

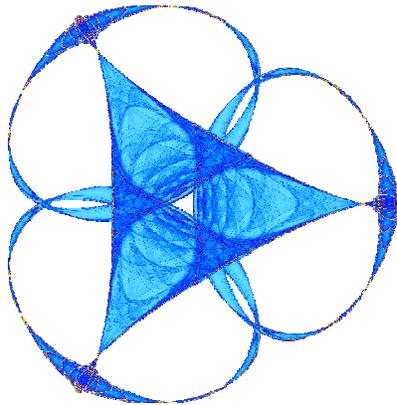
INVISCID LIMIT OF STOCHASTIC DAMPED 2D NAVIER-STOKES EQUATIONS

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**IMA Preprint Series #2414**

(December 2012)



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# Inviscid limit of stochastic damped 2D Navier-Stokes equations

Hakima Bessaih\* & Benedetta Ferrario†

## Abstract

We consider the inviscid limit of the stochastic damped 2D Navier-Stokes equations. We prove that, when the viscosity vanishes, the stationary solution of the stochastic damped Navier-Stokes equations converges to a stationary solution of the stochastic damped Euler equation and that the rate of dissipation of enstrophy converges to zero. In particular, this limit obeys an enstrophy balance. The rates are computed with respect to a limit measure of the unique invariant measure of the stochastic damped Navier-Stokes equations.

**MSC2010:** 60G10, 60H30, 35Q35.

**Keywords:** Inviscid limits, enstrophy balance, stationary processes, invariant measures.

## 1 Introduction

In this paper, we are interested in the equations of motion of incompressible fluids in a bounded domain of  $\mathbb{R}^2$ . In particular, we consider the Euler or Navier-Stokes equations damped by a term proportional to the velocity. Damping terms in two dimensional turbulence studies have been considered to model pumping due to friction with boundaries. Numerical studies of two dimensional turbulence employ devices to remove the energy that piles up at the large scales, and damping is the most common such device. We refer to [16, 8] for a physical motivation of the model and to [1, 18, 19] for a mathematical analysis of the deterministic damped Navier-Stokes equations and to [4, 5] for the stochastic damped Euler equations.

These stochastic damped equations are given by

$$(1) \quad \begin{cases} du + [-\nu\Delta u + (u \cdot \nabla)u + \gamma u + \nabla p]dt = dw \\ \nabla \cdot u = 0 \end{cases}$$

The non negative coefficients  $\nu$  and  $\gamma$  are called kinematic viscosity and sticky viscosity, respectively. The unknowns are the velocity  $u$  and the pressure  $p$ . Suitable boundary conditions have to be considered.

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For a fixed  $\gamma > 0$ , if  $\nu > 0$  these are called the stochastic damped Navier-Stokes equations, whereas if  $\nu = 0$  they are the stochastic damped Euler equations. If  $\gamma = 0$  and  $\nu = 0$ , we refer to [3, 7, 9, 22, 20] for an analysis of the existence and/or uniqueness of solutions and to [11] where some dissipation of enstrophy arguments are discussed in Besov spaces.

Turbulence theory investigates the behavior of certain quantities as the viscosity  $\nu$  vanishes. In particular, in the two dimensional setting one is interested in understanding what happens to the balance equation of energy and enstrophy (in the stationary regime) as the viscosity vanishes. D. Bernard [2] suggested that there is no anomalous dissipation of enstrophy in damped and driven Navier-Stokes equations; Constantin and Ramos [10] proved that there is no anomalous dissipation neither of energy nor of enstrophy as  $\nu \rightarrow 0$  for the deterministic damped Navier-Stokes equations in the whole plane. Some similar questions were suggested by Kupiainen [21] for the stochastic case. Therefore we address the same problem when the forcing term is of white noise type. Tools from stochastic analysis are very useful to investigate the same problem studied in [10], giving a rigorous meaning to the averages of velocity and vorticity. Indeed, using stochastic PDE's allows to express the stationary regime by means of an invariant measure, whereas in the deterministic setting the stationary regime is described by taking time averages on the infinite time interval.

In this paper we shall prove that in the stationary regime system (1) has no anomalous dissipation neither of energy nor of enstrophy as  $\nu \rightarrow 0$ . However, we shall be working in a finite two dimensional spatial domain and not in the whole plane; this answers one of the questions posed by Kupiainen in [21] about the behaviour of the stochastic damped Navier-Stokes equations on a torus for vanishing viscosity.

As far as the content of the paper is concerned, in Section 2 we introduce some functional spaces, the equations in their vorticity formulation and the assumptions on the noise term. We also introduce the classical properties of the nonlinear term associated to these equations. Section 3 is devoted to the well posedness of the stochastic 2D damped Navier-Stokes equations, where some uniform estimates are computed. Starting from a known result of existence and uniqueness of the invariant measure, we provide a balance law for the enstrophy. The vanishing viscosity limit is studied in Section 4 and stationary solutions are constructed by means of a tightness argument providing a balance relation for these stationary solutions. Using these results, we provide a proof of no anomalous of enstrophy and energy for the stochastic damped 2D Navier-Stokes equations.

## 2 Notations and hypothesis

Let the spatial domain  $D$  be the square  $[-\pi, \pi]^2$ ; periodic boundary conditions are assumed. A basis of the space  $L^2(D)$  with periodic boundary conditions is  $\{e_k\}_{k \in \mathbb{Z}^2}$ ,  $e_k(x) = \frac{1}{2\pi} e^{ik \cdot x}$ , whereas a basis for the space of periodic vector fields which are square integrable and divergence free is  $\{\frac{k^\perp}{|k|} e_k\}_{k \in \mathbb{Z}^2}$ . Actually we consider  $k \neq (0, 0)$ , since if  $u$  is a solution of system (1) then also  $u + c$  is a solution for any  $c \in \mathbb{R}$ . Therefore we consider velocity fields with vanishing mean value.

Let  $\mathbb{Z}_0^2 = \mathbb{Z}^2 \setminus \{(0, 0)\}$ , and  $\mathbb{Z}_+^2 = \{k = (k_1, k_2) \in \mathbb{Z} : k_1 > 0\} \cup \{k =$

$(0, k_2) \in \mathbb{Z}^2 : k_2 > 0\}$ . Given  $x = (x_1, x_2) \in \mathbb{R}^2$  we denote by  $|x|$  its norm:  $|x| = \sqrt{(x_1)^2 + (x_2)^2}$ . Given  $y = \Re y + i\Im y \in \mathbb{C}$  we denote by  $|y|$  its absolute value and by  $\bar{y}$  its complex conjugate:  $|y| = \sqrt{(\Re y)^2 + (\Im y)^2}$ ,  $\bar{y} = \Re y - i\Im y$ .

For any  $a \in \mathbb{R}$  we define the Hilbert space

$$H^a = \left\{ f = \sum_{k \in \mathbb{Z}_0^2} f_k e_k(x) : \sum_{k \in \mathbb{Z}_0^2} |f_k|^2 |k|^{2a} < \infty \right\}$$

with scalar product

$$\langle f, g \rangle_{H^a} = \sum_{k \in \mathbb{Z}_0^2} |k|^{2a} f_k \bar{g}_k;$$

we set

$$\|f\|_{H^a}^2 = \sum_{k \in \mathbb{Z}_0^2} |k|^{2a} |f_k|^2.$$

For a vector  $f = (f_1, f_2)$  we set

$$\|f\|_{H^a}^2 = \|f_1\|_{H^a}^2 + \|f_2\|_{H^a}^2.$$

In particular, for scalar functions we have  $\|f\|_{H^0}^2 = \|f\|_{L^2(D)}^2$  and  $\|f\|_{H^1}^2 = \|\nabla f\|_{H^0}^2$ .

The space  $H^a$  is compactly embedded in the space  $H^b$  if  $a > b$ .

Given a separable Hilbert space  $X$ , for  $\alpha > 0$  and  $p \geq 1$  we define the Banach space

$$W^{\alpha,p}(0, T; X) = \left\{ f \in L^p(0, T; X) : \int_0^T \int_0^T \frac{\|f(t) - f(s)\|_X^p}{|t - s|^{1+p\alpha}} dt ds < \infty \right\}$$

and we set

$$\|f\|_{W^{\alpha,p}(0,T;X)}^p = \int_0^T \|f(t)\|_X^p dt + \int_0^T \int_0^T \frac{\|f(t) - f(s)\|_X^p}{|t - s|^{1+p\alpha}} dt ds.$$

Let  $(\Omega, F, P)$  be a complete probability space, with expectation denoted by  $\mathbb{E}$ . We assume that the stochastic forcing term in (1) is of the form

$$w = w(t, x) = \sum_{k \in \mathbb{Z}_0^2} \sqrt{q_k} \beta_k(t) \frac{k^\perp}{|k|} e_k(x).$$

Here  $\{\beta_k\}_{k \in \mathbb{Z}_+^2}$  is a sequence of independent complex-valued standard Brownian motions on  $(\Omega, F, P)$ ; moreover  $\beta_{-k} = -\bar{\beta}_k$  and  $q_k = q_{-k}$  for any  $k \in \mathbb{Z}_+^2$ . The non negative coefficients  $q_k$  are assumed to satisfy the condition

$$(2) \quad Q_r := \sum_{k \in \mathbb{Z}_0^2} |k|^2 q_k < \infty.$$

In the 2D setting it is convenient to introduce the (scalar) vorticity

$$\xi = \nabla^\perp \cdot u \equiv \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}.$$

System (1) corresponds to

$$(3) \quad \begin{cases} d\xi + [-\nu\Delta\xi + \gamma\xi + u \cdot \nabla\xi]dt = dw^{curl} \\ \xi = \nabla^\perp \cdot u \end{cases}$$

obtained by taking the curl of both sides of the first equation of (1). Periodic boundary conditions have to be added to this system. The noise is  $w^{curl}(t, x) = i \sum_{k \in \mathbb{Z}_0^2} \sqrt{q_k} |k| \beta_k(t) e_k(x)$ ; classical results are

$$(4) \quad \mathbb{E} \|w^{curl}(t)\|_{H^0}^2 = 2tQ_r \quad \forall t \geq 0$$

and

$$(5) \quad \mathbb{E} \|w^{curl}\|_{W^{\alpha,p}(0,T;H^0)}^p \leq C(\alpha,p)(T^{1+p/2} + 1)Q_r^{p/2}$$

for any  $\alpha \in (0, \frac{1}{2})$  and  $p \geq 2$ . Here and henceforth,  $C(\alpha,p)$  denotes a positive constant depending on the specified parameters.

Knowing the vorticity  $\xi$  we recover the velocity  $u$  by solving the elliptic equation

$$(6) \quad -\Delta u = \nabla^\perp \xi.$$

This means that if  $\xi(x) = \sum_k \xi_k e_k(x)$ , then  $u(x) = -i \sum_k \frac{k^\perp}{|k|^2} \xi_k e_k(x)$ .

We present basic properties of the bilinear term in the 2D setting.

**Lemma 2.1** *If  $\xi = \nabla^\perp \cdot u$ , then*

$$(7) \quad \int_D [u \cdot \nabla \xi] \phi \, dx = - \int_D [u \cdot \nabla \phi] \xi \, dx$$

for all  $\xi, \phi \in H^1$ , and

$$(8) \quad \int_D [u \cdot \nabla \xi] \xi \, dx = 0$$

for all  $\xi \in H^1$ .

**Proof.** The first relation (7) is easily obtained by integrating by parts. Then, (8) is the particular case of (7) for  $\phi = \xi$ .  $\square$

**Lemma 2.2** *There exists a positive constant  $C$  such that*

$$(9) \quad \left| \int_D (u \cdot \nabla) v \cdot \psi \, dx \right| \leq C \|u\|_{L^4(D)} \|v\|_{L^4(D)} \|\psi\|_{H^1}$$

for all divergence free vectors with the regularity specified in the r.h.s., and for any  $a > 1$

$$(10) \quad \left| \int_D u \cdot \nabla \xi \, \phi \, dx \right| \leq C \|u\|_{H^0} \|\xi\|_{H^1} \|\phi\|_{H^a},$$

$$(11) \quad \left| \int_D u \cdot \nabla \xi \, \phi \, dx \right| \leq C \|u\|_{H^0} \|\xi\|_{H^{1+a}} \|\phi\|_{H^0}$$

for all functions with the regularity specified in the r.h.s., where  $\xi = \nabla^\perp \cdot u$ .

**Proof.** The key relationship for (9) is

$$\int_D [u \cdot \nabla] v \cdot \psi \, dx = - \int_D [u \cdot \nabla] \psi \cdot v \, dx$$

assuming sufficient regularity for  $u, v, \psi$ ; this is obtained by integrating by parts. Then we get the estimate by Hölder inequality and this is extended by density to vectors with the specified regularity. For (10) we use Hölder inequality and the continuous embedding  $H^a \subset L^\infty(D)$  for  $a > 1$ . Similarly we obtain the latter estimate.  $\square$

### 3 The stochastic damped Navier-Stokes equations

The well posedness of the stochastic damped 2D Navier-Stokes equations

$$(12) \quad \begin{cases} d\xi^\nu + [-\nu \Delta \xi^\nu + u^\nu \cdot \nabla \xi^\nu + \gamma \xi^\nu] dt = dw^{curl} \\ \xi^\nu = \nabla^\perp \cdot u^\nu \end{cases}$$

is very similar to the case when  $\gamma = 0$ . Here we assume periodic boundary conditions with period box  $[-\pi, \pi]^2$ .

The proof of existence of a unique solution for square summable initial vorticity is the same as the proof for square summable initial velocity that can be found in [13], where the proof is performed for  $\gamma = 0$ . Similar proofs can also be found in [3, 7] with some uniform estimates with respect to the viscosity  $\nu$ . Here, we improve the basic estimate for  $\gamma > 0$ , useful in the analysis of the limit as  $\nu \rightarrow 0$ .

**Theorem 3.1** *Let  $\gamma, \nu > 0$ ,  $T > 0$ ,  $\mathbb{E} \|\xi^\nu(0)\|_{H^0}^p < \infty$  for some  $p \geq 2$  and assumption (2) be satisfied. Then, there exists a process  $\xi^\nu$  with paths in  $C([0, T], H^0) \cap L^2([0, T]; H^1)$   $P$ -a.s., which is a Feller Markov process in  $H^0$  and is the unique solution for (12) with initial data  $\xi^\nu(0)$ . Moreover, there exists a positive constant  $C(p)$ , independent of  $\nu$ , such that*

$$(13) \quad \mathbb{E} \sup_{0 \leq t \leq T} \|\xi^\nu(t)\|_{H^0}^p \leq C(p).$$

*In particular, the constant  $C(p)$  depends on  $\gamma, p, Q_r, \mathbb{E} \|\xi^\nu(0)\|_{H^0}^p, T$ .*

**Proof.**

The proof of the existence of solutions, which is quite classical requires some Galerkin approximation of  $\xi^\nu$ , say  $\xi^{\nu, n}$ , for which a priori estimates are proved uniformly in  $n$ . Using a subsequence of  $\xi^{\nu, n}$  which converges in the weak or weak-star topologies of appropriate spaces, one can then prove that there exists a solution to (12). The proof of uniqueness and Feller property is standard and hence omitted. To ease notation, we replace the Galerkin approximation by the limit process  $\xi^\nu$  to obtain the required a priori estimates uniformly in  $n$  and  $\nu$ .

Let  $\nu > 0$  and  $t \in [0, T]$ ; applying Itô formula to  $\|\xi^\nu(t)\|_{H^0}^p$  we deduce that

$$\begin{aligned} d\|\xi^\nu(t)\|_{H^0}^p &\leq p\|\xi^\nu(t)\|_{H^0}^{p-2}\langle \xi^\nu(t), d\xi^\nu(t) \rangle + \frac{1}{2}p(p-1)\|\xi^\nu(t)\|_{H^0}^{p-2}Q_r dt \\ &= \nu p\|\xi^\nu(t)\|_{H^0}^{p-2}\langle \xi^\nu(t), \Delta \xi^\nu(t) \rangle dt - \gamma p\|\xi^\nu(t)\|_{H^0}^{p-2}\langle \xi^\nu(t), \xi^\nu(t) \rangle dt \\ &\quad - p\|\xi^\nu(t)\|_{H^0}^{p-2}\langle \xi^\nu(t), u^\nu(t) \cdot \nabla \xi^\nu(t) \rangle dt + p\|\xi^\nu(t)\|_{H^0}^{p-2}\langle \xi^\nu(t), dw^{curl}(t) \rangle \\ &\quad + \frac{1}{2}p(p-1)\|\xi^\nu(t)\|_{H^0}^{p-2}Q_r dt. \end{aligned}$$

Using (8) and integrating over  $(0, s)$ ,  $s \in [0, T]$ , we get that

$$(14) \quad \begin{aligned} \|\xi^\nu(s)\|_{H^0}^p + \nu p \int_0^s \|\xi^\nu(r)\|_{H^0}^{p-2} \|\xi^\nu(r)\|_{H^1}^2 dr + \gamma p \int_0^s \|\xi^\nu(r)\|_{H^0}^p dr &\leq \|\xi^\nu(0)\|_{H^0}^p \\ + p \int_0^s \|\xi^\nu(r)\|_{H^0}^{p-2} \langle \xi^\nu(r), dw^{curl}(r) \rangle + \frac{Q_r}{2} p(p-1) \int_0^s \|\xi^\nu(r)\|_{H^0}^{p-2} dr. \end{aligned}$$

On the other side, using first Burkholder-Davis-Gundy inequality and then Hölder inequality, we have that

$$\begin{aligned} &p\mathbb{E} \left( \sup_{0 \leq s \leq t} \int_0^s \|\xi^\nu(r)\|_{H^0}^{p-2} \langle \xi^\nu(r), dw^{curl}(r) \rangle \right) \\ &\leq p\mathbb{E} \left( \int_0^t \|\xi^\nu(r)\|_{H^0}^{2p-2} Q_r dr \right)^{1/2} \\ &\leq p\mathbb{E} \left( \sup_{0 \leq s \leq t} \|\xi^\nu(s)\|_{H^0}^{p/2} \left( Q_r \int_0^t \|\xi^\nu(r)\|_{H^0}^{p-2} dr \right)^{1/2} \right) \\ &\leq \frac{1}{2} \mathbb{E} \sup_{0 \leq s \leq t} \|\xi^\nu(s)\|_{H^0}^p + \frac{Q_r}{2} p^2 \mathbb{E} \int_0^t \|\xi^\nu(r)\|_{H^0}^{p-2} dr. \end{aligned}$$

Hence

$$(15) \quad \begin{aligned} &p\mathbb{E} \left( \sup_{0 \leq s \leq t} \int_0^s \|\xi^\nu(r)\|_{H^0}^{p-2} \langle \xi^\nu(r), dw^{curl}(r) \rangle \right) + \frac{Q_r}{2} p(p-1) \int_0^t \|\xi^\nu(r)\|_{H^0}^{p-2} dr \\ &\leq \frac{1}{2} \mathbb{E} \sup_{0 \leq s \leq t} \|\xi^\nu(s)\|_{H^0}^p + \frac{Q_r}{2} p(2p-1) \mathbb{E} \int_0^t \|\xi^\nu(r)\|_{H^0}^{p-2} dr \\ &\leq \frac{1}{2} \mathbb{E} \sup_{0 \leq s \leq t} \|\xi^\nu(s)\|_{H^0}^p + \epsilon \mathbb{E} \int_0^t \|\xi^\nu(r)\|_{H^0}^p dr + C(\epsilon, p, Q_r)t, \end{aligned}$$

for an arbitrary  $\epsilon > 0$ . Collecting all the estimates, taking expectation in (14) we get

$$(16) \quad \begin{aligned} &\frac{1}{2} \mathbb{E} \sup_{0 \leq s \leq t} \|\xi^\nu(s)\|_{H^0}^p + \nu p \mathbb{E} \int_0^t \|\xi^\nu(s)\|_{H^0}^{p-2} \|\xi^\nu(s)\|_{H^1}^2 ds + \gamma p \mathbb{E} \int_0^t \|\xi^\nu(s)\|_{H^0}^p ds \\ &\leq \mathbb{E} \|\xi^\nu(0)\|_{H^0}^p + \epsilon \mathbb{E} \int_0^t \|\xi^\nu(s)\|_{H^0}^p ds + C(\epsilon, p, Q_r)t. \end{aligned}$$

Choosing  $\epsilon = \frac{\gamma p}{2}$  we obtain

$$\frac{1}{2} \mathbb{E} \sup_{0 \leq s \leq t} \|\xi^\nu(s)\|_{H^0}^p \leq \mathbb{E} \|\xi^\nu(0)\|_{H^0}^p + \bar{C}(\gamma, p, Q_r)t,$$

This implies (13). Moreover, we also have

$$\nu p \mathbb{E} \int_0^t \|\xi^\nu(s)\|_{H^0}^{p-2} \|\xi^\nu(s)\|_{H^1}^2 ds + \gamma p \mathbb{E} \int_0^t \|\xi^\nu(s)\|_{H^0}^p ds \leq \tilde{C}(p, t, Q_r, E \|\xi^\nu(0)\|_{H^0}^p).$$

□

**Remark 3.2** *The process  $\xi^\nu$  solves system (12) in the following sense: for all  $t \in [0, \infty)$  and  $\phi \in H^a$  with  $a > 1$ , we have*

$$\begin{aligned} & \int_D \xi^\nu(t, x) \phi(x) dx + \nu \int_0^t \int_D \nabla \xi^\nu(s, x) \cdot \nabla \phi(x) dx ds \\ & + \int_0^t \int_D u^\nu(s, x) \cdot \nabla \xi^\nu(s, x) \phi(x) dx ds + \gamma \int_0^t \int_D \xi^\nu(s, x) \phi(x) dx ds \\ & = \int_D \xi^\nu(0, x) \phi(x) dx + \int_D w^{curl}(t, x) \phi(x) dx \quad P - a.s. \end{aligned}$$

The trilinear term is well defined thanks to (6) and (10).

For any  $\gamma > 0$  one can prove existence and uniqueness of the invariant measure for system (12), following the lines of the proofs for the 2D Navier-Stokes equation (the case  $\gamma = 0$ ). Indeed, Krylov-Bogoliubov method provides a way to prove the existence of an invariant measure; this applies for a wide class of noises. On the other side, uniqueness is a more delicate question. We just recall the best result of uniqueness of the invariant measure, proved by Hairer and Mattingly [17]. They assume that the noise acts on first few modes, i.e.

$$(17) \quad \left\{ \begin{array}{l} \exists \mathcal{Z} \text{ finite} : q_k \neq 0 \quad \forall k \in \mathcal{Z}, \quad q_k = 0 \quad \forall k \notin \mathcal{Z} \\ \text{where } \mathcal{Z} \text{ has to be chosen in such a way that} \\ \quad \bullet \text{ it contains at least two elements with different norms} \\ \quad \bullet \text{ the integer linear combinations of elements of } \mathcal{Z} \text{ generates } \mathbb{Z}^2 \end{array} \right.$$

Actually the kind and the number of forced modes, i.e. the elements of  $\mathcal{Z}$ , is chosen independently of the viscosity.

We summarize the result.

**Theorem 3.3** *Let  $\gamma > 0$ . If (17) holds, then for any  $\nu > 0$  system (12) has a unique invariant measure  $\mu^\nu$ . Moreover it is ergodic, i.e.*

$$(18) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \varphi(\xi^\nu(t)) dt = \int \varphi d\mu^\nu \quad \text{in } L^2(\Omega)$$

for any  $\varphi \in \mathcal{C}_b(H^0)$  and initial vorticity in  $H^0$ . Finally

$$(19) \quad \nu \int \|\nabla \xi\|_{H^0}^2 d\mu^\nu(\xi) + \gamma \int \|\xi\|_{H^0}^2 d\mu^\nu(\xi) = \frac{Q_r}{2}.$$

The latter equality is obtained by means of Itô formula applied to  $d\|\xi(t)\|_{H^0}^2$ .

**Remark 3.4** *i) All the previous results hold true when  $D$  is a smooth bounded domain in  $\mathbb{R}^2$ , under suitable boundary conditions.*

*ii) For other conditions granting the uniqueness of the invariant measure see e.g. [15, 12, 14]. Anyway, our results hold when the noise is such that the support of the unique invariant measure  $\mu^\nu$  for system (12) is a subset of  $H^1$ , so that (19) is meaningful. Moreover, for any invariant measure  $\mu^\nu$  obtained by Krylov-Bogoliubov method we have that*

$$(20) \quad \int \|\xi\|_{H^0}^{p-2} \|\xi\|_{H^1}^2 d\mu^\nu(\xi) < \infty$$

for any  $\nu > 0$  and  $p \geq 2$ . This is proved by means of Itô formula as in the proof of Theorem 3.1.

Now, we fix the family of the unique invariant measures, as given in Theorem 3.3, and consider the limit of vanishing viscosity.

**Corollary 3.5** *Let  $\gamma > 0$ . Then the family of invariant measures  $\{\mu^\nu\}_{\nu>0}$  is tight in  $H^{-s}$  for any  $s > 0$ ; in particular there exists a measure  $\mu^0$  in  $H^{-s}$  such that*

$$\mu^\nu \longrightarrow \mu^0 \quad \text{weakly in } H^{-s}$$

as  $\nu \longrightarrow 0$ .

**Proof.** From (19) we have

$$\int \|\xi\|_{H^0}^2 d\mu^\nu(\xi) \leq \frac{Q_r}{2\gamma}$$

uniformly in  $\nu \in (0, \infty)$ . Then, using that  $H^0$  is compactly embedded in  $H^{-s}$  we get tightness by means of the Chebyshev inequality.  $\square$

## 4 The vanishing viscosity limit

When  $\nu = 0$ , we deal with the stochastic damped Euler equations

$$(21) \quad \begin{cases} d\xi^0 + [u^0 \cdot \nabla \xi^0 + \gamma \xi^0] dt = dw^{curl} \\ \xi^0 = \nabla^\perp \cdot u^0 \end{cases}$$

with periodic boundary conditions, as before.

We are going to prove that this system has a stationary solution whose marginal at fixed time is the measure  $\mu^0$  and that the following balance equation holds:

$$\gamma \int \|\xi\|_{H^0}^2 d\mu^0(\xi) = \frac{1}{2} Q_r.$$

Finally, considering the limit in the balance equation (19) we get

$$\lim_{\nu \rightarrow 0} \nu \int \|\nabla \xi\|_{H^0}^2 d\mu^\nu(\xi) = 0.$$

This means that in the limit of vanishing viscosity the damped stochastic equations (12) have no dissipation of enstrophy.

We begin by dealing with the stationary stochastic process  $\xi^\nu$  whose law at any fixed time is the measure  $\mu^\nu$  of Theorem 3.3, and take the limit of vanishing viscosity. We have

**Proposition 4.1** *Let  $s > 0$ . The sequence  $\{\xi^\nu\}_{\nu>0}$  of stationary processes solving (12) has a subsequence converging, as  $\nu \rightarrow 0$ , in  $L^2([0, T]; H^{-s}) \cap C([0, T]; H^{-2-2s})$  (a.s.) to a process, which solves the damped Euler system (21). Moreover, the paths of the limit process belong (a.s.) to  $C([0, T]; H_w^0) \cap L^\infty([0, T]; H^0)$ , and the limit process is a stationary process in  $H^0$ .*

**Proof.** The proof is based on two steps: first we show that the sequence  $\{\xi^\nu\}_{\nu>0}$  is tight; then we pass to the limit in a suitable way and get that the limit process is a weak solution of system (21). Notice that we find a weak solution to system (21) (in the probabilistic sense), whereas system (12) has a unique strong solution.

Actually, the tightness and the convergence of the stationary processes  $\xi^\nu$  have already been done in [5] for equation (12) with a multiplicative noise. For the reader's convenience we recall the basic steps of the proof; the details can be found in [3, 5].

Writing equation (12) in the integral form

$$\xi^\nu(t) = \xi^\nu(0) + \nu \int_0^t \Delta \xi^\nu(s) ds - \int_0^t u^\nu(s) \cdot \nabla \xi^\nu(s) ds - \gamma \int_0^t \xi^\nu(s) ds + w^{curl}(t),$$

by usual estimations and bearing in mind estimate

$$\mathbb{E} \sup_{0 \leq t \leq T} \|\xi^\nu(t)\|_{H^0}^4 \leq C(4)$$

from Theorem 3.1, one gets that there exist constants  $C$  and  $C(p)$  such that

$$\begin{aligned} \mathbb{E} \left\| \int_0^\cdot \Delta \xi^\nu(s) ds \right\|_{W^{1,2}(0,T;H^{-2})}^2 &\leq C \\ \mathbb{E} \left\| \int_0^\cdot u^\nu(s) \cdot \nabla \xi^\nu(s) ds \right\|_{W^{1,2}(0,T;H^{-2-s})}^2 &\leq C \quad \text{by (7) and (11)} \\ \mathbb{E} \left\| \int_0^\cdot \xi^\nu(s) ds \right\|_{W^{1,2}(0,T;H^0)}^2 &\leq C \\ \mathbb{E} \|w^{curl}\|_{W^{\alpha,p}(0,T;H^0)}^p &\leq C(p) \quad \text{by (2)} \end{aligned}$$

for some (and all)  $s > 0$ ,  $\alpha \in (0, \frac{1}{2})$  and  $p \geq 2$ . Therefore

$$\sup_{\nu \in (0,1)} \mathbb{E} \|\xi^\nu\|_{W^{\alpha,2}(0,T;H^{-2-s})}^2 < \infty.$$

Let  $\xi^\nu$  be the  $\mu^\nu$ -stationary process solving equation (12). Using that the space  $L^2(0, T; H^0) \cap W^{\alpha,2}(0, T; H^{-2-s})$  is compactly embedded in  $L^2(0, T; H^{-s})$ , it follows from the previous estimates that the sequence of processes  $\{\xi^\nu\}_\nu$  is tight in  $L^2(0, T; H^{-s})$ . On the other hand, using that both the spaces  $W^{1,2}(0, T; H^{-2-s})$  and  $W^{\alpha,p}(0, T; H^{-2-s})$  with  $\alpha p > 1$  are compactly embedded in  $C([0, T]; H^{-2-2s})$ , we get tightness in  $C([0, T]; H^{-2-2s})$ .

Let us endow  $L_{loc}^2(0, \infty; H^{-s})$  by the distance

$$d_2(\xi, \zeta) = \sum_{n=1}^{\infty} s^{-n} \min(\|\xi - \zeta\|_{L^2(0,n;H^{-s})}, 1)$$

and  $C([0, \infty); H^{-2-2s})$  by the distance

$$d_{\infty}(\xi, \zeta) = \sum_{n=1}^{\infty} s^{-n} \min(\|\xi - \zeta\|_{C([0,n];H^{-2-2s})}, 1).$$

We have that the sequence  $\{\xi^{\nu}\}_{\nu}$  is tight in  $L_{loc}^2(0, \infty; H^{-s}) \cap C([0, \infty); H^{-2-2s})$ .

Prokhorov and Skorohod theorems allow to get there exists a basis  $(\tilde{\Omega}, \tilde{F}, \tilde{P})$  and on this basis,  $L_{loc}^2(0, \infty; H^{-s}) \cap C([0, \infty); H^{-2-2s})$ -valued random variables  $\tilde{\xi}^0, \tilde{\xi}^{\nu}$ , such that  $\mathcal{L}(\tilde{\xi}^{\nu}) = \mathcal{L}(\xi^{\nu})$  on  $L_{loc}^2(0, \infty; H^{-s}) \cap C([0, \infty); H^{-2-2s})$ , and

$$(22) \quad \lim_{n \rightarrow \infty} \tilde{\xi}^{n, \nu_n} = \tilde{\xi}^0 \quad \text{in } L_{loc}^2(0, \infty; H^{-s}) \cap C([0, \infty); H^{-2-2s}), \tilde{P} - a.s.$$

for a subsequence with  $\lim_{n \rightarrow \infty} \nu_n = 0$ .

The fact that the process  $\tilde{\xi}^0$  solves system (21) is classical. Indeed, considering  $s = \frac{1}{2}$  we have that  $\tilde{\xi}^{\nu} \rightarrow \tilde{\xi}^0$  in  $L_{loc}^2(0, \infty; H^{-1/2})$ ; this means, according to (6), that  $\tilde{u}^{\nu} \rightarrow \tilde{u}^0$  in  $L_{loc}^2(0, \infty; H^{1/2})$ . Since  $H^{1/2} \subset L^4(D)$ , we get by estimates similar to (9) that the quadratic term  $[\tilde{u}^{\nu} \cdot \nabla] \tilde{u}^{\nu}$  converges weakly to  $[\tilde{u}^0 \cdot \nabla] \tilde{u}^0$ , i.e.

$$\int_D \int_0^t [\tilde{u}^{\nu} \cdot \nabla] \tilde{u}^{\nu} \cdot \psi \, ds \, dx \longrightarrow \int_D \int_0^t [\tilde{u}^0 \cdot \nabla] \tilde{u}^0 \cdot \psi \, ds \, dx \quad \tilde{P} - a.s.$$

for all  $t$  finite and  $\psi \in [H^1]^2$ . For this it is enough to write

$$\begin{aligned} & \int_D \{[\tilde{u}^{\nu} \cdot \nabla] \tilde{u}^{\nu} \cdot \psi - [\tilde{u}^0 \cdot \nabla] \tilde{u}^0 \cdot \psi\} \, dx \\ &= + \int_D [(\tilde{u}^{\nu} - \tilde{u}^0) \cdot \nabla] \tilde{u}^{\nu} \cdot \psi \, dx + \int_D [\tilde{u}^0 \cdot \nabla] (\tilde{u}^{\nu} - \tilde{u}^0) \cdot \psi \, dx. \end{aligned}$$

In addition,  $\tilde{\xi}^{\nu}$  and  $\xi^{\nu}$  have the same law; then  $\tilde{\xi}^{\nu}$  is a stationary process. By the convergence  $\tilde{P}$ -a.s. in  $C([0, T]; H^{-2-2s})$  we get that also  $\tilde{\xi}^0$  is a stationary process.

Finally, from (13) we have that

$$\tilde{\xi}^0 \in L^{\infty}([0, T]; H^0) \quad \tilde{P} - a.s.$$

Then almost each path  $\tilde{\xi}^0 \in C([0, T]; H^{-2-2s}) \cap L^{\infty}(0, T; H^0)$ , for  $T < \infty$ ; thus it is weakly continuous in  $H^0$ , i.e. we have for any  $\phi \in H^0$

$$\lim_{t \rightarrow t_0} \langle \tilde{\xi}^0(t), \phi \rangle_{H^0} = \langle \tilde{\xi}^0(t_0), \phi \rangle_{H^0} \quad \tilde{P} - a.s.$$

and for any  $t \in [0, T]$

$$\|\tilde{\xi}^0(t)\|_{H^0} \leq \|\tilde{\xi}^0\|_{L^{\infty}(0, T; H^0)} \quad \tilde{P} - a.s.$$

(see [23] p 263). Hence, for every  $t \geq 0$ , the mapping  $\tilde{\omega} \mapsto \tilde{\xi}^0(t, \tilde{\omega})$  is well defined from  $\tilde{\Omega}$  to  $H^0$  and it is weakly measurable. Since  $H^0$  is a separable Banach space, it is strongly measurable (see [24] p 131). Therefore, it is meaningful to speak about the law of  $\tilde{\xi}^0(t)$  in  $H^0$ . The stationarity of  $\tilde{\xi}^0$  in  $H^0$  has to be understood in this sense.  $\square$

Let us denote by  $\tilde{\xi}^0$  the stationary process solving (21), as given in Proposition 4.1. We have

**Proposition 4.2** *For any time  $t$*

$$(23) \quad \gamma \tilde{\mathbb{E}} \|\tilde{\xi}^0(t)\|_{H^0}^2 = \frac{1}{2} Q_r.$$

**Proof.** Bearing in mind Theorem 3.1 we have  $\tilde{\mathbb{E}} \sup_{0 \leq t < \infty} \|\tilde{\xi}^\nu(t)\|_{H^0}^2 \leq C(2)$ ; hence

$$\tilde{\mathbb{E}} \|\tilde{\xi}^\nu(t)\|_{H^0}^2 \leq C(2).$$

This bound implies

$$\tilde{\xi}^\nu(t) \longrightarrow \tilde{\xi}^0(t) \quad \text{weakly in } L^2(\tilde{\Omega} \times D);$$

for the limit we have

$$(24) \quad \tilde{\mathbb{E}} \|\tilde{\xi}^0(t)\|_{H^0}^2 \leq \liminf_{\nu \rightarrow 0} \tilde{\mathbb{E}} \|\tilde{\xi}^\nu(t)\|_{H^0}^2 \leq C(2).$$

By working on the first equation of (21), Itô formula for  $d\|\tilde{\xi}^\nu(t)\|_{H^0}$  provides

$$(25) \quad \|\tilde{\xi}^0(t)\|_{H^0}^2 + \gamma \int_0^t \|\tilde{\xi}^0(s)\|_{H^0}^2 ds = \|\tilde{\xi}^0(0)\|_{H^0}^2 + \frac{t}{2} Q_r + \int_0^t \langle \tilde{\xi}^0(s), d\tilde{w}^{curl}(s) \rangle_{H^0},$$

$\tilde{P}$ -a.s. For this we have used (8).

Taking expectation and using stationarity we get (23).  $\square$

By taking suitable subsequences we have that  $\mu^0$  is the law of  $\tilde{\xi}^0(t)$  for any time  $t$ . Hence, equation (23) can be rewritten as

$$\gamma \int \|\xi\|_{H^0}^2 d\mu^0(\xi) = \frac{1}{2} Q_r.$$

**Remark 4.3** *At this point, we are not able to prove that  $\mu^0$  is an invariant measure for the system (21). In fact, the transition semigroup associated to (21) can not be defined in  $H^0$ : existence of a solution holds for initial vorticity in  $H^0$  but uniqueness requires stronger assumptions (see [4] and [6]). But to get the Feller and Markov properties in a space smaller than  $H^0$  is not trivial. Some work in progress in that direction is being made by the current authors.*

Now we have our main result

**Theorem 4.4** *For any  $\gamma > 0$ , we have*

$$(26) \quad \lim_{\nu \rightarrow 0} \nu \int \|\nabla \xi\|_{H^0}^2 d\mu^\nu(\xi) = 0.$$

**Proof.** Let us write the balance equation (19) in terms of the stationary process  $\xi^\nu$ , at any fixed time  $t$ :

$$(27) \quad \nu \mathbb{E} \|\nabla \xi^\nu(t)\|_{H^0}^2 + \gamma \mathbb{E} \|\xi^\nu(t)\|_{H^0}^2 = \frac{Q_r}{2}.$$

Considering the weak limit as in Proposition 4.1 and 4.2 we have

$$(28) \quad \begin{aligned} \limsup_{\nu \rightarrow 0} \nu \tilde{\mathbb{E}} \|\nabla \tilde{\xi}^\nu(t)\|_{H^0}^2 &= \frac{1}{2} Q_r - \gamma \liminf_{\nu \rightarrow 0} \tilde{\mathbb{E}} \|\tilde{\xi}^\nu(t)\|_{H^0}^2 \\ &\leq \frac{1}{2} Q_r - \gamma \tilde{\mathbb{E}} \|\tilde{\xi}^0(t)\|_{H^0}^2 \text{ by (24)} \\ &= 0 \text{ by (23)}. \end{aligned}$$

This gives (26). □

From this result we obtain the convergence of the mean enstrophy.

**Corollary 4.5** *For any  $\gamma > 0$ , we have*

$$(29) \quad \lim_{\nu \rightarrow 0} \int \|\xi\|_{H^0}^2 d\mu^\nu(\xi) = \int \|\xi\|_{H^0}^2 d\mu^0(\xi).$$

**Proof.** We consider the limit as  $\nu \rightarrow 0$  in (19); then use (26) and (23). □

**Remark 4.6** *All the results proved for the enstrophy  $\xi$  can be repeated and hence hold for the velocity  $u$ ; norms of one order less of regularity are involved and therefore the proofs are even easier. This means in particular that for the stochastic damped 2D Navier-Stokes equations, there is no anomalous dissipation of energy as  $\nu \rightarrow 0$  and energy balance equation holds for  $\nu > 0$  and also  $\nu = 0$ .*

**Acknowledgment:** The work of H. Bessaih was supported in part by the GNAMPA-INDAM project "Professori Visitatori". We would like to thank the hospitality of the Department of Mathematics of the University of Pavia where part of this research started and the IMA in Minneapolis where the paper has been finalized.

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