# ONE-DIMENSIONAL TWO-PHASE GENERALIZED FORCHHEIMER FLOWS OF INCOMPRESSIBLE FLUIDS

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# ONE-DIMENSIONAL TWO-PHASE GENERALIZED FORCHHEIMER FLOWS OF INCOMPRESSIBLE FLUIDS

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Abstract. We derive a non-linear system of parabolic equations to describe the one-dimensional two-phase generalized Forchheimer flows of incompressible, immiscible fluids in porous media, with the presence of capillary forces. Under relevant constraints on relative permeabilities and capillary pressure, non-constant steady state solutions are found and classified into sixteen types according to their monotonicity and asymptotic behavior. For a steady state whose saturation can never attain either value 0 or 1, we prove that it is stable with respect to a certain weight. This weight is a function comprised of the steady state, relative permeabilities and capillary pressure. The proof is based on specific properties of the steady state, weighted maximum principle and Bernstein's estimate.

## 1. Introduction

In this article we consider the filtration of two incompressible, immiscible fluids, namely, phase 1 and phase 2 in porous media. The main characteristics of the two-phase flow are saturation $S_i$ and velocity $u_i$ for of each $i$th phase, $i = 1, 2$. We study an idealized model of one-dimensional (1-D) flow in the unbounded domain $\mathbb{R} = (-\infty, \infty)$. This type of idealization is used by engineers and biologists for modeling flows in the relatively long and thin fractures, or between two parallel long horizontal wells, or in microchannels, etc. Each phase is subject to a generalized Forchheimer equation which is a nonlinear relation between its velocity, gradient of the pressure and relative permeability. The phases' pressures, in general, are not equal, hence resulting in capillary forces. For two-phase flows, both relative permeabilities and capillary function depend on the saturations. Those dependences are usually obtained from experimental data and are key characteristics of corresponding mathematical models (see e.g. [3]).

Multi-phase flows for the Darcy model are intensively studied in experiments and by numerical simulations (c.f. [16, 19] and references therein). Developed numerical algorithms and methods are nowadays used in many industrial packages such as ECLIPSE by Schlumberger and VIP by Halliburton. Among non-Darcy models, the two-term Forchheimer equation is studied numerically in [19].

From mathematical point of view, for Darcy equation of two-phase flows, the problem can be simplified using the so-called global pressure and total velocity. Using this reduction, previous works [1,4,6–9,14,15] show the interior regularity or existence or uniqueness of weak solutions.

In case of Forchheimer flows, which is the interest of this paper, the above reduction is not applicable due to its nonlinearity, c.f. Eq. (2.2) below. In addition to this difficulty, the dependence of capillary pressure and relative permeabilities on the saturations brings another complication. Namely, measurements can be less precise and exhibit extreme singularities when the saturation reaches its threshold values. Therefore new approach to

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the problem and appropriate interpretations of properties of the capillary pressure and relative permeabilities are needed in order to make mathematical analysis feasible.

In this current work, we consider the one-dimensional incompressible two-phase generalized Forchheimer flows. (For single-phase generalized Forchheimer flows in any dimensional space, see e.g. [2, 10–12, 18].) We will focus on the solvability of the stationary problem and the stability of the obtained steady states. In section 2, we derive system (2.14) of nonlinear partial differential equations (PDE) for the model. Basic (and physically relevant) assumptions (A and B) are made on the capillary pressure and relative permeabilities. In section 3, we study the steady state solution $S(x)$ by solving a non-linear ordinary differential equation (ODE) with the constraint $S(x) \in [0,1]$ for all $x \in \mathbb{R}$. The nonlinearity in the ODE is formed by the relative permeabilities and capillary pressure explicitly. Theorem 3.1 proves the existence of non-constant steady states and classifies all of them based on their monotonicity and asymptotic behavior at infinity (c.f. Table 1). Their graphs are sketched in Figures 1–8. However, even with the same set of parameters, namely, $(c_1, c_2, s_0)$, we have the dichotomy or quadrichotomy for the solution's type. A new set of conditions are found to determine the exact type of solution (c.f. Theorem 3.4). Those conditions are specific relations between relative permeabilities and capillary pressure near the limiting values $S = 0, 1$. In section 4, we derive the linearized system of equations near a "never trivial" steady state $(u_1^*, u_2^*, S_*(x))$ obtained in section 3. We decouple it to have a scalar parabolic equation for the perturbed saturation $\sigma(x,t)$, c.f. Eq. (4.8). The perturbed velocities can be easily retrieved by relations (4.9) and (4.10). The coefficients of this equation are functions derived from $S_*(x)$, permeability and capillary pressure, and blow up when $|x| \to \infty$. A weighted maximum principle is proved with the weight function depending on the steady state. Utilizing this crucial property, we establish the corresponding weighted stability for the perturbation $\sigma(x,t)$ and also derive long time estimates for its weighted $L^\infty$-norm. Moreover, the stability for velocities (on bounded intervals) is obtained by using Bernstein's estimate technique.

## 2. Formulation of the problem

We model an infinite 1-D porous medium by the real line $\mathbb{R}$ and assume that its porosity is a constant $\phi$ between 0 and 1. Each position $x$ in the medium is considered to be occupied by two fluids called phase 1 (for example, water) and phase 2 (for example, oil).

Denote the saturation, density, velocity, and pressure for each phase $i = 1, 2$ by $S_i$, $\rho_i$, $u_i$, and $p_i$, respectively.

The saturations naturally satisfy

$$0 \le S_1, S_2 \le 1 \quad \text{and} \quad S_1 + S_2 = 1. \tag{2.1}$$

Each phase's velocity is assumed to obey the generalized Forchheimer equation:

$$g_i(|u_i|)u_i = -\tilde{f}_i(S_i)\partial_x p_i, \quad i = 1, 2, \tag{2.2}$$

where $\tilde{f}_i(S_i)$ is the relative permeability for the $i$th phase, and $g_i$ is a generalized polynomial of the form

$$g_i(s) = a_0 s^{\alpha_0} + a_1 s^{\alpha_1} + \ldots + a_N s^{\alpha_N}, \quad s \ge 0, \tag{2.3}$$

with $N \ge 0$, $a_0 > 0$, $a_1, \ldots a_N \ge 0$, $\alpha_0 = 0 < \alpha_1 < \ldots \alpha_N$, all $\alpha_1, \ldots \alpha_N$ are real numbers. The above $N$, $a_j$, $\alpha_j$ in (2.3) depend on each $i$.

Conservation of mass commonly holds for each of the phases:

$$\partial_t(\phi \rho_i S_i) + \partial_x(\rho_i u_i) = 0, \quad i = 1, 2. \tag{2.4}$$

Due to incompressibility of the phases, i.e. $\rho_i = const. > 0$, Eq. (2.4) is reduced to

$$\phi\partial_t S_i + \partial_x u_i = 0, \quad i = 1, 2. \tag{2.5}$$

Let $p_c$ be the capillary pressure between two phases, more specifically,

$$p_1 - p_2 = p_c. \tag{2.6}$$

Hereafterward, we denote $S = S_1$. The relative permeabilities and capillary pressure are re-denoted as functions of $S$, that is, $\tilde{f}_1(S_1) = f_1(S)$, $\tilde{f}_2(S_2) = f_2(S)$ and $p_c = p_c(S)$. Then (2.2) and (2.6) become

$$g_i(|u_i|)u_i = -f_i(S)\partial_x p_i, \quad i = 1, 2, \tag{2.7}$$

$$p_1 - p_2 = p_c(S). \tag{2.8}$$

By scaling time, we can mathematically consider $\phi = 1$. From (2.1) and (2.5):

$$S_t = -u_{1x}, \quad S_t = u_{2x}. \tag{2.9}$$

Let $G_i(u) = g_i(|u|)u$, for $u \in \mathbb{R}$, $i = 1, 2$. Then each $G_i$ is a strictly increasing function from $\mathbb{R}$ onto $\mathbb{R}$, with $G_i(0) = 0$. Then by (2.7),

$$G_i(u_i) = -f_i(S)p_{ix}, \quad \text{or,} \quad p_{ix} = -\frac{G_i(u_i)}{f_i(S)}. \tag{2.10}$$

Taking derivative in $x$ of the equation (2.8) we have

$$p_{1x} - p_{2x} = p_c'(S)S_x. \tag{2.11}$$

Substituting (2.10) into (2.11) yields

$$\frac{G_2(u_2)}{f_2(S)} - \frac{G_1(u_1)}{f_1(S)} = p_c'(S)S_x,$$

hence

$$S_x = G_2(u_2)F_2(S) - G_1(u_1)F_1(S), \tag{2.12}$$

where

$$F_i(S) = \frac{1}{p_c'(S)f_i(S)}, \quad i = 1, 2. \tag{2.13}$$

In summary, we obtain (2.1), (2.9) and (2.12). We rewrite them here as a PDE system for the scalar unknowns $S = S(x, t)$, $u_1 = u_1(x, t)$ and $u_2 = u_2(x, t)$:

$$0 \le S \le 1, \qquad x \in \mathbb{R}, \quad t \ge 0, \tag{2.14a}$$

$$S_t = -u_{1x}, \quad S_t = u_{2x}, \qquad x \in \mathbb{R}, \quad t > 0, \tag{2.14b}$$

$$S_x = G_2(u_2)F_2(S) - G_1(u_1)F_1(S), \qquad x \in \mathbb{R}, \quad t > 0. \tag{2.14c}$$

This is the system of our interest and the rest of paper is devoted to studying it.

Below, we make basic assumptions on the relative permeabilities and capillary pressure. These are not physics laws, but rather our interpretation of experimental data (c.f. [3]), especially of those obtained in [5]. They cover certain scenarios of two-phase fluids in reality.

**Assumption A.**

$$f_1, f_2 \in C([0, 1]) \cap C^1((0, 1)), \tag{2.15a}$$

$$f_1(0) = 0, \quad f_2(1) = 0, \tag{2.15b}$$

$$f_1'(S) > 0, \quad f_2'(S) < 0 \text{ on } (0, 1). \tag{2.15c}$$

**Assumption B.**

$$p_c \in C^1((0, 1)), \quad p_c'(S) > 0 \text{ on } (0, 1), \tag{2.16a}$$

$$\lim_{S\to 0} p_c'(S) f_1(S) = \lim_{S\to 1} p_c'(S) f_2(S) = +\infty. \tag{2.16b}$$

Consequently, (2.15b) and (2.16b) give

$$\lim_{S\to 0} p_c'(S) = +\infty, \quad \lim_{S\to 1} p_c'(S) = +\infty. \tag{2.17}$$

By (2.16b), $F_1$ and $F_2$ can now be extended to functions of class $C([0,1]) \cap C^1((0,1))$ and satisfy

$$F_1(0) = F_1(1) = F_2(0) = F_2(1) = 0. \tag{2.18}$$

Therefore the right hand side of (2.14c) is well-defined for all $S \in [0,1]$.

Note that

$$\lim_{S\to 0^+} \frac{F_1(S)}{F_2(S)} = \lim_{S\to 1^-} \frac{F_2(S)}{F_1(S)} = \infty. \tag{2.19}$$

## 3. Steady state solutions

Our first step in understanding system (2.14) is to find its time-independent solutions, that is, those of the form

$$(S, u_1, u_2) = (S(x), u_1(x), u_2(x)), \quad x \in \mathbb{R}. \tag{3.1}$$

We obtain

$$0 \le S \le 1, \qquad x \in \mathbb{R}, \tag{3.2a}$$

$$u_1' = u_2' = 0, \qquad x \in \mathbb{R}, \tag{3.2b}$$

$$S' = G_2(u_2)F_2(S) - G_1(u_1)F_1(S), \qquad x \in \mathbb{R}. \tag{3.2c}$$

By (3.2b), $u_i(x) = u_i^* = const.$ for $i = 1, 2$. Let $c_i = G_i(u_i^*)$, $i = 1, 2$, then

$$u_i^* = G_i^{-1}(c_i). \tag{3.3}$$

Hereafter, we will consider $c_1$ and $c_2$ as parameters for our steady state solutions $(u_1^*, u_2^*, S)$. We rewrite (2.12) as

$$S' = F(S) \overset{\text{def}}{=\!=} c_2 F_2(S) - c_1 F_1(S). \tag{3.4}$$

Note that (3.4) always has two trivial equilibria $S(x) \equiv 0$ and $S(x) \equiv 1$.

If $c_1 = c_2 = 0$ then

$$u_1^* = u_2^* = 0, \quad S = const. \tag{3.5}$$

If $c_1 = 0$, $c_2 \neq 0$ then $u_1^* = 0$, $u_2^* \neq 0$ and

$$S' = c_2 F_2(S). \tag{3.6}$$

If $c_1 \neq 0$, $c_2 = 0$ then $u_1^* \neq 0$, $u_2^* = 0$ and

$$S' = -c_1 F_1(S). \tag{3.7}$$

Now consider $c_1 c_2 \neq 0$. We look for nontrivial equilibrium $S = const. = s^* \in (0,1)$ of (3.4), that is,

$$c_2 F_2(s^*) - c_1 F_1(s^*) = 0, \text{ or equivalently, } \frac{f_1(s^*)}{f_2(s^*)} = \frac{c_1}{c_2}. \tag{3.8}$$

By Assumption A, the function $f \overset{\text{def}}{=\!=} f_1/f_2$ is strictly increasing and maps $(0,1)$ onto $(0,\infty)$. Hence we can solve

$$s^* = f^{-1}(c_1/c_2) \quad \text{provided} \quad c_1 c_2 > 0. \tag{3.9}$$

The solution of (3.4) can be determined by an initial data $s_0$, hence we study the following constrained initial value problem (IVP):

$$S' = F(S), \quad S(0) = s_0, \quad 0 \le S(x) \le 1. \tag{3.10}$$

We have the following existence theorem for (3.10) which, more importantly, also categorizes all possible non-constant steady states.

**Theorem 3.1.** *Consider $c_1^2 + c_2^2 > 0$. Let $s_0 \in (0,1)$ and, in case $c_1c_2 > 0$, $s_0 \ne s^*$, where $s^*$ is the equilibrium in (3.9). Then there exists a unique solution $S(x) \in C^1(\mathbb{R})$ to (3.10). This solution has the following properties:*

*There exist unique $x_\ell \in [-\infty, 0)$ and $x_r \in (0, \infty]$ and $s_\ell, s_r \in \{0, 1, s^*\}$ such that*

$$\lim_{x \to x_\ell} S(x) = s_\ell \quad \text{and} \quad \lim_{x \to x_r} S(x) = s_r \tag{3.11}$$

*and*

$$s_\ell < S(x) < s_r \quad \text{for all} \quad x_\ell < x < x_r. \tag{3.12}$$

*If $x_\ell > -\infty$, resp. $x_r < \infty$, then*

$$S(x) = s_\ell \text{ for } x \le x_\ell, \text{ resp. } S(x) = s_r \text{ for } x \ge x_r. \tag{3.13}$$

*Moreover,*

$$S(x) \text{ is strictly monotone in } (x_\ell, x_r). \tag{3.14}$$

*These steady state solutions are categorized into 16 types which are listed in Table 1 and are sketched in Figures 1–8.*

| $c_1, c_2$ | $s_0$ | Type | $(x_\ell, x_r)$ | $s_\ell$ | $s_r$ | $S'(x)$ on $(x_\ell, x_r)$ |
|---|---|---|---|---|---|---|
| $c_1, c_2 > 0$ | $(s^*, 1)$ | 1A | $(-\infty, \infty)$ | $s^*$ | 1 | $+$ |
| | $(0, s^*)$ | 1B | $(-\infty, \infty)$ | $s^*$ | 0 | $-$ |
| | $(s^*, 1)$ | 1C | $(-\infty, x_*)$ | $s^*$ | 1 | $+$ |
| | $(0, s^*)$ | 1D | $(-\infty, x_*)$ | $s^*$ | 0 | $-$ |
| $c_1, c_2 < 0$ | $(s^*, 1)$ | 2A | $(-\infty, \infty)$ | 1 | $s^*$ | $-$ |
| | $(0, s^*)$ | 2B | $(-\infty, \infty)$ | 0 | $s^*$ | $+$ |
| | $(s^*, 1)$ | 2C | $(-x_*, \infty)$ | 1 | $s^*$ | $-$ |
| | $(0, s^*)$ | 2D | $(-x_*, \infty)$ | 0 | $s^*$ | $+$ |
| $c_1 \le 0 < c_2$ or $c_1 < 0 = c_2$ | $(0, 1)$ | 3A | $(-\infty, \infty)$ | 0 | 1 | $+$ |
| | | 3B | $(-x'_*, x_*)$ | | | |
| | | 3C | $(-\infty, x_*)$ | | | |
| | | 3D | $(-x_*, \infty)$ | | | |
| $c_1 > 0 \ge c_2$ or $c_1 = 0 > c_2$ | $(0, 1)$ | 4A | $(-\infty, \infty)$ | 1 | 0 | $-$ |
| | | 4B | $(-x'_*, x_*)$ | | | |
| | | 4C | $(-x_*, \infty)$ | | | |
| | | 4D | $(-\infty, x_*)$ | | | |

TABLE 1. Steady state solutions (Theorem 3.1). Notation: $x'_*, x_*$ are two numbers in $(0, \infty)$.

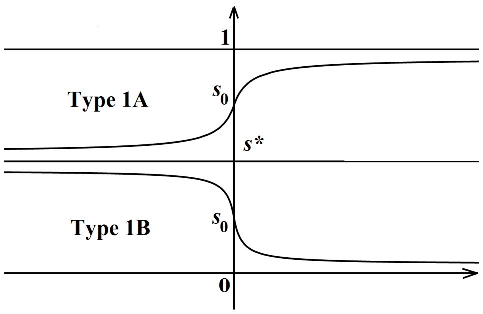


Figure 1. Types 1A & 1B.

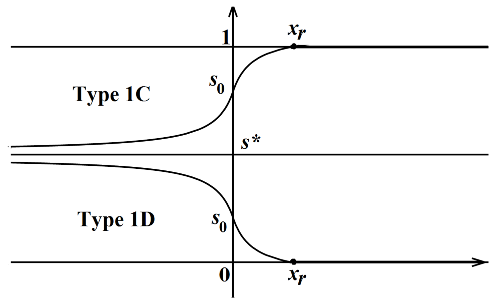


Figure 3. Types 1C & 1D.

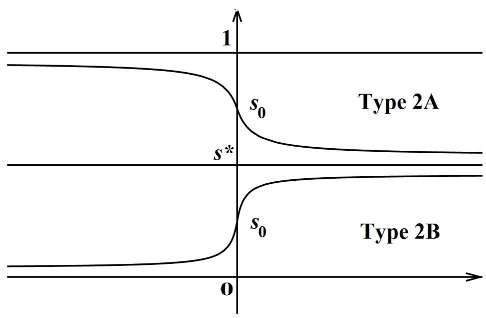


Figure 2. Types 2A & 2B.

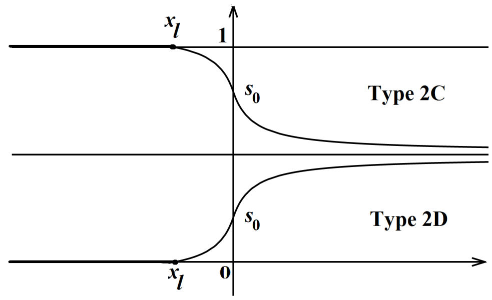


Figure 4. Types 2C & 2D.

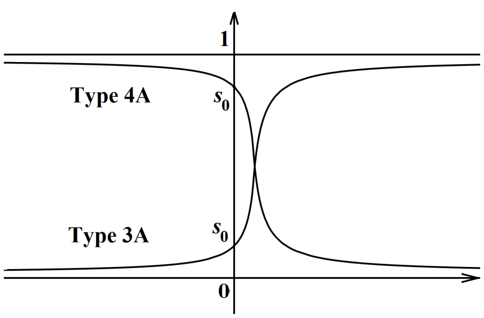


Figure 5. Types 3A & 4A.

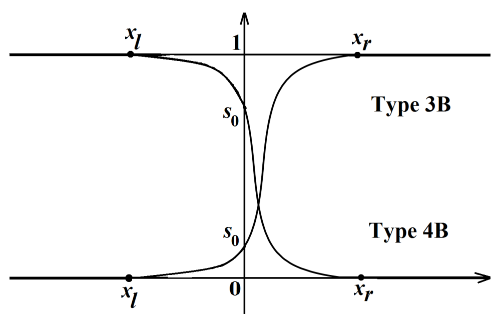


Figure 6. Types 3B & 4B.

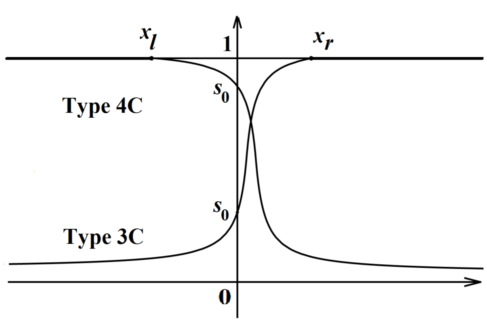


Figure 7. Types 3C & 4C.

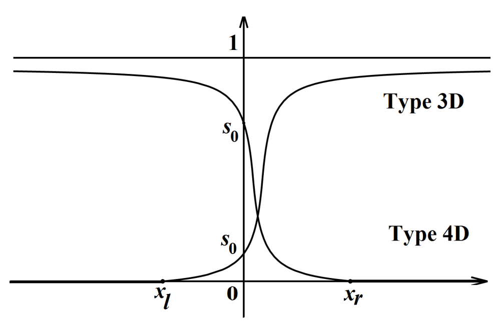


Figure 8. Types 3D & 4D.

We call solutions of types 1A–1B, 2A–2B, 3A–4A *never trivial solutions*, and solutions of types 1C–1D, 2C–2D, 3B–3D, 4B–4D *partially trivial solutions*.

*Proof of Theorem 3.1.* We consider different scenarios below.

**Case $\mathbf{c_1, c_2 > 0}$.** By (3.9), Eq. (3.10) has a unique equilibrium $s^* \in (0,1)$ and by the strict increase of $f$, we have

$$F(S) < 0 \text{ if } 0 < S < s^* \text{ and } F(S) > 0 \text{ if } s^* < S < 1. \tag{3.15}$$

Let $s_0 \in (s^*, 1)$. Under Assumptions A and B, $F'(S)$ is continuous on $(0,1)$. Consider (3.10) for $(x, S) \in \mathbb{R} \times (0,1)$. By fundamental theorems on the existence and dynamics of solutions to one-dimensional scalar differential equations (c.f. Theorems 3.3.2 and 3.3.3 in [17]), there exists a solution $S(x)$ for $x \in (-\infty, x_r)$, some $x_r \in (0, \infty]$ which satisfies $S(x) \in (s^*, 1)$ for $x \in (-\infty, x_r)$ and

$$\lim_{x\to-\infty} S(x) = s^* \quad \text{and} \quad \lim_{x\to x_r} S(x) = 1 \text{ in either case } x_r < \infty \text{ or } x_r = \infty. \tag{3.16}$$

By (3.15), $S'(x) = F(S(x)) > 0$ on $(-\infty, x_r)$, hence $S(x)$ is strictly increasing on $(-\infty, x_r)$. Let $x_\ell = -\infty$, $s_\ell = s^*$ and $s_r = 1$. We have proved (3.11), (3.12) and (3.14).

If $x_r = \infty$, we have solution of type 1A.

In case $x_r < \infty$, we extend $S(x)$ from $(-\infty, x_r)$ to $\mathbb{R}$ by defining $S(x) = 1$ for $x \geq x_r$. Then $S(x)$ satisfies (3.13). Note that at $x_r$, we have

$$\lim_{x\to x_r^-} S'(x) = \lim_{x\to x_r^-} F(S(x)) = F(1) = 0.$$

Therefore we can easily see that $S(x)$ belongs to $C^1([0,\infty))$ and satisfies (3.10) on $\mathbb{R}$. This solution is of type 1C.

For the uniqueness, if $x_r = \infty$ then $F \in C^1((0,1))$ implies that the solution is unique. Consider $x_r < \infty$. Let $S(x)$ be the above solution and $\tilde{S}(x)$ be another solution of (3.10). By uniqueness of the solution on $(-\infty, x_r)$, we have $S(x) = \tilde{S}(x)$ on $(-\infty, x_r)$. By continuity, $\tilde{S}(x_r) = S(x_r) = 1$. Note that we still have $\tilde{S}(x) \in (s^*, 1]$ for all $x \in \mathbb{R}$. Hence $\tilde{S}'(x) \geq 0$ for $x \in (x_r, \infty)$, which implies $1 \geq \tilde{S}(x) \geq \tilde{S}(x_r) = 1$ for $x \in (x_r, \infty)$. Therefore $\tilde{S} \equiv 1$ on $[x_r, \infty)$. We conclude that $S \equiv \tilde{S}$ on $\mathbb{R}$.

Now let $s_0 \in (0, s^*)$, the proof is similar and results in solutions of types 1B and 1D.

**Case $\mathbf{c_1 < 0 < c_2}$.** Note in this case that $s^*$ does not exist and $F(S) > 0$ for all $S \in (0,1)$. First, the solution $S(x) \in (0,1)$ exists on the maximal interval $(x_\ell, x_r)$ containing 0. Using the above arguments for $x > 0$ we have two possibilities for $x_r$. Similar arguments are used to deal with $x < 0$; we have two possibilities for $x_\ell$. Together these give rise to four types of solutions: 3A, 3B, 3C, 3D. We omit the details.

**Other cases.** The case $c_1, c_2 < 0$ or $c_1 > 0 > c_2$ or $c_1 = 0, c_2 \neq 0$ or $c_1 \neq 0, c_2 = 0$ can be proved in an analogous way. $\square$

Although Theorem 3.1 classifies all possible non-constant steady states, it does not indicate which type of solution we have for a given set of parameters $(c_1, c_2, s_0)$. For example, when $c_1, c_2 > 0$ and $s_0 > s^*$, one does not know whether solution is of type 1A or 1C. Below, we find sufficient conditions on $f_1$, $f_2$ and $p_c'$ that *determine* the types of solutions in Table 1. The following lemma is a building block in classifying a particular solution.

**Definition 3.2.** *We say $h(S) \sim g(S)$ as $S \to a^+$, resp. $S \to a^-$, if there exist $\delta > 0$ and $C > 0$ such that*

$$C^{-1}h(S) \leq g(S) \leq Ch(S) \text{ for all } a \leq S < a + \delta, \text{ resp. } a - \delta < S \leq a. \tag{3.17}$$

**Lemma 3.3.** *Let $S(x) \in C^1([0,\infty))$ be the solution of the following constrained IVP:*

$$S'(x) = -g(S(x)), \quad x \in (0, \infty), \tag{3.18a}$$

$$S(0) = s_0 > 0 \text{ and } S(x) \geq 0 \text{ for all } x \in (0, \infty), \tag{3.18b}$$

*where* $g(\cdot) \in C([0,\infty)) \cap C^1((0,\infty))$ *satisfies*

$$g(0) = 0, \quad g(S) > 0 \text{ for } S > 0. \tag{3.19}$$

*Assume there is a function* $h : [0,1) \to [0,\infty)$ *such that*

$$g \sim h \quad \text{as } S \to 0^+. \tag{3.20}$$

(i) *If*

$$\limsup_{S\to 0^+} h'(S) < \infty, \tag{3.21}$$

*then* $S(x)$ *is positive for all* $x \in [0,\infty)$.

(ii) *If*

$$\liminf_{S\to 0^+} (h^2(S))' > 0, \tag{3.22}$$

*then there exists* $x_* > 0$ *such that*

$$S(x) \in (0, s_0] \quad \text{for all } x \in [0, x_*) \text{ and } S(x) = 0 \quad \text{for all } x \in [x_*, \infty). \tag{3.23}$$

*Proof.* Note that the existence and uniqueness of such a solution $S(x)$ are similar to Theorem 3.1. Also, $S(x) \in [0, s_0)$ for all $x > 0$.

(i) We prove the assertion by contradiction. Suppose it does not hold true. By the local existence theorem, there is $x_0 > 0$ such that $S(x) \to 0$ as $x \to x_0^-$ and $S(x) > 0$ for all $x \in [0, x_0)$. By (3.20) and (3.19), $h(S) > 0$ for sufficiently small $S > 0$ and $\lim_{S\to 0^+} h(S) = 0$. Define $Y(x) = h(S(x))$. Then there is $0 < \delta < x_0$ such that for $x_0 - \delta \leq x < x_0$, we have

$$0 < C_1^{-1} Y(x) \leq g(S(x)) \leq C_1 Y(x), \tag{3.24}$$

$$h'(S(x)) < C_2, \tag{3.25}$$

where $C_1$, $C_2$ are positive numbers, and

$$\lim_{x\to x_0^-} Y(x) = 0. \tag{3.26}$$

For $x \in (x_0 - \delta, x_0)$:

$$\begin{aligned} Y'(x) &= h'(S)S' = -h'(S)g(S) \\ &\geq -C_2 g(S) \geq -C_2 C_1 Y = -C_3 Y, \end{aligned} \tag{3.27}$$

where $C_3 = C_1 C_2 > 0$. This yields

$$Y(x) \geq Y(x_0 - \delta) e^{-C_3(x - x_1)}, \quad x \in [x_0 - \delta, x_0). \tag{3.28}$$

Taking $x \to x_0^-$ and using (3.26), we obtain $0 \geq Y(x_0 - \delta)e^{-C_3\delta} > 0$ which is a contradiction. Therefore our assertion must hold true.

(ii) By the local existence, $S(x) > 0$ of (3.18) on a maximal interval $[0, x_*)$, where $0 < x_* \leq \infty$.

**Claim:** $x_* < \infty$.

*Proof of the Claim:* Suppose $x_* = \infty$, then $S(x) > 0$ for all $x > 0$. Since $S'(x) = -g(S(x)) < 0$, the function $S(x)$ is strictly decreasing on $[0,\infty)$. Hence $\lim_{x\to\infty} S(x) = a$ exists and is non-negative. If $a > 0$ then $S'(x) \leq -g(a)/2 < 0$ for sufficiently large $x$. Consequently, $\lim_{x\to\infty} S(x) = -\infty$, which is impossible. Therefore $a = 0$.

By (3.22) and (3.20), there exist $x_0 > 0$ and $C_0, C_1 > 0$ such that for all $x > x_0$, we have $(h^2)'|_{S(x)} \ge C_0$ and $C_1^{-1} g(S(x)) \ge h(S(x)) \ge C_1 g(S(x))$. Hence we have for $x > x_0$ that $h'(S(x)) > 0$ and

$$\big[h(S(x))\big]' = h'(S(x))S'(x) = -h'(S(x))g(S(x)) \le -C_1 h'(S(x))h(S(x)). \tag{3.29}$$

Integrating (3.29) from $x_0$ to $x$ gives

$$h(S(x)) - h(S(x_0)) \le -\frac{C_1}{2}\int_{x_0}^{x} (h^2)'|_{S(y)} dy \le -\frac{C_1 C_0}{2}(x - x_0). \tag{3.30}$$

Letting $x \to \infty$, we have, on one hand, $h(S(x)) \to h(0) = 0$, while on the other hand, $h(S(x)) \to -\infty$ due to (3.30). This is a contradiction. Therefore $x_*$ must be finite, hence the Claim is true.

Since $g \in C^1((0,1))$ we must have $S(x_*) = \lim_{x \to x_*^-} S(x) = 0$, otherwise the positive solution can be extended beyond $x_*$. Because $S'(x) \le 0$, the function $S(x)$ is decreasing, hence for $x > x_*$, $0 = S(x_*) \ge S(x) \ge 0$. Therefore $S \equiv 0$ on $[x_*, \infty)$. The proof is complete. □

The following conditions on the functions $F_1$ and $F_2$ will be referred to in our considerations:

$$\limsup_{S \to 0^+} F_1'(S) < \infty, \tag{3.31}$$

$$\liminf_{S \to 1^-} F_2'(S) > -\infty, \tag{3.32}$$

$$\liminf_{S \to 0^+} (F_1^2(S))' > 0, \tag{3.33}$$

$$\limsup_{S \to 1^-} (F_2^2(S))' < 0. \tag{3.34}$$

Note from (2.18) that the function $F(S)$ in (3.4) possesses the properties:

$$F \in C([0,1]) \cap C^1((0,1)) \text{ and } F(0) = F(1) = 0.$$

**Theorem 3.4.** *For $c_1 c_2 \ne 0$, the sufficient conditions for the solution types in Theorem 3.1 are listed in the two tables below:*

| *Parameters\Type* | *1A/2A* | *1C/2C* | *1B/2B* | *1D/2D* |
|---|---|---|---|---|
| $c_1, c_2 > 0/c_1, c_2 < 0$ | *(3.32)* | *(3.34)* | *(3.31)* | *(3.33)* |
| $s_0$ | $s_0 \in (s^*, 1)$ | | $s_0 \in (0, s^*)$ | |

| *Parameters\Type* | *3A/4A* | *3B/4B* | *3C/4C* | *3D/4D* |
|---|---|---|---|---|
| $c_1 < 0 < c_2/c_1 > 0 > c_2$ | *(3.31),(3.32)* | *(3.33),(3.34)* | *(3.31),(3.34)* | *(3.32),(3.33)* |
| $s_0$ | $s_0 \in (0,1)$ | | | |

These two tables are read in the way shown by the following examples. From the first table, second column: if $c_1, c_2 > 0$, $s_0 \in (s^*, 1)$ and (3.32) holds then the solution is of type 1A. From the second table, last column: if $c_1 > 0 > c_2$, $s_0 \in (0,1)$ and (3.32),(3.33) hold then the solution is of type 4D.

*Proof of Theorem 3.4.* Note from (2.19) that

$$|c_2|F_2(S) \pm |c_1|F_1(S) \sim F_2(S) \text{ as } S \to 1^-, \tag{3.35}$$

$$|c_1|F_1(S) \pm |c_2|F_2(S) \sim F_1(S) \text{ as } S \to 0^+. \tag{3.36}$$

**Case $\mathbf{c_1 > 0, c_2 > 0}$.** Let $s^* \in (0,1)$ be the equilibrium in (3.9).

Consider $s_0 > s^*$. From Table 1, we have $s_0 < S(x) \le 1$ and, by (3.15), $F(S(x)) \ge 0$ for all $x > 0$. By (3.35),

$$F(S) \sim F_2(S) \text{ as } S \to 1^-. \tag{3.37}$$

Define $X(x) = 1 - S(x)$. Then

$$X'(x) = -S'(x) = -F(S(x)) = -F(1 - X(x)) = -g(X(x)), \tag{3.38}$$

where $g(X) = F(1 - X)$ defined for $X \in [0, 1 - s^*]$. Then $g(0) = F(1) = 0$ and $g(X) > 0$ for all $X \in (0, 1 - s^*)$. Note that $X(0) = 1 - s_0 \in (0, 1 - s^*)$.

Let $h(X) = F_2(1 - X)$. Then by (3.37),

$$h(X) \sim g(X) \text{ as } X \to 0^+. \tag{3.39}$$

If (3.32) holds then

$$\limsup_{X\to 0^+} h'(X) = \limsup_{X\to 0^+}(-F_2'(1 - X)) = -\liminf_{S\to 1^-} F_2'(S) < \infty. \tag{3.40}$$

By Lemma 3.3(i) applied to Eq. (3.38), $X(x)$ is positive for all $x \in [0, \infty)$, which implies $S(x) = 1 - X(x)$ is less than 1 for $x \in [0, \infty)$. Therefore solution $S(x)$ is of type 1A.

If (3.34) holds then

$$\liminf_{X\to 0^+}(h^2(X))' = -\limsup_{X\to 0^+} 2F_2(1 - X)F_2'(1 - X) = -\limsup_{S\to 1^-}(F_2^2(S))' > 0. \tag{3.41}$$

By Lemma 3.3(ii) applied to Eq. (3.38), $X(x) = 0$ for all $x \ge x_*$ for some $x_* > 0$. Hence $S(x) = 1$ for all $x \ge x_*$. Therefore solution $S(x)$ of type 1C.

Consider $s_0 < s^*$. From Table 1 again, $S(x) \in [0, s_0)$ and $F(S(x)) \le 0$ for $x > 0$. We rewrite (3.10) in the form

$$S' = -\big[-F(S)\big] = -\big[c_1F_1(S) - c_2F_2(S)\big], \tag{3.42}$$

with $-F(S) > 0$ for $S \in (0, s^*)$. By (3.36),

$$-F(S) \sim F_1(S) \text{ as } S \to 0^+. \tag{3.43}$$

Applying Lemma 3.3 directly to (3.42), we have solution type 1B in the case of (3.31), and type 1D in the case of (3.33).

**Case $\mathbf{c_1 < 0, c_2 < 0}$.** With $s^* \in (0, 1)$ defined by (3.9), we have

$$F(S) < 0 \text{ if } S > s^* \text{ and } F(S) > 0 \text{ if } S < s^*.$$

Consider $s_0 > s^*$. In this case $S(x) \in (s_0, 1]$ and $F(S(x)) \le 0$ for all $x \in \mathbb{R}$.

Let $X(x) = 1 - S(-x)$. Then

$$X'(x) = S'(-x) = F(S(-x)) = F(1 - X(x)) = -g(X(x)), \tag{3.44}$$

where $g(X) = -F(1 - X)$ for $X \in [0, 1 - s^*]$. We have $X(0) = 1 - s_0 \in (0, 1 - s^*)$, $g(0) = -F(1) = 0$ and $g(X) > 0$ for all $X \in (0, 1 - s^*)$.

Let $h(X) = F_2(1 - X)$. Then by (3.35), $-F(S) \sim F_2(S)$ as $S \to 1^-$, hence $h(X) \sim g(X)$ as $X \to 0^+$.

If (3.32) holds then, again, we have (3.40). By Lemma 3.3(i) applied to Eq. (3.44), $X(x) > 0$ for $x \in [0, \infty)$ yielding $S(x) < 1$ for $x \in (-\infty, 0]$. Therefore $S(x)$ is of type 2A.

If (3.34) holds then we have (3.41). By Lemma 3.3(ii), $X(x) = 0$ for all $x \ge x_*$ for some $x_* > 0$, hence $S(x) = 1$ for all $x \le -x_*$. Thus $S(x)$ is of type 2C.

**Case $\mathbf{c_1c_2 < 0}$.** The proofs are similar when the equilibrium $s^* \in (0, 1)$ does not exist and we apply Lemma 3.3 (via simple transformations as above) to deal with both $x < 0$ and $x > 0$. □

In the case $c_1 = 0$ or $c_2 = 0$ but not both, to determine the specific type of solution for a given $s_0$, we need to consider the following four conditions in addition to (3.31), (3.32), (3.33) and (3.34):

$$\limsup_{S\to 0^+} F_2'(S) < \infty, \tag{3.45}$$

$$\limsup_{S\to 0^+} (F_2^2(S))' > 0, \tag{3.46}$$

$$\liminf_{S\to 1^-} F_1'(S) > -\infty, \tag{3.47}$$

$$\liminf_{S\to 1^-} (F_1^2(S))' < 0. \tag{3.48}$$

Similar to, but simpler than, Theorem 3.4, by applying Lemma 3.3 to equations (3.6) and (3.7), we have:

**Theorem 3.5.** *Let $c_1c_2 = 0$ with $c_1^2 + c_2^2 > 0$ and let $s_0 \in (0, 1)$. Then the sufficient conditions for each solution type of 3A–3D and 4A–4D are listed in the following table:*

| $c_1, c_2 \setminus$ *Type* | *3A/4A* | *3B/4B* | *3C/4C* | *3D/4D* |
|---|---|---|---|---|
| $c_2 > 0/c_2 < 0$ | *(3.45), (3.32)* | *(3.46), (3.34)* | *(3.45), (3.34)* | *(3.46), (3.32)* |
| $c_1 < 0/c_1 > 0$ | *(3.31), (3.47)* | *(3.33), (3.48)* | *(3.31), (3.48)* | *(3.33), (3.47)* |

## 4. Linear stability

In this section, we study the stability of a steady state solution $(S_*(x), u_1^*, u_2^*)$ of the system (2.14) with the parameters $c_1 = G_1(u_1^*)$, $c_2 = G_2(u_2^*)$ and $s_0 = S_*(0)$ being fixed throughout.

Linearizing equations (2.14b) and (2.14c) at $(S_*(x), u_1^*, u_2^*)$ yields

$$\begin{cases} \sigma_t = -v_{1x} \\ \sigma_t = v_{2x} \\ \sigma_x = \gamma_2 v_2 - \gamma_1 v_1 - \gamma\sigma, \end{cases} \tag{4.1}$$

where $\sigma(x, t), v_1(x, t)$, and $v_2(x, t)$ are the new unknown functions,

$$\gamma_i(x) = G_i'(u_k^*)F_i(S_*(x)), \quad i = 1, 2, \tag{4.2}$$

$$\gamma(x) = -G_2(u_2^*)F_2'(S)\big|_{S=S_*(x)} + G_1(u_1^*)F_1'(S)\big|_{S=S_*(x)} = -F'(S)\big|_{S=S_*(x)}, \tag{4.3}$$

with $F_i(S)$ defined by (2.13) and $F(S)$ defined by (3.4).

Roughly speaking, a solution $(\sigma, v_1, v_2)(x, t)$ of (4.1) is an approximation of the difference

$$\Big(S(x, t) - S_*(x), u_1(x, t) - u_1^*, u_2(x, t) - u_2^*\Big),$$

where $(S, u_1, u_2)(x, t)$ is a solution of (2.14) presumed to be close to the steady state $(S_*(x), u_1^*, u_2^*)$.

Let $v = v_1 + v_2$, the total velocity. From the third equation of (4.1), we have

$$v_2 = \frac{\sigma_x + \gamma_1 v + \gamma\sigma}{\gamma_1 + \gamma_2}. \tag{4.4}$$

Substituting (4.4) into the second equation of (4.1), we obtain a scalar equation for $\sigma(x, t)$:

$$\sigma_t = \Big(\frac{\sigma_x + \gamma_1 v + \gamma\sigma}{\gamma_1 + \gamma_2}\Big)_x. \tag{4.5}$$

From the first two equations of (4.1) we have $v' = 0$. Here we study the flows having zero total velocity at $x = \pm\infty$, that is, $v(-\infty, t) = v(+\infty, t) = 0$ for all $t$. Hence $v(x,t) = 0$ for all $x$ and $t$ and consequently

$$\sigma_t = \left(\frac{\sigma' + \gamma\sigma}{\gamma_1 + \gamma_2}\right)_x. \tag{4.6}$$

Denoting $d_i = G_i'(u_i^*)$, for $i = 1, 2$, and

$$\alpha(x) = \frac{1}{\gamma_1(x) + \gamma_2(x)} = \frac{1}{d_2 F_2(S_*(x)) + d_1 F_1(S_*(x))}, \tag{4.7}$$

we rewrite (4.6) as

$$\sigma_t = \big[\alpha(x)(\sigma_x + \gamma(x)\sigma)\big]_x, \tag{4.8}$$

and rewrite (4.4) as

$$v_2 = \alpha(\sigma_x + \gamma\sigma). \tag{4.9}$$

Since $v \equiv 0$, the function $v_1$ is simply

$$v_1 = -v_2. \tag{4.10}$$

Thus the system (4.1) is deduced to (4.8), (4.9) and (4.10). Since the equation (4.8) for $\sigma(x,t)$ is decoupled from $v_1, v_2$, we solve (4.8) first and then retrieve $v_1, v_2$ by (4.9) and (4.10).

Therefore we study the following Cauchy problem for $\sigma(x,t)$:

$$\begin{cases} \sigma_t = \big[\alpha(\sigma_x + \gamma\sigma)\big]_x & \text{on } \mathbb{R} \times (0,\infty), \\ \sigma\big|_{t=0} = \sigma_0(x) & \text{on } \mathbb{R}. \end{cases} \tag{4.11}$$

Below, we will focus on the case $S_*(x) \neq s^*$. We state our main results in the following Theorems 4.1 and 4.2. First is the stability for the saturation:

**Theorem 4.1.** *Assume $F_1, F_2 \in C^2((0,1))$ and let $S_*(x)$ be a steady state solution of type 1A, or 1B, 2A, 2B, 3A, 4A obtained in Theorem 3.1 and specified in Table 1. Suppose $\sigma_0(x) \in C(\mathbb{R})$ satisfies*

$$M_0 \stackrel{\text{def}}{=\!=} \sup_{x\in\mathbb{R}} \left|\frac{\sigma_0(x)}{F(S_*(x))}\right| < \infty. \tag{4.12}$$

(i) *Then there exists a solution $\sigma(x,t) \in C^{2,1}_{x,t}(\mathbb{R}\times(0,\infty)) \cap C(\mathbb{R}\times[0,\infty))$ of the Cauchy problem (4.11).*

(ii) *This solution is unique in the class of solutions $\sigma(x,t) \in C^{2,1}_{x,t}(\mathbb{R}\times(0,\infty)) \cap C(\mathbb{R}\times[0,\infty))$ that satisfy the following growth condition:*

$$\sup_{(x,t)\in\mathbb{R}\times[0,T]} \left|\frac{\sigma(x,t)}{F(S_*(x))}\right| < \infty \quad \textit{for any } T > 0. \tag{4.13}$$

(iii) *Furthermore,*

$$\sup_{(x,t)\in\mathbb{R}\times[0,\infty)} \left|\frac{\sigma(x,t)}{F(S_*(x))}\right| \le M_0, \tag{4.14}$$

*consequently,*

$$\lim_{|x|\to\infty} \sigma(x,t) = 0 \quad \textit{for any } t \ge 0, \tag{4.15}$$

*and*

$$\lim_{t\to\infty} \left(\sup_{x\in\mathbb{R}} \left|\frac{\sigma(x,t)}{F(S_*(x))}\right|\right) \textit{ exits and belongs to } [0, M_0]. \tag{4.16}$$

**Note.** Statement (4.15) shows the asymptotic behavior of the solution as $|x|\to\infty$ for each fixed $t\ge 0$, while statement (4.16) shows its asymptotic behavior as $t\to\infty$ uniformly in $x$.

Second, we have the stability for velocities:

**Theorem 4.2.** *Let $S_*(x)$ and $\sigma_0(x)$ be as in Theorem 4.1 and $F_1,F_2\in C^3((0,1))$. For any $L>0$, one has*

$$\sup_{|x|\le L}|v_1(x,t)| = \sup_{|x|\le L}|v_2(x,t)| \le M_L\Big(1+\frac{1}{\sqrt{t}}\Big)\sup_{x\in\mathbb{R}}|\sigma_0(x)| \tag{4.17}$$

*for any $t>0$, where $M_L>0$ depends on $L$ and $S_*(x)$.*

The rest of the paper is mainly devoted to proving Theorems 4.1 and 4.2.

Recall that we are under Assumptions A and B. Then $d_1,d_2>0$ and

$$\alpha(x)>0 \quad \text{for all } x\in\mathbb{R}. \tag{4.18}$$

Hence (4.8) is a parabolic equation. We start with transforming this equation to a more convenient form, as far as the maximum principle is concerned. Let

$$w(x,t)=\sigma(x,t)e^{\int_0^x\gamma(y)dy}. \tag{4.19}$$

We have

$$\alpha w_x = (\alpha\sigma_x+\sigma\gamma\alpha)e^{\int_0^x\gamma dy}, \tag{4.20}$$

$$w_t=\sigma_t e^{\int_0^x\gamma dy} = e^{\int_0^x\gamma dy}(\alpha\sigma_x+\alpha\gamma\sigma)_x. \tag{4.21}$$

By product rule,

$$w_t=\Big(e^{\int_0^x\gamma dy}(\alpha\sigma_x+\alpha\gamma\sigma)\Big)_x-(\alpha\sigma_x+\alpha\gamma\sigma)\gamma e^{\int_0^x\gamma dy}=(\alpha w_x)_x-\gamma(\alpha w_x).$$

Hence Eq. (4.8) deduces to

$$w_t-(\alpha w_x)_x+\gamma(\alpha w_x)=0. \tag{4.22}$$

Also, from (4.9) and (4.20), we have

$$v_2=\alpha w_x e^{-\int_0^x\gamma(y)dy}. \tag{4.23}$$

We will proceed by studying (4.22) first and then drawing conclusions for the solutions $\sigma,v_1,v_2$ via relations (4.19), (4.23) and (4.10). For the first step, we study equation (4.22) by its own rights with non-specific functions $\alpha$ and $\gamma$, that is, they are general functions *not* necessarily defined by (4.7) and (4.3). Such generality is assumed from here up to Theorem 4.7.

Therefore, we consider the Cauchy problem:

$$\begin{cases} \mathcal{L}w \overset{\text{def}}{=\!=} w_t-(\alpha w_x)_x+\gamma(\alpha w_x)=0 & \text{in } \mathbb{R}\times(0,\infty),\\ w\big|_{t=0}=w_0(x) & \text{in } \mathbb{R}, \end{cases} \tag{4.24}$$

where $w_0(x)$ is a given initial data, and $\alpha(x)>0$, $\gamma(x)$ are non-specific functions.

For the maximum principle, we need to construct barrier functions. As in [13], we seek for a positive super-solution $W(x,t)$, i.e., $\mathcal{L}W\ge 0$, of the form:

$$W(x,t)\overset{\text{def}}{=\!=}[\varphi(x)+kt]e^{\beta t} \quad \text{for } x\in\mathbb{R},\ t\ge 0, \tag{4.25}$$

where $k,\beta$ are non-negative numbers, and the function $\varphi(x)$ will be decided later. We have

$$\mathcal{L}W=\Big[k+\beta\varphi+\beta kt-(\alpha\varphi')'+\gamma\alpha\varphi'\Big]e^{\beta t}.$$

Setting $Y(x) = \alpha(x)\varphi'(x)$, we obtain

$$\mathcal{L}W = (-Y' + \gamma Y + k) + \beta\varphi + \beta kt.$$

Because $\beta, k \geq 0$, in order to have $W > 0$ and $\mathcal{L}W \geq 0$ it is sufficient that $\varphi(x) > 0$ and $Y(x)$ satisfies

$$-Y' + \gamma Y + k = 0.$$

Solving this ODE for $Y(x)$ we select

$$Y(x) = k\int_0^x e^{\int_y^x \gamma(z)dz}dy. \tag{4.26}$$

Since $\varphi'(x) = Y(x)/\alpha(x)$, we obtain

$$\varphi(x) = \varphi_0 + \int_0^x \frac{Y(y)}{\alpha(y)}dy, \quad \text{where } \varphi_0 \text{ is chosen to be positive.} \tag{4.27}$$

Since $xY(x) > 0$ and $\alpha(x) > 0$, the function $\varphi(x)$ defined by (4.27) satisfies $\varphi(x) > 0$ for all $x \in \mathbb{R}$.

We summarize the above discussion into the following lemma:

**Lemma 4.3.** *Given any positive constants $k$, $\beta$ and $\varphi_0$, the function $W(x,t)$ in (4.25) is a positive super-solution of (4.22), where $\varphi(x) \in C^2(\mathbb{R})$ is the positive function explicitly defined by (4.27) and (4.26).*

Hereafter we fix $\varphi_0 = k = \beta = 1$.

We obtain the following maximum principle for solutions of (4.22):

**Theorem 4.4.** *Let $T > 0$. Suppose $w \in C^{2,1}_{x,t}(\mathbb{R}\times(0,T]) \cap C(\mathbb{R}\times[0,T])$ solves (4.22) and is bounded, i.e.,*

$$|w(x,t)| \leq M \quad \text{for all } x \in \mathbb{R} \text{ and } t \in [0,T],$$

*for some constant $M > 0$. Assume the function $\varphi(x)$ defined in (4.27) satisfies*

$$\lim_{|x|\to\infty} \varphi(x) = \infty. \tag{4.28}$$

*Then*

$$\sup_{(x,t)\in\mathbb{R}\times[0,T]} |w(x,t)| \leq \sup_{x\in\mathbb{R}} |w(x,0)|.$$

*Proof.* The proof is standard by using the barrier function $W(x,t)$ in Lemma 4.3. For the sake of completeness, it is presented here.

Fix $\mu > 0$ and define the auxiliary function

$$u(x,t) = w(x,t) - \mu W(x,t), \quad x \in \mathbb{R},\ t \geq 0. \tag{4.29}$$

We have $u \in C^{2,1}_{x,t}(\mathbb{R}\times(0,T]) \cap C(\mathbb{R}\times[0,T])$ and

$$\mathcal{L}u = \mathcal{L}w - \mu\mathcal{L}W = -\mu\mathcal{L}W \leq 0.$$

Fix $y \in \mathbb{R}$. Let $L > |y|$ and define $U_{L,T} = (-L,L)\times(0,T]$ and its parabolic boundary $\Gamma_{L,T} = \big([-L,L]\times\{0\}\big) \cup \big(\{\pm L\}\times[0,T]\big)$. Applying the maximum principle to function $u$ and domain $U_{L,T}$, we get

$$\max_{\bar U_{L,T}} u = \max_{\Gamma_{L,T}} u. \tag{4.30}$$

For any $x \in \mathbb{R}$:

$$u(x,0) = w(x,0) - \mu W(x,0) = w(x,0) - \mu\varphi(x) \leq w(x,0). \tag{4.31}$$

For $|x| = L$ and $0 \le t \le T$:

$$\begin{aligned} u(x,t) &= w(x,t) - \mu W(x,t) \le w(x,t) - \mu(\varphi(x) + kt)e^{\beta t} \\ &\le w(x,t) - \mu\varphi(x) \le M - \mu\varphi(x). \end{aligned} \tag{4.32}$$

Since $\lim_{|x|\to\infty} \varphi(x) = \infty$, by taking $L$ sufficiently large, we have from (4.31) and (4.32) that

$$u(x,t) \le \sup_{x\in\mathbb{R}} w(x,0), \quad \text{for all } (x,t) \in \Gamma_{L,T}. \tag{4.33}$$

It follows (4.30) and (4.33) that

$$u(y,t) \le \sup_{x\in\mathbb{R}} w(x,0) \quad \text{for all } 0 \le t \le T. \tag{4.34}$$

Letting $\mu \to 0$ yields

$$w(y,t) \le \sup_{x\in\mathbb{R}} w(x,0) \quad \text{for all } 0 \le t \le T.$$

To complete the proof, we repeat the above arguments for $(-w)$. □

**Remark 4.5.** *The above maximum principle can be proved for a larger class of "slowly growing" solutions, depending on asymptotics of $\varphi(x)$ as $|x| \to \infty$. Having in mind our particular application, we only formulated Theorem 4.4 for the restrictive class of bounded solutions.*

Applying this maximum principle and the classical local existence result we obtain:

**Theorem 4.6.** *Assume (4.28). Suppose $\alpha \in C^2(\mathbb{R})$ and $\gamma \in C^1(\mathbb{R})$. Let $w_0(x)$ be a continuous and bounded function on $\mathbb{R}$. Then:*

(i) *There exists a solution $w(x,t) \in C^{2,1}_{x,t}(\mathbb{R} \times (0,\infty)) \cap C(\mathbb{R} \times [0,\infty))$ of the Cauchy problem (4.24).*

(ii) *This solution is unique in class of locally (in time) bounded solutions, i.e., the class of solutions $w(x,t)$ such that*

$$\sup_{(x,t)\in\mathbb{R}\times[0,T]} |w(x,t)| < \infty \quad \textit{for any } T > 0. \tag{4.35}$$

(iii) *Furthermore,*

$$\sup_{(x,t)\in\mathbb{R}\times[0,\infty)} |w(x,t)| \le \sup_{x\in\mathbb{R}} |w_0(x)|. \tag{4.36}$$

*Proof.* Rewrite the PDE in (4.24) as

$$\alpha w_{xx} + (\alpha' - \gamma\alpha)w_x - w_t = 0$$

for convenient comparison to equation (1.1) in [13]. Let $T$ be any positive number. It is easy to check that all conditions in Theorem 4, p.474 of [13] are satisfied and hence there exists a solution $w^{(T)}$ of (4.24) on the time interval $[0,T]$ and this solution is bounded. By the maximum principle in Theorem 4.4,

$$|w^{(T)}(x,t)| \le \max_{x\in\mathbb{R}} |w_0(x)|, \quad x \in \mathbb{R},\ 0 \le t \le T. \tag{4.37}$$

The uniqueness of $w^{(T)}$ among bounded solutions on the time interval $[0,T]$ follows from the mentioned maximum principle and linearity of operator $\mathcal{L}$.

For an integer $n \ge 1$, let $w^{(n)}(x,t)$ be the above unique solution on $[0,n]$. We define $w(x,t)$ for $t \in [0,\infty)$ as follows: for each $t > 0$, let

$$w(x,t) = w^{(n)}(x,t),$$

where $n$ is an integer greater than $t$. By the uniqueness of $w^{(n)}$, this $w(x,t)$ does not depend on $n$ and hence is well-defined. Clearly, $w(x,t)$ is the a solution of the Cauchy problem (4.24) (for all $t>0$) and, thanks to (4.37), we have (4.36).

The uniqueness of this solution $w(x,t)$ among solutions described in (ii) comes from the uniqueness of the bounded solution on $[0,T]$ for any $T>0$. □

For the spatial derivative, we apply Bernstein's a priori estimate technique (c.f. [13]) to bound $w'(x,t)$ for $x$ in a bounded interval and all $t>0$.

**Theorem 4.7.** *Let $\alpha(x)\in C^3(\mathbb{R})$, $\gamma(x)\in C^2(\mathbb{R})$ such that (4.28) holds. Suppose $w(x,t)$ is a solution of the Cauchy problem (4.24). Then for any $L>0$,*

$$|w_x(x,t)|\le M_L\Big(1+\frac{1}{\sqrt{t}}\Big)\max_{x\in\mathbb{R}}|w(x,0)| \quad \text{on } [-L,L]\times(0,\infty), \tag{4.38}$$

*where $M_L>0$ depends on $L,\alpha,\gamma$.*

*Proof.* The conditions on $\alpha$ and $\gamma$ imply $w_x\in C^{2,1}_{x,t}(\mathbb{R}\times(0,\infty))$.

Let $\delta>0$ and $G_{L,\delta}=[-L-1,L+1]\times(\delta,1+\delta]$ and define the auxiliary function

$$\tilde w(x,t)=\tau\Phi(x)(w_x)^2+Nw^2,$$

where $N>0$ will be chosen later and

$$\tau=t-\delta\in(0,1],\quad \Phi(x)=((L+1)^2-x^2)^2.$$

Then following the calculations in Theorem 1, p.450 in [13] we have

$$\begin{aligned}\mathcal{L}\tilde w &\le 2\tau\Phi\alpha' w_x w_{xx}+2\tau\Phi(\alpha'-\gamma\alpha)'(w_x)^2-2\tau\Phi\alpha(w_{xx})^2\\ &\quad-(\tau\mathcal{L}(\Phi)-\Phi)(w_x)^2-4\tau\alpha\Phi' w_x w_{xx}-2N\alpha w_x^2\\ &=2\tau\Phi\Big[\alpha' w_x w_{xx}-\alpha(w_{xx})^2-2\Phi' w_x w_{xx}\Big]\\ &\quad+\Big[2\tau\Phi(\alpha'-\gamma\alpha)'-(\tau\mathcal{L}(\Phi)-\Phi)-2N\alpha\Big](w_x)^2.\end{aligned}$$

By Cauchy's inequality and grouping terms, we have

$$\begin{aligned}\mathcal{L}\tilde w &\le 2\tau\Phi\Big[-\frac{\alpha}{2}(w_{xx})^2+\frac{\alpha'^2}{\alpha}w_x^2+\frac{4\Phi'^2}{\alpha}w_x^2\Big]\\ &\quad+\Big[2\tau\Phi(\alpha'-\gamma\alpha)'-(\tau\mathcal{L}(\Phi)-\Phi)-2N\alpha\Big](w_x)^2\\ &\le\Big\{2\tau\Phi\Big[\frac{\alpha'^2}{\alpha}+\frac{4\Phi'^2}{\alpha}+(\alpha'-\gamma\alpha)'\Big]-(\tau\mathcal{L}(\Phi)-\Phi)-2N\alpha\Big\}(w_x)^2.\end{aligned}$$

Since $\tau\in[0,1]$ and functions $\Phi,\alpha\in C^2$ and $\gamma\in C^1$, we obtain

$$\mathcal{L}\tilde w\le\big\{M-2N\alpha_0\big\}(w_x)^2, \tag{4.39}$$

where $M>0$ and $\alpha_0=\min\{\alpha(x):|x|\le L+1\}>0$. Hence there is $N$ sufficiently large depending on $L,\alpha,\alpha',(\alpha'-\gamma\alpha)$ such that

$$\mathcal{L}\tilde w\le 0 \quad \text{on } G_{L,\delta}.$$

By the standard maximum principle for bounded domains,

$$\max_{\bar G_{L,\delta}}\tilde w=\max\big\{\tilde w(x,t):|x|\le L+1, t=\delta \text{ or } |x|=L+1, \delta\le t\le\delta+1\big\}.$$

Note that $\tau\Phi(x) = 0$ when $t = \delta$ or $|x| = L+1$. Hence for $(x,t) \in \bar{G}_{L,\delta}$,

$$\begin{aligned}\tilde{w}(x,t) &\le N \max\{w^2(x,\delta), |x| \le L+1\}\\ &\quad + N\max\{w^2(x,t), |x| = L+1, t \in [\delta, 1+\delta]\}.\end{aligned}$$

Applying the maximum principle in Theorem 4.4 to $w$ we have

$$\tilde{w}(x,t) \le 2N \max_{x\in\mathbb{R}} w^2(x,0) \quad \text{for } |x| \le L+1,\ t \in [\delta, \delta+1]. \tag{4.40}$$

Let $|x| \le L$, then $\Phi(x) \ge 1$. If $t \in (0,1]$, let $\delta = t/2$, then $t = 2\delta \in [\delta, \delta+1]$, and hence by (4.40):

$$\frac{t}{2} w_x(x,t)^2 = (t-\delta)\Phi(x) w_x(x,t)^2 \le \tilde{w}(x,t) \le 2N \max_{x\in\mathbb{R}} w^2(x,0).$$

Thus we obtain (4.38) for $t \in (0,1]$.

If $t > 1$, let $\delta = t - 1/2$, then $t = \delta + 1/2 \in [\delta, \delta+1]$ and, again, by (4.40):

$$\frac{1}{2} w_x(x,t)^2 = (t-\delta)\Phi(x) w_x(x,t)^2 \le \tilde{w}(x,t) \le 2N \max_{x\in\mathbb{R}} w^2(x,0).$$

Therefore (4.38) is proved for $t > 1$. The proof is complete. □

We now return to the Cauchy problem (4.24) with the functions $\alpha$ and $\gamma$ specifically defined by (4.7) and (4.3). To apply Theorems 4.6 and 4.7 in this case, we need to check the essential condition (4.28) on function $\varphi(x)$. First, we calculate $\varphi(x)$ explicitly in terms of the steady state solution $S_*(x)$.

Recall that $S_*(x)$ is strictly monotone and $F(S)$ has only one sign for $S$ in the range of $S_*(x)$. We use the change of variable $\xi = S_*(z)$ for $z \in \mathbb{R}$. Then

$$d\xi = S_*'(z)dz = F(S_*(z))dz = F(\xi)dz,$$

$$\gamma(z) = c_1F_1'(S_*(z)) - c_2F_2'(S_*(z)) = c_1F_1'(\xi) - c_2F_2'(\xi) = -F'(\xi).$$

We have

$$e^{\int_y^x \gamma(z)dz} = e^{\int_{S_*(y)}^{S_*(x)} -\frac{F'(\xi)}{F(\xi)} d\xi} = \frac{|F(S_*(y))|}{|F(S_*(x))|} = \frac{F(S_*(y))}{F(S_*(x))}. \tag{4.41}$$

Using formula (4.41), we find that

$$\begin{aligned}Y(x) &= \int_0^x e^{\int_y^x \gamma(z)dz} dy = \int_0^x \frac{F(S_*(y))}{F(S_*(x))} dy\\ &= \frac{1}{F(S_*(x))}\int_0^x S_*'(y)dy = \frac{S_*(x) - s_0}{F(S_*(x))}.\end{aligned}$$

Therefore $\varphi(x)$ in (4.27) is explicitly expressed as

$$\varphi(x) = 1 + \int_0^x \frac{[S_*(z) - s_0]\cdot[d_1F_1(S_*(z)) + d_2F_2(S_*(z))]}{F(S_*(z))} dz. \tag{4.42}$$

**Proposition 4.8.** *Let $S_*(x)$ be a steady state solution as in Theorem 4.1. Let $\alpha(x)$ and $\gamma(x)$ be defined by (4.7) and (4.3), respectively. Then Condition (4.28) holds true.*

*Proof.* **Case $c_1c_2 \ne 0$.** We consider solution $S_*(x)$ of Type 1A first when $c_1, c_2 > 0$, $s_0 = S_*(0) > s^*$. We have from (4.42) that

$$\lim_{x\to\infty} \varphi(x) - 1 = \int_0^\infty \frac{[S_*(z) - s_0][d_1F_1(S_*(z)) + d_2F_2(S_*(z))]}{F(S_*(z))} dz$$

By the fact $\lim_{x\to\infty} S_*(x) = 1$ and property (2.19) we have

$$\lim_{x\to\infty} \frac{[S_*(x)-s_0][d_1F_1(S_*(x))+d_2F_2(S_*(x))]}{F(S_*(x))}$$
$$= \lim_{x\to\infty} \frac{[S_*(x)-s_0]\Big[d_1\frac{F_1(S_*(x))}{F_2(S_*(x))}+d_2\Big]}{c_2 - c_1\frac{F_1(S_*(x))}{F_2(S_*(x))}} = \frac{d_2(1-s_0)}{c_2} > 0.$$

Therefore,

$$\int_0^\infty \frac{[S_*(z)-s_0][d_1F_1(S_*(z))+d_2F_2(S_*(z))]}{F(S_*(z))}dz = \infty,$$

which implies $\lim_{x\to\infty}\varphi(x) = \infty$. Now

$$\lim_{x\to-\infty}\varphi(x) - 1 = \int_{-\infty}^0 \frac{[s_0 - S_*(z)][d_1F_1(S_*(z))+d_2F_2(S_*(z))]}{F(S_*(z))}dz.$$

When $z\to-\infty$, we have $S_*(z)\to s^*$, $F(S_*(z))\to F(s^*) = 0$ and

$$[s_0 - S_*(z)][d_1F_1(S_*(z))+d_2F_2(S_*(z))] \to [s_0 - s^*][d_1F_1(s^*)+d_2F_2(s^*] > 0.$$

Therefore $\lim_{x\to-\infty}\varphi(x) = \infty$ provided

$$\int_{-\infty}^0 \frac{dz}{F(S_*(z))} = \infty. \tag{4.43}$$

It suffices to prove (4.43). We write explicitly

$$F(S_*(z)) = c_2F_2(S_*(z)) - c_1F_1(S_*(z))$$
$$= \frac{[c_2f_1(S_*(z)) - c_1f_2(S_*(z))] - [c_2f_1(s^*) - c_1f_2(s^*)]}{p_c'(S_*(z))f_1(S_*(z))f_2(S_*(z))}.$$

Applying the Mean Value Theorem to the numerator gives

$$F(S_*(z)) = \frac{c_2f_1'(\xi) - c_1f_2'(\xi)}{p_c'(S_*(z))f_1(S_*(z))f_2(S_*(z))}(S_*(z) - s^*)$$

where $\xi$ lies between $S_*(z)$ and $s^*$. This implies

$$\frac{1}{F(S_*(z))} = \frac{p_c'(S_*(z))f_1(S_*(z))f_2(S_*(z))}{[c_2f_1'(\xi) - c_1f_2'(\xi)]}\cdot\frac{1}{S_*(z)-s^*}.$$

Since $c_1, c_2 > 0$ and $f_1, f_2 \in C^1([s^*, s_0])$ with $f_1' > 0$, $f_2' < 0$, we have

$$C_1^{-1} \le c_2f_1'(S) - c_1f_2'(S) \le C_1 \quad \text{for all } S\in[s^*, s_0],$$

for some $C_1 \in (0,\infty)$. Also,

$$C_2^{-1} \le p_c'(S), f_1(S), f_2(S) \le C_2 \quad \text{for all } S\in[s^*, s_0],$$

for some $C_2\in(0,\infty)$. Hence

$$\int_{-\infty}^0 \frac{dz}{F(S_*(z))} \ge \int_{-\infty}^0 \frac{C_1^{-1}C_2^{-3}}{S_*(z)-s^*}dz = \infty.$$

This proves (4.43) and completes the proof for Type 1A. The proofs of (4.28) for Type 1B, 2A, 2B, 3A and 4A are similar and are omitted.

**Case** $c_1c_2 = 0$**,** $c_1^2 + c_2^2 > 0$**.** Consider solution $S_*(x)$ of Type 3A when $c_2 > 0$. On one hand, since $\lim_{x\to\infty} S_*(x) = 1$ and

$$\begin{aligned}&\lim_{x\to\infty} \frac{[S_*(x) - s_0][d_1F_1(S_*(x)) + d_2F_2(S_*(x))]}{F(S_*(x))}\\ &= \lim_{x\to\infty} \frac{[S_*(x) - s_0]}{c_2}\Big[d_1\frac{F_1(S_*(x))}{F_2(S_*(x))} + d_2\Big] = \frac{d_2(1 - s_0)}{c_2} > 0,\end{aligned}$$

we still have $\lim_{x\to\infty} \varphi(x) = \infty$. On the other hand, $\lim_{x\to-\infty} S_*(x) = 0$ and

$$\begin{aligned}&\lim_{x\to-\infty} \frac{[s_0 - S_*(x)][d_1F_1(S_*(x)) + d_2F_2(S_*(x))]}{F(S_*(x))}\\ &= \lim_{x\to-\infty} \frac{s_0}{c_2}\Big[d_1\frac{F_1(S_*(x))}{F_2(S_*(x))} + d_2\Big] = \infty,\end{aligned}$$

then we have $\lim_{x\to\infty} \varphi(x) = \infty$.

The proofs for $S_*(x)$ of type 3A ($c_1 < 0$) and type 4A ($c_2 < 0$ or $c_1 > 0$) are similar. □

We are now ready to prove Theorems 4.1 and 4.2.

**Proof of Theorem 4.1**. Using expression (4.41), we have for $x \in \mathbb{R}$ that

$$\begin{aligned}w(x,t) &= \sigma e^{\int_0^x \gamma(y)dy} = \sigma(x,t)\frac{F(s_0)}{F(S_*(x))},\\ \sigma(x,t) &= w(x,t)\frac{F(S_*(x))}{F(s_0)},\end{aligned} \tag{4.44}$$

where $s_0 = S_*(0)$. With these two relations, the assertions in (i), (ii) and (4.14) of (ii) follow directly from (i), (ii) and (iii) of Theorem 4.6, noting that condition (4.28) is met by the virtue of Proposition 4.8.

By (4.14),

$$|\sigma(x,t)| \le M_0|F(S_*(x))| \quad \forall x \in \mathbb{R},\ \forall t \ge 0. \tag{4.45}$$

Since $S_*(x) \to s^*$ or 1 or 0 as $x \to \pm\infty$ and and $F$ is continuous on $[0,1]$ with $F(s^*) = F(0) = F(1) = 0$, we have $\lim_{|x|\to\infty} F(S_*(x)) = 0$. Then (4.15) follows this and (4.45).

For each $t \ge 0$ and $T > 0$, applying the maximum principle in Theorem 4.4 to interval $[t, t+T]$ in place of $[0,T]$, we have that the mapping

$$t \mapsto \sup_{x\in\mathbb{R}}\Big|\frac{\sigma(x,t)}{F(S_*(x))}\Big| = \frac{\sup_{x\in\mathbb{R}}|w(x,t)|}{|F(s_0)|}$$

is a decreasing function from $[0,\infty)$ to $[0, M_0]$, and hence assertion (4.16) follows. □

**Proof of Theorem 4.2**. Since $v_1 = -v_2$, it suffices to give proof for $v_2$. Note from (4.23) and (4.41) that

$$v_2 = \alpha w_x\frac{F(S_*(x))}{F(s_0)}.$$

Note that for $|x| \le L$ there is $C_0 > 0$ depending on $L$ such that

$$C_0^{-1} \le \alpha(x), |F(S_*(x))| \le C_0. \tag{4.46}$$

Therefore, we have for $|x| \le L$ that

$$|v_2(x,t)| \le C_L|w_x(x,t)|, \quad t > 0, \tag{4.47}$$

and

$$|w(x,0)| \le C_L|\sigma(x,0)|, \tag{4.48}$$

where $C_L > 0$. Then (4.17) results directly from (4.38). □

For the special case of constant steady state $S_*(x) \equiv s^* \in (0,1)$ with either $c_1c_2 \neq 0$ or $c_1 = c_2 = 0$, we have $\alpha \equiv \alpha_* > 0$ and $\gamma \equiv \gamma_* \in \mathbb{R}$. Equation (4.8) simply is

$$\sigma_t = \alpha_*(\sigma_{xx} + \gamma_*\sigma_x). \tag{4.49}$$

We make the change of dependent variable

$$u(x,t) = e^{cx+\alpha_* c^2 t}\sigma(x,t) \text{ with } c = \gamma_*/2,$$

then $u_t - \alpha_* u_{xx} = 0$, that is, $u(x,t)$ satisfies the heat equation. Thus one can obtain Theorems 4.1 and 4.2 for this case with the weight $e^{cx}$ instead of $1/F(S_*(x))$. However, we quickly state and prove the following result directly for $\sigma(x,t)$ without weights.

**Theorem 4.9.** *Let $\sigma_0(x)$ be a continuous and bounded function on $\mathbb{R}$. Then:*

(i) *There exists a unique bounded solution $\sigma(x,t) \in C^{2,1}_{x,t}(\mathbb{R}\times(0,\infty)) \cap C(\mathbb{R}\times[0,\infty))$ of the Cauchy problem*

$$\begin{cases} \sigma_t = \alpha_*(\sigma_{xx} + \gamma_*\sigma_x) & x \in \mathbb{R}, \quad t \in (0,\infty), \\ \sigma(x,0) = \sigma_0(x) & x \in \mathbb{R}. \end{cases} \tag{4.50}$$

(ii) *This solution $\sigma(x,t)$ satisfies the following maximum principle*

$$\sup_{(x,t)\in\mathbb{R}\times[0,\infty)} |\sigma(x,t)| \le \sup_{x\in\mathbb{R}} |\sigma_0(x)|, \tag{4.51}$$

*and its corresponding velocities satisfy for any $L > 0$ that*

$$\sup_{|x|\le L} |v_1(x,t)| = \sup_{|x|\le L} |v_2(x,t)| \le M_L\Big(1 + \frac{1}{\sqrt{t}}\Big) \sup_{x\in\mathbb{R}} |\sigma_0(x)| \tag{4.52}$$

*for all $t \in (0,\infty)$, where $M_L > 0$ depends on $L$.*

(iii) *Assume additionally that $\sigma_0'(x)$ is a continuous and bounded function on $\mathbb{R}$. Then*

$$\sup_{x\in\mathbb{R}} |v_1(x,t)| = \sup_{x\in\mathbb{R}} |v_2(x,t)| \le M \sup_{x\in\mathbb{R}} (|\sigma_0'(x)| + |\sigma_0(x)|). \tag{4.53}$$

*Proof.* The statements (i) and (ii) are results from Theorems 4.6 and 4.7 applied to the Cauchy problem (4.24) with $\alpha \equiv \alpha_*$ and $\gamma \equiv -\gamma_*$.

Note that $\sigma'$ satisfies (4.49), hence the maximum principle in part (ii) applies to $\sigma'$, thus yielding

$$\sup_{(x,t)\in\mathbb{R}\times[0,\infty)} |\sigma_x(x,t)| \le \sup_{x\in\mathbb{R}} |\sigma_0'(x)|. \tag{4.54}$$

Since $v_1 = -v_2 = -\alpha_*(\sigma_x + \gamma_*\sigma)$, (4.53) follows (4.51) and (4.54). □

**Remark 4.10.** *Above, we studied the linear stability of only never trivial steady state solution. Our analysis is based on the specific properties of this type of solution. This study, however, does not include other partially trivial steady state solutions obtained in Section 3. For those steady states, the linearized system (4.1) may not be defined for all $x$ (see the function $\gamma(x)$), and the coefficient function $\alpha(x)$ in the corresponding parabolic equation (4.8) has singularity at finite point $x$. We will deal with this problem in our future work.*

**Remark 4.11.** *In this paper, we assume that the limiting values $S = 0, 1$ can be reached. However, our results can be easily restated to cover the case when we have restriction $\underline{S} \le S \le \overline{S}$, where $\underline{S}$ and $\overline{S}$ are two threshold values in $(0,1)$.*

**Remark 4.12.** *Although sixteen types of steady states obtained in section 3 are mathematically valid, not all of them have been observed in experimental or field data. It is interesting to know whether all of them are physically feasible or which are more relevant than the others.*

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