this result to the case of N masses.

Fix $b \in \mathbb{R}$ and let $q \in L^2_{loc}(b,\infty)$. We begin with the representation of the solution u. As a preliminary step, let $D = \{(x,t)|0 < x < t < \infty\}$. Consider the Goursat problem:

$$k_{tt}(x,t) - k_{xx}(x,t) + q(x+b)k(x,t) = 0,\ (x,t) \in D, \tag{2.14}$$

$$k(0,t) = 0,\ k(x,x) = -\frac{1}{2}\int_0^x q(\eta+b)d\eta. \tag{2.15}$$

Proposition 3 *Fix $b \in \mathbf{R}$. For $q \in H^N_{loc}(b,\infty)$, the system (2.14, 2.15) has a unique generalized solution, denoted $k(b^+;x,t)$, such that $k(b^+;\cdot,\cdot) \in C^N(D)$, with*

$$k_x(b^+;\cdot,s), k_s(b^+;\cdot,s), k_x(b^+;x,\cdot), k_s(b^+;x,\cdot) \in H^N_{loc}. \tag{2.16}$$

The partial derivatives listed in (2.16) depend continuously in H^N_{loc} on the parameters x,s. The equation in (2.14) holds almost everywhere, and the boundary conditions are satisfied in a classical sense.

For a proof of this, the reader is referred to [7]. In that paper, the result is proven with H^N replaced by L^1, but the proof easily extends to our setting. In our case $q \in C^\infty$, and hence $k(b^+;x,t)$ is jointly C^∞ in (x,t).

We now solve the system

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + q(x)u = 0,\ t \in (0,\infty),\ x \in (b,\infty), \tag{2.17}$$

$$u(b,t) = f(t),\ t > 0, \tag{2.18}$$

$$u(x,t) = 0,\ x > b,\ t \leq 0. \tag{2.19}$$

The following holds by direct calculation:

Proposition 4 *Let k be as in Proposition 3.*

a) Suppose $f \in C^2_*$. Then the problem 2.17, 2.18, 2.19 has unique solution $w^f(b^+;x,t)$, with

$$w^f(b^+;x,t) = f(t-x+b) + \int_{s=x-b}^t k(b^+;x-b,s)f(t-s)ds; \tag{2.20}$$

$w^f \in H^2((b,\infty)\times(0,T))$, Eq. 2.17 is satisfied almost everywhere, and the boundary and initial conditions are satisfied in a classical sense.

b) For $f \in L^2_*$, the function $w^f(b^+;x,t)$ defined above gives a solution to Eq. 2.17 in the distribution sense, 2.18 holds a.e. t, and 2.19 holds for all x. Furthermore, $w^f \in C([0,T];L^2(b,\infty))$.

Remark on notation: the superscript on b^+ in $w^f(b^+;x,t)$ serves to indicate that the associated wave will propagate to the right; and for the same reason we use the + superscript on the kernel $k(b^+;x,t)$; similarly below $k(b^-;x,t)$ will be the kernel associated to waves propagating to the left.
Remark: Because $f(t)=0$ for $t<0$, (2.20) can be rewritten as

$$w^f(b^+;x,t)=\begin{cases} f(t-x+b)+\int_{s=x-b}^{t}k(b^+;x-b,s)f(t-s)ds, & x-b<t,\\ 0, & x-b\geq t.\end{cases}$$

Setting $u(x,t)=w^f(0^+;x,t)$ as in (2.20), Theorem 4 clearly holds for $t<a$.

To consider $t\geq a$, we must study the interaction between the wave w^f and the mass. In preparation for this, note that similar to Proposition 4, the system

$$\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}+q(x)u=0,\ t\in(-\infty,\infty),\ x\in(-\infty,a), \tag{2.21}$$

$$u(a,t)=\tilde{f}(t-a),\ t>0, \tag{2.22}$$

$$u(x,t)=0,\ x<a,\ t\leq 0. \tag{2.23}$$

is solved by

$$w^g(a^-;x,t)=\tilde{f}(t+x-2a)+\int_{s=a-x}^{t-a}k(a^-;a-x,s)\tilde{f}(t-a-s)ds, \tag{2.24}$$

where $k(a^-;x,s)$ is the obvious analogue to $k(b^+;x,s)$. We note in this case $k(a^-;0,s)=0$.

Returning to Theorem 4, for $t<\min(3a,2\ell-a)$ we define $h(t-a)=u(a,t)$; thus clearly $h(s)=0$ for $s\leq 0$. Then for $t\in[a,\min(2a,\ell))$ we have for $x>a$ by Proposition 4

$$u(x,t)=h(t-x)+\int_{s=x-a}^{t-a}k(a^+;x-a,s)h(t-a-s)ds. \tag{2.25}$$

Define $g(t)$ by

$$g(t-a)=h(t-a)-f(t-a)-\int_{s=a}^{t}k(0^+;a,s)f(t-s)ds.$$

Then for $x<a$, we have

$$\begin{aligned} u(x,t) &= f(t-x)+\int_{s=x}^{t}k(0^+;x,s)f(t-s)ds \\ &\quad +g(t+x-2a)+\int_{s=a-x}^{t-a}k(a^-;a-x,s)g(t-a-s)ds \qquad (2.26)\\ &= f(t-x)+\int_{s=x}^{t}k(0^+;x,s)f(t-s)ds \end{aligned}$$

$$+h(t+x-2a)-f(t+x-2a)-\int_{s=a}^{t+x-a}k(0^+;a,s)f(t+x-a-s)ds$$
$$+\int_{s=a-x}^{t-a}k(a^-;a-x,s)\left(h(t-s-a)-f(t-s-a)-\int_{r=a}^{t-s}k(0^+;a,r)f(t-s-r)dr\right)ds. \quad (2.27)$$

Thus g represents the wave reflected off the mass, while h represents the transmitted wave. Note that by the definition of g, (2.15) and (2.26), we have that the continuity condition, (2.5), is satisfied. The condition (2.6) implies by (2.25),(2.27):

$$\begin{aligned}
Mh''(t-a) &= -2h'(t-a)+\int_{s=0}^{t-a}\frac{\partial k}{\partial x}(a^+;0,s)h(t-a-s)ds \\
&+ 2f'(t-a)+2k(0^+;a,a)f(t-a)-\int_{s=a}^{t}\frac{\partial k}{\partial x}(0^+;a,s)f(t-s)ds \\
&+ \int_{s=a}^{t}\frac{\partial k}{\partial s}(0^+,a,s)f(t-s)ds \\
&+ \int_{s=0}^{t-a}\frac{\partial k}{\partial x}(a^-;0,s)\left(h(t-a-s)-f(t-a-s)-\int_{r=a}^{t-s}k(0^+,a,r)f(t-s-r)dr\right)ds.
\end{aligned} \quad (2.28)$$

We now discuss the existence and regularity of the function h.

Lemma 1 *Let $T>0$.*

A) Given $f\in C^2_*$, there exists a unique $h\in C^3_*$ solving (2.28) for all $t\le T$.

B) Define the mapping S by

$$(Sf)(t)=h(t). \quad (2.29)$$

Then S is well defined, and extends to a bounded and boundedly invertible linear mapping $L^2_*\to H^1_*$.

C) S extends to a bounded and boundedly invertible linear mapping $H^j_*\to H^{j+1}_*$ for any positive integer j.

Proof:

We begin with part A, so $f\in C^2(-\infty,T)$ with $f(t)=0$ for $t<0$.

We now show that a solution to (2.28) exists and is in C^3. We rewrite (2.28) as

$$Mh''(t-a)+2h'(t-a)=\psi(t-a)+2f'(t-a)+\phi^2(t-a)+\phi^3(t-a), \quad (2.30)$$

where

$$\begin{aligned}
\phi^2(t-a) &= 2k(0^+;a,a)f(t-a), \\
\phi^3(t-a) &= -\int_{s=a}^{t}\frac{\partial k}{\partial x}(0^+;a,s)f(t-s)ds+\int_{s=a}^{t}\frac{\partial k}{\partial s}(0^+;a,s)f(t-s)ds \\
&- \int_{s=0}^{t-a}\frac{\partial k}{\partial x}(a^-;0,s)\left(f(t-a-s)+\int_{r=a}^{t-s}k(0^+,a,r)f(t-s-r)dr\right)ds,
\end{aligned} \quad (2.31)$$

and

$$\psi(t-a) = \int_{s=0}^{t-a} \left(\frac{\partial k}{\partial x}(a^-; 0, s) + \frac{\partial k}{\partial x}(a^+; 0, s) \right) h(t-a-s) ds.$$

Integrating once and using $h'(0) = h(0) = f(0) = 0$, we get for $t \geq 0$

$$Mh'(t) + 2h(t) = 2f(t) + \int_{s=0}^{t} \left(\psi(s) + \phi^2(s) + \phi^3(s) \right) ds. \tag{2.32}$$

Integrating again, we get

$$h(t) = \frac{1}{M} \int_{s=0}^{t} e^{\frac{2}{M}(s-t)} \left(2f(s) + \int_{r=0}^{s} (\psi(r) + \phi^2(r) + \phi^3(r)) dr \right) ds. \tag{2.33}$$

Define

$$\Phi(t) = \frac{1}{M} \int_{s=0}^{t} e^{\frac{2}{M}(s-t)} \left(2f(s) + \int_{r=0}^{s} (\phi^2(r) + \phi^3(r)) dr \right) ds. \tag{2.34}$$

Define the operator K by

$$(Kp)(t) = \frac{1}{M} \int_{s=0}^{t} e^{\frac{2}{M}(s-t)} \int_{r=0}^{s} \int_{w=0}^{r} \left(\frac{\partial k}{\partial x}(a^-; 0, s) + \frac{\partial k}{\partial x}(a^+; 0, s) \right) p(r-w) dw dr ds. \tag{2.35}$$

Thus by (2.33),

$$(I - K)h = \Phi, \tag{2.36}$$

and formally h is solved by

$$h = \sum_{n=0}^{\infty} K^n \Phi. \tag{2.37}$$

We now prove the convergence of this series. It is easy to show for $t < T$

$$|K\Phi(t)| \leq t \| \left(\frac{\partial k}{\partial x}(a^-; 0, \cdot) + \frac{\partial k}{\partial x}(a^+; 0, \cdot) \right) \|_{L^1(0,t)} \|\Phi\|_\infty = Ct\|\Phi\|_\infty,$$

where $C = \|\frac{\partial k}{\partial x}(a^-; 0, \cdot) + \frac{\partial k}{\partial x}(a^+; 0, \cdot)\|_{L^1(0,t)}$, and then inductively that

$$|K^n \Phi(t)| \leq \frac{C^n \|\Phi\|_\infty t^n}{n!}.$$

This shows that the series converges uniformly on compact sets in t, and the solution h is continuous in t. That $h \in C^3$ follows from a bootstrapping argument using (2.36). That $h(t) = 0$ for $t \leq 0$ follows from (2.33). Uniqueness is clear from (2.37). This completes the proof of part A.

Thus for f satisfying the hypotheses of part A, the operator $Sf = h$ is well defined. It is easy to see that S is linear. We now prove the boundedness claimed in part B. Let $\{f_j\}$ be a sequence of functions in C^2_*, and such that f_j converges in L^2_* to a function f. We define corresponding sequences of functions $\{\phi^i_j\}$, with $i = 1, 2$, and using (2.34) we define $\{\Phi_j\}$. By (2.34), Φ is well defined for $f \in L^2_*$ and it is easy to see that $\Phi_j \to \Phi$ in L^2_*.

Now define $h_j = \sum_{n=0}^{\infty} K^n \Phi_j$, with K as in (2.35). By the proof of part A, we have

$$|h_j(t) - h_k(t)| = |\sum_{n=0}^{\infty} K^n(\Phi_j - \Phi_k)(t)| \leq \|\Phi_j - \Phi_k\|_\infty \sum_{n=0}^{\infty} \frac{C^n t^n}{n!}.$$

Since t is bounded, this proves that $j \to \sum_{n=0}^{\infty} K^n \Phi_j$ is a Cauchy sequence in C^0, so there exists $h \in C^0(0,T)$ such that h_j converges uniformly to h. Furthermore, h will satisfy (2.33) and hence is a generalized solution to (2.30). The boundedness of S now easily follows from examination of (2.33).

Next, we prove the invertibility of S in part B. By the Open Mapping Theorem, if suffices to prove S is a bijection. Fix $h \in H^1_*(-\infty, T)$. In what follows, terms that are uniquely determined by h will be denoted $F(t)$. We then rewrite (2.32):

$$F(t) = 2f(t) + \int_{s=0}^{t} \phi^2(s) + \phi^3(s) ds. \tag{2.38}$$

We can show that the integral terms on the right hand side can each be expressed in the form $\int_0^t f(s)K(s,t)ds$ with K continuous. We prove this for one such integral term, namely (recalling the ϕ^3 is given by (2.31))

$$\int_{s=0}^{t} \int_{r=0}^{s} \frac{\partial k}{\partial x}(a^-;0,r) \int_{w=a}^{s+a-r} k(0^+,a,w) f(s+a-r-w) dw dr ds$$

$$= \int_{s=0}^{t} \int_{r=0}^{s} \frac{\partial k}{\partial x}(a^-;0,r) \int_{w=0}^{s-r} k(0^+,a,w+a) f(s-r-w) dw dr ds.$$

With the change of variables $\eta = r + w,\ r = r$, the right hand side becomes

$$\int_{s=0}^{t} \int_{\eta=0}^{s} f(s-\eta) K_1(\eta) d\eta ds, \tag{2.39}$$

with

$$K_1(\eta) = \int_{r=0}^{\eta} \frac{\partial k}{\partial x}(a^-;0,r) k(0^+,a,\eta - r + a) dr.$$

With another change variables, (2.39) equals

$$\int_{r=0}^{t} f(r) \left(\int_{s=r}^{t} K_1(s-r) ds \right) dr,$$

as desired. The other integral terms on the right hand side of (2.38) can be treated similarly. Thus we can rewrite (2.38) as a Volterra equation of the second kind,

$$F(t) = f(t) + \int_0^t f(s) K(s,t) ds.$$

To prove injectivity of S, note that if $h = 0$ then $F(t) = 0$, and by properties of Volterra equations we conclude $f = 0$. Since h uniquely determines $F(t)$, surjectivity also follows.

Remark: The solution h to (2.28) exists for all t, but gives the position of the mass only for $t < \min(3a, 2\ell - a)$, ie. until a reflected wave reaches $x = a$ either from the left or from the right.

Remark: The equations for u, (2.20), (2.27), (2.25), all show that for $t < \ell$, the wave propagates at unit speed throughout the interval despite the presence of the mass at $x = a$. The unit speed of propagation will continue to hold for larger times too, and this will play a key role in our study of the control problem.

Remark: It is important to note that all the calculations above are local, so these result will extend to the N mass case.

Remark: In ([14], Prop. 2.6) it is claimed that $h'(t)$ is continuous for $f \in L^2(0,T)_*$, but (2.32) shows that this need not be the case.

Remark: Lemma 1 together with (2.27) show that a wave reflected off the mass will have the same regularity as the incoming wave. More precisely, if a wave reaches $x = a$ at time T, and if near the mass $x \to u(x, T - \epsilon) \in H^j_{loc}$ for small ϵ, then $x \to u(x, T + \epsilon) \in H^j_{loc}$. We now show the same property for a reflection off an endpoint. For simplicity of exposition, we choose reflection at $x = \ell$ at time $t = \ell$. For $t \leq \ell$ and $x > a$, by (2.25) the function

$$w^1_*(x,t) := h(t-x) + \int_{s=x-a}^{t-a} k(a^+; x-a, s) h(t-a-s) ds \tag{2.40}$$

solves the relevant equations of System A. However, for $t > \ell$, this expression no longer satisfies $u(\ell, t) = 0$. We correct for this by adding the term that models the wave's reflection, denoted $w^2_*(x,t)$, that uniquely solves the system

$$\begin{aligned} \frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial x^2} + q(x)v &= 0, \ t > \ell, \ x < \ell, \\ v(\ell, t) &= -w^0_*(\ell, t), \\ v(x,t) &= 0, \ t < \ell. \end{aligned}$$

By the analogue of (2.24), we have

$$w^2_*(x,t) = -w^1_*(\ell, t + x - \ell) - \int_{s=\ell-x}^{t-\ell} k(\ell^-; \ell - x, s) w^1_*(\ell, t-s) ds. \tag{2.41}$$

We then have $u = w^1_* + w^2_*$ for (x,t) sufficiently close to (ℓ, ℓ). For fixed t, it is also clear that $w^1_* \in H^j_*$ implies $w^2_* \in H^j_*$.

Summing up, when a wave moves to either an end point or a mass, its reflection will have the same regularity as the incoming wave. Also, by Lemma 1, a wave transmitted across a mass will be one Sobolev order more regular than the incoming wave. By applying these two principles, the proof of Theorem 4 is completed for larger times; the details are left to the reader.

2.2 Regularity and compatibility conditions for N masses

In this section we consider the System A_N, which we rewrite:

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + q(x)u &= 0,\ t \in (0,T),\ x \in (0,\ell) - \{a_j\}_{j=1}^N, \\
u(x,t) &= 0,\ t \le 0, \\
u(a_j^-,t) &= u(a_j^+,t),\ j = 1,\ldots,N, \\
M_j u_{tt}(a_j,t) &= u_x(a_j^+,t) - u_x(a_j^-,t), \\
u(0,t) &= f(t), \\
u(\ell,t) &= 0.
\end{aligned}$$

In what follows, we denote the solution u to System A_N by u^f. We prove the following extension of Proposition 1.

Proposition 5 *A) Suppose $T \in (a_j, a_{j+1}], j = 0, ...N$. Then*

$$\{u^f(\cdot,T) : f \in L^2(0,T)\} \subset \{\phi \in L^2(0,a_1) \times H^1(a_1,a_2) \times ...H^j(a_j,T) : \phi(t) = ...\phi^{(j-1)}(t) = 0,\ \forall t \ge T\}.$$

B) For $T > \ell$,

$$\{u^f(\cdot,T) : f \in L^2(0,T)\} \subset \tilde{W} := \{\phi \in L^2(0,a_1) \times H^1(a_1,a_2) \times ...H^n(a_N,\ell) : \phi(l) = 0\}.$$

C) For $i = 0,1,2$,

$$u(x,t) \in C^i([0,T], H^{-i}(0,a_1) \oplus H^{1-i}(a_1,a_2) \oplus ...H^{j-i}(a_j,T)).$$

Proof: Parts A and B are essentially proven in the previous subsections, where we demonstrated that a reflected (from a mass or a boundary) wave has the same regularity as an incident wave, and a transmitted wave is one unit more regular in the Sobolev scale (see, e.g. representations (2.25), (2.26), (2.27), and Lemma 1). Part C follows easily from the representation of the solution; the details are left to the reader.

To further clarify the properties of the waves in our system, we now consider certain compatibility conditions that must hold at the masses. Since H^1 functions are continuous, it follows that

$$u^f(a_j^-,t) = u^f(a_j^+,t),\ j = 2,...,N. \tag{2.42}$$

Also, using Proposition 5 part C along with the wave equation, we have first order compatibility conditions

$$\begin{aligned}
u_x^f(a_j^-,t) &= u_x^f(a_j^+,t) - M_j u_{tt}^f(a_j^+,t) \\
&= u_x^f(a_j^+,t) - M_j[u_{xx}^f(a_j^+,t) - q(a^+)u^f(a^+,t)] \\
&= u_x^f(a_j^+,t) - M_j[u_{xx}^f(a_j^+,t) - q(a)u^f(a^+,t)],\ j = 3,...,N. \qquad (2.43)
\end{aligned}$$

The proof of (2.43) can also be adapted to prove compatibility conditions on higher order derivatives, provided $u^f(x,t)$ is sufficiently regular. For instance, for $j \geq 4$ we have $u_{xx}(a_j^-,t) = u_{xx}(a_j^+,t)$, and for $j \geq 5$ we have

$$u_{xxx}(a_j^-,t) = u_{xxx}(a_j^+,t) - M_j[u_{xxxx}(a_j^+,t) - (qu)_{xx}(a_j^+,t)].$$

We now consider boundary conditions that must be satisfied by u at $x = \ell$. In addition to $u(x,\ell^-) = 0$, the wave equation induces conditions on higher order derivatives provided u is sufficiently smooth. For instance,

$$\begin{aligned} u^f_{xx}(\ell^-,t) &= u^f_{tt}(\ell^-,t) - q(\ell)u^f(\ell^-,t) \\ &= 0, \ \forall t > 0, \end{aligned}$$

and similarly

$$u^f_{xxxx}(\ell^-,t) = -2q'(\ell)u^f_x(\ell^-,t). \tag{2.44}$$

In what follows, we will say that a function $\phi(x)$ satisfies the *Condition C* if it satisfies

$$\phi(a_j^-) = \phi(a_j^+), j \geq 2, \ \phi'(a_j^-) = \phi'(a_j^+) - M_j[\phi_{xx}(a_j^+) - q(a_j)\phi(a_j)] \ j \geq 3, \ \phi(\ell^-) = 0,$$

and higher order conditions as necessary.

Similar arguments show that u_t must also satisfy compatibility conditions, for instance

$$u^f_t(a_j^-,t) = u^f_t(a_j^+,t), \ j = 3,...,N, \ \text{and}$$

$$u^f_{xt}(a_j^-,t) = u^f_{xt}(a_j^+,t) - M_j[u^f_{xxt}(a_j^+,t) - q(a)u^f_t(a^+,t)], \ j = 4,...,N,$$

and so on. Similarly, for $N \geq 2$ we have u_t satisfying:

$$\frac{\partial u_t}{\partial x}(\ell^-,t) = 0, \ , \ \forall t > 0.$$

We will say that a function $\psi(x)$ satisfies the *Condition C'* if

$$\psi(a_j^-) = \psi(a_j^+), j \geq 3, \ \psi'(a_j^-) = \psi'(a_j^+) - M_j[\psi_{xx}(a_j^+) - q(a)\psi(a)] \ j \geq 4, \ \psi(\ell^-) = 0,$$

and higher order conditions as necessary.

We now define the spaces W_0, W_{-1} as in Section 1, with norms

$$\|v\|^2_{W_i} = \sum_{j=0}^{N} \|v\|^2_{H^{i+j}(a_j,a_{j+1})}.$$

Then Theorem 1 follows immediately.

3 Shape and velocity controllability results

To prove Theorem 3, parts B and D, we require a preliminary result that we formulate now. Consider the vibrating string with no mass:

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + q(x)u &= 0,\ t \in (0,T),\ x \in (0,\ell), \\ u(x,t) &= 0,\ t \leq 0, \\ u(0,t) &= g(t), \\ u(\ell,t) &= 0. \end{aligned} \tag{3.45}$$

Lemma 2 *A) Suppose $\phi \in H^N(0,\ell)$ satisfies the boundary conditions at $x=\ell$ associated to W_0. Let $T>\ell$, and let $\delta = T-\ell$. Then there exists $g \in C_0^\infty(0,2\delta)$ such that*

$$\frac{\partial^j u}{\partial x^j}(\ell^-,T) = \phi^{(j)}(\ell^-),\ j=0,...,N-1.$$

B) Suppose $\psi \in H^{N-1}(0,\ell)$ satisfies the boundary conditions at $x=\ell$ associated to W_{-1}. Let $T>\ell$, and let $\delta = T-\ell$. Then there exists $g \in C_0^\infty(0,2\delta)$ such that

$$\frac{\partial^j u_t}{\partial x^j}(\ell^-,T) = \psi^{(j)}(\ell^-),\ j=0,...,N-1.$$

Proof: We prove part A; the simple adaptation for part B is left to the reader. For any smooth function $g(t)$, for (x,t) sufficiently close to (ℓ,ℓ), we have by analogues of (2.40) and (2.41) that $u(x,t) = w_*^1(x,t) + w_*^2(x,t)$ with

$$w_*^1(x,t) = g(t-x) + \int_{s=x}^{t} k(0^+;x,s)g(t-s)ds$$

and

$$w_*^2(x,t) = -g(t+x-2\ell) - \int_{s=\ell}^{t+x-\ell} k(0^+;\ell,s)g(t+x-\ell-s)ds$$
$$- \int_{s=\ell-x}^{t-\ell} k(\ell^-;\ell-x,s)[g(t-s-\ell) + \int_{r=\ell}^{t-s} k(0^+;\ell,r)g(t-s-r)dr]ds.$$

A tedious but straightforward calculation shows that

$$\frac{\partial^j u}{\partial x^j}(\ell,T) = \sum_{k=0}^{j} C_{j,k} g^{(k)}(T-\ell) + \int_{s=0}^{T-\ell} K_j(s)g(T-\ell-s)ds,\ j=0,...,N-1,$$

with $C_{j,k}$ various constants and $K_j(s) \in C^\infty[0,T-\ell]$. We define continuous linear functionals $\tau_j : C_0^\infty(0,2\delta) \to \mathbb{R}$ by

$$\tau_j(g) = \sum_{k=0}^{j} C_{j,k} g^{(k)}(T-\ell) + \int_{s=0}^{T-\ell} K_j(s)g(T-\ell-s)ds.$$

Since $u(\ell^-, t) = u_{xx}(\ell^-, t) = 0$, we have $\tau_0 = \tau_2 = 0$, and by (2.44) we have $\tau_4 = -2g'(\ell)\tau_1$. More generally, linear dependence relations for a subset of $\{\tau_j\}$ are in one to one correspondence with the boundary conditions associated to W_0 at $x = \ell$. Let D be the dimension of the $V := \text{span}(\tau_0, ..., \tau_{N-1})$. Let $\sigma : \{1, ..., D\} \to \{0, ..., N-1\}$ be an injection such that $\{\tau_{\sigma(j)}\}_{j=1}^D$ is a basis of V. We claim the mapping $\tau : C_0^\infty(0, 2\delta) \to \mathbb{R}^D$ given by $\tau(g) = (\tau_{\sigma(1)}(g), ..., \tau_{\sigma(D)}(g))^T$ is onto. In fact, if not, then there would exist a non-zero vector $\mathbf{c} = (c_1, ..., c_D)^T \in \mathbb{R}^D$ with

$$0 = \mathbf{c} \cdot \tau(g) = \sum_{j=1}^{D} c_j \tau_{\sigma(j)}(g), \ \forall g \in C_0^\infty(0, 2\delta).$$

This would contradict the linear independence of $\{\tau_{\sigma(j)}\}_{j=1}^D$.

Finally, given ϕ satisfying the hypotheses of the lemma, we find $g \in C_0^\infty(0, 2\delta)$ solving

$$\tau_{\sigma(j)}(g) = \phi^{(j)}(\ell^-), \ j = 1, ..., D.$$

The remaining boundary conditions will automatically satisfied because of the linear dependence relations.

We now prove Theorem 3, which we restate for the reader's convenience.

Theorem 3

A) Suppose $T \le \ell$. For any $\phi \in W_0^T$, there exists a unique $f \in L^2(0, T)$ such that $u^f(x, T) = \phi(x)$. Furthermore,

$$\|u^f(\cdot, T)\|_{W_0} \asymp \|f\|_{L^2(0,T)}. \tag{3.46}$$

B) For $T > \ell$,

$$\{u^f(\cdot, T) : f \in L^2(0, T)\} = W_0.$$

C) Suppose $T \le \ell$. For any $\psi \in W_{-1}^T$, there exists a unique $f \in L^2(0, T)$ such that $u_t^f(x, T) = \psi(x)$. Furthermore,

$$\|u_t^f(\cdot, T)\|_{W_{-1}} \asymp \|f\|_{L^2(0,T)}. \tag{3.47}$$

D) For $T > \ell$,

$$\{u_t^f(\cdot, T) : f \in L^2(0, T)\} = W_{-1}.$$

Proof:

We prove parts A and B; the simple adaptations for parts C,D are left to the reader. To prove Part A, fix $j \in \{0, \ldots, N\}$ and $T \in (a_j, a_{j+1}]$. We will prove that for any $\phi(x) \in W_0^T$, the equation

$$u^f(x, T) = \phi(x) \tag{3.48}$$

has a unique solution $f \in L^2(0, T)$. The key role in proving controllability is played by the wave v^f that has no reflections up to the moment T. Following (2.25) we can present it in the form

$$v^f(x, T) = h_i(T - x) + \int_{x-a_i}^{T-a_i} k(a_i^+; x - a_i, s) h_i(T - a_i - s) ds, \ i = 0, \ldots, j+1, \ a_i < x < a_{i+1}, \tag{3.49}$$

where we put $h_0(t) = f(t)$ and $h_{i+1}(t) = (Sh_i)(t)$. We set $\Lambda = 2\min|a_{i+1} - a_i|, \ i = 0, \dots, N$, and will solve the equation (3.48) by steps of the length at most Λ. This means that we will move by such steps along the $x-$axis from the right to the left starting at the point $x = T$.

Step 1. We solve the equation (3.48) for $x \in (\max(T - \Lambda, a_j), T)$. On this interval, by the definition of Λ and unit speed of propagation, we have $u^f = v^f$.

There are two possible cases: (a) $T - \Lambda \geq a_j$, (b) $T - \Lambda \in [a_{j-2}, a_j)$.

Case a: we have $u^f(x,T) = v^f(x,T)$ for $x > T - \Lambda$, hence the equation (3.48) reduces to

$$\phi(x) = h_j(T - x) + \int_{x-a_j}^{T-a_j} k(a_j^+; x - a_j, s) h_j(T - a_j - s) ds, \ T - \Lambda < x < T. \qquad (3.50)$$

This is a Volterra equation of the second kind, hence ϕ uniquely determines the function $h_j(t)$ on the interval $[0, \Lambda)$, and its regularity is the same as the regularity of $\phi(x)$. Using Lemma 1 we can conclude there exists a unique $f \in L^2(0, \Lambda)$ such that $f(t) = (S^{-j}h_j)(t)$ for $t \in [0, \Lambda)$.

Case b: $u^f(x,T) = v^f(x,T)$ for $x > a_j$, so we first use (3.50) on the interval $x \in (a_j, T]$ to find $h_j(t)$ on the interval $[0, T - a_j)$. Thus we find $f(t) = (S^{-j}h_j)(t)$ on the same time interval.

Let c_1 equal Λ in case 1, and $T - a_j$ in case 2.

Step 2

$$f_1(t) = \begin{cases} f(t), & t < c_1, \\ 0, & t \geq c_1. \end{cases}$$

Then $\phi_2(x) := \phi(x) - u^{f_1}(x,T) \in W^T$ is supported on $[0, T - c_1]$, and hence is in W^{T-c_1}. Then we can repeat the argument in Step 1 to find c_2 along with the unique f_2 supported in $[c_1, c_1 + c_2]$ such that $u^{f_2}(x,T) = \phi_2(x)$ for $x \in [T - c_1 - c_2, T - c_1]$. Thus $u^{f_1+f_2}(x,T) = \phi(x)$ for $x \in [T - c_1 - c_2, T]$

Step 3 We repeat the arguments of step 1 and step 2 as often as necessary, thus solving for f. This completes the proof of part A.

We now prove (3.46). Suppose there exists a sequence $\{f^n\}$, with $f \in L^2(0,T)$ and $\{\phi^n = u^{f^n}(x,T)\}$, such that $\|\phi^n\|_{W_0} \to 0$. We will prove $\|f^n\|_{L^2(0,T)} \to 0$.

Consider Step 1 in part A. Suppose $h_j^n(t)$ solves (3.50), i.e.

$$\phi^n(x) = h_j^n(T - x) + \int_{x-a_j}^{T-a_j} k(a_j^+; x - a_j, s) h_j^n(T - a_j - s) ds, \ T - c_1 < x < T. \qquad (3.51)$$

By assumption, we have

$$\|\frac{d^m}{dx^m}\phi^n\|_{L^2(T-c_1,T)} \to 0, \quad m = 0, ..., j.$$

We apply this with $m = 0$ to (3.51), which is a Volterra equation of the second kind. Arguing as in the proof of Lemma 1 part B, we deduce $\|h_j^n\|_{L^2(0,c_1)} \to 0$. An inductive argument applied to x-derivatives of (3.51) then easily gives

$$\|\frac{d^m}{dt^m}h_j^n\|_{L^2(0,c_1)} \to 0, \quad m = 0, ..., j.$$

It then follows by Lemma 1 that $\|f^n\|_{L^2(0,c_1)} \to 0$.

We now use the ideas in Step 2 to finish the proof. Specifically, let

$$f_1^n(t) = \begin{cases} f^n(t), & t < c_1, \\ 0, & t \geq c_1. \end{cases}$$

Then $\phi_2^n(x) := \phi^n(x) - u^{f_1^n}(x,T) \in W_0^T$ is supported in $[0, T - c_1]$, and $\|\phi_2^n\|_{W_0} \to 0$. Thus we can repeat the argument above on ϕ_2^n. Now iterating this argument as many times as necessary, as in Step 2, we get $\|f^n\|_{L^2(0,T)} \to 0$. This proves (3.46).

We now prove part B. Let $\phi \in W_0$. Let $T = \ell + \delta$, where we can assume without loss of generality $\delta \in (0, \Lambda/4)$. Recall $\ell_N = \ell - a_N$. Consider the system

$$\begin{aligned} \frac{\partial^2 \tilde{u}}{\partial t^2} - \frac{\partial^2 \tilde{u}}{\partial x^2} + q(x)\tilde{u} &= 0, \ t \in (0, \ell_N + \delta), \ x \in (a_N, \ell), \\ \tilde{u}(x,t) &= 0, \ t \leq 0, \\ \tilde{u}(a_N, t) &= g(t), \\ \tilde{u}(\ell, t) &= 0. \end{aligned}$$

By Lemma 2, there exists $g(t) \in C_0^\infty(0, 2\delta)$ such that

$$\frac{\partial^j \tilde{u}^g}{\partial x^j}(\ell^-, \ell_N + \delta) = \phi^{(j)}(\ell^-), \ j = 0, ..., N-1.$$

Since g is in the range of S^N, we can set $f_g = S^{-N} g \in L^2(0, 2\delta)$, so that $\phi(x) - u^{f_g}(x,T) \in W_0^\ell$. Thus we can apply the part A to find $f \in L^2(0,T)$ supported in (δ, T) solving $u^f(x,T) = \phi(x) - u^{f_g}(x,T)$. Then $f + f_g$ is the desired control.

Remark: Using the ideas in this section, we can prove the following shape controllability result. Assume the hypotheses of Lemma 2. Then there exists $f \in H_*^N(0,T)$ satisfying $u^f(x,T) = \phi(x)$. Theorems proving that highly regular target functions can be generated by highly regular controls have been proven in the context of $full$ controllability in a number of works, see [12] and references therein.

4 Spectrum and Riesz bases

4.1 Spectral theory of system

Consider the eigenvalue problem which arises by applying separation of variables to System A_N:

$$\begin{aligned} -\phi''(x) + q(x)\phi(x) &= \lambda^2 \phi(x), \ x \in (0,\ell) - \cup_{j=1}^N \{a_j\}, \\ \phi(0) &= \phi(\ell) = 0, \\ \phi(a_j^-) &= \phi(a_j^+), \\ \phi'(a_j^+) &= \phi'(a_j^-) - M_j \lambda^2 \phi(a_j), \ j = 1, \ldots, N. \end{aligned} \tag{4.52}$$

Let $\Gamma = \{\lambda_n\}_{n=1}^{\infty}$ be the set of eigenfrequencies of System (4.52), listed in increasing order. It is easy to show that the eigenfrequencies are simple. In fact, suppose ϕ_1, ϕ_2 solve the system (4.52) for the same λ. Then by scaling, we can also assume $\phi_1'(0) = \phi_2'(0)$. Thus $v = \phi_1 - \phi_2$ satisfies the $-v''(x) + q(x)v(x) = \lambda^2 v(x)$ on $(0, a_1)$ with $v(0) = v'(0) = 0$. By standard theory of differential equations, we conclude $v(x) = 0$ on $(0, a_1)$, and hence $\phi_1(x) = \phi_2(x)$ on $[0, a_1]$. Applying the same argument to ϕ_1, ϕ_2 on the subinterval $[a_1, \ell]$, we deduce $\phi_1 = \phi_2$ on $[a_1, a_2]$. Iterating, we get $\phi_1 = \phi_2$.

The set $\{(\lambda_n)^2\}$ can be realized as the spectrum of a self-adjoint operator as follows. First we introduce

$$\mathcal{H}_0 := L^2_M(0, \ell) = \{u \in L^2(0, \ell) : \ u(a_j) \in \mathbb{R}\},$$

where the norm in $L^2_M(0, \ell)$ is defined as

$$\int_0^{\ell} |u(x)|^2 dx + \sum_{j=1}^{N} M_j |u(a_j)|^2.$$

Denote by $< *, * >_M$ the associated inner product. This space is canonically isomorphic to $L^2(0, \ell) \oplus_{i=1}^{N} \mathbb{R}$. We define quadratic form

$$Q(u, v) = \sum_{j=0}^{N} \int_{a_j}^{a_{j+1}} u'(x)v'(x) + q(x)u(x)v(x)dx,$$

with quadratic form domain

$$\mathcal{Q} = \{u \in L^2_M(0, \ell) : \ u|_{(a_j, a_{j+1})} \in H^1(a_j, a_{j+1}), \ u(a_j^-) = u(a_j) = u(a_j^+) \ \forall j, \ \text{and} \ u(0^+) = u(\ell^-) = 0\}.$$

Associated with this semibounded, closed quadratic form is the self adjoint operator A, with operator domain

$$D(A) = \{u \in \mathcal{Q} : \ Au \in L^2_M(0, \ell)\}.$$

Then for $u \in D(A)$,

$$Au(x) = \begin{cases} -u''(x), & x \neq a_j, \ j = 1, ..., N, \\ \frac{1}{M_j}(u'(a_j^-) - u'(a_j^+)), & x = a_j, \ j = 1, ..., N. \end{cases}$$

Letting $\{\phi_n\}$ be the set of normalized eigenfunctions of A, by simplicity of the spectrum and the self-adjointness of A we have that this set is orthonormal with respect to $< *, * >_M$. It is easy to check that ϕ_n will solve the system (4.52).

We use the spectral representation to create a scale of Sobolev spaces:

$$\mathcal{H}_p = \{u(x) = \sum_{n=0}^{\infty} a_n \phi_n(x) \ : \ ||u||_p^2 = \sum_{n=0}^{\infty} |a_n|^2 |\lambda_n|^{2p} < \infty\}, \ p \in \mathbb{R}.$$

Thus $\mathcal{H}_2 = D(A)$. Associated to these spaces are various symmetric compatibility conditions at $x = a_j$. To give an example, any eigenfunction ϕ is in $\mathcal{H}_n$ for all n, so that for each j we have

$$\frac{1}{M_j}(\phi'(a_j^-) - \phi'(a_j^+)) = \lambda^2 \phi(a_j) = \phi''(a_j^-) = \phi''(a_j^+).$$

It is necessary to study in detail the eigenfunctions in the case $q = 0$. Thus consider the system

$$\begin{aligned} -\phi''(x) &= \lambda^2\phi(x),\ x \in (0,\ell) - \cup_{j=1}^{N} a_j, \\ \phi(0) &= \phi(\ell) = 0, \\ \phi(a_j^-) &= \phi(a_j^+), \\ \phi'(a_j^+) &= \phi'(a_j^-) - M_j\lambda^2\phi(a_j),\ j = 1, \dots, N. \end{aligned} \tag{4.53}$$

We solve for ϕ by the following procedure. On the interval $(0, a_1)$ we set $\phi(x) = \sin(\lambda x)$. We then obtain $\phi(a_1)$ and $\phi'(a_1^+)$ from equation (4.53). Then on the interval (a_1, a_2) we have

$$\phi(x) = \phi(a_1)\cos(\lambda(x - a_1)) + \frac{\phi'(a_1^+)}{\lambda}\sin(\lambda(x - a_1)). \tag{4.54}$$

Clearly, we can then iteratively solve for ϕ on (a_j, a_{j+1}) for each $j = 0, ..., N$. We can then define

$$G(\lambda) = \phi(\ell).$$

Of course, $G(\lambda) = 0$ whenever λ is an eigenvalue. We wish to examine some other properties of G. We begin with the following:

Lemma 3 *For each j, $j = 1, \dots, N + 1$,*

i)

$$\phi(a_j) = \sum_{n=0}^{j-1} b_n(\lambda)\lambda^n,$$

where for each n, the $b_n(\lambda)$ is a sum of products of the form

$$K_n \prod_{k=0}^{j-1} T_k(\lambda(a_{k+1} - a_k)),$$

where K is a constant and $T_k(z)$ equals either $\sin(z)$ or $\cos(z)$. Furthermore

$$b_{j-1}(\lambda) = (-1)^{j-1}\left(\prod_{k=1}^{j-1} M_k\right)\prod_{k=1}^{j}\sin((a_k - a_{k-1})\lambda). \tag{4.55}$$

Here we use the convention $\prod_1^0 = 1$.

ii)

$$\phi'(a_j^+)/\lambda = \sum_{n=0}^{j} \tilde{b}_n(\lambda)\lambda^n,$$

where for each n, the $\tilde{b}_n(\lambda)$ is a sum of products of the form

$$K_n \prod_{k=0}^{j} T_k(\lambda(a_{k+1} - a_k)),$$

where K_n is a constant and $T_k(z)$ equals either $\sin(z)$ *or* $\cos(z)$. *Furthermore,*

$$\tilde{b}_j(\lambda) = (-1)^j \prod_{k=1}^{j} M_k \sin((a_k - a_{k-1})\lambda). \tag{4.56}$$

iii) For λ close to 0, $\phi(a_j) = \sin(\lambda a_j) + O(\lambda^2)$ and $\phi'(a_j^+) = \lambda\cos(\lambda a_j) + O(\lambda^2)$.

Proof of Lemma: First, we simultaneously prove i) and ii) by induction on j. First, for $j = 1$, by (4.52) we have

$$\phi(a_1) = \sin(\lambda a_1), \quad \phi'(a_1^+)/\lambda = \cos(\lambda a_1) - M_1\lambda\sin(\lambda a_1), \tag{4.57}$$

so the lemma holds in this case. Assume now the result holds for some $j < N + 1$. On the interval $[a_j, a_{j+1}]$, we have

$$\phi(x) = \phi(a_j)\cos(\lambda(x - a_j)) + \frac{\phi'(a_j^+)}{\lambda}\sin(\lambda(x - a_j)). \tag{4.58}$$

Thus, setting $x = a_{j+1}$ and applying the inductive hypothesis,

$$\phi(a_{j+1}) = (-1)^j\lambda^j \prod_{k=1}^{j} M_k \sin(\lambda(a_k - a_{k-1})) \cdot \sin(\lambda(a_{j+1} - a_j)) + O(\lambda^{j-1}).$$

Furthermore, applying the inductive hypothesis again together with (4.58),

$$\begin{aligned}
\phi'(a_{j+1}^+) &= \phi'(a_{j+1}^-) - M_{j+1}\lambda^2\phi(a_{j+1}) \\
&= -M_{j+1}\lambda^2\phi(a_{j+1}) + O(\lambda^{j+1}) \\
&= (-1)^{j+1}\prod_{k=1}^{j+1} M_k \sin(\lambda(a_k - a_{k-1})\ \lambda^{j+2} + O(\lambda^{j+1}).
\end{aligned} \tag{4.59}$$

Induction has been established for (4.55),(4.56). The proof of the rest of i), ii) follows easily from (4.58) and (4.59) together with

$$\phi'(x) = -\lambda\phi(a_j)\sin(\lambda(x - a_j)) + \lambda\frac{\phi'(a_j^+)}{\lambda}\cos(\lambda(x - a_j)).$$

We will now prove iii), again by induction. For $j = 1$, the results hold by (4.57). Now assume the result holds for $j < N + 1$. By the inductive hypothesis and (4.58), followed by a trigonometric identity, we have

$$\begin{aligned}
\phi(a_{j+1}) &= (\sin(\lambda a_j) + O(\lambda^2))\cos(\lambda(a_{j+1} - a_j)) + (\cos(\lambda a_j) + O(\lambda))\sin(\lambda(a_{j+1} - a_j)) \\
&= \sin(\lambda a_{j+1}) + O(\lambda^2).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\phi'(a_{j+1}^+) &= \phi'(a_{j+1}^-) - \lambda^2 M_{j+1}\phi(a_{j+1}) \\
&= \phi(a_j)(-\lambda\sin(\lambda(a_{j+1} - a_j))) + \phi'(a_j^+)(\cos(\lambda(a_{j+1} - a_j))) + O(\lambda^2) \\
&= \lambda[(-\sin(\lambda a_j)\sin(\lambda(a_{j+1} - a_j))) + \cos(\lambda a_j)\cos(\lambda(a_{j+1} - a_j))] + O(\lambda^2) \\
&= \lambda\cos(\lambda a_{j+1}) + O(\lambda^2).
\end{aligned}$$

This completes the proof of iii).

Corollary 1

a) The function $G(\lambda)/\lambda$ is an entire function of exponential type ℓ on the upper and lower half planes.

b) The roots of $G(\lambda)/\lambda$ are precisely the eigenvalues of System (4.53).

c) Let $\lambda = x + iy$, $x, y \in \mathbb{R}$. Then there exists $y_0 > 0$ such that if $|y| > y_0$, then there exist constants $C_0, C_1 > 0$ such that

$$C_0 < \frac{|G(\lambda)|}{1 + |\lambda|^N} < C_1, \ \forall x.$$

Proof:

It is clear by construction that the roots of $G(\lambda)$ are the eigenvalues of System (4.53) together with $\lambda = 0$. By Lemma 3 part iii), $\lim_{\lambda \to 0} G(\lambda)/\lambda = \ell$. Part b) of the corollary now follows. That $G(\lambda)/\lambda$ is an entire function also follows from Lemma 3.

To complete the proof of part a, let $\lambda = re^{i\theta}$. Fix θ and suppose r is large. For $T(z)$ equalling either $\sin(z)$ or $\cos(z)$,

$$|T(\lambda(a_{j+1} - a_j))| \sim e^{r(a_{j+1}-a_j)|\sin(\theta)|}.$$

Thus, using the notation of Lemma 3 ,

$$\prod_{k=0}^{N} |T_k(\lambda(a_{k+1} - a_k))| \sim e^{r\ell|\sin(\theta)|}.$$

Part a) now follows from the parts i), ii) of Lemma 3 .

To prove part c), we use $G(\lambda) = u(a_{N+1})$ in Lemma 3 part i) to show

$$|\frac{G(\lambda)}{\lambda^N} - (-1)^N \prod_{k=1}^{N} M_k \prod_{k=1}^{N+1} \sin((a_k - a_{k-1})\lambda)| = O(\lambda^{-1}).$$

Next, it is easy to show that there exists $y_0 > 0$ such that if $|y| > y_0$,

$$\prod_{k=1}^{N+1} |\sin((a_k - a_{k-1})(x + iy))| \asymp 1,$$

this estimate being uniform in x. The desired inequalities now follow easily.

We now compute the asymptotics of the eigenfrequencies.

Let $\Gamma_0 = \{\gamma_n\}_{n=1}^{\infty}$ be the set of eigenfrequencies of System (4.52) with $q(x) = 0$ listed in increasing order.

Corollary 2 *We have*

$$\Gamma_0 = \cup_{j=0}^{N} \{\gamma_m^{(j)}\}_{m=1}^{\infty},$$

where for each j,

$$|\gamma_m^{(j)} - \frac{\pi m}{a_{j+1} - a_j}| = O(1/m).$$

Proof. Using Lemma 3 part i and $u(\ell) = 0$, we get that $\{\gamma_m\}$ is the solution set for an equation of the form

$$\prod_{k=1}^{N+1} \sin((a_k - a_{k-1})\lambda) = \sum_{n=0}^{N-1} b_n(\lambda)\lambda^{n-N},$$

where $b_n(\lambda)$ are analytic functions which are bounded near the real axis. The estimate easily follows from Rouche's Theorem.

Proposition 6 *Then there exists a constant $C > 0$ such that*

$$|\lambda_n - \gamma_n| < \frac{C}{n}, \ \forall j.$$

Proof: The quadratic forms associated to the systems (4.53), resp. (4.52) are

$$Q_0(u) = \sum_{k=0}^{N} \int_{x=a_k}^{a_{k+1}} |u'(x)|^2 + \sum_{k=1}^{N} M_k |u(a_k)|^2,$$

resp. $Q(u) = Q_0(u) + \int_0^\ell q(x)|u(x)|^2 dx$, with quadratic form domain for both Q and Q_0 being $\mathcal{Q}$. Suppose K_0, K_1 satisfy $K_0 < q(x) < K_1$ for almost all x. Then for all u in $\mathcal{Q}$,

$$Q_0(u) + K_0 \leq Q(u) \leq Q_0(u) + K_1.$$

Using a standard mini-max argument (see [17]), we get

$$\gamma_j^2 + K_0 \leq \lambda_j^2 \leq \gamma_j^2 + K_1, \ \forall j.$$

The proposition easily follows.

4.2 Riesz bases

Definition Assume $\{\mu_j\}$ is a non-repeating sequence. The generalized divided difference (GDD) of order zero for $\{e^{i\mu_n t}\}$ is $[e^{i\mu_1 t}](t) := e^{i\mu_1 t}$. The GDD of order $n-1$ is given by

$$[e^{i\mu_1 t}, \dots, e^{i\mu_n t}] = \frac{[e^{i\mu_1 t}, \dots, e^{i\mu_{n-1} t}] - [e^{i\mu_2 t}, \dots, e^{i\mu_n t}]}{\mu_1 - \mu_n}, \ \mu_1 \neq \mu_n.$$

One then easily derives the formula

$$[e^{i\mu_1 t}, \dots, e^{i\mu_n t}] = \sum_{k=1}^{n} \frac{e^{i\mu_k t}}{\prod_{j\neq k}(\mu_k - \mu_j)}.$$

It is shown in [4] that the functions $[e^{i\mu_1 t}]$, ..., $[e^{i\mu_1 t}, \dots, e^{i\mu_n t}]$ depend on the parameters μ_j continuously and symmetrically.

Let $\mathbb{K} = \{\pm 1, \pm 2, ...\}$. To use divided differences in our framework, it is convenient to extend the mapping $j \mapsto \lambda_j$ from $\mathbb{N}$ to $\mathbb{K}$ using

$$\lambda_{-j} = -\lambda_j, \ \forall j \in \mathbb{N},$$

and similarly for $j \mapsto \gamma_j$.

We cite the following facts from [4]. For any $z \in \mathbb{C}$, denote by $D_z(r)$ the disk with center z and radius r. Let $G_0^{(p)}(r)$, $p = 1, 2, ...$ be the connected components of the union $\cup_{z\in\Gamma_0} D_z(r)$. Write $\Gamma_0^{(p)}(r)$ for the subsequence of Γ_0 lying in $G_0^{(p)}$, $\Gamma_0^{(p)} := \{\gamma_n | \gamma_n \in G_0^{(p)}(r)\}$. By Corollary 2, Γ_0 can be decomposed into the union of $N+1$ uniformly discrete sets, which we label $\Gamma_{0,j}$. Let

$$\delta_j := \inf_{\gamma\neq\mu;\gamma,\mu\in\Gamma_{0,j}} |\gamma - \mu|, \quad \delta := \min_j \delta_j.$$

Then for $r < r_0 := \frac{\delta}{2N+2}$, the number $\mathcal{N}^{(p)}(r)$ of elements of $\Gamma_0^{(p)}$ is at most $N+1$.

It is now convenient to rewrite Γ_0:

$$\Gamma_0 = \{\gamma_k^p \ : \ p \in \mathbb{K}, k = 1, ..., \mathcal{N}^{(p)}, \ \gamma_k^p \in G_0^{(p)}(r)\}.$$

Denote by $\mathcal{E}^0$ be the family of associated exponential GDD:

$$\mathcal{E}^0 = \mathcal{E}^0(r) := \cup_{p\in\mathbb{K}}\{[e^{i\gamma_1^p t}], [e^{i\gamma_1^p t}, e^{i\gamma_2^p t}], \dots, [e^{i\gamma_1^p t}, \dots, e^{i\gamma^p_{\mathcal{N}^{(p)}} t}]\}.$$

Proposition 7 *Let σ be any $N-1$ element subset of $\mathcal{E}^0$. Define*

$$\mathcal{E}^0_\sigma := \mathcal{E}^0 \setminus \sigma.$$

Then $\mathcal{E}^0_\sigma$ forms a Riesz basis of $L^2(0, 2\ell)$.

Proof: We apply Theorem 3 from [4]. Thus it suffices to prove

i) the set $\{\gamma_j^p\}$ is relatively uniformly discrete, and

ii) there exists an entire function $F(\lambda)$ of exponential type with indicator diagram of width 2ℓ and zeros at $\lambda = \gamma_j$ of multiplicity 1 such that for some real h, the function $x \mapsto F(x+ih)$ satisfies the Muckenhoupt condition. For the relevant facts on indicator diagrams, the reader is referred to [3].

In our case, condition i) has already been established. To prove that condition ii) holds, consider the function

$$F_0(\lambda) = \frac{G(\lambda)}{\lambda \prod_{j\in\sigma}(\lambda - \lambda_j)}.$$

Then by Lemma 3 parts i), ii), F_0 is of exponential type with indicator diagram of width 2ℓ. Further, by Corollary 1 part C, there exists $y_0 > 0$, and $K_2 > K_1 > 0$ such that for $h > y_0$, $K_1 < |F_0(x+ih)|^2 < K_2$. It follows that $F_0(x+ih)$ satisfies the Muckenhoupt condition.

We now partition Γ analogously to Γ_0: $\Gamma = \cup_{p\in\mathbb{K}} \cup_{k-1}^{\mathcal{N}^{(p)}} \lambda_k^p$, and use the family of exponentials $\{e^{i\lambda_k^p t}\}$ to define a family $\mathcal{E}$ of GDD. Then,

Proposition 8 *Let σ be any $N-1$ element subset of $\mathbb{K}$. Then $\mathcal{E}_\sigma = \mathcal{E} \setminus \sigma$ forms a Riesz basis of $L^2(0, 2\ell)$.*

Proof: We apply Theorem 3 from [4]. Thus it suffices to prove conditions i, ii from the proof of the previous proposition.

In our case, condition i follows easily from Proposition 6 and Corollary 2. To prove that condition ii holds, consider the function

$$F(\lambda) = \prod_{j \in \mathbb{K} \setminus \sigma} \left(1 - \frac{\lambda}{\lambda_j}\right).$$

Then by [15, Ch. V, Sec. 4], F is of exponential type with indicator diagram of width 2ℓ. Furthermore, let F_0, h be as in the proof of Proposition 7. Then by ([1], Lemma 4) along with Proposition 6, we have

$$|F(x+ih)| \asymp |F_0(x+ih)|,$$

so that $F(x+ih)$ satisfies the Muckenhoupt condition.

4.3 More on spectral asymptotics

We conclude this section with a brief discussion on the asymptotics of $\phi_n'(0)$, where ϕ_n is a eigenfunction associated to the system (4.52). In the absence of any mass, if we assume $\{\pm\lambda_n^2, \tilde{\phi}_n(x)\}$ are the eigenvalues and associated normalized eigenfunctions, it is well known that $\lambda_n \asymp |\tilde{\phi}_n'(0)|$. In the case of attached masses, this is not generally the case. For simplicity we set $N = 1$, $q = 0$, $\ell = 1$ and $a_1 = a$ in what follows. We construct the eigenfunctions with the same procedure as in the beginning of this section, hence for $x \in (0, a)$ we have $\phi_n(x) = \sin(\lambda x)$, hence

$$\phi_n'(0) = \lambda_n.$$

We now compute $\eta_n := \|\phi_n\|_{L^2}$. Easily we see $\int_0^a |\phi_n|^2 \asymp a/2$. By (4.57), (4.58),

$$\int_a^1 |\phi_n|^2 = \int_a^\ell |\sin(\lambda_n x) - M\lambda_n \sin(\lambda_n a)\sin(\lambda_n(x-a))|^2 dx.$$

By Corollary 2, the spectrum consists of $\{\lambda_n^{(0)}\} \cup \{\lambda_n^{(1)}\}$, with $\lambda_n^{(0)} \asymp \pi n/a$ and $\lambda_n^{(1)} \asymp \pi n/(1-a)$. If $\lambda_n \sim \pi j/a$, then the integral above is bounded. However, if $\lambda_n \sim \pi j/(1-a)$, then this integral is unbounded unless $\frac{a}{1-a}$ is an integer.

In conclusion, if $\{\pm\lambda_n^2, \tilde{\phi}_n(x)\}$ are eigenvalues and associated normalized eigenfunctions in our case, then we can split the frequencies into $\{\lambda_n^0\}$, associated with $(0, a)$, and $\{\lambda_n^1\}$ associated with $(a, 1)$, with corresponding eigenfunctions denoted $\tilde{\phi}_n^0$, resp. $\tilde{\phi}_n^1$. Let η_n^0, resp. η_n^1, be associated with λ_n^0, resp. λ_n^1 Then we have

$$\begin{aligned} (\tilde{\phi}_n^0)'(0) &= \lambda_n^0/\eta_n^0 \asymp n, \\ (\tilde{\phi}_n^1)'(0) &= \lambda_n^1/\eta_n^1. \end{aligned} \tag{4.60}$$

5 Proof of Full controllability

In this section we prove Theorem 2. In what follows, we denote the frequencies of the system (4.52) by $\Gamma = \{\lambda_k^p : p \in \mathbb{N}, k = 1, ..., \mathcal{N}^{(p)}\}$, with ϕ_k^p the corresponding orthonormal eigenfunctions. We will assume for simplicity that the eigenvalues satisfy $(\lambda_k^p)^2 > 0$. If this were not the case in what follows, it would suffice to replace $\sin(\lambda t)/\lambda$ by $\sinh(|\lambda|t)/|\lambda|$ in the case $(\lambda)^2 < 0$, and by t in the case $(\lambda)^2 = 0$.

We present the solution of System A_N in the form of the series

$$u(x,t) = \sum_{p=1}^{\infty} \sum_{k=1}^{\mathcal{N}^{(p)}} a_k^p(t)\, \phi_k^p(x). \tag{5.61}$$

In what follows, we will sometimes drop the superscript p for readability. We wish to find f solving

$$u^f(x,T) = u_0(x),\ u_t^f(x,T) = u_1(x). \tag{5.62}$$

For any $f \in L^2(0,T)$, standard calculations using the weak solution formulation (see, eg. [3] Ch. 3) give for each p,

$$a_k(t) = \frac{\phi_k'(0)}{\lambda_k} \int_0^t f(\tau)\, \sin \lambda_k(t-\tau)\, d\tau,\ a_k'(t) = \phi_k'(0) \int_0^t f(\tau)\, \cos \lambda_k(t-\tau)\, d\tau. \tag{5.63}$$

Denote $s_k(x) = \sin(\lambda_k x)$ and $c_k(x) = \cos(\lambda_k x)$. We set $a_k = a_k(T), b_k = a_k'(T)$, and

$$\alpha_k = \frac{a_k \lambda_k}{\phi_k'(0)},\ \beta_k = \frac{b_k}{\phi_k'(0)}. \tag{5.64}$$

Let $< *, * >_T$ be the standard complex inner product on $L^2(0,T)$. Let $f^T(t) = f(T-t)$, so $\int_0^T f(t) \sin(\lambda(T-t))dt =< f^T, \sin(\lambda t) >_T$. Then (5.61) for $t = T$ can be written as

$$\alpha_k^p =< f^T, s_k^p >_T,\ p \in \mathbb{N},\ k = 1, ..., \mathcal{N}^{(p)}. \tag{5.65}$$

Similarly, we can have

$$\beta_n^p =< f^T, c_n^p >_T,\ p \in \mathbb{N},\ k = 1, ..., \mathcal{N}^{(p)}. \tag{5.66}$$

Using $e^{it} = \cos(t) + i \sin(t)$, we get the following equations which hold for all $p \in \mathbb{N},\ k = 1, ..., \mathcal{N}^{(p)}$:

$$\begin{aligned} i\alpha_k^p + \beta_k^p &= < f^T, e^{-i\lambda_k^p t} >_T, \\ -i\alpha_k^p + \beta_k^p &= < f^T, e^{i\lambda_k^p t} >_T . \end{aligned}$$

Now we extend the domain of $k \to \lambda_k^p$ to $\mathbb{K}$ by setting $\lambda_k^{-p} = -\lambda_k^p$ for $p \geq 1$. Similarly we set $\alpha_k^{-p} = -\alpha_k^p$, and $\beta_k^p = \beta_k^{|p|}$ for all $p \in \mathbb{K}$. Define γ_k^p by

$$\gamma_k^p = (-i\alpha_k^p + \beta_k^p),\ \forall\ p \in \mathbb{K},\ k = 1, ..., \mathcal{N}^{(p)}.$$

Then solving the control problem (5.62) is equivalent to solving for $f \in L^2(0,T)$ in the following moment problem:

$$< f^T, e^{i\lambda_k^p t} >_T = \gamma_k^p,\ p \in \mathbb{K},\ k = 1, ..., \mathcal{N}^{(p)}, \tag{5.67}$$

where $\{\gamma_j\}$ is an appropriate space. To find the appropriate space, we have to study the properties of the exponential family $\{e^{i\lambda_k^p t}\}$.

Recall

$$\mathcal{E} = \cup_{p\in\mathbb{K}}\{[e^{i\lambda_1^p t}], [e^{i\lambda_1^p t}, e^{i\lambda_2^p t}], \dots, [e^{i\lambda_1^p t}, \dots, e^{i\lambda_{\mathcal{N}^{(p)}}^p t}]\}.$$

We define $\mathcal{S}$ and $\mathcal{C}$ to be corresponding families of divided differences of s_n and c_n. Thus,

$$\mathcal{S} = \cup_{p\in\mathbb{N}}\{[s_1^p(t)], \dots, [s_1^p(t), \dots, s_{\mathcal{N}^{(p)}}^p(t)]\},$$

$$\mathcal{C} = \cup_{p\in\mathbb{N}}\{[c_1^p(t)], \dots, [c_1^p(t), \dots, c_{\mathcal{N}^{(p)}}^p(t)]\}.$$

Recall that if σ_e is a $N-1$ element subset of $\mathcal{E}$, then $\mathcal{E}\setminus\sigma_e$ forms a Riesz basis on $L^2(0,2\ell)$. It will be convenient to choose σ as follows. Define integer Q by $Q = (N-1)/2$ for N odd, and $Q = N/2$ for N even. Let

$$\sigma_e = \{[e^{i\lambda_1^p t}]\ :\ p = \pm1, ..., \pm Q\}.$$

Thus for any N, $\mathcal{E}\setminus\sigma_e$ forms a Riesz *sequence* on $L^2(0,T)$ for any $T \geq 2\ell$. Setting

$$\sigma_s = \{[s_1^p(t)]\ :\ p = 1, ..., Q\} \text{ and } \sigma_s' = \{[c_1^p(t)]\ :\ p = 1, ..., Q\},$$

it then follows from ([2], Lemma 5.1) that $\mathcal{S}\setminus\sigma_s$ and $\mathcal{C}\setminus\sigma_s'$ form Riesz sequences in $L^2(0,T)$ for $T \geq \ell$.

We wish to rewrite the moment problems above in terms of our GDD, but first we need to develop some notation. For λ_k^p as above and $a_1^p, ..., a_n^p \in \mathbb{C}$, we construct divided differences of these *numbers* iteratively by $[a_1^p]' = a_1^p$, and

$$[a_1^p, \dots, a_n^p]' = \frac{[a_1^p, \dots, a_{n-1}^p]' - [a_2^p, \dots, a_n^p]'}{\lambda_1^p - \lambda_n^p}.$$

It is easy to see that

$$[a_1^p, \dots, a_n^p]' = \sum_{k=1}^n \frac{a_k^p}{\prod_{j\neq k}(\lambda_k^p - \lambda_j^p)}. \tag{5.68}$$

Lemma 4 *For $n = 1, ..., \mathcal{N}^{(p)}$,*

$$a_n^p = \sum_{k=1}^n [a_1^p, ..., a_k^p]' \prod_{j=1}^{k-1}(\lambda_n^p - \lambda_j^p).$$

Here we use the convention that $\prod_{j=1}^0(\lambda_n^p - \lambda_j^p) = 1$.

Proof: For readability, we drop the p superscript in what follows. By (5.68) we have

$$a_n = \prod_{j=1}^{n-1}(\lambda_n - \lambda_j)\left([a_1, ..., a_n]' - \sum_{k=1}^{n-1} \frac{a_k/(\lambda_k - \lambda_n)}{\prod_{j\in\{1,...,(n-1)\}-k}(\lambda_k - \lambda_j)}\right). \tag{5.69}$$

An algebra exercise shows that for any m,

$$\sum_{k=1}^{m} \frac{a_k/(\lambda_k - \lambda_n)}{\prod_{j\leq m; j\neq k}(\lambda_k - \lambda_j)} = \frac{1}{\lambda_m - \lambda_n}\left([a_1, ..., a_m]' - \sum_{k=1}^{m-1} \frac{a_k/(\lambda_k - \lambda_n)}{\prod_{j\leq (m-1); j\neq k}(\lambda_k - \lambda_j)}\right).$$

Applying this repeatedly to (5.69) gives the lemma.

Returning to the proof of the theorem, we rewrite (5.65), with $t = T$, in the form

$$[\alpha_1^p, ..., \alpha_k^p]' =< f^T, [s_1^p, ..., s_k^p] >, \quad p \in \mathbb{N},\ k = 1, ..., \mathcal{N}^p, \tag{5.70}$$

and (5.66), with $t = T$, in the form

$$[\beta_1^p, ..., \beta_k^p]' =< f^T, [c_1^p, ..., c_k^p] >, \quad p \in \mathbb{N},\ k = 1, ..., \mathcal{N}^p. \tag{5.71}$$

For any p, let $\mathcal{N} = \mathcal{N}^{(p)}$. By Lemma 4, dropping the superscript p for readability.

$$\begin{aligned} a_1\phi_1 + \ldots a_{\mathcal{N}}\phi_{\mathcal{N}} &= \sum_{n=1}^{\mathcal{N}} \phi_n(\sum_{k=1}^{n}[a_1, ..., a_k]' \prod_{j=1}^{k-1}(\lambda_n - \lambda_j)) \\ &= \sum_{k=1}^{\mathcal{N}}[a_1, ..., a_k]'\ (\sum_{j=k}^{\mathcal{N}} \phi_j(\prod_{l=1}^{k-1}(\lambda_j - \lambda_l))). \end{aligned}$$

Thus, with $\eta_k = \lambda_k/\phi_k'(0)$ hence $a_k^p = \alpha_k^p/\eta_k^p$, we can rewrite (5.61) as

$$\begin{aligned} u(x, T) &= \sum_{p=1}^{\infty}\sum_{k=1}^{\mathcal{N}^p} a_k^p \phi_k^p \\ &= \sum_{p=1}^{\infty}\sum_{k=1}^{\mathcal{N}^p} \alpha_k^p \phi_k^p/\eta_k^p \\ &= \sum_{p=1}^{\infty}\sum_{k=1}^{\mathcal{N}^p}[\alpha_1^p, \ldots, \alpha_k^p]'(\sum_{j=k}^{\mathcal{N}^p} \frac{\phi_j^p}{\eta_j^p}\prod_{l=1}^{k-1}(\lambda_j^p - \lambda_l^p)). \end{aligned} \tag{5.72}$$

Claim:

$$\{\sum_{j=k}^{\mathcal{N}^p} \frac{\phi_j^p}{\eta_j^p}\prod_{l=1}^{k-1}(\lambda_j^p - \lambda_l^p) \mid p \geq 1,\ k = 1, ..., \mathcal{N}^{(p)}\} \text{ forms a Riesz basis of } W_0.$$

Proof of claim: Let $u_0 \in W_0$, and suppose for the moment $T > \ell$. By Theorem 3, part B, there exists $f \in L^2(0, T)$ such that $u^f(x, T) = u_0(x)$. We introduce simplifying notations and rewrite (5.70) in the form

$$\nu_n =< f^T, S_n >_T, \quad n \in \mathbb{N}, \tag{5.73}$$

and rewrite (5.72) in the form

$$u_0(x) = u(x,T) = \sum_{n\in\mathbb{N}} \nu_n \psi_n. \tag{5.74}$$

We can choose the notation so that $\sigma_s = \{S_1, ..., S_Q\}$. Since ϕ_n form an orthogonal basis in $L^2(0,\ell)$, then u can be uniquely represented as series (5.61), and hence as series (5.74). For now, we have not yet shown that $\nu_n \in \ell^2$. However, $\mathcal{E}$ is a Riesz sequence on $L^2(0,2T)$, and hence $\mathcal{S}$ is a Riesz sequence in $L^2(0,T)$ (see [5]). Thus the formula $\nu_n =< f^T, S_n >_T$ shows that $\{\nu_n\} \in \ell^2$.

Now we use that $\{S_n : n \geq Q+1\}$ is a Riesz sequence for $L^2(0,\ell)$. It follows from this that we can find $g \in L^2(0,\ell)$ solving

$$< g^T, S_n >_\ell = \nu_n,\ n \geq Q+1, \ \text{ and } u^g(x,\ell) = \sum_{n=Q+1}^{\infty} \nu_n \psi_n(x).$$

Proposition 5 part A now implies $\sum_{n=Q+1}^{\infty} \nu_n \psi_n \in W_0^\ell$, and Theorem 3 implies

$$\|g\|_{L^2(0,\ell)} \asymp \| \sum_{n=Q+1}^{\infty} \nu_n \psi_n \|_{W_0}.$$

Because $\{S_n : n \geq Q+1\}$ is a Riesz sequence,

$$\sum_{Q+1}^{\infty} |\nu_n|^2 \asymp ||g||^2_{L^2(0,\ell)}.$$

Combining, we get

$$|| \sum_{n=Q+1}^{\infty} \nu_n \psi_n ||^2_{W_0} \asymp \sum_{n=Q+1}^{\infty} |\nu_n|^2.$$

Let $\tilde{W}$ be the closed subspace $\{\sum_{Q+1}^{\infty} z_n \psi_n :\ z_n \in \ell^2\}$. It is easily verified that $\phi_k^p \in W_0$ for all p, k, so the same holds for ψ_n for all n, and so $\tilde{W} \subset W_0$. By construction, functions $\psi_1, ..., \psi_Q$ are linearly independent, and it is not hard to show they are not in $\tilde{W}$. It follows that

$$\begin{aligned} || \sum_{n\in\mathbb{N}} \nu_n \psi_n ||^2_{W_0} &\asymp \sum_{n=1}^{Q} |\nu_n|^2 + || \sum_{n\geq Q+1} \nu_n \psi_n ||^2_{W_0} \\ &\asymp \sum_{n=1}^{\infty} |\nu_n|^2. \end{aligned} \tag{5.75}$$

It follows that the series in (5.74) converges in W_0. In fact, for any $P > 0$, $\sum_1^P \nu_n \psi_n \in W_0$. Applying the argument above, we get $|| \sum_{n\geq P} \nu_n \psi_n ||^2_{W_0} \asymp \sum_{n=P}^{\infty} |\nu_n|^2$. From (5.75), $\{\psi_n\}$ forms a Riesz basis in W_0.

We similarly find a Riesz basis for W_{-1}. Arguing as above,

$$u_t(x,T) = \sum_{p=1}^{\infty}\sum_{k=1}^{\mathcal{N}^p}[\beta_1^p,\dots,\beta_k^p]'(\sum_{j=k}^{\mathcal{N}^p}\phi_j^p(\phi_k^p)'(0)\prod_{l=1}^{k-1}(\lambda_j^p-\lambda_l^p)).$$

Claim 2:

$$\{\sum_{j=k}^{\mathcal{N}^p}\phi_j^p(\phi_k^p)'(0)\prod_{l=1}^{k-1}(\lambda_j^p-\lambda_l^p) \mid p\geq 1,\ k=1,...,\mathcal{N}^{(p)}\} \text{ forms a Riesz basis of } \ W_{-1}.$$

The proof of this is similar to the proof of Claim 1, and is left to the reader.

We complete the proof of the theorem as follows. If $(u_0,u_1)\in W_0\times W_{-1}$, then we have by Claim 1 along with (5.72) that

$$\sum_{p=1}^{\infty}\sum_{k=1}^{\mathcal{N}^{(p)}}|[\alpha_1^p,\dots,\alpha_k^p]'|^2 \asymp \|u_0\|_{W_0}^2.$$

Furthermore, Claim 2 implies

$$\sum_{p=1}^{\infty}\sum_{k=1}^{\mathcal{N}^{(p)}}|[\beta_1^p,\dots,\beta_k^p]'|^2 \asymp \|u_1\|_{W_{-1}}^2.$$

It is easy to see that for all $p\in\mathbb{K}$, and all $k=1,..,\mathcal{N}^{(p)}$,

$$[\beta_1^p,\dots,\beta_k^p]'+i[\alpha_1^p,\dots,\alpha_k^p]'=[\gamma_1^p,\dots,\gamma_k^p]'.$$

We rewrite (5.67) as

$$[\gamma_1^p,\dots,\gamma_k^p]' =< f^T,[e^{i\lambda_1^p t},\dots,e^{i\lambda_k^p t}] >_T;\ p\in\mathbb{K},\ k=1,...,\mathcal{N}^{(p)}.$$

Since $\mathcal{E}$ is a Riesz sequence on $L^2(0,T)$ for $T>2\ell$, and a Riesz basis for $T=2\ell$ when $N=2$, there exists a solution $f\in L^2(0,T)$ to the moment problem with

$$\|f\|_{L^2(0,T)}^2 \asymp \sum_{p,k}|[\gamma_1^p,\dots,\gamma_k^p]'|^2 \asymp \|u_0\|_{W_0}^2+\|u_1\|_{W_{-1}}^2.$$

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