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An Observation About Aumann Correlated Equilibria Points

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Abstract: *in this paper we study the conditions for which an Aumann equilibrium point is also a Nash point in the original game.*

Key words: Aumann equilibrium point, Nash equilibrium point, fiber.

Comments with related topics

The correlated strategies have been used in some earliest contributions as for example in Luce and all. On the other hand Marchi in [3] in the year 1969 consider and study the generalization of the minimax theorem in zero sum two person game. This was obtained in successful way by studying exhaustly the fiber bundle that appeared in the relation of the product of probabilities and the joint probability set. This is performed with a natural embedding. In this paper the mathematics is rather complicated for the requirements of the equations. Hence we use the term cooperative strategy rather than the correlated strategies. The paper with the only example of two strategies of $\left|\Sigma_1\right| = \left|\Sigma_2\right| = 2$ with different payoff or the two players is more simple that the just remarked. Here we found a very important feature without the concrete study of the fiber bundle and its study the fiber in each point. This is summarized in the fundamental inequalities (3). The dimension of the fiber is just the number of free $z's$. The paper just mentioned was generalized only for the probability aspect in Marchi [4] in the year 1972. No relation with the game is considered.

Introduction

In this note we are concerning with same aspects of game theory of n-players. As it is very well know in the non cooperative games the main concept introduced as solution is the equilibrium point of Nash [6]. In two elegant proofs besides the introduction he proved the existence of an equilibrium points in the mixed extension of a finite n-person games.

After some years Aumann introduced the strong equilibria for correlated points, but he was very successful with the correlated equilibrium n-person non-cooperative games.

We need the following concepts in order to present the new correlated equilibria. Let $N = \{1,2\}$ be the set of players and for each player $i \in \{1,2\}$, Σ_i is the set non empty set of pure strategies that for our purpose $\left|\Sigma_i\right| = 2$. Consider the set

$$\sum_1 \times \sum_2 = \left\{ z \in \Re^2 \, / \, z = (i,j), i \in \Sigma_1, j \in \Sigma_j \right\}$$

and the set of all joint probability

$$\overline{\sum_1 \times \sum_2} = \left\{ z \in \Re^4 \, / \, z(i,j) \geq 0, \sum_{i,j} z(i,j) = 1 \right\}$$

This set has a very strong relation with the set of product probabilities of the finite sets $\sum_1$ and $\sum_2$:

$$\overline{\sum_1} \times \overline{\sum_2} = \left\{ (x,y) \in \Re^4 \, / \, x(i) \geq 0, y(j) \geq 0, \sum_{i \in \Sigma_1} x(i) = 1, \sum_{j \in \Sigma_2} y(j) = 1 \right\}$$

This relationship might be viewed in the paper by Marchi [] where is exhaustive studied such subject for low dimension. For completeness and help for the reader we remember the Nash equilibrium concept.

For completeness and help for the reader we remember the Nash equilibrium concept. We indicate the game by $\Gamma = \{\Sigma_i, A_i, i \in N\}$, where A_1 and $A_2 : \Sigma_1 \times \Sigma_2 \to \Re$ are the pure payoff functions, then the mixed extension is given by $\overline{\Gamma} = \{\overline{\Sigma}_i, E_i, i \in N\}$, where E_i, for each *I*, is the simple expectation function defined as

$$E_k(x,y) = \sum_{i,j=1}^{2} A_k(i,j) x(i) y(j)$$

for $k = 1,2$. A Nash point is a 2-tupla of vectors $\bar{x}, \bar{y}$ such that

$$E_1(\bar{x},\bar{y}) \geq E_1(x,\bar{y}) \qquad \forall x \in \overline{\Sigma}_1 \tag{1}$$
$$E_2(\bar{x},\bar{y}) \geq E_2(\bar{x},y) \qquad \forall y \in \overline{\Sigma}_2$$

Now in general an Aumann correlated equilibria point is a point $z \in \overline{X_l \Sigma_l}$ such that

$$\sum_{\sigma_{N-\{i\}} \in \overline{X_{l \neq i} \Sigma_l}} z(\sigma_{N-\{i\}}, \sigma_i)\left(A_i(\sigma_{N-\{i\}}, \sigma_i) - A_i(\sigma_{N-\{i\}}, \tau_i)\right) \geq 0, \ \forall \sigma_i, \tau_i \in \Sigma_i,$$

where σ_i and τ_i are pure strategies of player $i \in N$. This was introduced by Aumann in 1974. The meaning of it was also given by Aumann as a recommendation by a referee to each player obtaining an optimum without to abide by it. A first formal proof was provided by Hart-Smeidlher [2] in 1989, with an elegant proof using new results in two person zero sum game.

At this point we must to emphasize that an Aumann correlated equilibrium is Nash in pure strategies in the following game

$$\Gamma_2 = \left\{ \sum_i, B_i^2, i \in N \right\}$$

where

$$B_i^2(\sigma_i) = \sum_{\sigma_{N-\{i\}}} z(\sigma_{N-\{i\}}, \sigma_i) A_i(\sigma_{N-\{i\}}, \sigma_i)$$

which is not the original.

We now wish to compute in general the Aumann correlated equilibrium. The set of all correlated equilibria is a non-empty compact and convex polyhedron.

Consider $N = \{1,2\}$ then the equations defining the Aumann correlated equilibrium with $a_{ij} = A_1(i,j)$, $b_{ij} = A_2(i,j)$ and $z(i,j) = z_{ij}$ are

$$z_{11}(a_{11} - a_{21}) + z_{12}(a_{12} - a_{22}) \geq 0 \qquad i = 1$$
$$z_{21}(a_{21} - a_{11}) + z_{22}(a_{22} - a_{12}) \geq 0 \qquad i = 2$$
$$z_{11}(b_{11} - b_{12}) + z_{21}(b_{21} - b_{22}) \geq 0 \qquad j = 1 \tag{2}$$
$$z_{12}(b_{12} - b_{11}) + z_{22}(b_{22} - b_{21}) \geq 0 \qquad j = 2$$

Now we will consider a rather important feature: when a correlated equilibrium of Aumann projects in a natural way in a Nash point in the original game. The first case that we analyze is when the Nash equilibrium is in both players completely mixed. Then from (1) and a standard form:

$$E_1(\bar{x},\bar{y}) = E_1(1,\bar{y}) = E_1(2,\bar{y}) \geq E_1(x,\bar{y}) \qquad \forall x$$
$$E_2(\bar{x},\bar{y}) = E_2(\bar{x},1) = E_2(\bar{x},2) \geq E_1(\bar{x},y) \qquad \forall y$$

or

$$a_{11}\bar{y} + a_{21}(1-\bar{y}) = a_{12}\bar{y} + a_{22}(1-\bar{y})$$

from here we obtain

$$\bar{y} = \frac{a_{12}-a_{22}}{a_{11}+a_{22}-a_{12}-a_{21}} = \frac{a_{12}-a_{22}}{\delta_A}$$

On the other hand the z´s, x´s and y´s are related by a natural projection, namely

$$\begin{aligned} z_{11}+z_{12} &= \bar{y} \\ z_{21}+z_{22} &= 1-\bar{y} \\ z_{11}+z_{21} &= \bar{x} \\ z_{22}+z_{12} &= 1-\bar{x} \end{aligned} \qquad (3)$$

or

$$\begin{aligned} z_{12} &= y - z_{11} \\ z_{21} &= x - z_{11} \\ z_{22} &= 1-x-y-z_{11} \end{aligned}$$

Now since $z = (z_{11},z_{12},z_{21},z_{22})$, $(x,1-x)$ and $(y,1-y)$ are probability vectors then

$$\begin{aligned} 0 \leq z_{12} &= y - z_{11} \leq 1 \\ 0 \leq z_{21} &= x - z_{11} \leq 1 \\ 0 \leq z_{22} &= 1-x-y-z_{11} \leq 1 \end{aligned}$$

which implies

$$\begin{aligned} z_{11} &\leq y \\ z_{11} &\leq x \\ z_{11}+x+y &\leq 1 \end{aligned}$$

or

$$x+y+1 \leq z_{11} \leq min\{x,y\} \qquad (4)$$

From (2) we obtain the four inequalities

$$z_{12} \geq \frac{a_{21}-a_{12}}{a_{12}-a_{22}} z_{11}$$

$$z_{21} \geq \frac{b_{12}-b_{11}}{b_{21}-b_{22}} z_{11}$$

and the numerators and the denominators both positives or both negative

$$z_{22} \geq \frac{a_{11}-a_{21}}{a_{22}-a_{12}} z_{21} \geq \frac{a_{11}-a_{21}}{a_{22}-a_{12}} \frac{b_{12}-b_{11}}{b_{21}-b_{22}} z_{11}$$

$$z_{22} \geq \frac{b_{21}-b_{12}}{b_{22}-b_{21}} z_{12} \geq \frac{b_{21}-b_{12}}{b_{22}-b_{21}} \frac{a_{21}-a_{12}}{a_{12}-a_{22}} z_{11}$$

We realize that the last inequalities are the same. Now replacing from the first and the second term of (3) into the first and third inequalities of (2) we obtain

$$z_{11} \geq \frac{a_{22}-a_{12}}{\delta_A} y = \frac{(a_{22}-a_{12})(a_{22}-a_{21})}{\delta_A^2}$$

and for the remaining equations

$$z_{11} \geq \frac{(b_{22}-b_{12})(b_{22}-b_{21})}{\delta_B^2}$$

or summarizing

$$max\left\{\frac{(a_{22}-a_{12})(a_{22}-a_{21})}{\delta_A^2}, \frac{(b_{22}-b_{12})(b_{22}-b_{21})}{\delta_B^2}, \frac{a_{22}-a_{21}}{\delta_A} + \frac{b_{22}-b_{12}}{\delta_B} + 1\right\}$$

$$\leq z_{11} \leq min\left\{\frac{b_{22}-b_{12}}{\delta_B}, \frac{a_{22}-a_{21}}{\delta_A}\right\}.$$

And in this way we obtain the following important result: if we have a completely mixed Nash and it comes from any Aumann correlated equilibrium fulfilling (5) then it projects to the given Nash.

Now we present an example fulfilling all the requirements. Take δ_A and δ_B positive then $\delta_A = a_{11} + a_{22} - a_{12} - a_{21} > 0$, then $a_{22} - a_{21}$ has to be non negative. The same for $\delta_B = b_{11} + b_{22} - b_{12} - b_{21} > 0$, therefore it must be $b_{22} - b_{12} \geq 0$. From the previous considerations, we have $a_{22} = a_{21} + \varepsilon_1$ with $\varepsilon_1 \geq 0$. Then $a_{11} \geq a_{21} + a_{12} - a_{22}$ $\geq a_{12} - \varepsilon_1$. From (5) one of the terms $\frac{b_{22}-b_{12}}{\delta_B}, \frac{a_{22}-a_{21}}{\delta_A}$ must be less to one since they are assumed mixed equilibrium points. For simplicity we take both terms less than one. From this remark besides we must have that

$$\frac{(a_{22}-a_{12})(a_{22}-a_{21})}{\delta_A^2} \leq \frac{a_{22}-a_{21}}{\delta_A}$$

then $\frac{a_{22}-a_{12}}{\delta_A} \leq 1$ or $a_{11} = a_{21} + \bar{\varepsilon}_1$ with $\bar{\varepsilon}_1 \geq 0$. Similarly for the second player we obtain $b_{11} + b_{22} - b_{12} - b_{21} > 0$, $b_{22} = b_{12} + \varepsilon_2$ with $\varepsilon_2 \geq 0$, $b_{11} \geq b_{12} - \varepsilon_2$ and finally $b_{11} = b_{12} + \bar{\varepsilon}_2$ with $\bar{\varepsilon}_2 \geq 0$.

It is important to realize that in (5) we have considered the comparison among the analogous terms in the minimum and maximum. On the other hand, we must to consider the cross inequalities as follows

$$\frac{(b_{22}-b_{12})(b_{22}-b_{21})}{\delta_B^2} \leq \frac{a_{22}-a_{21}}{\delta_A}$$

and

$$\frac{(a_{22}-a_{12})(a_{22}-a_{21})}{\delta_A^2} \leq \frac{b_{22}-b_{12}}{\delta_B}$$

operating from the previous one, we gets

$$(a_{22}-a_{12})(b_{22}-b_{21}) \leq \delta_A \delta_B$$

or

$$(a_{12}-a_{22}) \leq \delta_B \frac{\bar{\varepsilon}_1}{\bar{\varepsilon}_2}$$

and from the second inequality we get

$$(a_{22}-a_{12})(b_{22}-b_{21}) \leq \delta_A \delta_B$$

or

$$\left(b_{12}-b_{22}\right) \le \delta_A \frac{\bar{\varepsilon}_2}{\bar{\varepsilon}_1}.$$

As a numerical example we present the following payoff matrices

$$A=\begin{pmatrix} 1 & 2 \\ 1{,}5 & 1{,}3 \end{pmatrix} \text{ and } B=\begin{pmatrix} 1 & 3 \\ 1{,}5 & 1 \end{pmatrix}$$

then we get $\delta_A = 0{,}7 > 0$, $\delta_B = 2{,}5 > 0$, $a_{22} - a_{21} = 1$, $b_{22} - b_{12} = 2$, $a_{11} - a_{12} = 0{,}2$, $b_{11} - b_{21} = 0{,}5$, $a_{11} - a_{21} = 0{,}5$ and $b_{11} - b_{12} = 0{,}5$.

In order to give an exhaustive explanation of the whole topics we present the computation of the completely mixed Nash equilibrium, considering

$$E_1\left(1,\bar{y}\right) = E_1\left(2,\bar{y}\right)$$

and

$$E_2\left(\bar{x},1\right) = E_2\left(\bar{x},2\right)$$

from here we get $\bar{x} = 4/5$ and $\bar{y} = 5/6$, then

$$\bar{x} + \bar{y} - 1 \le z_{11} \le min\{\bar{x},\bar{y}\}$$

$$\frac{4}{5} + \frac{5}{6} - 1 \le z_{11} \le min\left\{\frac{4}{5},\frac{5}{6}\right\}$$

$$0{,}63 \le z_{11} \le 0{,}8$$

and therefore for the extreme $z_{11} = 0{,}63 = 19/30$ we get $z_{21} = 1/6$, $z_{12} = 1/5$ and $z_{22} = 0$, when we consider $z_{11} = 4/5$ we get $z_{21} = 0$, $z_{12} = 1/30$ and $z_{22} = 1/6$.

In this way we have proved that for all z_{11} such that $0{,}63 \le z_{11} \le 0{,}8$ the Aumann correlated equilibrium points project to the Nash $\left(4/5, 5/6\right)$.

The case where the Nash point is not completely mixed is much easier that the just study.

Comments

As we have mention in the text, we point out that the first exact introduction explicit computations of correlated point were presented by Marchi in [3]. Here it was extended the minimax theorem when we take into account correlated strategies. In such a contribution besides of the minimax theory it was studied the natural fiber bundle of the relationships among the product of mixed probabilities with the joint strategies. In the way that we found in this contribution, the need of study the faces and extreme points of the fibers is immediately given by considering the inequality

$$\bar{x} + \bar{y} - 1 \le z_{11} \le min\{\bar{x},\bar{y}\}.$$

In general in Marchi [3] it was extended the just fiber bundle for arbitrary dimension and axes and in Marchi and Morillas [5] in functional analysis. We make very clear that the notion of correlated equilibrium points in general was first introduced in the literature by Aumann in [1] and proved rigorously mathematically by Hart – Schmeidler in [2]. In similar way as we studied here it is possible to analyse the posteriorly introduced coarse equilibrium points.

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