A FAMILY OF STEADY TWO-PHASE GENERALIZED FORCHHEIMER FLOWS AND THEIR LINEAR STABILITY ANALYSIS

By

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# A FAMILY OF STEADY TWO-PHASE GENERALIZED FORCHHEIMER FLOWS AND THEIR LINEAR STABILITY ANALYSIS

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Abstract. We model multi-dimensional two-phase flows of incompressible fluids in porous media using generalized Forchheimer equations and the capillary pressure. Firstly, we find a family of steady state solutions whose saturation and pressure are radially symmetric and velocities are rotation-invariant. Their properties are investigated based on relations between the capillary pressure, each phase's relative permeability and Forchheimer polynomial. Secondly, we analyze the linear stability of those steady states. The linearized system is derived and reduced to a parabolic equation for the saturation. This equation has a special structure depending on the steady states which we exploit to prove two new forms of the lemma of growth of Landis-type in both bounded and unbounded domains. Using these lemmas, qualitative properties of the solution of the linearized equation are studied in details. In bounded domains, we show that the solution decays exponentially in time. In unbounded domains, in addition to their stability, the solution decays to zero as the spatial variables tend to infinity. The Bernstein technique is also used in estimating the velocities. All results have a clear physical interpretation.

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## 1. Introduction

In this paper, we study two-phase flows of incompressible fluids in porous media with each phase subjected to a Forchheimer equation. Forchheimer equations are often used by engineers to take into account the deviation from Darcy's law in case of high velocity, see e.g. [4, 20]. The standard Forchheimer equations are two-term law with quadratic nonlinearity, three-term law with cubic nonlinearity, and power law with a non-integer power less than two (see again [4, 20]). These models are extended to the generalized Forchheimer equation of the form

$$g(|\mathbf{u}|)\mathbf{u} = -\nabla p, \tag{1.1}$$

where $\mathbf{u}(\mathbf{x}, t)$ is the velocity field, $p(\mathbf{x}, t)$ is the pressure, and $g(s)$ is a generalized polynomial of arbitrary order (integer or non-integer) with positive coefficients. This equation was intensively analyzed for single-phase flows from mathematical and applied point of view in [3, 11–13, 15]. Its study for two-phase flows was later initiated in [14]. Regarding two-phase flows in porous media, it is always a challenging subject even for Darcy's law. Their models involve a complicated system of

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nonlinear partial differential equations (PDE) for pressures, velocities, densities and saturations with many parameters such as porosity, relative permeability functions and capillary pressure function. Current analysis of two-phase Darcy flows in literature is mainly focused on the existence of weak solutions [6–8] and their regularity [1, 2, 9, 17, 18]. However, questions about the stability and dynamics are not answered. The nonlinearity of the relative permeabilities and capillary pressure and their imprecise characteristics near the extreme values make it hard to analyze the modeling PDE system. The two-phase generalized Forchheimer flows are even more difficult due to the additional nonlinearity in the momentum equation. For example, unlike the Darcy flows, there is no Kruzkov-Sukorjanski transformation [17] to convert the system to a convenient form for the total velocity. Therefore, new methods are needed for the Forchheimer flows. In [14], we study the one-dimensional case using a novel approach. We will develop the techniques in [14] further to investigate the multi-dimensional case in this article.

We consider $n$-dimensional two-phase flows in porous media with constant porosity $\phi$ between 0 and 1. Here the dimension $n$ is greater or equal to 2, even though in practice we only need $n = 2, 3$. Each position $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ in the medium is considered to be occupied by two fluids called phase 1 (for example, water) and phase 2 (for example, oil).

Saturation, density, velocity, and pressure for each $i$th-phase ($i = 1, 2$) are $S_i \in [0, 1]$, $\rho_i \geq 0$, $\mathbf{u_i} \in \mathbb{R}^{\mathbf{n}}$ and $p_i \in \mathbb{R}$, respectively. The saturation functions naturally satisfy

$$S_1 + S_2 = 1. \tag{1.2}$$

Each phase's velocity is assumed to obey the generalized Forchheimer equation:

$$g_i(|\mathbf{u}_i|)\mathbf{u}_i = -\tilde{f}_i(S_i)\nabla p_i, \quad i = 1, 2, \tag{1.3}$$

where $\tilde{f}_i(S_i)$ is the relative permeability for the $i$th phase, and $g_i$ is of the form

$$g_i(s) = a_0 s^{\alpha_0} + a_1 s^{\alpha_1} + \ldots + a_N s^{\alpha_N}, \quad s \geq 0, \tag{1.4}$$

with $N \geq 0$, $a_0 > 0$, $a_1, \ldots a_N \geq 0$, $\alpha_0 = 0 < \alpha_1 < \ldots \alpha_N$, all $\alpha_1, \ldots \alpha_N$ are real numbers. The above $N$, $a_j$, $\alpha_j$ in (1.4) depend on each $i$. We call $g_i(s)$ in (1.4) the Forchheimer polynomial of (1.3).

Conservation of mass commonly holds for each of the phases:

$$\partial_t(\phi\rho_i S_i) + \mathrm{div}(\rho_i \mathbf{u}_i) = 0, \quad i = 1, 2. \tag{1.5}$$

Due to incompressibility of the phases, i.e. $\rho_i = const. > 0$, Eq. (1.5) is reduced to

$$\phi\partial_t S_i + \mathrm{div}\,\mathbf{u}_i = 0, \quad i = 1, 2. \tag{1.6}$$

Let $p_c$ be the capillary pressure between two phases, more specifically,

$$p_1 - p_2 = p_c. \tag{1.7}$$

Hereafterward, we denote $S = S_1$. The relative permeabilities and capillary pressure are re-denoted as functions of $S$, that is, $\tilde{f}_1(S_1) = f_1(S)$, $\tilde{f}_2(S_2) = f_2(S)$ and $p_c = p_c(S)$. Then (1.3) and (1.7) become

$$g_i(|\mathbf{u}_i|)\mathbf{u}_i = -f_i(S)\nabla p_i, \quad i = 1, 2, \tag{1.8}$$

$$p_1 - p_2 = p_c(S). \tag{1.9}$$

By scaling time, we can mathematically consider, without loss of generality, $\phi = 1$.

By (1.2) and (1.6):

$$S_t = -\mathrm{div}\,\mathbf{u}_1, \quad S_t = \mathrm{div}\,\mathbf{u}_2. \tag{1.10}$$

For $i = 1, 2$, define the function $\mathbf{G}_i(\mathbf{u}) = g_i(|\mathbf{u}|)\mathbf{u}$ for $\mathbf{u} \in \mathbb{R}^n$. Then by (1.8),

$$\mathbf{G}_i(\mathbf{u}_i) = -f_i(S)\nabla p_i, \quad \text{or,} \quad \nabla p_i = -\frac{\mathbf{G}_i(\mathbf{u}_i)}{f_i(S)}. \tag{1.11}$$

Taking gradient of the equation (1.9) we have

$$\nabla p_1 - \nabla p_2 = p_c'(S)\nabla S. \tag{1.12}$$

Substituting (1.11) into (1.12) yields

$$\frac{g_2(|\mathbf{u}_2|)\mathbf{u}_2}{f_2(S)} - \frac{g_1(|\mathbf{u}_1|)\mathbf{u}_1}{f_1(S)} = p_c'(S)\nabla S,$$

hence

$$F_2(S)g_2(|\mathbf{u}_2|)\mathbf{u}_2 - F_1(S)g_1(|\mathbf{u}_1|)\mathbf{u}_1 = \nabla S,$$

where

$$F_i(S) = \frac{1}{p_c'(S)f_i(S)}, \quad i = 1, 2. \tag{1.13}$$

In summary we study the following PDE system for $\mathbf{x} \in \mathbb{R}^n$ and $t \in \mathbb{R}$:

$$0 \le S = S(\mathbf{x}, t) \le 1, \tag{1.14a}$$

$$S_t = -\mathrm{div}\,\mathbf{u}_1, \tag{1.14b}$$

$$S_t = \mathrm{div}\,\mathbf{u}_2, \tag{1.14c}$$

$$\nabla S = F_2(S)\mathbf{G}_2(\mathbf{u}_2) - F_1(S)\mathbf{G}_1(\mathbf{u}_1). \tag{1.14d}$$

This paper is devoted to studying system (1.14). We will obtain a family of non-constant steady states with particular geometric properties. Specifically, the saturation and pressure are functions of $|\mathbf{x}|$, while each phase's velocity is $\mathbf{x}$ multiplied by a radial scalar function. Their properties, particularly, the behavior as $|\mathbf{x}| \to \infty$, will be obtained. For the stability study, we linearize system (1.14) at these steady states. We deduce from this linearized system a parabolic equation for the saturation. In bounded domains, we establish the lemma of growth in time and prove the exponential decay of its solutions in sup-norm as time $t \to \infty$. In unbounded domains, we prove the maximum principle and the stability. Furthermore, we show that the solutions go to zero as the spatial variables tend to infinity.

The paper is organized as follows. In section 2 we find the family of non-constant steady states described above. Various sufficient conditions are given for their existence in unbounded domains (Theorems 2.2). Their asymptotic behavior as $|\mathbf{x}| \to \infty$ is studied in details. In section 3, we linearize the originally system at the obtained steady states. We derive a parabolic equation for the saturation which will become the focus of our study. It is then converted to a convenient form for the study of sup-norm of solutions. Such a conversion is possible thanks to the special structure of the equation and of the steady states. Preliminary properties of the coefficient functions of this linearized equation are presented. Section 4 is focused on the study of the linearized equation for saturation in bounded domains. We prove the asymptotic stability results (Theorems 4.8 and 4.9) by utilizing a variation of Landis's lemma of growth in time variable (Lemma 4.3). The Bernstein's a priori estimate technique is used in proving interior continuous dependence of the velocities on the initial and boundary data (Proposition 4.7). In section 5, we study the linearized equation in an (unbounded) outer domain. The maximum principle (Theorem 5.2) is proved and used to obtain the stability of the zero solution (Theorems 5.10 and 5.11, part (ii)). We also prove a lemma of growth in the spatial variables (Lemma 5.5) by constructing particular barriers (super-solutions) using the specific structure of the linearized equation for saturation (Lemma 5.4). Using this, we prove a dichotomy theorem on the solution's behavior (Lemma 5.6), and ultimately show that the solution, on any finite time interval, decays to zero as $|\mathbf{x}| \to \infty$. For time tending to infinity, we find an increasing, continuous function $r(t) > 0$ with $r(t) \to \infty$ as $t \to \infty$ such that along any curve $\mathbf{x}(t)$ with $|\mathbf{x}(t)| \ge r(t)$, the solution goes to zero. (See Theorems 5.10 and 5.11, part (iii).) It is worth mentioning that the asymptotic stability in sup-norm in section 4 and behavior of the solution at spatial infinity have their own merits in the qualitative theory of linear parabolic equations.

## 2. Special steady states

In this section we find and study steady states which processes some symmetry.

Assume $p_i$ and $S$ are radial functions. We can write

$$p_i(\mathbf{x},t) = p_i(r,t), \quad S(\mathbf{x},t) = S(r,t), \quad \text{where } r = |\mathbf{x}| = \big(\sum_{i=1}^n x_i^2\big)^{1/2}. \tag{2.1}$$

Denote $\mathbf{e}_r = \mathbf{x}/|\mathbf{x}|$. By (1.8),

$$g_i(|\mathbf{u}_i|)\mathbf{u}_i = -f_i(S)\frac{\partial p_i}{\partial r}\cdot\frac{\mathbf{x}}{r} = -f_i(S)\frac{\partial p_i}{\partial r}\mathbf{e}_r. \tag{2.2}$$

Noting in (2.2) that $f_i(S)\frac{\partial p_i}{\partial r}$ is radial, then clearly $|\mathbf{u}_i|$ is also radial and we have

$$\mathbf{u}_i = u_{ir}\mathbf{e}_r, \quad \text{where } u_{ir} = \mathbf{u}_i\cdot\mathbf{e}_r = u_{ir}(r,t). \tag{2.3}$$

Therefore

$$\operatorname{div}\mathbf{u}_i = \frac{1}{r^{n-1}}\frac{\partial}{\partial r}(r^{n-1}u_{ir}) \tag{2.4}$$

and, from (1.14d),

$$F_2(S)g_2(|\mathbf{u}_2|)\mathbf{u}_2 - F_1(S)g_1(|\mathbf{u}_1|)\mathbf{u}_1 = \nabla S = \frac{\partial S}{\partial r}\mathbf{e}_r. \tag{2.5}$$

Taking the scalar product of both sides of (2.5) with $\mathbf{e}_r$ we obtain

$$G_2(u_{2r})F_2(S) - G_1(u_{1r})F_1(S) = \frac{\partial S}{\partial r}, \tag{2.6}$$

where

$$G_i(u) = g_i(|u|)u \quad \text{for } u\in\mathbb{R}. \tag{2.7}$$

We will study $S(r,t)$ and $u_i(r,t) \stackrel{\text{def}}{=\!=} u_{ir}$ $(i=1,2)$ as functions of independent variables $(r,t)\in(0,\infty)\times\mathbb{R}$. The system (1.14) becomes

$$0\le S\le 1, \tag{2.8a}$$

$$\frac{\partial S}{\partial t} = -r^{1-n}\frac{\partial}{\partial r}(r^{n-1}u_1), \tag{2.8b}$$

$$\frac{\partial S}{\partial t} = r^{1-n}\frac{\partial}{\partial r}(r^{n-1}u_2), \tag{2.8c}$$

$$\frac{\partial S}{\partial r} = G_2(u_2)F_2(S) - G_1(u_1)F_1(S). \tag{2.8d}$$

We make basic assumptions on the relative permeabilities and capillary pressure.

**Assumption A.**

$$f_1, f_2 \in C([0,1])\cap C^1((0,1)), \tag{2.9a}$$

$$f_1(0)=0, \quad f_2(1)=0, \tag{2.9b}$$

$$f_1'(S)>0, \quad f_2'(S)<0 \text{ on } (0,1). \tag{2.9c}$$

**Assumption B.**

$$p_c'\in C^1((0,1)), \quad p_c'(S)>0 \text{ on } (0,1). \tag{2.10}$$

We find steady state solutions $(S,u_1,u_2) = (S(r),u_1(r),u_2(r))$ for system (2.8) in the domain $[r_0,\infty)$ for a fixed $r_0>0$.

From (2.8b), we have $\frac{d}{dr}(r^{n-1}u_i)=0$, hence

$$u_i(r) = c_i r^{1-n}, \quad \text{where } c_i = \mathit{const.}, \quad i=1,2. \tag{2.11}$$

Substituting (2.11) into (1.14d) yields

$$S' = G_2(c_2r^{1-n})F_2(S) - G_1(c_1r^{1-n})F_1(S) \quad \text{for } r>r_0. \tag{2.12}$$

The rest of this section is devoted to studying the following initial value problem with constraints:

$$S' = F(r,S(r)) \quad \text{for } r>r_0, \quad S(r_0)=s_0, \quad 0<S(r)<1. \tag{2.13}$$

where $s_0$ is always a number in $(0,1)$ and

$$F(r,S(r)) = G_2(c_2 r^{1-n})F_2(S) - G_1(c_1 r^{1-n})F_1(S).$$

First we state a standard local existence theorem.

**Theorem 2.1.** *There exist a maximal interval of existence $[r_0, R_{\max})$, where $R_{\max} \in (r_0, \infty]$, and a unique solution $S \in C^1([r_0, R_{\max}))$ of (2.13) on $(r_0, R_{\max})$. Moreover, if $R_{\max}$ is finite then either*

$$\lim_{r\to R_{\max}^-} S(r) = 0 \quad \text{or} \lim_{r\to R_{\max}^-} S(r) = 1. \tag{2.14}$$

*Proof.* Under Assumption B, $F(r,S)$ is continuous and locally Lipschitz for the second variable for all $r \in (r_0, \infty)$, $S \in (0,1)$. The existence of the unique solution $S \in C^1([r_0, R_{\max}); (0,1))$ on the maximal interval $[0, R_{\max})$ is classical.

Assume $R_{\max} < \infty$. For given $0 < \varepsilon \le \varepsilon_0 \overset{\text{def}}{=\!=} \min\{1/4, R_{\max}/2\}$, let $K = [r_0, R_{\max}] \times [\varepsilon, 1-\varepsilon]$. We claim that there is $R_\varepsilon \ge r_0$ such that $(r, S(r)) \notin K$ for all $r \in (R_\varepsilon, R_{\max})$. Suppose not, then there is the sequence $r_i \to R_{\max}$ as $i \to \infty$ such that $(r_i, S(r_i)) \in K$ for all $i$. Choose $N > 0$ such that for all $i \ge N$,

$$\{(r,S) : |r - r_i| \le \varepsilon/2 \text{ and } |S - S(r_i)| \le \varepsilon/2\} \subset K',$$

where $K' = [r_0, R_{\max} + \varepsilon/2] \times [\varepsilon/2, 1-\varepsilon/2]$. According to the local Existence and Uniqueness theorem (Theorem 3.1 p. 18 in [10]) the solution starting at point $(r_i, S(r_i))$ exists on the interval $[r_i, r_i + d)$, where $d = \min\{\frac{1}{L}, \frac{\varepsilon}{2}, \frac{\varepsilon}{2M}\}$ with $M = \max_{K'} |F(r,S)|$ and $L$ being the Lipschitz constant for $F$ in $K'$. Note that $d$ is independent of $i$. Let $i$ be sufficiently large such that $r_i + d > R_{\max}$, then solution $S(r)$ exists beyond $R_{\max}$ which is a contradiction to maximality of $R_{\max}$. Hence our claim is true. Now using the continuity of $S(r)$ we have

$$\text{either } S(r) > 1-\varepsilon, \forall r \in (R_\varepsilon, R_{\max}) \text{ or } S(r) < \varepsilon, \forall r \in (R_\varepsilon, R_{\max}). \tag{2.15}$$

In particular, for $\varepsilon = \varepsilon_0$ we have either (a) $S(r) > 1 - \varepsilon_0, \forall r \in (R_{\varepsilon_0}, R_{\max})$, or (b) $S(r) < \varepsilon_0, \forall r \in (R_{\varepsilon_0}, R_{\max})$. In case (a), it is easy to see from (2.15) that for $0 < \varepsilon < \varepsilon_0$, $S(r) > 1 - \varepsilon, \forall r \in (R'_\varepsilon, R_{\max})$ where $R'_\varepsilon = \max\{R_{\varepsilon_0}, R_\varepsilon\}$. Thus, $\lim_{r\to R_{\max}^-} S(r) = 1$. Similarly, for the case (b) we have $\lim_{r\to R_{\max}^-} S(r) = 0$. The proof is complete. □

Next, we are interested in the case $R_{\max} = \infty$. First, we find sufficient conditions for that. We need to make the following assumptions on the relative permeabilities and capillary pressure:

$$\lim_{S\to 0} p_c'(S) f_1(S) = \lim_{S\to 1} p_c'(S) f_2(S) = +\infty. \tag{2.16}$$

These are our interpretation of experimental data (c.f. [4]), especially of those obtained in [5]. They cover certain scenarios of two-phase fluids in reality.

By (1.13) and (2.16), $F_1$ and $F_2$ can now be extended to functions of class $C([0,1]) \cap C^1((0,1))$ and satisfy

$$F_1(0) = F_1(1) = F_2(0) = F_2(1) = 0. \tag{2.17}$$

Therefore the right hand side of (1.14d) is well-defined for all $S \in [0,1]$. Note that

$$\lim_{S\to 0^+} \frac{F_1(S)}{F_2(S)} = \lim_{S\to 1^-} \frac{F_2(S)}{F_1(S)} = \infty. \tag{2.18}$$

The following additional conditions on $F_1$ and $F_2$ will be referred to in our considerations:

$$\limsup_{S\to 0^+} F_1'(S) < \infty, \tag{2.19}$$

$$\liminf_{S\to 1^-} F_1'(S) > -\infty. \tag{2.20}$$

$$\liminf_{S\to 1^-} F_2'(S) > -\infty. \tag{2.21}$$

$$\limsup_{S\to 0^+} F_2'(S) < \infty, \tag{2.22}$$

**Theorem 2.2.** *Assume (2.16) and $c_1^2 + c_2^2 > 0$. Then $R_{\max}$ in Theorem 2.1 is infinity, that is, the solution $S(r)$ of (2.13) exists on $[r_0, \infty)$, in the following cases*

*Case 1a.* $c_2 \le 0 < c_1$ *and (2.19).* *Case 1b.* $c_1 = 0 > c_2$ *and (2.22).*

*Case 2a.* $c_1 \le 0 < c_2$ *and (2.21).* *Case 2b.* $c_2 = 0 > c_1$ *and (2.20).*

*Case 3.* $c_1, c_2 > 0$ *and (2.19), (2.21).*

*Case 4.* $c_1, c_2 < 0$.

*Proof.* Suppose $R_{\max} < \infty$. We consider the following four cases.

**Case 1.** $c_2 \le 0 \le c_1$. We provide the proof of Case 1a, while Case 1b can be proved similarly. We have $F(r, S) < 0$ for all $r \in [r_0, R_{\max})$. Thus $S' < 0$ for all $r \in [r_0, R_{\max})$. By Theorem 2.1,

$$\lim_{r \to R_{\max}^-} S(r) = 0. \tag{2.23}$$

Note that $G_1(c_1 r^{1-n})$ and $G_2(c_2 r^{1-n})$ are bounded, and $G_1(c_1 r^{1-n})$ is bounded below by a positive number on $[r_0, R_{\max}]$. Combining these facts with relation (2.18), we infer that there are $\delta > 0$ and $C_1, C_2 > 0$ such that for $r \in [0, R_{\max})$ and $S \in (0, \delta)$,

$$-C_1 F_1(S) \le F(r, S) \le -C_2 F_1(S). \tag{2.24}$$

By (2.23), there is $r_1 \in (0, R_{\max})$ such that $S(r) < \delta$ for all $r \in [r_1, R_{\max})$. Define $Y(r) = F_1(S(r))$. By (2.19), there are $\tilde{r} \in (r_1, R_{\max})$ and $C_3 > 0$

$$F_1'(S(r)) < C_3 \text{ for all } r \in (\tilde{r}, R_{\max}). \tag{2.25}$$

For $r \in (\tilde{r}, R_{\max})$, using (2.24) we have

$$Y'(r) = F_1'(S)S' = F_1'(S)F(r, S) \ge -CF_1'(S)F_1(S) > -C_4 F_1(S) = -C_4 Y(r), \tag{2.26}$$

where $C > 0$, $C_4 = CC_3 > 0$. Thus (2.26) gives

$$Y(r) \ge Y(\tilde{r}) e^{-C_4(r - \tilde{r})}, \quad r \in [\tilde{r}, R_{\max}). \tag{2.27}$$

We have from (2.23) and (2.17) that

$$\lim_{r \to R_{\max}^-} Y(r) = 0. \tag{2.28}$$

Let $r \to R_{\max}^-$ in (2.27) and using (2.28), we obtain $0 \ge Y(\tilde{r}) e^{-C_4(R_{\max} - \tilde{r})} > 0$ which is a contradiction.

**Case 2.** $c_1 \le 0 \le c_2$. Both Case 2a and 2b are proved similarly. Consider Case 2a. Since $F(r, S) > 0$ for all $r \in [r_0, R_{\max})$, $S' > 0$ for all $r \in [r_0, R_{\max})$ therefore by Theorem 2.1, $\lim_{r \to R_{\max}^-} S(r) = 1$.

Let $X = 1 - S$. Then $\lim_{r \to R_{\max}^-} X(r) = 0$ and

$$X' = -S' = -F(r, 1 - X) = \tilde{F}(r, X) = G_1(c_1 r^{1-n}) \tilde{F}_1(X) - G_2(c_2 r^{1-n}) \tilde{F}_2(X), \tag{2.29}$$

where $\tilde{F}_i(X) = F_i(1 - X)$. Similar to the proof of Case 1a, there are $\delta > 0$ and $C_1, C_2 > 0$ such that

$$-C_1 \tilde{F}_2(X) \le \tilde{F}(r, X) \le -C_2 \tilde{F}_2(X), \tag{2.30}$$

for all $r \in [r_0, R_{\max}]$ and $X \in (0, \delta)$. Note that condition (2.21) is equivalent to $\limsup_{X \to 0^+} \tilde{F}_2'(X) < \infty$. Repeating the proof in Case 1a with $\tilde{F}_2$ instead of $F_1$ leads to a contradiction.

**Case 3.** According to Theorem 2.1 we have two cases.

(i) Case $\lim_{r \to R_{\max}^-} S(r) = 0$. By (2.18) there are constants $C_1, C_2 > 0$ and $\delta > 0$ such that

$$-C_1 F_1(S) \le F(r, S) \le -C_2 F_1(S).$$

for all $r \in [r_0, R_{\max}]$ and $S \in (0, \delta)$. Also, there is $r_1 \in (0, R_{\max})$ such that $S(r) < \delta$ for all $r \in (r_1, R_{\max})$. Then the exact argument for Case 1a yields a contradiction.

(ii) Case $\lim_{r \to R_{\max}^-} S(r) = 1$. By (2.18), there $\delta > 0$ and $C_1, C_2 > 0$ such that

$$C_1 F_2(S) \le F(r, S) \le C_2 F_2(S)$$

for all $r \in [r_0, R_{\max}]$ and $S \in (1 - \delta, 1)$. Then the proof is proceeded similar to Case 2a under condition (2.21) to obtain a contradiction.

**Case 4.** Again, according to Theorem 2.1 we have two cases.

(i) Case $\lim_{r\to R_{\max}^-} S(r) = 0$. By (2.18), there are $\delta > 0$ and $C_1, C_2 > 0$ such that

$$0 < C_1F_1(S) \le F(r, S) \le C_2F_1(S)$$

for all $r \in [r_0, R_{\max}]$ and $S \in (0, \delta)$. Let $r_1$ be as in **Case 3**(i). Then for $r \in (r_1, R_{\max})$ we have $S'(r) > 0$, and hence $S(r) \ge S(r_1) > 0$ which contradicts the fact $\lim_{r\to R_{\max}^-} S(r) = 0$.

(ii) Case $\lim_{r\to R_{\max}^-} S(r) = 1$. By (2.18), there are $\delta > 0$ and $C_1, C_2 > 0$ such that

$$-C_1F_2(S) \le F(r, S) \le -C_2F_2(S) < 0$$

for all $r \in [r_0, R_{\max}]$ and $S \in (1-\delta, 1)$. There is $r_1 \in (r_0, R_{\max})$ such that $S(r) \in (1-\delta, 1)$ for all $r \in (r_1, R_{\max})$. Thus $S'(r) < 0$ for all $r \in (r_1, R_{\max})$ which gives $S(r) \le S(r_1)$. Letting $r \to R_{\max}$ yields $1 \le S(r_1) < 1$. This is a contradiction.

From all the above contradictions, we must have $R_{\max} = \infty$ and the proof is complete. □

To study $S(r)$ as $r \to \infty$, for the solution $S(r)$ in the Theorem 2.2 we will need function $h(r) \in (0, 1)$ such that

$$G_2(c_2r^{1-n})F_2(h(r)) - G_1(c_1r^{1-n})F_1(h(r)) = 0. \tag{2.31}$$

To prove existence of such function consider $c_1c_2 \ne 0$. Then (2.31) is equivalent to

$$\frac{f_1(h(r))}{f_2(h(r))} = \frac{c_1g_1(|c_1|r^{1-n})}{c_2g_2(|c_2|r^{1-n})}.$$

Since $f \stackrel{\text{def}}{=\!=} f_1/f_2$ is strictly increasing and maps $(0, 1)$ onto $(0, \infty)$, we can solve

$$h(r) = f^{-1}\Big(\frac{c_1g_1(|c_1|r^{1-n})}{c_2g_2(|c_2|r^{1-n})}\Big) \quad \text{provided} \quad c_1c_2 > 0. \tag{2.32}$$

Note that

$$\lim_{r\to\infty} h(r) = s^* \stackrel{\text{def}}{=\!=} f^{-1}\Big(\frac{c_1a_1^0}{c_2a_2^0}\Big) \in (0, 1). \tag{2.33}$$

Let $\xi(r) = r^{1-n} \in (0, \infty)$. We rewrite $h(r)$ as

$$h(r) = f^{-1}\Big(Q(\xi(r))\Big) \quad \text{where} \quad Q(\xi) = \frac{c_1g_1(|c_1|\xi)}{c_2g_2(|c_2|\xi)} \quad \text{for } \xi > 0. \tag{2.34}$$

**Theorem 2.3.** *If solution $S(r)$ of (2.13) exists in $[r_0, \infty)$, then there exists $R > r_0$ such that solution $S(r)$ is monotone on $(R, \infty)$, and, consequently, $\lim_{r\to\infty} S(r)$ exists.*

*Proof.* If $c_1c_2 \le 0$ then all $r \ge r_0$ either $S' \ge 0$ or $S' \le 0$. Thus $S(r)$ is monotone on $[r_0, \infty)$.

Consider the case $c_1c_2 > 0$. Then $h(r)$ in (2.34) exists. We rewrite $Q(\xi)$ as

$$Q(\xi) = \frac{c_1}{c_2} \cdot \frac{\sum_{i=0}^{m_1} a_i\xi^{\alpha_i}}{\sum_{i=0}^{m_2} b_i\xi^{\beta_i}}. \tag{2.35}$$

where $a_i, b_i > 0$, $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_{m_1}$, $0 = \beta_0 < \beta_1 < \cdots < \beta_{m_2}$ .

If $Q' \equiv 0$ then $h(r) \equiv s^*$ is an equilibrium. It is easy to see that if $s_0 > (<) s^*$ then $S(r) > (<) s^*$ for all $r$, hence $S(r)$ is monotone on $r \in [r_0, \infty)$.

Now we consider $Q' \ne 0$. A simple calculation gives

$$Q'(\xi) = \frac{c_1}{\xi c_2}\Big(\sum_{i=0}^{m_2} b_i\xi^{\beta_i}\Big)^{-2}\sum_{i=1}^{m_3} A_i\xi^{\gamma_i},$$

where $m_3 \ge 1$, $A_i \ne 0, 0 < \gamma_1 < \gamma_2 < \cdots < \gamma_{m_3}$. Note that $Q'(\xi)$ has the same sign as $A_1$ for $\xi > 0$ sufficiently small. Combining this with the fact $f' > 0$, we have that $A_1h'(r) < 0$ for all $r > R$, where $R > 0$ is a sufficiently large number.

**Claim 1.** There is $\tilde{R} > R$ such that $S'(r) \ge 0$ on $(\tilde{R}, \infty)$ or $S'(r) \le 0$ on $(\tilde{R}, \infty)$.

Then the theorem's statements obviously follow Claim 1.

To prove Claim 1 we consider the following cases.

**Case 1:** $A_1 < 0$. Then $h(r)$ is increasing in $[R,\infty)$ and, hence, $h(r) < s^*$ for all $r \ge R$.

Case 1A: $S(r) \ge h(r)$ for all $r > R$. Then $S' \ge 0$ for all $r > R$ or $S' \le 0$ for all $r > R$.

Case 1B: There exists $R_1 > R$ such that $S(R_1) < h(R_1)$.

+ Case 1B(i): $F(r,S) > 0 \Leftrightarrow S > h(r)$. Then $S' > 0$ if $S(r) > h(r)$ and $S' < 0$ if $S(r) < h(r)$. It is easy to see that $S(r) < h(R_1) \le h(r)$ for all $r > R_1$. Therefore $S'(r) < 0$ for all $r > R_1$.

+ Case 1B(ii): $F(r,S) < 0 \Leftrightarrow S > h(r)$. Then $S' < 0$ if $S(r) > h(r)$ and $S' > 0$ if $S(r) < h(r)$.

**Claim 2.** $S(r) \le h(r)$ for all $r \ge R_1$ and hence $S'(r) \ge 0$ for all $r > R_1$.

Suppose Claim 2 is false. Then there is $R_2 > R_1$ such that $S(R_2) > h(R_2)$. There is $\tilde{r} \in (R_1, R_2)$ such that $S(\tilde{r}) = h(\tilde{r})$. Hence, $S$ is decreasing on $(\tilde{r}, R_2)$, $S(R_2) \le S(\tilde{r}) = h(\tilde{r}) \le h(R_2)$. This is a contradiction.

**Case 2:** $A_1 > 0$. Then $h(r)$ is decreasing in $[R,\infty)$ and $h(r) > s^*$ for all $r \ge R$ .

Case 2A: $S(r) \le h(r)$ for all $r > R$. Then $S' \le 0$ for all $r > R$ or $S' \ge 0$ for all $r > R$.

Case 2B: There exists $R_1 > R$ such that $S(R_1) > h(R_1)$.

+ Case 2B(i): $F(r,S) > 0 \Leftrightarrow S > h(r)$. Then $S' > 0$ if $S(r) > h(r)$ and $S' < 0$ if $S(r) < h(r)$. Similar to Case 1B(i), $h(r) < h(R_1) < S(r)$ for all $r > R_1$. Therefore $S'(r) > 0$ for all $r > R_1$.

+ Case 2B(ii): $F(r,S) < 0 \Leftrightarrow S > h(r)$. Then $S' < 0$ if $S(r) > h(r)$ and $S' > 0$ if $S(r) < h(r)$. Similar to Case 1B(ii), $S(r) \ge h(r)$ for all $r \ge R_1$. Therefore $S'(r) \le 0$ for all $r > R_1$.

From the above considerations, we see that Claim 1 holds true and the proof is complete. □

Let $s_\infty = \lim_{r\to\infty} S(r)$ in Theorem 2.3. Note that $s_\infty \in [0,1]$.

**Lemma 2.4.** *For $n = 2$ and $c_1^2 + c_2^2 > 0$, if $s_\infty$ is neither $0$ nor $1$ then $s_\infty$ must be $s^*$.*

*Proof.* Assume $s_\infty \ne 0, 1$. We prove by contradiction. Suppose $s_\infty \ne s^*$. Then

$$c_3 \stackrel{\text{def}}{=\!=} |F_2(s_\infty) a_2^0 c_2 - F_1(s_\infty) a_1^0 c_1| > 0. \tag{2.36}$$

For any $R > r_0$, We write $S(r) = I_1(R) + I_2(R)$ where

$$I_1(R) = s_0 + \int_{r_0}^{R} F(z, S(z))dz \quad \text{and} \quad I_2(R) = \int_R^r F(z,S(z))dz.$$

For sufficiently large $R$ and $r > R$

$$|I_2(R)| = \int_R^r F(z,S(z))dz \ge \frac{c_3}{2}\int_R^r z^{-1}dz = \frac{c_3}{2}(\ln r - \ln R).$$

Therefore

$$|S(r)| \ge \frac{c_3}{2}(\ln r - \ln R) - I_1(R) \to \infty \text{ as } r \to \infty.$$

Thus $S(r)$ is unbounded which contradicts the fact $S(r) \in (0,1)$. Hence $s_\infty = s^*$. □

Using Lemma 2.4 we can drastically reduce the range of $s_\infty$ in case $n = 2$.

**Theorem 2.5.** *Let $n = 2$ and $c_1^2 + c_2^2 > 0$. Suppose $S(r)$ is a solution of (2.13) on $[r_0, \infty)$.*

(i) *If $c_1 \le 0$ and $c_2 \ge 0$ then $s_\infty = 1$.*

(ii) *If $c_1 \ge 0$ and $c_2 \le 0$ then $s_\infty = 0$.*

(iii) *If $c_1, c_2 < 0$ then $s_\infty = s^*$.*

(iv) *If $c_1, c_2 > 0$ then $s_\infty \in \{0, 1, s^*\}$.*

*Proof.* (i) In this case, $S'(r) > 0$ for all r, hence $S(r) > s_0$. This implies $s_\infty \ne 0$. In addition, $s^*$ does not exist. Therefore, by Lemma 2.4, $s_\infty$ must be 1.

(ii) The proof is similar to that of (i).

(iii) We have $F(r,S) < 0$ for $S < h(r)$ and $F(r,S) > 0$ for $S < h(r)$. Thus, it is easy to see that $s_\infty$ cannot be $0, 1$. By Lemma 2.4, $s_\infty$ must be $s^*$.

(iv) This is a direct consequence of Lemma 2.4. □

In general, we do not know the value of $s_\infty$ based on $s_0$. However, in some particular cases, we can determine the range of $s_\infty$.

**Example 2.6.** We consider the following special $g_i$'s:

$$g_i(u) = a_i + b_i u^\alpha \quad \text{where } a_i > 0,\ b_i > 0,\ \text{for } i = 1, 2 \text{ and } \alpha > 0. \tag{2.37}$$

We have from (2.34) when $c_1 c_2 > 0$ that

$$Q'(\xi) = \frac{c_1 \Delta}{c_2(a_2 + b_2|c_2|^\alpha \xi)^2} \text{ with } \Delta = a_2 b_1 |c_1|^\alpha - a_1 b_2 |c_2|^\alpha.$$

We now detail the range of $s_\infty$ case by case.
**Case** $n > 2$**.**

A. $c_1, c_2 > 0$.
  A1. $\Delta < 0$.
    (i) $s_0 > s^*$. Then $s_\infty \in (s_0, 1]$.
    (ii) $h(r_0) \le s_0 \le s^*$. Then $s_\infty \in [0, 1]$.
    (iii) $s_0 < h(r_0)$. Then $s_\infty \in [0, s_0)$.
  A2. $\Delta > 0$.
    (i) $s_0 > h(r_0)$. Then $s_\infty \in (s_0, 1]$.
    (ii) $s^* \le s_0 \le h(r_0)$. Then $s_\infty \in [0, 1]$.
    (iii) $s_0 < s^*$. Then $s_\infty \in [0, s_0)$.
  A3. $\Delta = 0$.
    (i) $s_0 > s^*$. Then $s_\infty \in (s_0, 1]$.
    (ii) $s_0 = s^*$. Then $s_\infty = s^*$.
    (iii) $s_0 < s^*$. Then $s_\infty \in [0, s_0)$.
B. $c_1, c_2 < 0$.
  B1. $\Delta < 0$.
    (i) $s_0 > s^*$. Then $s_\infty \in (h(r_0), s_0)$.
    (ii) $h(r_0) \le s_0 \le s^*$. Then $s_\infty \in (h(r_0), s^*]$.
    (iii) $s_0 < h(r_0)$. Then $s_\infty \in (s_0, s^*]$.
  B2. $\Delta > 0$.
    (i) $s_0 > h(r_0)$. Then $s_\infty \in [s^*, s_0)$.
    (ii) $s^* \le s_0 \le h(r_0)$. Then $s_\infty \in [s^*, h(r_0))$.
    (iii) $s_0 < s^*$. Then $s_\infty \in (s_0, h(r_0))$.
  B3. $\Delta = 0$.
    (i) $s_0 > s^*$. Then $s_\infty \in [s^*, s_0)$.
    (ii) $s_0 = s^*$. Then $s_\infty = s^*$.
    (iii) $s_0 < s^*$. Then $s_\infty \in (s_0, s^*]$.
C. $c_1 \le 0 < c_2$ or $c_1 < 0 = c_2$. Then $s_\infty \in (s_0, 1]$.
D. $c_2 \le 0 < c_1$ or $c_1 = 0 > c_2$. Then $s_\infty \in [0, s_0)$.

Verifications of the cases above are presented in the Appendix.
**Case** $n = 2$**.** We use the analysis in A, which is still valid for $n = 2$, to explicate the case $c_1, c_2 > 0$ in Theorem 2.5. Let $s_m = \min\{h(r_0), s^*\}$ and $s_M = \max\{h(r_0), s^*\}$.

(i) $s_0 > s_M$. Then $s_\infty = 1$.
(ii) $s_m \le s_0 \le s_M$. Then $s_\infty \in \{0, 1, s^*\}$.
(iii) $s_0 < s_m$. Then $s_\infty = 0$.

## 3. Linearization

We study the linear stability of a steady state solution $(\mathbf{u}_1^*(\mathbf{x}), \mathbf{u}_2^*(\mathbf{x}), S_*(\mathbf{x}))$ of system (1.14). The formal linearizion of system (1.14) at $(\mathbf{u}_1^*(\mathbf{x}), \mathbf{u}_2^*(\mathbf{x}), S_*(\mathbf{x}))$ is

$$\sigma_t = -\text{div } \mathbf{v}_1, \tag{3.1a}$$

$$\sigma_t = \text{div } \mathbf{v}_2, \tag{3.1b}$$

$$\nabla\sigma = F_2(S_*)\mathbf{G}_2'(\mathbf{u}_2^*)\mathbf{v}_2 + F_2'(S_*)\sigma\mathbf{G}_2(\mathbf{u}_2^*) - \Big(F_1(S_*)\mathbf{G}_1'(\mathbf{u}_1^*)\mathbf{v}_1 + F_1'(S_*)\sigma\mathbf{G}_1(\mathbf{u}_1^*)\Big). \tag{3.1c}$$

Above, the unknowns are $\sigma(\mathbf{x},t) \in \mathbb{R}$, $\mathbf{v}_1(\mathbf{x},t), \mathbf{v}_2(\mathbf{x},t) \in \mathbb{R}^n$. A solution $(\sigma, \mathbf{v}_1, \mathbf{v}_2)$ of (3.1) is considered as an approximation of the difference between a solution $(S(\mathbf{x},t), \mathbf{u}_1(\mathbf{x},t), \mathbf{u}_2(\mathbf{x},t))$ of (1.14) and the steady state $(\mathbf{u}_1^*(\mathbf{x}), \mathbf{u}_2^*(\mathbf{x}), S_*(\mathbf{x}))$ in (3.2). The system (3.1) is obtained by utilizing Taylor expansions in (1.14) at $(\mathbf{u}_1^*, \mathbf{u}_2^*, S_*)$ with respect to variables $\mathbf{u}_1, \mathbf{u}_2, S$ and then neglecting non-linear terms. In theory of ordinary differential equations, linearizion has direct connections with the stability of steady states. In PDE theory, this is not always the case. Nonetheless, in many scenarios, stability of the linearized equations lead to the stability of the original ones. In this article we only focus on the stability for the linearized system (3.1).

We consider, particularly, the steady states obtained in the previous section, that is,

$$\mathbf{u}_1^*(\mathbf{x}) = c_1|\mathbf{x}|^{-n}\mathbf{x}, \quad \mathbf{u}_2^*(\mathbf{x}) = c_2|\mathbf{x}|^{-n}\mathbf{x}, \quad S_*(\mathbf{x}) = \hat{S}(|\mathbf{x}|), \tag{3.2}$$

where $c_1, c_2$ are constants and $\hat{S}(r)$ is a solution of (2.13).

Let $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$. Adding equation (3.1a) to (3.1b) gives

$$\text{div } \mathbf{v} = 0. \tag{3.3}$$

Assume $\mathbf{v} = \mathbf{V}(\mathbf{x},t) \in \mathbb{R}^n$, where $\mathbf{V}(\mathbf{x},t)$ is a given function. We have

$$\mathbf{v}_1 = \mathbf{V} - \mathbf{v}_2, \tag{3.4}$$

hence (3.1c) provides

$$\nabla\sigma = \sigma\mathbf{b} + \underline{\mathbf{B}}\mathbf{v}_2 - \mathbf{c}, \tag{3.5}$$

where

$$\underline{\mathbf{B}} = \underline{\mathbf{B}}(\mathbf{x}) = F_2(S_*)\mathbf{G}_2'(\mathbf{u}_2^*) + F_1(S_*)\mathbf{G}_1'(\mathbf{u}_1^*), \tag{3.6}$$

$$\mathbf{b} = \mathbf{b}(\mathbf{x}) = F_2'(S_*)\mathbf{G}_2(\mathbf{u}_2^*) - F_1'(S_*)\mathbf{G}_1(\mathbf{u}_1^*), \tag{3.7}$$

$$\mathbf{c} = \mathbf{c}(\mathbf{x},t) = F_1(S_*)\mathbf{G}_1'(\mathbf{u}_1^*)\mathbf{V}(\mathbf{x},t). \tag{3.8}$$

The $n \times n$ matrix $\underline{\mathbf{B}}$ is invertible (see Lemma 3.2 below), and we denote its inverse by

$$\underline{\mathbf{A}} = \underline{\mathbf{A}}(\mathbf{x}) = \underline{\mathbf{B}}^{-1}(\mathbf{x}). \tag{3.9}$$

Solving for $\mathbf{v}_2$ from (3.5) we obtain

$$\mathbf{v}_2 = \underline{\mathbf{A}}(\nabla\sigma - \sigma\mathbf{b}) + \underline{\mathbf{A}}\mathbf{c}. \tag{3.10}$$

Substituting (3.10) into (3.1b) gives

$$\sigma_t = \nabla \cdot \Big[\underline{\mathbf{A}}(\nabla\sigma - \sigma\mathbf{b})\Big] + \nabla \cdot (\underline{\mathbf{A}}\mathbf{c}). \tag{3.11}$$

Then (3.11), (3.4) and (3.10) is our linearized system for (1.14) at the steady state $(\mathbf{u}_1^*(\mathbf{x}), \mathbf{u}_2^*(\mathbf{x}), S_*(\mathbf{x}))$.

**Remark 3.1.** *In our approach, the total velocity $\mathbf{V}$ and hence the vector function $\mathbf{c}$ are supposed to be known, whereas the phase velocities $\mathbf{v}_i$ $(i = 1, 2)$ are the unknowns. Therefore, our results below can be considered as the qualitative study of the flow depending on the property of the total velocity. Such restriction, however, is justified in practice or in case $\mathbf{V}$, as a perturbation, itself is radial. In the latter consideration, by (3.3), $\mathbf{V} = \mathbf{V}(t)$ is totally determined by its boundary values.*

We will focus on studying classical solutions of (3.11). For such purpose, the maximum principle plays an important role. Although there is not an obvious maximum principle for (3.11), we can convert it to an equation for which there is one. We proceed as follows. Rewrite vector function $\mathbf{b}(\mathbf{x})$ explicitly as

$$\mathbf{b}(\mathbf{x}) = \Big(F_2'(S_*(\mathbf{x}))g_2(\frac{|c_2|}{|\mathbf{x}|^{n-1}})\frac{c_2}{|\mathbf{x}|^n} - F_1'(S_*(\mathbf{x}))g_1(\frac{|c_1|}{|\mathbf{x}|^{n-1}})\frac{c_1}{|\mathbf{x}|^n}\Big)\mathbf{x} = \lambda(|\mathbf{x}|)\mathbf{x}, \tag{3.12}$$

where

$$\lambda(r) = F_2'(\hat{S}(r))g_2(\frac{|c_2|}{r^{n-1}})\frac{c_2}{r^n} - F_1'(\hat{S}(r))g_1(\frac{|c_1|}{r^{n-1}})\frac{c_1}{r^n}. \tag{3.13}$$

By defining

$$\Lambda(\mathbf{x}) = \frac{1}{2}\int_{r_0^2}^{|\mathbf{x}|^2} \lambda(\sqrt{\xi})d\xi = \int_{r_0}^{|\mathbf{x}|} r\lambda(r)dr, \tag{3.14}$$

we have for $\mathbf{x} \neq 0$ that

$$\mathbf{b}(\mathbf{x}) = \nabla\Lambda(\mathbf{x}). \tag{3.15}$$

Substituting this relation into (3.11) we obtain

$$\begin{aligned}\sigma_t &= \nabla\cdot\Big[\underline{\mathbf{A}}(\nabla\sigma - \sigma\nabla\Lambda)\Big] + \nabla\cdot(\underline{\mathbf{A}}\mathbf{c}) = \nabla\cdot\Big[e^{\Lambda}\underline{\mathbf{A}}\nabla(e^{-\Lambda}\sigma)\Big] + \nabla\cdot(\underline{\mathbf{A}}\mathbf{c})\\ &= e^{\Lambda}\nabla\cdot\Big[\underline{\mathbf{A}}\nabla(e^{-\Lambda}\sigma)\Big] + e^{\Lambda}\nabla\Lambda\cdot\Big[\underline{\mathbf{A}}\nabla(e^{-\Lambda}\sigma)\Big] + \nabla\cdot(\underline{\mathbf{A}}\mathbf{c}).\end{aligned}$$

Let

$$w(\mathbf{x},t) = e^{-\Lambda(\mathbf{x})}\sigma(\mathbf{x},t). \tag{3.16}$$

Then $w$ satisfies

$$w_t = e^{-\Lambda}\sigma_t = \nabla\cdot\Big(\underline{\mathbf{A}}\nabla w\Big) + \nabla\Lambda\cdot\underline{\mathbf{A}}\nabla w + e^{-\Lambda}\nabla\cdot(\underline{\mathbf{A}}\mathbf{c}). \tag{3.17}$$

Using relation (3.15) again yields

$$w_t - \nabla\cdot(\underline{\mathbf{A}}\nabla w) - \mathbf{b}\cdot\underline{\mathbf{A}}\nabla w = e^{-\Lambda}\nabla\cdot(\underline{\mathbf{A}}\mathbf{c}). \tag{3.18}$$

For the velocities, we have from (3.10) and (3.16) that

$$\mathbf{v}_2 = \underline{\mathbf{A}}\big[\nabla(e^{\Lambda}w) - e^{\Lambda}w\mathbf{b}\big] + \underline{\mathbf{A}}\mathbf{c} = \underline{\mathbf{A}}\big[e^{\Lambda}\nabla w + we^{\Lambda}\nabla\Lambda - e^{\Lambda}w\mathbf{b}\big] + \underline{\mathbf{A}}\mathbf{c}.$$

Thus,

$$\mathbf{v}_2 = e^{\Lambda}\underline{\mathbf{A}}\nabla w + \underline{\mathbf{A}}\mathbf{c}. \tag{3.19}$$

We will proceed by studying (3.18) first and then drawing conclusions for $\sigma, \mathbf{v}_1, \mathbf{v}_2$ via the relations (3.16), (3.19) and (3.4).

In the following, we present some properties of $\underline{\mathbf{B}}$, $\underline{\mathbf{A}}$ and $\mathbf{b}$. They have some structures and estimates which are crucial for our next sections. These are based on the special form of the steady state $(\mathbf{u}_1^*, \mathbf{u}_2^*, S_*)$.

Denote by $\underline{\mathbf{I}}_n$ the $n\times n$ identity matrix. Consider $c_1^2 + c_2^2 > 0$ and $\mathbf{x}\neq 0$. We have for $i = 1,2$ that

$$\mathbf{G}_i'(\mathbf{u}_i^*) = g_i(|\mathbf{u}_i^*|)\underline{\mathbf{I}}_n + g_i'(|\mathbf{u}_i^*|)\frac{\mathbf{u}_i^*(\mathbf{u}_i^*)^T}{|\mathbf{u}_i^*|} = g_i(|c_i|\,|\mathbf{x}|^{1-n})\underline{\mathbf{I}}_n + g_i'(|c_i|\,|\mathbf{x}|^{1-n})|c_i|\,|\mathbf{x}|^{-1-n}\mathbf{x}\mathbf{x}^T. \tag{3.20}$$

Since these matrices are symmetric, so is $\underline{\mathbf{B}}$. For each $i = 1,2$ and arbitrary $\mathbf{z}\in\mathbb{R}^n$,

$$\mathbf{z}^T\mathbf{G}_i'(\mathbf{u}_i^*)\mathbf{z} = g_i(|c_i|\,|\mathbf{x}|^{1-n})|\mathbf{z}|^2 + g_i'(|c_i|\,|\mathbf{x}|^{1-n})|c_i|\,|\mathbf{x}|^{-1-n}|\mathbf{x}\cdot\mathbf{z}|^2.$$

Define

$$\beta = \beta(\mathbf{x}) \stackrel{\text{def}}{=\!=} \sum_{i=1}^{2} F_i(S_*(\mathbf{x})) g_i(|c_i|\,|\mathbf{x}|^{1-n}), \tag{3.21}$$

$$\gamma = \gamma(\mathbf{x}) \stackrel{\text{def}}{=\!=} \sum_{i=1}^{2} F_i(S_*(\mathbf{x})) g_i'(|c_i|\,|\mathbf{x}|^{1-n})|c_i|\,|\mathbf{x}|^{1-n}. \tag{3.22}$$

Then

$$\beta|\mathbf{z}|^2 \le \mathbf{z}^T\underline{\mathbf{B}}\mathbf{z} \le (\beta+\gamma)|\mathbf{z}|^2. \tag{3.23}$$

The first inequality in (3.23) proves that $\mathbf{z}^T\underline{\mathbf{B}}\mathbf{z} > 0$ for all $\mathbf{z} \neq 0$. Therefore, $\underline{\mathbf{B}}$ is positive definite and hence it is invertible. Since $\underline{\mathbf{B}}$ is symmetric, so is its inverse $\underline{\mathbf{A}}$. Thus, we have:

**Lemma 3.2.** *For any $c_1^2 + c_2^2 > 0$ and $\mathbf{x} \neq 0$, matrices $\underline{\mathbf{B}}(\mathbf{x})$ and $\underline{\mathbf{A}}(\mathbf{x})$ are symmetric, invertible and positive definite.*

Since matrix $\underline{\mathbf{B}}$ is symmetric and positive definite, it has positive eigenvalues $\lambda_1(\underline{\mathbf{B}}) \le \lambda_2(\underline{\mathbf{B}}) \le \cdots \le \lambda_n(\underline{\mathbf{B}})$. We have

$$\lambda_1(\underline{\mathbf{B}}) = \min_{\mathbf{z}\neq 0} \frac{\mathbf{z}^T\underline{\mathbf{B}}\mathbf{z}}{|\mathbf{z}|^2} \quad \text{and} \quad \lambda_n(\underline{\mathbf{B}}) = \max_{\mathbf{z}\neq 0} \frac{\mathbf{z}^T\underline{\mathbf{B}}\mathbf{z}}{|\mathbf{z}|^2}. \tag{3.24}$$

It follows from (3.24) and (3.23) that

$$\beta \le \lambda_1(\underline{\mathbf{B}}) \le \lambda_n(\underline{\mathbf{B}}) \le \beta + \gamma. \tag{3.25}$$

By the Spectral Theorem,

$$\lambda_1(\underline{\mathbf{A}}) = \frac{1}{\lambda_n(\underline{\mathbf{B}})} \ge \frac{1}{\beta+\gamma} \quad \text{and} \quad \lambda_n(\underline{\mathbf{A}}) = \frac{1}{\lambda_1(\underline{\mathbf{B}})} \le \frac{1}{\beta}. \tag{3.26}$$

We now consider $0 < r_0 \le |\mathbf{x}| < R_{\max}$. Let $a_0^{(i)} = g_i(0)$ for $i = 1, 2$, and define

$$d_0 = \min\{a_0^{(1)}, a_0^{(2)}\}, \quad d_1 = d_1(r_0) = \sum_{i=1}^{2} g_i(|c_i| r_0^{1-n}), \tag{3.27}$$

$$d_2 = d_2(r_0) = \sum_{i=1}^{2} g_i(|c_i| r_0^{1-n})|c_i| r_0^{1-n}, \quad d_3 = d_3(r_0) = \sum_{i=1}^{2} g_i'(|c_i| r_0^{1-n})|c_i| r_0^{1-n}, \tag{3.28}$$

$$d_4 = d_4(r_0) = d_1 + d_3. \tag{3.29}$$

Then

$$d_0 \sum_{i=1}^{2} F_i(S_*(\mathbf{x})) \le \beta(\mathbf{x}) \le d_1 \sum_{i=1}^{2} F_i(S_*(\mathbf{x})) \quad \text{and} \quad \gamma(\mathbf{x}) \le d_3 \sum_{i=1}^{2} F_i(S_*(\mathbf{x})). \tag{3.30}$$

By (3.23), (3.26) and (3.30),

$$d_0|\mathbf{z}|^2 \sum_{i=1}^{2} F_i(S_*(\mathbf{x})) \le \mathbf{z}^T\underline{\mathbf{B}}(\mathbf{x})\mathbf{z} \le d_4|\mathbf{z}|^2 \sum_{i=1}^{2} F_i(S_*(\mathbf{x})), \tag{3.31}$$

$$\frac{1}{d_4 \sum_{i=1}^{2} F_i(S_*(\mathbf{x}))} \le \lambda_1(\underline{\mathbf{A}}) \le \lambda_n(\underline{\mathbf{A}}) \le \frac{1}{d_0 \sum_{i=1}^{2} F_i(S_*(\mathbf{x}))}. \tag{3.32}$$

Applying (3.24) to matrix $\underline{\mathbf{A}}$, we have

$$\mathbf{z}^T\underline{\mathbf{A}}(\mathbf{x})\mathbf{z} \ge \lambda_1(\underline{\mathbf{A}})|\mathbf{z}|^2 \ge \frac{|\mathbf{z}|^2}{d_4 \sum_{i=1}^{2} F_i(S_*(\mathbf{x}))} \quad \forall \mathbf{z} \in \mathbb{R}^{\mathbf{n}}. \tag{3.33}$$

Denote by $|\underline{\mathbf{A}}|$ and $\|\underline{\mathbf{A}}\|_{\rm op}$ the Euclidean and operator norms of matrix $\underline{\mathbf{A}}$, respectively. Then

$$|\underline{\mathbf{A}}| \le c_0 \|\underline{\mathbf{A}}\|_{\rm op} = c_0 \lambda_n(\underline{\mathbf{A}}), \tag{3.34}$$

for some constant $c_0 > 0$. Thus,

$$|\underline{\mathbf{A}}(\mathbf{x})| \le \frac{c_0}{d_0 \sum_{i=1}^2 F_i(S_*(\mathbf{x}))} \quad \forall |\mathbf{x}| \in [r_0, R_{\max}). \tag{3.35}$$

For the boundedness of $\mathbf{b}$, we have

$$|\mathbf{b}(\mathbf{x})| \le \sum_{i=1}^2 \Big[ |F_i'(\hat S(|\mathbf{x}|))| g_i(|c_i||\mathbf{x}|^{1-n}) |c_i||\mathbf{x}|^{1-n} \Big] \le d_2 \sum_{i=1}^2 |F_i'(\hat S(|\mathbf{x}|))| \quad \forall |\mathbf{x}| \in [r_0, R_{\max}). \tag{3.36}$$

From (3.14) and (3.13),

$$\Lambda(\mathbf{x}) = \int_{r_0}^{|\mathbf{x}|} r\lambda(r) dr = \int_{r_0}^{|\mathbf{x}|} \Big[ F_2'(\hat S(r)) G_2(c_2 r^{1-n}) - F_1'(\hat S(r)) G_1(c_1 r^{1-n}) \Big] dr. \tag{3.37}$$

Then

$$|\Lambda(\mathbf{x})| \le d_2 \int_{r_0}^{|\mathbf{x}|} \Big[ |F_1'(\hat S(r))| + |F_2'(\hat S(r))| \Big] dr \quad \forall |\mathbf{x}| \in [r_0, R_{\max}). \tag{3.38}$$

Also, matrix $\underline{\mathbf{B}}$ has the following special property:

$$\underline{\mathbf{B}}(\mathbf{x})\mathbf{x} = \sum_{i=1}^2 \Big\{ F_i(\hat S(|\mathbf{x}|)) \left[ g_i(|c_i||\mathbf{x}|^{1-n}) + g_i'(|c_i||\mathbf{x}|^{1-n}) |c_i||\mathbf{x}|^{1-n} \right] \Big\} \mathbf{x} = \phi(|\mathbf{x}|)\mathbf{x}, \tag{3.39}$$

where

$$\phi(r) = \sum_{i=1}^2 F_i(\hat S(r)) \left[ g_i(|c_i| r^{1-n}) + g_i'(|c_i| r^{1-n}) |c_i| r^{1-n} \right]. \tag{3.40}$$

Since $g_i' \ge 0$,

$$\phi(r) \ge d_0 [F_1(\hat S(r)) + F_2(\hat S(r))] \quad \forall r \in [r_0, R_{\max}). \tag{3.41}$$

Since $g_i(s)$ and $g_i'(s)s$ are increasing on $[0,\infty)$, we have

$$\phi(r) \le d_4 [F_1(\hat S(r)) + F_2(\hat S(r))] \quad \forall r \in [r_0, R_{\max}). \tag{3.42}$$

We now discuss the regularity of the involved functions. For $D \subset \mathbb{R}^n \times \mathbb{R}$, we define class $C^m_{\mathbf{x}}(D)$ as the set of functions $f(\mathbf{x},t) \in C(D)$ whose partial derivatives with respect to $\mathbf{x}$ up to order $m$ are continuous in $D$. The class $C^m_t$ is defined similarly and $C^{m,m'}_{\mathbf{x},t} = C^m_{\mathbf{x}} \cap C^{m'}_t$.

Note that

$$\frac{\partial \underline{\mathbf{A}}}{\partial x_i} = -\underline{\mathbf{A}} \frac{\partial \underline{\mathbf{B}}}{\partial x_i} \underline{\mathbf{A}}. \tag{3.43}$$

By definitions (3.6), (3.7), (3.8) and relation (3.43), we easily obtain:

**Lemma 3.3.** *Assume $F_1, F_2 \in C^m((0,1))$ for some $m \ge 1$. Let $R \in (r_0, R_{\max})$ and denote*

$$\mathcal{O} = \{\mathbf{x} : r_0 < |\mathbf{x}| < R\}.$$

(i) *Then $\underline{\mathbf{B}}, \underline{\mathbf{A}} \in C^m(\bar{\mathcal{O}})$, $\mathbf{b} \in C^{m-1}(\bar{\mathcal{O}})$ and $\Lambda \in C^m(\bar{\mathcal{O}})$.*

(ii) *If, in addition, $\mathbf{V} \in X(\mathcal{O} \times (0,\infty))$ then $\mathbf{c} \in X(\mathcal{O} \times (0,\infty))$, where $X$ can be $C^m$ or $C^m_{\mathbf{x}}$ or $C^m_t$.*

## 4. Case of bounded domain

In this section, we study the linear stability of the obtained steady flows in section 2 on bounded domains. More specifically, we investigate the stability of the trivial solution for the linearized system (3.1). The key instrument in proving the asymptotic stability is a Landis-type lemma of growth (see [19]). To prove such a lemma we use specific structures of the coefficients of equation (3.18) to construct singular sub-parabolic functions. These are motivated by the so-called $F_{s,\beta}$ functions introduced in [19].

Let $r_0 > 0$ be fixed throughout. We consider in this section an open, bounded set $U$ in $\mathbb{R}^n \setminus \bar{B}_{r_0}$. We fix $R > 0$ such that $U \subset \mathscr{U} \stackrel{\text{def}}{=} B_R \setminus \bar{B}_{r_0}$. Denote $\Gamma = \partial U$, $D = U \times (0,\infty)$ and $\mathscr{D} = \mathscr{U} \times (0,\infty)$.

We consider a steady state $(u_1^*(\mathbf{x}), u_2^*(\mathbf{x}), S_*(\mathbf{x}))$ as in (3.2) with $c_1^2 + c_2^2 > 0$. Recall that (3.11), (3.4) and (3.10) is our linearized system for (1.14). We study the equation for $\sigma(\mathbf{x},t)$ first. More specifically, we study the following initial-boundary value problem (IBVP):

$$\begin{cases} \sigma_t = \nabla \cdot \Big[\underline{\mathbf{A}}(\nabla\sigma - \sigma\mathbf{b})\Big] + \nabla\cdot(\underline{\mathbf{A}}\mathbf{c}) & \text{on } U\times(0,\infty),\\ \sigma = g(\mathbf{x},t) & \text{on } \Gamma\times(0,\infty),\\ \sigma = \sigma_0(\mathbf{x}) & \text{on } U\times\{t=0\}. \end{cases} \tag{4.1}$$

Regarding the initial and boundary data in (4.1), we always assume that

$$\sigma_0 \in C(\bar{U}),\ g\in C(\Gamma\times[0,\infty)) \text{ and } \sigma_0(\mathbf{x}) = g(\mathbf{x},0) \text{ on } \Gamma. \tag{4.2}$$

Assume that

$$0 < \underline{s} \le \hat{S}(r) \le \bar{s} < 1 \quad \forall r\in[r_0,R], \quad \text{where } \underline{s} \text{ and } \bar{s} \text{ are constants.} \tag{4.3}$$

Assumption (4.3) is valid for any solution $\hat{S}$ in Theorem 2.1 with $R_{\max} > R$, in particular, when $R_{\max} = \infty$ as in Theorem 2.2. Under constraint (4.3) and Assumptions A and B, we easily see the following facts. Let

$$\mu_1 = \sum_{i=1}^2 \max_{\underline{s}\le s\le\bar{s}} F_i(s), \quad \mu_2 = \sum_{i=1}^2 \min_{\underline{s}\le s\le\bar{s}} F_i(s), \quad \mu_3 = \sum_{i=1}^2 \max_{\underline{s}\le s\le\bar{s}} |F_i'(s)|. \tag{4.4}$$

Then $\mu_1$, $\mu_2$ and $\mu_3$ are positive numbers.

From (3.33) and (4.3) follows that

$$\mathbf{z}^T\underline{\mathbf{A}}(\mathbf{x})\mathbf{z} \ge \frac{|\mathbf{z}|^2}{C_0} \quad \forall \mathbf{x}\in\bar{\mathscr{U}},\ \mathbf{z}\in\mathbb{R}^n, \tag{4.5}$$

where $C_0 = d_4\mu_1$.

From (3.35), (3.36) and (4.3), we get

$$|\underline{\mathbf{A}}(\mathbf{x})| \le \frac{c_0}{C_1} \quad \text{and} \qquad |\mathbf{b}(\mathbf{x})| \le C_2 \quad \forall \mathbf{x}\in\bar{\mathscr{U}}, \tag{4.6}$$

where $c_0$ is in (3.34), $C_1 = d_0\mu_2$ and $C_2 = d_2\mu_3$.

For the smoothness, by Lemma 3.3,

$$\underline{\mathbf{B}}, \underline{\mathbf{A}} \in C^1(\bar{\mathscr{U}}) \quad \text{and} \quad \mathbf{b}\in C(\bar{\mathscr{U}}). \tag{4.7}$$

First, we consider the the existence of classical solutions of (4.1). We use the known result from theory of linear parabolic equations in [16]. This will require certain regularity of the coefficients of (4.1). Those requirements, in turn, can be formulated in terms of functions $F_1$ and $F_2$, thanks to Lemma 3.3.

**Condition (E1)**. $F_1, F_2 \in C^7((0,1))$ and $V \in C^6_{\mathbf{x}}(\bar{D})$; $V_t \in C^3_{\mathbf{x}}(\bar{D})$.

**Theorem 4.1** ( [16]). *Assume **(E1)**, then there exists a unique solution $\sigma \in C(\bar{D}) \cap C^{2,1}_{\mathbf{x},t}(D)$ of problem (4.1).*

Note that we did not attempt to optimize Condition **(E1)**. As seen below, the study of qualitative properties of solution $\sigma$ will require much less stringent conditions than **(E1)**.

Now we turn to the stability, asymptotic stability and structural stability issues. Our main tool is the maximum principle. As discussed in the previous section, we use the transformation (3.16) to convert the PDE in (4.1) to a more convenient form (3.18). Define the differential operator on the left-hand side of (3.18) by

$$\mathcal{L}w = \partial_t w - \nabla \cdot (\underline{\mathbf{A}}\nabla w) - \mathbf{b} \cdot \underline{\mathbf{A}}\nabla w. \tag{4.8}$$

Corresponding to (4.1), the IBVP for $w(\mathbf{x}, t)$ is

$$\begin{cases} \mathcal{L}w = f_0 & \text{in } U \times (0, \infty), \\ w(\mathbf{x}, 0) = w_0(\mathbf{x}) & \text{in } U, \\ w(\mathbf{x}, t) = G(\mathbf{x}, t) & \text{on } \Gamma \times (0, \infty), \end{cases} \tag{4.9}$$

where $w_0(\mathbf{x})$ and $G(\mathbf{x}, t)$ are given initial data and boundary data, respectively, and $f_0(\mathbf{x}, t)$ is a known function. We will obtain results for solution $w$ of (4.9) and then reformulate them in terms of solution $\sigma$ of the original problem (4.1).

Since the existence and uniqueness issues are settled in Theorem 4.1, our main focus now is the qualitative properties of solution $w$ of (4.9). For these, we only need properties (4.5), (4.6), the special structure of equation (4.1), and the assumption that the classical solution in $C(\bar{D}) \cap C^{2,1}_{\mathbf{x},t}(D)$ already exists. The fine properties of the solutions obtained below have their own merit in the theory of linear parabolic equations.

It follows from (4.5) and (4.6) that the maximum principle holds for any classical solution of $\mathcal{L}w \le (\ge) 0$ in $D$. To obtain better estimates for solutions, especially as $t \to \infty$, we use the following barrier function. Define

$$W(\mathbf{x}, t) = \begin{cases} t^{-s} e^{-\frac{\varphi(\mathbf{x})}{t}} & \text{if } t > 0, \\ 0 & \text{if } t \le 0, \end{cases} \tag{4.10}$$

where the number $s > 0$ and the function $\varphi(\mathbf{x}) > 0$ will be decided later. Then

$$\mathcal{L}W = t^{-s-2} e^{-\frac{\varphi}{t}} \Big\{ t\big( - s + \nabla \cdot (\underline{\mathbf{A}}\nabla\varphi) + \mathbf{b} \cdot \underline{\mathbf{A}}\nabla\varphi \big) + \varphi - (\underline{\mathbf{A}}\nabla\varphi) \cdot \nabla\varphi \Big\}.$$

Thus, $\mathcal{L}W \le 0$ if

$$s \ge \nabla \cdot (\underline{\mathbf{A}}\nabla\varphi) + \mathbf{b} \cdot \underline{\mathbf{A}}\nabla\varphi \quad \text{and} \quad \varphi \le (\underline{\mathbf{A}}\nabla\varphi) \cdot \nabla\varphi. \tag{4.11}$$

We will choose $\varphi$ to satisfy

$$\underline{\mathbf{A}}\nabla\varphi = \kappa_0 \mathbf{x}, \tag{4.12}$$

where $\kappa_0$ is a positive constant selected later. Equivalently, with the use of (3.39),

$$\nabla\varphi = \kappa_0 \underline{\mathbf{A}}^{-1}\mathbf{x} = \kappa_0 \underline{\mathbf{B}}\mathbf{x} = \kappa_0 \phi(|\mathbf{x}|)\mathbf{x}, \tag{4.13}$$

where $\phi(r)$ is defined by (3.40). By (3.41), (4.3) and (4.4),

$$\phi(r) \ge d_0\mu_2 = C_1 \quad \text{for } r_0 \le r \le R. \tag{4.14}$$

By (3.42), (4.3) and (4.4),

$$\phi(r) \le d_4\mu_1 = C_0 \quad \text{for } r_0 \le r \le R. \tag{4.15}$$

Define for $\mathbf{x} \in \bar{\mathscr{U}}$ the function

$$\varphi(\mathbf{x}) = \kappa_0 \Big( \varphi_0 + \int_{r_0}^{|\mathbf{x}|} r\phi(r)dr \Big), \quad \text{where } \varphi_0 = \frac{C_0 r_0^2}{2} \text{ and } \kappa_0 = \frac{C_0}{2C_1}. \tag{4.16}$$

Then $\varphi(\mathbf{x})$ satisfies both equations (4.12) and (4.13). We have for $\mathbf{x} \in \bar{\mathscr{U}}$ that

$$0 < \varphi(\mathbf{x}) \le \kappa_0 \Big( \varphi_0 + C_0 \int_{r_0}^{|\mathbf{x}|} rdr \Big) = \frac{\kappa_0 C_0}{2} |\mathbf{x}|^2. \tag{4.17}$$

Applying (4.12), (4.13), and then (3.31) and (4.4) we obtain

$$(\underline{\mathbf{A}}\nabla\varphi)\cdot\nabla\varphi = \kappa_0^2\mathbf{x}^T\underline{\mathbf{B}}\mathbf{x} \geq d_0\kappa_0^2\Big(\sum_{i=1}^{2} F_i(\hat{S}(|\mathbf{x}|))\Big)|\mathbf{x}|^2 \geq d_0\kappa_0^2\mu_2|\mathbf{x}|^2 = \kappa_0^2 C_1|\mathbf{x}|^2 = \frac{\kappa_0 C_0}{2}|\mathbf{x}|^2 \tag{4.18}$$

Then we have from (4.17) and (4.18) that $\varphi \leq (\underline{\mathbf{A}}\nabla\varphi)\cdot\nabla\varphi$ in $\mathscr{U}$, which is the second requirement in (4.11). On the other hand, by (4.12) and (4.6),

$$\nabla\cdot(\underline{\mathbf{A}}\nabla\varphi) + \mathbf{b}\cdot\underline{\mathbf{A}}\nabla\varphi = \kappa_0(\nabla\cdot\mathbf{x} + \mathbf{b}\cdot\mathbf{x}) \leq \kappa_0(n + C_2R). \tag{4.19}$$

Select

$$s = s_R \stackrel{\text{def}}{=\!=} \kappa_0(n + C_2R). \tag{4.20}$$

Then we have $s \geq \nabla\cdot(\underline{\mathbf{A}}\nabla\varphi) + \mathbf{b}\cdot(\underline{\mathbf{A}}\nabla\varphi)$ in $\mathscr{U}$, which is the first requirement in (4.11). Thus, we obtain $\mathcal{L}W \leq 0$ in $\mathscr{U}\times(0,\infty)$. For further references, we formulate this as a lemma.

**Lemma 4.2.** *With parameter $s = s_R$ selected as in (4.20) and function $\varphi$ defined by (4.16), the function $W(\mathbf{x},t)$ in (4.10) belongs to $C^{2,1}_{\mathbf{x},t}(\mathscr{D})\cap C(\bar{\mathscr{D}})$ and satisfies $\mathcal{L}W \leq 0$ in $\mathscr{D}$.*

Above, the regularity of $W(\mathbf{x},t)$ follows the fact that $\varphi(\mathbf{x}) \geq \kappa_0\varphi_0 > 0$ for $\mathbf{x}\in\bar{\mathscr{U}}$.

We now establish this section's key lemma of growth. We fix $s = s_R$ by (4.20) and also the following two parameters

$$q = \frac{\kappa_0 C_0}{2s} \quad \text{and} \quad \eta_0 = \Big(\frac{r_0}{R}\Big)^{2s}, \tag{4.21}$$

and denote $D_1 = U\times(0, qR^2]$.

**Lemma 4.3** (Lemma of growth in time)**.** *Assume $w(\mathbf{x},t)\in C^{2,1}_{\mathbf{x},t}(D_1)\cap C(\bar{D}_1)$. If*

$$\mathcal{L}w \leq 0 \ \textit{on}\ D_1 \quad \textit{and} \quad w \leq 0 \ \textit{on}\ \Gamma\times(0,qR^2), \tag{4.22}$$

*then*

$$\max\{0, \sup_U w(\mathbf{x}, qR^2)\} \leq \frac{1}{1+\eta_0}\max\{0, \sup_U w(\mathbf{x},0)\}. \tag{4.23}$$

*Proof.* (i) Let $M = \max\{0, \sup_{\bar{D}_1} w\}$. By (4.22) and maximum principle, we have

$$M = \max\{0, \sup_{\bar{U}} w(\mathbf{x},0)\}. \tag{4.24}$$

Let $W(\mathbf{x},t)$ be as in (4.10) and define the auxiliary function

$$\tilde{W}(\mathbf{x},t) = M[1 - \eta W(\mathbf{x},t)],$$

where constant $\eta > 0$ will be specified later. Our intention is to prove that

$$\tilde{W}(\mathbf{x},t) \geq w(\mathbf{x},t) \quad \text{for all } (\mathbf{x},t)\in\bar{D}_1. \tag{4.25}$$

By Lemma 4.2, $\mathcal{L}W \leq 0$ in $D_1$, hence, $\mathcal{L}\tilde{W} \geq 0$ in $D_1$. By maximum principle, it suffices to show that

$$\tilde{W}(\mathbf{x},t) \geq w(\mathbf{x},t) \quad \text{for all } (\mathbf{x},t)\in\partial_p D_1 = \big[\bar{U}\times\{0\}\big]\cup\big[\Gamma\times(0,qR^2]\big]. \tag{4.26}$$

On the base $\bar{U}\times\{0\}$, function $W(\mathbf{x},0)$ vanishes, hence,

$$\tilde{W}(\mathbf{x},0) = M \geq w(\mathbf{x},0).$$

On the side boundary $\Gamma\times(0,qR^2]$, additional analysis is required. First observe for $\mathbf{x}\in\bar{\mathscr{U}}$ that $\varphi(\mathbf{x}) \geq \kappa_0\varphi_0 = \frac{\kappa_0 C_0 r_0^2}{2}$. Therefore,

$$\tilde{W}(\mathbf{x},t) = M\left[1 - \eta t^{-s}e^{-\frac{\varphi(\mathbf{x})}{t}}\right] \geq M\left[1 - \eta t^{-s}e^{-\frac{\kappa_0 C_0 r_0^2}{2t}}\right] \quad \text{in } \bar{\mathscr{U}}\times[0,\infty). \tag{4.27}$$

Let $h_0(t) = t^{-s} e^{-\frac{\kappa_0 C_0 r_0^2}{2t}}$ for $t \ge 0$. By elementary calculations, the maximum of $h_0(t)$ is attained at $t_0 = \frac{\kappa_0 C_0 r_0^2}{2s}$. By letting

$$\eta = \frac{1}{\max_{[0,\infty)} h_0(t)} = \frac{1}{h_0(t_0)} = \Big(\frac{e\kappa_0 C_0 r_0^2}{2s}\Big)^s, \tag{4.28}$$

we get from (4.27) that $\tilde W(\mathbf{x},t) \ge M[1-\eta h_0(t_0)] = 0$ in $\bar{\mathscr{U}} \times [0,\infty)$. Particularly,

$$\tilde W(\mathbf{x},t) \ge 0 \ge w(\mathbf{x},t) \quad \text{on } \Gamma \times (0, qR^2].$$

Thus, the comparison in (4.26) holds and, therefore, (4.25) is proved.

We now estimate $\tilde W(\mathbf{x},t)$. By (4.17), for $(\mathbf{x},t) \in D$ we have

$$\tilde W(\mathbf{x},t) \le M\left[1 - \eta t^{-s} e^{-\frac{\kappa_0 C_0 |\mathbf{x}|^2}{2t}}\right] \le M\left[1 - \eta t^{-s} e^{-\frac{\kappa_0 C_0 R^2}{2t}}\right].$$

Let $h_1(t) = t^{-s} e^{-\frac{\kappa_0 C_0 R^2}{2t}}$ for $t > 0$. Then $t_1 = \frac{\kappa_0 C_0 R^2}{2s} = qR^2$ is the critical point and

$$h_1(t_1) = (qR^2)^{-s} e^{-\frac{\kappa_0 C_0}{2q}} \ge \Big(\frac{2s}{e\kappa_0 C_0 R^2}\Big)^s.$$

Letting $t = t_1$ in (4.25), we have

$$w(\mathbf{x},t_1) \le \tilde W(\mathbf{x},t_1) \le M\left[1 - \eta\Big(\frac{2s}{e\kappa_0 C_0 R^2}\Big)^s\right] = M(1-\eta_0) \le \frac{M}{1+\eta_0}, \tag{4.29}$$

and, hence, (4.23) follows. □

Using Lemma 4.3, we show the decay, as $t \to \infty$, of solution $w(\mathbf{x},t)$ of the IBVP (4.9) in the homogeneous case, i.e., when $f_0 \equiv 0$ and $G \equiv 0$.

**Proposition 4.4** (Homogeneous problem). *Assume $w(\mathbf{x},t) \in C^{2,1}_{\mathbf{x},t}(D) \cap C(\bar D)$ satisfies*

$$\mathcal{L} w = 0 \text{ in } D \quad \text{and} \quad w = 0 \text{ on } \Gamma \times (0,\infty). \tag{4.30}$$

*Then*

$$-e^{-\eta_1 t} \inf_U |w(\mathbf{x},0)| \le w(\mathbf{x},t) \le (1+\eta_0) e^{-\eta_1 t} \sup_U |w(\mathbf{x},0)| \quad \forall (\mathbf{x},t) \in D, \tag{4.31}$$

*where $\eta_1 = \frac{\ln(1+\eta_0)}{qR^2}$.*

*Proof.* Let $k \in \mathbb{N}$. Applying Lemma 4.3 with $D_1$ being replaced by $U \times (T_{k-1}, T_k]$ gives

$$\max\{0, \sup_U w(\mathbf{x}, kqR^2)\} \le \frac{1}{1+\eta_0} \max\{0, \sup_U w(\mathbf{x}, (k-1)qR^2)\}.$$

By induction in $k$, we obtain

$$\max\{0, \sup_U w(\mathbf{x}, kqR^2)\} \le \frac{1}{(1+\eta_0)^k} \max\{0, \sup_U w(\mathbf{x}, 0)\}. \tag{4.32}$$

Now applying (4.32) to function $-w$ instead of $w$, we obtain

$$\min\{0, \inf_U w(\mathbf{x}, kqR^2)\} \ge \frac{1}{(1+\eta_0)^k} \min\{0, \inf_U w(\mathbf{x}, 0)\}. \tag{4.33}$$

For any $t > 0$, there is an integer $k \ge 0$ such that $t \in (T_k, T_{k+1}]$ where $T_k = kqT^2$. By (4.30) and maximum principle for domain $U \times (T_k, T_{k+1}]$, and then using (4.32) we have

$$\begin{aligned} w(\mathbf{x},t) &\le \max\{0, \sup_U w(\mathbf{x}, T_k)\} \le (1+\eta_0)^{-k} \max\{0, \sup_U w(\mathbf{x}, 0)\} \\ &= (1+\eta_0) e^{-\eta_1 T_{k+1}} \sup_U |w(\mathbf{x},0)| \le (1+\eta_0) e^{-\eta_1 t} \sup_U |w(\mathbf{x},0)|. \end{aligned} \tag{4.34}$$

Similarly, using (4.33) instead of (4.32) we have

$$\begin{aligned} w(\mathbf{x},t) &\ge \min\{0,\inf_U w(\mathbf{x},T_k)\} \ge (1+\eta_0)^{-k}\min\{0,\inf_U w(\mathbf{x},0)\}\\ &\ge -e^{-\eta_1 T_k}\inf_U |w(\mathbf{x},0)| \ge -e^{-\eta_1 t}\inf_U |w(\mathbf{x},0)|. \end{aligned} \tag{4.35}$$

Therefore, (4.31) follows (4.34) and (4.35). □

Next, we consider the non-homogeneous case for the IBVP (4.9). Similar to (4.2), we always consider

$$w_0 \in C(\bar U),\ G\in C(\Gamma\times[0,\infty)) \text{ and } w_0(\mathbf{x}) = G(\mathbf{x},0) \text{ on } \Gamma. \tag{4.36}$$

**Proposition 4.5** (Non-homogeneous problem)**.** *Assume $f_0\in C(\bar D)$ and*

$$\Delta_1 \overset{\text{def}}{=\!=} \sup_{U\times(0,\infty)} |f_0(\mathbf{x},t)| + \sup_{\Gamma\times(0,\infty)} |G(\mathbf{x},t)| < \infty \tag{4.37}$$

*There is a positive constant $C$ such that if $w(\mathbf{x},t)\in C^{2,1}_{\mathbf{x},t}(D)\cap C(\bar D)$ is a solution of (4.9), then*

$$|w(\mathbf{x},t)| \le C\big[e^{-\eta_1 t}\sup_U |w_0(\mathbf{x})| + \Delta_1\big] \quad \forall(\mathbf{x},t)\in D, \tag{4.38}$$

*where $\eta_1>0$ is defined in Proposition 4.4.*

*Proof.* Denote $T_k = kqR^2$ for any integer $k\ge 0$. Let $k\in\mathbb{N}$ and

$$v_k(\mathbf{x},t) = w(\mathbf{x},t) - \Delta_1(t-T_{k-1}+1) \quad \text{ for } (\mathbf{x},t)\in \bar U\times[T_{k-1},T_k]. \tag{4.39}$$

Then $v_k$ satisfies

$$\mathcal{L}v_k = \mathcal{L}w - \Delta_1 = f_0 - \Delta_1 \le 0 \quad \text{in } U\times(T_{k-1},T_k],$$

and

$$v_k(\mathbf{x},t)\le 0 \text{ on } \Gamma\times(T_{k-1},T_k).$$

Applying Lemma 4.3 to function $v_k$, we have

$$\max\{0,\sup_U v_k(\mathbf{x},T_k)\} \le \frac{1}{1+\eta_0}\max\{0,\sup_U v_k(\mathbf{x},T_{k-1})\}. \tag{4.40}$$

Note that $v_k(\mathbf{x},T_k) = w(\mathbf{x},T_k) - \Delta_1(qR^2+1)$ and $v_k(\mathbf{x},T_{k-1}) = w(\mathbf{x},T_{k-1})-\Delta_1 \le w(\mathbf{x},T_{k-1})$. Hence,

$$\begin{aligned} \max\{0,\sup_U w(\mathbf{x},T_k)\} &\le \max\{0,\sup_U v_k(\mathbf{x},T_k)\} + \Delta_1(qR^2+1)\\ &\le \frac{1}{1+\eta_0}\max\{0,\sup_U w(\mathbf{x},T_{k-1})\} + \Delta_1(qR^2+1). \end{aligned}$$

Iterating this inequality gives

$$\begin{aligned} \max\{0,\sup_U w(\mathbf{x},T_k)\} &\le \frac{1}{(1+\eta_0)^k}\max\{0,\sup_U w(\mathbf{x},0)\} + \Delta_1(qR^2+1)\sum_{j=0}^{k-1}\frac{1}{(1+\eta_0)^j}\\ &\le \frac{1}{(1+\eta_0)^k}\max\{0,\sup_U w(\mathbf{x},0)\} + \frac{\Delta_1(1+qR^2)(1+\eta_0)}{\eta_0}. \end{aligned} \tag{4.41}$$

By using the relation (4.39) between $v_k(\mathbf{x},t)$ and $w(\mathbf{x},t)$, maximum principle for function $v_k(\mathbf{x},t)$, and estimate (4.41), we have for any $t\in[T_{k-1},T_k]$ with $k\ge 1$ that

$$\begin{aligned} w(\mathbf{x},t) &\le v_k(\mathbf{x},t) + \Delta_1(1+qR^2) \le \max\{0,\sup_U w(\mathbf{x},T_{k-1})\} + \Delta_1(1+qR^2)\\ &\le (1+\eta_0)^{-k+1}\max\{0,\sup_U w(\mathbf{x},0)\} + \frac{\Delta_1(1+qR^2)(1+\eta_0)}{\eta_0} + \Delta_1(1+qR^2)\\ &\le (1+\eta_0)^{-\frac{t}{qR^2}+1}\sup_U |w(\mathbf{x},0)| + \frac{2\Delta_1(1+qR^2)(1+\eta_0)}{\eta_0}. \end{aligned}$$

Therefore,

$$w(\mathbf{x},t) \le C\big[e^{-\eta_1 t}\sup_U |w(\mathbf{x},0)| + \Delta_1\big]. \tag{4.42}$$

Similarly, we obtain the same estimate for $(-w)$ and hence, (4.38) follows. □

For the asymptotic behavior of $w(\mathbf{x},t)$ as $t\to\infty$, we have the following.

**Corollary 4.6.** *Assume $f_0 \in C(\bar D)$ is bounded and*

$$\Delta_2 \stackrel{\text{def}}{=\!=} \limsup_{t\to\infty}\left[\sup_{\mathbf{x}\in U}|f_0(\mathbf{x},t))| + \sup_{\mathbf{x}\in\Gamma}|G(\mathbf{x},t)|\right] < \infty. \tag{4.43}$$

*There exists $C = C(\eta_0, q, R, M) > 0$ such that if $w(\mathbf{x},t)\in C^{2,1}_{\mathbf{x},t}(D)\cap C(\bar D)$ solves (4.9), then*

$$\limsup_{t\to\infty}\left[\sup_{\mathbf{x}\in U}|w(\mathbf{x},t)|\right] \le C\Delta_2. \tag{4.44}$$

*Proof.* Note that

$$\sup_U |w_0(\mathbf{x})| + \sup_D |f_0(\mathbf{x},t)| + \sup_{\Gamma\times(0,\infty)} |G(\mathbf{x},t)| < \infty.$$

Then by Proposition 4.5, $w(\mathbf{x},t)$ is bounded on $\bar D$. Let $\varepsilon > 0$. From our assumption there is $t_0 > 0$ such that

$$\sup_{U\times[t_0,\infty)} |f_0(\mathbf{x},t))| + \sup_{\Gamma\times[t_0,\infty)} |G(\mathbf{x},t)| < \Delta_2 + \varepsilon.$$

Applying Lemma 4.5 to the domain $U\times(t_0,\infty)$ we obtain

$$|w(\mathbf{x},t)| \le C[e^{-\eta_1(t-t_0)}\sup_{\mathbf{x}\in U}|w(\mathbf{x},t_0)| + \Delta_2 + \varepsilon]. \tag{4.45}$$

Therefore, passing $t\to\infty$ and then $\varepsilon\to 0$ in (4.45) yields (4.44). □

Next, we estimate $|\nabla w(\mathbf{x},t)|$ by using Bernstein's technique (c.f. [16]).

**Proposition 4.7.** *Assume $f_0\in C(\bar D)$, $\nabla f_0 \in C(D)$, (4.37) and*

$$\Delta_3 \stackrel{\text{def}}{=\!=} \sup_D |\nabla f_0| < \infty. \tag{4.46}$$

*For any $U' \Subset U$ there is $\tilde M > 0$ such that if $w(\mathbf{x},t)\in C^{2,1}_{\mathbf{x},t}(D)\cap C(\bar D)$ is a solution of (4.9) that also satisfies $w\in C^3_{\mathbf{x}}(D)$ and $w_t \in C^1_{\mathbf{x}}(D)$, then*

$$|\nabla w(\mathbf{x},t)| \le \tilde M\Big[1 + \frac{1}{\sqrt t}\Big]\Big[e^{-\eta_1 t}\sup_U |w(\mathbf{x},0)| + \Delta_1 + \sqrt{\Delta_3}\Big] \quad \forall (\mathbf{x},t)\in U'\times(0,\infty). \tag{4.47}$$

*Proof.* Note that $\nabla w \in C^{2,1}_{\mathbf{x},t}(D)$. By using finite covering of compact set $U'$, it suffices to prove (4.47) for $\mathbf{x}$ in some ball inside $U$. Consider a ball $B_\delta(\mathbf{x}_*) = \{\mathbf{x} : |\mathbf{x}-\mathbf{x}^*| \le \delta\} \Subset U$ with some $\mathbf{x}_*\in U$ and $\delta > 0$. Let $t_0 > 0$, define in the cylinder $G_\delta \stackrel{\text{def}}{=\!=} B_\delta(\mathbf{x}_*)\times(t_0, 1+t_0]$ the following auxiliary function

$$\tilde w(\mathbf{x},t) = \tau\Phi(\mathbf{x})|\nabla w|^2 + Nw^2 + N_1(1+t_0-t), \tag{4.48}$$

where

$$\tau = t - t_0 \in (0,1], \quad \Phi(\mathbf{x}) = (\delta^2 - |\mathbf{x}-\mathbf{x}_*|^2)^2. \tag{4.49}$$

The numbers $N, N_1 \ge 0$ will be chosen later. We rewrite the operator $\mathcal{L}$ as

$$\mathcal{L}w = w_t - \sum_{i,j=1}^n a_{ij}(\mathbf{x})\partial_i\partial_j w - \tilde{\mathbf{b}}\cdot\nabla w, \tag{4.50}$$

where $\tilde{\mathbf{b}}(\mathbf{x}) = (\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_n) \stackrel{\text{def}}{=\!=} \nabla \cdot \underline{\mathbf{A}} + \underline{\mathbf{A}}\mathbf{b}$. Then following the calculations in Theorem 1 of section 2 on page 450 in [16] we have

$$\begin{aligned}\mathcal{L}\tilde{w} \le 2\tau\Phi\Big\{ &\sum_{i,j,k=1}^n \frac{\partial a_{ij}}{\partial x_k}\frac{\partial w}{\partial x_k}\frac{\partial^2 w}{\partial x_i \partial x_j} + \sum_{i,k=1}^n \frac{\partial \tilde{b}_i}{\partial x_k}\frac{\partial w}{\partial x_k}\frac{\partial w}{\partial x_i} - \sum_{i,j,k=1}^n a_{ij}\frac{\partial^2 w}{\partial x_k \partial x_i}\frac{\partial^2 w}{\partial x_k \partial x_j}\Big\}\\ &- (\tau\mathcal{L}(\Phi) - \Phi)|\nabla w|^2 - 4\tau \sum_{i,j,k=1}^n a_{ij}\frac{\partial \Phi}{\partial x_i}\frac{\partial w}{\partial x_k}\frac{\partial^2 w}{\partial x_k \partial x_j} - 2N\sum_{i,j=1}^n a_{ij}\frac{\partial w}{\partial x_i}\frac{\partial w}{\partial x_j}\\ &- 2\tau\Phi\sum_{k=1}^n \frac{\partial f_0}{\partial x_k} - 2Nwf_0 - N_1.\end{aligned} \tag{4.51}$$

We estimate the right-hand side of (4.51) term by term. Let $\varepsilon > 0$. The numbers $K_i$, for $i = 1, 2, 3\ldots$, used in the calculations below are all positive and independent of $w$. We denote the matrix of second derivatives of $w$ by $\nabla^2 w$, and denote its Euclidean norm by $|\nabla^2 w|$. Note that $\underline{\mathbf{A}}$, $\mathbf{b}$ and $\tilde{\mathbf{b}}$ are bounded in $B_\delta(\mathbf{x}^*)$. This and Cauchy-Schwarz inequality imply

$$\begin{aligned}2\tau\Phi\sum_{i,j,k=1}^n \frac{\partial a_{ij}}{\partial x_k}\frac{\partial w}{\partial x_k}\frac{\partial^2 w}{\partial x_i \partial x_j} &\le 2C\tau\Phi|\nabla w||\nabla^2 w|^2 \le \varepsilon^{-1}K_1|\nabla w|^2 + 2\varepsilon\tau\Phi|\nabla^2 w|^2,\\ -(\tau\mathcal{L}(\Phi) - \Phi)|\nabla w|^2 + 2\tau\Phi\sum_{i,k=1}^n \frac{\partial \tilde{b}_i}{\partial x_k}\frac{\partial w}{\partial x_k}\frac{\partial w}{\partial x_i} &\le K_2|\nabla w|^2.\end{aligned}$$

Since $\underline{\mathbf{A}}$ is positive definite,

$$\sum_{i,j,k=1}^n a_{ij}\frac{\partial^2 w}{\partial x_k \partial x_i}\frac{\partial^2 w}{\partial x_k \partial x_j} \ge K_3|\nabla^2 w|^2, \quad \sum_{i,j=1}^n a_{ij}\frac{\partial w}{\partial x_i}\frac{\partial w}{\partial x_j} \ge K_3|\nabla w|^2.$$

Also, we have

$$-4\tau\sum_{i,j,k=1}^n a_{ij}\frac{\partial \Phi}{\partial x_i}\frac{\partial w}{\partial x_k}\frac{\partial^2 w}{\partial x_k \partial x_j} \le \varepsilon^{-1}K_4|\nabla w|^2 + 2\varepsilon\tau|\nabla\Phi|^2|\nabla^2 w|^2,$$

$$-2\tau\Phi\sum_{k=1}^n \frac{\partial f_0}{\partial x_k} \le K_5\Delta_3,$$

and by using estimate (4.38) for $w$,

$$-2Nwf_0 \le K_6\Delta_1 N\big[e^{-\eta_1 t_0}\sup_U |w(\mathbf{x},0)| + \Delta_1\big].$$

Combining the above estimates, we obtain from (4.51) that

$$\begin{aligned}\mathcal{L}\tilde{w} \le\ & 2\tau\Phi\Big(2\varepsilon + \varepsilon\frac{|\nabla\Phi|^2}{\Phi} - K_3\Big)|\nabla^2 w|^2 + \Big(K_2 + \varepsilon^{-1}(K_1 + K_4) - 2NK_3\Big)|\nabla w|^2\\ &+ K_5\Delta_3 + K_6\Delta_1 N\big[e^{-\eta_1 t_0}\sup_U |w(\mathbf{x},0)| + \Delta_1\big] - N_1.\end{aligned}$$

Since $|\nabla\Phi|^2/\Phi \le 16\delta^2$, we have

$$\begin{aligned}\mathcal{L}\tilde{w} \le\ & 2\tau\Phi\Big(K_7\varepsilon - K_3\Big)|\nabla^2 w|^2 + \Big(K_2 + K_8\varepsilon^{-1} - 2NK_3\Big)|\nabla w|^2\\ &+ (K_5 + K_6 N)\big[\Delta_3 + \Delta_1 e^{-\eta_1 t_0}\sup_U |w(\mathbf{x},0)| + \Delta_1^2\big] - N_1.\end{aligned} \tag{4.52}$$

In (4.52), choose $\varepsilon = K_3/K_7$ and $N = [K_2 + K_8\varepsilon^{-1}]/(2K_3)$, then take

$$N_1 = (K_5 + K_6 N)(\Delta_1 e^{-\eta_1 t_0}\sup_U |w(\mathbf{x},0)| + \Delta_1^2 + \Delta_3).$$

We find that $\mathcal{L}\tilde{w} \le 0$ in $G_\delta$. Applying the maximum principle gives

$$\max_{\bar{G}_\delta} \tilde{w} = \max\big\{\tilde{w}(\mathbf{x},t) : (\mathbf{x},t) \in B_\delta(\mathbf{x}_*) \times \{t_0\} \cup \partial B_\delta(\mathbf{x}_*) \times [t_0, t_0+1]\big\}. \tag{4.53}$$

Note that $\tau\Phi(\mathbf{x}) = 0$ when $t = t_0$ or $\mathbf{x} \in \partial B_\delta(\mathbf{x}_*)$. Hence (4.53) implies,

$$\max_{\bar{G}_\delta} \tilde{w} \le N \max_{B_\delta(\mathbf{x}_*)} w^2(\mathbf{x}, t_0) + N \max_{\partial B_\delta(\mathbf{x}_*)\times[t_0,t_0+1]} w^2(\mathbf{x},t) + N_1. \tag{4.54}$$

Using estimate (4.38) for the first two terms on the right-hand side of (4.54) we obtain

$$\begin{aligned}\max_{\bar{G}_\delta} \tilde{w} &\le 2K_9 N\Big[e^{-\eta_1 t_0} \sup_U |w(\mathbf{x},0)| + \Delta_1\Big]^2 + N_1 \le K_{10}\Big[e^{-2\eta_1 t_0}\sup_U |w(\mathbf{x},0)|^2 + \Delta_1^2 + \Delta_3\Big]\\ &\le C\Big[e^{-2\eta_1 t}\sup_U |w(\mathbf{x},0)|^2 + \Delta_1^2 + \Delta_3\Big].\end{aligned}$$

Now, we consider $\mathbf{x} \in B_{\delta/2}(\mathbf{x}_*)$. If $t \in (0,1]$ let $t_0 = t/2$, then $t = 2t_0 \in [t_0, 1+t_0]$ and hence

$$\begin{aligned}\frac{t}{2}|\nabla w(\mathbf{x},t)|^2 \min_{B_{\delta/2}(\mathbf{x}_*)} \Phi(\mathbf{x}) &\le (t-t_0)\Phi(\mathbf{x})|\nabla w(\mathbf{x},t)|^2\\ &\le \tilde{w}(\mathbf{x},t) \le C\Big[e^{-2\eta_1 t}\sup_U |w(\mathbf{x},0)| + \Delta_1^2 + \Delta_3\Big]. \end{aligned}\tag{4.55}$$

If $t > 1$ let $t_0 = t - 1/2$, then $t \in [t_0, 1+t_0]$ and hence

$$\begin{aligned}\frac{1}{2}|\nabla w(\mathbf{x},t)|^2 \min_{B_{\delta/2}(\mathbf{x}_*)} \Phi(\mathbf{x}) &\le (t-t_0)\Phi(\mathbf{x})|\nabla w(\mathbf{x},t)|^2\\ &\le \tilde{w}(\mathbf{x},t) \le C\Big[e^{-2\eta_1 t}\sup_U |w(\mathbf{x},0)| + \Delta_1^2 + \Delta_3\Big]. \end{aligned}\tag{4.56}$$

Since $\min_{B_{\delta/2}(\mathbf{x}_*)} \Phi(\mathbf{x}) > 0$, it follows (4.55) and (4.56) that

$$|\nabla w(\mathbf{x},t)| \le M(\delta)\Big[1 + \frac{1}{\sqrt{t}}\Big]\Big[e^{-\eta_1 t}\sup_U |w(\mathbf{x},0)| + \Delta_1 + \sqrt{\Delta_3}\Big] \tag{4.57}$$

for $\mathbf{x} \in B_{\delta/2}(\mathbf{x}_*)$ and $t > 0$. Then using a finite covering of $U'$, we obtain (4.47) from (4.57). □

We return to the IBVP (4.1) for $\sigma(\mathbf{x},t)$ now. Recall that the existence and uniqueness of the solution $\sigma$ were already addressed in Theorem 4.1.

**Theorem 4.8.** *Assume **(E1)** and*

$$\Delta_4 \overset{\text{def}}{=\!=} \sup_D(|\mathbf{V}(\mathbf{x},t)| + |\nabla\mathbf{V}(\mathbf{x},t)|) + \sup_{\Gamma\times[0,\infty)} |g(\mathbf{x},t)| < \infty. \tag{4.58}$$

*Then the solution $\sigma(\mathbf{x},t)$ of the IBVP (4.1) satisfies*

$$\sup_{\mathbf{x}\in U} |\sigma(\mathbf{x},t)| \le C\Big[e^{-\eta_1 t}\sup_U |\sigma_0(\mathbf{x})| + \Delta_4\Big] \quad \textit{for all } t > 0. \tag{4.59}$$

*Moreover,*

$$\limsup_{t\to\infty}\Big[\sup_{\mathbf{x}\in U} |\sigma(\mathbf{x},t)|\Big] \le C\Delta_5, \tag{4.60}$$

*where*

$$\Delta_5 = \limsup_{t\to\infty}\Big[\sup_{\mathbf{x}\in U}(|\mathbf{V}(\mathbf{x},t)| + |\nabla\mathbf{V}(\mathbf{x},t)|) + \sup_{\mathbf{x}\in\Gamma} |g(\mathbf{x},t)|\Big]. \tag{4.61}$$

*Proof.* Let $w(\mathbf{x},t) = \sigma(\mathbf{x},t)e^{-\Lambda(\mathbf{x})}$, $f_0(\mathbf{x},t) = e^{-\Lambda(\mathbf{x})}\nabla\cdot(\underline{\mathbf{A}}(\mathbf{x})\mathbf{c}(\mathbf{x},t))$, $G(\mathbf{x},t) = e^{-\Lambda(\mathbf{x})}g(\mathbf{x},t)$ and $w_0(\mathbf{x}) = e^{-\Lambda(\mathbf{x})}\sigma_0(\mathbf{x})$. Then $w(\mathbf{x},t)$ solves (4.9). We observe from (3.38) that

$$|\Lambda(\mathbf{x})| \le d_2\mu_3(R-r_0) \quad \forall \mathbf{x}\in\mathscr{U}. \tag{4.62}$$

Combining with the boundedness of $\|\underline{\mathbf{A}}\|_{C^1(\mathscr{U})}$, we have

$$|f_0(\mathbf{x},t)| \le C(|\mathbf{V}(\mathbf{x},t)| + |\nabla\mathbf{V}(\mathbf{x},t)|) \quad \forall(\mathbf{x},t)\in D. \tag{4.63}$$

Thanks to these relations, the assumptions in Proposition 4.5 hold, thus, the assertions (4.59) and (4.60) follow directly from (4.38) and (4.44). □

For the velocities, we have the following result.

**Theorem 4.9.** *Assume* ***(E1)*** *and*

$$\Delta_6 \overset{\text{def}}{=\!=} \sup_D(|\mathbf{V}(\mathbf{x},t)| + |\nabla\mathbf{V}(\mathbf{x},t)| + |\nabla^2\mathbf{V}(\mathbf{x},t)|) < \infty \text{ and } \Delta_7 \overset{\text{def}}{=\!=} \sup_{\Gamma\times[0,\infty)}|g(\mathbf{x},t)| < \infty. \tag{4.64}$$

*Then for any $U' \Subset U$, there is a positive number $\tilde{M}$ such that for $i=1,2$, and $t>0$,*

$$\sup_{\mathbf{x}\in U'}|\mathbf{v}_i(\mathbf{x},t)| \le \tilde{M}\Big(1+\frac{1}{\sqrt{t}}\Big)\Big[e^{-\eta_1 t}\sup_U|\sigma_0(\mathbf{x})| + \Delta_6 + \sqrt{\Delta_6} + \Delta_7\Big]. \tag{4.65}$$

*Consequently, if*

$$\lim_{t\to\infty}\Big\{\sup_{\mathbf{x}\in U}(|\mathbf{V}(\mathbf{x},t)| + |\nabla\mathbf{V}(\mathbf{x},t)| + |\nabla^2\mathbf{V}(\mathbf{x},t)|) + \sup_{\mathbf{x}\in\Gamma}|g(\mathbf{x},t)|\Big\} = 0, \tag{4.66}$$

*then for any $\mathbf{x}\in U$,*

$$\lim_{t\to\infty}\mathbf{v}_1(\mathbf{x},t) = \lim_{t\to\infty}\mathbf{v}_2(\mathbf{x},t) = 0. \tag{4.67}$$

*Proof.* Note that solution $\sigma(\mathbf{x},t)$ of (4.1) satisfies $\sigma\in C^3_{\mathbf{x}}(D)$ and $\sigma_t\in C^2_{\mathbf{x}}(D)$. Let $w, f_0, G, w_0$ be the same as in Theorem 4.8. Using the estimate of $\nabla w$ in Lemma 4.7 and formula (3.19), we easily obtain estimate (4.65) for $\mathbf{v}_2$. Then the estimate for $\mathbf{v}_1$ follows this and (3.4). The proof of (4.67) is similar to that of (4.44). We take $U' = B_\delta(\mathbf{x})$ such that $U'\Subset U$. For $T>0$, let

$$\Delta_{6,T} = \sup_{U\times[T,\infty)}(|\mathbf{V}(\mathbf{x},t)| + |\nabla\mathbf{V}(\mathbf{x},t)| + |\nabla^2\mathbf{V}(\mathbf{x},t)|) \text{ and } \Delta_{7,T} = \sup_{\Gamma\times[T,\infty)}|g(\mathbf{x},t)|.$$

Use (4.65) for all $t>T$ and $\Delta_{6,T}$, $\Delta_{7,T}$ in place of $\Delta_6$, $\Delta_7$. Then let $T\to\infty$ noting that $\Delta_{6,T}\to 0$ and $\Delta_{7,T}\to 0$. □

**Remark 4.10.** *The key ingredient of the above asymptotic results is Lemma 4.3, the lemma of growth in time. It is worth mentioning that this result can be extended to more general parabolic equations in more general domains $D$ in $\mathbb{R}^{n+1}$ rather than just cylindrical-in-time domains $D = U\times(0,\infty)$.*

## 5. Case of unbounded domain

We will analyze the linear stability of the steady flows from section 2 in an unbounded, outer domain $U = \mathbb{R}^n\setminus\bar{\Omega}$, where $\Omega$ is a simply connected, open, bounded set containing the origin. To emphasize the ideas and techniques, we consider the simple case $\Omega = B_{r_0}$ for some $r_0>0$.

For $R>r>0$, denote $\mathcal{O}_r = \mathbb{R}^n\setminus\bar{B}_r$, $\mathcal{O}_{r,R} = B_R\setminus\bar{B}_r$, and denote their closures by $\bar{\mathcal{O}}_r$ and $\bar{\mathcal{O}}_{r,R}$, respectively. Then $U = \mathcal{O}_{r_0}$. Let $\Gamma = \partial U = \{\mathbf{x} : |\mathbf{x}| = r_0\}$ and $D = U\times(0,\infty)$.

For $T>0$ we denote $U_T = U\times(0,T]$, then its closure is $\bar{U}_T = \bar{U}\times[0,T]$ and its parabolic boundary is $\partial_p U_T = [\bar{U}\times\{0\}]\cup[\Gamma\times(0,T]]$.

Same as in section 4, we consider a steady state $(u_1^*(\mathbf{x}), u_2^*(\mathbf{x}), S_*(\mathbf{x}))$ in (3.2) with $c_1^2+c_2^2>0$ and $\hat{S}(r)$ exists for all $r\ge r_0$. We assume throughout this section that

$$0 < \underline{s} \le \hat{S}(r) \le \bar{s} < 1 \quad \forall r\ge r_0, \text{ where } \underline{s}, \bar{s} = const. \tag{5.1}$$

For instance, in one of the cases in Theorem 2.2 if the limit $s_\infty \stackrel{\rm def}{=\!=} \lim_{r\to\infty} \hat{S}(r)$, which exists according to Theorem 2.3, belongs to the interval $(0,1)$ then (5.1) holds.

The problems of our interest are (4.1) and its transformed form (4.9).

Let $\mu_i$, for $i=1,2,3$, and $C_j$, for $j=0,1,2$, be defined as in section 4 (see (4.4), (4.5) and (4.6)). Thanks to condition (5.1), which plays the role of (4.3) in section 4, the main properties (4.5), (4.6) and (4.7) still hold with $\mathscr{U}=\mathcal{O}_{r_0,R}$ being replaced by $\mathscr{U}=U=\mathcal{O}_{r_0}$.

5.1. **Maximum principle for unbounded domain.** We establish the maximum principle for equation $\mathcal{L}w=0$ in the domain $U$ with operator $\mathcal{L}$ defined by (4.8). For $T>0$, we construct a barrier function $W(\mathbf{x},t)$ of the form:

$$W(\mathbf{x},t) \stackrel{\rm def}{=\!=} (T-t)^{-s} e^{\frac{\varphi(\mathbf{x})}{T-t}} \quad \text{for } (\mathbf{x},t)\in \mathcal{O}_{r_0,R}\times(0,T), \tag{5.2}$$

where constant $s>0$ and function $\varphi(\mathbf{x})>0$ will be decided later. Elementary calculations give

$$\mathcal{L}W = (T-t)^{-s-2} e^{\frac{\varphi}{T-t}} \Big\{ (T-t)\big(s - \nabla\cdot(\underline{\mathbf{A}}\nabla\varphi) - \mathbf{b}\cdot\underline{\mathbf{A}}\nabla\varphi\big) + \varphi - (\underline{\mathbf{A}}\nabla\varphi)\cdot\nabla\varphi \Big\}.$$

Then $\mathcal{L}W\ge 0$ if

$$s \ge \nabla\cdot(A\nabla\varphi) + \mathbf{b}\cdot\underline{\mathbf{A}}\nabla\varphi \quad \text{and} \quad \varphi \ge (\underline{\mathbf{A}}\nabla\varphi)\cdot\nabla\varphi. \tag{5.3}$$

Similar to section 4, we choose

$$\varphi(\mathbf{x}) = \kappa_1\Big(\varphi_1 + \int_{r_0}^{|\mathbf{x}|} r\phi(r)dr\Big), \quad \text{where } \varphi_1 = \frac{C_1 r_0^2}{2} > 0 \text{ and } \kappa_1 = \frac{C_1}{2C_0}, \tag{5.4}$$

and function $\phi$ is defined by (3.40). As in Lemma 4.2, we have

$$\underline{\mathbf{A}}\nabla\varphi = \kappa_1\mathbf{x} \quad \text{and} \quad \nabla\varphi = \kappa_1\phi(|\mathbf{x}|)\mathbf{x}. \tag{5.5}$$

By (3.41), $\phi(r)\ge d_0\mu_2 = C_1>0$. Then

$$\varphi(\mathbf{x}) \ge \kappa_1\Big(\varphi_1 + C_1\int_{r_0}^{|\mathbf{x}|} r dr\Big) = \frac{\kappa_1 C_1}{2}|\mathbf{x}|^2. \tag{5.6}$$

Also, we see from (5.5) and (3.31) that

$$(\underline{\mathbf{A}}\nabla\varphi)\cdot\nabla\varphi = \kappa_1^2\mathbf{x}^T\underline{\mathbf{B}}\mathbf{x} \le d_4\kappa_1^2|\mathbf{x}|^2\sum_{i=1}^2 F_i(S_*(\mathbf{x})) \le \kappa_1^2 d_4\mu_1|\mathbf{x}|^2 = \kappa_1^2 C_0|\mathbf{x}|^2 = \frac{\kappa_1 C_1}{2}|\mathbf{x}|^2. \tag{5.7}$$

Then we have from (5.6) and (5.7) that

$$\varphi(\mathbf{x}) \ge (A\nabla\varphi)\cdot\nabla\varphi. \tag{5.8}$$

By (4.6) and (5.5), we have

$$\nabla\cdot(\underline{\mathbf{A}}\nabla\varphi) + \mathbf{b}\cdot\underline{\mathbf{A}}\nabla\varphi \le \kappa_1(n + C_2|\mathbf{x}|) \le C_3(1+|\mathbf{x}|), \quad \text{where } C_3 = \kappa_1(n+C_2).$$

Select

$$s = s_R \stackrel{\rm def}{=\!=} C_3(1+R), \tag{5.9}$$

then

$$s \ge \nabla\cdot(\underline{\mathbf{A}}\nabla\varphi) + \mathbf{b}\cdot\underline{\mathbf{A}}\nabla\varphi \quad \text{in } \mathcal{O}_{r_0,R}. \tag{5.10}$$

Therefore $\mathcal{L}W\ge 0$ in $\mathcal{O}_{r_0,R}\times(0,T)$. We summarize the above arguments in the following lemma.

**Lemma 5.1.** *Let $T>0$, $R>r_0$ and let the function $\varphi$ be defined by (5.4). Then for $s=s_R$ in (5.9), the function $W(\mathbf{x},t)$ in (5.2) belongs to $C^{2,1}_{\mathbf{x},t}(D)\cap C(\bar{D})$ and satisfies $\mathcal{L}W\ge 0$ in $\mathcal{O}_{r_0,R}\times(0,T)$.*

Using the above barrier function $W(x,t)$, we have the following maximum principle.

**Theorem 5.2.** *Let $T > 0$ and $w(\mathbf{x}, t)$ be a bounded function in $C^{2,1}_{\mathbf{x},t}(U_T) \cap C(\bar{U}_T)$ that solves $\mathcal{L}w = f_0$ in $U_T$, where $f_0 \in C(\bar{U}_T)$. Then*

$$\sup_{\bar{U}_T} |w(\mathbf{x}, t)| \le \sup_{\partial_p U_T} |w(\mathbf{x}, t)| + (T + 1) \sup_{\bar{U}_T} |f_0|. \tag{5.11}$$

*Proof.* Given any $(\mathbf{x}_0, t_0) \in U \times (0, T)$. Let $\delta > 0$ such that $t_0 < T - \delta$. Let $M = \sup_{\bar{U}_T} |w(\mathbf{x}, t)|$ and $N = \sup_{\bar{U}_T} |f_0|$ which are finite numbers. Let $\mu > 0$ be arbitrary. Select $R > 0$ sufficiently large such that

$$T^{-C_3(1+R)} e^{\frac{\kappa_1 C_1 R^2}{2T}} > M/\mu. \tag{5.12}$$

Denote $\mathcal{C} = \mathcal{O}_{r_0,R} \times (0, T - \delta]$. Then $(\mathbf{x}_0, t_0) \in \mathcal{C}$. Let $W(\mathbf{x}, t)$ be as in Lemma 5.1. We define the auxiliary function

$$u(\mathbf{x}, t) = w(\mathbf{x}, t) - N(t + 1) - \mu W(\mathbf{x}, t), \quad (\mathbf{x}, t) \in \mathcal{C}. \tag{5.13}$$

We have $u \in C^{2,1}_{\mathbf{x},t}(\mathcal{C}) \cap C(\mathcal{C})$ and, thanks to Lemma 5.1, function $u$ satisfies

$$\mathcal{L}u = f_0 - N - \mu \mathcal{L}W \le 0 \quad \text{in } \mathcal{C}.$$

By the maximum principle,

$$\max_{\bar{\mathcal{C}}} u = \max_{\partial_p \mathcal{C}} u. \tag{5.14}$$

Let us evaluate $u(x, t)$ on the parabolic boundary $\partial_p \mathcal{C}$. For any $\mathbf{x} \in \mathcal{O}_{r_0,R}$,

$$u(\mathbf{x}, 0) \le w(\mathbf{x}, 0) - \mu W(\mathbf{x}, 0) = w(\mathbf{x}, 0) - \mu T^{-s} e^{\frac{\varphi(\mathbf{x})}{T}} \le w(\mathbf{x}, 0). \tag{5.15}$$

For $|\mathbf{x}| = r_0$ and $0 \le t \le T - \delta$,

$$u(\mathbf{x}, t) \le w(\mathbf{x}, t) - \mu W(\mathbf{x}, t) \le w(\mathbf{x}, t). \tag{5.16}$$

For $|\mathbf{x}| = R$ and $0 \le t \le T - \delta$, we have from (5.6), (5.9) and (5.12) that

$$u(\mathbf{x}, t) \le w(\mathbf{x}, t) - \mu (T - t)^{-s} e^{\frac{\varphi(\mathbf{x})}{T - t}} \le M - \mu T^{-C_3(1+R)} e^{\frac{\kappa_1 C_1 R^2}{2T}} \le 0. \tag{5.17}$$

Hence, we have from (5.14), (5.15),(5.16) and (5.17) that

$$\max_{\bar{\mathcal{C}}} u(\mathbf{x}, t) \le \max\{0, \sup_U w(\mathbf{x}, 0), \sup_{\Gamma \times [0,T]} w(\mathbf{x}, t)\}. \tag{5.18}$$

In particular, it follows from (5.18) that

$$u(\mathbf{x}_0, t_0) \le \max\{0, \sup_{\partial_p U_T} w\}. \tag{5.19}$$

Now, letting $\mu \to 0$ in (5.13) yields

$$w(\mathbf{x}_0, t_0) - N(t_0 + 1) \le \max\{0, \sup_{\partial_p U_T} w\} \le \sup_{\partial_p U_T} |w|.$$

Hence,

$$w(\mathbf{x}_0, t_0) \le \sup_{\partial_p U_T} |w| + N(T + 1).$$

Repeating the above arguments for $(-w)$ gives

$$|w(\mathbf{x}_0, t_0)| \le \sup_{\partial_p U_T} |w| + N(T + 1) \tag{5.20}$$

for any $(\mathbf{x}_0, t_0) \in U \times (0, T)$. Therefore, (5.11) follows. □

We study the following IBVP (4.9) for $w(\mathbf{x}, t)$.

**Condition (E2)**. $F_1, F_2 \in C^7((0, 1))$, $w_0 \in C(\bar{U})$, $G \in C(\Gamma \times [0, \infty))$ and $G(\mathbf{x}, 0) = w_0(\mathbf{x})$ on $\Gamma$.

**Theorem 5.3.** *Assume* ***(E2)****,* $f_0 \in C^5_{\mathbf{x}}(\bar D)$*,* $\partial_t f_0 \in C^3_{\mathbf{x}}(\bar D)$*. Suppose* $w_0(\mathbf{x})$*,* $G(\mathbf{x},t)$ *and* $f_0(\mathbf{x},t)$ *are bounded functions. Then,*

(i) *There exists a solution* $w(\mathbf{x},t) \in C^{2,1}_{\mathbf{x},t}(D) \cap C(\bar D)$ *of (4.9) .*

(ii) *This solution is unique in class of locally (in time) bounded solutions, i.e., the class of solutions* $w(\mathbf{x},t)$ *such that*

$$\sup_{U\times[0,T]} |w(\mathbf{x},t)| < \infty \quad \text{for any } T > 0. \tag{5.21}$$

(iii) *Furthermore, for* $(\mathbf{x},t) \in D$*,*

$$|w(\mathbf{x},t)| \le \Delta_8 + \Delta_9(t+1), \tag{5.22}$$

*where*

$$\Delta_8 = \max\{\sup_U |w_0(\mathbf{x})|, \sup_{\Gamma\times[0,\infty)} |G(\mathbf{x},t)|\} \quad \text{and} \quad \Delta_9 = \sup_D |f_0|. \tag{5.23}$$

*Proof.* We rewrite equation in the non-divergent form

$$\mathcal{L}w = \frac{\partial w}{\partial t} - \sum_{i,j=1}^n a_{ij}\frac{\partial^2 w}{\partial x_i \partial x_j} - \sum_{i,j=1}^n \left[(a_{ij})_{x_i} + a_{ij}b_i\right]\frac{\partial w}{\partial x_j} = 0.$$

Thanks to Theorem 4 p.474 of [16] and the maximum principle in Theorem 5.2, one can prove (i), (ii) and (iii) using similar arguments presented in Theorem 4.6 of [14]. We omit the details. □

5.2. **Lemma of growth in spatial variables.** We now study the behavior of the solutions as $|\mathbf{x}| \to \infty$. This requires a different type of lemma of growth and a new barrier function.

Let $R > 0$ and $\ell \ge R + r_0$. Denote

$$\mathcal{O}_R(\ell) = \mathcal{O}_{\ell-R,\ell+R} = \{\mathbf{x} \in \mathbb{R}^n : ||\mathbf{x}| - \ell| < R\} \quad \text{and} \quad \mathcal{S}_\ell = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| = \ell\}. \tag{5.24}$$

Define the barrier function

$$\mathcal{W}(\mathbf{x},t) = \frac{1}{(t+1)^s} e^{-\frac{\psi(\mathbf{x})}{t+1}} \quad \text{for } |\mathbf{x}| \ge r_0,\ t \ge 0, \tag{5.25}$$

where parameter $s > 0$ and function $\psi > 0$. Then

$$\mathcal{L}\mathcal{W} = (t+1)^{-s-2} e^{-\frac{\psi(\mathbf{x})}{t+1}} \Big\{(t+1)\big[-s + \nabla\cdot(\underline{\mathbf{A}}\nabla\psi) + \mathbf{b}\cdot\underline{\mathbf{A}}\nabla\psi\big] + \psi - (\underline{\mathbf{A}}\nabla\psi)\cdot\nabla\psi\Big\}. \tag{5.26}$$

Hence, $\mathcal{L}\mathcal{W} \le 0$ if

$$s \ge \nabla\cdot(\underline{\mathbf{A}}\nabla\psi) + \mathbf{b}\cdot\underline{\mathbf{A}}\nabla\psi \quad \text{and} \quad \psi \le (\underline{\mathbf{A}}\nabla\psi)\cdot\nabla\psi. \tag{5.27}$$

Denote $\boldsymbol{\xi}(\mathbf{x}) = \ell\mathbf{x}/|\mathbf{x}|$. We will choose $\psi$ such that

$$\underline{\mathbf{A}}\nabla\psi = \kappa_2(\mathbf{x} - \boldsymbol{\xi}) \quad \text{for some } \kappa_2 > 0.$$

By (3.39) and (3.40),

$$\nabla\psi = \kappa_2\underline{\mathbf{A}}^{-1}(\mathbf{x} - \boldsymbol{\xi}) = \kappa_2\underline{\mathbf{B}}\mathbf{x}(|\mathbf{x}| - \ell)/|\mathbf{x}| = \kappa_2\phi(|\mathbf{x}|)(|\mathbf{x}| - \ell)\mathbf{x}/|\mathbf{x}|. \tag{5.28}$$

Select

$$\psi(\mathbf{x}) = \kappa_2\int_\ell^{|\mathbf{x}|}(r-\ell)\phi(r)dr, \quad \text{where } \kappa_2 = \frac{C_0}{2C_1} \tag{5.29}$$

and function $\phi$ is defined by (3.40). For all $\mathbf{x} \in \mathcal{O}_R(\ell)$, we have from (3.42) that

$$\psi(\mathbf{x}) \le \kappa_2 C_0\int_\ell^{|\mathbf{x}|}(r-\ell)dr = \frac{\kappa_2 C_0}{2}(|\mathbf{x}| - \ell)^2. \tag{5.30}$$

By (3.31),

$$(\underline{\mathbf{A}}\nabla\psi)\cdot\nabla\psi = \kappa_2^2(\mathbf{x}-\boldsymbol{\xi})^T\underline{\mathbf{B}}(\mathbf{x})(\mathbf{x}-\boldsymbol{\xi}) \ge d_0\kappa_2^2|\mathbf{x}-\boldsymbol{\xi}|^2\sum_{j=1}^2 F_j(S_*(\mathbf{x}))$$
$$\ge \kappa_2^2 C_1(|\mathbf{x}|-\ell)^2 = \frac{\kappa_2 C_0}{2}(|\mathbf{x}|-\ell)^2.$$

Hence this and (5.30) give $\psi \le (\underline{\mathbf{A}}\nabla\psi)\cdot\nabla\psi$, that is, the second condition in (5.27). Also,

$$\nabla\cdot(\underline{\mathbf{A}}\nabla\psi)+\mathbf{b}\cdot(\underline{\mathbf{A}}\nabla\psi) = \kappa_2\Big[\nabla\cdot(\mathbf{x}-\boldsymbol{\xi})+\mathbf{b}\cdot(\mathbf{x}-\boldsymbol{\xi})\Big] = \kappa_2\Big[n-(n-1)\frac{\ell}{|\mathbf{x}|}+\mathbf{b}\cdot(\mathbf{x}-\boldsymbol{\xi})\Big]$$
$$\le \kappa_2(n+|\mathbf{b}|R).$$

Then by (4.6),

$$\nabla\cdot(\underline{\mathbf{A}}\nabla\psi)+\mathbf{b}\cdot(\underline{\mathbf{A}}\nabla\psi) \le \kappa_2(n+C_2R) \le C_3(1+R), \tag{5.31}$$

where $C_3 = \kappa_2(n+C_2)$. By selecting

$$s = s_R \overset{\text{def}}{=\!=} C_3(1+R), \tag{5.32}$$

we have $s \ge \nabla\cdot(\underline{\mathbf{A}}\nabla\psi)+\mathbf{b}\cdot(\underline{\mathbf{A}}\nabla\psi)$ which is the first condition in (5.27). Therefore $\mathcal{L}W \le 0$ in $\mathcal{O}_R(\ell)\times(0,\infty)$. We have proved:

**Lemma 5.4.** *Given any $R>0$ and $\ell \ge R+r_0$. Let $s=s_R$ be defined by (5.32) and the function $\psi$ be defined by (5.29). Then the function $\mathcal{W}(\mathbf{x},t)$ in (5.25) belongs to $C^{2,1}_{\mathbf{x},t}(D)\cap C(\bar{D})$ and satisfies $\mathcal{L}\mathcal{W}\le 0$ on $\mathcal{O}_R(\ell)\times(0,\infty)$.*

Next is the lemma of growth in the spatial variables.

**Lemma 5.5.** *Given $T>0$, let*

$$R = R(T) = C_4(1+T), \tag{5.33}$$

$$\eta_0 = \eta_0(T) = \Big(1-\frac{1}{2^{C_5(T+1)}}\Big)\frac{1}{(T+1)^{2C_5(T+1)}}, \tag{5.34}$$

*where $C_4 = \max\{1, \frac{8C_3}{\kappa_2 e C_0}\}$ and $C_5 = C_3C_4$. Suppose $w(\mathbf{x},t)\in C^{2,1}_{\mathbf{x},t}(U_T)\cap C(\bar{U}_T)$ satisfies $\mathcal{L}w\le 0$ on $U_T$ and $w(\mathbf{x},0)\le 0$ on $\bar{U}$. Let $\ell$ be any number such that $\ell\ge R+r_0$, then*

$$\max\big\{0, \sup_{\mathcal{S}_\ell\times[0,T]} w(\mathbf{x},t)\big\} \le \frac{1}{1+\eta_0}\max\big\{0, \sup_{\bar{\mathcal{O}}_R(\ell)\times[0,T]} w(\mathbf{x},t)\big\}. \tag{5.35}$$

*Proof.* Denote

$$M_\ell = \max\big\{0, \sup_{\bar{\mathcal{O}}_R(\ell)\times[0,T]} w(\mathbf{x},t)|\big\} \quad\text{and}\quad m_\ell = \max\big\{0, \sup_{\mathcal{S}_\ell\times[0,T]} w(\mathbf{x},t)\big\}.$$

Let $\mathcal{W}$ be defined as in Lemma 5.4. Let $\eta>0$ chosen later and define

$$\widetilde{W}(\mathbf{x},t) = M_\ell(1-\mathcal{W}(\mathbf{x},t)+\eta),$$

then $\mathcal{L}\widetilde{\mathcal{W}}\ge 0$ in $\mathcal{O}_R(\ell)\times(0,T]$. We have

$$\widetilde{W}(\mathbf{x},0) = M_\ell(1-e^{-\psi(\mathbf{x})}+\eta) \ge 0 \ge w(\mathbf{x},0). \tag{5.36}$$

By (5.30), $\psi(\mathbf{x})\le \kappa_2C_0R^2/2$ when $|\mathbf{x}|=\ell\pm R$, hence

$$\widetilde{W}(\mathbf{x},t)|_{|\mathbf{x}|=\ell\pm R} \ge M_\ell\Big(1-(t+1)^{-s}e^{-\frac{\kappa_2C_0R^2}{2(t+1)}}+\eta\Big). \tag{5.37}$$

Let $f(z) = z^{-s}e^{-\frac{\kappa_2 C_0 R^2}{2z}}$ for $z \ge 0$. Select $\eta = \max_{[0,\infty)} f(z)$. Elementary calculations show $\eta = (\frac{2s}{\kappa_2 e C_0 R^2})^s$. Then $t \in [0,T]$, it follows (5.37) that

$$\widetilde{W}(\mathbf{x},t)|_{|\mathbf{x}|=\ell\pm R} \ge M_\ell \ge \max\{0, w(\mathbf{x},t)|_{|\mathbf{x}|=\ell\pm R}\}. \tag{5.38}$$

From (5.36), (5.38) and maximum principle we obtain

$$\widetilde{W}(\mathbf{x},t) \ge w(\mathbf{x},t) \quad \text{on } \bar{\mathcal{O}}_R(\ell)\times(0,T).$$

Particularly,

$$\widetilde{W}(\mathbf{x},t) \ge w(\mathbf{x},t) \quad \text{on } \mathcal{S}_\ell\times(0,T). \tag{5.39}$$

Moreover, since $\psi(\mathbf{x}) = 0$ when $|\mathbf{x}| = \ell$, $\mathcal{W}(\mathbf{x},t) \ge \frac{1}{(T+1)^s}$ thus

$$\widetilde{W}(\mathbf{x},t)|_{|\mathbf{x}|=\ell} \le M_\ell\Big[1 - \frac{1}{(T+1)^s} + \eta\Big]. \tag{5.40}$$

Since $R \ge 1$, we easily estimate

$$\eta = \Big[\frac{2C_3(1+R)}{\kappa_2 e C_0 R^2}\Big]^{C_3(1+R)} \le \Big(\frac{4C_3 R}{\kappa_2 e C_0 R^2}\Big)^{C_3(1+R)} \le \Big(\frac{C_4}{2R}\Big)^{C_3(1+R)}.$$

Hence

$$\begin{aligned}\frac{1}{(T+1)^s} - \eta &\ge \frac{1}{(T+1)^{C_3(1+R)}} - \Big(\frac{C_4}{2R}\Big)^{C_3(1+R)} = \Big(1-\frac{1}{2^{C_3(R+1)}}\Big)\frac{1}{(T+1)^{C_3(1+R)}}\\ &\ge \Big(1-\frac{1}{2^{C_3R}}\Big)\frac{1}{(T+1)^{2C_3R}} = \Big(1-\frac{1}{2^{C_5(T+1)}}\Big)\frac{1}{(T+1)^{2C_5(T+1)}} = \eta_0.\end{aligned} \tag{5.41}$$

From (5.39), (5.40) and (5.41) we obtain $(1-\eta_0)M_\ell \ge m_\ell$, thus, $M_\ell \ge \frac{m_\ell}{1-\eta_0} \ge (1+\eta_0)m_\ell$, which gives (5.35). $\square$

**Lemma 5.6.** *Let $T>0$ and $R$, $\eta_0$ and $w(\mathbf{x},t)$ be as in Lemma 5.5. For $i \ge 1$, let*

$$\bar{m}_i = \max\big\{0, \sup_{\mathcal{S}_{r_0+iR}\times[0,T]} w(\mathbf{x},t)\big\}. \tag{5.42}$$

*Part A (Dichotomy for one cylinder). Then for any $i \ge 1$, we have either of the following cases.*

(a) *If $\bar{m}_{i+1} \ge \bar{m}_{i-1}$, then $\bar{m}_{i+1} \ge (1+\eta_0)\bar{m}_i$.*
(b) *If $\bar{m}_{i-1} \ge \bar{m}_{i+1}$, then $\bar{m}_{i-1} \ge (1+\eta_0)\bar{m}_i$.*

*Part B (Dichotomy for many cylinders). For any $k \ge 0$, we have the following two possibilities:*

(i) *There is $i_0 \ge k+1$ such that $\bar{m}_{i_0+j} \ge (1+\eta_0)^j\bar{m}_{i_0}$ for all $j \ge 0$.*
(ii) *For all $j \ge 0$, $\bar{m}_{k+j} \le (1+\eta_0)^{-j}\bar{m}_k$.*

*Proof.* Part A. By maximum principle,

$$\begin{aligned}\sup_{\bar{\mathcal{O}}_R(r_0+iR)\times[0,T]} w(\mathbf{x},t) &\le \max\big\{\sup_{\mathcal{S}_{r_0+(i\pm1)R}\times[0,T]} w(\mathbf{x},t), \sup_{\bar{\mathcal{O}}_R(r_0+iR)} w(\mathbf{x},0)\big\}\\ &\le \max\big\{\sup_{\mathcal{S}_{r_0+(i\pm1)R}\times[0,T]} w(\mathbf{x},t), 0\big\} \le \max\{\bar{m}_{i-1}, \bar{m}_{i+1}\}.\end{aligned}$$

Hence,

$$\sup_{\bar{\mathcal{O}}_R(r_0+iR)\times[0,T]} w(\mathbf{x},t) \le \max\{\bar{m}_{i-1}, \bar{m}_{i+1}\}. \tag{5.43}$$

Let $\ell = r_0 + iR$. Applying Lemma 5.5 and (5.43), we obtain

$$\bar{m}_i \le \frac{1}{1+\eta_0}\max\big\{0, \sup_{\bar{\mathcal{O}}_R(r_0+iR)\times[0,T]} w(\mathbf{x},t)\big\} \le \frac{1}{1+\eta_0}\max\{\bar{m}_{i-1}, \bar{m}_{i+1}\}.$$

Then the statements (a) and (b) obviously follow.

Part B. For $i < j$, define the cylinder

$$\mathcal{C}_{i,j} = \mathcal{O}_{r_0+iR,r_0+jR} \times (0,T) = \{(\mathbf{x},t) : r_0 + iR < |\mathbf{x}| < r_0 + jR,\ t \in (0,T)\}.$$

We say that (a) and (b) above are two cases for cylinder $\mathcal{C}_{i-1,i+1}$.

Let $k \geq 0$. By Part A, we have either of the following two cases.

**Case 1.** There is $i_0 \geq k$ such that Case (a) holds for $\mathcal{C}_{i_0,i_0+2}$, that is,

$$\bar{m}_{i_0+2} \geq \bar{m}_{i_0} \quad \text{and} \quad \bar{m}_{i_0+2} \geq (1+\eta_0)\bar{m}_{i_0+1}. \tag{5.44}$$

Then applying Part A to $\mathcal{C}_{i_0+1,i_0+3}$ we have either

$$\text{Case (a) holds for } \mathcal{C}_{i_0+1,i_0+3}, \text{ which gives } \bar{m}_{i_0+3} \geq \bar{m}_{i_0+1} \text{ and } \bar{m}_{i_0+3} \geq (1+\eta_0)\bar{m}_{i_0+2}, \tag{5.45}$$

or

$$\text{Case (b) holds for } \mathcal{C}_{i_0+1,i_0+3}, \text{ which gives } \bar{m}_{i_0+1} \geq \bar{m}_{i_0+3} \text{ and } \bar{m}_{i_0+1} \geq (1+\eta_0)\bar{m}_{i_0+2}. \tag{5.46}$$

Observe that (5.44) and (5.46) hold simultaneously if only if

$$\bar{m}_{i_0} = \bar{m}_{i_0+1} = \bar{m}_{i_0+2} = \bar{m}_{i_0+3} = 0, \tag{5.47}$$

which is a special case of (5.45). Hence we always have Case (a) for the next cylinder $\mathcal{C}_{i_0+1,i_0+3}$. Then by induction, Case (a) holds for the cylinders $\mathcal{C}_{i_0+j-1,i_0+j+1}$ for all $j \geq 1$. Thus,

$$\bar{m}_{i_0+j+1} \geq (1+\eta_0)\bar{m}_{i_0+j} \geq (1+\eta_0)^2\bar{m}_{i_0+j-1} \geq \ldots \geq (1+\eta_0)^j\bar{m}_{i_0+1}. \tag{5.48}$$

Re-indexing $i_0+1$ by $i_0$ in (5.48), we obtain (i).

**Case 2.** For all $i \geq k$, Case (b) holds for $\mathcal{C}_{i,i+2}$, that is, $\bar{m}_i \geq (1+\eta_0)\bar{m}_{i+1}$ for all $i \geq k$. Therefore,

$$\bar{m}_k \geq (1+\eta_0)\bar{m}_{k+1} \geq (1+\eta_0)^2\bar{m}_{k+2} \geq \ldots \geq (1+\eta_0)^j\bar{m}_{k+j}, \tag{5.49}$$

which implies (ii). □

Using the above dichotomy, we obtain the behavior of a sub-solution $w$ as $|\mathbf{x}| \to \infty$.

**Proposition 5.7.** *Assume $w \in C^{2,1}_{\mathbf{x},t}(U_T) \cap C(\bar{U}_T)$ satisfies $w(\mathbf{x},0) \leq 0$ in $U$, $\mathcal{L}w \leq 0$ on $U_T$, and $w(\mathbf{x},t)$ is bounded on $\bar{U}_T$. Then*

$$\limsup_{r\to\infty}\Big(\sup_{\mathcal{S}_r\times[0,T]} w(\mathbf{x},t)\Big) \leq 0. \tag{5.50}$$

*Proof.* Let $\bar{m}_i$ be defined as in Lemma 5.6.

**Case 1**: There are infinitely many $i$ such that $\bar{m}_i = 0$. Then there is a sequence $\{i_l\}$ increasing to $\infty$ as $l \to \infty$ such that $\bar{m}_{i_l} = 0$ for all $l \geq 1$. Then by maximum principle for cylinder $\mathcal{C}_{i_l,i_{l+1}}$ we have $w(\mathbf{x},t) \leq 0$ on $\mathcal{C}_{i_l,i_{l+1}}$ for all $l \geq 1$. Therefore $w(\mathbf{x},t) \leq 0$ in $\{|\mathbf{x}| \geq r_0 + i_1R\} \times [0,T]$. This gives (5.50).

**Case 2**: There are only finitely many $i$ such that $\bar{m}_i = 0$. Then there is $N > 0$ such that $\bar{m}_i > 0$ for all $i \geq N$. We apply part B of Lemma 5.6 to $k = N$. If (i) holds, then there is $i_0 \geq N+1$ such that $\bar{m}_{i_0+j} \geq (1+\eta_0)^j\bar{m}_{i_0} > 0$ for all $j \geq 0$; thus, $\lim_{j\to\infty}\bar{m}_{i_0+j} = \infty$ which contradicts $w(\mathbf{x},t)$ being bounded on $U_T$. Hence we must have (ii), that is, for all $j \geq 0$, $\bar{m}_{N+j} \leq (1+\eta_0)^{-j}\bar{m}_N$. Therefore, $\lim_{j\to\infty}\bar{m}_{N+j} = 0$ which, in combining with (5.43), proves (5.50). □

As for solutions of the IBVP (4.9) in a finite time interval, we have the following.

**Theorem 5.8.** *Let $w \in C^{2,1}_{\mathbf{x},t}(U_T) \cap C(\bar{U}_T)$ be a bounded solution of (4.9) on $U_T$ with $f_0 \in C(\bar{U}_T)$. If*

$$\lim_{|\mathbf{x}|\to\infty} w_0(\mathbf{x}) = 0, \tag{5.51}$$

$$\lim_{|\mathbf{x}|\to\infty}\sup_{0\leq t\leq T}|f_0(\mathbf{x},t))| = 0, \tag{5.52}$$

*then*

$$\lim_{r\to\infty}\Big(\sup_{\mathcal{S}_r\times[0,T]}|w(\mathbf{x},t)|\Big) = 0. \tag{5.53}$$

*Proof.* Note that $w_0 \in C(\bar{U})$, $G \in C(\Gamma \times [0,T])$. By Theorem 5.2, $w(\mathbf{x},t)$ is bounded on $\bar{U}_T$. Let $\varepsilon$ be an arbitrary positive number. There is $\tilde{r}_0 > 0$ such that for $|\mathbf{x}| > \tilde{r}_0$ we have

$$|w_0(\mathbf{x})| < \varepsilon \quad \text{and} \quad \sup_{0\le t\le T} |f_0(\mathbf{x},t)| < \varepsilon. \tag{5.54}$$

Let $\tilde{w} = \pm w - \varepsilon(t+1)$ then $\tilde{w}$ is bounded on $\bar{U}_T$ and $\mathcal{L}\tilde{w} < 0$ on $\mathcal{O}_{\tilde{r}_0} \times (0,T]$, and $\tilde{w}(\mathbf{x},0) \le 0$ on $\bar{\mathcal{O}}_{\tilde{r}_0}$. Applying Proposition 5.7 to $\tilde{w}$ with $r_0$ being replaced by $\tilde{r}_0$ gives

$$\limsup_{r\to\infty}\Big(\sup_{\mathcal{S}_r\times[0,T]} \tilde{w}(\mathbf{x},t)\Big) \le 0.$$

This implies

$$\limsup_{r\to\infty}\Big(\sup_{\mathcal{S}_r\times[0,T]} [\pm w(\mathbf{x},t)]\Big) \le \varepsilon(T+1).$$

Therefore,

$$\limsup_{r\to\infty}\Big(\sup_{\mathcal{S}_r\times[0,T]} |w(\mathbf{x},t)|\Big) \le \varepsilon(T+1).$$

Letting $\varepsilon \to 0$ we obtain (5.53). □

We now consider problem (4.9) for all $t > 0$ under condition (5.51). Although it is not known whether $\lim_{t\to\infty} w(\mathbf{x},t)$ exists for each $\mathbf{x}$, we prove in the corollary below that such limit is zero along some curve $\mathbf{x}(t)$ which goes to infinity as $t \to \infty$.

**Corollary 5.9.** *Let $w(\mathbf{x},t) \in C^{2,1}_{\mathbf{x},t}(D) \cap C(\bar{D})$ be a bounded solution of (4.9) on $D$ with $f_0 \in C(\bar{D})$. Assume $w_0 \in C(\bar{U})$ satisfies (5.51), $G \in C(\Gamma \times [0,\infty))$ is bounded, and (5.52) holds for each $T > 0$. Then there exists an increasing, continuous function $r(t) > 0$ satisfying $\lim_{t\to\infty} r(t) = \infty$ such that*

$$\lim_{t\to\infty}\Big(\sup_{\mathbf{x}\in\bar{\mathcal{O}}_{r(t)}} |w(\mathbf{x},t)|\Big) = 0. \tag{5.55}$$

*Proof.* By Theorem 5.8, there exists a strictly increasing sequence $\{r_k\}_{k=1}^\infty$ of positive numbers such that $\lim_{k\to\infty} r_k = \infty$ and

$$\sup_{\{\mathbf{x}:|\mathbf{x}|\ge r_k\}\times[0,k]} |w(\mathbf{x},t)| < \frac{1}{k}. \tag{5.56}$$

Let $r(t)$ be the piecewise linear function passing through the points $(k, r_{k+1})$ then $r(t)$ is increasing and $r(t) \to \infty$ as $t \to \infty$. By (5.56), for each $k$ we have

$$\sup\{|w(\mathbf{x},t)| : k \le t \le k+1, |\mathbf{x}| \ge r(t)\} \le \sup_{\{\mathbf{x}:|\mathbf{x}|\ge r_{k+1}\}\times[0,k+1]} |w(\mathbf{x},t)| < \frac{1}{k+1}.$$

Taking $k \to \infty$ we obtain (5.55). □

We now return to the IBVP (4.1) for $\sigma$. We will use the transformation $\sigma = we^{\Lambda}$. To compare $\sigma$ and $w$, we need to estimate $\Lambda(\mathbf{x})$. Recall from (3.37) that

$$\Lambda(\mathbf{x}) = \int_{r_0}^{|\mathbf{x}|} \tilde{F}(r)dr, \text{ where } \tilde{F}(r) = F_2'(\hat{S}(r))g_2\Big(\frac{|c_2|}{r^{n-1}}\Big)\frac{c_2}{r^{n-1}} - F_1'(\hat{S}(r))g_1\Big(\frac{|c_1|}{r^{n-1}}\Big)\frac{c_1}{r^{n-1}}.$$

For $R$ sufficiently large and $r \ge R$, we have $|\tilde{F}(r)| \le Cr^{1-n}$. Then we have in the case $n \ge 3$ that $|\tilde{F}(r)| \le Cr^{-2}$, hence $|\Lambda(\mathbf{x})| \le C_6$ for all $|\mathbf{x}| \ge r_0$, and

$$0 < C_7^{-1} \le e^{\Lambda(\mathbf{x})} \le C_7 \quad \forall |\mathbf{x}| \ge r_0. \tag{5.57}$$

**Theorem 5.10.** *Let $n \geq 3$. Assume **(E1)** and*

$$\Delta_{10} \stackrel{\text{def}}{=\!=} \max\{\sup_U |\sigma_0(\mathbf{x})|, \sup_{\Gamma\times[0,\infty)} |g(\mathbf{x},t)|\} < \infty, \tag{5.58}$$

$$\Delta_{11} \stackrel{\text{def}}{=\!=} \sup_D |\nabla\cdot(\underline{\mathbf{A}}(\mathbf{x})\mathbf{c}(\mathbf{x},t))| < \infty. \tag{5.59}$$

*Then,*

(i) *There exists a solution $\sigma(\mathbf{x},t) \in C^{2,1}_{\mathbf{x},t}(D)\cap C(\bar D)$ of problem (4.1). This solution is unique in class of solutions $\sigma(\mathbf{x},t)$ that satisfy*

$$\sup_{U\times[0,T]} |\sigma(\mathbf{x},t)| < \infty \quad \textit{for any } T>0. \tag{5.60}$$

(ii) *There is $C>0$ such that for $(\mathbf{x},t)\in D$,*

$$|\sigma(\mathbf{x},t)| \leq C\big[\Delta_{10} + \Delta_{11}(t+1)\big]. \tag{5.61}$$

(iii) *In addition, if*

$$\lim_{|\mathbf{x}|\to\infty} \sigma_0(\mathbf{x}) = 0 \quad \textit{and} \quad \lim_{|\mathbf{x}|\to\infty} \sup_{0\leq t\leq T} |\nabla\cdot(\underline{\mathbf{A}}(\mathbf{x})\mathbf{c}(\mathbf{x},t))| = 0 \textit{ for each } T>0, \tag{5.62}$$

*then*

$$\lim_{r\to\infty}\Big(\sup_{\mathcal{S}_r\times[0,T]} |\sigma(\mathbf{x},t)|\Big) = 0 \quad \textit{for any } T>0, \tag{5.63}$$

*and furthermore, there is a continuous, increasing function $r(t)>0$ with $\lim_{t\to\infty} r(t)=\infty$ such that*

$$\lim_{t\to\infty}\Big(\sup_{\mathbf{x}\in\bar{\mathcal{O}}_{r(t)}} |\sigma(\mathbf{x},t)|\Big) = 0. \tag{5.64}$$

*Proof.* Let $w_0(\mathbf{x}) = \sigma_0(\mathbf{x})e^{-\Lambda(\mathbf{x})}$, $G(\mathbf{x},t) = g(\mathbf{x},t)e^{-\Lambda(\mathbf{x})}$ and $f_0(x,t) = e^{-\Lambda(\mathbf{x})}|\nabla\cdot(\underline{\mathbf{A}}(\mathbf{x})\mathbf{c}(\mathbf{x},t))|$. Thanks to (5.57) and (5.58), we have

$$\max\{\sup_U |w_0(\mathbf{x})|, \sup_{\Gamma\times[0,\infty)} |w(\mathbf{x},t|\} \leq C\Delta_{10},$$

$$\sup_D |f_0| \leq C\Delta_{11}.$$

Then statements in (i), (ii) and (iii) follow directly from Theorems 5.3 and 5.8, and Corollary 5.9 for problem (4.9), the relation $\sigma(\mathbf{x},t) = w(\mathbf{x},t)e^{\Lambda(\mathbf{x})}$ and the boundedness of $e^{\Lambda(\mathbf{x})}$ in (5.57). We omit the details. □

As a consequence of (5.64), for any continuous curve $\mathbf{x}(t)$ with $|\mathbf{x}(t)| \geq r(t)$, one has

$$\lim_{t\to\infty} \sigma(\mathbf{x}(t),t) = 0. \tag{5.65}$$

The case $n=2$ is treated next with some restriction on the steady state.

**Theorem 5.11.** *Let $n=2$ and $\hat S(r)$ be a solution of (2.13) with $c_1, c_2 < 0$. Assume **(E1)** and*

$$\Delta_{12} \stackrel{\text{def}}{=\!=} \max\{\sup_U e^{-\Lambda(\mathbf{x})}|\sigma_0(\mathbf{x})|, \sup_{\Gamma\times[0,\infty)} |g(\mathbf{x},t)|\} < \infty, \tag{5.66}$$

$$\Delta_{13} \stackrel{\text{def}}{=\!=} \sup_D e^{-\Lambda(\mathbf{x})}|\nabla\cdot(\underline{\mathbf{A}}(\mathbf{x})\mathbf{c}(\mathbf{x},t))| < \infty. \tag{5.67}$$

*Then the following statements hold true.*

(i) *There exists a solution $\sigma(\mathbf{x},t) \in C^{2,1}_{\mathbf{x},t}(D)\cap C(\bar D)$ of problem (4.1). This solution is unique in class of solutions $\sigma(\mathbf{x},t)$ that satisfy*

$$\sup_{U\times[0,T]} e^{-\Lambda(\mathbf{x})}|\sigma(\mathbf{x},t)| < \infty \quad \textit{for any } T>0. \tag{5.68}$$

(ii) *There is $C>0$ such that for $(\mathbf{x},t)\in D$,*

$$|\sigma(\mathbf{x},t)|\le C\big[\Delta_{12}+\Delta_{13}(t+1)\big].$$

(iii) *Statement* (iii) *of Theorem 5.10 holds true if condition (5.62) is replaced by*

$$\lim_{|\mathbf{x}|\to\infty} e^{-\Lambda(\mathbf{x})}\sigma_0(\mathbf{x})=0 \quad \textit{and} \quad \lim_{|\mathbf{x}|\to\infty}\sup_{0\le t\le T} e^{-\Lambda(\mathbf{x})}|\nabla\cdot(\underline{\mathbf{A}}(\mathbf{x})\mathbf{c}(\mathbf{x},t))|=0 \textit{ for each } T>0. \tag{5.69}$$

*Proof.* According to Theorem 2.5, $\lim_{r\to\infty}\hat S(r)=s^*\in(0,1)$, where $s^*$ is defined in (2.33). The proof consists of two steps.

*Step 1.* We show that statements (i)–(iii) hold true under the following condition

$$F_2'(s^*)a_2^0c_2-F_1'(s^*)a_1^0c_1<0. \tag{5.70}$$

Let $c_4=-(F_2'(s^*)a_2^0c_2-F_1'(s^*)a_1^0c_1)>0$. We have for any $R>r_0$ and $|\mathbf{x}|>R$ that

$$\Lambda(\mathbf{x})=\int_{r_0}^R\tilde F(r)dr+\int_R^{|\mathbf{x}|}\tilde F(r)dr=I_1(R)+I_2(R).$$

For sufficiently large $R_0>r_0$, we have for $|\mathbf{x}|>R_0$ that

$$I_2(R_0)\le\frac12\int_{R_0}^{|\mathbf{x}|}\big(F_2'(\hat S(r))a_2^0c_2-F_1'(\hat S(r))a_1^0c_1\big)r^{-1}dr\le-\frac14\int_{R_0}^{|\mathbf{x}|}c_4r^{-1}d\xi\le 0.$$

Obviously, $I_1(R_0)$ is finite. This gives $e^{\Lambda(\mathbf{x})}\le C_8<\infty$ for all $|\mathbf{x}|\ge r_0$. Thus,

$$|\sigma|\le C_9|w|\quad\text{with constant } C_9>0. \tag{5.71}$$

Setting $w(\mathbf{x},t)=\sigma(\mathbf{x},t)e^{-\Lambda(\mathbf{x})}$, we have $\mathcal{L}w=f_0$, where $f_0$ is as in Theorem 5.10. Then (i)–(iii) easily follow Theorems 5.3, 5.8, Corollary 5.9 and relation (5.71).

*Step 2.* Now, it suffices to show that condition (5.70) is satisfied with $c_1,c_2<0$. On the one hand, we have from (2.33) that

$$\frac{a_1^0c_1}{a_2^0c_2}=f(s^*)=\frac{f_1}{f_2}(s^*)=\frac{F_2(s^*)}{F_1(s^*)}.$$

Then $a_1^0c_1F_1(s^*)=a_2^0c_2F_2(s^*)\overset{\text{def}}{=\!=}\mathcal{A}\ne 0$. On the other hand,

$$F_2'(s^*)a_2^0c_2-F_1'(s^*)a_1^0c_1=\mathcal{A}\Big[\frac{F_2'(s^*)}{F_2(s^*)}-\frac{F_1'(s^*)}{F_1(s^*)}\Big]=\mathcal{A}\frac{F_1(s^*)}{F_2(s^*)}\Big(\frac{F_2}{F_1}\Big)'(s^*).$$

The assumptions on $f_1$ and $f_2$ provide $(F_2/F_1)'(s^*)=(f_1/f_2)'(s^*)>0$ and $F_1(s^*),F_2(s^*)>0$. Since $c_1,c_2<0$, we have $\mathcal{A}<0$ and, hence, $F_2'(s^*)a_2^0c_2-F_1'(s^*)a_1^0c_1<0$. The proof is complete. ☐

**Remark 5.12.** *Similar to Theorem 4.9, we can use Bernstein's technique to estimate $\mathbf{v}_1(\mathbf{x},t)$ and $\mathbf{v}_2(\mathbf{x},t)$ uniformly in $\mathbf{x}\in U'\Subset U$. We do not provide details here.*

## Appendix A

We give proof to the statements on the range of $s_\infty$ in Example 2.6. Recall that $s_\infty\in[0,1]$.

In the case $\Delta=0$ of A and B, $h(r)\equiv s^*$ is the equilibrium and the conclusions are clear. Also, for C and D, $S(r)$ is monotone and the statements easily follow. We focus on the remaining cases.

A. $c_1,c_2>0$. Note that $F(r,S)>0$ iff $S>h(r)$, hence $S'(r)>0$ iff $S(r)>h(r)$.

- $\Delta<0$. Then $h(r)$ increases and $h(r)<s^*$ for all $r$. Consider $s_0>s^*$. Then $S(r)>s^*>h(r)$ for all $r$. It follows that $S(r)$ is strictly increasing which implies $s_\infty>s_0$. Now, consider $s_0<h(r_0)$. Then $S(r)<h(r)$ for all $r$, thus $S(r)$ is strictly decreasing and, therefore, $s_\infty<s_0$.
- $\Delta>0$. In this case, $h(r)$ is decreasing, and $h(r)>s^*$ for all $r$. Then the arguments are the same as in the case $\Delta<0$.

B. $c_1,c_2<0$. Observe that $F(r,S)>0$ iff $S<h(r)$, hence $S'(r)>0$ iff $S(r)<h(r)$.

- $\Delta < 0$. Then $h(r)$ is increasing and $h(r) < s^*$ for all $r$.

  We prove (iii) first when $s_0 < h(r_0)$. Exactly the same as Claim 2 in the proof of Theorem 2.3, we have $S(r) \le h(r) < s^*$ for all $r$. Thus $S(r)$ is increasing on $[r_0, \infty)$. Hence $s_\infty \in [s_0, s^*]$. Since $S(r)$ is strictly increasing for $r$ near $r_0$, we have $s_\infty > s_0$.

  We prove (ii). Consider the subcase $h(r_0) < s_0 \le s^*$. Then there exists $r_1 > r_0$ such that $S(r) > h(r)$ for $r < r_1$ and $S(r_1) = h(r_1)$. Similar arguments to (iii), we have $S(r_1) \le S(r) \le h(r)$ for all $r < r_1$. Hence $s_\infty \le s^*$ and $s_\infty \ge h(r_1) > h(r_0)$.

  In the particular case $s_0 = h(r_0)$, one can show that $h(r_0) \le S(r) \le h(r)$ for all $r > r_0$. If $S(r) \equiv h(r)$ then $s_\infty = s^*$. Otherwise, there is $r_1 > r_0$ and such that $h(r_0) \le S(r_1) < h(r_1)$. Similar to (iii) with $r_0$ playing the role of $r_1$, we have $s_\infty \in (S(r_1), s^*]$. Hence $s_0 \in (h(r_0), s^*]$.

  Finally, we prove (i) when $s_0 > s^*$. Clearly, $S(r) < s_0$ for all $r > r_0$. If $s_0 > S(r) > s^*$ for all $r > r_0$ then we have $S(r)$ strictly deceasing and $s_\infty \in [s^*, s_0)$. Otherwise, there is $r_1$ such that $S(r_1) = s^*$. Then using (ii) we obtain $s_\infty \in (h(r_0), s_*]$.
- $\Delta > 0$. Then $h(r)$ is decreasing, and $h(r) > s^*$ for all $r$. The proof is similar to the case $\Delta < 0$.

## References


[1] H. W. Alt and E. DiBenedetto. Flow of oil and water through porous media. *Astérisque*, (118):89–108, 1984. Variational methods for equilibrium problems of fluids (Trento, 1983).

[2] H. W. Alt and E. DiBenedetto. Nonsteady flow of water and oil through inhomogeneous porous media. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 12(3):335–392, 1985.

[3] E. Aulisa, L. Bloshanskaya, L. Hoang, and A. Ibragimov. Analysis of generalized Forchheimer flows of compressible fluids in porous media. *J. Math. Phys.*, 50(10):103102, 44, 2009.

[4] J. Bear. *Dynamics of Fluids in Porous Media.* Dover, New York, 1972.

[5] R. H. Brooks and A. Corey. Hydraulic properties of porous media. *Hydrol. Pap, Colo. State Univ., Forst Collins*, (3), 1964.

[6] S. Brull. Two compressible immiscible fluids in porous media: the case where the porosity depends on the pressure. *Adv. Differential Equations*, 13(7-8):781–800, 2008.

[7] C. Cancès. Finite volume scheme for two-phase flows in heterogeneous porous media involving capillary pressure discontinuities. *M2AN Math. Model. Numer. Anal.*, 43(5):973–1001, 2009.

[8] C. Cancès, T. Gallouët, and A. Porretta. Two-phase flows involving capillary barriers in heterogeneous porous media. *Interfaces Free Bound.*, 11(2):239–258, 2009.

[9] E. DiBenedetto, U. Gianazza, and V. Vespri. Continuity of the saturation in the flow of two immiscible fluids in a porous medium. *Indiana Univ. Math. J.*, 59(6):2041–2076, 2010.

[10] J. K. Hale. *Ordinary differential equations.* Robert E. Krieger Publishing Co. Inc., Huntington, N.Y., second edition, 1980.

[11] L. Hoang and A. Ibragimov. Structural stability of generalized Forchheimer equations for compressible fluids in porous media. *Nonlinearity*, 24(1):1–41, 2011.

[12] L. Hoang and A. Ibragimov. Qualitative study of generalized Forchheimer flows with the flux boundary condition. *Adv. Diff. Eq.*, 17(5–6):511–556, 2012.

[13] L. Hoang, A. Ibragimov, T. Kieu, and Z. Sobol. Stability of solutions to generalized Forchheimer equations of any degree. 2012. submitted.

[14] L. T. Hoang, A. Ibragimov, and T. T. Kieu. One-dimensional two-phase generalized Forchheimer flows of incompressible fluids. *J. Math. Anal. Appln.*, 401(2):921–938, 5 2013.

[15] L. T. Hoang, T. T. Kieu, and T. V. Phan. Properties of generalized Forchheimer flows in porous media. 2013. submitted.

[16] A. M. Il′in, A. S. Kalashnikov, and O. A. Oleĭnik. Second-order linear equations of parabolic type. *Tr. Semin. im. I. G. Petrovskogo*, (21):9–193, 341, 2001.

[17] S. N. Kružkov and S. M. Sukorjanskiĭ. Boundary value problems for systems of equations of two-phase filtration type; formulation of problems, questions of solvability, justification of approximate methods. *Mat. Sb. (N.S.)*, 104(146)(1):69–88, 175–176, 1977.

[18] S. N. Kruzhkov. Uniqueness of the solutions of mixed problems for a degenerate system of the theory of two-phase filtration. *Vestnik Moskov. Univ. Ser. I Mat. Mekh.*, (2):28–33, 95, 1985.

[19] E. M. Landis. *Second order equations of elliptic and parabolic type*, volume 171 of *Translations of Mathematical Monographs.* American Mathematical Society, Providence, RI, 1998. Translated from the 1971 Russian original by Tamara Rozhkovskaya, With a preface by Nina Ural′tseva.

[20] B. Straughan. *Stability and wave motion in porous media*, volume 165 of *Applied Mathematical Sciences*. Springer, New York, 2008.

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